Notes for the XVIII Escuela Venezolana de Matematicas
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INTERACTING PARTICLE SYSTEMS: RENORMALIZATION AND MULTI-SCALE ANALYSIS

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Chapter 1. Two basic examples and some tools.

1.1 Introduction.

This course is oriented to introduce the audience to a circle of ideas which are basic for very powerful methods, known as "block arguments", "multiscale analysis", and "renormalization transformation". They are used in a big variety of areas ranging from mathematics, classical and quantum physics, chemistry and biology, and, depending on a group or tradition, may be found under several different names.

We will be restricted to the mathematical side of the theory, in particular to its applications in probabilistic context, focusing on a special type of random spatial processes, called percolative systems.

Percolative systems form a large class of random spatial processes with a huge number of interacting components, where interesting "global" (or macroscopic) phenomena can be expressed in a

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natural way in terms of paths of local (or microscopic) events that "percolate" through space (or space-time). Standard examples are spatial epidemics models, models of porous materials, ad-hoc wireless networks etc. Less obvious (but at least as important) examples are statistical mechanics models like magnetization or localization of waves, where percolation plays a more subtle role. The models usually have several parameters and an important problem is whether there is a phase transition (a dramatic change of the global behavior at some "critical" choice of the parameters), and how the system behaves at and near the critical point. So-called renormalization tools and scaling ideas are key ingredients to handle such problems.

All renormalization group studies have in common the idea of re-expressing the parameters which define a model in terms of some other set of parameters, which presumably have simpler interaction, at the same time keeping unchanged those properties of the model, which are of interest. This is usually achieved via a delicate coarse-graining procedure (sometimes also called spin-block transformation). Informally speaking, it means that a new system is constructed whose components correspond to large space-time blocks in the original system. This rescaling can be repeated so that we get level 1 blocks, level 2 blocks etc. Interesting phenomena in the original system can often be expressed in terms of the simultaneous occurrence, at all levels, of a related phenomenon. There are many choices one can make in such a procedure, and it is not an easy task to make a right choice: the essence lies in the art to select a small number of key variables and postulate proper relations among them. Such relations should behave well under the above mentioned scaling operations. How efficient such coarse-graining may be largely depends on the internal structure of the system, and in particular how precise description can be obtained at the length scale which is called the correlation length of the model, i.e. the length scale at which the overall properties of the microscopic (original) variables begin to differ markedly from macroscopic properties.

Several apparently simple mathematical models, such as percolation, contact process and self-avoiding random walk, are paradigms for the study of critical phenomena in statistical mechanics. Although these models (as many variations of them) have been studied by mathematicians for about fifty years, exciting new developments continue to occur.

The notes have been given some self-consistency, but material requires familiarity with basic facts in probability.

1.2 The percolation model. Some basic facts and tools.

Percolation refers typically to the phenomenon of propagation of a fluid (water, oil, gas) in a porous medium (rock, protection mask). Searching for a mathematical formulation that could cope with the basic features, Broadbent and Hammersley [11] proposed in 1957 a simple stochastic model which became known as the *percolation model*, on which there exists now a vaste literature, both physical and mathematical. The basic idea is to think of pores and/or channels in the microscopic structure of the medium which could be *open* or *closed* in a random fashion, e.g. independently, with given probability.

The first natural problem concerns the evaluation of the probability of finding arbitrarily long paths through which the fluid is able to propagate. To formulate it mathematically one may take a graph $\mathbb{G} = (\mathbb{V}, \mathbb{E})$, where $\mathbb{V}$ denotes the set of vertices or sites (pores in the medium) and $\mathbb{E}$ is the set of bonds or edges (channels or passageways). Given two numbers $p(e), p(v)$ in the interval $[0,1]$ we take each pore (channel) to be open or closed with probability $p(v)$ ($p(e)$, respectively), independently of anything else. A path in $\mathbb{G}$ refers to an alternating sequence $x_0, e_0, x_1, e_1, \dots, e_{n-1}, x_n$, where $x_0, \dots, x_n$ are distinct vertices, and $e_i = \langle x_i, x_{i+1} \rangle$ is the edge with endvertices x_i and x_{i+1}. In this case we say that the path *connects* x_0 and x_n and has *length* n. The path is said to be *open* when all the x_i and e_i are open.

A natural example is the d-dimensional hypercube lattice $\mathbb{Z}^d$ with nearest neighbour edges i.e. $\mathbb{E}^d = \{\langle x, y\rangle : \|x-y\|_1 = 1\}$. The probability space can be taken as $(\Omega, \mathcal{A}, P)$, where $\Omega = \{0,1\}^{\mathbb{Z}^d} \times \{0,1\}^{\mathbb{E}^d}$, $\mathcal{A}$ is the σ-field generated by the cylinder sets in Ω, that is, the events that depend on only finitely many vertices and edges, and $P = P_{p(v)} \times P_{p(e)}$, where P_p corresponds to the Bernoulli product measure with density p (on the corresponding space). Thus $\omega(u) = 1$ indicates that u is open, where u could be a vertex or an edge.

We get the standard bond percolation model by taking the pores to be all open, so that only the channels are involved, each one can be open or closed with probabilities p and $1-p$, respectively, where p is a number in the interval $[0,1]$. In this case $\Omega = \{0,1\}^{\mathbb{E}^d}$ and $P = P_p$, the product measure with $P_p(\omega(e) = 1) = p$ for each $e \in \mathbb{E}^d$. Analogously, making $p(e) = 1$ we have the site percolation model, in which case $\Omega = \{0,1\}^{\mathbb{Z}^d}$.

To fix things we restrict now to the standard bond percolation model on $\mathbb{Z}^d$.[1] Two sites x and y are said to be connected if there exist an open path that connects x to y, and we denote this as $x \leftrightarrow y$. We complete the definition by saying $x \leftrightarrow x$, for any x. This defines an equivalence relation and $C(x)$ denotes the class which contains x, which we call the *open cluster* of x. In other words, deleting the closed edges, the original graph is transformed into a random graph, of which the sets $C(x)$ become the connected components.

The first basic questions concern the size of $C(x)$ and the possible existence of an infinite open path. Since the model is homogeneous we take $x = 0$, the origin in $\mathbb{Z}^d$ and set $\theta(p) = \theta(p, d) :=$

[1]We write simply $\mathbb{Z}^d$ for $(\mathbb{Z}^d, \mathbb{E}^d)$.

$P_p(C(0)$ is infinite), so that we can write

$$\theta(p,d) = P_p(|C(0)| = \infty) = 1 - \sum_{n=1}^{\infty} P_p(|C(0)| = n),$$

$|C(0)|$ denoting the cardinality of $C(0)$.

Using coupling methods we easily see that this function is increasing in p. While for $d = 1$, it is zero unless $p = 1$, if $d \geq 2$ the model shows the possibility or not of percolation through the random medium, depending on p. This is expressed in the following basic result.

Theorem 1.1 . *For $d \geq 2$, there exists a critical value $0 < p_c(d) < 1$ so that*

$$\theta(p,d) = \begin{cases} = 0 & \text{if } p < p_c(d) \\ > 0 & \text{if } p > p_c(d) \end{cases}$$

We now develop the basic tools and prove this result.

Duality. Planar duality is a very useful concept. For a graph that can be drawn in the plane in a way that the edges intersect only at the vertices, also called *planar graph*, there is a dual that can be defined as follows: on each face of the graph we put a vertex of the dual; we connect two such vertices with an edge if and only if the corresponding faces share an edge in the original graph. In the particular case of $\mathbb{Z}^2$, this gives simply $\mathbb{Z}^2_* = \{(m+\frac{1}{2}, n+\frac{1}{2})\colon (m,n) \in \mathbb{Z}^2\}$, with edges among neighbors, isomorphic to the original lattice. (See Figure 1.1.)

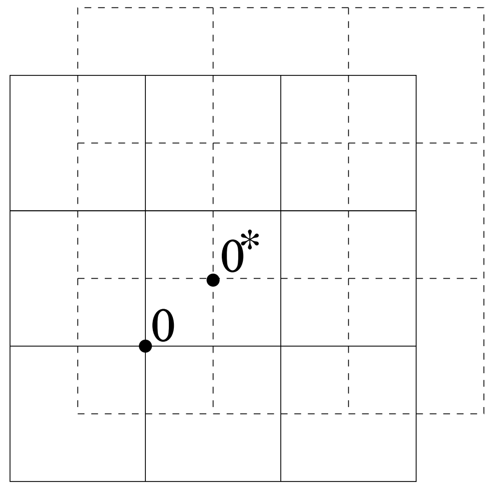

FIGURE 1.1

There is a one to one relation between edges in both graphs, since each edge e in $\mathbb{Z}^2$ is crossed by a unique edge e^* in the dual graph. This induces a bond percolation model in the dual by setting e^* is open (closed) if and only if e to be open (closed, respectively), as in Figure 1.2.

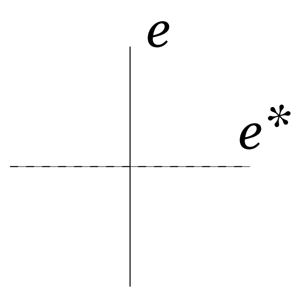

FIGURE 1.2

Coupling Coupling in probability theory refers to a (joint) construction of two or more random processes on the same probability space. It appears as a very useful tool for establishing comparisons, for proving convergences, etc.

Let $(U(e): e \in \mathbb{E}^d)$ be a family of independent random variables, each of them uniformly distributed in the interval $[0,1]$, with $\mathbb{P}$ denoting the probability measure on the underlying space. (For instance, we can take $\Sigma = [0,1]^{\mathbb{E}^d}$ the canonical space with its Borel σ-field, $\mathbb{P}$ the product measure, and $U(e)$ defined as the coordinate map $\sigma \mapsto \sigma(e)$.) On this underlying space we construct a joint realization of the bond percolation models in $\mathbb{Z}^d$, for all p varying in $[0,1]$, by setting $\omega_p(e) = 1$ or 0 according to $U(e) < p$ or not. The family $(\omega_p(e): e \in \mathbb{E}^d)$ gives the bond percolation model with parameter p, in the sense that its distribution on $\{0,1\}^{\mathbb{Z}^d}$ is P_p. When referring to this particular construction, also called *standard coupling*, we say that e is p-open if $\omega_p(e) = 1$, and p-closed otherwise.

Remark 1.2 . (i) The above construction makes evident the monotonicity in p since for $p_1 \le p_2$ we have $\omega_{p_1}(e) \le \omega_{p_2}(e)$, for each edge e. In particular, it shows that $\theta(p)$ increases with p.

We may set

$$p_c(d) = \sup\{p \in [0,1]: \theta(p) = 0\}.$$

and, according to the previous remark, the proof of Theorem 1.1 follows at once if we show that $0 < p_c(d) < 1$.

Before we proceed to the proof notice that by embedding $\mathbb{Z}^d$ into $\mathbb{Z}^{d+1}$ as the space generated by the first d coordinates, we see that $\theta(p,d)$ increases in d and thus $p_c(d+1) \le p_c(d)$. The proof of Theorem 1.1 thus reduces to the following proposition.

Proposition 1.3 .
(i) $p_c(d) > 0$ *for any* d.
(ii) $p_c(2) < 1$.

Proof.
Proof of part (i). Let us fix $d \ge 2$ and omit it from the notation. Clearly, for $C(0)$ to be infinite we must have open paths of

arbitrarily length starting at the origin, so that

$$\theta(p) = P_p(|C(0)| = \infty) \le P_p(N(n) \ge 1) \le E_p(N(n)),$$

where $N(n)$ is the number of open paths starting at the origin having length n. But, any given path of length n has probability p^n to be open, so that

$$E_p(N(n)) = p^n \sigma(n)$$

where $\sigma(n)$ counts the number of paths of length n starting at the origin.

A simple counting argument yields $\sigma(n) \le 2d(2d-1)^{n-1}$, which allows us to conclude that $\theta(p) = 0$ for $p < (2d-1)^{-1}$.

Proof of part (ii). This may be achieved through a contour argument, with roots in an article of R. Peierls (1936) on the Ising model. For this we consider the dual graph $\mathbb{Z}^2_*$ obtained by shifting the original lattice by $(1/2, 1/2)$, and the one-to-one relation between edges in both graphs, with the induced bond percolation model in the dual, as described above.

At this point it is convenient to recall the definition of a path given before, and in the same setup, if $x_0, e_0, x_1, \dots, e_{n-1}, x_n$ is a path of length n and $e_n = \langle x_n, x_0 \rangle$ is an edge, we say that $x_0, e_0, x_2, \dots, e_{n-1}, x_n, e_n, x_0$ forms a circuit (of length $n+1$), which is said to be open (closed) if all the edges are open (closed, respectively). The finiteness of $C(0)$ is equivalent to the existence of a closed circuit in $\mathbb{Z}^2_*$ around the origin. The geometrical picture is clear: being $C(0)$ finite, we consider its edge boundary, formed by those edges $\langle x, y \rangle$ with $x \in C(0), y \notin C(0)$. By construction these edges are closed and the set of their dual edges contains a circuit around the origin (for a full proof see [25]). The situation is illustrated in Figure 1.3.

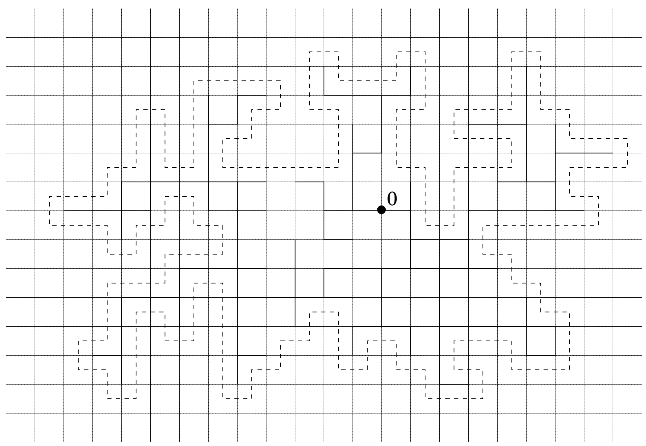

FIGURE 1.3

Thus

$$P_p(|C(0)| < \infty)$$
$$\leq P_p(\exists \text{ a closed circuit in } \mathbb{Z}^2_* \text{ enclosing the origin })$$

and taking into account the self-duality

$$P_p(|C(0)| < \infty) \leq \sum_{\gamma \circ 0} P_p(\gamma \text{ is closed }),$$

the sum running over all closed circuites around the origin, denoted by $\gamma \circ 0$. Given any circuit with length n the probability of being closed is $(1-p)^n$ and so, if we write $\beta(n)$ for the number of circuits of length n that encircle the origin we see that

$$P_p(|C(0)| < \infty) \le \sum_{n \ge 4} \beta(n)(1-p)^n \le \sum_{n \ge 4} \frac{n}{3}(3(1-p))^n, \quad (1.1)$$

where we again have used a simple counting argument to see that $\beta(n) \le n3^{n-1}$. To see this notice that such a circuit must have an edge $\langle (k+1/2, -1/2), k+1/2, 1/2) \rangle$ with $k \in \{0, \dots, n-1\}$ and starting from this one we have at most three ways to choose the next one. Since the r.h.s. of (1.1) tends to zero as $p \nearrow 1$, we see that $P_p(|C(0)| < \infty) < 1$ for $p < 1$ large enough. □

Remark 1.4 Two observations concerning the previous proof are in order:
(a) The argument used in part (i) can indeed be slightly modified to see that $\chi(p) := E_p(|C(0)|) < \infty$, for $p < (2d-1)^{-1}$. This is outlined in *Exercise 1.*
(b) The argument used in part (ii) can be improved to yield that $p_c(2) \le 2/3$. This is outlined in *Exercise 2.*

Putting together the previous statements we see that $(2d-1)^{-1} \le p_c(d) \le 2/3$, for any $d \ge 2$. The exact value of $p_c(d)$ depends on the dimension, and it is known only in the case $d = 2$ by a celebrated theorem of Kesten, proven originally in [24]. The inequality $p_c(2) \ge 1/2$ was proven by Harris in 1960, in a fundamental article for the development of percolation theory, where various important ideas and techniques were introduced. It is known that $p_c(d) \sim (2d)^{-1}$ as $d \to \infty$, cf. [26]).

Setting

$$\psi(p) := P_p(\text{there exists an infinite open cluster}),$$

Theorem 1.1 implies that $\psi(p) = 0$ for $p < p_c(d)$. On the other hand, the occurrence or not of this event is not affected when we change the configuration at finitely many edges, i.e., it is a tail event. Thus, by Kolmogorov 0-1 law, we have that $\psi(p) = 0$ or 1, according to $\theta(p) = 0$ or $\theta(p) > 0$. It can be seen that $\theta(\cdot)$ is continuous except possibly at p_c. A natural question involves the behaviour at criticality: is $\theta(p_c) = 0$? This is equivalent to the continuity at $p = p_c$. (See Sec. 1.2.4.) The answer is known to be affirmative for $d = 2$ and $d \geq 19$, so that in these cases the graph of the function $\theta(\cdot)$ looks somehow as the picture on the left in Figure 1.4. One may conjecture the same is true at all dimensions, but the problem remains open in general.

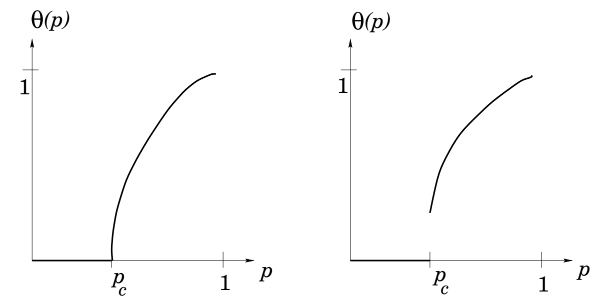

FIGURE 1.4 PERCOLATION PROBABILITY

1.2.1. Monotonicity. FKG and BK inequalities

On the space $\Omega = \{0, 1\}^{\mathbb{E}^d}$ we consider the partial order given by $\omega \leq \omega'$ if $\omega(e) \leq \omega'(e)$ for all edges e, i.e. whenever an edge is open in ω it is also open in ω'.

A random variable X on Ω is said to be *increasing* if $X(\omega) \leq X(\omega')$ whenever $\omega \leq \omega'$. If $-X$ is increasing we say that X is *decreasing*. We say that an event $A \in \mathcal{A}$ is increasing (decreasing) if its indicator function $\mathbf{1}_A$ is increasing (decreasing, resp.), where $\mathbf{1}_A(\omega) = 1$ if $\omega \in A$ and 0 otherwise. Thus, A is increasing iff its complement, A^c, is decreasing. Some natural examples of increasing events are: $\{x \leftrightarrow y\}$, $\{|C(0)| = \infty\}$, $\{\exists$ an infinite open path$\}$.

The coupling of all the measures P_p defined through the configurations ω_p right after Theorem 1.1 makes evident a monotonicity in the parameter p, cf. Remark 1.2. It implies the following statement whose proof is left as and exercise. (See *Exercise 3.*)

Proposition 1.5 . *Let X be an increasing random variable on $(\Omega, \mathcal{A})$. If $p_1 \leq p_2$ in the interval $[0,1]$ and $E_{p_i}(X)$ exists for $i = 1,2$, then $E_{p_1}(X) \leq E_{p_2}(X)$. In particular, if A is an increasing event, the function $P_p(A)$ increases in p.*

A very useful property held by the measures P_p is the Harris-FKG inequality:

Theorem 1.6 . (Harris-FKG inequality) (a) *If X and Y are bounded increasing random variables in $(\Omega, \mathcal{A})$, then*

$$E_p(XY) \geq E_p(X)E_p(Y) \tag{1.2}$$

(b) *If A and B are increasing events in $(\Omega, \mathcal{A})$, then*

$$P_p(A \cap B) \geq P_p(A)P_p(B). \tag{1.3}$$

The above theorem says that increasing events (variables) are positively correlated under the measures P_p. In particular (1.3) can be stated in terms of the conditional probability:

$$P_p(A \mid B) \geq P_p(A),$$

i.e. knowing that B occurs increases the chances for A to occur. For product measures this was first proven by Harris in [21]. The property was later investigated for a more general class of measures of special importance in statistical mechanical models for ferromagnetic interactions. The extension is due to Fortuin, Kasteleyn and Ginibre (1971), see [17], and it is usually named simply FKG inequality (or FKG property).

Proof of Theorem 1.6. Part (b) follows at once from part (a) by taking $X = \mathbf{1}_A$ and $Y = \mathbf{1}_B$. To prove (a) let us first consider the case when X and Y are cylinder random variables, i.e. they depend on the state variables at finitely many edges $e_1, e_2, \ldots, e_n$. We may proceed by induction on n. For $n = 1$ we have $X = f(\omega(e_1)), Y = g(\omega(e_1))$ where f, g are increasing functions on $\{0, 1\}$. We see that

$$(f(x) - f(y))(g(x) - g(y)) \geq 0$$

for any values of $x, y \in \{0, 1\}$. Averaging for x, y independent and distributed as $\omega(e_1)$ under P_p, we get

$$2(E_p(XY) - E_p(X)E_p(Y)) \geq 0$$

as we wanted to check. For the induction step we let $k \geq 1$ and assume the result holds for any $n \leq k$. Let now X, Y be increasing functions that depend on $\omega(e_1), \ldots, \omega(e_{k+1})$. We then have:

$$\begin{aligned} E_p(XY) &= E_p\left(E_p(XY \mid \omega(e_1), \ldots, \omega(e_k))\right) \\ &\geq E_p\left(E_p(X \mid \omega(e_1), \ldots, \omega(e_k)) E_p(Y \mid \omega(e_1), \ldots, \omega(e_k))\right) \\ &\geq E_p\left(E_p(X \mid \omega(e_1), \ldots, \omega(e_k))\right) E_p\left(E_p(Y \mid \omega(e_1), \ldots, \omega(e_k))\right) \\ &= E_p(X)E_p(Y), \end{aligned} \tag{1.4}$$

where the equalities follow from the definition of conditional expectation, the first inequality follows from the case $n = 1$, once we notice that for fixed $\omega(e_1),\dots,\omega(e_k)$, X and Y increase in $\omega(e_{k+1})$. The second inequality follows from the induction assumption since $E_p(X \mid \omega(e_1),\dots,\omega(e_k))$ and $E_p(Y \mid \omega(e_1),\dots,\omega(e_k))$ are increasing in $\omega(e_1),\dots,\omega(e_k)$.

To treat the general case we use an approximation procedure. Let X, Y be increasing, with finite second moment, and let $e_1, e_2, \dots$ be an enumeration of $\mathbb{E}^d$. Taking $X_n = E_p(X \mid \omega(e_1),\dots,\omega(e_n))$ and analogously for Y_n we see that both X_n and Y_n are increasing and cylinder, so that

$$E_p(X_n Y_n) \geq E_p(X_n)E_p(Y_n). \tag{1.5}$$

We can apply the martingale convergence theorem (see e.g. [12]) to conclude that $X_n \to X$ and $Y_n \to Y$ both a.s. and in L^2 under P_p, from where we see (check this as *Exercise 4*) that the r.h.s. in (1.5) converges to $E_p(X)E_p(Y)$ and that the l.h.s. tends to $E_p(XY)$, concluding the proof. $\square$

To illustrate further the use of monotonicity we now describe an inequality which, if compared to (b) in Theorem 1.6, goes in opposite direction. It is due to van den Berg and Kesten(1985), and goes under the name of BK inequality. Let A and B be two increasing events in $\Omega = \{0,1\}^{\mathbb{E}^d}$. We denote by $A \circ B$ the set of configurations ω for which there exist two open (edge) disjoint paths, one of them guaranteeing the occurrence of A and the other implying the occurrence of B. The event $A \circ B$ is called *disjoint occurrence* of A and B.

Theorem 1.7 . (BK inequality)

If A and B are increasing cylinder events in $\{0,1\}^{\mathbb{E}^d}$, then

$$P_p(A \circ B) \leq P_p(A)P_p(B). \tag{1.6}$$

Remark. Notice that $A \circ B \subset A \cap B$ and that $A \circ B$ is also increasing if so are A and B. The inequality (1.6) extends to the disjoint occurrence of any finite number of increasing cylinder events. (Check this as *Exercise 5.*)

A natural application of this inequality comes when we take $A = \{x \leftrightarrow y\}$, $B = \{u \leftrightarrow v\}$ for $x, y, u, v \in \mathbb{Z}^d$, in which case the occurrence of $A \circ B$ means the existence of two disjoint open paths, connecting x to y and u to v respectively. In this case, (1.6) becomes

$$P_p(\{x \leftrightarrow y\} \circ \{u \leftrightarrow v\} \mid \{u \leftrightarrow v\}) \leq P_p(\{x \leftrightarrow y\}). \qquad (1.7)$$

We shall not prove (1.6) but simply discuss a basic idea behind (1.7). Let us start by restricting first to a finite sublattice $\mathbb{G}$ of $\mathbb{Z}^d$ and noticing that when we condition to the occurrence of an open path from x to y in $\mathbb{G}$ we are given a positive information. This would increase the (conditional) probability of any other increasing event as e.g. $\{u \leftrightarrow v\}$ in the same graph. But while considering the disjoint occurrence we are forced to avoid one of the existing open paths connecting x to y. That this is enough to make the conditional probability smaller as given in (1.7) is not obvious.

A basic idea for proving (1.7) goes as this: having fixed $\mathbb{G}$ as in the previous paragraph and fixing any edge e let us modify the graph by replacing e by two edges that connect the same vertices as e, call them e_1 and e_2 and change P_p by making the edges e_1, e_2 open or closed with probability p and $1-p$ respectively, independently among themselves and independently of the rest. While considering the event $\{x \leftrightarrow y\} \circ \{u \leftrightarrow v\}$ in this new model, the open paths from x to y are not allowed to use e_1 and those from u to v cannot use e_2. The probability is not smaller than the original one. We proceed recursively, replacing each edge f in $\mathbb{G}$ by two parallel ones f_1, f_2 and modifying P_p as before. Consider the probabilities of $\{x \leftrightarrow y\} \circ \{u \leftrightarrow v\}$ in these models where the edges f_1 (f_2) cannot be used to connect x to y (u to v, respectively). These probabilities do not decrease along the process. Since $\mathbb{G}$ is finite, when we have finished to replace all original edges we shall have two independent copies of the original model, so that we get (1.7) in $\mathbb{G}$. For a formal proof we refer to [19], or to the original article [8].

Remark. (Reimer inequality) A version of the BK inequality holds also for general cylinder events A and B, provided $A \circ B$ is replaced by

$$\begin{aligned} A\Box B := \{\omega \colon \exists \text{ two disjoint open paths } \gamma = \{e_1, \dots, e_m\}, \\ \gamma' = \{e'_1, \dots, e'_n\} \text{ so that } C(\omega,\gamma) \subset A, C(\omega,\gamma') \subset B\}, \end{aligned} \tag{1.8}$$

where $C(\omega, S) = \{\omega' \colon \omega'(e) = \omega(e)\ \forall e \in S\}$, with S a finite subset of $\mathbb{E}^d$. Notice that if A and B are increasing cylinder events, then $A\Box B = A \circ B$. The extension was conjectured by van den Berg and Kesten, and it was proven by Reimer [36]:

$$P_p(A\Box B) \le P_p(A)P_p(B) \tag{1.9}$$

for A and B cylinder events in $\{0,1\}^{\mathbb{E}^d}$. One may observe that if A is increasing and B is decreasing, then $A\Box B = A \cap B$. From this and replacing B by its complement B^c, we see that (1.9) also extends the Harris-FKG inequality for cylinder events. We shall not use the more general formulation of Reimer. (For the proof and comments see [19].)

1.2.2. Russo's formula.

The content of this subsection has a close relation to ideas and tools coming from *reliability theory*. A concrete motivation comes from the problem in the next subsection, where we ask if $\chi(p) = E_p(|C(0)|) < +\infty$ for any $p < p_c$ (recall Exercise 1). Having this in mind one needs a way to see how does the quantity

$$g_p(n) := P_p(0 \leftrightarrow \partial S(n)) \tag{1.10}$$

decay in n, for any $p < p_c$, and where $S(n) = \{x : \|x\|_1 \le n\}$. The basic point is to have a way to compare $g_p(n)$ for different values of p below the critical parameter.

Notation. If A is a cylinder event, we let Δ_A denote the minimal (finite) set of edges on whose values does the occurrence (or not) of A depend. This is called the *support* of A.

More generally, let A be an increasing cylinder event on $\Omega = \{0,1\}^{\mathbb{E}^d}$. The function $P_p(A)$ is increasing in p and we need to get a handle on how fast does it increase. Recalling the previously defined coupling we see that for $\delta > 0$ small

$$P_{p+\delta}(A) - P_p(A) = \mathbb{P}(\omega_p \notin A, \omega_{p+\delta} \in A). \tag{1.11}$$

For this not to vanish, the discrepancy set $X_{p,\delta} = \{e \in \Delta_A : p \le U(e) < p + \delta\}$ must be non-empty. But its cardinality $|X_{p,\delta}|$ has a Binomial distribution with parameters $|\Delta_A|$ and δ. Thus $\mathbb{P}(|X_{p,\delta}| \ge 2) = o(\delta)$, so that when considering the leading term in (1.11) we have only the case when $X_{p,\delta} = \{e\}$ with $e \in \Delta_A$. Moreover, it is necessary that given the rest of the configuration ω_p (i.e., at all other edges but e), the value at edge e has an essential aspect to determine the occurrence (or not) of A, in the following way:

Given A and a configuration ω one says that the edge e is *pivotal* for (A, ω) if $\mathbf{1}_A(\omega) \ne \mathbf{1}_A(\omega')$ where ω' is the configuration obtained from ω by changing the value at e. We see at once that this property does not depend on the value $\omega(e)$, so that the event $\{e \text{ is pivotal for } A\} := \{\omega : e \text{ is pivotal for } (A, \omega)\}$ is independent of the state at e. We may write

$$
\begin{aligned}
&P_{p+\delta}(A) - P_p(A) = \\
&\sum_{e\in\Delta_A} \mathbb{P}(\omega_p \notin A, \omega_{p+\delta} \in A), X_{p,\delta} = \{e\}) + o(\delta) = \\
&\sum_{e\in\Delta_A} \mathbb{P}(e \text{ is pivotal for } (A,\omega_p), p \le U(e) < p+\delta, X_{p,\delta} = \{e\}) + o(\delta) \\
&= \sum_{e\in\Delta_A} \mathbb{P}(e \text{ is pivotal for } (A,\omega_p), p \le U(e) < p+\delta) + o(\delta),
\end{aligned}
$$

where the last $o(\delta)$ term is the sum of the previous one with

$$\sum_{e\in\Delta_A} \mathbb{P}(e \text{ is pivotal for } (A,\omega_p), p \le U(e) < p+\delta, X_{p,\delta} \neq \{e\}).$$

Since the property of e being pivotal for (A,ω) does not depend on the value of $\omega(e)$, we get

$$P_{p+\delta}(A) - P_p(A) = \delta \sum_{e\in\Delta_A} P_p(e \text{ is pivotal for } A) + o(\delta),$$

from where the next theorem follows.

Theorem 1.8 . *If A is an increasing cylinder event, then $P_p(A)$ is differentiable in p and*

$$\frac{d}{dp} P_p(A) = E_p(N(A)), \tag{1.12}$$

where $N(A)(\omega)$ is the number of pivotal edges for (A,ω).

To get the announced comparison we observe for $P_p(\omega(e) = 1, e \text{ is pivotal for } A) = pP_p(e \text{ is pivotal for } A)$. Thus, if A is an increasing cylinder event, (1.12) implies that

$$\begin{aligned}
\frac{d}{dp}P_p(A) &= \frac{1}{p}\sum_{e\in\Delta_A} P_p(\omega(e)=1, e \text{ is pivotal for } A) \\
&= \frac{1}{p}\sum_{e\in\Delta_A} P_p(A)P_p(e \text{ is pivotal for } A \mid A) \\
&= \frac{1}{p}P_p(A)E_p(N(A)\mid A),
\end{aligned}$$

which we may write as

$$\frac{d}{dp}\log P_p(A) = \frac{1}{p}E_p(N(A)\mid A).$$

Integrating it we have

$$P_{p_2}(A) = P_{p_1}(A)\exp\left\{\int_{p_1}^{p_2}\frac{1}{p}E_p(N(A)\mid A)dp\right\}, \tag{1.13}$$

for $0 \le p_1 < p_2 \le 1$, and then using the straightforward bound $N(A) \le |\Delta_A|$, we get

$$P_{p_2}(A) \le P_{p_1}(A)(p_2/p_1)^{|\Delta_A|} \tag{1.14}$$

Remark. There is a close relation between the type of analysis leading to Russo's formula with relations and techniques used in reliability theory. This is very natural, when we interpret the graph as a network with the state $\omega(e)$ of an edge representing the reliability of a given link in the network. For a brief summary see Sec. 2.5 in [19] and the references therein, including the books of Barlow and Proschan [4,5] and the article by Moore and Shannon [32].

1.2.3. The subcritical phase.

Exercise 1 shows that the expected size of $C(0)$, denoted by $\chi(p)$, is finite for p small. This is an increasing function of p and one naturally asks if $\chi(p) < \infty$ for all $p < p_c$. The answer is affirmative, eliminating the possibility of another critical point here. This result was proven by Menshikov [33] and also independently by Aizenman and Barsky [1]. We now describe briefly the main steps of Menshikov's argument (with improvement by Kesten).

Writing S_n for the $\|\cdot\|_1$-ball in $\mathbb{Z}^d$, $S_n = \{x \in \mathbb{Z}^d\colon \|x\|_1 \le n\}$ and $\partial S_n = \{x \in \mathbb{Z}^d\colon \|x\|_1 = n\}$ its boundary, let $A_n = \{0 \leftrightarrow \partial S_n\}$, the event that there exists an open path from the origin to ∂S_n. The question is answered through the following theorem:

Theorem 1.9 . *If $p < p_c(d)$ then it exists $\psi_p > 0$ so that*

$$P_p(A_n) \le e^{-n\psi_p} \qquad \textit{for all } n. \tag{1.15}$$

This theorem says that the radius of $C(0)$ has an exponentially decaying tail. Since $|S(n)| \le c_d(n+1)^d$ for a constant c_d that depends only on the dimension, we see that $[|C(0)| \ge c_d(n+1)^d] \subset A_n$ and consequently:

$$P_p(|C(0)| \ge n) \le e^{-n^{1/d}\tilde{\psi}_p} \qquad \text{for all } n, \tag{1.16}$$

for suitable $\tilde{\psi}_p > 0$ and any $p < p_c$.

Proof. Let $g_p(n) := P_p(A_n)$ as above. For the moment we only know that $g_p(n) \downarrow \theta(p)$, as $n \uparrow \infty$, and $\theta(p) = 0$ for $p < p_c$. From Russo's formula, expressed in (1.13), we see that for $0 < u < v < 1$

$$g_u(n) \le g_v(n) \exp\left\{ -\int_u^v E_p(N(A_n) \mid A_n) dp \right\}. \tag{1.17}$$

The basic point is that for $p < p_c$ and large n, upon conditioning on the "rare" event A_n we should get a large number of pivotal edges, consistent with some sparsity of the set $C(0)$. One might expect to have a sort of roughly linear (in n) lower bound for $E_p(N(A_n) \mid A_n)$. To see this, conditioning on A_n we consider the pivotal edges $e_1, \ldots, e_N$, which then have to be open (since A_n is increasing). Each open path connecting 0 to ∂S_n must contain each of these edges always in the same order.

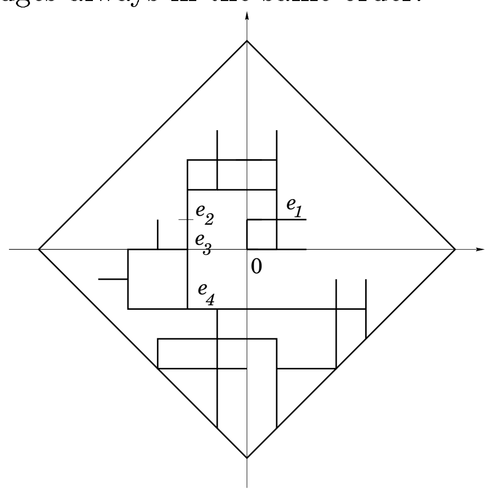

FIGURE 1.5

We see that the set $C(0)$ looks like sort of "sausages", connected by these pivotal edges, and we want a lower bound on the expected number of such sausages. Let us call ρ_i the radius of the i-th such sausage, that is, $\rho_i = \|x_i - y_{i-1}\|_1$ where $y_0 = 0$ and x_i, y_i are the vertices connected by e_i (in the proper order), for $i = 1, \ldots, N$. (See Figures 1.5, 1.6) Notice that

$$[(\rho_1 + 1) + \cdots + (\rho_j + 1) \leq n] \subset [N(A_n) \geq j]. \tag{1.18}$$

Therefore, for any $0 < p < 1$

$$P_p(N(A_n) \geq j \mid A_n) \geq P_p(\rho_1 + \cdots + \rho_j \leq n - j \mid A_n). \tag{1.19}$$

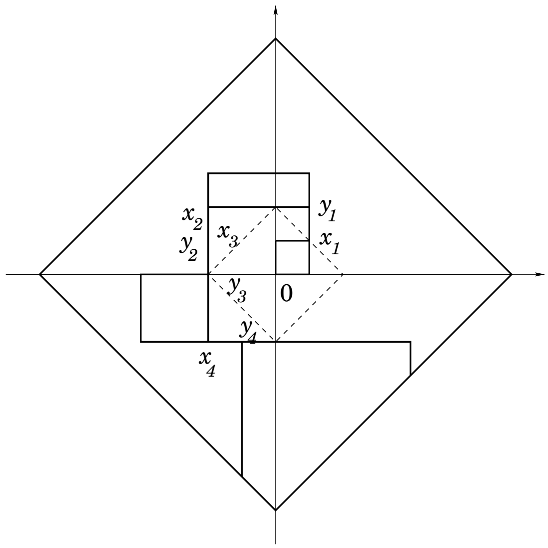

FIGURE 1.6

The first key point comes from a comparison on the distribution of the $\rho_i, i = 1, \ldots, N$ upon conditioning on A_n with N i.i.d. variables $R_1, \ldots, R_N$, distributed as the radius of $C(0)$ under P_p (defined as ∞ if $C(0)$ is infinite), and which is a finite random variable if $p < p_c$. One can prove that

Lemma 1.10 . *Under the above conditions we have:*

$$P_p(\rho_j \leq r_j \mid A_n, \rho_1 = r_1, \ldots, \rho_{j-1} = r_{j-1}) \geq P(R_j \leq r_j) \quad (1.20)$$

for each n, j, *provided* $r_1 + \cdots + r_j \leq n - j$. *Moreover,*

$$P_p(\rho_1 + \cdots + \rho_j \leq n - j \mid A_n) \geq P(R_1 + \cdots + R_j \leq n - j). \quad (1.21)$$

This lemma, whose proof we postpone to the end of the argument, brings connection to a classical result from renewal theory. Observe that from (1.18) and (1.21) we have

$$P_p(N(A_n) \geq j \mid A_n) \geq P(\tilde{R}_1 + \cdots + \tilde{R}_j \leq n)$$

where $\tilde{R}_i = 1 + \min\{R_i, n\}$. Thus, if $T = \min\{j \geq 1 \colon \tilde{R}_1 + \cdots + \tilde{R}_j > n\}$ one has

$$[T > j] = [\tilde{R}_1 + \cdots + \tilde{R}_j \leq n], \tag{1.22}$$

and collecting (1.19) (1.21) and (1.22) we get

$$\begin{aligned} E_p(N(A_n) \mid A_n) &= \sum_{j \geq 1} P_p(N(A_n) \geq j \mid A_n) \\ &\geq \sum_{j \geq 1} P(T > j) = E(T) - 1. \end{aligned} \tag{1.23}$$

Recall now the classical Wald identity (see [12])

$$E(\tilde{R}_1 + \cdots + \tilde{R}_T) = E(T)E(\tilde{R}_1).$$

Since $\tilde{R}_1 + \cdots + \tilde{R}_T > n$ by definition, we have

$$E(T) > n/E(\tilde{R}_1) = n/\sum_{k=0}^{n} g_p(k), \tag{1.24}$$

since $1 = g_p(0)$ and

$$E(\min\{R_1, n\}) = \sum_{k=1}^{n} P(R_1 \geq k) = \sum_{k=1}^{n} g_p(k).$$

Putting together (1.17), (1.23) and (1.24) we see that

$$g_u(n) \leq g_v(n) \exp\left(- \int_u^v \left(\frac{n}{\sum_{k=0}^{n} g_p(k)} - 1\right) dp \right)$$

for any $0 < u < v < 1$, and using the monotonicity of $p \mapsto g_p(k)$ we get

$$g_u(n) \leq g_v(n) \exp\left(-(v-u)\left(\frac{n}{\sum_{k=0}^{n} g_v(k)} - 1\right) \right). \tag{1.25}$$

With (1.25) in hands, the issue becomes how we get that $\sum_k g_p(k) < \infty$ for each $p < p_c$, which clearly would imply (1.15). An important step is the following analytical result, obtained with the use of (1.25) and whose proof is omitted in this course. (See [19].)

Lemma 1.11 . *If $p < p_c$ there exists $\gamma(p) > 0$ so that*

$$g_p(n) < \gamma(p)n^{-1/2} \quad \textit{for all} \quad n. \tag{1.26}$$

Having (1.26) we can use (1.25) again to see that if $u < v < p_c$ then

$$g_u(n) \leq g_v(n) \exp\Big((v-u) - C(v-u)n^{1/2}\Big),$$

from where it follows that $\sum_k g_p(k) < \infty$ for any $p < p_c$, and the proof of Theorem 1.9. □

Proof of Lemma 1.10. We first check that (1.21) follows from (1.20). This is a standard argument in stochastic comparison. We write

$$\begin{aligned} P_p(\rho_1 + \cdots + \rho_j \leq n-j \mid A_n) \\ &= \sum_{i=1}^{n-j} P_p(\rho_1 + \cdots + \rho_{j-1} = i, \rho_j \leq n-j-i \mid A_n) \\ &\geq \sum_{i=1}^{n-j} P_p(\rho_1 + \cdots + \rho_{j-1} = i \mid A_n) P_p(R_j \leq n-j-i) \\ &\times P_p(\rho_1 + \ldots, \rho_{j-1} + R_j \leq n-j \mid A_n), \end{aligned} \tag{1.27}$$

where the random variable R_j is independent of the edge configuration on S_n and is distributed as the radius of $C(0)$. (For convenience we may have enlarged the probability space so as to support an i.i.d. sequence $R_k, k \geq 1$ distributed as in the lemma, independent of the edge configuration ω. We still call P_p the basic measure.) Clearly we can iterate (1.27) and arrive to (1.21), as claimed.
The proof of (1.20) uses the BK inequality.

We need to prove that

$$P_p(\{\rho_j > r_j\} \cap B \cap A_n) \leq P_p(R_1 > r_j) P_p(B \cap A_n), \tag{1.28}$$

where $B = \{\rho_i = r_i, i = 1, \dots, j-1\}$. When $j = 1$ this reduces to

$$P_p(\{\rho_1 > r_1\} \cap A_n) \le P_p(R_1 > r_1) P_p(A_n). \tag{1.29}$$

Notice that $\{R_1 > r_1\} = A_{r_1+1}$ and observe that if $\rho_1 > r_1$ then there exist at least two edge disjoint open paths from the origin to the boundary of S_{r_1+1}. It follows that $\{\rho_1 > r_1\} \cap A_n \subset A_{r_1+1} \circ A_n$ and (1.29) follows at once from the BK inequality.

To consider the case $j \ge 2$ we decompose B according to all possible sausages (with their open edges) up to the vertex y_{j-1}: on B we consider the set of vertices and open edges which can be reached from the origin without using the edge e_{j-1} to which we add e_{j-1} and the marked vertex y_{j-1}. This gives a graph γ and $y(\gamma) = y_{j-1}$ is the marked vertex. We decompose B according to the realization of γ and call B_γ the corresponding sub-event. Then

$$P_p(A_n \cap B) = \sum_\gamma P_p(A_n \mid B_\gamma) P_p(B_\gamma) \tag{1.30}$$

and

$$P_p(\{\rho_j > r_j\} \cap A_n \cap B) = \sum_\gamma P_p(\{\rho_j > r_j\} \cap A_n \mid B_\gamma) P_p(B_\gamma). \tag{1.31}$$

Now, $P_p(A_n \mid B_\gamma)$ coincides with the probability that there is an open path from y_{j-1} to ∂S_n without touching other vertices of γ, i.e. without using edges from the previous sausages, including the pivotal edge and the boundary edges of each sausage, an event which we write as $\{y_{j-1} \leftrightarrow \partial S_n \text{ off } \gamma\}$.

Similarly $P_p(\{\rho_j > r_j\} \cap A_n \mid B_\gamma)$ coincides with the probability that there exist two edge disjoint paths from y_{j-1} to $\partial S(y_{j-1}, r_j + 1)$[2] and from y_{j-1} to ∂S_n without using edges from γ. From the

[2] $S(y,r) = \{x \colon \|x-y\|_1 \le r\}$ and $\partial S(y,r) = \{x \colon \|x-y\|_1 = r\}$

BK inequality we have

$$\begin{aligned} &P_p(\{\rho_j > r_j\} \cap A_n \mid B_\gamma) \\ &\le P_p(y_{j-1} \leftrightarrow \partial S(y_{j-1}, r_j+1) \text{ off } \gamma) P_p(y_{j-1} \leftrightarrow \partial S_n \text{ off } \gamma) \\ &\le P_p(A_{r_j+1}) P_p(A_n \mid B_\gamma), \end{aligned}$$

since

$$\begin{aligned} &P_p(y_{j-1} \leftrightarrow \partial S(y_{j-1}, r_j+1) \text{ off } \gamma) \\ \le &P_p(y_{j-1} \leftrightarrow \partial S(y_{j-1}, r_j+1)) = P_p(A_{r_j+1}). \end{aligned}$$

Taking (1.30) and (1.31) into account we get (1.28), thus concluding the proof. □

Remarks.
(i) While the previous analysis provides only a sub-exponential decay on the cluster size distribution in the subcritical phase, cf. (1.16), it is known that $P_p(|C(0)| \ge n) \le e^{-a(p)n}$ if $p < p_c$ for any d. This can be proven also with the help of BK inequality, by showing that $E_p(\exp t|C(0)|) < \infty$ for t positive and small, as proven by Aizenman and Newman in [3]. It can be combined with subadditivity, which is another tool that will appear later in these classes to obtain further results. In the subcritical phase the model exhibits a standard (volume scale) large deviation principle. (See [19] for details.)
(ii) Using Theorem 1.9, one can see that the function χ_p is infinitely differentiable in the subcritical region. Instead, using the exponential decay of the distribution of $|C(0)|$, the analyticity of χ_p for $p \in (0, p_c)$ can be proven. (See [19].)
(iii) The connectivity between two sites x and y is defined as $\tau_p(x, y) := P_p(x \leftrightarrow y)$. Theorem 1.9 implies at once that it decays exponentially in $\|x - y\|_1$, if $p < p_c$.

1.2.4. The supercritical phase.

We turn to the consideration of some basic features of the supercritical phase ($p > p_c$). Since $\theta(p) > 0$ we see that an infinite open cluster exists with probability one, as it follows from the ergodicity of the measure. A first natural question is related to the number $\mathcal{N}$ of infinite open clusters. In the supercritical phase this is a random variable with values in $\{1, 2, \dots\} \cup \{\infty\}$. It is invariant by translations on the basic space, and again the ergodicity of the measure P_p implies that $\mathcal{N}$ is P_p *a.s.* constant, for each p. An important classical result ([2],[13]) tells us that whenever $\theta(p) > 0$ there is exactly one infinite cluster. We examine it now in the context of $\mathbb{Z}^d$. It holds for a certain class of graphs, but generally this is not true.

Theorem 1.12 . *If $\theta(p) > 0$ then $P_p(\mathcal{N} = 1) = 1$*

This result can be seen in two steps, stated below:

Proposition 1.13 . *$P_p(2 \le \mathcal{N} < \infty) = 0$ for any p.*

Proof. This is due to Newman and Schulman [34]. We already know that there exists k_p constant so that $P_p(\mathcal{N} = k_p) = 1$ and we want to see that $2 \le k_p < \infty$ is impossible. Let $B(n) = \{-n, \dots, n\}^d$. Assuming that $1 \le k_p < \infty$, we can consider the event

$$A_n = \{\text{all infinite clusters intersect } B(n)\},$$

and see that $P_p(A_n \cap \{\mathcal{N} = k_p\}) \to 1$ as n tends to infinity. In particular, $P_p(A_n) > 0$ for n large.

But A_n is independent on the state of the edges having both end-vertices in $B(n)$. On the other hand, making all these edges open would generate a unique infinite cluster for the realizations in A_n. Thus, if D_n is the event that all edges with both vertices in $B(n)$ are open, we have, for any n:

$$P_p(\mathcal{N} = 1) \ge P_p(A_n \cap D_n) = P_p(A_n) P_p(D_n).$$

Since $P_p(D_n) > 0$ for all n and $P_p(A_n) > 0$ for n large, we see that $P_p(\mathcal{N} = 1) > 0$, implying that $k_p = 1$. $\square$

Proposition 1.14 . *$P_p(\mathcal{N} \geq 3) = 0$ for any p.*

Proof. This result was initially proven by Aizenman, Kesten and Newman in [2]. The argument sketched below is due to Burton and Keane (see [13]). Of course it is enough to consider $0 < p < 1$. It is geometrical, and the following notion plays an important role. If $\mathbb{G} = (\mathbb{V}, \mathbb{E})$ is a connected graph, a vertex $x \in \mathbb{V}$ is said to be a *triple point* if: (a) there are three incident edges to x; (b) if we remove x and the three incident edges we get a graph with three connected components. (See Figure 1.7.)

We apply this concept to the graph determined by an infinite open cluster. A vertex $x \in \mathbb{Z}^d$ is a *special triple point* if: (a) it belongs to an infinite open cluster; (b) it has three adjacent open edges in such a way that if x and these edges are removed, the cluster splits exactly into three infinite open clusters. Let T_x be the event that x is a special triple point.

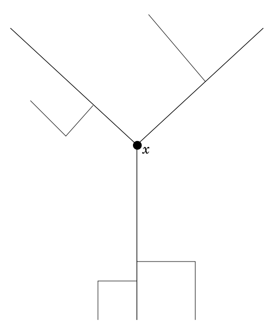

FIGURE 1.7 TRIPLE POINT

The argument has two pieces:

(i) If $k_p \geq 3$ one can see that $P_p(T_0) > 0$.

By the translation invariance of the measure, $P_p(T_x)$ is the same for all x. Thus, if $P_p(T_0) > 0$ we can take $c > 0$ for which the number of special triple points in $B(n)$ is at least cn^d, with positive probability, for each $n \geq 1$.

On the other hand, the argument sketched below implies that, if for a given realization ω we had order n^d special triple points in $B(n-1)$, then we would be forced to find the same number of distinct points in $\partial B(n)$. But for large n this is impossible in $\mathbb{Z}^d$. A way to make this argument precise goes as follows:

(ii) Let $\mathbb{G}$ be a connected graph as above, where one has $x_1, \ldots, x_r$ as distinct triple points. Let us write $V_1(x_i), V_2(x_i), V_3(x_i)$ for the connected components of $\mathbb{G}$ after deletion of x_i and its three incident edges. Among the $3r$ sets $V_1(x_i), V_2(x_i), V_3(x_i), i = 1, \ldots, r$, at least $r+2$ of them are disjoint.

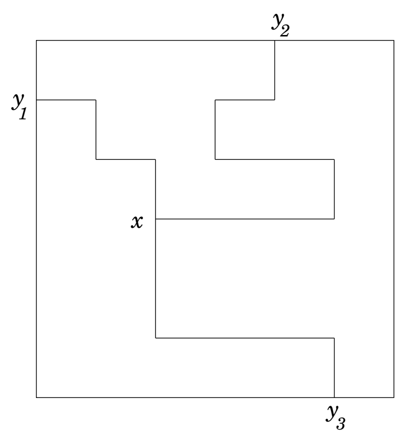

FIGURE 1.8

To verify the previous statement one may first check (which one can do by induction on r) that there must exist i and two branches, say $V_1(x_i), V_2(x_i)$, which do not intersect $\{x_k, k \neq i\}$. From this one can see that removing the vertices in $V_3(x_i)$ together with the edges they touch, gives another graph for which the points $x_k, k \neq i$ are still triple points, and finally apply this last statement to get (ii). (See [18], [19].)

We leave as exercise to see that from (ii) above we get indeed that the number of special triple points in $B(n-1)$ cannot exceed the number of sites in $\partial B(n)$. (Hint: apply the statement for each of the connected components of such special triple points, using only the bonds with at least one endvertex in $B(n-1)$.)

Let us now verify (i) above. Write $\mathbb{E}_{B(n)}$ for the set of edges with both endvertices in $B(n)$. Let $\mathcal{N}_n$ be the number of infinite open clusters that intersect $B(n)$ and let $\mathcal{N}_n^0$ be the number of open clusters intersecting $B(n)$ when we take as closed all the edges in $\mathbb{E}_{B(n)}$. Therefore $P(\mathcal{N}_n^0 \geq 3) \geq P(\mathcal{N}_n \geq 3)$ which converges to $P(\mathcal{N} \geq 3) = 1$ as $n \to \infty$. In particular, we can fix an n so that $P(\mathcal{N}_n^0 \geq 3) > 0$. Moreover, we can see that: (a) for a realization ω in $\{\mathcal{N}_n^0 \geq 3\}$ we can select three distinct points $y_1 = y_1(\omega)$, $y_2 = y_2(\omega)$, $y_3 = y_3(\omega)$ (selected in some predetermined way, if more such points do exist) that belong to three distinct open clusters without using the edges in $\mathbb{E}_{B(n)}$; (b) $\{\mathcal{N}_n^0 \geq 3\}$ is independent of the state at the edges in $\mathbb{E}_{B(n)}$.

Given y_1, y_2, y_3 we can take three paths that connect the origin to each of these points, and use only edges in $\mathbb{E}_{B(n)}$ in a way that the origin is the only common site to any two of them, and each touches $\partial B(n)$ only at the corresponding y_i. Let $T_{0,n}$ be the event that the edges along these paths are open and the other edges in $\mathbb{E}_{B(n)}$ are closed. (See Figure 1.8.) This has a positive probability, since $0 < p < 1$. Noticing that $T_{0,n} \cap \{\mathcal{N}_n^0 \geq 3\} \subset T_0$ we conclude that $P_p(T_0) > 0$ for each $0 < p < 1$. □

Some consequences of Theorem 1.12 and further important results.

1. For $p > p_c$ one has $\tau_p(x,y) := P_p(x \leftrightarrow y) \geq (\theta(p))^2$.

This property follows quite easily from Theorem 1.12. Indeed we have

$$\tau_p(x,y) \geq P_p(x, y \text{ belong to a same infinite open cluster}).$$

From the uniqueness of the infinite open cluster we see that the above probability coincides with $P_p(x \leftrightarrow \infty, y \leftrightarrow \infty)$. Applying the FKG inequality we get the statement.

2. The function $p \mapsto \theta(p)$ is continuous in the interval $(p_c, 1]$.

The left-continuity can be obtained as an application of the uniqueness result. A proof along these lines is due to van den Berg and Keane [7] and uses the coupling through the configurations ω_p, mentioned earlier. (See [19].) The right continuity is valid in the all interval $[0, 1]$ and the argument is simpler: the functions $g_n(p) = P_p(0 \leftrightarrow \partial B(n))$ are clearly continuous in p and decrease to $\theta(p)$ as $n \to \infty$; this implies that $\theta(p)$ is upper semi-continuous; since it is also increasing in p we get the right-continuity. (See [37], [19].)

3. The continuity of the function $p \mapsto \theta(p)$ at $p = p_c$ is therefore equivalent to $\theta(p_c) = 0$. This is known to be true for $d = 2$ and for $d \geq 19$. (See [19].)

4. In the supercritical phase the cluster size distribution has a sub-exponential decay. (See [19].) For the full large deviation analysis see the monograph [14].)

Some results on the case $d = 2$. The bond percolation model on $\mathbb{Z}^2$ is quite special, due to the possibility of exploiting the self-duality of the graph, as described in the beginning of Section 1.2. Recall that for the dual percolation model, we say that a given edge e is open (closed) if and only if the edge e^* in the dual which cross it is open (closed) respectively.

One important example where duality plays an important role is the determination of p_c in this special case:

Theorem 1.15 . (Kesten) $p_c(\mathbb{Z}^2) = 1/2$.

This result is one of the most important millstones in the history of percolation. Kesten's proof, presented in [24], was the crowning achievement of four papers published over a period of 21 years. In 1960 Harris proved that $\theta(\frac{1}{2}) = 0$, see Lemma below, and then in 1978 two independent but largely equivalent works by Russo, and by Seymour and Welsh provided the necessary ingredients on the mean cluster size. Kesten showed how to build on their arguments to obtain the full result.

Lemma 1.16 . (Harris) *For bond percolation on* $\mathbb{Z}^2$ *it is the case that* $\theta(\frac{1}{2}) = 0$. *In particular,* $p_c(\mathbb{Z}^2) \geq \frac{1}{2}$.

Remark. Lemma 1.16 was first proved using arguments of some geometric complexity in a remarkable paper by Harris [21]. This article contained also several of the techniques used in later important developments of percolation. As a byproduct, Harris obtained the uniqueness of the infinite cluster in two dimensions. We now present a simple and elegant proof that $\theta(\frac{1}{2}) = 0$, due to Zhang (1988). It uses the uniqueness of the infinite cluster, given in Theorem 1.12.

Proof of Lemma 1.16. We follow the argument of Yu Zhang. Let us suppose that $\theta(1/2) > 0$. For each positive integer n, let $A_{(E,n)}$ (respectively $A_{(D,n)}$, $A_{(S,n)}$, $A_{(I,n)}$), denote the event that some vertex on the left (respectively right, superior, inferior) side of the square $\Lambda_n = [0,n]^2$ belongs to an infinite open path $\mathbb{Z}^2$ which does not contain any other vertex of Λ_n. Clearly, the events $A_{(E,n)}$, $A_{(D,n)}$, $A_{(S,n)}$, $A_{(I,n)}$) are increasing, all have the same probability, and their union is the event that some vertex in Λ_n belongs to an open infinite cluster. Due to the assumption that $\theta(1/2) > 0$, such an infinite open cluster exists with probability one, so that

$$P_{1/2}(A_{(E,n)} \cup A_{(D,n)} \cup A_{(S,n)} \cup A_{(I,n)}) \longrightarrow 1 \qquad \text{as } n \to \infty. \quad (1.32)$$

The "square-root trick" given below implies that

$$\lim_{n \to +\infty} P_{1/2}(A_{(u,n)}) = 1 \qquad \text{for } u = E, D, S, I. \quad (1.33)$$

Remark. (Square-root trick) If $A_1, A_2, \ldots, A_m$ are increasing events and have the same probability, then

$$P_p(A_1) \geq 1 - \left\{1 - P_p(\cup_{i=1}^m A_i)\right\}^{1/m}.$$

To see this, notice that using the FKG inequality we have

$$1 - P_p(\cup_{i=1}^m A_i) = P_p(\cap_{i=1}^m A_i^c) \geq \prod_{i=1}^m P_p(A_i^c) = (1 - P_p(A_1))^m.$$

Due to (1.33) we can choose N so that

$$P_{1/2}(A_{(u,N)}) > \frac{7}{8} \qquad \text{for } u = E, D, S, I. \tag{1.34}$$

Moving to the dual lattice $\mathbb{Z}_*^2 = \{(m + \frac{1}{2}, n + \frac{1}{2}) \colon (m,n) \in \mathbb{Z}^2\}$, we define the dual box $\Lambda_n^* = \{x + (1/2, 1/2) \colon 0 \leq x_1, x_2 \leq n\}$.

Let $A_{(E,n)}^*$ (respectively $A_{(D,n)}^*$, $A_{(S,n)}^*$, $A_{(I,n)}^*$) denote the event that some vertex on the left (respectively right, superior, inferior) side of Λ_n^* belongs to an infinite closed path in $\mathbb{Z}_*^2$ not containing any other vertex of Λ_n^*. Each edge in $\mathbb{Z}_*^2$ is closed with probability 1/2, so that

$$P_{1/2}(A_{(u,N)}^*) = P_{1/2}(A_{(u,N)}) > \frac{7}{8} \qquad \text{for } u = E, D, S, I. \tag{1.35}$$

Consider now the event $A = A_{(E,N)} \cap A_{(D,N)} \cap A_{(S,N)}^* \cap A_{(I,N)}^*$ where there exist open paths in $\mathbb{Z}^2 \setminus \Lambda_N$ touching the left and right sides of Λ_N, and infinite closed paths in $\mathbb{Z}_*^2 \setminus \Lambda_N^*$ touching the superior and inferior sides of Λ_N^* cf. Figure 1.9.

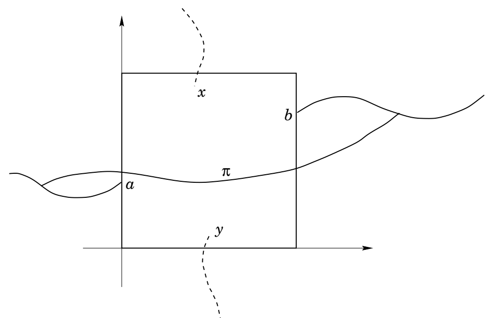

FIGURE 1.9

By (1.34) and (1.35), we see that the probability that A doesn't occur, $P(A^c)$, satisfies

$$P_{1/2}(A^c) < \frac{1}{2},$$

and so $P_{1/2}(A) > \frac{1}{2}$. If A occurs, then $\mathbb{Z}^2 \backslash \Lambda_N$ contains two disjoint infinite open clusters, since the involved clusters are physically separated by an infinite closed path in the dual; each open path in $\mathbb{Z}^2 \setminus \Lambda_N$ connecting these two clusters would contain an edge that cross a closed edge in the dual, but such an edge cannot exist. Similarly, for the realizations in A, the graph $\mathbb{Z}^2_* \backslash \Lambda^*_N$ contains two infinite closed clusters, physically separated by an infinite open path in $\mathbb{Z}^2 \setminus \Lambda_N$. Now, the full lattice $\mathbb{Z}^2$ cannot contain (almost surely) more than one infinite open cluster and, so that there must exist (almost surely in A) an open connection between the above mentioned infinite open clusters. By the geometry of the situation (see Figure 1.9) this connection must cross Λ_N and thus forms a barrier for possible connections in the dual connecting the two infinite closed clusters. Therefore, almost surely in A, the dual lattice must contain two or more infinite closed clusters. This last event has probability zero, due to our assumption that $\theta(1/2) > 0$ and Theorem 1.12. Thus $P_{1/2}(A) = 0$, in contradiction with the previous deduction, and the initial hypothesis that $\theta(1/2) > 0$ must be incorrect, concluding the proof. □

1.3. Oriented percolation.

There are many reasons to impose some extra modifications in the percolation models defined in the previous section. Practical motivation comes from many areas, chemistry, field theory and life-sciences being constant sources of this type of requirements. One of the most traditional model from this family is oriented percolation. To define this model we will assume that all edges of the graph $\mathbb{Z}^d$ have "north-east" orientation, i.e. all edges are oriented in the direction of increasing coordinate. As before, we can consider site or bond models, and the unique difference between usual (non-oriented) and oriented model is that a path from x to y can move along edges only in the direction prescribed to each edge, i.e. one could think that oriented edge $x \rightarrowtail x + e_1$ is closed if one wants to traverse this edge in direction from $x + e_1$ to x. The open oriented cluster of x, here denoted as C_x, is defined as the set of all those vertices y for which there is an open oriented path from x to y, with the understanding that $C_x = \{x\}$ if there is no open path starting at x.

There are many parallels between results for oriented percolation and those for ordinary percolation. On the other hand the corresponding proofs may differ greatly. But not only this makes these two models substantially different from each other: the oriented percolation models belong to a different "universality class" than regular percolation. This means that the nature of the infinite cluster and the behavior of the system for p near p_c (defined below) is different in the two cases. We will not discuss these issues in details here.

Let us restrict ourselves for the moment to $\mathbb{Z}^2$. We rotate our picture $\pi/4$ counterclockwise, and let $\widetilde{\mathbb{Z}}^2 = \{(x,y) \in \mathbb{Z}^2 : x+y \text{ is even},\ y \geq 0\}$. We define

$$\xi_n^0 = \{x : (x,n) \in \widetilde{\mathbb{Z}}^2 \text{ and } (0,0) \text{ is connected to } (x,n) \text{ by an open oriented path}\}.$$

ξ_n^0 is a random subset of $\{-n,\dots,n\}$, and we can think of the model as a discrete time growth process, with the vertical coordinate indicating the time. Let

$$r_n^0 = \sup \xi_n^0, \quad \text{and} \quad \ell_n^0 = \inf \xi_n^0, \quad (\sup \emptyset = -\infty,\ \inf \emptyset = +\infty),$$

which are called the right (resp. left) edge of the process.

Considering the standard Bernoulli case as before, i.e. the sites or edges are open (closed) with probability p ($1-p$, respectively), the process exhibits as before a phase transition, in the sense that there exists[3] $p_c \in (0,1)$ and the for $p > p_c$ there is positive probability that $\xi_n^0 \neq \emptyset$, for all n, and as before in this case an infinite oriented open cluster exists, with probability one.

An important property of this process, which will be discussed in more details in the next section in the context of the contact process, is that for $p > p_c$ there is a positive constant $\alpha(p)$ so that, with probability one, if the open oriented cluster of the origin reaches infinity (i.e. a.s. on the event $\{\xi_n^0 \neq \emptyset, \forall n\}$), we have $\lim_n n^{-1} r_n^0 = \alpha(p)$ and $\lim_n n^{-1} \ell_n^0 = -\alpha(p)$. In other words, if the oriented cluster is infinite, it must grow linearly.

1.4 The Harris contact process. Some basic facts and tools.

[3]The actual value of p_c differs, of course, in the site or bond versions.

We now discuss in some detail another example of "classical" percolative system sharing many features with (oriented) percolation. It was introduced by Harris in [22] and conceived with biological interpretation. This is a continuous time Markov process taking values on $\mathcal{P}(\mathbb{Z}^d)$, the set of all subsets of $\mathbb{Z}^d$, and informally described as follows: particles are distributed in $\mathbb{Z}^d$ in such a way that each site is either empty or occupied by at most one particle. The evolution is Markovian: each particle disappears after an exponential waiting time of rate one, independently of all the rest; at any time, each particle has the possibility to create a new particle at each empty neighboring site, with rate λ, also independently of everything else.

In the biological interpretation, occupied sites correspond to infected individuals, while empty ones correspond to healthy individuals. We have set the recovery rate as unit, and λ is the rate of propagation of the infection in each direction. With that in mind, the process may be thought as a very simplified mathematical model for the spread of an infection. There are various extensions with extra enriching ingredients.

Notation. Identifying each $\xi \subset \mathbb{Z}^d$ with its indicator function $\mathbf{1}_\xi$, we may think of the process as taking values on $\mathcal{X} := \{0,1\}^{\mathbb{Z}^d}$, and freely shift between the two. We then write $\xi(x,t) = 1$ or 0, according to $x \in \xi(t)$ or not, where $t \geq 0$ denotes the time.

To be rigorous, one needs a precise definition and construction of such an infinite system. Based on the informal description, if the initial state has infinitely many particles, the process should leave it instantaneously; infinitely many changes should happen during any arbitrarily small time interval, contrarily to the case of a Markov chain. Nevertheless, as the rates are bounded and the propagation happens only through neighbors, the evolution at any given finite collection of sites, and during a fixed finite time interval, does not depend on what happens "too far away". This allows a formal mathematical construction of the infinite system as limit of finite ones, which can be achieved through semigroup theory, using Hille-Yosida theorem or the so-called martingale problem method. (See [29].)

The contact process allows also a very useful and more explicit construction through a random graph in the space-time diagram $\mathbb{Z}^d \times \mathbb{R}_+$, from which several properties are more readily seen. This graphical construction is due to Harris and we now describe it.

For each $x \in \mathbb{Z}^d$ consider $(\tau_n^x)_{n\in\mathbb{N}}$, $(\tau_n^{x,x+\mathbf{e}_i})_{n\in\mathbb{N}}$, $(\tau_n^{x,x-\mathbf{e}_i})_{n\in\mathbb{N}}$ for $i = 1, \ldots, d$, the arrival times of $2d+1$ independent Poisson processes, where (τ_n^x) has rate 1 and the others have rate $\lambda > 0$, which is the parameter of the model. ($\mathbf{e}_i$ denotes the canonical unitary vector in i^{th} direction.) All such Poisson processes, also as x varies on $\mathbb{Z}^d$, are independently taken. For each $x, y \in \mathbb{Z}^d$ with $||x-y||_1 = 1$, and $n \geq 1$, we draw arrows in $\mathbb{Z}^d \times \mathbb{R}_+$, from $(x, \tau_n^{x,y})$ to $(y, \tau_n^{x,y})$. Secondly, we put down a cross sign ($\times$) at each of the points (x, τ_n^x), $n \geq 1$. A segment linking (x, s) to (x, t) is called a time segment and has the orientation from (x, s) to (x, t) if $s < t$. Given two points (x, s) and (y, t) in $\mathbb{Z}^d \times \mathbb{R}_+$, with $s < t$, we shall say that there is a path from (x, s) to (y, t) if there is a connected chain of oriented time segments and arrows in the constructed random graph, which links (x, s) to (y, t), without going through any cross sign, and following the orientation of time segments and arrows. (See Figure 1.10.)

The basic contact process with parameter λ and starting at $A \subset \mathbb{Z}^d$ is defined as follows: $\xi^A(0) = A$, and if $t > 0$

$$\begin{aligned}\xi^A(t) = \{y \in \mathbb{Z}^d\colon &\text{there is a path from } (x,0) \text{ to } (y,t) \\ &\text{for some } x \in A\}.\end{aligned} \tag{1.36}$$

(A is the initial configuration. $(\xi^A(t)\colon t \geq 0)$ denotes the all process)

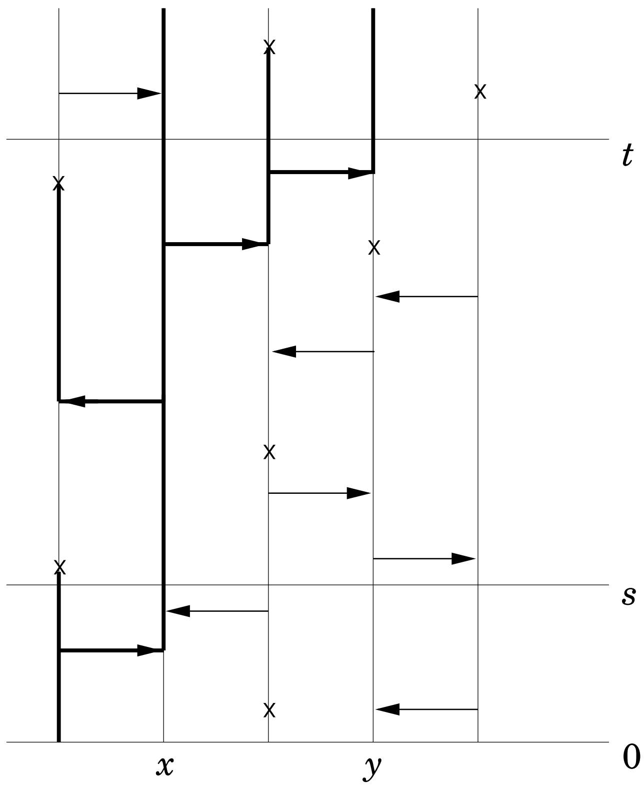

FIGURE 1.10

Remark. On the space $\mathcal{X}$ we consider the usual product topology, for the discrete topology on $\{0,1\}$. Based on a system of independent Poisson processes (τ_n^x), $(\tau_n^{x,y})$ (also called "marks"), one can verify that the graphical construction yields, almost surely, trajectories $t \mapsto \xi^A(t)$ in the Skorohod space $D([0,+\infty),\mathcal{X})$ of paths on $\mathcal{X}$ which are right-continuous, with left-limits. If P_A denotes the law of $\xi^A(\cdot)$ on $D([0,+\infty),\mathcal{X})$, then it can be seen that $A \mapsto P_A(B)$ is a Borel measurable function, for each Borel set B in the Skorohod space.

Several basic properties, are readily seen from the graphical construction. Let $s > 0$ and consider the "marks" in $\mathbb{Z}^d \times [s,+\infty)$. This is still a Poisson system just as before, independent of the "marks" in $\mathbb{Z}^d \times [0,s)$. (Notice that the probability of finding a "mark" at a fixed time s is zero.) Therefore, if for $A \subset \mathbb{Z}^d$ we set:

$$ {}_{(s)}\xi^A(t) = \{y \in \mathbb{Z}^d\text{:there is a path from } (x,s) \text{ to } (y,s+t) \text{ for some } x \in A\}, \tag{1.37} $$

then ${}_{(s)}\xi^A(\cdot)$ is a basic contact process starting from A. Moreover, given s:

$$ \xi^A(s+\cdot) = {}_{(s)}\xi^{\xi^A(s)}(\cdot), \quad \text{a.s.}, \tag{1.38} $$

and we immediately get the Markov property of the contact process.

The same random graph may be used to construct several contact processes, providing a natural coupling of them. This includes the restriction of the contact process to various volumes $\Lambda \subset \mathbb{Z}^d$ (e.g. cubes, slabs or cylinders). In this case we follow the previous definition, considering only the paths entirely contained in $\Lambda \times \mathbb{R}_+$. Such restricted processes are denoted by $\xi^A_\Lambda(t)$. For $\Lambda = B(N) = \{-N,\dots,N\}^d$, $N \geq 1$, we write $\xi^A_N(t)$.
Notation. If $A \subset \mathbb{Z}^d$, $\xi^A_\Lambda(t)$ stands for $\xi^{A\cap\Lambda}_\Lambda(t)$. $\mathcal{X}_\Lambda = \{0,1\}^\Lambda$.

Remark 1.17 $\xi^A_N(\cdot)$ is a continuous time Markov chain with values on the finite set $\mathcal{P}(B(N))$ (identified with $\mathcal{X}_{B(N)}$). From the graphical construction one easily verifies the following finite volume approximation: given $t_0 < +\infty$ and a finite set $B \subset \mathbb{Z}^d$, we have

$$ \lim_{N\to+\infty} \sup_{0\leq t\leq t_0} P(\xi^A(t) \cap B \neq \xi^A_N(t) \cap B) = 0, $$

for any $A \subset \mathbb{Z}^d$. On the other side, as $t \to +\infty$, the infinite system $\xi^A(t)$ behaves quite differently than $\xi_N^A(t)$. This is part of the discussion below.

From the previous construction we get the following monotonicity relations to hold a.s. for all $t > 0$:

$$\begin{aligned} &\text{(i)} \quad \xi_N^A(t) \subset \xi^A(t), \quad \text{if} \quad A \subset B(N), \\ &\text{(ii)} \quad \xi_N^A(t) \subset \xi_N^B(t), \quad \text{if} \quad A \subset B \subset B(N), \\ &\text{(iii)} \quad \xi^A(t) \subset \xi^B(t), \quad \text{if} \quad A \subset B \subset \mathbb{Z}^d. \end{aligned} \tag{1.39}$$

(Items (ii) and (iii) correspond to what is called attractiveness.)

Notation. When the initial configuration is the maximal one we will omit the superscript: $\xi_N(t) = \xi_N^{B(N)}(t)$, $\xi(t) = \xi^{\mathbb{Z}^d}(t)$.

Concerning the ergodic behaviour of the contact process, the situation for $\xi_N^A(\cdot)$ is trivial: it is a finite-state Markov chain, the empty set is an absorbing state which is reached from any initial state. That is, letting

$$T_N^A := \inf\{t > 0\colon \xi_N^A(t) = \emptyset\}, \tag{1.40}$$

then $T_N^A \le T_N^{B(N)} < +\infty$ a.s. In particular, the unique invariant measure is $\delta_\emptyset$, the Dirac pointmass at $\emptyset$. The process is ergodic, in the sense that for any A, $\xi_N^A(t)$ converges in law to $\delta_\emptyset$, as $t \to \infty$.

The infinite system shows a different behaviour, exhibiting what can be called dynamical phase transition. This was one of the reasons for the big interest it raised. Of course, $\emptyset$ keeps being an absorbing state, but there is a critical value λ_c (depending on d) so that for $\lambda > \lambda_c$ there exists a non-trivial invariant measure $\mu^{(\lambda)} \neq \delta_\emptyset$. It appears as the limit in law of $\xi(t)$, when $t \to +\infty$. (This is the analogue of existence of percolation in the model of Sec. 1.3 and we shall address it nextly.)

For N large the chain $\xi_N(\cdot)$ may be seen as a suitable truncation (perturbation) of the infinite system, and $\mu^{(\lambda)}$ gives place to a metastable state. If $\lambda > \lambda_c$, the process restricted to $B(N)$ is able to survive for a time exponentially large (in $|B(N)|$). Its metastable behaviour can be dynamically characterized, which includes the asymptotic unpredictability of $T_N/E(T_N)$. (See [35] and references therein.)

Monotonicity and attractiveness. Recall from subsection 1.2.1 the coordinatewise order on $\mathcal{X}$: $\xi \le \xi'$ means that $\xi(x) \le \xi'(x)$ for all $x \in \mathbb{Z}^d$, which corresponds to the inclusion in $\mathcal{P}(\mathbb{Z}^d)$. It induces a partial order on the space of probability measures on $\mathcal{X}$: if μ_1 and μ_2 are probability measures on $(\mathcal{X}, \mathcal{B}(\mathcal{X}))$ we say that μ_1 is stochastically smaller than μ_2, and write $\mu_1 \prec \mu_2$, if and only if $\mu_1(f) \le \mu_2(f)$ for each continuous and increasing function $f\colon \mathcal{X} \to \mathbb{R}$. (*Notation:* $\mu(f) := \int f d\mu$.)

The following classical result gives an equivalent coupling property, which extends the construction in Section 1.2.

Proposition 1.18 . *Let μ_1, μ_2 be probability measures on $\mathcal{X} = \{0,1\}^{\mathbb{Z}^d}$. The following are equivalent:*

(i) There exists a measure $\bar\mu$ on $\mathcal{X} \times \mathcal{X}$ with marginals μ_1 and μ_2 i.e.,

$$\bar\mu(A \times \mathcal{X}) = \mu_1(A), \quad \bar\mu(\mathcal{X} \times A) = \mu_2(A) \quad \textit{for all} \quad A \in \mathcal{B}(\mathcal{X}), \tag{1.41}$$

and such that

$$\bar\mu\{(\xi, \xi')\colon \xi \le \xi'\} = 1. \tag{1.42}$$

(ii) $\mu_1 \prec \mu_2$.

The important part is the implication $(ii) \Rightarrow (i)$. For a proof and further discussion see e.g. Th. 2.4, Ch. II of [29] or the references therein. The implication $(i) \Rightarrow (ii)$ is trivial (and used in subsection 1.2.1).

Recalling (1.39), we see that the use of the same "marks" in the graphical construction of the contact process provides examples of a direct verification of the coupling property (i) above. In this way the proof of this proposition is not essential for the following discussion, though this is a very useful result in a more general context.

It is proper to observe that $\prec$ indeed defines a partial order on the space $\mathcal{M}_1(\mathcal{X})$, of the probability measures on $(\mathcal{X}, \mathcal{B}(\mathcal{X}))$. (Verify it as exercise.)

Going back to the contact process, we see from (1.39)-(iii) that whenever $A \subset B$, the law of $\xi^A(t)$ is stochastically smaller than that of $\xi^B(t)$. The same holds for the process restricted to a given volume. The property of order preservation is usually called attractiveness or stochastic monotonicity.

Remark. If $(\Omega, \mathcal{A}, P)$ is a probability space on which the Poisson processes of "marks" were constructed and μ is a probability measure on $(\mathcal{X}, \mathcal{B}(\mathcal{X}))$, the contact process $\xi^\mu(t)$ with initial distribution μ may be constructed on the space $\tilde{\Omega} = \mathcal{X} \times \Omega$ with the product σ-algebra, replacing P by $\mu \times P$. That is, we first choose an initial A according to μ and then run the evolution $\xi^A(\cdot)$, according to the given "marks". In this way $P(\xi^\mu(\cdot) \in B) = \int P(\xi^\eta(\cdot) \in B)\mu(d\eta)$, where B is a Borel set on the path space. Having two initial measures μ_1 and μ_2 coupled cf. (1.42), we may use the same "marks" to couple the processes (on $\bar{\Omega} = \mathcal{X} \times \mathcal{X} \times \Omega$, $\bar{P} = \bar{\mu} \times P$) in a way that with probability one $\xi^{\mu_1}(t) \subset \xi^{\mu_2}(t)$, for all $t \geq 0$. By Proposition 1.18 we see that if $\mu_1 \prec \mu_2$, then the law of $\xi^{\mu_1}(t)$ is stochastically smaller than that of $\xi^{\mu_2}(t)$.

From (1.38) and (1.39) we see that the laws of $\xi(t)$ stochastically decrease in t. In particular, we get the existence of a probability $\mu^{(\lambda)}$ on $\mathcal{X}$ to which they converge, as $t \to +\infty$. The attractiveness implies that any weak limit of the laws of $\xi^A(t_n)$, as

$t_n \to +\infty$, is stochastically smaller than $\mu^{(\lambda)}$. Thus, we have the following equivalencies:

(i) $\mu^{(\lambda)} = \delta_\emptyset$;

(ii) $\xi^A(t)$ converges in law to $\delta_\emptyset$ for any initial A (ergodicity);

(iii) $\lim_{t\to+\infty} P\big(x \in \xi(t)\big) = 0 \quad$ for any $\quad x \in \mathbb{Z}^d$.

This is a general feature of attractive processes on $\{0, 1\}^{\mathbb{Z}^d}$: there is always a maximal (and a minimal) invariant measure and ergodicity is equivalent to their coincidence.

The graphical construction yields, for any $s, t \geq 0$, a coupling of $\xi(s+t)$ and $\xi(t)$ that is concentrated on $\{(\eta, \eta') : \eta \subset \eta'\}$, cf. (1.38) and (1.39). Letting $s \to +\infty$ (and using compactness) we get a measure $\bar{\mu}_t$ concentrated on the upper triangle $\{(\eta, \eta') : \eta \subset \eta'\}$, whose first marginal is $\mu^{(\lambda)}$ and the second is the law of $\xi(t)$. In other words, $\mu^{(\lambda)}$ is stochastically smaller than the law of $\xi(t)$ and $\bar{\mu}_t$ is a coupling as in Proposition 1.18.

Harris Inequality. An important property relating probability and order has already appeared in subsection 1.2.1: we say that a probability measure μ on $\mathcal{X}$ is FKG, or that it has positive correlations, if $\mu(fg) \geq \mu(f)\,\mu(g)$ for all functions $f, g : \mathcal{X} \to \mathbb{R}$, increasing and continuous.

We have already seen in 1.2.1 that the product measures on $\mathcal{X}$ are important examples of FKG measures. Here we briefly discuss this question in the context of dynamics.

Without a quite explicit knowledge of the measure, it is generally hard to verify the FKG property. Therefore, it is interesting to know whether a given stochastic dynamics preserves this property, that is, if the initial measure is FKG, so is the law of the process at any time $t > 0$. We would typically apply this to any fixed configuration, or more generally, to any product measure at time zero.

The following relation of the FKG property and attractive dynamics was set by Harris in [23]: an attractive continuous time Markov chain on a finite partially ordered state space preserves the FKG property if and only if it allows jumps only between comparable states.

We see at once that such property applies to the contact process on a finite volume Λ, or to any attractive chain on $\mathcal{X}_\Lambda = \{0,1\}^\Lambda$ for which only one spin changes at each jump. Recalling Remark 1.17, this extends to the infinite volume contact process. It implies that if $t \geq 0$ and f, g are increasing cylinder functions, then

$$E\big(f(\xi(t))g(\xi(t))\big) \geq E\big(f(\xi(t))\big)E\big(g(\xi(t))\big).$$

Letting $t \to +\infty$ in the previous inequality, we see that $\mu^{(\lambda)}$ has positive correlations. Moreover, applying the Markov property and the attractiveness we see that also the law of $(\xi(t_1), \ldots, \xi(t_k))$ is FKG on the product space of k copies of $\mathcal{X}$, for any $k \geq 1$, $t_1, \ldots, t_k \geq 0$.

The next variant of the previous statement is sometimes useful. For Λ and $\tilde{\Lambda}$ finite, consider the pair $(\xi_\Lambda(t), \xi_{\tilde{\Lambda}}(t))$ constructed on the same basic graph. This is an attractive process with respect to the coordinatewise order on $\mathcal{X}_\Lambda \times \mathcal{X}_{\tilde{\Lambda}}$. We see that only jumps between comparable configurations are allowed. Applying again Harris result and the approximation through finite volumes, we conclude that for any $\Lambda, \tilde{\Lambda} \subset \mathbb{Z}^d$, if f and g are increasing and continuous, then

$$E\left(f(\xi_\Lambda(t))g(\xi_{\tilde{\Lambda}}(t))\right) \geq E\left(f(\xi_\Lambda(t))\right)E\left(g(\xi_{\tilde{\Lambda}}(t))\right). \tag{1.43}$$

As in the previous paragraph, this extends to functions on the time evolution.

One may also verify (1.43) and its extension along the time evolution by setting it in terms of the percolation graph: consider first a discrete approximation based on the following random graph: for $\delta > 0$ small, $x \in \mathbb{Z}^d, k \geq 0$, arrows $(x, k\delta) \to (x, (k+1)\delta)$ are set with probability $1 - \delta$ and arrows $(x, k\delta) \to (y, (k+1)\delta)$ are set with probability $\lambda\delta$ if $\|y-x\|_1 = 1$, with full independence among different arrows. Using the FKG property of the product measure, as in subsection 1.2.1, and then letting $\delta \to 0$, one recovers (1.43).

Self-duality. Self-duality is a quite special property, stated as follows: if $A,\ B \subset \mathbb{Z}^d$ and $t > 0$, then

$$P(\xi^A(t) \cap B \neq \emptyset) \;=\; P(\xi^B(t) \cap A \neq \emptyset). \tag{1.44}$$

The above relation is particularly useful if exactly one among the sets A or B is finite. For finite B, $\xi^B(\cdot)$ is a continuous time Markov chain, and (1.44) describes the law of the infinite system $\xi^A(t)$ in terms of such chain.

Using the graphical construction, (1.44) follows from the time reversal property for a system of independent Poisson processes: we fix t positive and consider the restriction of the random graph to $\mathbb{Z}^d \times [0,t]$; reversing the direction of all arrows and of the time, and keeping the same $\times$-"marks", we get another graph, through which we define for $0 \le s \le t$:

$$\begin{aligned}\hat{\xi}^{A,t}(s) = \{x \in \mathbb{Z}^d\colon \exists \text{ a reversed path from some } (y,t),\\ y \in A, \text{ to } (x, t-s)\}.\end{aligned} \tag{1.45}$$

By a simple property of Poisson processes, the law of $(\hat{\xi}^{A,t}(s)\colon 0 \le s \le t)$ is the same as that of $(\xi^A(s)\colon 0 \le s \le t)$. On the other hand, the construction implies at once that $\xi^B(t) \cap A \neq \emptyset$ if and only if $\hat{\xi}^{A,t}(t) \cap B \neq \emptyset$, proving (1.44). (Notice that t is fixed in this construction.)

An important special case of (1.44) comes from choosing $A = \mathbb{Z}^d$, B finite:

$$P(\xi(t) \cap B \neq \emptyset) = P(\xi^B(t) \neq \emptyset) = P(T^B > t),$$

where $T^B := \inf\{t > 0\colon \xi^B(t) = \emptyset\}$ (setting $\inf\emptyset = +\infty$). In particular, letting $t \to +\infty$ we get

$$\mu^{(\lambda)}\{\eta\colon \eta \cap B \neq \emptyset\} = P(T^B = +\infty), \tag{1.46}$$

and taking $B = \{0\}$ in the last equation: $\rho_\lambda := \mu^{(\lambda)}\{\eta : 0 \in \eta\} = P(T^{\{0\}} = +\infty)$.

Dynamical Phase Transition. From the random graph construction and the translation invariance of the Poisson system of "marks", we see that $\mu^{(\lambda)}$ is translation invariant. In particular $\mu^{(\lambda)} = \delta_\emptyset$ if and only if $\rho_\lambda = 0$.

The graphical construction yields also the monotonicity in λ. Namely, if $\lambda < \lambda'$ then the law of $\xi^A(t)$ with rate λ is stochastically smaller than that with rate λ'. To see this, we may simply start with the random graph corresponding to λ', and construct a new graph by keeping the same $\times$-"marks" and for the arrows, each one is kept (disregarded) with probability λ/λ' ($1 - \lambda/\lambda'$, resp.) independently of all the rest. The process constructed on the new graph is clearly smaller than that on the initial one and corresponds to a contact process with rate λ. In particular, if $\lambda < \lambda'$ we have $\mu^{(\lambda)} \prec \mu^{(\lambda')}$. Setting

$$\lambda_c^d = \sup\{\lambda : \rho_\lambda = 0\}, \tag{1.47}$$

the previous discussion tells us:
(i) if $\lambda < \lambda_c^d$, then $\mu^{(\lambda)} = \delta_\emptyset$ and $P(T^B = +\infty) = 0$, for any finite set B;
(ii) if $\lambda > \lambda_c^d$, then $\mu^{(\lambda)} \neq \delta_\emptyset$ and $P(T^{\{0\}} = +\infty) > 0$.

A priori $\lambda_c^d \in [0, +\infty]$, and one of the above alternatives could be empty. An immediate comparison of the cardinality of $\xi^{\{0\}}(t)$ with a branching process whose birth rate is $2\lambda d$, and whose death rate is one, shows that $\lambda_c^d \geq \frac{1}{2d}$. This is a too crude comparison (at least for d small). Another simple lower bound is $\lambda_c^1 \geq 1$, obtained by comparing the diameter of $\xi^{\{0\}}(t)$ (for $d = 1$) with a random walk which increases by one with rate 2λ and decreases by one with rate 2, whenever it is larger than or equal to 2; this walk has a negative drift if $\lambda < 1$, which would force the set $\xi^{\{0\}}$ to be reduced to a point infinitely many times a.s. on the set $\{T^{\{0\}} = +\infty\}$; due to the positive death rate this forces a.s. extinction. Duality can be exploited in a more elaborated fashion to get $\lambda_c^d \geq \frac{1}{2d-1}$. (See [29].)

That $\lambda_c^d < +\infty$ constitutes an important classical result, initially proven by Harris. For any value of d it is known that $\lambda_c^d \leq \frac{2}{d}$, as proven by Holley and Liggett. (See Cor. 4.4, Ch. VI of [29].) For sharper approximations of λ_c^d see p. 128 of [31] and references therein.

The behaviour at λ_c^d had been a challenging problem. Using dynamical renormalization techniques, Bezuidenhout and Grimmett [9] have shown that $\rho_{\lambda_c} = 0$.

The measure $\mu^{(\lambda)}$ is also ergodic with respect to the translations. (See [29].) Moreover, it exhibits exponential decay of correlations. This is closely related to the speed of convergence of the dynamics as $t \to +\infty$, which plays an important role in the verification of metastability for these processes, cf. Ch. 4 of [35].

Ergodic behaviour. A first natural question concerns the full description of the set of invariant measures and convergence to equilibrium in the supercritical phase. Due to peculiar properties of the random graph on $\mathbb{Z} \times [0, +\infty)$, involving crossing of paths, the one-dimensional case is very special, for which reason the basic problems concerning the ergodic behaviour were settled much earlier in this case.

To keep the discussion as simple as possible, we first discuss the case $d = 1$ (postponing the multi-dimensional case to Chapter 4).

The characterization of $\delta_\emptyset$ and $\mu^{(\lambda)}$ as the unique extremal invariant measures constitutes a fundamental fact. In the case of a one dimensional contact process, more complete information on the ergodic behaviour is summarized by the following classical results due to Durrett ([15]).

Complete Convergence Theorem. If $\lambda > \lambda_c^1$, $A \subset \mathbb{Z}$, and

$f\colon \mathcal{P}(\mathbb{Z}) \to \mathbb{R}$ is continuous, then:

$$\lim_{t\to\infty} E(f(\xi^A(t))) = P(T^A < \infty)\, f(\emptyset) + P(T^A = +\infty)\, \mu^{(\lambda)}(f).$$

In other words, $\xi^A(t)$ converges in law to $P(T^A < \infty)\,\delta_\emptyset + P(T^A = +\infty)\,\mu^{(\lambda)}$.

Pointwise Ergodic Theorem. Under the same conditions as above:

$$\frac{1}{t}\int_0^t f(\xi^A(s))ds \to f(\emptyset)\, \mathbf{1}_{[T^A<\infty]} + \mu^{(\lambda)}(f)\, \mathbf{1}_{[T^A=+\infty]} \quad \text{a.s.}$$

Sketch of proof. For the full proofs see Th. 2.28 and Th. 2.33, Ch. VI of [29]. We sketch the basic arguments for the proof of the complete convergence theorem: considering $A = \{0\}$, the statement becomes equivalent to:

$$\begin{aligned} &\lim_{t\to+\infty} P(\xi^{\{0\}}(t) \cap B \neq \emptyset, T^{\{0\}} > t) \\ &\quad= \mu^{(\lambda)}\{\xi\colon \xi \cap B \neq \emptyset\} P(T^{\{0\}} = +\infty), \end{aligned} \tag{1.48}$$

for any finite set B.

The proof sketched here works only for $d = 1$ and heavily uses the "edge" processes: $r_t := \max \xi^{\mathbb{Z}\cap(-\infty,0]}(t)$ and $\ell_t := \min \xi^{\mathbb{Z}\cap[0,+\infty)}(t)$. ($\xi^{\mathbb{Z}\cap(-\infty,0]}(t)$ and $\xi^{\mathbb{Z}\cap[0,+\infty)}(t)$ are a.s. non-empty, and r_t and ℓ_t are well defined, integer valued.) Since the random graph on which the process is constructed lies on the plane, we can take advantage of its nearest neighbour character (horizontal arrows only among nearest neighbours) and use crossing properties of paths to see that $\{T^{\{0\}} > t\} = \{\ell_s \le r_s, \forall s \le t\}$, and that on this event $\xi^{\{0\}}(t) \cap [\ell_t, r_t] = \xi(t) \cap [\ell_t, r_t]$.

A crucial point is the existence, for $\lambda > \lambda_c^1$, of a constant $\alpha(\lambda) > 0$ so that $\lim_{t\to+\infty} \frac{r_t}{t} = \alpha(\lambda) = -\lim_{t\to+\infty} \frac{\ell_t}{t}$ a.s., i.e., a linear growth condition. Kingman-Liggett subadditive ergodic theorem (Th. 2.6, Ch. VI of [29]) is used for verification of the a.s. (and L_1) convergence. Further analysis is needed to determine that $\alpha(\lambda) > 0$ for $\lambda > \lambda_c^1$. (See Th. 2.19 and Th. 2.27, Ch. VI of [29].)

Given these ingredients and having fixed a finite set B, we have

$$\xi^{\{0\}}(t) \cap B = \xi(t) \cap B \qquad \text{on} \qquad \{[\ell_t, r_t] \supset B, T^{\{0\}} > t\},$$

and the indicator function of this last event tends a.s. to that of $\{T^{\{0\}} = +\infty\}$. Therefore one needs to argue that the conditional distribution of $\xi(t)$ given $\{T^{\{0\}} = +\infty\}$ tends to $\mu^{(\lambda)}$. If not the conditioning, this would be just the definition of $\mu^{(\lambda)}$. Using the attractiveness we see that the conditioning does not spoil the limit: fix s and take $t > s$, so that by (1.38) and (1.39)-(iii) we have $P(\xi(t) \cap B \neq \emptyset, \mathcal{E}_s) \leq P(\xi(t-s) \cap B \neq \emptyset) P(\mathcal{E}_s)$, where $\mathcal{E}_s = \{T^{\{0\}} > s\}$ or its complement. Thus $\limsup_{t\to+\infty} P(\xi(t) \cap B \neq \emptyset, \mathcal{E}_s) \leq \mu^{(\lambda)}\{\xi : \xi \cap B \neq \emptyset\} P(\mathcal{E}_s)$ for any s, in each of the two cases. But the sum on the l.h.s. (for $\mathcal{E}_s$ and its complement) tends to $\mu^{(\lambda)}\{\xi : \xi \cap B \neq \emptyset\}$ entailing that $\lim_{t\to+\infty} P(\xi(t) \cap B \neq \emptyset, T^{\{0\}} > s) = \mu^{(\lambda)}\{\xi : \xi \cap B \neq \emptyset\} P(T^{\{0\}} > s)$, for each s. Since $P(s < T^{\{0\}} < +\infty)$ tends to zero as $s \to +\infty$ it is simple to conclude the announced convergence of the conditional distribution.

The extension to any A finite is simple; the conclusion then follows by attractiveness and since $P(T^A = +\infty)$ tends to 1 as the cardinality $|A|$ tends to infinity, cf. Remark 1.19 below. $\square$

The proof of the pointwise ergodic theorem combines an argument analogous to the one we have just sketched with the standard reasoning based on the ergodic theorem for the stationary process $\xi^{\mu^{(\lambda)}}$. We omit it.

Extensions of these results to $d \geq 2$ require other tools. The validity of the complete convergence theorem for $d \geq 2$ and any $\lambda > \lambda_c^d$ is a consequence of the results in [9] to be briefly mentioned in Chapter 4.

Remark 1.19 Let $d = 1$ and $\lambda > \lambda_c^1$. Among the sets A of a given finite cardinality n, the survival probability $P(T^A = +\infty)$ is minimized if A is a "block" $\{1, \ldots, n\}$. In fact, if $A = \{x_1, \ldots, x_n\}$ and $B = \{y_1, \ldots, y_n\}$ are such that $0 < x_{i+1} - x_i \leq y_{i+1} - y_i$ for each $i = 1, \ldots, n-1$, then

$$P(T^A = +\infty) \leq P(T^B = +\infty). \tag{1.49}$$

Being the death rate constant, (1.49) should be a consequence of the way $\xi^A(t)$ growths, through empty neighbours of occupied sites. It may be verified through the following coupling argument, due to Liggett (cf. Th. 1.9, Ch.VI of [29]): it is possible to couple ξ^A and ξ^B in such a way that for any $t < T^A$ not only $|\xi^B(t)| \geq |\xi^A(t)|$, but also $\xi^B(t)$ is more spread out, as initially. That is, if $\varphi_t \colon \xi^A(t) \to \xi^B(t)$ associates to the i^{th} element of $\xi^A(t)$ (usual order) the i^{th} one in $\xi^B(t)$, then $|\varphi_t(x) - \varphi_t(y)| \geq |x - y|$. For this, set $\varphi_0(x_i) = y_i$ for each i; having defined the joint evolution up to a given time t, with the desired property, let the sites $x \in \xi^A(t)$ and $\varphi_t(x)$ use the same exponential death "clocks"; those elements of $\xi^B(t)$ which are not in the image of φ_t have independent death "clocks". Consider the collection of independent exponential "clocks" which determine the birth on the empty neighbours of $\xi^A(t)$: if the first change is a birth at $x \pm 1 \notin \xi^A(t)$ (due to an arrow from $x \in \xi^A(t)$), we simultaneously create a particle in ξ^B, at $\varphi_t(x) \pm 1$ (which is possible!); births at empty neighbours of unpaired sites in ξ^B occur independently. At each birth time s we update φ_s so that the i^{th} element of $\xi^A(s)$ (usual order) continues to correspond to the i^{th} one in $\xi^B(s)$, for each $i \leq |\xi^A(s)|$.

Chapter 2. One-dimensional contact process: rates of convergence.

Time orientation not only makes the analogy between contact process and oriented percolation very strong, but, generally speaking, many tools and non-rigorous methods could be rigorously formalized in the case of oriented percolative systems, transforming heuristic arguments into proofs. Here we will present an important example of this type.

As it was already mentioned in the preceding section, Kingman's ergodic theorem implies existence of the asymptotic shape in the supercritical phase. Here we will see a much more refined fact related to the edge process, namely the large deviations estimate:

Proposition 2.1 . *If* $\lambda > \lambda_c$ *and* $a < \alpha(\lambda)$*, then* $\lim_{t\to+\infty} \frac{1}{t} \log P(r_t < at)$ *exists and is negative.*

We will not give a detailed proof of the above proposition, nevertheless below we give all necessary details of the renormalization procedure leading to its proof.

Arguments of such type were first applied to percolation in the works of Kesten, Russo and Seymour and Welsh. The application to the one-dimensional contact process were made by Durrett and Griffeath, and extended by Gray to a quite general class of attractive one-dimensional systems. The method has shown to be useful in many applications and further developments. The basic point is that by changing the scale one is able to compare the original system to an oriented one-dependent percolation model where the density of "open" sites is arbitrarily close to 1. The use of contour arguments becomes possible and provides very useful information for the original process at any supercritical parameter λ.

In this short discussion, aiming to point out the basic scheme only, we follow [29], where the construction of Durrett and Griffeath has been presented with modifications proposed in Gray.

For full details we refer to any of these texts.

Definition 2.2 Let $\widetilde{\mathbb{Z}}^2_+ = \{(i,j) \in \mathbb{Z} \times \mathbb{Z}_+ : i+j \text{ is even}\}$. Sites (i,j), $(i',j') \in \widetilde{\mathbb{Z}}^2_+$ are declared neighbours[4] iff $||(i,j)-(i',j')||_1 := |i-i'|+|j-j'| = 2$. A one-dependent oriented percolation model on $\widetilde{\mathbb{Z}}^2_+$, with parameter p, refers to any set of random variables $\{U_{(i,j)} : (i,j) \in \widetilde{\mathbb{Z}}^2_+\}$ such that:
(i) $P(U_{(i,j)} = 1) = p$ and $P(U_{(i,j)} = 0) = 1-p$ for each (i,j);
(ii) If $F \subset \widetilde{\mathbb{Z}}^2_+$ and $||(i,j)-(i',j')||_1 > 2$ for each distinct $(i,j), (i',j') \in F$, then the random variables $\{U_{(i,j)} : (i,j) \in F\}$ are independent.
An open oriented path in this model is a sequence of sites (i_1, j_1), ..., (i_m, j_m) with $j_{k+1} = j_k + 1$, $|i_{k+1} - i_k| = 1$, if $1 \le k \le m-1$, such that $U_{(i_k, j_k)} = 1$ if $1 \le k \le m$. (That is, site (i,j) is "open" if $U_{(i,j)} = 1$; otherwise it is closed.)

The attribute "one-dependent" means that $U_{(i,j)}$, $(i,j) \in F$ are independent when F does not contain a pair of neighbours. (If $j \ge 2$, (i,j) has 8 neighbours in $\widetilde{\mathbb{Z}}^2_+$.)

One says that percolation from the origin occurs if it exists an infinite open oriented path starting at $(0,0)$, i.e, if $C_{(0,0)} := \{(i,j) : \exists \text{ open oriented path from } (0,0) \text{ to } (i,j)\}$ is infinite. The interpretation of a fluid flowing along channels through open sites is a natural one, being the reason for the expression "(i,j) is wet" $(i,j) \in C_{(0,0)}$. Occurrence of percolation means that "the fluid reaches infinitely many sites".

In other words, $\widetilde{\mathbb{Z}}^2_+$ is made into a graph with oriented edges, connecting (i,j) to each of $(i \pm 1, j+1)$, on which one considers site percolation with the described dependence.

Remark 2.3 In the setup of Definition 2.2 we need the following applications of Peierls type contour arguments (p close to one), which are crucial for Proposition 2.1. We follow Th. 3.19 and Th. 3.21, Ch. VI of [29]:

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[4]The minimal $||\cdot||_1$- distance between distinct points in $\widetilde{\mathbb{Z}}^2_+$ is 2.

(i) $P(C_{(0,0)}$ is infinite $) > 0$ if p is sufficiently near to 1.

Proof. If $U_{(0,0)} = 1$ and $C_{(0,0)}$ is finite, we may consider a "closed" contour which blocks percolation. (Contours are defined in the dual lattice.) Equivalently, we may think of the region $D := \cup_{(i,j)\in C_{(0,0)}} D_{(i,j)} \subset \mathbb{R}^2$, i.e., each site (i,j) is replaced by the diamond $D_{(i,j)} := \{(x,y) \in \mathbb{R}^2 \colon |x-i| + |y-j| \le 1\}$. Non-occurrence of percolation implies that $\mathbb{R}^2 \setminus D$ has a unique infinite component. Its boundary Γ consists of an even number (at least 4) of "segments", i.e., any of the four segments forming the boundary of a diamond $D_{(i,j)}$. Let γ be a fixed realization of Γ with $2n$ "segments", each of which belongs to the boundary of a unique "wet" $D_{(i,j)}$. Exactly n "segments" are upper (left or right) boundary of the corresponding "wet" $D_{(i,j)}$, implying that $U_{(i-1,j+1)} = 0$ (upper left) or $U_{(i+1,j+1)} = 0$ (upper right). We see that γ determines a set of at least $n/2$ different sites on which $U_\cdot = 0$ (the same site corresponds to at most two different upper "segments"); since each site has at most 8 different neighbours, we extract (in a deterministic procedure, using a given fixed total order in $\widetilde{\mathbb{Z}}^2_+$) a set F with cardinality at least $n/18$, containing no pairs of neigbhours, with $U_{(i,j)} = 0$ for each $(i,j) \in F$. Thus, $P(\Gamma = \gamma \mid U_{(0,0)} = 1) \le (1-p)^{n/18}$ and since there are at most 3^{2n-2} different possible γ with $2n$ "segments", we get $P(C_{(0,0)}$ is finite $) \le (1-p) + p\sum_{n\ge 2} 3^{2n-2}(1-p)^{n/18} < 1$, for p close to one. $\square$

(ii) Set $\tilde{r}_n := \max\{j\colon \exists$ open oriented path in $\widetilde{\mathbb{Z}}^2_+$ from $(m,0)$ to (j,n) for some $m \le 0\}$. If $\tilde{a} < 1$, $P(\tilde{r}_n \le \tilde{a}n) \le ce^{-\tilde{c}n}$ for suitable $c, \tilde{c} > 0$ and all $n \ge 1$, provided p is close enough to 1.

Proof. Let us consider

$$W = \{(i,j) \in \widetilde{\mathbb{Z}}^2_+ \colon \exists \text{ open oriented path from } (m,0) \text{ to } (i,j) \text{ for some } m \le 0\},$$

and again it is convenient to replace W by $V = \cup_{(i,j)\in W} D_{(i,j)} \subset \mathbb{R}^2$. Given $n \ge 1$ with probability one, the set $\mathbb{R} \times [0,n] \setminus V$ has only one infinite component, whose boundary we denote by Γ. The number of "segments" in Γ is of the form $n+2m$ for some $m \geq 0$. Each such "segment" is contained in the boundary of exactly one $D_{(i,j)}$, with $(i,j) \in W$, and is classified according to its position in this diamond, as upper left, upper right, lower left and lower right. For any fixed m, and calling n_{ul}, n_{ur}, n_{ll}, and n_{lr} the number of segments of the corresponding type, one has:

$$n_{lr}+n_{ll}+n_{ul}+n_{ur}=n+2m$$
$$n_{lr}+n_{ll}-n_{ul}-n_{ur}=\tilde{r}_n,$$

and so $2(n_{ul}+n_{ur})=n-\tilde{r}_n+2m$, implying that $\{\tilde{r}_n<\tilde{a}n\} \subset \{n_{ul}+n_{ur}>\frac{2m+n(1-\tilde{a})}{2}\}$. Taking into account the same observations as in the previous remark, and since the number of possible different realizations of Γ with $n+2m$ segments is at most 3^{n+2m} one ends up with $P(\tilde{r}_n<\tilde{a}n) \leq \sum_{m\geq 0} 3^{n+2m}(1-p)^{\frac{2m+n(1-\tilde{a})}{36}} < 3^{-n+1}$, provided $1-p<3^{-72/(1-\tilde{a})}$. $\square$

Remark. The reduction from $2n$ to $n/18$ on the number of "segments" (bonds in the dual lattice) contributing to the estimate comes from: the orientation, the fact that we have a site model, and that we cannot use nearest neighbour sites.

It remains to see how a rescaling procedure relates the supercritical contact process to a one-dependent percolation model, with p close to one. This brief discussion follows Sec. 3, Ch. VI of [29]. One again uses the existence and positivity of an asymptotic drift, $\alpha(\lambda)$, for the "edge" process r_t, if $\lambda>\lambda_c$, as previously discussed. Let $\alpha=\alpha(\lambda)$, $0<\beta<\alpha/3$, and choose a parameter $M>0$ such that $M\beta/2$ and $M\alpha$ are integers. For $(i,j) \in \widetilde{\mathbb{Z}}^2_+$, let us define the event $\mathcal{E}_{(i,j)}$ as the set of those realizations in the graphical construction such that for each of given parallelograms $R_{(i,j)}$ and $L_{(i,j)}$ there is a path lying completely within the given parallelogram and connecting its bottom edge to the top one. $R_{(i,j)}$ and $L_{(i,j)}$ are translates by $(Mi(\alpha-\beta), Mj)$ of $R_{(0,0)}$ and $L_{(0,0)}$, respectively, where:

$$R_{(0,0)} = \{(x,t) \in \mathbb{Z} \times [0, M(1+\beta/\alpha)]\colon \\ \alpha t - 3M\beta/2 \le x \le \alpha t - M\beta/2\},$$
$$L_{(0,0)} = \{(x,t)\colon (-x,t) \in R_{(0,0)}\}.$$

Clearly $P(\mathcal{E}_{(i,j)})$ is the same for each $(i,j) \in \widetilde{\mathbb{Z}}^2_+$. The event $\mathcal{E}_{(i,j)}$ depends only on the portion of the graph inside the rectangle

$$A_{(i,j)} := [Mi(\alpha-\beta) - M(\alpha+\beta/2), Mi(\alpha-\beta) + M(\alpha+\beta/2)] \\ \times [Mj, M(j+1+\beta/\alpha)].$$

Since $0 < \beta < \alpha/3$, $A_{(i,j)} \cap A_{(i',j')} = \emptyset$ whenever $|(i,j)-(i',j')|_1 > 2$, implying condition (ii) in Definition 2.2 for variables $U_{(i,j)} := \mathbf{1}_{\mathcal{E}_{(i,j)}}$. To escape away from criticality as previously mentioned, one verifies that for fixed $\alpha = \alpha(\lambda)$ and $0 < \beta < \alpha/3$

$$\lim_{M \to +\infty} P(\mathcal{E}_{(0,0)}) = 1. \tag{2.1}$$

The parallelograms $L_{(i,j)}, R_{(i,j)}$ for $(i,j) \in \widetilde{\mathbb{Z}}^2_+$ were constructed to "match" properly, cf. Figure 2.1, in such a way that the existence of an open oriented path $(i_1,j_1), \dots, (i_m,j_m)$ in S implies the existence of a path in the graph of the contact process from the bottom of $R_{(i_1,j_1)}$ to the top of $R_{(i_m,j_m)}$ and of $L_{(i_m,j_m)}$. In particular, percolation from $(0,0)$ guarantees the survival of the contact process starting from $A := [-3M\beta/2, 3M\beta/2] \cap \mathbb{Z}$, and Remark 2.3 provides upper bound for $P(T^A < +\infty)$, as one checks straightforwardly. It remains to prove (2.1).

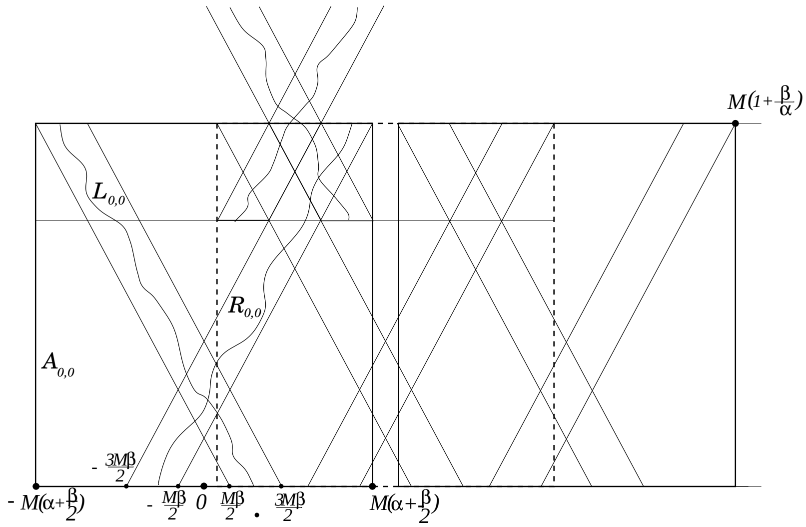

FIGURE 2.1

Proof of (2.1). For fixed α and β as above, let $\mathcal{E}$ be the set of configurations exhibiting a path which lies inside $R_{(0,0)}$ and connects the bottom of $R_{(0,0)}$ to its top. It suffices to show that $\lim_{M\to+\infty} P(\mathcal{E}) = 1$. For $n \in \mathbb{Z}$, let r_t^n denote the right edge of the contact process starting from $\xi(0) = (-\infty, n] \cap \mathbb{Z}$, so that $r_t^0 = r_t$ and the processes $(r_t^n\colon t \geq 0)$ and $(n + r_t\colon t \geq 0)$ have the same law. From this and the a.s. convergence of r_t/t to α, as $t \to +\infty$, we see that taking e.g. $M_0 = [M\beta/3] + 1$ and $t_0 = M(1 + \beta/\alpha)$, one gets

$$\lim_{M\to+\infty} P\left\{ r_t^{-M\beta/2 - M_0} < \alpha t - \frac{\beta M}{2}, \forall t \leq t_0 \right\} = 1,$$

as well

$$\lim_{M\to+\infty} P\left\{ r_{t_0}^{-M\beta/2 - M_0} > M\alpha - \frac{\beta M}{2} + M_0 \right\} = 1.$$

On the intersection of these two events the graph of $r_t^{-M\beta/2-M_0}$, $0 \le t \le t_0$ does not get to the right of $R_{(0,0)}$ and reaches its top at least M_0 sites to the right of the left extreme.

To guarantee the existence of a path as desired, it suffices to impose that $r^{-M\beta/2-M_0}$ does not touch the left boundary of $R_{(0,0)}$. By itself this is the more complicated estimate of the lower tail of r_t. An observation due to Gray transforms it into a control on the upper tail: under the given conditions, if $r^{-M\beta/2-M_0}$ arrives to the top of $R_{(0,0)}$ after touching its left boundary, the average drift must be larger than α. To prove the statement let $\mathcal{D}_n$ be the set of realizations of the graph such that there is a path connecting the vertical space-time segment $\{-3\beta M/2+n\} \times [(n-1)/\alpha, n/\alpha)$ to the horizontal strip $\{(m,t_0): m \ge M\alpha - \beta M/2 + M_0\}$. The intersection of

$$\{r_t^{-M\beta/2-M_0} < \alpha t - \beta M/2, \forall t \le t_0, r_{t_0}^{-M\beta/2-M_0} > M\alpha - \beta M/2 + M_0\},$$

with $\cap_{n=1}^{M(\alpha+\beta)} \mathcal{D}_n^c$ is contained in $\mathcal{E}$, and it remains to show that $\sum_{n=1}^{M(\alpha+\beta)} P(\mathcal{D}_n)$ tends to zero as $M \to +\infty$. Looking at the reversed graph from time t_0 we see that

$$\begin{aligned} P(\mathcal{D}_{M(\alpha+\beta)-n}) &= P(r_s > M_0 + n \text{ for some } s \in [n/\alpha, (n+1)/\alpha)) \\ &\le P(r_{n/\alpha} > 3M_0/4 + n) \\ &\quad + P(\sup_{s \in [n/\alpha,(n+1)/\alpha)} r_s - r_{n/a} > M_0/4). \end{aligned}$$

By the attractiveness $r_{t+u} \le r_u + r_{u,t}$ where, cf. (1.38), $r_{u,t} := \max {}_{(u)}\xi^{(-\infty,r_u]\cap\mathbb{Z}}(t) - r_u$ is independent of r_u, and has the same law as r_t. Using this and the comparison of r_t with a Poisson process with rate λ, we immediately see that the second term of the above decomposition decreases exponentially in M, uniformly in n. It remains to prove that

$$\lim_{M \to +\infty} \sum_{n=1}^{M(\alpha+\beta)} P(r_{n/\alpha} > 3M_0/4 + n) = 0. \tag{2.2}$$

Since Er_t/t tends to α as $t \to +\infty$, given $\varepsilon > 0$ to be chosen later, we can take $t = t_\varepsilon$ so that $Er_t < (\alpha + \varepsilon)t$. Previous observation tells that the variable r_{kt} is stochastically smaller than a sum of k i.i.d random variables distributed as r_t. But r_t is dominated by a Poisson random variable, so that $E(e^{\theta r_t}) < +\infty$ for $\theta > 0$, and we may see that if $t = t_\varepsilon$ is picked as above, there exists $\gamma_\varepsilon > 0$ so that

$$P(r_{kt} \geq kt(\alpha + \varepsilon)) \leq e^{-\gamma_\varepsilon k},$$

for all $k \geq 1$. From this and choosing ε so that $\varepsilon(1 + \beta/\alpha) < \beta/4$ we see that given $\delta > 0$ we can take n_δ (independent of M) so that the sum for $n \geq n_\delta$ in (2.2) is smaller than δ. Taking M large we control the remaining terms. □

Proof of Proposition 2.1. The existence of the limit is a consequence of the previous observation that $r_{t+s} \leq r_s + r_{s,t}$, which entails

$$P(r_{t+s} \leq a(t+s)) \geq P(r_s \leq as)P(r_t \leq at).$$

We must check that the limit is strictly negative. With α, β, and M as in the previous construction and $\tilde{r}_n$ cf. Remark 2.3, we have $r_{nM} \geq (\alpha - \beta)M\tilde{r}_n - 3\beta M/2$. It suffices to take $\beta < \alpha/3$ so that $a < \alpha - \beta$, in which case we would take any $\frac{a}{\alpha-\beta} < \tilde{a} < 1$ and M large enough for the renormalized model to fit into item (ii) of Remark 2.3, according to (2.1). For example, $\beta < \alpha/3 \wedge (\alpha - a)/2$ with the choice $\tilde{a} \in (\frac{2a}{\alpha+a}, 1)$ works. □

The usefulness of the renormalization procedure is that the choice of M large enough allows to successfully use simple contour methods. As one sees it leads to a poor control on the limit in Proposition 2.1.

Remark. The previous construction allows to see that if $\lambda > \lambda_c$ also the restricted process $\xi^{\{0\}}_{[0,+\infty)}$ has positive probability of survival, i.e., the two processes have the same critical parameter.

(For large p the related one-dependent oriented site percolation model has positive probability of survival even if restricted to $\{(i,j) \in \widetilde{\mathbb{Z}}^2_+ : i \geq 0\}$.)

Chapter 3. Application of dynamical renormalization to percolation.

The lectures covered by this chapter are based on Secs. 7.1 and 7.2 of [19].

In the previous chapter we have already seen examples of renormalization arguments applied to percolation models. The block construction used there was fixed deterministically. Another possibility that has shown to be a very powerful tool, as consequence of the flexibility it brings in, is the use of dynamically constructed blocks.

In this chapter we learn how this method was used by Grimmett and Marstrand to get a better understanding of the multidimensional ($d \geq 3$) supercritical phase in percolation. The basic ideas in [20] are similar to those used by Barsky, Grimmett and Newman in [6].

Let $d \geq 3$ and consider the two-dimensional slab of thickness k

$$\mathbb{S}_k = \mathbb{Z}^2 \times \{0, \dots, k\}^{d-2}. \tag{3.1}$$

We may consider the bond percolation problem on this graph, that is, only the edges $e \in \mathbb{E}^d$ with endvertices in $\mathbb{S}_k$ are considered, and let us write $p_c(\mathbb{S}_k)$ for the critical parameter. We see at once that $p_c(2) = p_c(\mathbb{S}_0) \geq p_c(\mathbb{S}_1) \geq \cdots \geq p_c(d) \equiv p_c$. We may let

$$p_c^{\text{slab}} := \lim_{k \to \infty} p_c(\mathbb{S}_k), \tag{3.2}$$

which automatically satisfies $p_c^{\text{slab}} \geq p_c$. That equality indeed holds constitutes one of the important results which has been obtained with the help of a dynamical block construction.

Before getting into the proof, let us think a little on the information provided by this result: if $p > p_c$ then we can find k so large that $p > p_c(\mathbb{S}_k)$, which guarantees that almost surely an infinite open cluster exists in $\mathbb{S}_k$. Using the translation invariance of the measure P_p we get infinite open clusters on all translates of S_k. This gives a good geometric information on the structure of the open cluster in $\mathbb{Z}^d$.

The basic idea is to construct random blocks of a given size so that:
(i) As the size tends to infinity, the probability of any given block being “good” tends to one.
(ii) The events that distinct blocks are good/bad are independent.
(iii) An infinite path of good blocks implies percolation in the original model.

Having this in mind and comparing with a two-dimensional site percolation, one tries to prove that a successful path as in (iii) can be constructed within $\mathbb{S}_k$ (k sufficiently large). **Notation.** $B(k) = \{-k, \ldots, k\}^d$.

Theorem 3.1 . *(a) Let $d \geq 2$ and let F be an infinite connected subset of $\mathbb{Z}^d$ with $p_c(F) < 1$. Then, for all $\eta > 0$ there exists a positive integer k so that*

$$p_c(2kF + B(k)) \leq p_c(\mathbb{Z}^d) + \eta.$$

(b) If $d \geq 3$ then $p_c^{\text{slab}} = p_c$.

Proof. We consider only the case $F = \{(x_1, \ldots, x_d) : x_3 = \cdots = x_d = 0\}$, for simplicity. In this case $2kF + B(k) = \{(x_1, \ldots, x_d)$: $-k \leq x_i \leq k, i = 3, \ldots, d\}$ and so $p_c(2kF + B(k)) = p_c(\mathbb{S}_{2k})$. To simplify further the notation we look at the details when $d = 3$, but the arguments apply to general case.

The initial construction of the proof goes as follows:
1. Assume $\theta(p) > 0$. Given $\varepsilon > 0$ we can take m so that

$$P_p(B(m) \leftrightarrow \infty) > 1 - \varepsilon, \tag{3.3}$$

i.e.,

$$P_p(\text{ an infinite open cluster intersects } B(m)) > 1 - \varepsilon. \tag{3.4}$$

2. Let $k \geq 1$. We can take $n > 2m$ large so that

$$P_p(B(m) \text{ is connected to at least } k \text{ points in } \partial B(n)) > 1 - 2\varepsilon. \tag{3.5}$$

3. If k is chosen large enough one can ensure that with probability at least $1 - 3\varepsilon$ there is a point $x \in \partial B(n)$ so that $B(m) \leftrightarrow x$ and x is connected to another translate of $B(m)$ lying on the surface of $B(n)$ and having all edges open (*seed*). This can be done for each face of $B(n)$ and it provides a first step of the procedure.
4. Starting from these new seeds adjacent to the faces of $B(n)$ one tries to iterate the procedure in a way that it stays restricted to a suitable $\mathbb{S}_N$, and being able to control the probability it fails in each step. One of the main difficulty comes from the negative information (closed edges found on the way) that one acquires in the process, and which is handled with the help of coupling arguments with a slightly larger density.

Before entering the details we fix some notation: (see Figure 3.1) e_j denotes the jth canonical unitary vector.

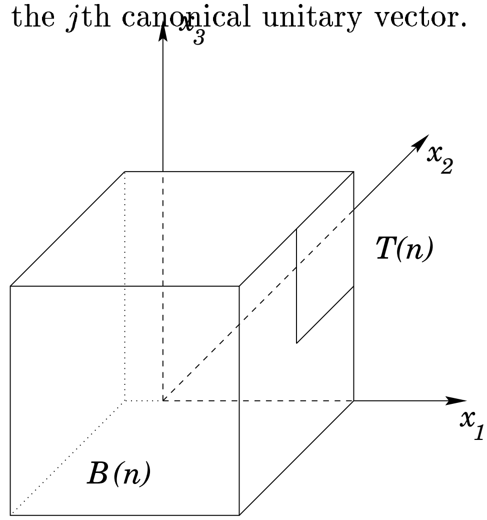

FIGURE 3.1

$$F(n) = \{x \in \partial B(n)\colon x_1 = n\},$$

called face of $B(n)$,

$$T(n) = \{x \in \partial B(n)\colon x_1 = n, x_j \geq 0 \text{ if } j \geq 2\},$$

and for $m, n \geq 1$:

$$T(m,n) = \cup_{j=1}^{2m+1}\{j\mathrm{e}_1 + T(n)\}.$$

A box $x + B(m)$ is said to be a *seed* (for a given configuration ω) if $\omega(e) = 1$ for all edges e within this box.

If $2m < n$, set

$$K(m,n) = \{x \in T(n)\colon \langle x, x+\mathrm{e}_1\rangle \text{ is open, } x + \mathrm{e}_1 \text{ lies in a seed within } T(m,n)\}. \tag{3.6}$$

With such ingredients we may state the first lemma: (See Figure 3.2.)

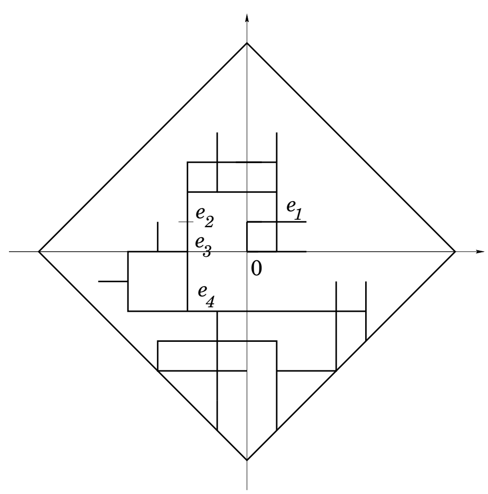

FIGURE 3.2

Lemma 3.2 . *If $\theta(p) > 0$ and $\eta > 0$, there exist $m = m(p,\eta), n = n(p,\eta)$ such that $n > 2m$ and*[5]

$$P_p(B(m) \leftrightarrow K(m,n) \text{ in } B(n)) > 1 - \eta. \tag{3.7}$$

Proof. Since $\theta(p) > 0$, an infinite open cluster exists a.s., so that

$$P_p(B(m) \leftrightarrow \infty) \to 1 \text{ as } m \to \infty.$$

Pick m so that

$$P_p(B(m) \leftrightarrow \infty) \geq 1 - (\eta/3)^{24},$$

and pick M so that

$$pP_p(B(m) \text{ is a seed}) > 1 - (\eta/2)^{1/M}.$$

Assume $2n+1 = j(2m+1)$ for an integer j, and partition $T(n)$ into squares of length $2m$. Let $V(n) = \{x \in T(n) \colon x \leftrightarrow B(m) \text{ in } B(n)\}$. If $|V(n)| \geq (2m+1)^2 M$ then there exist at least M such squares to which $B(m)$ is connected in $B(n)$. Take M of them in a fixed order; for each one (called S), let x be its smallest element in $V(m)$ for some given order, and ask if the edge $\langle x, x+\mathrm{e}_1\rangle$ is open and the box $\cup_{j=1}^{2m+1}(j\mathrm{e}_1 + S)$ is a seed. We get

$$\begin{aligned} &P_p(B(m) \leftrightarrow K(m,n) \text{ in } B(n)) \\ \geq &P_p(|V(n)| \geq (2m+1)^2 M)(1 - (1 - pP_p(B(m) \text{ is a seed}))^M) \\ \geq &P_p(|V(n)| \geq (2m+1)^2 M)(1 - \eta/2). \end{aligned} \tag{3.8}$$

[5] $B \leftrightarrow C$ in D means that there is an open path contained in D and connecting a point in B to a point in C.

To bound from below the probability on the right hand side, we recall that $\partial B(n)$ has 24 disjoint translates of $T(n)$. Thus, from the FKG inequality:

$$P_p(|V(n)| < (2m+1)^2M)^{24} \le P(|Z(n)| < 24(2m+1)^2M),$$

where $Z(n) = \{x \in \partial B(n) \colon x \leftrightarrow \partial B(m) \text{ in } B(n)\}$.

Letting $\ell = 24(2m+1)^2M$, we write

$$\begin{aligned} P_p(|Z(n)| < \ell) &\le P_p(|Z(n)| < \ell, B(m) \leftrightarrow \infty) + P_p(B(m) \not\leftrightarrow \infty) \\ &\le P_p(1 \le |Z(n)| < \ell) + (\eta/3)^{24}. \end{aligned}$$

On the event $\{|Z(n)| < \ell\}$ there are at most 3ℓ edges exiting $B(n)$ from $Z(n)$. If all of them are closed we have $Z(n+1) = \emptyset$, implying that

$$(1-p)^{3\ell} P_p(1 \le |Z(n)| < \ell) \le P_p(Z(n) \ne \emptyset, Z(n+1) = \emptyset) =: \varepsilon_n$$

which tends to zero as $n \to \infty$. Therefore

$$\begin{aligned} P_p(|V(n)| < (2m+1)^2M) \le& (P(|Z(n)| < \ell))^{1/24} \\ \le& ((1-p)^{-3\ell}\varepsilon_n + (\eta/3)^{24})^{1/24} \qquad (3.9) \\ \le& \eta/2 \end{aligned}$$

for n large ($n = n(p, \eta)$ much larger than $m = m(p, \eta)$). From (3.8) and (3.9) we get (3.7) and conclude the proof. □

Some care is needed when we want to iterate the procedure since while searching for the seeds in $T(m, n)$ we collect both positive (open edges) as well as negative information (closed edges). This prevents us from continuing just with FKG inequality. At this point there is an important trick (called "sprinkling") which comes from considering slightly larger success probability $p + \delta$. To explain it we keep in mind the uniform coupling and the notion of p-openness set in Chapter 1: $(\Sigma, \mathcal{B}(\Sigma), \mathbb{P}))$ a space on which the system of uniform variables $\{U(e)\}_{e\in\mathbb{E}^3}$ is defined, and e said to be p-open if and only if $U(e) < p$.

Notation. If $V \subset \mathbb{Z}^3$ we write $\partial^+ V = \{x \in \mathbb{Z}^3 \setminus V \colon \exists y \in V, \|x - y\| = 1\}$, the exterior vertex boundary. $\Delta V = \{\langle x, y\rangle \colon x \in V, y \notin V, \|x-y\| = 1\}$, also called the exterior edge boundary. $\mathbb{E}_V = \{\langle x, y\rangle \colon x, y \in V, \|x-y\| = 1\}$. ($\mathbb{E}_V = \mathbb{E}^3$ when $V = \mathbb{Z}^3$.)

Lemma 3.3 . *If $\theta(p) > 0$, given ε, δ positive we can find integers $m = m(p, \varepsilon, \delta), n = n(p, \varepsilon, \delta), n > 2m$ verifying the following property:*

Let R be a set such that $B(m) \subset R \subset B(n)$, $(R \cup \partial^+ R) \cap T(n) = \emptyset$ and let $\beta \colon \Delta R \cap \mathbb{E}_{B(n)} \to [0, 1-\delta]$. Set

$$\begin{aligned} G =& \{\exists \text{ path joining } R \text{ to } K(m,n), p\text{- open outside } \Delta R, \\ &\quad \text{and } (\beta(e)+\delta)\text{-open at its unique edge } e \text{ lying in } \Delta R\} \\ H =& \{\text{all edges } e \text{ in } \Delta R \cap \mathbb{E}_{B(n)} \text{ are } \beta(e)\text{-closed}\}. \end{aligned}$$

Then $\mathbb{P}(G \mid H) > 1 - \varepsilon$.

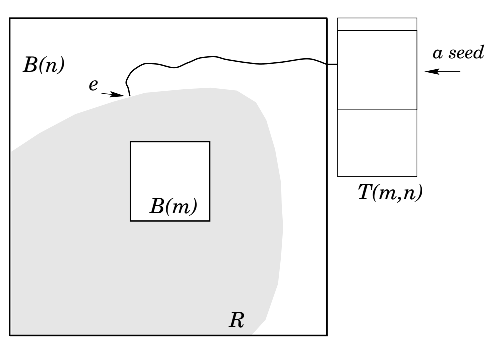

FIGURE 3.3

Proof. (See Figure 3.3 for an illustration.) Let t be an integer large enough so that

$$(1-\delta)^t < \varepsilon/2 \tag{3.10}$$

and then pick η so that

$$0 < \eta < \frac{\varepsilon}{2}(1-p)^t. \tag{3.11}$$

Apply Lemma 3.2 for this η to get m, n such that $n > 2m$ and

$$P_p(B(m) \leftrightarrow K(m,n) \text{ in } B(n)) > 1-\eta.$$

Fix R and β as in the statement. A path from $B(m)$ to $K(m,n)$ must contain a path from ∂R to $K(m,n)$. Thus

$$P_p(\partial R \leftrightarrow K(m,n) \text{ in } B(n)) > 1-\eta.$$

Let $K \subset T(n)$ and

$$Z(K) = \{\langle x,y\rangle : x \in R, y \in B(n) \setminus R, \exists \text{ open path from } y \text{ to } K \\ \text{using no edges of } \mathbb{E}_R \cup \Delta R\}.$$

We want to see that if $P_p(\partial B(m) \leftrightarrow K \text{ in } B(n))$ is large then $Z(K)$ must be large. This is clear, since any path from ∂R to K must pass by $Z(K)$, and so

$$\begin{aligned}(1-p)^t P_p(|Z(K)| \leq t) \leq & P_p(\text{all edges in } Z(K) \text{ are closed}) \\ \leq & P_p(\partial R \not\leftrightarrow K \text{ in } B(n)).\end{aligned}$$

We now apply this to $K = K(m,n)$ defined by (3.6), and we get

$$\begin{aligned}P_p(|Z(K(m,n))| \leq t) \leq & (1-p)^{-t} P_p(\partial R \not\leftrightarrow K(m,n) \text{ in } B(n)) \\ \leq & (1-p)^{-t}\eta \\ \leq & \varepsilon/2.\end{aligned}$$

Consider the coupling of various p models as before. Conditioned on $Z = Z(K(m,n))$, the random variables $U(e), e \in Z$ independent and uniformly distributed in $[0,1]$, so that

$$\begin{aligned}
&\mathbb{P}(\text{all edges } e \text{ in } Z \text{ are } (\beta(e)+\delta)\text{-closed} \mid H)\\
\leq &\mathbb{P}(|Z| \leq t \mid H) + (1-\delta)^t\\
= &P_p(|Z| \leq t) + (1-\delta)^t\\
\leq &\varepsilon/2 + \varepsilon/2 = \varepsilon.
\end{aligned}$$

□

The next ingredient for the proof of Theorem 3.1 is the construction of a renormalized process, as indicated, and which will be compared to the site percolation model. Before getting to it, one should recall that if F is a connected subset of $\mathbb{Z}^d$ then $p_c(F) < 1$ if and only if $p_c^{site}(F) < 1$. More precisely (see [19], Sec. 1.6):

Lemma 3.4 . *Let F be an infinite connected subset of $\mathbb{Z}^d$. Then*

$$p_c(d) \leq p_c(F) \leq p_c^{site}(F) \leq 1 - (1 - p_c(F))^{2d}. \tag{3.12}$$

We now turn to the proof of Theorem 3.1 and describe the renormalized construction. Let $\eta > 0$ be small so that $0 < \eta < (1-p_c)p_c$ and $p = p_c + \eta/2,\ \delta = \eta/12,\ \varepsilon = (1 - p_c^{site}(F))/24$.

Thus $p > p_c$ and so $\theta(p) > 0$. We apply Lemma 3.3 for ε, δ as above and let m, n be given by that Lemma. Let $N = m + n + 1$. $2N$ will be the side-length of the blocks used in the renormalization procedure, and we shall check the statement of Theorem 3.1 with $k = 2N$. For this, let

$$B_x = B_x(N) = 4Nx + B(N) \quad x \in \mathbb{Z}^d \quad \textit{(site-boxes)} \tag{3.13}$$

The set of vertices will be $\{4Nx : x \in \mathbb{Z}^d\}$. If x and y are adjacent then B_x and B_y are named adjacent. Those $Nz + B(N)$, with

$z \in \mathbb{Z}^d$ such that exactly one component of z is not divisible by 4 are called *bond-boxes.* If this component is divisible by 2 the box is called *half-way box.* See Figure 3.4.

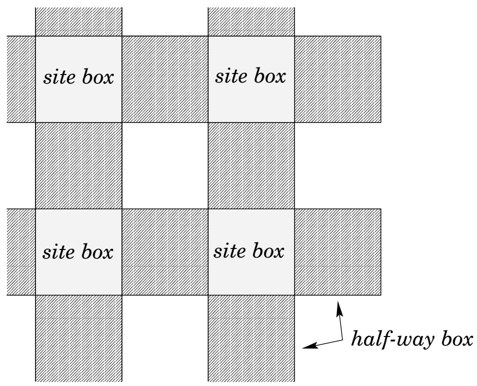

FIGURE 3.4

Site boxes will correspond to renormalized sites. They will be examined recursively in an order which will depend on the previous steps (random). For this reason the procedure is called "dynamical renormalization". A fixed ordering of the edges with endvertices in F is given as before $\{f_1, f_2, \dots\}$. The procedure along which site boxes B_x are to be examined will involve suitably defined random variables $\{Y(x) : x \in F\}$.

The event $\{B_0$ is occupied$\}$ is represented in Figure 3.6 (drawn in two dimensions, but keeping in mind that $d = 3$). Its construction and the probability estimates involve an increasing family of edge subsets $\mathcal{E}_1, \mathcal{E}_2, \dots, \mathcal{E}_8$ on which we are acquiring information along the procedure. This information will be stored in the form of Lemma 3.3, for suitable β_k, γ_k (e is $\beta_k(e)$-closed, $\gamma_k(e)$-open), and $\beta_k \uparrow, \gamma_k \downarrow$ in k.

Let $B_0 = B_0(N)$, $\mathcal{E}_1 = \mathbb{E}_{B(m)}$, and

$$\beta_1(e) = 0, \text{ for all } e \in \mathbb{E}^3$$
$$\gamma_1(e) = \begin{cases} p \text{ if } e \in \mathcal{E}_1 \\ 1 \text{ otherwise.} \end{cases} \tag{3.14}$$

At the first step one requires $B(m)$ to be a seed, or equivalently, that $\beta_1(e) \le U(e) < \gamma_1(e)$ for all e.

Notation. Two edges e, f are called adjacent, if they share one endvertex. This is denoted by $e \approx f$, and it gives a graph (with the associated notions). If $\mathcal{E} \subset \mathbb{E}^d$ we write

$$\Delta\mathcal{E} = \{f \in \mathbb{E}^3 \setminus \mathcal{E} : f \approx e \text{ for some } e \in \mathcal{E}\}.$$

For $j = 1, 2, 3$ and $\tau = \pm$, let $L_j^\tau : \mathbb{Z}^3 \to \mathbb{Z}^3$ be an isometry of $\mathbb{R}^3$ that preserves $\mathbb{Z}^3$, and such that $L_j^\tau 0 = 0, L_j^\tau \mathrm{e}_1 = \tau \mathrm{e}_j$; assume L_1^+ is the identity and that the set $\{L_j^\tau, \tau = \pm, j = 1, 2, 3\}$ is closed under composition.

$\mathcal{E}_2$ will be a random set of edges to be defined below. For this, set

$$Z_1 = B(n) \cup \bigcup_{\substack{j=1,2,3 \\ \tau=\pm}} L_j^\tau(T(m,n)) \tag{3.15}$$

Consider all edge paths π with vertices in Z_1 such that:

- its first edge f lies in $\Delta\mathcal{E}_1$, and it is $\beta_1(f) + \delta$-open;
- all other edges of π lie outside $\mathcal{E}_1 \cup \Delta\mathcal{E}_1$ and are p-open.

Writing $\tilde{\mathcal{E}}_1$ for the set of edges which belong to such paths, let

$$\mathcal{E}_2 = \mathcal{E}_1 \cup \tilde{\mathcal{E}}_1, \tag{3.16}$$

and let $\mathcal{R}_2$ be the set of endvertices of those edges belonging to $\mathcal{E}_2$. Let also

$$K_j^\tau(m,n) = \{z \in L_j^\tau(T(n)) : \langle z, z + \tau \mathrm{e}_j \rangle \text{ is } p\text{-open and lies in some seed within } L_j^\tau(T(m,n))\}. \tag{3.17}$$

We can apply Lemma 3.3 to $R = B(m)$ and $\beta = \beta_1$ to get

$$\mathbb{P}(\mathcal{R}_2 \cap K_j^\tau(m,n) \neq \emptyset \mid B(m) \text{ is a seed }) > 1 - \varepsilon, \tag{3.18}$$

so that

$$\mathbb{P}(\mathcal{R}_2 \cap K_j^\tau(m,n) \neq \emptyset, \forall \tau, j \mid B(m) \text{ is a seed }) > 1 - 6\varepsilon. \tag{3.19}$$

Let us now set:

$$\beta_2(e) = \begin{cases} \beta_1(e) \quad \text{if } e \notin \mathbb{E}_{Z_1}; \\ \beta_1(e) + \delta \quad \text{if } e \in \Delta\mathcal{E}_1 \setminus \mathcal{E}_2; \\ p \quad \text{if } e \in (\Delta\mathcal{E}_2 \setminus \Delta\mathcal{E}_1) \cap \mathbb{E}_{Z_1}; \\ 0 \quad \text{otherwise,} \end{cases} \tag{3.20}$$

and

$$\gamma_2(e) = \begin{cases} \gamma_1(e) \quad \text{if } e \in \mathcal{E}_1; \\ \beta_1(e) + \delta \quad \text{if } e \in \Delta\mathcal{E}_1 \cap \mathcal{E}_2; \\ p \quad \text{if } e \in \mathcal{E}_2 \setminus (\mathcal{E}_1 \cup \Delta\mathcal{E}_1); \\ 1 \quad \text{otherwise,} \end{cases} \tag{3.21}$$

so that $\beta_2(e) \leq U(e) < \gamma_2(e)$ for all e, on the event $[\mathcal{R}_2 \cap K_j^\tau(m,n) \neq \emptyset, \forall \tau, i, j]$. The occurrence of this event represents the second step being successful.

Notice that if E_2 is an arbitrary edge subset and R_2 is the corresponding vertex set, we can find $A \in \sigma(U(e){:}\, e \in \mathbb{E}_{R_2})$ such that

$$\begin{aligned} &\{\sigma \in \Sigma{:}\, \mathcal{E}_1 = \mathbb{E}_{B(m)}, \mathcal{E}_2 = E_2\} \\ =& A \cap \{\beta_{2,\mathcal{E}_1 = \mathbb{E}_{B(m)}, \mathcal{E}_2 = E_2}(e) \leq U(e), \text{ for } e \in \Delta R_2\} \end{aligned}$$

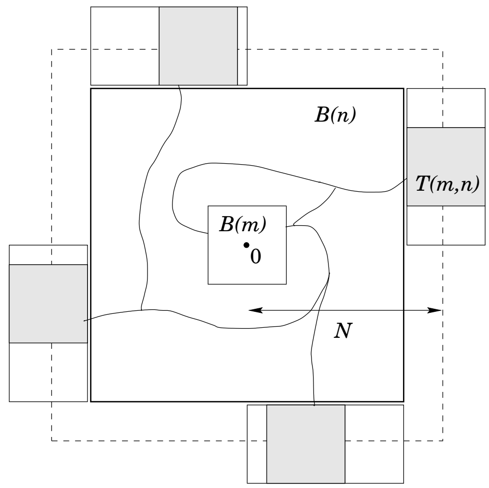

FIGURE 3.5

If these first two steps of the procedure are successful (see Figures 3.5 and 3.7) we proceed further trying to link the obtained seed in each $L_j^\tau(T(m,n))$ to a new seed in the half-way box $2\tau N e_j + B(N)$. If this is successful in all six directions, as explained below, we will say that the renormalized site B_0 is occupied. (See Figure 3.6.)

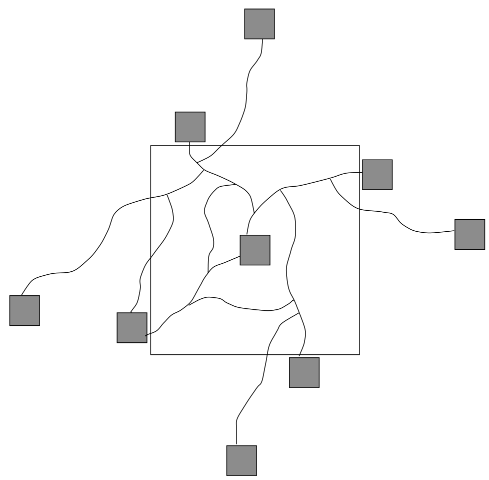

FIGURE 3.6 OCCUPIED RENORMALIZED SITE

To fix ideas take $\tau = +, j = 1$, so that we are looking at $2N\mathrm{e}_1 + B(N)$. Let $b_2 + B(m)$ be the earliest seed (smaller b_2 in lexicographic order) with all its edges in $\mathcal{E}_2 \cap \mathbb{E}_{T(m,n)}$. Then $b_2 + B(m) \subset N\mathrm{e}_1 + B(N)$. Since all coordinates of b_2 are positive, we see that $b_2 + T(m,n) \not\subset 2N\mathrm{e}_1 + B(N)$. Therefore we replace $T(m,n)$ by

$$T^*(m,n) = \cup_{j=1}^{2m+1}(j\mathrm{e}_1 + T^*(n)), \text{ where}$$
$$T^*(n) = \{x \in \partial B(n) \colon x_1 = n, x_j \le 0, \text{ for } j = 2,3\},$$

in such a way that $b_2 + T^*(m,n) \subset 2N\mathrm{e}_1 + B(N)$. Then one sets

$$Z_2 = b_2 + (B(n) \cup T^*(m,n)) \tag{3.22}$$

and defines the following random sets of edges and vertices.

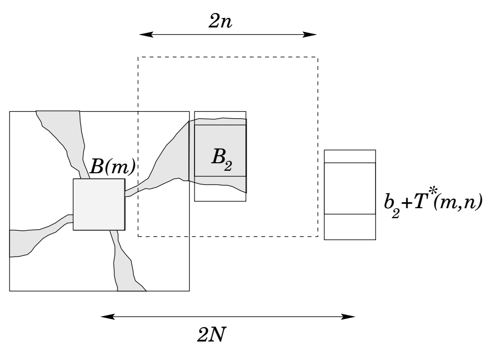

FIGURE 3.7

$\tilde{\mathcal{E}}_2$ will be the set of the edges that belong to paths in Z_2, such that:

- its first edge f lies in $\Delta\mathcal{E}_2$ and it is $\beta_2(f)+\delta$-open;
- other edges are outside $\mathcal{E}_2 \cup \Delta\mathcal{E}_2$ and are p-open.

One sets $\mathcal{E}_3 = \mathcal{E}_2 \cup \tilde{\mathcal{E}}_2$, $\mathcal{R}_3$ is the set of endvertices for edges belonging to $\mathcal{E}_3$, and

$$K^*(m,n) = \{z \in b_2 + T^*(m,n) \colon \langle z, z+\mathrm{e}_1\rangle \text{is } p\text{-open}, z+\mathrm{e}_1 \\ \text{lies in some seed contained in } b_2 + T^*(m,n)\}.$$

Applying Lemma 3.3: condition on $\mathcal{E}_1, \mathcal{E}_2$, and $R = \mathcal{R}_2 \cap (b_2 + B(n))$, with $\beta(\cdot)$ given by the restriction of $\beta_2(\cdot)$ to $\Delta R \cap \mathbb{E}_{b_2+B(n)}$, and we see that

$$\mathbb{P}(\mathcal{R}_3 \cap K^*(m,n) \neq \emptyset \mid \mathcal{E}_1, \mathcal{E}_2) > 1-\varepsilon. \tag{3.23}$$

Thus, we may introduce the new functions $\beta_3(\cdot) \le \gamma_3(\cdot)$:

$$\beta_3(e) = \begin{cases} \beta_2(e) & \text{if } e \notin \mathbb{E}_{Z_1\cup Z_2}; \\ \beta_2(e)+\delta & \text{if } e \in (\Delta\mathcal{E}_2 \setminus \mathcal{E}_3) \cap \mathbb{E}_{Z_1\cup Z_2}; \\ p & \text{if } e \in (\Delta\mathcal{E}_3 \setminus \Delta\mathcal{E}_2) \cap \mathbb{E}_{Z_1\cup Z_2}; \\ 0 & \text{otherwise,} \end{cases} \tag{3.24}$$

and

$$\gamma_3(e) = \begin{cases} \gamma_2(e) & \text{if } e \in \mathcal{E}_2; \\ \beta_1(e) + \delta & \text{if } e \in \Delta\mathcal{E}_2 \cap \mathcal{E}_3; \\ p & \text{if } p \in \mathcal{E}_3 \setminus (\mathcal{E}_2 \cup \Delta\mathcal{E}_2); \\ 1 & \text{otherwise.} \end{cases} \tag{3.25}$$

It follows that given a set of edges $E_3 \supset E_2$ and the corresponding set of vertices R_3 there exists $A \in \sigma(U(e) \colon e \in E_3)$ such that

$$\begin{aligned} &\{\mathcal{E}_1 = \mathbb{E}_{B(m)}, \mathcal{E}_2 = E_2, \mathcal{E}_3 = E_3\} \\ =&A \cap \{\beta_{3,\mathcal{E}_1=\mathbb{E}_{B(m)},\mathcal{E}_2=E_2,\mathcal{E}_3=E_3}(e) \le U(e), \text{ for } e \in \Delta R_3\}. \end{aligned}$$

Proceeding with half-way boxes in the other directions, we say that the renormalized site B_0 *is occupied* if we succeed in all directions, see Figure 3.5. From (3.19), (3.23), and its analogues in the other directions, we see that

$$\begin{aligned} &\mathbb{P}(B_0 \text{ is occupied } \mid B(m) \text{ is a seed }) \\ >&(1-6\varepsilon)(1-\varepsilon)^6 \ge 1 - 12\varepsilon = (1 + p_c^{site}(F))/2, \end{aligned} \tag{3.26}$$

due to the definition of ε (right after the statement of Lemma 3.4). It is important to observe that at the end of this procedure we have

$$\beta_8(e) \le \gamma_8(e) \le p + 6\delta \quad \text{for } e \in \mathcal{E}_8. \tag{3.27}$$

No edge lies in more than seven translates of $B(n)$ and from the construction one gets

$$\beta_8(e) \le \gamma_8(e) \le p + 6\delta = p + \eta/2 \qquad \text{for } e \in \mathcal{E}_8. \tag{3.28}$$

Since $U(e) < \gamma_8(e)$ we see that it is $(p + \eta/2)$-open.

Having settled what it means for the renormalized site B_0 to be occupied, we see how this renormalized process can be dynamically constructed so as to yield the comparison with a standard site percolation model on F. Such dynamical procedure[6] is now sketched. (See Figure 3.8.)

Consider $\mathbb{E}_F$ ordered in any fixed way: $f_1, f_2, \ldots$, and construct a sequence of pairs $(A_k, \tilde{A}_k)$, $k = 1, 2, \ldots$ of subsets of F:

- $A_1 = \{0\}, \tilde{A}_1 = \emptyset$ if B_0 is occupied.
- $A_1 = \emptyset, \tilde{A}_1 = \{0\}$ if B_0 is not occupied.

Having defined $(A_1, \tilde{A}_1), \ldots, (A_k, \tilde{A}_k)$, consider the subset E of $\mathbb{E}_F$ of those edges with one endvertex in A_k and the other outside $A_k \cup \tilde{A}_k$.

If $E = \emptyset$, then $A_{k+1} = A_k, \tilde{A}_{k+1} = \tilde{A}_k$, i.e., the process stops.

If $E \neq \emptyset$, let f be its smallest element and let x_{k+1} be the endvertex of f which is outside $A_k \cup \tilde{A}_k$. We then set:

- $A_{k+1} = A_k \cup \{x_{k+1}\}, \tilde{A}_{k+1} = \tilde{A}_k$ if $B_{x_{k+1}}$ is occupied;
- $A_{k+1} = A_k, \tilde{A}_{k+1} = \tilde{A}_k \cup \{x_{k+1}\}$ if $B_{x_{k+1}}$ is not occupied.

Thus $A_1 \subset A_2 \subset \ldots$; $\tilde{A}_1 \subset \tilde{A}_2 \subset \ldots$ and we let

$$A_\infty = \cup_{k\geq 1} A_k, \quad \tilde{A}_\infty = \cup_{k\geq 1} \tilde{A}_k.$$

The set A_∞ can be thought as part of the occupied renormalized cluster of B_0 and $\tilde{A}_\infty = \partial^+ A_\infty$ its exterior vertex boundary (with the convention that $\partial^+\emptyset = \{0\}$).

Remark 3.5

(a) If "B_x is occupied" (or not) is replaced by "x is open" (or closed) independently with probability $\tilde{p}$ ($1-\tilde{p}$, respectively), the construction yields the site percolation cluster of the origin under the Bernoulli measure on $\{0,1\}^F$ with parameter $\tilde{p}$.

(b) Given any family of $\{0,1\}$-valued random variables $(Y(x), x \in F)$ the construction is applicable with "B_x is occupied" replaced by "$Y(x) = 1$".

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[6] Very similar to the usual proof of the last inequality Lemma 3.4 that we omitted.

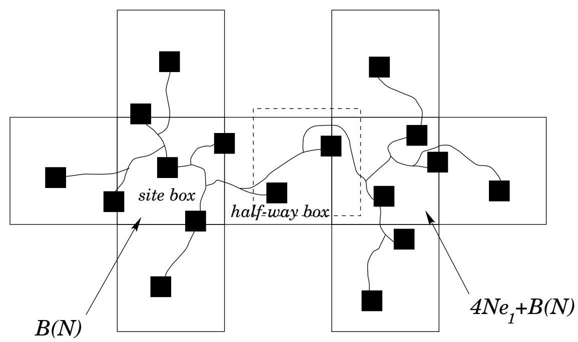

FIGURE 3.8 DYNAMICAL CONSTRUCTION (RENORMALIZED)

We need to examine the above construction and to establish a comparison with the simple independent process. We start assuming that B_0 is occupied, since otherwise the process stops at once, and consider the smallest edge f which has endvertices 0 and $x \in F$. To fix ideas we assume it to be $\langle 0, \mathrm{e}_1 \rangle$ and examine if B_{e_1} is occupied, which is done as follows:

(i) try to link the seed in the half-way box $2N\mathrm{e}_1 + B(N)$ to the site-box $4N\mathrm{e}_1 + B(N)$;

(ii) provided (i) is successful, we then try to link such seed to the other five half-way boxes that touch $4N\mathrm{e}_1 + B(N)$.

We say that B_{e_1} is occupied if both (i) and (ii) are successful. The procedure at each of these steps is as follows:

For step (i) we proceed in two sub-steps: in each of them we first look for seeds of the type $b + B(m)$ located on the proper quarter faces of boxes, in order to compensate the effect resulting from the fact that the starting seed is not at the center of the half-way box $2Ne_1 + B(N)$, in the first sub-step (of the bond-box $3Ne_1 + B(N)$, in the second sub-step).

In step (ii) we start from the seed $b^* + B(m)$ obtained in (i) and try to construct the seeds in the other five half way boxes in a suitable way: look for seeds on the surface of $4Ne_1 + B(N)$ with vertices whose first coordinate is not smaller than that of b^* (to guarantee they have not being examined yet). If successful, one tries to link this new seed to each of the five corresponding half-way boxes.

From the previous analysis we get the following probability estimate:

$$\mathbb{P}(B_{e_1} \text{ is occupied } \mid B_0 \text{ is occupied}) > (1 + p_c^{site}(F))/2. \quad (3.29)$$

The process is continued as described above. Calling $\gamma := (1 + p_c^{site}(F))/2 > p_c^{site}(F)$, we see that for $k \geq 1$, having constructed $(A_1, \tilde{A}_1), \ldots, (A_k, \tilde{A}_k)$, then we have

$$\mathbb{P}\left(B_{x_{k+1}} \text{ is occupied} \mid (A_1, \tilde{A}_1), \ldots, (A_k, \tilde{A}_k)\right) \geq \gamma, \quad (3.30)$$

in the notation described above.

This allows us to use a comparison with an (independent) site percolation model on F, and it is convenient to use the standard coupling as in Section 1.2, now applied to site variables: consider $(U(x), x \in F)$ i.i.d. random variables uniformly distributed on the interval $[0, 1]$.

The site $0 \in F$ is declared to be *red* if $U(0) < \mathbb{P}(B_0 \text{ is occupied})$, and *black* otherwise, in which case the process stops. If 0 is declared red, we start the construction described above with A_k as the set of sites colored red, and $\tilde{A}_k$ the black ones up to the k-th step, for each k. Given $(A_1, \tilde{A}_1), \ldots, (A_k, \tilde{A}_k)$, the site x_{k+1} is colored red or black according to $U(x_{k+1}) < \mathbb{P}(B_{x_{k+1}} \text{ is occupied} \mid (A_1, \tilde{A}_1), \ldots, (A_k, \tilde{A}_k))$, or not. In this way we clearly have the distribution of these last random sets $(A_1, \tilde{A}_1), (A_2, \tilde{A}_2), \ldots$ under

$\mathbb{P}$ is the same as that of the previous ones. The set $\tilde{A}_\infty$ coincides with the exterior vertex boundary of A_∞ and moreover, for $x \in F$ to belong to $\tilde{A}_\infty$ we must have $U(x) \geq \gamma$, i.e. site x must be γ-closed. Consequently we see that the cluster of γ-open sites which contains 0 cannot intersect $\tilde{A}_\infty$ and so it must be contained in A_∞. Since $\gamma > p_{c,site}(F)$ we have that

$$\mathbb{P}(|A_\infty| = \infty) > 0. \tag{3.31}$$

Finally notice that (3.31) implies that $4NF + B(2N)$ contains and infinite $(p+\eta)$-open path, so that $p_c(4NF + B(2N)) \leq p + \eta$, concluding the proof of Theorem 3.1. □

The comparison of this last step is usually summarized as follows:

Lemma 3.6 . *Let $Y(x), x \in F$ be a family of $\{0,1\}$-valued random variables, and consider the previous dynamical construction $(A_k, \tilde{A}_k), k \geq 1$, cf. Remark 3.5 (b). If there exists a constant $\gamma > p_c^{site}(F)$ so that*

$$P(Y(x_{k+1}) = 1 \mid (A_1, B_1), \ldots, (A_k, B_k)) \geq \gamma, \quad \text{for all } k, \tag{3.32}$$

then $P(|A_\infty| = \infty) > 0$.

Chapter 4. Half-space percolation and multi-dimensional contact process.

The conjecture that for ordinary percolation $\theta(p_c) = 0$ up to now remains one of the major open problems for $3 \leq d \leq 18$. However, multiple attempts to prove it brought to the interesting results on their own, and led to the development of new techniques. Here we stop with an example, which played an important role in the history of percolation theory, and served as a basis for further developments, namely for the proof that multi-dimensional contact process dies out at the critical point, and in particular was the triggering work for the results of the preceding chapter.

In fact, the method of the last chapter comes close to proving that $\theta(p_c) = 0$ for general d. Indeed, assume that $\theta(p) > 0$ and $\eta > 0$. Then there is an event, call it A, defined in a finite box B such that:

i) $P_p(A) > 1 - \epsilon$, for some prescribed small $\epsilon > 0$;

ii) the fact i) implies that $\theta(p + \eta) > 0$.

Assume, that one can prove this fact with $\eta = 0$, and assume also that $\theta(p_c) > 0$. Then $P_{p_c}(A) > 1 - \epsilon$. But since B is finite, it implies that $P_p(A)$ is continuous function of p. Therefore there exists $p' < p_c$, such that $P_{p'}(A) > 1 - \epsilon$. It follows thus by ii) that $\theta(p') > 0$. This contradicts the definition of p_c, and therefore $\theta(p_c) = 0$.

Going back to the contact process, recall that several arguments in the Chapter 2 (and in the quoted proofs) required that $d = 1$. Due to the planarity of the graphical construction, one could take advantage of path intersection properties. Together with the subadditive ergodic theorem, this was a key fact in the proof of basic theorems. One example is the linear growth of $\xi_t^{\{0\}}$ given its survival, used in Durrett-Griffeath's renormalization procedure.

The replacement of a priori fixed blocks in the renormalization procedure by a more flexible dynamical construction represented a breakthrough in the analysis of percolative systems. It is due to Barsky, Grimmett and Newman [6], who considered high-dimensional (non-oriented) percolation and to Bezuidenhout and Grimmett [9], who constructed a variant of the procedure for the higher dimensional contact process.

4.1. Percolation in half-spaces.

Barsky, Grimmett and Newman [6] considered the following problem: Let $d \geq 2$, and let $\mathbb{H} = \mathbb{Z}^{d-1} \times \mathbb{Z}_+$. $\mathbb{H}$ is called "half-space", and we write

$$\theta_{\mathbb{H}}(p) = P_p(0 \leftrightarrow \infty \text{ i}n\ \mathbb{H})$$

for the corresponding probability, and

$$p_c(\mathbb{H}) = \sup\{p : \theta_{\mathbb{H}}(p) = 0\}$$

for the critical probability of $\mathbb{H}$. It follows from the main theorem of the last Chapter, by taking $F = \mathbb{H}$ that $p_c(\mathbb{H}) = p_c$.

Theorem 4.1 . *Let $d \geq 2$. We have that $\theta_{\mathbb{H}}(p_c) = 0$.*

We will outline basic steps of the proof. For simplicity take $d = 3$.
Blocks. Let L, H be positive integers and define the block $B(L, H)$ by

$$B(L, H) = [-L, L]^2 \times [0, H].$$

We define "an underside" U, "a top" T, and "sides" S of the brick in the following way:

$$U = U(L, H) = [-L, L]^2 \times \{0\},$$
$$T = T(L, H) = [-L, L]^2 \times \{H\},$$
$$S = S(L, H) = \{x \in B(L, H) : |x_j| = L \text{ for some } j \in \{1, 2\}\}.$$

The top T may be divided into four congruent "subfacets":

$$T_1 = [0, L]^2 \times \{H\}, \quad T_2 = [0, L] \times [-L, 0] \times \{H\},$$
$$T_3 = [-L, 0]^2 \times \{H\}, \quad T_4 = [-L, 0] \times [0, L] \times \{H\}.$$

Analogously, the set S may be divided into four "facets", each of which may be divided into two "sub-facets" congruent to $\{L\} \times [0, L] \times [0, H]$. We will denote the eight corresponding regions by $S_1, S_2, \ldots, S_8$.

Let m be a positive integer. We define the two-dimensional region

$$b_k(m) = [-m, m]^{k-1} \times \{0\} \times [-m, m]^{3-k} \quad \text{for } k = 1, 2, 3,$$

and we designate as squares all translates $x + b_k(m)$ for $x \in \mathbb{Z}^3$ and $k = 1, 2, 3$. A square $x + b_k(m)$ is called a seed if all edges pairs of vertices in the square are open. To each $x \in S \cup U$ we associate a square $b(x) = b(x, m, L, H)$ having x at its center, in the following manner: to any $x \in T$ we associate the square $b(x) = x + b_3(m)$. If $x \in S \setminus T$, we find some i, such that $x \in S_i$, and we define $k(x)$ such that $b(x) = x + b_{k(x)}(m)$ is parallel to S_i; for x belonging to more than one S_i, we pick one of these according to some predetermined rule. Let $b(0) = b_3(m)$, a square having the origin at its center.

Assume that $L \geq m$ and $H \geq 2m$. The block $B(L, H)$ is called *good* if:

a) there exists $x_i \in S_i$ for $1 \leq i \leq 8$, and $y_j \in T$ for $1 \leq j \leq 4$, such that every $b(x_i)$ and every $b(y_j)$ is a seed;

b) for every i, there exists an open path of $B(L, H)$ joining $b(x_i)$ to $b(0)$ using no edges in the underside U;

c) for every j, there exists an open path of $B(L, H)$ joining $b(y_j)$ to $b(0)$ using no edges in the underside U.

Lemma 4.2 . *Suppose that $\theta_{\mathbb{H}}(p_c) > 0$. If $\eta > 0$, there exist integers m, L, H, satisfying $m \geq 1, L \geq m$, and $H \geq 2m$, such* *that*

$$P_{p_c}(B(L,H) \textit{ is good}) > 1 - \eta.$$

Now, using the construction similar to the previous chapter, one shows

Lemma 4.3 . *There exists a strictly positive number ν such that the following holds. Let $0 < p < 1$. Suppose that $m, L, H,$ are positive integers satisfying $m \geq 1, L \geq m$, and $H \geq 2m$, such that*

$$P_{p_c}(B(L,H) \textit{ is good}) > 1 - \nu. \tag{4.1}$$

Then

$$\theta_{\mathbb{H}}(p) > 0.$$

Thoerem 4.1 is a consequence of Lemmas 4.2 and 4.3: suppose $\theta_{\mathbb{H}}(p_c) > 0$, and let ν be given as in Lemma 4.3. By Lemma 4.2, there exist integers m, L, H, satisfying $m \geq 1, L \geq m$, and $H \geq 2m$, such that (4.1) holds with $p = p_c$. The event $\{B(L,H)$ is good$\}$ depends on the states of a finite set of edges only, whence its probability under P_p is a continuous function of p. Therefore there exists $p' < p_c$ such that

$$P_{p_c}\{B(L,H) \text{ is good}\} > 1 - \nu.$$

By Lemma 4.3 we have that $\theta_{\mathbb{H}}(p') > 0$, and it follows by contradiction that $\theta_{\mathbb{H}}(p_c) = 0$.

4.2 Contact process.

Taking advantage of the flexibility of the construction in the proof of Lemma 4.3, Bezuidenhout and Grimmett proved that if the contact process survives on $\mathbb{Z}^d \times [0, +\infty)$ with positive probability, the same holds on a sufficiently deep space-time slab $\Lambda_K^{d-1} \times \mathbb{Z} \times [0, +\infty)$, where $\Lambda_K = \{-K, \ldots, K\}$ as before.

The statement is based on the validity of a finite volume condition: if $\lambda > \lambda_c^d$, $\varepsilon > 0$, and M is any given (large) number, one can find r, L, T so that starting from $\Lambda_r^d \times \{0\}$, the (space-time) process restricted to $B_{L,T} = \Lambda_L^d \times [0, T]$ not only survives up to time T with probability at least $1 - \varepsilon$, but also produces at least M infected points on $\Lambda_L^d \times \{T\}$ and at least M well separated infected points on each side of $B_{L,T}$, i.e., points connected to $\Lambda_r^d \times \{0\}$ through a path contained in $B_{L,T}$. (In fact, one can find such many points on each orthant of the top and of the sides of $B_{L,T}$.) Having many well separated points one is able to grow again a suitable translate of $\Lambda_r^d \times \{0\}$, using only paths contained in $\Lambda_L^{d-1} \times [L, 2L] \times [0, 2T]$. (The FKG inequality plays an important role in the verification of this property.)

Using this kind of estimate and successive restarting, one is able to compare the process on a slab $\Lambda_{2L}^{d-1} \times \mathbb{Z} \times [0, +\infty)$ to a suitable one-dependent oriented bond percolation on $S = \{(u, v) \in \mathbb{Z} \times \mathbb{Z}_+, u + v \text{ is even}\}$, each bond $(u, v) \to (u \pm 1, v + 1)$ being open with probability very near to one. For this, if $k \in \mathbb{N}$, one considers the sets

$$\mathcal{L}^{\pm} = \Lambda_{2L}^{d-1} \times \{(x, t) \colon 0 \le t \le (2k+2)T, -5L \pm \frac{L}{2T} t \le x \le 5L \pm \frac{L}{2T} t\},$$

and for each $x \in \Lambda_{2L}^{d-1} \times [-2L, 2L]$ and $t \in [0, 2T]$, let $\mathcal{E}^{\pm}(x, t)$ be the event that $(x, t) + \Lambda_r^d \times \{0\}$ is connected inside $\mathcal{L}^{\pm}$ to each point in some translate $(y, s) + \Lambda_r^d \times \{0\}$, with $(y, s) \in ((\pm k - 2)L, (\pm k + 2)L] \times [2kT, (2k + 2)T]$. The basic estimate iterated k times tells us that given $\delta > 0, k \in \mathbb{N}$, we can take L and T so that $P(\mathcal{E}^{\pm}(x, t)) \ge 1 - \delta$, for all such (x, t). Using "seeds" (fully infected translates of initial $\Lambda_r^d \times \{0\}$) located in $(ukL\mathbf{e}_d, 2vkT) + \Lambda_L^{d-1} \times \Lambda_{2L} \times [0, 2T]$, with $(u, v) \in S$, one gets the previously mentioned comparison, and the survival (with positive probability) in the space-time region

$$\mathcal{L} = \cup_{(u,v) \in S} \{(ukL\mathbf{e}_d, 2vkT) + (\mathcal{L}^+ \cup \mathcal{L}^-)\}.$$

($\mathbf{e}_d$ denotes the canonical unit vector in $\mathbb{R}^d$)

We will not discuss this dynamical renormalization procedure here, referring reader to the original article [9], or [30], where modified version is presented. An important feature is that once the basic growth condition involves only a finite volume, it is continuous in λ. This implies one of the important conclusions in: the critical contact process dies out.

Chapter 5. Multi-scale analysis at work.

In this chapter we look at two examples where multi-scale renormalization techniques are applied to the study of percolative systems in the presence of disorder.

In usual situations the homogeneous system exhibits at least two different types of behaviour (called phases), obtained by varying one or more parameters: an ordered phase, characterized by the existence of long range order in the system, and a localized phase, characterized by the decay of some correlation functions. For instance, in the independent homogeneous percolation model, by varying the parameter p we have either subcritical behaviour (diameter of open cluster decays exponentially fast) or supercritical behaviour, with the presence of an infinite open cluster, and finally the critical point p_c.

In the presence of disorder, each phase may manifest itself in infinitely many arbitrarily large regions, where the system's parameters will be in the range characteristic of these phases.

5.1 The percolation model in a dependent environment.

Let us consider the following oriented site percolation model: on the graph

$$\widetilde{\mathbb{Z}}^2_+ = \{(x,y) \in \mathbb{Z} \times \mathbb{Z}_+ : x+y \text{ is even}\},$$

the lines $H_i := \{(x,y) \in \widetilde{\mathbb{Z}}^2_+ , y = i\}$ are first declared to be *bad* or *good*, independently of each other, with probabilities δ and $1-\delta$, respectively; sites on good lines are open with probability p_G, and on bad lines they are open with probability p_B, independently of each other, given the configuration of lines. More formally: on a suitable probability space $(\Omega, \mathcal{A}, P)$, we take a Bernoulli sequence $\xi = (\xi_i : i \in \mathbb{Z}_+)$ with $P(\xi_i = 1) = \delta = 1 - P(\xi_i = 0)$, which determines if H_i is bad or good, accordingly. Given the configuration ξ

we have occupation variables $(\eta_z : z \in \widetilde{\mathbb{Z}}^2_+)$ which are conditionally independent given ξ, and $P(\eta_z = 1 \mid \xi) = p_B = 1 - P(\eta_z = 0 \mid \xi)$ if $z \in H_i$ with $\xi_i = 1$, and $P(\eta_z = 1 \mid \xi) = p_G = 1 - P(\eta_z = 0 \mid \xi)$ if $z \in H_i$ with $\xi_i = 0$. If $\eta_z = 1$ the site z is open, and otherwise it is closed. We consider the oriented site model, i.e. an *open oriented path* is a path directed upwards (northwest-northeast) all of whose vertices (or sites) are open. (With respect to the usual presentation of oriented percolation on $\mathbb{Z}_+ \times \mathbb{Z}_+$, as in Sec. 1.3, there has been a rotation of $\pi/4$ counterclockwise.)

The interesting situation is when $p_G > p_c$, the critical probability for homogeneous oriented site percolation on $\mathbb{Z}_+ \times \mathbb{Z}_+$, and p_B a small positive number. Given $p_G > p_c$ and $p_B > 0$ we ask if $\delta > 0$ may be taken small enough so that there is percolation, that is, a positive probability of percolating to infinity from the origin. The answer is positive, as stated in the next theorem, due to Kesten, Sidoravicius and Vares ([27]).

Theorem 5.1 . *In the setup described above, let*

$$\Theta(p_G, p_B, \delta) \;=\; P(C_0 \textit{ is infinite }),$$

where C_0 denotes the oriented open cluster of the origin. Then if $p_G > p_c$ and $p_B > 0$ we can find $\delta_0 = \delta_0(p_G, p_B) > 0$ so that $\Theta(p_g, p_B, \delta) > 0$ for all $\delta \le \delta_0$.

The proof of this theorem has some quite involved features. To avoid them but still discuss the main aspects of the multi-scale renormalization method, in these lectures we consider a much simpler model, where the configuration of bad/good lines is deterministic and has an hierarchical structure. Moreover, we prove only a weaker version, for p_G close to one. (The extension from the case of large p_G to any super-critical value is not hard.)

The hierarchical model.

We begin with the description of the *hierarchical* binary sequence ξ^L which gives the lines configuration, and where L is a suitably large integer. We first consider the sequence $\tilde{\xi}^L = \{\tilde{\xi}^L_n\}_{n\geq 1} \in \mathbb{N}^{\mathbb{N}}$ by setting:

$$\tilde{\xi}^L_n = \begin{cases} k & \text{if } L^k \mid n, \text{ and } L^{k+1} \nmid n; \\ 0 & \text{if } L \nmid n. \end{cases}$$

The binary sequence is defined as follows: $\xi^L_0 = 0$, and $\{\xi^L_n\}_{n\geq 1}$ is obtained from $\tilde{\xi}^L$ by replacing each element $\tilde{\xi}^L_n = k,\ k \geq 2$ of $\tilde{\xi}^L$ by a string of k consecutive ones: at the place where it appears an element $\tilde{\xi}^L_n = k,\ k \geq 2$, we remove it, and insert k consecutive ones, shifting the part $\{\tilde{\xi}^L_i\}_{i\geq n+1}$ of the original sequence $\tilde{\xi}^L$ by $k-1$ units to the right. The inserted string of k consecutive ones will be called *cluster*[7] *of mass* k, for any $k \geq 1$. This is a convergent procedure and we define ξ^L as the limiting binary sequence. All clusters are labelled in increasing order as $\{\mathcal{C}_i\}_{i\geq 1}$, and $m(\mathcal{C}_i)$ denotes the mass of the i^{th} cluster. Moreover by $\alpha_i = \alpha_i(\mathcal{C}_i)$ (resp. $\omega_i = \omega_i(\mathcal{C}_i)$) we denote the position of the first (resp. the last) 1 in the cluster. If $m(\mathcal{C}_i) = 1$ we have $\alpha_i = \omega_i$. The following property is very important for the construction: for any two clusters $\mathcal{C}$ and $\mathcal{C}'$ one has

$$d(\mathcal{C}, \mathcal{C}') \geq L^{m(\mathcal{C}) \wedge m(\mathcal{C}')}.$$

Construction of renormalized lattices.

The goal of this step is to construct a sequence of partitions $\{\mathbf{H}_k\}_{k\geq 0}$ of $\widetilde{\mathbb{Z}}^2_+$ into horizontal layers, which will be used for the

[7]Not to confuse with the notion of open cluster.

definition of renormalized sites along the multi-scale procedure. It is an iterative construction.

Level 0. To begin, we define 0-layers as the initial lines which characterize the environment: $H_{0,j} = H_j = \{(x,y) \in \widetilde{\mathbb{Z}}^2_+ , y = i\}$, for any $j \in \mathbb{Z}_+$. If $j \in \Gamma$ we say that 0-layer $H_{0,j}$ is *bad*, and otherwise it is called *good*. These names are justified by the fact that vertices which belong to good 0-layers are open with large probability (namely, p_G), and vertices which belong to bad 0-layers are open with small probability (namely p_B).

Level 1. Let $\{\mathcal{C}_j\}_{j\geq 1}$ be the family of all clusters labelled in increasing order, as before. Recall that $\alpha(\mathcal{C}_j)$ and $\omega(\mathcal{C}_j)$ are respectively, its start- and end- points. We set, for each $j \geq 1$:

$$\widetilde{\omega}^1_j = \begin{cases} \omega(\mathcal{C}_j) + 3, & \text{if} \quad m(\mathcal{C}_j) = 1; \\ \omega(\mathcal{C}_j), & \text{if} \quad m(\mathcal{C}_j) > 1, \end{cases} \tag{5.1}$$

$\widetilde{\alpha}^1_{j+1} = \widetilde{\omega}^1_j + 1$, and $\widetilde{\alpha}^1_1 = 2$ (By the definition of ξ^L, $m(\mathcal{C}_1) = 1$ and $\alpha(\mathcal{C}_1) = L$.) Thus, the intervals $[\widetilde{\alpha}^1_j, \widetilde{\omega}^1_j], j \geq 1$ give a partition of $\{2, 3, \dots\}$. For those j such that $m(\mathcal{C}_j) > 1$ we split the interval $[\widetilde{\alpha}^1_j, \widetilde{\omega}^1_j]$ into $[\widetilde{\alpha}^1_j, \alpha(\mathcal{C}_j) - 1]$ and $[\alpha(\mathcal{C}_j), \widetilde{\omega}^1_j]$, and we relabel the new partition of $\{2, 3, \dots\}$ as $[\alpha_{1,j}, \omega_{1,j}], j \geq 1$, in increasing order. To complete the decomposition of $\mathbb{Z}_+$ we write $\alpha_{1,0} = 0$ and $\omega_{1,0} = 1$. We then set, for $j \geq 0$:

$$\mathcal{H}_{1,j} = [\alpha_{1,j}, \omega_{1,j}] \qquad H_{1,j} = \bigcup_{s \in [\alpha_{1,j}, \omega_{1,j}]} H_{0,s}.$$

The layers $H_{1,j}$ are called 1-layers. Either the 1-layer contains at most one bad 0-layer, in which case we call it a *good* 1-layer, or it is the union of $k \geq 2$ consecutive bad 0-layers, and we call it a *bad* 1-layer.

Level 2. Among all clusters $\{\mathcal{C}_j\}_{j\geq 1}$ we now consider those which have mass at least 2, and temporarily rename them as $\tilde{\mathcal{C}}_j, j \geq 1$ (always in increasing order). Since $m(\tilde{\mathcal{C}}_j) \geq 2$, we know from the previous level that for each j there exist i_j so that $\tilde{\mathcal{C}}_j = [\alpha_{1,i_j}, \omega_{1,i_j}]$. For $j \geq 1$, we then set:

$$\widetilde{\omega}_j^2 = \begin{cases} \omega_{i_j+3}^1, & \text{if} \quad m(\tilde{\mathcal{C}}_j) = 2; \\ \omega_{i_j}^1, & \text{if} \quad m(\tilde{\mathcal{C}}_j) > 2, \end{cases}$$

$\widetilde{\alpha}_{j+1}^2 = \widetilde{\omega}_j^2 + 1$, and $\widetilde{\alpha}_1^2 = \alpha_{1,3}$ (by the construction $m(\tilde{\mathcal{C}}_1) = 2$). For those j such that $m(\tilde{\mathcal{C}}_j) > 2$ we split the interval $[\widetilde{\alpha}_j^2, \widetilde{\omega}_j^2]$ into $[\widetilde{\alpha}_j^2, \alpha(\tilde{\mathcal{C}}_j) - 1]$ and $[\alpha(\tilde{\mathcal{C}}_j), \widetilde{\omega}_j^2]$, and consider the obtained partition of $\{n \in \mathbb{Z}_+ : n \geq \alpha_{1,3}\}$, which we relabel in increasing order as $[\alpha_{2,j}, \omega_{2,j}], j \geq 1$. To complete it to a partition of $\mathbb{Z}_+$ we set $\alpha_{2,0} = 0, \omega_{2,0} = \omega_{1,2}$. We then set, for $j \geq 0$:

$$\mathcal{H}_{2,j} = [\alpha_{2,j}, \omega_{2,j}] \qquad H_{2,j} = \bigcup_{s \in [\alpha_{2,j}, \omega_{2,j}]} H_{0,s}.$$

The sets $H_{2,j}, j \geq 0$ are called 2-layers, and each 2-layer is the union of consecutive 1-layers. Analogously to the previous case, a 2-layer $H_{2,j}$ with $j \geq 1$, may consist of $m \geq 3$ consecutive bad lines, in which case we call it a *bad* 2-layer, or it contains at most one pair of consecutive bad lines with all the remaining 1-layers which form it been good ones, in which case it is called a *good* 2-layer. The 2-layer $H_{2,0}$ is kind of exceptional, being formed by three 1-layers: the exceptional 1-layer $H_{1,0}$ and two 1-layers $H_{1,1}, H_{1,2}$, which are necessarily good provided $L \geq 3$.

Level k. Having proceeded with the previous construction up to level $k-1$, we now consider, among all clusters $\{\mathcal{C}_j\}_{j \geq 1}$ those which have mass at least k, and temporarily rename them as $\tilde{\mathcal{C}}_j, j \geq 1$ (always in increasing order).

Since $m(\tilde{\mathcal{C}}_j) \geq k$, the previous construction shows that for each $j \geq 1$ there exists i_j so that $\tilde{\mathcal{C}}_j = [\alpha_{k-1,i_j}, \omega_{k-1,i_j}]$. We then set, for $j \geq 1$:

$$\widetilde{\omega}_j^k = \begin{cases} \omega_{k-1,i_j+3}, & \text{if} \quad m(\tilde{\mathcal{C}}_j) = k; \\ \omega_{k-1,i_j}, & \text{if} \quad m(\tilde{\mathcal{C}}_j) > k, \end{cases}$$

$\widetilde{\alpha}_{j+1}^k = \widetilde{\omega}_j^k + 1$, and finally $\widetilde{\alpha}_1^k = \alpha_{k-1,3}$. The sets $[\widetilde{\alpha}_j^k, \widetilde{\omega}_j^k], j \geq 1$ give a partition of $\{n \in \mathbb{Z}_+ : n \geq \alpha_{k-1,3}\}$. Again, when $m(\tilde{\mathcal{C}}_j) > k$ we split the interval $[\widetilde{\alpha}_j^k, \widetilde{\omega}_j^k]$ into $[\widetilde{\alpha}_j^k, \alpha(\tilde{C}_j) - 1]$ and $[\alpha(\tilde{\mathcal{C}}_j), \widetilde{\omega}_j^k]$. We relabel the new intervals in increasing order as $[\alpha_{k,j}, \omega_{k,j}], j \geq 1$ and finally we complete the family to a partition of $\mathbb{Z}_+$ by setting $\alpha_{k,0} = 0, \omega_{k,0} = \omega_{k-1,2}$. We then set for $j \geq 0$:

$$\mathcal{H}_{k,j} = [\alpha_{k,j}, \omega_{k,j}] \qquad H_{k,j} = \bigcup_{s \in [\alpha_{k,j}, \omega_{k,j}]} H_{0,s}.$$

The sets $H_{k,j}, j \geq 0$ are called k-layers. Each k-layer is the union of consecutive $(k-1)$-layers. We recursively see that if $j \geq 1$ either the k-layer $H_{k,j}$ is constituted by $m \geq k+1$ consecutive bad lines, in which case it is called a *bad* k-layer, or it contains at most one set of k consecutive bad lines (a bad $(k-1)$-layer), and all the remaining $(k-1)$-layers which form it are good $(k-1)$-layers, in which case $H_{k,j}$ is called a *good* k-layer. Again as before the k-layer $H_{k,0}$ is exceptional, being formed by the exceptional $(k-1)$-layer $H_{k-1,0}$ and two $(k-1)$-layers $H_{k-1,1}, H_{k-1,2}$ which are necessarily good if $L \geq 3$.

For any $k \geq 1$, the good k-layers are said to be of *type 2* when they do not contain any bad $(k-1)$-layer. These are the layers which are followed by a bad k-layer. The good k-layers which contain one bad $(k-1)$-layer are then said to be of *type 1.*

We now define renormalized sites $S_{i,j}^k$ with $(i,j) \in \widetilde{\mathbb{Z}}_+^2$, also called k-sites, for all $k \geq 0$. (See Figure 5.1.)

For a constant $c > 0$ to be fixed below (depending on p_G) we set:

Level 0. $S^0_{(i,j)} = (i,j)$, for $(i,j) \in \widetilde{\mathbb{Z}}^2_+$.
Level 1. For any $(i,j) \in \widetilde{\mathbb{Z}}^2_+$, we set

$$S^1_{(i,j)} = \left(\frac{i-1}{2}cL, \frac{i+1}{2}cL\right] \times \mathcal{H}_{1,j+1}. \tag{5.2}$$

Level k $(k \geq 2)$**.** We set, recursively, for any $k \geq 2$ and $(i,j) \in \widetilde{\mathbb{Z}}^2_+$:

$$\begin{aligned} S^k_{(i,j)} = \cup_{(x,y)} \Big\{ & S^{k-1}_{(x,y)} \colon S^{k-1}_{(x,y)} \subset \\ & \subset \Big(\frac{i-1}{2}(cL)^k - \frac{1}{2}(cL)^{k-1}, \frac{i+1}{2}(cL)^k\Big] \times \mathcal{H}_{k,j+1} \Big\}. \end{aligned} \tag{5.3}$$

The constant c is taken to satisfy the condition $c < 3r(p_G)/14$, where $r(p_G) > 0$ is the asymptotic slope in homogeneous oriented site percolation on $\mathbb{Z}^2$ with parameter $p_G > p_c$, as in Section 1.3 (see [16]). For simplicity of writing we assume $c^{-1} \in \mathbb{N}$ and $cL/2 \in \mathbb{N}$.
Remark. The layers $H_{k,0}$, which correspond to the union of two layers of each of the previous levels $0, \dots, k-1$ starting from the origin, will be used to define what we call a $(k-1)$-seed (see below) from which the system percolates into $S^k_{0,0}$ with large probability. (The number of 0-layers in $H_{k,0}$ is bounded from above by $c_4 L^{k-1}$, where c_4 is a suitable positive constant.)

The k-site $S^k_{(i,j)}$ is said to be *good*, when the corresponding $H_{k,j+1}$, cf. (5.3), is good. Similarly, a good k-site is said to be of *type 1* or *type 2* also according to the corresponding k-layer that contains it. Therefore, from the construction we have:

- A good k-site S^k of type 1 contains $L-4$ (or $L-3$, if contained in $H_{k,1}$) horizontal layers of good $(k-1)$-sites, forming what we call the *kernel* of S^k, denoted by $K(S^k)$. The kernel is followed by an horizontal layer of bad $(k-1)$-sites (part of the k consecutive bad lines in the k-layer), which is then followed by three horizontal layers of good $(k-1)$-sites. Observe that among the first $L-4$ (or $L-3$ according to the case) layers of good $(k-1)$-sites, all but the last one are formed of type 1 good $(k-1)$-sites, i.e. they necessarily intersect a bad $(k-2)$-layer, and the last one is formed of type 2 good $(k-1)$-sites, i.e. has no layer of bad $(k-2)$-sites.

- A good k-site S^k of type 2 contains no layer of bad $(k-1)$-sites. In this case it contains only $L-4$ layers of good $(k-1)$-sites, where again all layers but the last one are formed of type 1 good $(k-1)$-sites, and the last is formed of type 2 good $(k-1)$-sites. In this situation $K(S^k) = S^k$. By previous observation, S^k is of type 2 if and only the k-layer where it is contained is followed by a bad k-layer.

Notation.

(a) The 0-site with the coordinates $\bar{C}^k(S^k_{(i,j)}) = (\frac{i}{2}(cL)^k, \alpha_{k,j+1})$, will be called the *central* site of $S^k_{(i,j)}$.

(b) $B(S^k_{(i,j)}) = (\bar{C}^k(S^k_{(i,j)}) - \frac{3}{2}cL^{k-1}, \bar{C}^k(S^k_{(i,j)}) + \frac{3}{2}cL^{k-1}) \times \{\alpha_{k,j+1}\}$.

(c) The $(k-1)$-sites $S^{k-1} \subset S^k_{(i,j)}$ such that $S^{k-1} \cap B(S^k_{(i,j)}) \neq \emptyset$ will be called *centrally located* sites in $S^k_{(i,j)}$. (There are three such $(k-1)$-sites.)

(d) Let us also define

$$\mathcal{B}(S^k_{(i,j)}) = \left[\bar{C}^k(S^k_{(i,j)}) - \frac{1}{6}(cL)^{k-1}, \bar{C}^k(S^k_{(i,j)}) + \frac{1}{6}(cL)^{k-1}\right] \times \left[\alpha_{k-1,i_j+2}, \omega_{k-1,i_j+3}\right].$$

Observe, that $\mathcal{B}(S^k_{(i,j)}) \subset S^k_{(i,j-1)}$.

(d) Finally we consider the following rectangles:

$$
\begin{aligned}
D_l(S^k_{(i,j)}) &= \left[\frac{i-1}{2}(cL)^k + \frac{1}{12}(cL)^k, \frac{i-1}{2}(cL)^k + \frac{2}{12}(cL)^k\right] \\
&\quad \times \left[\alpha_{k-1,i_{j+1}+2}, \omega_{k-1,i_{j+1}+3}\right], \\
D_r(S^k_{(i,j)}) &= \left[\frac{i+1}{2}(cL)^k - \frac{2}{12}(cL)^k, \frac{i+1}{2}(cL)^k - \frac{1}{12}(cL)^k\right] \\
&\quad \times \left[\alpha_{k-1,i_{j+1}+2}, \omega_{k-1,i_{j+1}+3}\right], \\
D^K_l(S^k_{(i,j)}) &= \left[\frac{i}{2}(cL)^k + \frac{1}{12}(cL)^k, \frac{i}{2}(cL)^k + \frac{2}{12}(cL)^k\right] \\
&\quad \times \mathcal{H}_{k-1,i_{j+1}-1}, \\
D^K_r(S^k_{(i,j)}) &= \left[(\frac{i}{2}+1)(cL)^k - \frac{2}{12}(cL)^k, (\frac{i}{2}+1)(cL)^k - \frac{1}{12}(cL)^k\right] \\
&\quad \times \mathcal{H}_{k-1,i_{j+1}-1}, \\
D^K(S^k_{(i,j)}) &= \left[\frac{i}{2}(cL)^k + \frac{1}{12}(cL)^k, (\frac{i}{2}+1)(cL)^k - \frac{1}{12}(cL)^k\right] \\
&\quad \times \mathcal{H}_{k-1,i_{j+1}-1}, \\
\hat{D}_l(S^k_{(i,j)}) &= \left[\frac{i}{2}(cL)^k + \frac{1}{12}(cL)^k, \frac{i}{2}(cL)^k + \frac{2}{12}(cL)^k\right] \\
&\quad \times \mathcal{H}_{k-1,i_{j+1}+1}, \\
\hat{D}_r(S^k_{(i,j)}) &= \left[(\frac{i}{2}+1)(cL)^k - \frac{2}{12}(cL)^k, (\frac{i}{2}+1)(cL)^k - \frac{1}{12}(cL)^k\right] \\
&\quad \times \mathcal{H}_{k-1,i_{j+1}+1}, \\
\hat{D}(S^k_{(i,j)}) &= \left[\frac{i}{2}(cL)^k + \frac{1}{12}(cL)^k, (\frac{i}{2}+1)(cL)^k - \frac{1}{12}(cL)^k\right] \\
&\quad \times \mathcal{H}_{k-1,i_{j+1}+1}.
\end{aligned}
$$

We now introduce several key definitions: seed, s-passability (from a seed), and c-passability (from three centrally located sites). See Figure 5.1 for an illustration.

0-Seed. A 0-seed $Q^{(0)}$ is a set of three open sites in $\widetilde{\mathbb{Z}}^2_+$, disposed as follows: $Q^{(0)}_{i,j} = \{(i,j), (i+1, j+1), (i-1, j+1)\}$. (When the location is not important we will eliminate the subscript.) The sites $(i-1, j+1)$ and $(i+1, j+1)$ are called the active sites of $Q^{(0)}$, and we denote $A(Q^{(0)}) = \{(i-1, j+1), (i+1, j+1)\}$. The site (i,j) is called the root of $Q^{(0)}$, and we write $R(Q^{(0)}) = \{(i,j)\}$.

Passability at level 1. A good 1-site S^1 is said to be passable from a seed $Q^{(0)}$ if:

- There exist two rooted 0-seeds $Q_l(S^1)$ and $Q_r(S^1)$, located respectively in $D_l(S^1)$ and $D_l(S^1)$.
- $R(Q_l(S^1))$ and $R(Q_r(S^1))$ are connected to $A(Q^{(0)})$ by an open oriented path lying[8] in S^1. (In the usual application, the sites $A(Q^{(0)})$ are supposed to be located just below the 1-site S^1.)

Consequently $R(Q_l(S^1))$ and $R(Q_r(S^1))$ are also connected to $R(Q^{(0)}))$.

1-Seed. A 1-seed consists of three good 1-sites $S^1_{(i,j)}$, $S^1_{(i-1,j+1)}$, $S^1_{(i+1,j+1)}$ in such a way that:

- $S^1_{(i,j)}$ is passable from a given seed $Q^{(0)} = Q$;
- $S^1_{(i-1,j+1)}$ and $S^1_{(i+1,j+1)}$ are passable from $Q_l(S^1_{(i,j)})$ and from $Q_r(S^1_{(i,j)})$, respectively.

In this case,

$$Q^{(1)} = S^1_{(i,j)} \cup S^1_{(i-1,j+1)} \cup S^1_{(i+1,j+1)} \cup Q$$

is called a rooted 1-seed and we set

$$\begin{aligned} R(Q^{(1)}) =& R(Q), \\ A(Q^{(1)}) =& A(Q_l(S^1_{(i-1,j+1)})) \cup A(Q_r(S^1_{(i-1,j+1)})) \\ & \cup A(Q_l(S^1_{(i+1,j+1)})) \cup A(Q_r(S^1_{(i+1,j+1)})). \end{aligned}$$

[8]except possibly by the initial vertex in the path

The site $R(Q)$ is called the root of $Q^{(1)}$ and the sites in $A(Q^{(1)})$ are called the active sites of $Q^{(1)}$.

Passability at level k. A good k-site $S^k_{i,j}$ is said to be passable from the $(k-1)$-seed $Q^{(k-1)}$ if:
- There exist two $(k-1)$-seeds $Q_l(S^k)$ and $Q_r(S^k)$, located respectively at $D_l(S^k)$ and $D_r(S^k)$ of S^k.
- $R(Q_l(S^k))$ and $R(Q_r(S^k))$ are connected to $A(Q^{(k-1)})$ by an open oriented path of 0-sites lying entirely in S^k (except possibly by the first vertex of the path). (As before, for the applications the sites $A(Q^{(k-1)})$ are located just below S^k.)

Consequently, $R(Q_l(S^k))$ and $R(Q_r(S^k))$ are also connected to $R(Q^{(k-1)})$.

k-Seed. It consists of three good k-sites $S^k_{(i,j)}$, $S^k_{(i-1,j+1)}$, and $S^k_{(i+1,j+1)}$, in a way that:
- $S^k_{(i,j)}$ is passable from a given $(k-1)$-seed $Q^{(k-1)}$;
- $S^k_{(i-1,j+1)}$ and $S^k_{(i+1,j+1)}$ are passable from $Q_l(S^k_{(i,j)})$ and from $Q_r(S^k_{(i,j)})$, respectively.

In this case,

$$Q^{(k)} = S^k_{(i,j)} \cup S^k_{(i-1,j+1)} \cup S^k_{(i+1,j+1)} \cup Q^{(k-1)}$$

is called a (rooted) k-seed, and we set

$$\begin{aligned} R(Q^{(k)}) =& R(Q^{(k-1)}), \\ A(Q^{(k)}) =& A(Q_l(S^k_{(i-1,j+1)})) \cup A(Q_r(S^k_{(i-1,j+1)})) \\ & \cup A(Q_l(S^k_{(i+1,j+1)})) \cup A(Q_r(S^k_{(i+1,j+1)})). \end{aligned}$$

The site $R(Q^{(k)})$ in $\widetilde{\mathbb{Z}}^2_+$, called the root of $Q^{(k)}$; the sites in $A(Q^{(k)})$ are called the active sites of $Q^{(k)}$.

In the proof we still need the notion of centrally passable renormalized sites, which we define below.

A 0-site $(i,j) \in \widetilde{\mathbb{Z}}^2_+$ is centrally passable if and only if it is open.

A good 1-site S^1 is centrally passable if there exist two 0-seeds $Q_l(S^1)$ and $Q_r(S^1)$ located respectively in $D_l(S^1)$ and $D_l(S^1)$, as before, and such that $R(Q_l(S^1))$ and $R(Q_r(S^1))$ are connected by an open path lying in S^1 to at least one centrally located site of S^1.

A renormalized good k-site S^k is centrally passable if:
• at least one centrally located $(k-1)$-site $S^{k-1}_{i',j'}$ in S^k is centrally passable;
• there exist two $(k-1)$-seeds, $Q_l(S^k)$ and $Q_r(S^k)$ located at $D_l(S^k)$ and $D_r(S^k)$ respectively, and such that $R(Q_l(S^k))$ and $R(Q_r(S^k))$ are connected to $A(Q_l(S^{k-1}_{(i',j')}))$ or $A(Q_r(S^{k-1}_{(i',j')}))$.

We will say that two k-sites $S^k_{i,j}$ and $S^k_{i+1,j+1}$ ($S^k_{i-1,j+1}$ resp.) are connected, if $S^k_{i,j}$ is s- or c-passable, and $S^k_{i+1,j+1}$ ($S^k_{i-1,j+1}$ resp.) is passable from $Q_r(S^k_{i,j})$ ($Q_l(S^k_{i,j})$ resp.).

We also will say that $\tilde{S}^k$ belongs to an open cluster of S^k (denoted by U_{S^k})[9] if S^k is s- or c-passable, and there exists a sequence of k-sites $S^k = S^k_0,\ S^k_1, \ldots, S^k_n = \tilde{S}^k$ such that S^k_j is connected to S^k_{j-1} for all $1 \le j \le n$.

If Q is a fixed $(k-1)$-seed from which we check if S^k is s-passable or not, we may use $Q_B(S^k)$ for Q, in order to emphasize that.

The proof of existence of percolation for the hierarchical model will be built through a very special way to achieve passability, which will involve the next important concept.

Definition 5.2 Dense kernel Having fixed $\rho \in (1/2, 1)$, we say that:

Level 1. S^1 has dense kernel if

$$U_{Q_B(S^1)}\big|_{K(S^1)} \cap D^K(S^1)| \ge \frac{5}{6}\rho c L,$$

[9]Similarly for U_Q, if Q is a seed.

$$|U_{Q_B(S^1)}\big|_{K(S^1)} \cap D_l^K(S^1)| \geq \frac{1}{12}\rho c L,$$

and

$$|U_{Q_B(S^1)}\big|_{K(S^1)} \cap D_r^K(S^1)| \geq \frac{1}{12}\rho c L.$$

Analogously we define c-dense kernel.

Level k**.** S^k has dense kernel if

$$U_{Q_B(S^k)}\big|_{K(S^k)} \cap D^K(S^k)| \geq \frac{5}{6}\rho c L,$$

$$|U_{Q_B(S^k)}\big|_{K(S^k)} \cap D_l^K(S^k)| \geq \frac{1}{12} c L$$

and

$$|U_{Q_B(S^k)}\big|_{K(S^k)} \cap D_r^K(S^k)| \geq \frac{1}{12} c L.$$

Analogously we define c-dense kernel.

Notational Remark. From now on we say that S^k is s-passable, if it is s-passable and has dense kernel. Analogously for c-passable.

Observe, that if the k-site S^k has dense kernel, then each $(k-1)$-site that belongs to the cluster $U_{Q(S^k)}\big|_{K(S^k)}$ is also s-passable, and thus, each of these $(k-1)$-sites S^{k-1} lying on the top layer of the $K(S^k)$ (notice that in this case $S^{k-1} = K(S^{k-1})$), has at least $\rho c L$ sites S^{k-2} on its top layer which are s-passable, etc.. Therefore each S^k with dense kernel has at least $(\rho c L)^k$ 0-sites on its top layer, which are connected to the root of Q.

The good k-sites have been defined having in mind to achieve high crossing probabilities when the upwards (northwest-northeast) direction is considered. That is so because its piece of a bad $(k-1)$-layer corresponding to k consecutive bad lines is located on the top part of the k-site, so that process can grow well before meeting a very hard environment. As we shall see next, the proof uses a trick involving downwards crossing, and for this it is convenient to consider a reversed partition.

Passability in reversed direction.

Reversed sites. We begin with the horizonal layers at all scales.

Level 0. The 0-layers $\widehat{H}_{0,j} = H_j$.

Level 1. Take $\{\mathcal{C}_j\}_{j\geq 1}$ as before. We set, for each $j \geq 1$:

$$\widehat{\omega}_j^1 = \begin{cases} \alpha(\mathcal{C}_j) - 3, & \text{if} \quad m(\mathcal{C}_j) = 1; \\ \alpha(\mathcal{C}_j), & \text{if} \quad m(\mathcal{C}_j) > 1, \end{cases} \tag{5.4}$$

and $\widehat{\alpha}_j^1 = \widehat{\omega}_{j+1}^1 - 1$, so that we have a partition of $\{n \in \mathbb{Z}_+ : n \geq \widehat{\omega}_1^1\}$ (recall that $\widehat{\omega}_1^1 = L - 3$). When $m(\mathcal{C}_j) > 1$ we split the interval $[\widehat{\omega}_j^1, \widehat{\alpha}_j^1]$ into two pieces $[\widehat{\omega}_j^1, \omega(\mathcal{C}_j)] (= [\alpha(\mathcal{C}_j), \omega(\mathcal{C}_j)])$ and $[\omega(\mathcal{C}_j)+1, \widehat{\alpha}_j^1]$. We rewrite the new partition in increasing order as $[\widehat{\omega}_{1,j}, \widehat{\alpha}_{1,j}], j \geq 1$, and complete it to a partition of $\mathbb{Z}_+$ by setting $\widehat{\omega}_{1,0} = 0, \widehat{\alpha}_{1,0} = L - 4$.

For each $j \geq 0$ we set

$$\widehat{\mathcal{H}}_{1,j} = [\widehat{\omega}_{1,j}, \widehat{\alpha}_{1,j}] \qquad \widehat{H}_{1,j} = \bigcup_{s \in [\widehat{\omega}_{1,j}, \widehat{\alpha}_{1,j}]} \widehat{H}_{0,s}.$$

Level 2. As before, consider all clusters which have mass at least 2, and rename them as $\tilde{\mathcal{C}}_j, j \geq 1$ (always in increasing order).

From the previous construction, since $m(\tilde{\mathcal{C}}_j) \geq 2$ we know that for each j there exist i_j so that $\tilde{\mathcal{C}}_j = [\widehat{\omega}_{1,i_j}, \widehat{\alpha}_{1,i_j}]$. We define

$$\widehat{\omega}_j^2 = \begin{cases} \widehat{\alpha}_{1,i_j - 3}\,, & \text{if} \quad m(\tilde{\mathcal{C}}_j) = 2; \\ \widehat{\alpha}_{1,i_j}\,, & \text{if} \quad m(\tilde{\mathcal{C}}_j) > 2, \end{cases}$$

and $\widehat{\alpha}_j^2 = \widehat{\omega}_{j+1}^2 - 1$, so that we have a partition of $\{n \in \mathbb{Z}_+ : n \geq \widehat{\omega}_1^2\}$. When $m(\tilde{\mathcal{C}}_j) > 2$ we split the interval $[\widehat{\omega}_j^2, \widehat{\alpha}_j^2]$ into two pieces $[\widehat{\omega}_j^2, \omega(\mathcal{C}_j)] (= [\alpha(\mathcal{C}_j), \omega(\mathcal{C}_j)])$ and $[\omega(\mathcal{C}_j) + 1, \widehat{\alpha}_j^2]$. We rewrite the new partition in increasing order as $[\widehat{\omega}_{2,j}, \widehat{\alpha}_{2,j}]$, $j \geq 1$ and complete it to a partition of $\mathbb{Z}_+$ by setting $\widehat{\omega}_{2,0} = 0, \widehat{\alpha}_{2,0} = \widehat{\omega}_1^2$.

For each $j \geq 0$ we set

$$\widehat{\mathcal{H}}_{2,j} = [\widehat{\omega}_{2,j}, \widehat{\alpha}_{2,j}] \qquad \widehat{H}_{2,j} = \bigcup_{s \in [\widehat{\omega}_{2,j}, \widehat{\alpha}_{2,j}]} \widehat{H}_{0,s}.$$

Level k. Assuming the procedure completed up to level $(k-1)$ we consider all clusters which have mass at least k, and rename them as $\tilde{\mathcal{C}}_j, j \geq 1$ (always in increasing order).

From the previous construction, since $m(\tilde{\mathcal{C}}_j) \geq k$ we know that for each j there exist i_j so that $\tilde{\mathcal{C}}_j = [\widehat{\omega}_{k-1,i_j}, \widehat{\alpha}_{k-1,i_j}]$.

In this case we define

$$\widehat{\omega}_j^k = \begin{cases} \widehat{\alpha}_{k-1,i_j-3}, & \text{if} \quad m(\tilde{\mathcal{C}}_j) = k; \\ \widehat{\alpha}_{k-1,i_j}, & \text{if} \quad m(\tilde{\mathcal{C}}_j) > k, \end{cases}$$

and $\widehat{\alpha}_j^k = \widehat{\omega}_{j+1}^k - 1$, so that we have a partition of $\{n \in \mathbb{Z}_+ : n \geq \widehat{\omega}_1^k\}$. When $m(\tilde{\mathcal{C}}_j) > k$ we split the interval $[\widehat{\omega}_j^k, \widehat{\alpha}_j^k]$ into two pieces $[\widehat{\omega}_j^k, \omega(\mathcal{C}_j)](= [\alpha(\mathcal{C}_j), \omega(\mathcal{C}_j)])$ and $[\omega(\mathcal{C}_j) + 1, \widehat{\alpha}_j^k]$. We rewrite the new partition in increasing order as $[\widehat{\omega}_{k,j}, \widehat{\alpha}_{k,j}]$, $j \geq 1$ and complete it to a partition of $\mathbb{Z}_+$ by setting $\widehat{\omega}_{k,0} = 0, \widehat{\alpha}_{k,0} = \widehat{\omega}_1^2$. For each $j \geq 0$ we set

$$\widehat{\mathcal{H}}_{k,j} = [\widehat{\omega}_{k,j}, \widehat{\alpha}_{k,j}] \qquad \widehat{H}_{k,j} = \bigcup_{s \in [\widehat{\omega}_{k,j}, \widehat{\alpha}_{k,j}]} \widehat{H}_{0,s}.$$

The renormalized reverse k-sites $\widehat{S}_{i,j}^k$ with $(i,j) \in \widetilde{\mathbb{Z}}_+^2$ are defined as before, replacing $\mathcal{H}_{k,j+1}$ by $\widehat{\mathcal{H}}_{k,j+1}$. The notions of seed, and $\widehat{s}$ or $\widehat{c}$- (reverse) passability are defined similarly.

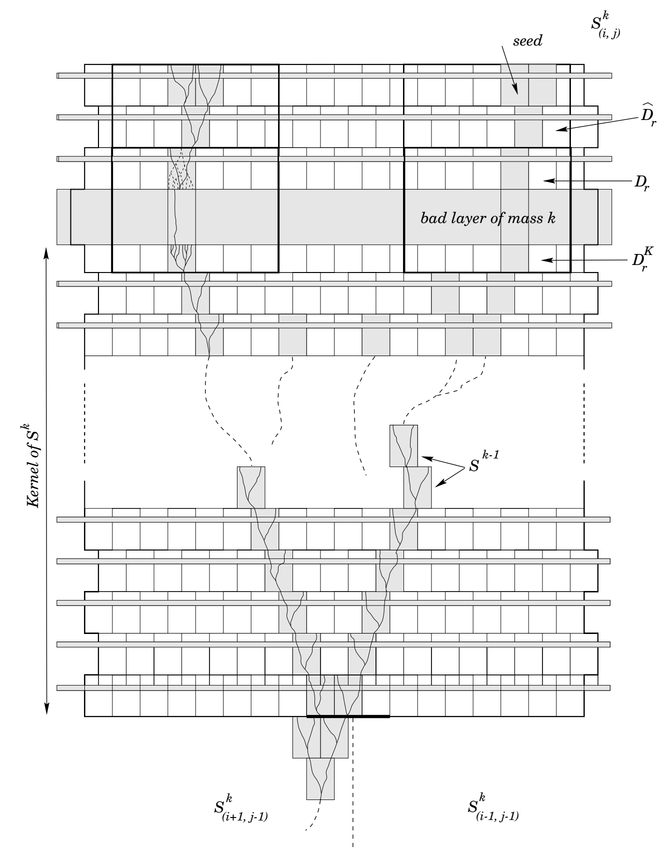

FIGURE 5.1 RENORMALIZED SITE

Probability estimates. Drilling.

Given $p > p_c$ we know from [16] (see also Sec. 1.3) that there exists an asymptotic density $\rho(p)$ for the oriented cluster in homogeneous oriented percolation model with parameter p. Moreover, $\lim_{p\to 1}\rho(p) = 1$. Thus we may take $p^* < 1$ so that $\rho(p) > 1/2$ for all $p > p^*$. (Of course $p_c < p^* < 1$.) Our arguments yield a proof of the theorem when $p_G > p^*$. (See Remark 5.4 below.) Let then $\epsilon > 0$ be small so that $\rho := \rho(p_G) - \epsilon > 1/2$, and set $\hat{\rho} = \rho - 1/2$.

Let

$$p_k = P_p(S^k \text{ is } s\text{-passable}), \qquad k \geq 1,$$

and let $q_k = 1 - p_k$. The following Lemma is the key ingredient of the proof:

Lemma 5.3 . *Given $p_B > 0$ and $p_G > p^*$, there exists L large enough such that*

$$q_k \leq q_{k-1}^2 \qquad \text{for all } k \geq 1, \tag{5.5}$$

and where $q_0 = 1 - p_G$.

The main idea and key steps of the proof will be discussed now.

The proof of the lemma is by induction on k. For any $k \geq 0$ and any $j \geq 1$ we will say that sites $S^k_{(i,j)}$ and $S^k_{(i',j)}$ are well separated, if $|i' - i| > 4$.

If $k = 1$ we can choose $L > 0$, large enough, such that:

$$P(S^1 \text{ has } s\text{-dense kernel from } Q_B(S^1)\, |Q_B(S^1) \text{ is a seed }) \geq 1 - \frac{q_0^2}{5}. \tag{5.6}$$

Before we proceed just a simple observation on how to get (5.6): given the 0-seed, one may use a standard Peierls argument for survival, and then the shape theorem for the homogeneous model with parameter p_G, as mentioned in Sec. 1.3.(See Remark 5.4 below.)

If S^1 has s-dense kernel, then each $D^K_l(S^1)$ and $D^K_r(S^1)$ contains at least $\frac{1}{12}\rho c L$ 0-sites connected to the origin (by oriented path contained in S^1) and thus at least $\frac{1}{48}\rho cL$ among them are well separated. We fix N such that

$$(1-p_0^3)^N \le \frac{q_0^2}{5}, \tag{5.7}$$

and in both $D_l^K(S^1)$ and $D_r^K(S^1)$, we group these well separated sites into N disjoint sets in such a way that each set contains at least $\frac{1}{48N}\hat{\rho}cL$ of such well separated 0-sites connected to the origin. The second required condition on L is such that

$$N(1-p_B)^{\frac{p^2}{32N}\frac{1}{6}\hat{\rho}cL} \le \frac{q_0^2}{5}.$$

(The role of p^2 will become more transparent in the next step.) This gives $1-\frac{q_0^2}{5}$ as a lower bound for the conditional (conditioned to the 0-seed) probability of the event:

$G_{D_l^K(S^1)} =$

[there exists at least one open path in each of N groups of well separated sites in $D_l^K(S^1)$, and which starts at one of such 0-site $S^0_{(i,j)}$, goes through the bad line, and ends at the open 0-site $S^0_{(i,j+1)}$].

Denote by $\mathcal{G}_{D_l^K(S^1)}$ the set of sites $S^0_{(i,j+1)}$ mentioned above, which are connected through the bad line to $S^0_{(i,j)}$ and then to the origin by and open path. Observe that sites of $\mathcal{G}_{D_l^K(S^1)}$ are well separated. This implies that for each site of $\mathcal{G}_{D_l^K(S^1)}$ lying after the bad line, we can check independently, if there is 0-seed above each of these sites or not. Thus if $G_{D_l^K(S^1)}$ occurs from (5.7) we get that

$$P\big(\exists \text{ seed } Q_l^0 \text{ with root } S^0_{(i,j+1)} \in \mathcal{G}_{D_l^K(S^1)} \,|Q_B(S^1) \text{ is a seed }\big) \ge 1-\frac{q_0^2}{5}.$$

Collecting all together we get

$$P\big(S^1 \text{ is } s\text{-passable from } Q_B(S^0)\ |Q_B(S^1) \text{ is a seed }\big) = p_1 \geq 1 - q_0^2.$$

Remark 5.4 The standard Peierls argument of counting the blocking contours, gives aq_0^3 (a fixed constant) as an upper bound for the probability not to survive from a given 0-seed; this is compatible with (5.6) if q_0 is small, so that putting together with the shape theorem and the asymptotic density result of standard oriented percolation we get (5.6). Thus, simply because we started with a seed that contains only 3 sites we might need a larger p_0 than just $p_0 > p^*$ (since we want $aq_0^3 < q_0^2/6$ say). To be able to go down to any $p_0 > p^*$ we have to take larger seeds, but there is no essential difference. It is important that once (5.6) holds for a given p_0 it holds for all $p > p_0$, as one can see from comparison (coupling arguments).

Assume now that (5.5) holds for $(k-1)$-sites, that is:

$$\begin{aligned} &P(S^{k-1} \text{ is } s\text{-passable from} \\ &\quad Q_B(S^{k-1}) \mid Q_B(S^{k-1}) \text{ is a } (k-1)\text{-seed }) \\ &\quad = p_{k-1} \geq 1 - q_{k-2}^2. \end{aligned} \tag{5.8}$$

Estimate (5.8) and the choice of $L > 0$ at the first step imply, cf. Remark 5.4, that:

$$\begin{aligned} &P(S^k \text{ has } s\text{-dense kernel} \\ &\quad \text{from } Q_B(S^{k-1})) \mid Q_B(S^{k-1}) \text{ is a } (k-1)\text{-seed }) \\ &\quad \geq 1 - \frac{q_{k-1}^2}{5}. \end{aligned} \tag{5.9}$$

If S^k has s-dense kernel we can find at least $\frac{1}{48}\hat{\rho}cL$ well separated $(k-1)$-sites S^{k-1} in $D_l^K(S^{k-1})$ as well as in $D_r^K(S^{k-1})$, which by themselves are s-dense (these sites coincide with their kernel). As in the first step we group these well separated sites S^{k-1} into N disjoint sets in each $D_l^K(S^k)$ and $D_r^K(S^k)$ in a way that each set contains at least $\frac{1}{48N}\hat{\rho}cL$ well separated $k-1$-sites connected to the seed of S^k.

Given $S^{k-1}_{(i,j)}$ in $D_l^K(S^k)$, the major difficulty is to produce with probability large enough a site $S^{k-1}_{(i,j+2)}$ on "the other side" of the $(k-1)$-bad layer, and which will play the role of s-passable site, from which we could continue our procedure and check whether there exists or not the $(k-1)$-seed $Q_l(S^k)$ starting either from $Q_l(S^{k-1}_{(i,j+2)})$ or $Q_r(S^{k-1}_{(i,j+2)})$. Thus we would like that the event

$$T(S^{k-1}_{(i,j+2)}) = \big[\text{either root of the seed } Q_l(S^{k-1}_{(i,j+2)}) \text{ or } Q_r(S^{k-1}_{(i,j+2)}) \\ \text{is connected to the origin by an open path of 0-sites}\big]$$

has large enough (conditional) probability. To do so, consider for each such $(k-1)$-site $S^{k-1}_{(i,j)}$ located in $D_l^K(S^k)$, the "forward" site $S^{k-1}_{(i,j+2)}$ and the reverse site $\widehat{S}^{k-1}_{(i,j+1)}$ located in $\hat{D}_l(S^k)$. The site $S^{k-1}_{(i,j)}$ is separated from $S^{k-1}_{(i,j+2)}$ and $\widehat{S}^{k-1}_{(i,j+1)}$ by the bad $(k-1)$-layer which consists of k consecutive bad lines. From the definition of forward and reverse partitions we have that $\widehat{S}^{k-1}_{(i,j+1)} \subset S^{k-1}_{(i,j+2)}$. (See Figure 5.2.)

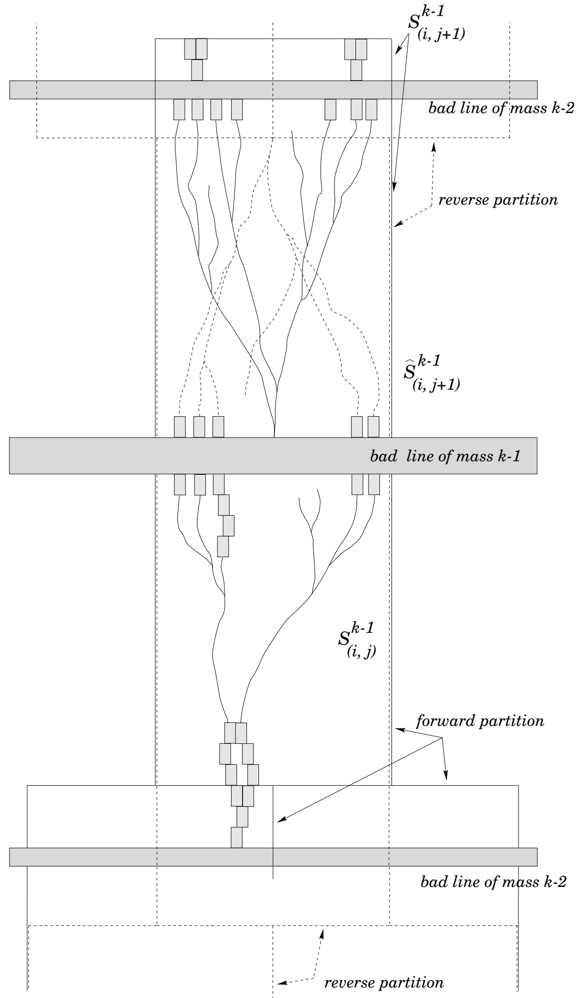

FIGURE 5.2

We notice that if the site $S^{k-1}_{(i,j)}$ is s-dense and the site $\hat{S}^{k-1}_{(i,j+1)}$ is $\widehat{c}$-dense (in reverse direction), then there are at least $\frac{1}{12}\hat{\rho}cL$ pairs of $(k-2)$-sites S^{k-2} and $\widehat{S}^{k-2}$ on opposite sides of bad layer, which are localized in $D^K_l(S^{k-1}_{(i,j)})$ and $D_l(\hat{S}^{k-1}_{(i,j+1)})$ respectively, and which are s- and $\widehat{s}$-dense, and moreover have the same first coordinate. Each such pair S^{k-2} and $\widehat{S}^{k-2}$ contains at least $\frac{1}{6}\hat{\rho}cL$ sites S^{k-3} and $\hat{S}^{k-3}$ on opposite sides of the bad line, which have the same first coordinate and are s- and $\widehat{s}$-dense. Continuing the argument we conclude that each pair $S^{k-1}_{(i,j)}$ and $\widehat{S}^{k-1}_{(i,j+1)}$, provided they are s- and c-dense respectively, contains at least $\frac{1}{2}(\frac{1}{6}\hat{\rho}cL)^{k-1}$ distinct pairs of 0-sites which either have the same (if k is odd), or different by -1 (if k is even) first coordinate. Denote this set of 0-sites by $E_l(S^{k-1}_{(i,j)}, \widehat{S}^{k-1}_{(i,j+1)})$, and by $E_{l-}(S^{k-1}_{(i,j)}, \widehat{S}^{k-1}_{(i,j+1)}) \subset E(S^{k-1}_{(i,j)}, \widehat{S}^{k-1}_{(i,j+1)})$ the sites which belong to $S^{k-1}_{(i,j)}$ and $E_{l+}(S^{k-1}_{(i,j)}, \widehat{S}^{k-1}_{(i,j+1)}) \subset E(S^{k-1}_{(i,j)}, \widehat{S}^{k-1}_{(i,j+1)})$ the sites which belong to $\widehat{S}^{k-1}_{(i,j+1)}$.

Now observe that if the site $S^{k-1}_{(i,j+2)}$ is c-passable and the site $\widehat{S}^{k-1}_{(i,j+1)}$ is $\widehat{c}$-dense then each 0-site of the set $A(Q_l(S^{k-1}_{(i,j+2)}))$ is connected by an open oriented path to each 0-site of the set $E_{l+}(S^{k-1}_{(i,j)}, \widehat{S}^{k-1}_{(i,j+1)})$ and each 0-site of the set $A(Q_r(S^{k-1}_{(i,j+2)}))$ is connected by an open path to each 0-site of the set $E_{r+}(S^{k-1}_{(i,j)}, \widehat{S}^{k-1}_{(i,j+1)})$. On the other side if S^k has s dense kernel, then each 0-site of $E_{l-}(S^{k-1}_{(i,j)}, \widehat{S}^{k-1}_{(i,j+1)})$ and $E_{r-}(S^{k-1}_{(i,j)}, \widehat{S}^{k-1}_{(i,j+1)})$ is connected to the origin by an open path. For each such pair $S^0_{(i',j')} \in E_{l-}(S^{k-1}_{(i,j)}, \widehat{S}^{k-1}_{(i,j+1)})$ and $S^0_{(i',j'+k+1)} \in E_{l+}(S^{k-1}_{(i,j)}, \widehat{S}^{k-1}_{(i,j+1)})$ (assuming that k is odd), we define the "straight" path $\pi^V(S^0_{(i',j')}, S^0_{(i',j'+k+1)}) = \{S^0_{(i'+(-1)^r,j'+r)}, r = 1,\dots,k\}$ connecting $S^0_{(i',j')}$ to $S^0_{(i',j'+k+1)}$ through the bad layer.

Observe also that if in each of N groups of $(k-1)$-sites at least one pair of 0-sites from the set $E_l(S^{k-1}_{(i,j)}, \widehat{S}^{k-1}_{(i,j+1)})$, for some (i,j), is connected through the bad layer, then it immediately implies that either root of the seed $Q_l(S^{k-1}_{(i,j+2)})$ or $Q_r(S^{k-1}_{(i,j+2)})$ is connected to the origin by an open path of 0-sites.

Taking into account that each group has at least $\frac{1}{48N}\hat{\rho}cL$ well separated $(k-1)$-sites, and the fact that the probability that for each such well separated site $S^{k-1}_{(i,j)}$ the site $S^{k-1}_{(i,j+2)}$ is c-passable and the site $\hat{S}^{k-1}_{(i,j+1)}$ is $\widehat{c}$-dense is bounded from below by p^2_{k-1}, we get that with probability converging to 1 super-exponentially fast in k we have at least $\frac{p^2_{k-1}}{2}\frac{1}{48N}\hat{\rho}cL$ well separated triples of $(k-1)$-sites $S^{k-1}_{(i,j)}$, $S^{k-1}_{(i,j+2)}$ and $\widehat{S}^{k-1}_{(i,j+1)}$ such that the second and the third site are c-passable and $\widehat{c}$-dense respectively. Each pair of sites $S^{k-1}_{(i,j)}$ and $\widehat{S}^{k-1}_{(i,j+1)}$ from this triple has at least $\frac{1}{2}(\frac{1}{6}\hat{\rho}cL)^{k-1}$ pairs of 0-sites with the same first coordinate, we get that the probability that at least one of "straight" paths π^v connecting pairs of 0-sites on opposite sites of the bad layer within the is bounded from below by

$$1-\left(1-p_B^k\right)^{\frac{p^2_{k-1}}{32N}(\frac{1}{6}\hat{\rho}cL)^k},$$

and thus the probability that in each of N blocks there exists at least one connecting path is bounded from below by

$$1-N\left(1-p_B^k\right)^{\frac{p^2_{k-1}}{32N}(\frac{1}{6}\hat{\rho}cL)^k}\geq 1-\frac{q^2_{k-1}}{5}.$$

On the other side from the choice of N we have:

$$(1-p^3_{k-1})^N\leq\frac{q^2_{k-1}}{5},$$

which implies that with probability at least $1-\frac{q^2_{k-1}}{5}$ we will find a $(k-1)$-seed among N sites S^{k-1} which are located right after the bad layer and have a seed whose root is connected to the origin.

All together this implies that

$$P(S^k \text{ is } s\text{-passable from } Q_B(S^k)\mid Q_B(S^k) \text{ is a } k\text{-seed })\geq 1-q^2_{k-1}.$$

and the lemma is proven by induction. □

To complete the proof of analogue of Theorem 5.1 for the hierarchical model, we observe that the probability to have the first 0-seed equals to p^3. From this 0-seed we get the passable 1-sites with probability at least p_1 and thus the probability that to have 1-seed on the top of the first seed we equals p_1^3. Continuing the recursive procedure we get that

$$P(\text{there exists an infinite cluster starting from the origin}) \\ \geq \prod_{k=0}^{+\infty} p_k^3 > 0,$$

which shows that if p_G is large enough, $p_B > 0$, and we take L sufficiently large, then there is percolation in the hierarchical model. Enlarging the seeds as mentioned above this argument yields the result for $p_G > p^*$. One needs to work further to extend it to all $p_G > p_c$.

5.2 The contact process in random environment.

The notes for this lecture are based on the article [10], by Bramson, Durrett and Schonmann.

Let us consider the following contact process on $\mathbb{Z}$: each site is independently declared to be *bad* or *good* with probabilities p and $1-p$, respectively. The birth parameter is assumed to be constant and equal to one at all sites, i.e., at an unoccupied site a particle is born with rate given by the number of occupied neigbhours. The death rate depends on the status of the site; a particle at a bad site dies with rate Δ and at good sites the death rate is δ. That is, we have a random environment $\omega = (\delta(x))_{x\in\mathbb{Z}}$ given by independent Bernoulli variables

$$P(\delta(x) = \Delta) = p \qquad P(\delta(x) = \delta) = 1 - p$$

and for any given realization ω (fixed once for all) we consider a inhomogeneous contact process ξ_t with birth rate $\lambda = 1$ and death rates given by the $\delta(x)$. The interesting situation corresponds to $\delta < \delta_c < \Delta$, where δ_c is the critical parameter in the homogeneous case. Thus, on the good regions we have a tendency to grow (supercritical) while in the bad regions the process tends to die out. Using a multi-scale analysis, Bramson, Durrett, and Schonmann [10] proved that no matter how small is the density of good sites, provided positive (any $p < 1$), and no matter how bad are the bad sites (any finite value of Δ) it is possible to make the good sites so good (i.e. one can take δ so small) in a way that $\xi_t^{\{0\}}$ survives with positive probability, for almost all realizations of the environment. As a by-product this brings a region of values of the parameters (Δ, δ, p) for which there is survival without linear growth, in contrast with what we have seen to hold in the standard homogeneous case. We concentrate only on their result on survival, to learn how multi-scale analysis can be successfully used for such problems. More general models can be considered by allowing the birth rates to be random as well. Liggett [30] and Klein [28] have given sufficient conditions for extinction, and Liggett [31] gives sufficient conditions for survival with positive probability, for almost ll realizations of the environment. Nevertheless, these conditions do not apply to the region of parameters considered in [10].

Theorem 5.5 . *For any $p < 1$, any $\Delta < \infty$ there exists $\delta_0(\Delta, p) > 0$ so that if $\delta < \delta_0(\Delta, p)$, then*

$$P^\omega(\xi_t^{\{0\}} \neq \emptyset, \forall t) > 0$$

for almost all $\omega = (\delta(x))_x$, where P^ω is the law of the inhomogeneous process corresponding to ω.

Remark. Before getting to the basic ideas of the proof given in [10] it is instructive to compare this result with that stated in

Theorem 5.1 of the previous section (and proven in a particular case for the hierarchical model). In the situation of Theorem 5.1, the good or bad regions are transversal to the direction of growth (time). In that frame one proves that when the frequency of bad lines is small there is survival. It is not hard to see that the result of Theorem 5.5 is not valid in that frame. A natural question (a hard exercise?) refers to an the analogue of Theorem 5.1 in the present frame. That is, instead of improving the good sites (making δ small), the frequency p is made small, in the frame of Theorem 5.5.

Basic idea of the proof.

The arguments of Bramson, Durrett and Schonmann, which we now study, involve a sequence of rapidly increasing numbers $N_1, N_2, \dots$ corresponding to spatial scales. N_1 will be chosen such that a single bad site surrounded by stretches of at least N_1 good sites does not bring difficulty. This leads to a *good level 1 site*. Each bad site not sharing this property originates what is called a *1-gap*, and to deal with them one moves to the next scale: a 1-gap that is surrounded by at least N_2 consecutive good level 1 sites, becomes a *good level 2 site*, and it should cause no problem. When a 1-gap does not have this property we get a *2-gap*, i.e. a 2-gap correspond to the situation of a 1-gap within less than N_2 good level 1 stretches), and one deals with them at the next scale.

By choosing the sequence (N_k) properly and reducing to the case of a small p, which can be done without big difficulty, one gets a situation where k-gaps eventually will stop to grow (at suitable k) so that from some space point y^* the picture will be that of k-gaps surrounded by N_k good stretches of level $k-1$. Fixing such a situation on can prove that starting at y^* the process survives for all times with a probability bounded away from zero, provided the sequence N_k was suitable chosen, and δ is small enough. Contrarily to the example in the previous section and following the more standard pattern in multi-scale blocking arguments, a proper choice will be

$$N_k = L^{(1.1)^k} \tag{5.10}$$

where L will be fixed and suitably large. Their argument has two parts:

(a) Show that gaps stop to grow.

(b) Get the inductive probability statement.

We start by turning precise the definition of gaps and blocks at all scales. To simplify the discussion we make the extra assumption that p is suitably small. This can be released by a one-step blocking argument:

Step 0. Given $\varepsilon > 0$ (to be determined) we take κ integer so that $p^\kappa < \varepsilon$, split $\mathbb{Z}_+$ into intervals $I_j = \{(j-1)\kappa, \dots, j\kappa - 1\}$, for $j = 1, 2, \dots$ and set

$$\omega_j^0 = \begin{cases} G & \text{if } \delta_n = \delta \text{ for some } n \in I_j, \\ B & \text{if } \delta_n = \Delta \text{ for all } n \in I_j. \end{cases}$$

In the following we assume that $\kappa = 1$, i.e. that p is small enough. The extension to the general case requires replacing the state at sites by the ω^0 configuration. It brings some minor modifications in the computations.

Thus, under the extra assumption:

$$\omega_n^0 = \begin{cases} G \text{ if } \delta_n = \delta; \\ B \text{ if } \delta_n = \Delta. \end{cases}$$

Step 1. Let $T_0^0 = 0, Y_0^1 = 0$ and for $n \geq 1$,

$$T_n^0 = \inf\{m > T_{n-1}^0 \colon \omega_m^0 = G\},$$
$$X_n^1 = T_n^0 - T_{n-1}^0 - 1, Y_n^1 = 1$$

That is, the random variables X_n^1 give the lengths of the successive runs of G, understood to be zero if we have two consecutive B.

We see at once that $\{X_n^1, n \geq 1\}$ are i.i.d. random variables. (The same for the Y_n^1, identically equal to one.) To visualize the definition, let us exemplify. If ω^0 is given by a configuration starting as

$$\omega^0 = (G, G, G, G, G, B, B, G, G, G, G, B, B, B, G, G, B, \ldots),$$

we see that

$$X_1^1 = 5,\ X_2^1 = 0,\ X_3^1 = 4,\ X_5^1 = 0,\ X_6^1 = 0,\ X_7^1 = 2, \ldots.$$

The intervals (T_{n-1}^0, T_n^0) in $\mathbb{Z}$, for $n \geq 1$, are called *1-blocks*. A single bad in between two blocks is called a *1-gap*. At level 1 one consider new variables

$$\omega^1(n) = \begin{cases} G \text{ if } X_n^1 > N_1; \\ B \text{ if } X_n^1 \leq N_1. \end{cases}$$

Step 2. From the configuration ω^1, take $T_0^1 = 0$, and for $n \geq 1$, one allows two attempts to start a *2-block*,[10] setting:

$$\begin{aligned} S_n^1 &= \begin{cases} T_{n-1}^1 + 1 \text{ if } \omega^1(T_{n-1}^1 + 1) = G \\ T_{n-1}^1 + 2 \text{ if } \omega^1(T_{n-1}^1 + 1) = B. \end{cases} \\ T_n^1 &= \inf\{m \geq S_n^1 \colon \omega^1(m) = B\} \\ U_n^1 &= T_n^1 - S_n^1, \end{aligned} \tag{5.11}$$

so that U_n^1 gives the length of the nth G-run in ω^1. To fixate the definitions let us take an example:

$$\omega^1 = (G, G, G, B, B, G, G, G, G, G, B, B, B, G, G, G, G, \ldots)$$

for which one has:

$$\begin{aligned} &S^1 = 1,\ S_2^1 = 6,\ S_3^1 = 13;\ \ T_1^1 = 4,\ T_2^1 = 11, \\ &T_3^1 = 13;\ U_1^1 = 3,\ U_2^1 = 5,\ U_3^1 = 0, \ldots. \end{aligned}$$

[10]This is an important feature.

The *2-blocks* are determined by the G-runs $[S^1_n, T^1_n - 1]$, as the union of all lower level blocks and gaps from the S^1_nth to the $(T^1_n - 1)$th 1-blocks. (It contains U^1_n 1-blocks.) A *2-gap* is what stays in between two consecutive 2-blocks. For $n \geq 1$ we set:

$$\begin{aligned} X^2(n) =& X^1(S^1_n) + Y^1(S^1_n) + X^1(S^1_n + 1) + Y^1(S^1_n + 1) + \dots \\ &+ Y^1(T^1_n - 2) + X^1(T^1_n - 1), \end{aligned} \tag{5.12}$$

$$Y^2(n) = \begin{cases} Y^1(T^1_n - 1) + X^1(T^1_n) + Y^1(T^1_n) & \text{if} \quad S^1_{n+1} = T^1_n + 1; \\ Y^1(T^1_n - 1) + X^1(T^1_n) + Y^1(T^1_n) & \\ \qquad + X^1(T^1_n + 1) + Y^1(T^1_n + 1) & \text{if} \quad S^1_{n+1} = T^1_n + 2, \end{cases} \tag{5.13}$$

so that $X^2(n)$ is the number of sites in the nth 2-block, and $Y^2(n)$ is the number of sites in the 2-gap that stays between the nth and the $(n+1)$th 2-block. We easily see that $(X^2(n) \colon n \geq 1)$ and $(Y^2(n) \colon n \geq 1)$ are i.i.d. sequences. There may be a gap before the first 2-block and we set

$$Y^2(0) = \begin{cases} Y^1(0) & \text{if} \quad S^1_1 = 1; \\ Y^1(0) + X^1(1) + Y^1(1) & \text{if} \quad S^1_1 = 2. \end{cases} \tag{5.14}$$

The level 2 variables become

$$\omega^2(n) = \begin{cases} G \text{ if } X^2(n) > N_2; \\ B \text{ if } X^2(n) \leq N_2. \end{cases} \tag{5.15}$$

From the configuration $(\omega^2_1, \omega^2_2, \dots)$ we repeat the previous procedure, successively.

Step $k+1$**.** Having defined the level k configuration $(\omega^k_1, \omega^k_2, \dots)$ we set $T^k_0 = 0$ and define $T^k_n, S^k_n, n \geq 1$ by (5.11) with the superscript 1 replaced by k. Similarly, $X^{k+1}(n), Y^{k+1}(n)$, and $Y^{k+1}(0)$ are then given by (5.12), (5.13) and (5.14) with the superscript 1 replaced by k.

A key step in the argument of [10] to control the growth of gaps is given by the Lemma below, whose proof shows why one should allow two attempts to start a k-block.

Lemma 5.6 . *If $\beta_i(k)$, $i = 1, 2$ are defined through the relations*

$$P(X^k(1) \le N_k) = L^{-\beta_1(k)} \qquad P(X^k(1) \le N_{k+1}) = L^{-\beta_2(k)} \tag{5.16}$$

and $\varepsilon > 0$ is chosen so that

$$\beta_1(k) \ge 30(1.3)^k,\ \beta_2(k) \ge 20(1.3)^k \tag{5.17}$$

for $k = 1$ then indeed (5.17) holds for all $k \ge 1$.

Proof. The basic ingredient is: if (5.17) holds for a given k, then: (I) $\beta_1(k+1) \ge 2\beta_2(k)$ and (II) $\beta_2(k+1) \ge 20(1.3)^{k+1}$. (I) follows after one checks that

$$\begin{aligned} L^{-\beta_1(k+1)} =& P(X^{k+1}(1) \le N_{k+1}) \\ \le& P(X^k(1) \vee X^k(2) \\ \le& N_{k+1}) \\ =& L^{-2\beta_2(k+1)}, \end{aligned}$$

where the first equality follows from the definition in (5.17), the second equality comes from the definition and the fact that $X^k(1)$ and $X^k(2)$ are i.i.d. The inequality is the main point and it follows from the inclusion

$$[X^k(1) \vee X^k(2) > N_{k+1}] \subset [X^{k+1}(1) > N_{k+1}].$$

which can be seen by splitting the event on the left according to: (i) $X^k(1) > N_{k+1}$, (ii) $N_k < X^k(1) \le N_{k+1}, X^k(2) > N_{k+1}$, or (iii) $X^k(1) \le N_k, X^k(2) > N_{k+1}$ and recalling, in the third case, that one gives two chances to start a $(k+1)$-block.

To check (II), since $X^k(1), X^k(2), \dots$ are i.i.d. and $N_k \ge 1$, one has

$$\begin{aligned} L^{-\beta_2(k+1)} &= P(X^{k+1}(1) \le N_{k+2}) \\ &\le P(\min_{1 \le j \le N_{k+2}} X^k(j) \le N_k) \\ &\le N_{k+2} L^{-\beta_1(k)} \\ &\le L^{(1.1)^{k+2} - 30(1.3)^k} \end{aligned}$$

which implies that $\beta_2(k+1) \geq 30(1.3)^k - (1.1)^{k+2} \geq 20(1.3)^{k+1}$.

From (I) and (II) above we see that as soon as (5.17) holds for $k=1$ it must hold for all $k \geq 1$. To see that it holds for $k=1$ it suffices to have

$$(1-p)^{L^{(1.1)}} \geq L^{39} \quad \text{and} \quad (1-p)^{L^{(1.1)^2}} \geq L^{-26}. \tag{5.18}$$

Having fixed L, this amounts to take p (or ε) small enough. □

In the proof given in [10], L is chosen as $L = 10^{300}$ (huge). From the previous lemma we see that

$$\sum_{k=1}^{\infty} P(X^k(1) \leq N_k) \leq \sum_{k=1}^{\infty} L^{-30(1.3)^k} < \infty$$

and by Borel-Cantelli lemma: there exists $K(\omega) < \infty$ a.s. so that $X_1^k(\omega) > N_k$ for all $k \geq K(\omega)$, i.e. $S_1^k = 1$, $Y^k(0) = Y^{K(\omega)}(0)$ for all $k \geq K(\omega)$. The main estimate from [10] gives that if $y^* = 1 + Y_0^{K(\omega)}$ then taking δ small one has

$$P(\xi_t^{y^*} \neq \emptyset, \forall t \geq 0) \geq 1/2 \tag{5.19}$$

Size of blocks and gaps. Set ν_k as the maximal number of sites possible in a k-gap. The construction gives $\nu_1 = 1$, $\nu_{k+1} \leq 3\nu_k + 2N_k$, for any $k \geq 1$. We recall $N_0 = L$, $N_k = N_{k-1}^{1.1} = L^{(1.1)^k}$, for $k \geq 1$. With this in mind one easily checks the following Lemma whose proof is left as exercise.

Lemma 5.7 . *For $L \geq 9^{10}$ one has $\nu_k \leq 3N_{k-1}$, $N_{k-1} \leq N_k/9$. In particular, $\nu_k \leq N_k/3$.*

Remark. For the probability estimates involved in the proof of (5.19) it may be technically convenient to break the k-blocks when they contain a too large number of $(k-1)$-blocks.

Lemma 5.8 . *Any k-block, with $k \geq 1$, may be broken into pieces that start and end with $(k-1)$-blocks, with the length of each piece being in the interval $[N_k, 3N_k]$.*

Proof. We proceed by induction. For $k = 1$ the conclusion is trivial, since a G run with length in $[jN_1, (j+1)N_1)$ can be broken into $(j-1)$ strings of length N_1 and one string of length in $[N_1, 2N_1)$. If a k-block has length larger than $3N_k$ we make a cut at the endpoint of the first $(k-1)$-block that stays after N_k. Considering that N_k by be in a $(k-1)$-gap we see that the length to the left of the cut is at most $N_k + \nu_k + 3N_{k-1}$ (using the induction assumption at the last point). We see that this is bounded from above by $N_k(1+1/27+3/9) = \frac{37}{27}N_k$. What is left to the right of the cut is then of length at least $(3-38/27)N_k > N_k$, which allows to conclude the proof. $\square$

Key probability estimates. In what follows let

$$M_k = \alpha^{-6N_{k-1}}, \qquad \text{for } k \geq 1,$$

where $\alpha > 0$ is taken suitably small:

(i) $\alpha \leq e^{-2\Delta}(1-e^{-1})$,

(ii) $\alpha < e^{-\sqrt{21}}$,

(iii) $\alpha < \alpha_0$, with α_0 to be determined from the estimates that follow.

If the intervals $[a,b]$ and $[c,d]$ are k-blocks separated by the k-gap (c,d), consider the events:

$$\begin{aligned} C_k &= \{(a,0) \to \{b\} \times [0, 3N_k M_{k-1}] \text{ in } [a,b] \times \mathbb{R}\} \\ D_k &= \{[a,b] \times \{0\} \to [a,b] \times \{2M_k\} \text{ in } [a,b] \times \mathbb{R}\} \qquad (5.20) \\ E_{k+1} &= \{\{a\} \times [0, M_k] \to \{d\} \times [0, M_k] \text{ in } [a,b] \times \mathbb{R}\}, \end{aligned}$$

where $S \to T$ in $[a,b] \times \mathbb{R}$ means that in the Harris graphical representation of the contact process, there is a path form a point in S to a point in T entirely contained in $[a,b] \times \mathbb{R}$.

Remark. Under the above conditions

$$M_k/M_{k-1} = \alpha^{-6(N_{k-1}-N_{k-2})} \geq \alpha^{-5N_{k-1}}, \quad 3N_kM_{k-1} \ll M_k.$$

Proposition 5.9 . *For $\delta > 0$ small enough the following holds: given the previous choice of L, $0 < \alpha < 1/27$, $\pi_1 = 1/4$ and $\pi_j = 36\alpha^{N_{j-2}/72}$ for $j \geq 2$, then for any $k \geq 1$*

$$\begin{aligned} p_k :=& P(C_k) \geq \prod_{i=1}^{k}(1-\pi_j) \\ q_k :=& 1 - P(D_k) \leq \alpha^{N_{k-1}} \\ r_{k+1} :=& 1 - P(E_{k+1}) \leq \alpha^{N_k}, \end{aligned}$$

wiith the understanding that the estimates hold for any k-blocks with size in the interval $[N_k, 3N_k]$.

Remark. With the notation and the conditions of Proposition 5.9 one checks that $\sum_{j\geq 1}\pi_j < 3/8$, and so $\prod_{j\geq 1}(1-\pi_j) > 5/8$. ($L \geq 2^{10}$ and $0 < \alpha < 1/2$ are sufficient here.)

Indeed, $N_{i+1} = N_iN_i^{1/10} \geq N_iL^{1/10} \geq 2N_i$ if $L \geq 2^{10}$, and since $N_i \geq 72$ we also have $N_{i+1}/72 \geq N_i/72 + 1$. Thus,

$$\begin{aligned} \sum_{j\geq 2}\pi_j \leq 36\alpha^{N_0/72}\sum_{j\geq 2}\alpha^{j-2} &= 36\alpha^{L/72}(1-\alpha)^{-1} \\ &\leq 72 \times 2^{-L/72} < 2^7 2^{-L/100} < 2^{-3}. \end{aligned}$$

It is convenient to see now that Theorem 5.5 will follow once one has Proposition 5.9.

Proof of Theorem 5.5. Let y^* be given by just before (5.19), and $k \geq 1$. Applying Proposition 5.9 to the k-block that starts at y^*, simply called $C_k = [y^*, b]$, we have that $P(C_k) \geq 5/8$, cf. the previous Remark. The same conditions on L, α guarantee that for the corresponding D_k^c one has $P(D_k^c) \leq \alpha^L \leq 1/8$. Since C_k and D_k are increasing events, the FKG inequality gives $P(C_k \cap D_k) \geq 1/2$. But $3N_k M_{k-1} < M_k$, so that the crossing lemma (see Ch. 2) implies that $\xi_t^{y^*}$ survives up to M_k on $C_k \cap D_k$. The argument works for any $k \geq 1$, so that the conclusion follows. □

Proof of Proposition 5.9. The proof is by induction, and it runs as follows: having the estimates for $P(C_k)$ and $P(D_k)$ one gets the estimate for $P(E_{k+1})$. Putting together the estimates for $P(D_k)$ and $P(E_{k+1})$ one gets those for $P(C_{k+1}), P(D_{k+1})$ completing the induction step. The basis of the induction goes as follows: notice that if $\delta = 0$ (i.e. the process does not die at the good sites) we would have $P(D_1) = 1$, and also $P(C_1) > 7/8$ provided $\alpha < \alpha_0$ small engouh. Given this, and since the events C_1, D_1 depend on a finite portion of Harris graphical construction, the probabilities are continuous in δ. In particular, we have the needed estimates, if δ is small enough.

Estimating r_{k+1} from p_k, q_k. In the previous notation set $\ell = c - b$. One then checks:

Lemma 5.10 . $P\{(b, 0) \to \{c\} \times \{\ell - 1, \ell\}\} \geq \alpha^\ell$.

Proof. Let A_m be the event that there occurs a birth from m to $m+1$ in the time interval [m,m+1), and no death during the time interval $((m-1)^+, m+1)$. then $A_0, \ldots, A_{\ell-1}$ are independent and $P(A_i) \geq \alpha$ for each $i = 0, \ldots, \ell - 1$. □

We apply this last lemma and recall that the gap (b, c) has length bounded by ν_k. For each time interval $[7jN_k M_{k-1}, 7(j+1)N_k M_{k-1})$ one tries to go from a to d as follows:

- Try to go from a to b during $[7jN_k M_{k-1}, (7j + 3)N_k M_{k-1}]$. The probability of succeeding is at least $1/2$.
- If one succeeds in the previous steo, then try to drill at once through (b, c);

- succeeding with the drilling we shall be at c by time $(7j + 3)N_k M_{k-1} + \nu_k \leq (7j+4)N_k M_{k-1}$ and will then have probability at least $1/2$ to cross $[c,d]$ before time $(7j+7)N_k M_{k-1}$.

The probability of succeeding in all the three steps is at least $\alpha^{\nu_k}/4$. The number of attempts at disposal is $\mu(k) = M_k/7N_k M_{k-1} \geq \alpha^{-5M_{k-1}}/7N_k$, so that we get:

$$r_{k+1} = 1 - P(E_{k+1}) \leq (1-\alpha^{\nu_k}/4)^{\mu(k)} \leq \exp\left(-\mu(k)\frac{\alpha^{\nu_k}}{4}\right)$$
$$\leq \exp\left(-\frac{\alpha^{-2N_{k-1}}}{28(N_{k-1})^{1.1}}\right).$$

On the last term we now use that $e^y \geq y^3/6$ if $y > 0$ we get

$$r_{k+1} \leq \exp\left(-\frac{8N_{k-1}^{1.9}(\log\frac{1}{\alpha})^3}{6\times 28}\right) = \exp\left(-\frac{N_{k-1}^{1.9}}{21}(\log\frac{1}{\alpha})^3\right)$$
$$\leq \exp\left(-N_k \log\frac{1}{\alpha}\right) \leq \alpha^{N_k}$$

provided $(\log\frac{1}{\alpha})^2 \geq 21$. For this we require $\alpha \leq e^{-\sqrt{21}}$.

Estimating p_{k+1}, q_{k+1} from q_k, r_{k+1}.

Given a $(k+1)$-block $[a,b]$, let v be the number of k-blocks that form it, so that

$$N_{k+1}/(3N_k + \nu_k) \leq v \leq 3N_{k+1}/N_k,$$

and recalling that $N_{k+1} = N_k^{1.1}, \nu_k \leq N_{k-1}/3$, we see that

$$\frac{3}{10}N_k^{1/10} \leq v \leq 3N_k^{1/10}. \tag{5.21}$$

Let these k-blocks be written as $[a_1, b_1], [a_2, b_2], \ldots, [a_v, b_v]$ listed from left to right, so that $a_1 = a$ and $b_v = b$. As in previous chapters, we may now consider a comparison with a one-dependent site percolation on

$$\mathcal{V}_1 = \{(i,j) : 1 \leq i \leq v, j \geq 1, i+j \text{ even } \},$$

where one declares (i,j) to be *open* if

$$\begin{aligned}
[a_i, b_i] \times \{(j-1)M_k\} &\to [a_i, b_i] \times \{(j+1)M_k\}, \\
\{b_i\} \times [jM_k, (j+1)M_k] &\to \{a_{i-1}\} \times [jM_k, (j+1)M_k], \\
\{a_i\} \times [jM_k, (j+1)M_k] &\to \{b_{i+1}\} \times [jM_k, (j+1)M_k],
\end{aligned}$$

with the understanding that when $i = 1$ ($i = v$) we omit the second (third) condition. Otherwise one says that (i,j) is *closed.* Then:

(A) $\rho := P((i,j) \text{ is closed }) \leq q_k + 2r_{k+1} \leq 3\alpha^{N_{k-1}}$.

(B) If $|i - i'| + |j - j'| > 2$ then the events "(i,j) is open" and "(i',j') is open" are independent.

(C) If there exists an open path $(i_1, 1), \ldots, (i_n, n)$ in $\mathcal{V}_1$ such that $i_1 = i, \ldots, i_n = j$, and $|i_s - i_{s-1}| = 1, s = 2, \ldots, n$, then in the contact process graphical construction we have $[a_i, b_i] \times \{0\} \to [a_j, b_j] \times \{(n+1)M_k\}$ in $[a,b] \times \mathbb{R}$.

Contours argument similar to those in [16], mentioned in Chapter 2, may be used at this point to yield: (see [10] for the full proof)

Lemma 5.11 . *If $\rho < 6^{-36}$ then the probability that there is no open path from $\{1, \ldots, v\} \times \{1\}$ to $\{1, \ldots, v\} \times \{K\}$ in $\mathcal{V}_1$ is bounded from above by $2K(3\rho^{1/36})^v$.*

Remark. Recall: $\rho \leq 3\alpha^{N_{k-1}} \leq 3\alpha^L < 32^{-L} \leq 6^{-36}$ for $\alpha < 1/2$ and L as previously chosen, and $K = M_{k+1} \geq 2M_{k+1}/M_k$. Applying the last lemma we get:

$$q_{k+1} \le 2M_{k+1}\left(3(3\alpha^{N_{k-1}})^{1/36}\right)^v \le 2.\alpha^{-6N_k}3^{37v/36}\alpha^{vN_{k-1}/36}.$$

Performing computations, and recalling (5.21) we see that

$$\begin{aligned} q_{k+1} &\le 2\cdot 3^{4N_k^{1/10}}\alpha^{-6N_k}\alpha^{N_{k-1}N_k^{1/10}/120} \\ &= 2\cdot 3^{4N_k^{1/10}}\alpha^{-6N_k}\alpha^{N_k^{1.11/1.1}/120} \\ &\le 2(81)^{N_k^{0.1}}\alpha^{2N_k} \le \alpha^{N_k}, \end{aligned}$$

provided $\alpha \le 1/81$ and where we used that $N_k^{1.11/1.1}/120 - 6N_k \ge 2N_k$, or equivalently $N_k^{0.01/1.1} \ge 120 \times 8$ which is true if $L \ge 1000^{100}$. This gives the estimate for q_{k+1}.

To prove the validity of the estimate for $P(C_{k+1})$ one uses the auxiliary site percolation model on $\mathcal{V}_2 = \{(i,j)\colon i,j \ge 1, i+j \text{ even }\}$, and the following analogous estimate whose proof is omitted see [16] or [10]):

Lemma 5.12 . *If $\rho < 6^{-72}$ and $v \le K/2$ then*

$$P((1,1) \to \{v\} \times [0,K] \text{ in } \mathcal{V}_2) \ge 1 - 12\rho^{1/32}.$$

Notice that if in the graphical construction of the contact process we have $(a_1, 0) \to \{b_1\} \times [0, 3N_kM_{k-1}]$ in $[a_1, b_1] \times \mathbb{R}$ (which corresponds to an event of the type C_k) and

$$F_{k+1} = \{\exists \text{ path from } (1,1) \text{ to } [v,\infty) \times \{3N_{k+1}\} \text{ in } \mathcal{V}_2\}$$

then again the property of crossing paths implies that C_{k+1} occurs at the $(k+1)$-block $[a,b]$:

$$C_{k+1} = \{(a,0) \to \{b\} \times [0, 3N_{k+1}M_k] \text{ in } [a,b] \times \mathbb{R}\}$$

and (i, j) in $\mathcal{V}_2$ corresponds to $[a_i, b_i] \times [(j-1)M_k, (j+1)M_k]$.

Since the events C_k, F_{k+1} are increasing, we can apply the FKG inequality, writing:

$$P(C_{k+1}) \geq P(C_k \cap F_{k+1}) \geq P(C_k)P(F_{k+1}) \geq p_k P(F_{k+1})$$

Checking the conditions we see that $\rho \leq 3\alpha^{N_{k-1}} \leq 3(2^{-L}) \leq 6^{-72}$, and $v \leq 3N_{k+1}/N_k \leq N_{k+1}/2$ so that

$$P(F_{k+1}) \geq 1 - 12(3\alpha^{N_{k-1}})^{1/72} \geq 1 - 36\alpha^{N_{k-1}/72}$$

completing the proof. $\square$

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