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On Kinematical Limits of Finite Integrals and Algebraic Properties of Integration in Anomalous Amplitudes

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"I don't wanna fight anymore imagine wars,
that only brought me darkness, destruction, and loss..."

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Abstract

We perform an investigation of the leading anomalous graphs in four and six dimensions of the topology of a triangle and box, respectively. The linear and log power counting in some of their integrals are handled using an alternative strategy called Implicit Regularization (Ireg). The idea is to preserve from the beginning the arbitrary routings and isolate them in convergent integrands employing an algebraic identity. The divergences are not explicitly integrated but are organized in formal expressions corresponding to surface terms and irreducible scalars. Finite parts are freely integrated and automatically functions of routing differences; they are projected in a class of functions whose integral representation obeys many recurrence relations that are useful in the study of symmetries. Over the set of fermionic amplitudes studied, a class of linear relations is established among their integrands, and we name them relations among green functions (Ragfs). Under integration, they represent linearity of integration; based on general and detailed computations, we determined how, in this set of amplitudes, such algebraic aspect is set by the kinematical limits of finite amplitude that appear in the Ragfs. The result of the constraint furnishes a value to the formal surface term incompatible with translational invariance in momentum space. This incompatibility rooted in features independent of the divergences can be interpreted as the cause of violation of at least one Ward identity since if the referred limit was zero, then all WIs could be kept. From the perspective drawn, we can localize the array of distinct results that can be obtained for the anomalous amplitudes based on which property is chosen to be maintained. The features investigated extend to other graphs with more external lines and other types of coupling, such as the pseudotensor and tensor vertexes in four dimensions and triangle topology; nevertheless, the violations of the Ragfs are not directly related to kinematical limits.

Resumo

É realizada uma investigação dos primeiros diagramas anômalos em quatro e seis dimensões da topologia de um triângulo e box, respectivamente. A contagem de potência linear e logarítmica em algumas de suas integrais é tratada usando uma estratégia alternativa chamada Regularização Implícita (Ireg). A ideia é preservar desde o início os rótulos da linhas internas arbitrários e isolá-los em integrandos convergentes empregando uma identidade algébrica. As divergências não são explicitamente integradas, mas são organizadas em expressões formais correspondentes a termos de superficie e integrais escalares irredutíveis. As partes finitas são integradas livremente e são automaticamente função das diferenças entre os rótulos. Estas são projetadas em uma classe de funções cuja representação integral obedece várias relações de recorrência que são úteis no estudo de simetrias. Sobre o conjunto de amplitudes fermiônicas estudadas, uma classe de relações lineares é estabelecida entre seus integrandos, e as denominamos relações entre funções de green (Ragfs). Sobre integração, representam a linearidade da integração; com base em cálculos gerais e detalhados, determinamos como, neste conjunto de amplitudes, tal aspecto algébrico é restrito pelos limites cinemáticos de amplitudes finitas que aparecem nas Ragfs. O resultado da restrição fornece um valor ao termo de superfície formal que é incompatível com a invariância de translação no espaço de momentos. Esta incompatibilidade que tem origem em características independentes das divergências pode ser interpretada como a causa da violação de pelo menos uma identidade de Ward, pois se o referido limite fosse zero, então todas as WIs poderiam ser mantidos. A partir da perspectiva traçada, podemos localizar o conjunto de resultados distintos que podem ser obtidos para as amplitudes anômalas com base na propriedade escolhida para ser mantida. As características investigadas estendem-se a outros grafos com mais linhas externas e outros tipos de acoplamento, como os vértices pseudotensores e tensores em quatro dimensões e topologia triangular; no entanto, as violações dos Ragfs não estão diretamente relacionadas com os limites cinemáticos.

Palavras-chave: Anomalias, Limites cinemáticos, Linearidade de integração, Divergências, Regularização Implícita.

Publications

This dissertation is based on the peer-reviewed publication of the author,

- J.F. Thuorst; L. Ebani; T.J. Girardi. Low-energy theorems and linearity breaking in anomalous amplitudes. *Annals Phys.* 468 (**2024**) 169725.

And an expanded pre-print version,

- L. Ebani; T. J. Girardi; J. F. Thuorst. Symmetries in one loop solutions: The AV, AVV, and $AVVV$ diagrams, from $2D$, $4D$, and $6D$ dimensions and the role of breaking integration linearity. (**2022**) *arXiv:2212.03309*.

The dissertation develops explicitly the technical tools needed for these publications.

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Introduction

Since their inception, anomalies have played an important role in quantum field theories (QFTs). Concisely, anomalies come from the quantum violation of symmetries present in the classical theory. This subject arose when the authors [3]-[6] attempted to build models with fermions coupled to axial currents. Afterwards, it resurfaced in two dimensions through the non-conservation of the axial current in two-point perturbative corrections [7]. In four dimensions, it manifests through the coupling of axial and vector currents in one fermionic loop, the ABJ anomaly of the triangle graph [8]-[10]. The presence of one anomalous term on the axial current divergence is responsible for the decay rate of some mesons [11], including the experimentally observed decay of the neutral pion into two photons [12].

The concept of anomaly received prominence due to the breaking of Ward Identities (WI), crucial in guaranteeing the renormalizability of gauge models [13]. Theories featuring spontaneous symmetry breaking, such as the Standard Model, resort to anomalous cancellation to circumvent this problem [14, 15]. This mechanism becomes fundamental for maintaining the consistency of the theory, also contributing to the prediction of particles as the top quark [16]. Some research lines suggest the need for a similar mechanism to establish a gauge theory in the gravitational context. Anomalies manifest when gravitational fields couple to fermions, with two gravitons contributing to the axial anomaly from a triangle diagram, see Kimura [17], Delbourgo and Salam [18].

This subject remains important in investigations within the domain of Kaluza-Klein theories, irrespective of renormalization [19, 20]. We stress its relevance regarding the breaking of diffeomorphism invariance in purely gravitational anomalies (without gauge coupling), see Gaume [21]. When interacting with photons and Weyl fermions, one also acknowledges violations of conformal symmetry in the propagation of gravitons [22, 23]. Furthermore, recent contributions have revisited the Weyl anomalies on the Pontryagin density contribution, mainly by Bonora in [24]-[28]. Lorentz anomalies can be interchanged with Einstein anomalies using the local Bardeen-Zumino polynomial [29], which transforms the consistency into a covariant form for anomalies, see [30, 31] for a simple application. Ultimately, anomalies are recognized as an intrinsic aspect of symmetries [32], establishing criteria for delimiting admissible field theories.

With this background established, we aim to elucidate some aspects relevant to the anomalies study, aspects that are fundamentally linked to explicit computation of perturbative amplitudes. For such, let us develop our investigation in a general model coupling fermions with boson fields of even and odd parity (spins zero and one), coupling without derivatives. The n-vertex polygon graphs of spin-1/2 internal propagators are the center of this analysis, being explored in two, four, and six dimensions. The corresponding amplitudes exhibit Dirac traces containing two gamma matrices beyond the space-time dimension, whose evaluation yields combinations involving the metric tensor and the Levi-Civita symbol. Hence, traces admit equivalent expressions that differ in their index arrangements, signs, and number of monomials. That only produces identities at first glance; however, subtle consequences emerge since the involved amplitudes are divergent. This feature led to many works developed in recent years, sometimes proposing rules to take these traces [33]-[37] and [38]-[40]. Part of our task is to shed light on this issue, and we use operations on general identities governing the Clifford algebra for such.

This outset is intimately linked to the divergent content of amplitudes. When dealing with linearly divergent structures, a shift in the integration variable requires compensation through non-zero surface terms [41, 16]. These objects bring coefficients depending on arbitrary routings attributed to internal momenta[1]. Although conservation sets routing differences as physical momenta, internal momenta remain arbitrary and might assume non-covariant expressions [42]. This feature represents a break in the translational invariance, violating a crucial requirement for establishing WIs and thus violating other symmetries. Alternatively, some regularization techniques [43, 44] partially preserve symmetries because they maintain translational invariance by eliminating surface terms.

Given the impossibility of satisfying all WIs in four dimensions [45], we attribute a special treatment to the axial triangle. That motivates the pursuit of odd correlators involving axial and vector vertices, the AV^n-type amplitudes in $2n$ dimensions. They are $(n+1)$-order tensors written as functions of n momentum variables, which leads to low-energy theorems derived from well-defined finite functions, an approach in this direction can be found in [46]. We obtain these theorems through momenta contractions over general tensors, achieving meaningful results regarding the anomaly's source and implications.

Such a perspective is associated with relations among Green functions (Ragfs), obtained from momenta contractions over amplitudes independently of prescriptions to evaluate divergences. These relations embody the linearity of the integration and are a central ingredient of the procedure adopted for our calculations. We use the set of tools proposed by O. A. Battistel in his Ph.D. thesis [47], later known as Implicit Regularization (IReg). Several investigations applied this strategy in even and odd dimensions [51]-[50] and multi-loops calculations [56]. Other works also have a similar approach [60]-[59].

This strategy uses an identity to expand propagators, allowing us to isolate divergent objects without modifying expressions derived from Feynman rules. Evaluating these objects is

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[1] The same surface terms appear within tensor integrals exhibiting logarithmic power counting, albeit without arbitrary coefficients.

unnecessary in the initial steps; hence, one can opt for a prescription at the end of the calculations. No choices are made for internal momenta; they feature arbitrary routings used along the work. Lastly, the organization of finite integrals is also a helpful feature [61, 62]. We improved its efficiency by developing a systematization through finite tensors and their properties.

By exploring general tensor forms, we show how the kinematical behavior of finite integrals links to anomalous contributions. Although violations are unavoidable, different prescriptions affect how they manifest within the calculations. Interpretations that set surface terms as zero make results symmetric for even amplitudes. Meanwhile, they lead to the already-known competition between gauge and chiral symmetries for anomalous amplitudes. We elucidate this point by studying Dirac traces and how they allow different results for the same amplitude. Differently, an interpretation adopting one (specific) finite value for surface terms implies that all trace manipulations provide a unique tensor. Although that preserves the linearity of integration, it induces violating terms for even and odd amplitudes. Our perspective on low-energy implications offers a clear understanding of this subject.

The thesis is organized in **two main parts**. First, the chapters 1, 2, and 3 with general character; second, the chapter 4, 5, and 6 where resides the application of the former ones.

In **Chapter 1**, we introduce a first set of notations that appear along the thesis, mainly in chapters 2 and 3. The last section of the chapter is exclusively dedicated to introducing reasons and meaning of arbitrary routings, additionally its connection with the same concept that appear in the literature, however, with other purposes.

In **Chapter 2**, the general model is set, Feynman rules for vertices and propagators established, and a brief and formal treatment of WIs is made. Then, the subsequent section lay down with detail the integrand Ragfs and the integrated Ragfs, for arbitrary masses also. The next section handles in the diagrammatic level the WIs and how they relate to Ragfs. It ends with three necessary conditions, connected to Ragfs and translation symmetry, for WIs to hold. The last section (2.3) constructs a general and symmetry independent low-energy theorem (LET) and discusses the WIs from a kinematical perspective. The deduction is made for arbitrary even dimension, in the course of chapters 4, 5, and 6 it is specialized to the appropriate context.

The **Chapter 3** is the computational powerhouse of the thesis, in there we return to the subject of arbitrary routings and argue for and about the strategy of analysis IReg. The chapter is the longest one because it furnishes quite specific cues to actually perform computations with the strategy, beyond the problems studied in the thesis, e.g., gravitational anomalies. In section (3.3), a systematic for generalized surface terms is approached, but we need only the most simple structures in the subsequent chapters. In section (3.4) we introduced a basis for finite functions and an efficient organization in terms of tensors resembling Feynman integrals. Then the next section is essentially examples of computation and application of notation, it can be left aside in a first moment. The practical results about reductions or recursions for finite parts, enabling all operations for verification of Ragfs, are developed in section (3.6). After that a dedicated derivation of a class of tensors we called sign-tensors is performed. Finally, we discuss some features of traces that will appear in the remaining chapters applied to particular amplitudes and dimensions. All derivations about finite parts where done in such a way they are valid for multiple masses and general dimension. The chapter has more information than what is necessary to follow with the investigation.

Then the second main part starts with **Chapter 4**. It applies the results of previous chapters specialized for equal masses in two dimensions for AV-VA correlator. The connection with even amplitudes permits a simple comparison of Ragfs for even and odd amplitudes.

The **Chapter 5** handle amplitudes in four dimensions, AVV-VAV-VVA-AAA. It is the most detailed application of the tools developed in the first part. It has at least one explicit example for most operations performed. The objective is to allow the reader to reproduce the conclusions. The low-energy theorems are replicated in relation to the general results in (2.3). Two exclusive sections are used to connect them with linearity of integration. The last section of the chapter discusses some features of symmetry violations and violation of linearity. It ends with a series of conclusions and comments about application of the methods for analysis of odd tensors applied to other types of coupling.

The **Chapter 6** extends the analysis to two four-point amplitudes, $AVVV$ and $VAAA$ emphasizing the general validity of the investigation. This chapter, again, ends with partial conclusions about how the tools apply to other types of coupling.

Finally, we conclude with **Chapter 7,** summarizing our findings and discussing one particular direction for future investigation.

As a guide for consultation we have the appendices: the finite functions and their reductions that are needed in two, four, and six dimensions are presented in **Appendix A.1**, **Appendix A.2,** and **Appendix A.3.** These appendices are a faster way to verify the results in dimension specific chapters. The **Appendices A.4** and **A.5** contain subamplitudes needed in four and six dimensions. The last **Appendix A.6** demonstrates a proposition about traces alluded in chapter 5.

Chapter 1

Notations, Conventions, and Preliminaries

> "*The difficulty, as in all this work, is to find a notation which is both concise and intelligible to at least two persons, of whom which may be an author*"—P. Matthews, A. Salam. Mod. Rev. 23 (1951) 314.

This chapter introduces a labelling system for tensors such as Minkowski vectors, metric tensor, Levi-Civita tensor, Dirac matrices, their products, and characteristic features like (anti)symmetrizations and hermicitictity in the first section. The second introduce a notation for the integrand and integrals, seizing the opportunity to digress over the uses of the notation, some of which appear in the investigation. Finally the third section will have an extensive discussion of the artifice of arbitrary routings. That section is an intermediary step between notation and the investigation per-se.

1.1 Indexing, Tensors and Dirac Matrices

First and foremost, the Riemannian manifold which refer any assertion in the body's text is the Minkowski one: the pair $\mathbb{M}^d = \left(\mathbb{R}^d, g\right)$, where g is the metric tensor whose expression in an orthonormal and coordinate basis is diagonal, $g_{\mu\nu} = \mathrm{diag}\left(1, -1, \cdots, -1\right)$ of signature $(1, d-1)$.

The sets of natural, integer, rational, real and complex numbers, $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$, collectively denoted by $\mathbb{F}$, will be understood as containing the neutral element of addition, when the zero need to be excluded it is indicated by $\mathbb{F}_* := \mathbb{F}\backslash\left\{0\right\}$. The units are the natural system, $c = \hbar = 1$.

Adopting some labelling system that has an alphabet whose members are Lorentz indices, namely

$$\mathfrak{L} = \left\{\alpha, \beta, \gamma, \ldots, \alpha_n, \beta_n, \ldots \mu_n, \nu_n \cdots : n \in \mathbb{N}_*\right\}, \tag{1.1}$$

and the set of sequences or strings of finite length denoted by $\mathfrak{L}^* = \{\mathrm{I} \,|\, \mathrm{I} : \mathbb{N} \to \mathfrak{L}, \quad |\mathrm{I}| < \infty\}$ with a product operation of concatenation or plain and simple juxtaposition, $\mathfrak{L}^* \times \mathfrak{L}^* \to \mathfrak{L}^*$. For example:

$$\mathrm{I} = (\mu_1, \mu_2) \equiv \mu_1\mu_2, \quad \mathrm{J} = (\alpha_1, \alpha_2) \equiv \alpha_1\alpha_2, \tag{1.2}$$

then we have their concatenation as

$$\mathrm{IJ} = (\mu_1, \mu_2, \alpha_1, \alpha_2) \equiv \mu_1\mu_2\alpha_1\alpha_2 = \mu_{12}\alpha_{12}. \tag{1.3}$$

Let it be the string of indexes[1] $\mathrm{I}_n = \left(\mu_{i_1}, \ldots, \mu_{i_n}\right) \equiv \mu_{i_1}\mu_{i_2}\cdots\mu_{i_n} \equiv \mu_{i_1 i_2\cdots i_n}$, which in principle is ordered, but the objects indexed by it may have symmetries, that will be pointed in a moment, which in some instances work just as a set of indices. The empty string will denote some scalar and the length (or depth) of it is given by $|\mathrm{I}_n| = n$. One author, R. Penrose, in *Spinor and Space-time V.1* [77], uses the term composite index. The importance to us is capturing the permutation and degeneracy features of their host objects. That author uses a more sophisticated structure, for us the indices will be just the components of the objects.

In a sequence of examples we shall introduce the other objects that are necessary. Let us begin by the very metric tensor: defining a symmetric tensor composed of its tensor products as follows

$$\begin{aligned} g_{\mathrm{I}_{2n}} \quad &: = \frac{1}{2^n n!}\sum_{\sigma\in S_{2n}}\prod_{i=1}^{n} g_{\mu_{\sigma(2i-1)\sigma(2i)}}, && (1.4)\\ g_{\mathrm{I}_{2n}} \quad &= \quad g_{\sigma(\mathrm{I}_{2n})}, \quad \sigma\left(\mathrm{I}_{2n}\right) := \mu_{\sigma(1)\ldots\sigma(2n)}, \quad \sigma \in S_{2n}, && (1.5)\end{aligned}$$

where the permutations σ are the set of bijections $\sigma : [1,n] \to [1,n] := \{1, \ldots, n\}$. In the formula above we can notice the merging we do with the indices $g_{\mu_{\sigma(2i-1)\sigma(2i)}} = g_{\mu_{\sigma(2i-1)}\mu_{\sigma(2i)}}$. That action is not a obligatory and it is used as a device to shrink some expression or to avoid repetition of symbols. The factor $(2^n n!)^{-1}$ in the definition is to cut off the degeneracy that summing over the $(2n)!$ permutations have. Hence, what we effectively have is a fully symmetric tensor with just the independent monomials which in this case are in number $|g_{\mathrm{I}_{2n}}| = (2n-1)!! = \prod_{i=1}^{n}(2i-1)$, the double factorial.

Example 1.1.1 *For four indices,* $\mathrm{I}_4 = \mu_{1234}$, *we have*

$$g_{\mathrm{I}_4} = g_{\mu_{1234}} = g_{\mu_{12}}g_{\mu_{34}} + g_{\mu_{13}}g_{\mu_{24}} + g_{\mu_{14}}g_{\mu_{23}}. \tag{1.6}$$

The next notational topic is traces of strings of Dirac matrices. They differ from the expression above by some signs and a factor, all given below. But first the definition:

$$\left\{\gamma_{\mu_1}, \gamma_{\mu_2}\right\}_+ = 2g_{\mu_{12}}\mathbf{1}; \quad \forall\mu \in \{0, \cdots, d-1\}, \gamma_\mu, \mathbf{1}\in\mathrm{Mat}\left(2^{d/2}, \mathbb{C}\right), \tag{1.7}$$

and we will omit reference to the identity matrix in general. The products of Dirac matrices can then be accommodated in our notational scheme as

$$\gamma_{\mathrm{I}_k} = \gamma_{\mu_{12\ldots k}} = \gamma_{\mu_1}\gamma_{\mu_2}\cdots\gamma_{\mu_k}. \tag{1.8}$$

Note that the order of indices in the string can be read from the notation used in the definition. The ordering can be read by context or by convention as we shall do soon.

---

[1] We shall use interchangeably the notation for sequences: $(\mu_i)_{i=1}^{l} = (\mu_1, \cdots \mu_l) = \mu_1 \cdots \mu_l = \mu_{1\cdots l} = \mathrm{I}_l$.

The traces of those strings obey the formulas

$$\begin{aligned} \mathrm{tr}\left(\gamma_{\mathrm{I}_{2n+1}}\right) &= 0, \text{ For even dimensions} && (1.9)\\ 2^{-d/2}\mathrm{tr}\left(\gamma_{\mathrm{I}_{2n}}\right) &= \frac{1}{2^n n!}\sum_{\sigma\in S_{2n}} \mathrm{sgn}\left(\sigma\right)\prod_{i=1}^{n} g_{\mu_{\sigma(2i-1)}\mu_{\sigma(2i)}} = \mathrm{pf}\left(\mathbf{M}\right). && (1.10) \end{aligned}$$

The parity function $\mathrm{sgn}\left(\sigma\right)$ is -1 if σ is a odd permutation or $+1$ if it is a even one. The last equality express the result as a pfaffian of a matrix. For a antisymmetric matrix of even dimensions the determinant can be written as the pfaffian square $\det\mathbf{M} = \mathrm{pf}^2\left(\mathbf{M}\right)$. The mentioned pfaffian satisfies a recurrence relation

$$\mathrm{pf}\left(\mathbf{M}\right) = \sum_{j=1,j\neq i}^{2n} \left(-1\right)^{i+j+1+\theta(i-j)} a_{ij}\mathrm{pf}\left(\mathbf{M}_{\hat{i}\hat{j}}\right) = \sum_{i=2}^{2n}\left(-1\right)^{i} a_{1i}\mathrm{pf}\left(\mathbf{M}_{\widehat{1}\widehat{i}}\right), \qquad (1.11)$$

where $\theta\left(i-j\right)$ is the Heaviside step function. This is the recurrence relation for the traces, and if one sets the elements of matrix $\mathbf{M}$ in the form

$$g_{i,j} := g_{\mu_i\mu_j} = g_{\mu_{ij}}\ , \qquad (1.12)$$

$$\mathbf{M} = \begin{pmatrix} 0 & g_{12} & g_{13} & \cdots & g_{1,2n}\\ -g_{12} & 0 & g_{23} & \cdots & g_{2,2n}\\ -g_{13} & -g_{23} & 0 & \cdots & \vdots\\ \vdots & \vdots & \vdots & 0 & g_{2n-1,2n}\\ -g_{1,2n} & & & -g_{2n-1,2n} & 0 \end{pmatrix}, \quad \mathrm{pf}\left(\mathbf{M}\right) = \frac{\mathrm{tr}\left(\gamma_{\mathrm{I}_{2n}}\right)}{\mathrm{tr}\left(\mathbf{1}\right)}. \qquad (1.13)$$

there will follow immediately the trace formula. The book of Caianiello, *Combinatorics and Renormalization in Quantum Field Theory* [79], contains more information about the uses of pfaffian in diverse themes of QFT. These remarks are being pointed since in the article [80] there are results which are just the beginning of more efficient way to compute lengthy traces and by different method the Kahane algorithm in [81] and generalization by Chisholm [82].

> "The proof of our final result is long and tedious, and even the statement of it is fraught with notational difficulties. We therefore explain it by an example.."—R. S. Chisholm.

The quotation does not mean we will endeavour in matters related to algorithms for traces or detailed and completely formal demonstrations in any subject we enter. However, following the spirit of the quotation we will try to exemplify our assertions and definitions as long as possible. Therefore, one example follow.

Example 1.1.2 *Take* $\gamma_{\mathrm{I}_4} = \gamma_{\mu_{1234}}$, *its trace reads*

$$2^{-d/2}\mathrm{tr}\left(\gamma_{\mathrm{I}_4}\right) = g_{\mu_{12}}g_{\mu_{34}} - g_{\mu_{13}}g_{\mu_{24}} + g_{\mu_{14}}g_{\mu_{23}}. \qquad (1.14)$$

About the common tensor appearances there are the epsilon tensor and antisymmetrized products of Clifford algebra generators. For the latter one, we have the definition:

Definition 1.1.3 *The skew-symmetrized or antisymmetrized product of* $0 \leq n \leq d$ *Dirac matrices is defined by*

$$\gamma_{[\mathrm{I}_n]} = \frac{1}{n!} \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma)\, \gamma_{\sigma(\mathrm{I}_n)}; \quad \sigma(\mathrm{I}_n) = \mu_{\sigma(1)\ldots\sigma(n)}. \tag{1.15}$$

For d *dimensions and* $n > d$, $\gamma_{[\mathrm{I}_{n>d}]} = 0$. *Since the tensor rank would necessarily ask for repetition of the numerical values of the indices in all components. For us the set of indices of some tensor are what they are, natural numbers. The empty string* I_0 *denotes the identity matrix,* $\gamma_{[\mathrm{I}_0]} = \mathbf{1}$.

To illustrate we have an example.

Example 1.1.4 *The most simple one has two indices, or the commutator of the generators:*

$$\gamma_{[\mathrm{I}_2]} = \gamma_{[\mu_{12}]} = \frac{1}{2}\left(\gamma_{\mu_{12}} - \gamma_{\mu_{21}}\right) = \frac{1}{2}\left[\gamma_{\mu_1}, \gamma_{\mu_2}\right] \tag{1.16}$$

About the signs, see the trivial fact that follows from the definition itself

$$\gamma^{[\mathrm{I}_r]} = \operatorname{sgn}(\sigma)\, \gamma^{[\sigma(\mathrm{I}_r)]}, \tag{1.17}$$

more caveats in the following.

For tensors that will turn up as numerators of integrals studied, numerators which are tensor powers of $K_i^\mu = k^\mu + k_i^\mu$, we define:

$$K_{\mathbf{i}}^{\mathrm{I}_n} = K_{i_1}^{\nu_1} K_{i_2}^{\nu_2} \cdots K_{i_n}^{\nu_n} = K_{i_1\cdots i_n}^{\nu_{12\cdots n}} \tag{1.18}$$

$$\mathrm{I}_n = \nu_{i_1 i_2 \cdots i_n}, \quad \mathbf{i} = i_1 i_2 \cdots i_n. \tag{1.19}$$

Notice that they are not symmetric for general $\mathbf{i}$,

$$K_{\mathbf{i}}^{\mathrm{I}_n} = K_{\sigma(\mathbf{i})}^{\sigma(\mathrm{I}_n)} \neq K_{\sigma(\mathbf{i})}^{\mathrm{I}_n} \neq K_{\mathbf{i}}^{\sigma(\mathrm{I}_n)}, \tag{1.20}$$

but when the sequence $\mathbf{i} = (i_r)_{r=1}^n$ is constant we just keep that constant as label, e.g. $\mathbf{i} = (1, \ldots, 1)$ for which $K_1^{\mathrm{I}_n} = \prod_{r=1}^n \left(k_{\mu_r} + k_{1\mu_r}\right)$ is the representation. Observe that the particular greek letter used and its placement to represent the tensor is largely an open choice.

Remark 1.1.5 *Now a moment for commentary and fixing some directions. A reasonable formalization though possible would imply too much energy spent since we only want a notation to shorten expressions and manipulations without loss of information or track of terms, signs, and factors. With this in mind we also adopt the following: Capital upright Roman letters will be employed for strings of Lorentz indices. Boldface lowercase Roman letters will represent multi-indices and other vectors but not Lorentz ones, in the lack of any explicit statement or convention to the contrary.*

It is not hard to grasp nor simple to confuse—the indexing system.

We must be careful when the strings represent contractions, in each case the same letter and number of indices must be adequate, and the relative position, like when the string is developed in the normal notation. In general, when no more than one string is contracted we use the letter C for the string and ν for its indices—remembering that this is a choice, if needed other letters are employed. An example of contraction will come when the chiral matrix enters the scene and is defined also.

As mentioned in the beginning the string of indices has naturally the concatenation product,

$$\mathrm{I}_r = \mu_{1\cdots r}, \quad \mathrm{C}_{d-r} = \nu_{1\cdots d-r}, \quad \mathrm{I}_r\mathrm{C}_{d-r} = \mu_{1\cdots r}\nu_{1\cdots d-r}. \tag{1.21}$$

The product is not commutative in general, and as example of behavior attached to some object we have $\varepsilon_{\mathrm{I}_r\mathrm{C}_{d-r}} = (-1)^{(d-r)r}\,\varepsilon_{\mathrm{C}_{d-r}\mathrm{I}_r}$. This follows from the definitional property

$$\begin{aligned} \varepsilon_{\mathrm{I}_d} &= \operatorname{sgn}(\sigma)\,\varepsilon_{\sigma(\mathrm{I}_d)}, & (1.22)\\ \varepsilon^{0,1,2\cdots d-1} &= -\varepsilon_{0,1,2\cdots d-1} = 1. & (1.23)\end{aligned}$$

Not all objects own such high symmetry, the role of the strings is to abbreviate in diverse forms the indices at play. We assume the indices of the indices in the sequence

$$\mathrm{I}_n = \nu_{i_1<i_2\cdots<i_n} \tag{1.24}$$

are in ascending order when the objects are fully antisymmetric and nothing more is said about. The notation will be allowed to be plastic enough to point out the important parameters, or operations, and assertions aimed at.

Clifford algebra generalities, returning to the subject of Dirac matrices. They are generators of an irreducible matrix representation of a Clifford algebra. For it we write down an important property they have, grading. For more informations consult the book by F. R. Harvey, *Spinors and Calibrations* [88]. As a vector space the algebra may be written as

$$\mathrm{Cl}_{1,d-1}\left(\mathbb{R}^d, g\right) = \bigoplus_{i=0}^{d}\mathrm{Cl}^{[i]}, \tag{1.25}$$

where the i in the i^{th} summand in the direct sum is the grade of the vectors in that sub-space. It corresponds to the number of the Dirac matrices being skew-symmetrized

$$\mathrm{Cl}^{[i]} = \operatorname{span}_{\mathbb{R}}\left\{\gamma_{[\mathrm{I}_i]}\right\}; \quad \gamma_{[\mathrm{I}_i]} \in \operatorname{Mat}\left(2^{d/2}, \mathbb{C}\right), \tag{1.26}$$

and their dimensions as vector spaces are given by

$$\dim\mathrm{Cl}^{[i]} = \binom{d}{i}; \quad \dim\mathrm{Cl}_{1,d-1} = \sum_{i=1}^{d}\binom{d}{i} = 2^d. \tag{1.27}$$

The highest grade sub-space is one-dimensional, its only generator is the chiral matrix defined as

$$\gamma_* = \frac{\mathrm{i}^{d/2-1}}{d!}\varepsilon_{\mathrm{C}_d}\gamma^{\mathrm{C}_d} = \frac{\mathrm{i}^{d/2-1}}{d!}\varepsilon_{\nu_1\cdots\nu_d}\gamma^{\nu_1\cdots\nu_d} = \mathrm{i}^{d/2-1}\gamma_0\gamma_1\cdots\gamma_{d-1}. \tag{1.28}$$

I call the reader to note that the last equality has explicit space-time indices appearing, it is the product of all generators—Dirac matrices. In even dimensions the chiral matrix as defined above has the properties

$$\gamma_*^\dagger = \gamma_*; \quad \gamma_*^2 = 1; \quad \left\{\gamma_*, \gamma_\mu\right\} = 0, \forall \mu \in \{0, \cdots, d-1\}\,. \tag{1.29}$$

The hermiticity property is representation dependent since hermiticity properties of the gamma matrices themselves depend on that. Any result derived assumes a representation unitarily equivalent to one that satisfies

$$\gamma^0\gamma^\mu\gamma^0 = \gamma^{\mu\dagger},$$

as the Dirac, or Weyl ones.

The Hodge star isomorphism is given by multiplication with the highest grade element

$$\gamma_*\gamma_{[\mathrm{I}_r]} = \mathrm{i}^{r(r-1)+d/2-1}\frac{\varepsilon_{\mathrm{I}_r\mathrm{C}_{d-r}}\gamma^{[\mathrm{C}_{d-r}]}}{(d-r)!} \tag{1.30}$$

Here we can see identical strings C_{d-r} accommodated in the form of contraction, again, and accompanying the free indices string I_d.

In this next step we have other types of tensors that appear in chapter 3, and which are formed not only by metric tensors. We are going to employ a notation similar to and first introduced in [85] by Passarino and Veltman, and thoroughly used by other authors like Davydychev and Tarasov in the refs. [86, 87].

Definition 1.1.6 *Multivariate symmetric tensor. Working with fully symmetric tensors T_i, whose rank maybe be bigger than two, we denote a fully symmetric tensor with unit normalization and formed by their tensor powers as*

$$\{[T_1]^{a_1}\cdots[T_r]^{a_r}\}^{\mathrm{I}}; \quad |\mathrm{I}| = \sum\nolimits_{i=1}^{r}\left[a_i \mathrm{rank}\left(T_i\right)\right]; \quad a_i \in \mathbb{N} \tag{1.31}$$

the set of its indices has depth $|\mathrm{I}|$ (number of indices present) as given above. It is the sum of all independent monomials with a_i copies of T_i all in product and fully symmetric in $|\mathrm{I}|$ indices.

To better grasp the definition (or the only way to this) let us work out some examples.

Example 1.1.7 *Choosing $T_{1\mu} = p_\mu$ and $T_{2\mu\nu} = g_{\mu\nu}$*

$$\left\{[T_1]^1\,[T_2]^1\right\}^{\mathrm{I}_3} = \{[g]\,[p]\}^{\mathrm{I}_3} = g_{\mu_{12}}p_{\mu_3} + g_{\mu_{13}}p_{\mu_2} + g_{\mu_{23}}p_{\mu_1} = \mathsf{k}^{p}_{\mathrm{I}_3}. \tag{1.32}$$

Another tensor with the same choices, but having one more factor in one tensor,

$$\begin{aligned}\{[g]\,[p]^2\}^{\mathrm{I}_4} &= g_{\mu_{12}}p_{\mu_3}p_{\mu_4} + g_{\mu_{13}}p_{\mu_2}p_{\mu_4} + g_{\mu_{14}}p_{\mu_2}p_{\mu_3} + g_{\mu_{23}}p_{\mu_1}p_{\mu_4} + g_{\mu_{24}}p_{\mu_1}p_{\mu_3} + g_{\mu_{34}}p_{\mu_1}p_{\mu_2} \\ &= \mathsf{k}^{pp}_{\mathrm{I}_4} = \mathsf{k}^{p^2}_{\mathrm{I}_4}.\end{aligned} \tag{1.33}$$

The variety of expressions are to illustrate the logic invoked by the notation. In the previous tensors, furnished as illustration, we introduce the notation $\{\mathsf{k}^{p}_{\mathrm{I}_3}, \mathsf{k}^{p^2}_{\mathrm{I}_4}\}$ that is a particular case of an adapted notation when metric and only vectors appear.

Definition 1.1.8 *Let it be $u, v \in \mathbb{R}^d$ then we define*

$$\begin{aligned} \mathsf{k}^{u^r v^s}_{\mathrm{I}_{r+s+2t}} &= \frac{1}{2^t t! s! r!} \sum_{\sigma \in S_{r+s+2t}} \left\{ \left[\prod_{i=1}^{r} u_{\mu_{\sigma(i)}} \right] \left[\prod_{j=r+1}^{s+r} v_{\mu_{\sigma(j)}} \right] \left[\prod_{l=s+r+1}^{2t-1} g_{\mu_{\sigma(l)} \mu_{\sigma(l+1)}} \right] \right\} \qquad (1.34) \\ r, s, t &\in \mathbb{N} \end{aligned}$$

In the expression above we have a fully symmetric tensor with u appearing s number of times, v appearing r times, and the remaining $2t$ indices are carried by metric tensor and then the sum runs over $(r+s+2t)!$ permutations. To end, there is the factor in front which enforces any independent monomial to turn up with unit coefficient. The reciprocal of that factor is the number independent of terms as well. Note San Serif font is used for k.

The connection with the notation of [86, 87] is expressed as

$$\mathsf{k}^{i^s J^r}_{\mathrm{I}_n} = \left\{ [k_i]^s \, [J]^r \, [g]^{\lfloor (n-r-s)/2 \rfloor} \right\}^{\mu_1 \cdots \mu_n} \qquad (1.35)$$

where $\lfloor m \rfloor$ is the closest integer less than m, the floor function. Note we suppressed the vector $u = k_i$ just keeping its index and dropped the square brackets. In our notation we must infer the number of metric tensors, which is easy in any case, and the power-like appearance reveal its efficiency in time. This notation was mainly construed by *T. Girardi and L. Ebani* in their Ph.d. investigations [90, 89]. This tensor notation can be used for more than two vectors, it has at most three layers

$$\mathsf{k}^{A_1^{a_1} A_2^{a2} \cdots A_n^{n} \leftarrow \text{The tensor powers}}_{\mu_1 \cdots \mu_j \leftarrow \text{The index set}} \quad .$$

The readiness to write some expressions justifies some of our notational devices, for example, the $2n$ dimensional traces

$$\begin{aligned} \mathrm{tr}\left(\gamma_* \gamma_{\mathrm{I}_{2n}}\right) &= 2^n \left(-i\right)^{n+1} \varepsilon_{\mathrm{I}_{2n}} \qquad (1.36) \\ \mathrm{tr}\left(\gamma_* \gamma_{\mathrm{I}_{2n+2}}\right) &= 2^n \left(-i\right)^{n+1} \sum_{\substack{a<b \\ a,b \in [1,2n+2]}} \left(-1\right)^{a+b+1} g_{\mu_{ab}} \varepsilon_{\mathrm{I}^{ab}_{2n}}; \quad \mathrm{I}^{ab}_{2n} = \left(\mu_c\right)^{2n}_{c=1, c \neq a,b} . \qquad (1.37) \end{aligned}$$

These formulae will be demonstrated in section (3.8).

This next content has the nature of an appendix, however, it is included here to exercise the notations. The Schouten identity: It is the fact that a fully antisymmetric tensor of rank $d+1$ in d-dimensions must be identically zero. Consider the sequence of indices $\mathrm{I}_d = (\mu_i)^d_{i=1}$ and a vector V_μ, the notation for the identity is given as follows

$$\varepsilon_{[\mathrm{I}_d} V_{\mu_{d+1}]} = \frac{1}{(d+1)!} \sum_{\sigma \in S_{d+1}} \mathrm{sgn}\left(\sigma\right) \varepsilon_{\sigma(\mathrm{I}_d)} V_{\mu_{\sigma(d+1)}} = 0. \qquad (1.38)$$

Only cyclic permutations are necessary, let us find it. For each permutation with $V_{\mu_{\sigma(d+1)}}$ fixed there are $d!$ terms which can be brought into the form

$$\operatorname{sgn}(\sigma)\, \varepsilon_{\mu_{\sigma(1)}\cdots\mu_{\sigma(i)}\cdots\mu_{\sigma(d)}} V_{\mu_{\sigma(d+1)}} = -\varepsilon_{\mu_1\cdots\mu_{i-1}\mu_{d+1}\mu_{i+1}\cdots\mu_d} V_{\mu_i}, \tag{1.39}$$

since it is equivalent to a transposition and any transposition has odd parity. Hence, we may write

$$\begin{aligned} \sum_{\sigma\in\mathcal{S}_{d+1}} \operatorname{sgn}(\sigma)\, \varepsilon_{\sigma(\mathrm{I}_d)} V_{\mu_{\sigma(d+1)}} &= d!\left\{\varepsilon_{\mathrm{I}_d} V_{\mu_{d+1}} - \sum_{i=1}^{d} \varepsilon_{\mathrm{S}_{i-1}\mu_{d+1}\mathrm{S}_{d-i}} V_{\mu_i}\right\} = 0 & (1.40)\\ \mathrm{S}_{i-1} &= (\mu_a)_{a=1}^{i-1}; \quad \mathrm{S}_{d-i} = (\mu_a)_{a=i+1}^{d}. & (1.41) \end{aligned}$$

Permuting μ_{d+1} to the right of S_{d-i} and concatenating them we get $\mu_{d+1}\mathrm{S}_{d-i} = (-1)^{d-i}\,\mathrm{S}_{d-i+1}$, then by transposing the strings $\mathrm{S}_{i-1}\mathrm{S}_{d-i+1} = (-1)^{(i-1)(d-i+1)}\,\mathrm{S}_{d-i+1}\mathrm{S}_{i-1}$ follows

$$\mathrm{S}_{i-1}\mu_{d+1}\mathrm{S}_{d-i} = (-1)^{i(d-i)+(i-1)}\,\mathrm{S}_{d-i+1}\mathrm{S}_{i-1}. \tag{1.42}$$

If 0 appear as the length of the string, it means the empty string. Putting together, simplifying some signs and including the first term, we get

$$(d+1)\,\varepsilon_{[\mathrm{I}_d} V_{\mu_{d+1}]} = \sum_{i=1}^{d+1} (-1)^{i(d+1-i)}\, \varepsilon_{\mathrm{S}_{d-i+1}\mathrm{S}_{i-1}} V_{\mu_i} = \sum_{\sigma\in\mathbb{Z}_{d+1}} \operatorname{sgn}(\sigma)\, \varepsilon_{\sigma(\mathrm{I}_d)} V_{\mu_{\sigma(d+1)}} = 0. \tag{1.43}$$

The sum reduces to cyclic permutations and if d is even, the sign is always positive; but in odd dimensions we have an oscillation due to $(-1)^{i.(d+1-i)} = (-1)^{i.d}$. Being the total result vanishing, the normalization does not matter, thus, we shall always write for any tensor carrying an additional set of indices $\mathrm{A}_{|J|} = (\alpha_j)_{j\in J}$, the following expression

$$\varepsilon_{[\mathrm{I}_d} V^{\mathrm{A}}_{\mu_{d+1}]} = 0 \tag{1.44}$$

The indices in $\mathrm{A}_{|J|}$ may be in contraction with the antisymmetrized ones.

Example 1.1.9 *In two dimensions a Schouten identity with one contraction can be written as*

$$\varepsilon_{[\mu_{12}} J^{\nu_1}_{2,\nu_1]} = \varepsilon_{\mu_{12}} J^{\nu_1}_{2,\nu_1} + \varepsilon_{\nu_1\mu_1} J^{\nu_1}_{2,\mu_2} + \varepsilon_{\mu_2\nu_1} J^{\nu_1}_{2,\mu_1} = 0,$$

the meaning of $J_2^{\mu_{12}}$ *will be established in 1.2.*

1.2 Integrands and Integrals

Rather late than never, let us consider some rational functions, Feynman integrands such as

$$\frac{K_{a_1}^{\mu_1}\cdots K_{a_m}^{\mu_m}}{\left(K_{b_1}^2 - m_{b_1}^2 + i0^+\right)\cdots\left(K_{b_n}^2 - m_{b_n}^2 + i0^+\right)}. \tag{1.45}$$

In the numerator we have the tensor powers of $K_i = k + k_i$ which were pondered about in the previous section, i.e., it may be written as $K_{\mathbf{a}}^{\mathrm{I}_m} = \prod_{r=1}^{m} K_{ar}^{\mu_r}$, when it is convenient to do so. The denominator factors receive the notation

$$D_i = \left(K_i^2 - m_i^2\right); \quad D_{i_1,\cdots,i_n} = \prod_{s=1}^{n} D_{i_s}, \tag{1.46}$$

Note that we have suppressed the prescription $+i0^+$ with the understand that masses carry a negative imaginary part responsible for the analytical structure of the amplitudes. We may also use a set notation sometimes $D_I = \prod_{i\in I} D_i$, where $I = \{i_1, \cdots, i_n\}$ is some appropriate set of indices.

Without lost of generality, we can bring all indices in the integrand above 1.45 to a reference one. By using $K_a = K_b + (k_a - k_b)$ we may write any of those rational functions as linear combinations of

$$\bar{j}_n^{\mathrm{I}_m} = \frac{K_a^{\mu_1} \cdots K_a^{\mu_m}}{\prod_{s=1}^{n} D_{b_s}} = \frac{K_a^{\mathrm{I}_m}}{D_{b_1 \ldots b_n}}, \tag{1.47}$$

we just require that $a \in \{b_1, \ldots b_n\}$.

From that, and obeying a logic explained in the sequel, we define their integrals as

$$\bar{J}_n^{\mathrm{I}_m} = \int_{\mathbb{R}^d} \mathrm{d}k \left[\bar{j}_n^{\mathrm{I}_m}\right] \equiv \int \frac{\mathrm{d}^d k}{c\,(d)} \frac{K_a^{\mathrm{I}_m}}{D_{b_1 \ldots b_n}}, \tag{1.48}$$

whose normalization of the measure has a number of conventions possible $\{c\,(d) = i\pi^{d/2}, c\,(d) = (2\pi)^{d/2}, c\,(d) = (2\pi)^{d}, \cdots\}$. Each has its advantages and can recovered quite simply. We will commonly choose the first in generic derivations, $c\,(d) = i\pi^{d/2}$, because it eliminate the common factor $i/\left(4\pi\right)^{d/2}$ that appear globally in integrals, mainly in the computational chapter 3. However, in some chapters we shall use other conventions, but it is indicated in the begining of the chapter what convention is adopted.

Here I call attention for a common usage where the integrand is identified by a lowercase letter and its integral gets an uppercase letter. This is going to happen with the green functions as well. The function of the overbar in notation for integrals will be discussed in chapter 3.

1.3 External and Internal Coordinates: Arbitrary Routings

This section has the heavy duty of motivating the, implicitly introduced, labelling of the integrals and amplitudes to be studied in the majority of the thesis. Some parts of this section uses a notation that will not turn up elsewhere; it may be understood as a standalone section, but integrated to the rest and indispensable in a certain sense.

In the sequel we shall handle sequences of distributions corresponding to free propagators in perturbation theory. As is common, the Fourier transform of these objects are better suited, since in position space they involve roots and Hankel functions, but are rational functions in momentum space.

In discussing the Fourier transform we try to motivate the device of arbitrary routings, which appear in the literature as the name of dual coordinates among others. Furthermore, we seize the opportunity to talk about the recurrent theme of translation invariance.

Let us start by considering the sequence of vectors

$$\begin{aligned} x &= (x_1,\dots,x_n) = (x_i)_{i=1}^n\,; \quad q = (q_i)_{i=1}^n\,; \quad r = (r_i)_{i=1}^n\,, \\ x_i, q_i, r_i &\in \mathbb{R}^d, \end{aligned} \tag{1.49}$$

where x will denote position space and q, and r momentum space. A graph is going to appear soon. Consider a sequence of functions $(F_i)_{i=1}^n$ which depend on relative coordinates $x_{i,j} := x_i - x_j$ arranged in a chain $F_1(x_{1,2})\,F_2(x_{2,3})\dots F_n(x_{n,n+1})$ and identify $x_{n+1} \equiv x_1$ to get a cycle. This is a typical object of our interest, the string of propagator-like functions. To lead it to the momentum space we Fourier transform it:

$$S(x) \;:\; = \prod_{i=1}^n F_i(z_i)\,; \quad z_i = x_{i,i+1}, \tag{1.50}$$

$$\langle r, z\rangle \;:\; = \sum\nolimits_{i=1}^n r_i \cdot z_i; \tag{1.51}$$

$$\check{F}_i(r_i) = \int_{\mathbb{R}^d} \mathrm{d}r_i e^{-ir_i\cdot z_i} F_i(z_i)\,, \tag{1.52}$$

$$S(x) = \int_{\mathbb{R}^{n\times d}} \mathrm{d}^n r \exp\left(-i\langle r, z\rangle\right) \prod_{i=1}^n \check{F}_i(r_i)\,, \tag{1.53}$$

where the measure in $\mathbb{R}^{n\times d}$ is expressed as $\mathrm{d}^n r = \prod_{i=1}^n \mathrm{d}r_i$, together with the abbreviation $\mathrm{d}r_i \equiv \mathrm{d}^d r_i / c(d)$. The global factors in this part will be suppressed.

Having written each factor in term of its inverse Fourier transform and performing a Fourier transform in the whole string of propagator functions, there follows

$$\hat{S}(q) = \int_{\mathbb{R}^{nd}} \mathrm{d}^n x e^{i\langle x,q\rangle} S(x) = \int_{\mathbb{R}^{nd}} \mathrm{d}^n r \int_{\mathbb{R}^{nd}} \mathrm{d}^n x \exp\left(i\langle x, q\rangle - i\langle r, z\rangle\right) \prod_{i=1}^n \check{F}_i(r_i)\,. \tag{1.54}$$

Then, in the exponential's argument, that is explicitly given as

$$\langle x, q\rangle - \langle r, z\rangle = \sum_{i=1}^n r_i \cdot x_i - \sum_{i=1}^n r_i \cdot x_{i,i+1}, \tag{1.55}$$

we split the second summation, use the modulo n labelling of x variables ($x_1 \equiv x_{n+1}$), and shift the summation index upwards to get

$$\sum_{i=1}^n r_i \cdot x_{i,i+1} = \sum_{i=1}^n r_i \cdot x_i - \sum_{i=1}^{n-1} r_i \cdot x_{i+1} + r_n \cdot x_1 = \sum_{i=1}^n r_i \cdot x_i - \sum_{i=1}^n r_{i-1} \cdot x_i. \tag{1.56}$$

With the understand that $r_0 \equiv r_n$, being that just a convenient convention to write a summation with no special term. Therefore, by $r_{i,j} := r_i - r_j$, we may write

$$\langle x, q\rangle + \langle r, z\rangle = \sum_{i=1}^n x_i \cdot (q_i - r_{i,i-1})\,. \tag{1.57}$$

Back to the transform we get

$$\begin{aligned}\hat{S}\left(q\right) &= \int_{\mathbb{R}^{n\times d}} \mathrm{d}^n r \prod_{j=1}^{n} \check{F}_j\left(r_j\right) \int_{\mathbb{R}^{n\times d}} \mathrm{d}^n x \prod_{i=1}^{n} \exp i[x_i \cdot (r_i - p_{i,i-1})] \\ &= \int_{\mathbb{R}^{n\times d}} \mathrm{d}^n r \prod_{i=1}^{n} \delta\left(q_i - r_{i,i-1}\right) \prod_{j=1}^{n} \check{F}_j\left(r_j\right).\end{aligned} \tag{1.58}$$

Now we achieved the point where we can free ourselves of most but one integration. The act of reducing the momentum constraints require a sequence of choices, which we want to elaborate over. Let us start by an example.

Example 1.3.1 *To reduce the integrations in the following expression*

$$\hat{S}\left(q\right) = \int_{\mathbb{R}^{n\times d}} \mathrm{d}r_1 \ldots \mathrm{d}r_n \delta\left(q_1 - r_{1,n}\right) \delta\left(q_2 - r_{2,1}\right) \ldots \delta\left(q_n - r_{n,n-1}\right) \check{F}_1\left(r_1\right) \ldots \check{F}_n\left(r_n\right),$$

we may begin by the integration in r_1 *which will asserts that* $r_1 = r_n + q_1$. *Then continuing by integrating (just integrating) in the sequence* $r_2, r_3 \ldots$, *we will get*

$$\begin{aligned}\hat{S}\left(q\right) &= \delta\left(q_1 + q_2 + \cdots + q_{n-1} + q_n\right) \int_{\mathbb{R}^d} \mathrm{d}r_n \\ &\quad \check{F}_1\left(r_n + q_1\right) \check{F}_2\left(r_n + q_1 + q_2\right) \ldots \check{F}_{n-1}\left(r_n + q_1 + \cdots + q_{n-2} + q_{n-1}\right) \check{F}_n\left(r_n\right).\end{aligned} \tag{1.59}$$

Note the very last integral escape, and we get the delta constraint over the total sum of the variables q_i. We kept the label r_n to differentiate of other options of integration. In the end, that variable is a dummy one, which we shall make a convention latter. And we already dropped the sum of all momenta in the last function in the string, $\check{F}_n\left(r_n\right)$. And what about other options? Let see one more example.

Example 1.3.2 *Starting with* r_2 *whose constraint imposes* $r_2 = r_1 + q_2$ *and then integrating* $r_3, r_4 \ldots$, *we get*

$$\hat{S}\left(q\right) = \delta(\textstyle\sum_{i=1}^{n} q_i) \displaystyle\int_{\mathbb{R}^d} \mathrm{d}r_1 \check{F}_1\left(r_1\right) \check{F}_2\left(r_1 + q_2\right) \ldots \check{F}_{n-1}\left(r_1 + q_2 + \cdots + q_{n-1}\right) \check{F}_n\left(r_1 - q_1\right). \tag{1.60}$$

Besides the dummy variable, we get a different integrand only by choice in the sequence of operations performed.

The glaring similarity would be completed as equality by translating the variable of integration $r_1 \to r_1 + q_1$. Before reaching this point, let us observe that the choices can assume quite general forms. Starting the integration in r_{i+1}, we get the expression

$$\hat{S}\left(q, i\right) = \delta(\textstyle\sum_{i=a}^{n} q_a) \displaystyle\int_{\mathbb{R}^d} \mathrm{d}r_i \prod_{j=1}^{n} \check{F}_j(r_i + \sum_{l=i+1}^{n+j} q_l), \tag{1.61}$$

with the notational understanding of periodicity for labelling system, that means

$$q_i = q_{n+i} \to q_1 = q_{n+1}, \tag{1.62}$$

$$q_i = q_{i-n} \to q_n = q_0. \tag{1.63}$$

Which simplify the work of writing the expression. Also observe that we put a label i in $\hat{S}(q,i)$ just to remind that, a priori, they are distinct functions of the external momenta $\{q_i\}$, which is the role this set of variables play. If all strings of functions were absolutely convergent there would be only one function.

As we have been mentioning, to compare any expression produced by the choices in the order of application of constraints, one needs to actively use the shift transformation. At this point it is introduced a tool for such aim, that will bring benefits and a cost.

Introducing an arbitrary set of coordinates $\{k_1 \ldots, k_n\}$ and a defining relation with the external momenta, $q_i = k_{i,i-1} = k_i - k_{i-1}$. For this relation to be valid for all $i \in \{1, \ldots n\}$ indices, we are called to notationally work with a extend set $\{k_0, k_1 \ldots, k_n\}$. Here we will not, a priori, impose periodicity in k_i. Then returning to the point, we write

$$\hat{S}(k) = \int_{\mathbb{R}^{nd}} \mathrm{d}^n r \left\{ \prod_{i=1}^{n} \delta\left(r_{i,i-1} - k_{i,i-1}\right) \right\} \prod_{l=1}^{n} \check{F}_l\left(r_l\right), \tag{1.64}$$

and by shifting $r_1 \to r_1 + k_1$, the argument in $\delta\left(r_{1,0} - k_{1,0}\right) \to \delta\left(-r_0 + (r_1 + k_0)\right)$ changes. Now, by the convention of r_i variables, $r_0 = r_n$, we keep integrating over $r_n, r_{n-1}, \cdots, r_2$, but we do not assume the convention $k_0 = k_n$ yet. The obtained expression is one of a general form, one that starts integrating in any r_{a-1}, it is given by

$$\begin{aligned} \hat{S}(k) &= \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (1.65) \\ &= \delta\left(k_{n,0}\right) \int_{\mathbb{R}^d} \mathrm{d}r_a \left\{ \prod_{i=1}^{a} \check{F}_i\left(r_a + k_i\right) \right\} \left\{ \prod_{i=a+1}^{n-1} \check{F}_i\left(r_a + k_i - k_{n,0}\right) \right\} \check{F}_n\left(r_a + k_0\right) \\ &\xrightarrow{\text{by the } \delta \text{ constraint, } k_0 = k_n} \int_{\mathbb{R}^d} \mathrm{d}r_a \prod_{i=1}^{n} \check{F}_i\left(r_a + k_i\right). \end{aligned}$$

The reason to not agree since the beginning in writing $q_1 = k_{1,0} = k_{1,n}$ is because the variables $\{k_i : 1 \leq i \leq n\}$ have the property that $k_{1,n} + \sum_{i=2}^{n} k_{i,i-1} = k_{1,n} + k_{n,1} \equiv 0$. The sum telescopes to the difference of the last and first term. In other words they hardwire total momentum conservation $\sum_{i=1}^{n} q_i = 0$.

The delta distributions constrains the traffic of momenta through the vertices, since the definition $q_i = k_{i,i-1}$ of the external momenta in terms of the internal ones says that the difference of momenta in adjacent edges (when $\hat{S}$ in represented by a graph) must be the same no matter the parametrization. The reminiscent overall delta $\delta\left(k_{n,0}\right)$ now allows to identify $k_0 \equiv k_n$. It is the incarnation of total momenta conservation. But notice that momenta conservations says nothing about the individual nature of the routings coordinates; moreover, the parametrizations exemplified in terms of external momenta are all coded in coordinate choices for k_i.

It is important to point out the generality of the routings. If it is expanded the d-dimensional delta distribution as

$$\delta\left(r_i - q_i\right) \equiv \delta^{(d)}\left(r_i - q_i\right) = \delta^{(1)}\left(r_{i,(0)} - q_{i,(0)}\right) \cdots \delta^{(1)}\left(r_{i,(d-1)} - q_{i,(d-1)}\right), \tag{1.66}$$

for in a next step reduce the constraints component-wise in each vector, we would not be capable to account for the arguments of a general propagator function F with a covariant expression, but k_i can, because they are not covariant quantities. They are labels whose only determinate values are their differences. *For the experienced practitioner of the field this may sound trivial, however, for the beginner these points perhaps may clarify some things.*

Summarizing: In effect, we 'broke up' the variables q_i and distributed their pieces among the 'propagator' functions, hoping for their future reassembling and no other source of functional dependence, all in an automatic manner. Such hope would be attained by translating the integration variables. The arbitrariness of the routings, therefore, replaces the translation symmetry, which would connect parametrizations of the internal edges by external momenta. This means that the possible freedom of translational symmetry is codified in the arbitrary choice of the routings. If such choices have no effect, we have the symmetry. On the other hand, their appearance is symptomatic of a cascade of other symmetry violations. The effects of this artifice of analysis will be deepened in the section handling extraction of information from formally divergent integrals and or amplitudes.

Let us simplify and fix some little changes of notation (essentially used throughout the text). The integration variable will be denoted by the letter k, and in general the expression acquire a form like the following

$$\begin{aligned} S\left(1,\cdots,n\right) &= \int_{\mathbb{R}^d} \mathrm{d}k \left[F_1\left(K_1\right)\cdots F_n\left(K_n\right)\right]; & (1.67)\\ K_i &= k+k_i; \quad k_i \in \left\{k_1,\ldots k_n\right\}, n\in\mathbb{N} & (1.68)\end{aligned}$$

In addition, we shall drop the symbols over $\hat{S}$, $\check{S}$ and distinguish the Fourier transforms from their conjugated transforms contextually, noticing whether the object refers to position or momentum space.

Definition 1.3.3 *The routing variables will have their difference denoted by*

$$p_{i,j} := k_i - k_j; \quad \sum_{i=1}^{n} p_{i,i-1} \equiv 0 \Leftrightarrow p_{1,n} + p_{n,1} = 0. \tag{1.69}$$

And its conjugate pair and generalizations

$$P_{i,j} = k_i + k_j; \quad P_{i_1 i_2 \ldots i_b} = \sum_{a=1}^{b} k_{i_a} \tag{1.70}$$

Remark 1.3.4 *In an amplitude* $p_{i,j}$ *relates with the external momenta very simply, we will set* $p_{i,i-1} = q_i$*, and use* $p_{i,j} = p_{i,l} + p_{l,j}$ *when necessary. As of the conjugate* $P_{i,j}$ *and its generalization, only one is really necessary because they basically represent the arbitrary routings. We can write the differences p is terms of the sums P, but not the opposite. Context shall bring more information.*

In literature, e.g. [75, 74], some descriptions used to describe the attitude of employing arbitrary routings are: dual variables trivialize the total momenta conservation, or make the momentum conservation manifest, or hardwire the property. Furnishing an unconstrained way to represent momentum conservation.

Dual variables are also called zone variables since we can divide the embedding of a planar graph by regions (in particular 1-loop graphs are planar). The regions are constructed extending external vertices to infinity, what divides the plane into a set of disconnected regions: the external ones unbounded and the internal ones bounded. Then proceeding by associating a vertex plus a momenta variable to each region, and connecting all of them by edges that crosses the internal edges of the original graph only once. Next the momentum flowing in each edge of the original graph are the differences of the momenta in the vertices of the "dual" graph [2]. Repeating the procedure for the dual graph results in the original graph, but with the external edges deleted. See this schematically in the fig. 1.1 below

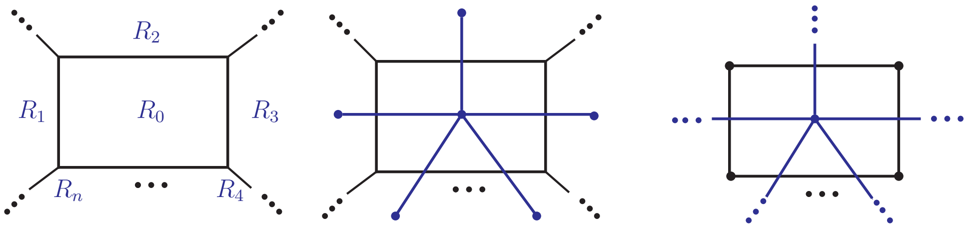

Figure 1.1: Dual Graph

Nowadays, the investigation of integrand properties per se, besides its integral, is a common theme in high multiplicity as higher loop approximations of QFTs. Those investigations seek to efficiently incorporate constraints in advanced computations for models like $4D$ $\mathcal{N} = 4$ SYM in the planar limit. For that aim, a series of new variables are introduced to treat amplitudes like MHV, i.e., maximum helicity-violating ones. They are dual coordinates, spinor-helicity variables, and momentum twistor variables to name the most salient ones, see [75, 78, 74].

Their objective is said to trivialize constraints like momenta conservation and on-shellness for external massless particles, both in the case of momentum twistors. In the case of dual variables new features are spotted as dual conformal symmetry and Yangin symmetry of classes of integrals, the Ph.d. thesis about the subject from J. Miczajka [83] is a rich source about the subject, in this author's opinion.

Nevertheless, the investigations cited handle finite integrals: dual conformal symmetry in massless integrals that are ultraviolet and infrared finite. Moreover, in the instance where there is a need to regulate an infrared singularity the use of some type of dimensional regularization breaks that symmetry, see the introduction of the article [76][3].

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[2] I have quoted the word dual because this graph is a little different from the formal definition of a dual graph.

[3] The article is one of the first full application of dual coordinate device without even name it like that, but is not the first the very first use of the device, this author could not find the very first occurence.

Our objective will be other: to study the implications of resorting to the natural freedom to route or label the edges of a graph by non-physical coordinates, like arbitrary routings (dual coordinates). This artifice already appeared wholly in the work of O. A. Battistel since his P.h.d. thesis in 1999 [47] and the paper, with M.C. Nemes, [70]. The context is the important subject of anomalies, in particular the correlators of currents which are bilinears in a spin one-half field, massive, and Dirac.

Being massive is not a conceptual downgrading, because the source of the massless-limit anomaly may be understood when observed from the massive instance. Now we pass to the model and our inquires.

Chapter 2

Model and Amplitudes

To appreciate the consequences of a set of fields and symmetries, classically, we would have to solve a set of partial differential equations that select an extremum of the action, plus a set of boundary conditions for solution uniqueness. Whereas quantum mechanically, we formally have a set of options to follow. For the past fifty years, Feynman's path integral has been the dominating method of going from a classical model of fields to a corresponding quantum theory. However, the companion strategy of canonical quantization never dwindled in its usefulness and insight. Today we have a symbiosis between both ideas, they are quite frequently mixed in textbooks and research articles.

Let it be a path integral or canonical approach, beyond free field models, perturbation theory is more often than not the weapon chosen to exhibit predictions for the models. Perturbation theory is most useful when the first few steps reveal the important features of the solution, and the remaining ones give small corrections. After classical tree-level effects, the most important from a general phenomenological point of view are the one-loop corrections. They comprise a functional which is the first power in the Planck constant, in the so-called loop expansion of QFT.

The tree level is a given input, schematically:

$$S_{\mathsf{classical}}\left[\Phi\right]=\Gamma_{\hbar^{0}}\left[\Phi\right]=\int_{\mathbb{R}^{d}}\mathrm{d}x\left[\mathcal{L}\left(\Phi,P\left(\partial\right)\Phi\right)\left(x\right)\right]. \tag{2.1}$$

The Lagrangian $\mathcal{L}$ is commonly a polynomial in its variables, and necessarily a polynomial for a set of models where renormalizability is required. The polynomial in the partial derivatives acts as $P_n\left(\partial\right)\Phi=\sum_{\alpha\in\mathbb{N}^d;|\boldsymbol{\alpha}|\leq n}c_{\boldsymbol{\alpha}}\partial^{\boldsymbol{\alpha}}\Phi$. For monomials of degree two in the fields (free part), the differential operators are at most of degree two for bosons and one for fermions. The quantized counterpart is symbolically given by

$$\mathrm{e}^{iW_{\mathsf{quantum}}}=\int\mathrm{d}\mu\mathrm{e}^{iS_{\mathsf{classical}}}. \tag{2.2}$$

The classical model from which follows the amplitudes through some choice of quantization, which we shall not discuss more than some general aspects, and the ingredients employed for the thesis can be, in generality, expressed as

$$S = \Gamma_{\hbar^0}\left[\Phi, \mathcal{J}\right] = \int_{\mathbb{R}^d} \mathrm{d}x \left\{ \mathcal{L}_{\mathsf{Dirac}}\left(\psi, \bar{\psi}\right) + \sum_{n=0}^{d/2} c_\Gamma \left[\mathcal{J}^{\Gamma}_{\mathrm{I}_n} \Phi^{\mathrm{I}_n}\right] \right\}. \tag{2.3}$$

The fermion Lagrangian $\mathcal{L}_{\mathsf{Dirac}}$ denoting the free part is given by

$$\mathcal{L}_{\mathsf{Dirac}} = \mathcal{L}_0 = \frac{1}{2}\bar{\psi}\left(i\overleftrightarrow{\not{\partial}} - 2m\right)\psi, \tag{2.4}$$

where $A\overleftrightarrow{\partial_\mu}B = A\partial_\mu B - \left(\partial_\mu A\right)B$. The other terms correspond to the interaction action

$$S_{\mathsf{int}} = \sum_{n=0}^{d/2} \int_{\mathbb{R}^d} \mathrm{d}x \left(c_\Gamma \mathcal{J}^{\Gamma}_{\mathrm{I}_n} \Phi^{\mathrm{I}_n}\right) = \sum_{n=0}^{d/2} \int_{\mathbb{R}^d} \mathrm{d}x c_\Gamma \left[\left(\bar{\psi}\Gamma_i\psi\right)_{\mathrm{I}_n} \Phi^{\mathrm{I}_n}\right]. \tag{2.5}$$

The coupling constants are denoted by $c_\Gamma \in \{c_S, c_{P,}c_V, c_A, c_{T,}c_{\tilde{T}}, \dots\}$. We will set them to unit, $c_\Gamma = 1$, since they do not participate in the deductions to come. Notwithstanding, they can be recovered in any case. The fermionic bilinears, which some are symmetry currents, Noether currents for a classical exact symmetry or an approximate one, are given by

$$\mathcal{J}^{\Gamma_i}_{\mathrm{I}}(x) = \left(\bar{\psi}\Gamma_i\psi\right)(x); \quad \Gamma_i \in \left\{\gamma_{[\mathrm{I}_k]} = \gamma_{[\mu_1 \dots \mu_k]} : 0 \leq k \leq d\right\}. \tag{2.6}$$

Where $\gamma_{[\mathrm{I}_k]}$ appeared in the definition (1.1.3). However, by the formula

$$\gamma_*\gamma_{[\mathrm{I}_r]} = i^{r(r-1)+d/2-1}\frac{\varepsilon_{\mathrm{I}_r\mathrm{C}_{d-r}}\gamma^{[\mathrm{C}_{d-r}]}}{(d-r)!}, \tag{2.7}$$

one may adopted a restricted set. They allow to write basis of grade bigger then $d/2$ as chiral matrix multiplied by the ones with grade $\leq d/2$. These identities may be used to compute traces as well.

Definition 2.0.5 *The vertices of our amplitudes will belong to the following set*

$$\Gamma_i \in \left\{\gamma_{[\mathrm{I}_n]}, \gamma_*\gamma_{[\mathrm{I}_n]}\right\} = \left\{1, \gamma_*, \gamma_\mu, \gamma_*\gamma_\mu, \gamma_{[\mu\nu]}, \gamma_*\gamma_{[\mu\nu]}\right\}, \tag{2.8}$$

where the tensor ones appear in subamplitudes as do the lower rank ones. They will denote the following currents

$$\begin{aligned} & \mathcal{J}^S;\ \mathcal{J}^P;\ \mathcal{J}^V_\mu;\ \mathcal{J}^A_\mu; \mathcal{J}^T_{\mu\nu};\ \mathcal{J}^{\widetilde{T}}_{\mu\nu} \\ = \quad & \bar{\psi}\psi;\ \bar{\psi}\gamma_*\psi;\ \bar{\psi}\gamma_\mu\psi;\ \bar{\psi}\gamma_{*\mu}\psi; \bar{\psi}\gamma_{[\mu\nu]}\ \psi;\ \bar{\psi}\gamma_{*[\mu\nu]}\psi, \end{aligned} \tag{2.9}$$

and the term current will be use for all of them.

Few comments are needed here. First, the set of bosonic fields are: scalar, pseudo-scalar, vector, pseudo-vector or axial, and antisymmetric tensor fields. In the respective order, they are denoted as

$$\Phi = \left(S(x), P(x), V_\mu(x), A_\mu(x), T_{\mu\nu}(x), \widetilde{T}_{\mu\nu}(x)\right). \tag{2.10}$$

The labels that will appear as upper index of amplitudes will accompany the ones for the vertex itself, but in our cases they will be denoted by

$$\Gamma_i \in \left\{ S, P, V, A, T, \widetilde{T} \right\}. \tag{2.11}$$

In correlation with the fields the densities $\mathcal{J}$ couple to. Next step is the green functions.

Definition 2.0.6 *Then the definition of the r-point*$(r = 1, 2 \ldots)$ *green functions, or correlators of currents,*

$$T^{\boldsymbol{\Gamma}}_{\mathrm{I}_n} = \left\langle \Omega \middle| \mathsf{T} \left\{ \mathcal{J}^{\Gamma_1}_{\mathrm{I}_{1,a_1}} (x_1) \, \mathcal{J}^{\Gamma_2}_{\mathrm{I}_{2,a_2}} (x_2) \cdots \mathcal{J}^{\Gamma_r}_{\mathrm{I}_{r,a_r}} (x_r) \right\} \middle| \Omega \right\rangle = \left\langle \prod\nolimits_{i=1}^{r} \mathcal{J}^{\Gamma_i}_{\mathrm{I}_{a_i}} (x_i) \right\rangle .$$

In the notation we have

$$\boldsymbol{\Gamma} = \Gamma_1 \cdots \Gamma_r = \Gamma_{1 \cdots r}, \quad \mathrm{I}_n = \mathrm{I}_{1,a_1} \cdots \mathrm{I}_{r,a_r}, \quad n = \sum\nolimits_{i=1}^{r} a_i,$$

the sequence $\boldsymbol{\Gamma} = (\Gamma_i)_{i=1}^{r}$ *will correlate with the sequence of indices that appear in the concatenation* $\mathrm{I}_n = \mathrm{I}_{1,a_1} \cdots \mathrm{I}_{r,a_r}$, *e.g., in the expression* $T^{AAVVSVT}_{\alpha\beta\delta\sigma\eta\rho\tau}$, *with* $\mathrm{I}_7 = \alpha\beta\delta\sigma\eta\rho\tau$, *the index* η *corresponds to the sixth vector vertex,* $\rho\tau$ *both correspond to the seventh and tensor vertex, and so on. The indices* a_r *refers to the tensor rank of the vertex. The time-ordering operator reads* $\mathsf{T} \left(\mathcal{O} (x_1) \cdots \mathcal{O} (x_n) \right) = \sum_{\sigma \in S_n} \prod_{i=1}^{n-1} \theta(x^{(0)}_{\sigma(i)} - x^{(0)}_{\sigma(i+1)}) \mathcal{O} \left(x_{\sigma(i)} \right)$.

Motivation for the Perturbative Expression: What follows is motivational in nature. Motivation for the perturbative expression to be investigated. Abbreviating the formal measure by $\mathrm{d}\mu_\psi = \mathrm{d}\psi \mathrm{d}\bar{\psi}$ we write the generating functional

$$Z (\Phi, \eta, \bar{\eta}) = \int \mathrm{d}\mu_\psi \exp i \int_{\mathbb{R}^d} \mathrm{d}x \left[\mathcal{L}_0 + \mathcal{L}_{\mathsf{int}} + \bar{\psi}\eta + \bar{\eta}\psi \right]. \tag{2.12}$$

By standard manipulations, this functional can be written as

$$Z (\Phi, \eta, \bar{\eta}) = \sum_{n \geq 0} \frac{i^n}{n!} \left\{ S_{\mathsf{int}} \left[\Phi, \mathcal{J} \left(\frac{\delta}{\delta\eta} \Gamma \frac{\delta}{\delta\bar{\eta}} \right) \right] \right\}^n Z (\eta, \bar{\eta}) \tag{2.13}$$

and the free field generator as

$$Z (\eta, \bar{\eta}) = \exp i \int_{\mathbb{R}^{2d}} \mathrm{d}x_1 \mathrm{d}x_2 \bar{\eta} (x_1) \, S_F (x_{12}) \, \eta (x_2) \, . \tag{2.14}$$

$S_F (x_1 - x_2)$ is the Feynman propagator, more discussion in the sequel. The formula above is a compilation of a functional whose derivatives gives the green functions

$$Z [\Phi] = \left(Z [\Phi, \eta, \bar{\eta}] \right)_{\eta = \bar{\eta} = 0} = \frac{1}{n!} \sum_{n \geq 1} Z_n (x_1, \ldots, x_n) \, \Phi (x_1) \cdots \Phi (x_n) \, . \tag{2.15}$$

It can be used to compute our correlator

$$\langle \mathcal{J} (x_1) \cdots \mathcal{J} (x_n) \rangle = \left. \frac{(-\mathrm{i})^n \, \delta^n Z [\Phi]}{\delta\Phi (x_1) \cdots \delta\Phi (x_n)} \right|_{\Phi = 0} = \frac{\int \mathrm{d}\mu_\psi \mathcal{J} (x_1) \cdots \mathcal{J} (x_n) \, e^{iS_0}}{\int \mathrm{d}\mu_\psi e^{iS_0}}.$$

These amplitudes are loop amplitudes since the composite current, bilinear in the fermion field, obliges the propagators coming from the functional derivatives to have identical arguments, in which case we have tadpoles, or to close in cycles. However, the cycles may be disconnected. To obtain only connected amplitudes, we need the connected generator

$$\Gamma\left[\Phi\right]=-i\log Z\left[\Phi\right]. \tag{2.16}$$

The traditional notation for effective action is employed, since by not integrating the fields Φ the connected generator will produce exactly the 1PI graphs[1]. The external edges, amputed by the Legendre transform, are simply not there, Φ are external fields. However, this is not loss of information because by turning on interacting boson fields, the amplitudes developed here will appear in that case with the same mathematical form, and that is our concern.[2]

Another way to reach loop amplitudes for the model is using the interaction-picture formula

$$\left\langle\Omega\left|\mathsf{T}\prod_{i=1}^{r}\mathcal{J}\left(x_i\right)\right|\Omega\right\rangle\sim\left\langle\Omega_0\left|\mathsf{T}\left\{\prod_{i=1}^{r}\mathcal{J}_0\left(x_i\right)\mathrm{e}^{-i\int\mathrm{d}^dz\mathcal{J}_0(z)\Phi(z)}\right\}\right|\Omega_0\right\rangle.$$

Thereby, the first term in the expansion followed by Wick theorem applied to the free operators produces the expressions we need. That means, products of propagators and vertices. We take only the 1PI graphs into account.

- **Amplitudes Definitions**

As pointed various paths leads to the cycle (loop) graphs or amplitudes here. In any case we have our spinor propagator in momentum space as

$$S\left(i\right)=S_F\left(K_i,m_i\right)=\frac{1}{\not{K}_i-m_i}=\frac{\not{K}_i+m_i}{D_i}. \tag{2.17}$$

Reminding the $K_i=k+k_i$ contains the integration variable and a routing. The amplitudes will start with their integrands

$$t_{\mathrm{I}}^{\mathbf{\Gamma}}\left(k,k_1,\ldots,k_n\right)=\mathrm{tr}\left[\Gamma_1S\left(1\right)\cdots\Gamma_nS\left(n\right)\right];\quad\mathbf{\Gamma}=\Gamma_1\ldots\Gamma_n. \tag{2.18}$$

The string of indices I will receive its depth in context $|\mathrm{I}_n|=n$, however, the common attitude is just express it as $\mu_{1\cdots n}$. Note the convention used for integrals $\bar{j}_k^{\mathrm{I}_l}$, about the lowercase and uppercase letters, is being extended here to the amplitudes. Therefore, the integrated amplitude reads

$$T_{\mathrm{I}}^{\mathbf{\Gamma}}\left(1,\ldots,n\right)=\int_{\mathbb{R}^d}\mathrm{d}k\left[t_{\mathrm{I}}^{\mathbf{\Gamma}}\left(k,k_1,\ldots,k_n\right)\right]. \tag{2.19}$$

Note we are stripping off any factor of $\mathrm{i}=\sqrt{-1}$ from propagators and vertices, plus setting aside minus signs from fermion loop rule. All retrievable if needed.

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[1] Graphs with only one connected component, whose number of connected components does not grow by deleting one edge.

[2] Keeping the sources for $\bar{\psi}$ and ψ then

$$\Gamma\left[\Phi,\psi,\bar{\psi}\right]=W\left[\Phi,\eta,\bar{\eta}\right]-\int_{\mathbb{R}^d}\mathrm{d}x\left[\bar{\psi}\eta+\bar{\eta}\psi\right];\quad\frac{\delta\Gamma}{\delta\psi}=\eta,\quad\frac{\delta\Gamma}{\delta\bar{\psi}}=-\eta.$$

Example 2.0.7 *The n^{th}-rank and odd amplitude of one axial vertex and remaining vector ones*

$$\begin{aligned} t^{AV\ldots V}_{\mu_1\cdots\mu_n}(1,\ldots,n) &= \operatorname{tr}\left[\gamma_*\gamma_{\mu_1}S(1)\,\gamma_{\mu_2}S(2)\ldots\gamma_{\mu_n}S(n)\right] \qquad (2.20)\\ \mathrm{I}_n &= \mu_{1\ldots n};\quad \Gamma_1 = A,\quad \Gamma_{n\geq 2} = V. \end{aligned}$$

Its integral gives the amplitude

$$T^{AV\ldots V}_{\mu_1\cdots\mu_n} = \int_{\mathbb{R}^d}\mathrm{d}k\left[t^{AV\ldots V}_{\mu_1\cdots\mu_n}\right]. \qquad (2.21)$$

The arguments are omitted in context. If the highest n-point expression being discussed does not explicitly show its variables, assume it is $(1,\cdots,n)$.

All external momenta are incoming and in our variables they get coded as

$$p_{i,j} = k_i - k_j = K_i - K_j;\quad q_i = p_{i,i-1};\quad i,j\in\{1,\cdots,n\}, \qquad (2.22)$$

the modulo n labelling is understood. They are represented in the graph of fig. (2.1)

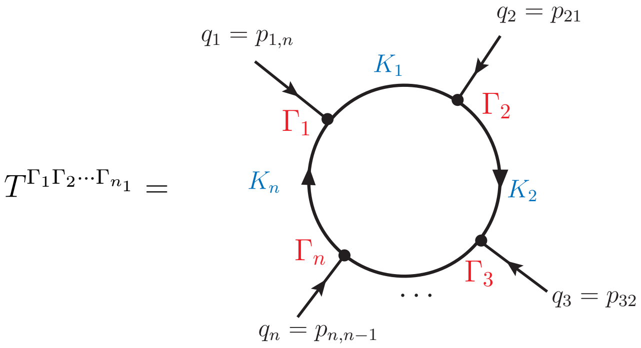


Figure 2.1: General diagram for the one-loop amplitudes of this work.

- **Symmetries**

Let us be direct here. Consider a more general current for two massive fermions $\mathcal{J}^{\Gamma}_{ab} = \bar{\psi}_a\Gamma\psi_b$. To have this coupling one needs a internal symmetry present, non-abelian in nature, connecting different species of fermions of masses m_a and m_b. In that scenario we would have semi-conservation laws. Using the free field motion's equations

$$(\mathrm{i}\partial\!\!\!/ - m_a)\,\psi_a = 0;\quad \bar{\psi}_a(\mathrm{i}\overleftarrow{\partial\!\!\!/} + m_a) = 0,$$

which are enough for our purposes, we derive the equations

$$\begin{aligned} \partial^{\mu}\mathcal{J}^{V}_{\mu,ab} &= \mathrm{i}\,(m_a - m_b)\,\bar{\psi}_a\psi_b = \mathrm{i}\,(m_a - m_b)\,\mathcal{J}^{S}_{ab} \qquad (2.23)\\ -\partial^{\mu}\mathcal{J}^{A}_{\mu,ab} &= \mathrm{i}\,(m_a + m_b)\,\bar{\psi}_a\gamma_*\psi_b = \mathrm{i}\,(m_a + m_b)\,\mathcal{J}^{P}_{ab}. \qquad (2.24) \end{aligned}$$

That for us have the same status of symmetry. Equal masses imply vector current conservation and vanishing masses in axial current conservation. We have introduced the feature of distinct masses to elaborate over certain aspects of our derivations. All the results investigated can directly extended to arbitrary masses.

These classical equations of motion, added to equal-time commutation relations, result in Ward-Takahashi identites (WI) for the correlator of definition (2.0.6). Formally, and for equal masses, we have

$$\begin{aligned}
\partial^{x_1}_{\mu_1}\left\langle \mathcal{J}^{A,\mu_1}\left(x_1\right)\mathcal{J}^{V}_{\mu_2}\left(x_2\right)\mathcal{J}^{V}_{\mu_3}\left(x_2\right)\cdots\right\rangle &= -2\mathrm{i}m\left\langle \mathcal{J}^{P}\left(x_1\right)\mathcal{J}^{V}_{\mu_2}\left(x_2\right)\mathcal{J}^{V}_{\mu_3}\left(x_2\right)\cdots\right\rangle, \\
\partial^{x_2}_{\mu_2}\left\langle \mathcal{J}^{A,\mu_1}\left(x_1\right)\mathcal{J}^{V}_{\mu_2}\left(x_2\right)\cdots\right\rangle &= 0.
\end{aligned}$$

The dots represent other vector of axial currents. Our discussion of WIs is brief, since it will be expanded in the very important section (2.2). For each of these correlators there are a set of diagrams to consider in order to investigate WIs and this links to the concept of relations among green functions (Ragfs) and linearity of integration. This prospects are material of the next two sections.

2.1 The iRagfs and IRagfs: Relations Among Green Functions

In this section we establish integrand Ragfs and integral Ragfs. The set of iRagfs are a set of identities, as the set of IRagfs are equations. However one may wonder: equations for what variable? For even-parity amplitudes IRagfs remain identities; nonetheless, in a class of odd-parity ones they are equations, which admit only one solution, see the eq. (5.111) for the answer in four dimensions. Without more explications, let us elaborate a while.

Starting by a general n-pt amplitude

$$t^{\mathbf{\Gamma}}_{\mathrm{I}_n}\left(1,\ldots,n\right) = \mathrm{tr}[\prod_{i=1}^{n}\Gamma_i S\left(i\right)];\quad \mathbf{\Gamma} = \Gamma_1\ldots\Gamma_n,$$

where the vertices $\mathbf{\Gamma}$ must carry Lorentz indices. The Ragfs are simply equations relating the integrands first and then integrals of the amplitudes. Let us start by the slash $\not{p}_{i.j}$ of a general momentum, it can be rewritten as inverse propagators and masses[3], explicitly

$$p^{\mu}_{i.j}\gamma_{\mu} = \not{p}_{i.j} = \not{K}_i - \not{K}_j = S^{-1}_F\left(i\right) - S^{-1}_F\left(j\right) + \left(m_i - m_j\right).$$

The artifice allows to relate green functions inside the classes of amplitudes, the relations are just consequences of defined operations of algebraic character.

Let us approach a sub-string in an integrand's amplitude, which is under trace operation and with the cyclic labelling understood, i.e.

$$\left(m_0, k_0, S_0\right) \equiv \left(m_n, k_n, S_n\right);\ \text{and}\quad \left(m_{n+1}, k_{n+1}, S_{n+1}\right) \equiv \left(m_1, k_1, S_1\right).$$

---

[3] Furthermore, it is similar to a WI for the tree level 1PI vector vertex Λ_{μ}.

With $S_i = S_F(i)$ to diminished the cluttered expressions. Let it be that sub-sequence of vertices and propagators given by

$$\cdots \Gamma_{i-1} S(i-1) \Gamma_i^{\mu_i} S(i) \Gamma_{i+1} S(i+1) \cdots . \tag{2.25}$$

By contracting with $p_{i,i-1}^{\mu_i}$, and focusing on $q_{i,\mu}\Gamma_i^{\mu}$ first, we have two cases that we denote by V and A superscript, they are simply

$$\begin{aligned} \left[q_i^{\mu}\Gamma_{\mu}^{V}\right] &= q_{i,\mu}\gamma^{\mu} = S_i^{-1} - S_{i-1}^{-1} + (m_i - m_{i-1}), & (2.26)\\ \left[q_i^{\mu}\Gamma_{\mu}^{A}\right] &= q_{i,\mu}\gamma_*\gamma^{\mu} = \gamma_* S_i^{-1} - \gamma_* S_{i-1}^{-1} + (m_i - m_{i-1})\gamma_*. & (2.27) \end{aligned}$$

When inserted on the string 2.25 they produce, for the a vector vertex, the following expression

$$\begin{aligned} \Gamma_{i-1}S_{i-1}\left[q_i^{\mu}\Gamma_{\mu}^{V}\right] S_i \Gamma_{i+1} S_{i+1} &= \Gamma_{i-1}S_{i-1}\Gamma_{i+1}S_{i+1} - \Gamma_{i-1}S_i\Gamma_{i+1}S_{i+1} & (2.28)\\ &\quad + (m_i - m_{i-1})\Gamma_{i-1}S_{i-1}\mathbf{1}S_i\Gamma_{i+1}S_{i+1}, \end{aligned}$$

and for an axial vertex

$$\begin{aligned} \Gamma_{i-1}S_{i-1}\left[q_i^{\mu}\Gamma_{\mu}^{A}\right] S_i\Gamma_{i+1}S_{i+1} &= \Gamma_{i-1}S_{i-1}\gamma_*\Gamma_{i+1}S_{i+1} - \Gamma_{i-1}\left[S_{i-1}\gamma_* S_{i-1}^{-1}\right] S_i\Gamma_{i+1}S_{i+1}\\ &\quad + (m_i - m_{i-1})\Gamma_{i-1}S_{i-1}\gamma_* S_i\Gamma_{i+1}S_{i+1}. & (2.29) \end{aligned}$$

For vector case one propagator gets reduced, and for distinct masses we have an additional term where the scalar vertex replaces the vector one, $V \to S$. Whereas in the second case transformations may happen. The last term of the axial contraction 2.29 will have the same n-pt degree of the source amplitude, tensor rank diminished by one unit and $A \to P$. The last step is the sandwich

$$S_{i-1}\gamma_* S_{i-1}^{-1} = -\left(\mathbf{1} + 2m_{i-1}S_{i-1}\right)\gamma_* \tag{2.30}$$

that changes the string to its final, close to final, form

$$\begin{aligned} \Gamma_{i-1}S_{i-1}\left[q_i^{\mu}\Gamma_{\mu}^{A}\right] S_i\Gamma_{i+1}S_{i+1} &= \Gamma_{i-1}S_{i-1}\left(\gamma_*\Gamma_{i+1}\right)S_{i+1} + \left(\Gamma_{i-1}\gamma_*\right)S_i\Gamma_{i+1}S_{i+1} & (2.31)\\ &\quad + (m_i + m_{i-1})\Gamma_{i-1}S_{i-1}\gamma_* S_i\Gamma_{i+1}S_{i+1}. & (2.32) \end{aligned}$$

However, let us observe that the products $\gamma_*\Gamma_{i+1}$ and $\Gamma_{i-1}\gamma_*$, in the first two terms, change of nature and sign. The first behave as

$$\gamma_*\left(\Gamma_{i+1}^{S}, \Gamma_{i+1}^{P}, \Gamma_{i+1}^{V}, \Gamma_{i+1}^{A}, \Gamma_{i+1}^{T}, \Gamma_{i+1}^{\widetilde{T}}\right) \to \left(\Gamma_{i+1}^{P}, \Gamma_{i+1}^{S}, \Gamma_{i+1}^{A}, \Gamma_{i+1}^{V}, \Gamma_{i+1}^{\widetilde{T}}, \Gamma_{i+1}^{T}\right) \tag{2.33}$$

and the second as

$$\left(\Gamma_{i-1}^{S}, \Gamma_{i-1}^{P}, \Gamma_{i-1}^{V}, \Gamma_{i-1}^{A}, \Gamma_{i-1}^{T}, \Gamma_{i-1}^{\widetilde{T}}\right)\gamma_* \to \left(\Gamma_{i-1}^{P}, \Gamma_{i-1}^{S}, -\Gamma_{i-1}^{A}, -\Gamma_{i-1}^{V}, \Gamma_{i-1}^{\widetilde{T}}, \Gamma_{i-1}^{T}\right). \tag{2.34}$$

Because the anticommutation with odd tensor-rank vertices. Thereby, the type of vertices may produce a sum of lower point functions in some cases and changes of nature.

In dimension $d = 2$, examples of non-trivial behavior are

$$q_1^{\mu_1}\left[t_{\mu_1}^{AP}\right] = \left[t^S(2)\right] + \left[t^S(1)\right] + (m_1 + m_2)\left[t^{PP}\right], \tag{2.35}$$

$$q_1^{\mu_1}\left[t_{\mu_1}^{VS}\right] = \left[t^S(2)\right] + \left[t^S(1)\right] + (m_1 - m_2)\left[t^{SS}\right], \tag{2.36}$$

$$q_1^{\mu_1}\left[t_{\mu_1}^{VP}\right] = \left[t^P(2)\right] + \left[t^P(1)\right] + (m_1 - m_2)\left[t^{SP}\right]. \tag{2.37}$$

Four our main amplitudes, decorated only with vector and axial vertices, we have the scheme

$$t_{\mathrm{I}}^{\mathbf{\Gamma}} = t_{\mu_1\cdots\mu_n}^{\Gamma_1\cdots\Gamma_n} \tag{2.38}$$

$$(p_{i,i-1})^{\mu_i}\left[t_{\mathrm{I}}^{\mathbf{\Gamma}}\right] = [t_{\mathrm{I}^i}^{\mathbf{\Gamma}^{i-1}}(\widehat{i-1})] - [t_{\mathrm{I}^i}^{\mathbf{\Gamma}^i}(\widehat{i})] + (m_i - (-1)^{c_i}\, m_{i-1})\, [t_{\mathrm{I}^i}^{\mathbf{\Gamma}_i}]. \tag{2.39}$$

Where $|\mathrm{I}| = n$, $|\mathrm{I}^i| = n - 1$, and the exponent $c_i = 1$ for $\Gamma_i = A$ and $c_i = 0$ for $\Gamma_i = V$. It is possible to arrange the indices in ascending order, i.e. $\mathrm{I}_{n-1}^i = \mu_{j_1<j_2\cdots<j_{n-1}}$ with $j_a \neq i$. The sequence of labels in $\mathbf{\Gamma}^{i-1}$ and $\mathbf{\Gamma}^i$ can differ by a cyclic permutation just for the relation involving contraction with $q_1^{\mu_1}$. Contrary to the general convention of Penrose [77], about contraction with a index inside a composite one, I will left the contraction index μ_i implicit inside the composite index I_{n+1}. Then, the symbol I_n^i is the string with μ_i projected out, or its simple omission $q^{\mu_i}\left(t_{\mathrm{I}_{n+1}}\right) = t_{\mathrm{I}_n^i} = t_{\dots\widehat{\mu_i}\dots}$. When it is absolutely clear that the particular index being contracted is part of that string, which is not hard to observe in context.

A sufficiently complex example illustrates better the matter of Ragfs. In six dimensions, $d = 6$, we handle boxes which have the following Ragfs: for the single axial box

$$p_{14}^{\mu_1}\left[t_{\mu_{1234}}^{AVVV}\right] = \left[t_{\mu_{234}}^{AVV}(2,3,4)\right] - \left[t_{\mu_{423}}^{AVV}(1,2,3)\right] + 2m\left[t_{\mu_{234}}^{PVVV}\right], \tag{2.40}$$

$$p_{21}^{\mu_2}\left[t_{\mu_{1234}}^{AVVV}\right] = \left[t_{\mu_{134}}^{AVV}(1,3,4)\right] - \left[t_{\mu_{134}}^{AVV}(2,3,4)\right], \tag{2.41}$$

$$p_{32}^{\mu_3}\left[t_{\mu_{1234}}^{AVVV}\right] = \left[t_{\mu_{124}}^{AVV}(1,2,4)\right] - \left[t_{\mu_{124}}^{AVV}(1,3,4)\right], \tag{2.42}$$

$$p_{43}^{\mu_4}\left[t_{\mu_{1234}}^{AVVV}\right] = \left[t_{\mu_{123}}^{AVV}(1,2,3)\right] - \left[t_{\mu_{123}}^{AVV}(1,2,4)\right], \tag{2.43}$$

and the triple axial one

$$p_{14}^{\mu_1}\left[t_{\mu_{1234}}^{VAAA}\right] = \left[t_{\mu_{234}}^{AAA}(2,3,4)\right] - \left[t_{\mu_{423}}^{AAA}(1,2,3)\right], \tag{2.44}$$

$$p_{21}^{\mu_2}\left[t_{\mu_{1234}}^{VAAA}\right] = \left[t_{\mu_{134}}^{VVA}(1,3,4)\right] - \left[t_{\mu_{134}}^{AAA}(2,3,4)\right] + 2m\left[t_{\mu_{134}}^{VPAA}\right], \tag{2.45}$$

$$p_{32}^{\mu_3}\left[t_{\mu_{1234}}^{VAAA}\right] = \left[t_{\mu_{124}}^{VAV}(1,2,4)\right] - \left[t_{\mu_{124}}^{VVA}(1,3,4)\right] + 2m\left[t_{\mu_{124}}^{VAPA}\right], \tag{2.46}$$

$$p_{43}^{\mu_4}\left[t_{\mu_{1234}}^{VAAA}\right] = \left[t_{\mu_{123}}^{AAA}(1,2,3)\right] - \left[t_{\mu_{123}}^{VAV}(1,2,4)\right] + 2m\left[t_{\mu_{123}}^{VAPA}\right]. \tag{2.47}$$

One more case, for distinct masses. Consider the $(n+1)$-pt amplitude

$$t_{\mu_1\cdots\mu_{n+1}}^{AV^n} = \mathrm{tr}\left[\gamma_*\gamma_{\mu_1}S(1)\,\gamma_{\mu_2}S(2)\dots\gamma_{\mu_{n+1}}S(n+1)\right], \tag{2.48}$$

its first contraction reads

$$q_1^{\mu_1}\left[t_{\mathrm{I}_{n+1}}^{AV^n}\right] = \left[t_{\mathrm{I}_n^1}^{AV^{n-1}}\left(\hat{1},\dots,n+1\right)\right] - \left[t_{\mathrm{I}_n^1}^{V^{n-1}A}\left(\sigma\left(1,\dots,\hat{n}\right)\right)\right] + (m_1 + m_n)\left[t_{\mathrm{I}_n^1}^{PV^n}\right], \tag{2.49}$$

or permuting the vertices in the second term of the r.h.s. to get

$$q_1^{\mu_1}\left[t_{\mathrm{I}_{n+1}}^{AV^n}\right]=\left[t_{\mathrm{I}_n^1}^{AV^{n-1}}\left(\hat{1},\ldots,n+1\right)\right]-\left[t_{\sigma\left(\mathrm{I}_n^1\right)}^{AV^{n-1}}\left(1,\ldots,\hat{n}\right)\right]+\left(m_1+m_{n+1}\right)\left[t_{\mathrm{I}_n^1}^{PV^n}\right], \tag{2.50}$$

where the permutation means $\sigma\left(\mathrm{I}_n^1\right)=\mu_{n,2,\cdots,n-1}$. For the other vector vertices, we have

$$\begin{aligned} q_i^{\mu_i}\left[t_{\mathrm{I}_{n+1}}^{AV^n}\right] &= \left[t_{\mathrm{I}_n^i}^{AV^{n-1}}\left(\hat{1},\ldots,n\right)\right]-\left[t_{\mathrm{I}_n^i}^{AV^{n-1}}\left(1,\ldots,\hat{n}\right)\right]+\left(m_i-m_{i-1}\right)\left[t_{\mathrm{I}_n^i}^{\mathbf{\Gamma}}\right], \\ 2 &\leq i\leq n+1. \end{aligned} \tag{2.51}$$

The vector of labels has the scalar label S in the i^{th} position, i.e. $(\Gamma_1\ldots\Gamma_{n+1})=(A,V\ldots,S_i,\ldots V)$. The string I_n^i just omits the index μ_i, and the hats in the arguments means the omission of the corresponding propagator. The same reasoning applies to even amplitudes, any perturbative green function carrying at least one Lorentz index have a set of Ragfs.

The set of relationships discussed we will call integrand Ragfs, abbreviated as *iRagfs*. After integration, we denote them as *IRagfs*. These identities have non-trivial behavior in integral form, IRagfs. They embody the linearity of integration when the operation holds, and for non-negative power counting they stand for the linearity of the functional which replaces it. Any regularization known assumes it can be distributed linearly over sums of terms it regularizes, if not explicitly at least implicitly.

The IRagfs for the specific class where $\Gamma_i\in\{A,V\}$, now reads

$$q_i^{\mu_i}\left[T_{\mathrm{I}}^{\mathbf{\Gamma}}\right]=[T_{\mathrm{I}^i}^{\mathbf{\Gamma}^{i-1}}(\widehat{i-1})]-[T_{\mathrm{I}^i}^{\mathbf{\Gamma}^i}(\widehat{i})]+\left(m_i+(-1)^{c_i}\,m_{i-1}\right)[T_{\mathrm{I}^i}^{\mathbf{\Gamma}_i}]. \tag{2.52}$$

Where $c_i=1$ if $\Gamma_i=V$ and $c_i=-1$ if $\Gamma_i=A$; what means that for vector vertices and equal masses the last term is not present. The relation with WIs, and anticipating the fact that if the IRagfs cannot hold Ward-Takahashi identities either, is the matter of next section.

2.2 IRafgs and Their WI Counterparts

A correlator, coming from wick contractions of free fields, has its 1PI contributions given by

$$\left\langle\mathcal{J}\left(x_1\right)\cdots\mathcal{J}\left(x_n\right)\right\rangle_{\mathsf{1PI}}\sim\sum_{\sigma\in S_n/\mathbb{Z}_n}T_{\mu_{\sigma(1)}\cdots\mu_{\sigma(1)}}^{\Gamma_{\sigma(1)}\cdots\Gamma_{\sigma(n)}}. \tag{2.53}$$

Hence, we will adopted the notation for the sums of crossed channels as

$$T_{\mathrm{I}}^{\Gamma_1\to\Gamma_{2\cdots n}}=\sum_{\sigma\in S_n/\mathbb{Z}_n}T_{\mu_{\sigma(1)}\cdots\mu_{\sigma(1)}}^{\Gamma_{\sigma(1)}\cdots\Gamma_{\sigma(n)}}=\sum_{\sigma\in S_n/\mathbb{Z}_n}T\left(\sigma\left(\mathbf{\Gamma}\right),\sigma\left(\mathrm{I}\right)\right). \tag{2.54}$$

where $\sigma\in S_n/\mathbb{Z}_n$ indicates the sum over non-cyclic permutations. For targeting a discussion of WIs we must treat the complete correlator. Therefore, in a sequence of fairly detailed arguments, some of which are basic ones, we finalize with the proposition that all algebraic relation are needed to state a WI.

Aiming some illustration we will route a specific expression by external momenta, the expression is an arbitrary 3-pt amplitude,

$$\left\langle\prod_{i=1}^{3}\left[\bar{\psi}\left(x_i\right)\Gamma_i\psi\left(x_i\right)\right]\right\rangle_{\mathsf{1PI}}=T^{\Gamma_{123}}\left(x_1,x_2,x_3\right)+T^{\Gamma_{132}}\left(x_1,x_3,x_2\right), \tag{2.55}$$

where the position space representation is the starting point (as of now I am not touching the definitional status of the representation). They are given by

$$\begin{aligned} T^{\Gamma_{123}}\left(x_1,x_2,x_3\right) &= \operatorname{tr}\left[\Gamma_1 S\left(x_{12}\right)\Gamma_2 S\left(x_{23}\right)\Gamma_3 S\left(x_{31}\right)\right], & (2.56)\\ T^{\Gamma_{132}}\left(x_1,x_3,x_2\right) &= \operatorname{tr}\left[\Gamma_1 S\left(x_{13}\right)\Gamma_3 S\left(x_{32}\right)\Gamma_2 S\left(x_{21}\right)\right]. & (2.57)\end{aligned}$$

We see that any simultaneous permutation of (x_i,Γ_i) return the same two combinations. The sum of their Fourier transforms instructs us to compute the same graph where the vertices and momenta are permuted, namely

$$T^{\Gamma_1\to\Gamma_2\Gamma_3}\left(q_1,q_2,q_3\right)=T^{\Gamma_1\Gamma_2\Gamma_3}\left(q_1,q_2,q_3\right)+T^{\Gamma_1\Gamma_3\Gamma_2}\left(q_1,q_3,q_2\right). \tag{2.58}$$

Any simultaneous permutations of (q_i,Γ_i) does the job, then, if nothing else interfere, this should in principle produces the same result. That is true for convergent expressions.

Let us go deeper in this point. Employing the formulas of the section 1.3, in examples 1.3.1 and 1.3.2, about reducing momenta constraints, and dropping the overall momenta conservation $\delta(\sum_{i=1}^{3}q_i)$, we get, by application of eq. (1.3.1), the following possibility

$$\mathcal{F}\left[T^{\Gamma_{123}}\left(x_1,x_2,x_3\right)\right]=\int_{\mathbb{R}^d}\mathrm{d}k\operatorname{tr}\left[\Gamma_1 S\left(k+q_1\right)\Gamma_2 S\left(k+q_1+q_2\right)\Gamma_3 S\left(k\right)\right]. \tag{2.59}$$

On the other hand, through (1.3.2) applied to the second graph follows another independent possibility

$$\mathcal{F}\left[T^{\Gamma_{132}}\left(x_1,x_3,x_2\right)\right]=\int_{\mathbb{R}^d}\mathrm{d}k\operatorname{tr}\left[\Gamma_1 S\left(k\right)\Gamma_3 S\left(k+q_2\right)\Gamma_2 S\left(k-q_1\right)\right]. \tag{2.60}$$

What is equivalent to reduce the momentum constraint in different order, and independently in each expression.

Motivated by this observation, that the arbitrary routing of amplitudes are also independent for distinct graphs differing by a permutation of the external data, i.e. vertices as a whole with its internal symmetry operators as well, we write the complete arbitrary expression for the amplitude

$$\begin{aligned} t^{\Gamma_1\to\Gamma_2\Gamma_3}\left(\left(k_i\right),\left(l_i\right)\right) &= t^{\Gamma_{123}}\left(k_1,k_2,k_3\right)+t^{\Gamma_{132}}\left(l_1,l_2,l_3\right), & (2.61)\\ T^{\Gamma_1\to\Gamma_2\Gamma_3}\left(\left(k_i\right),\left(l_i\right)\right) &= T^{\Gamma_{123}}\left(k_1,k_2,k_3\right)+T^{\Gamma_{132}}\left(l_1,l_2,l_3\right). & (2.62)\end{aligned}$$

For the direct channel we have

$$\left(q_1,q_2,q_3\right)=\left(p_{13},p_{21},p_{32}\right),\quad p_{ij}=k_i-k_j, \tag{2.63}$$

and for the crossed channel

$$(q_1, q_2, q_3) = (l_{13}, l_{32}, l_{21}) \,, \quad l_{i,j} := l_i - l_j . \tag{2.64}$$

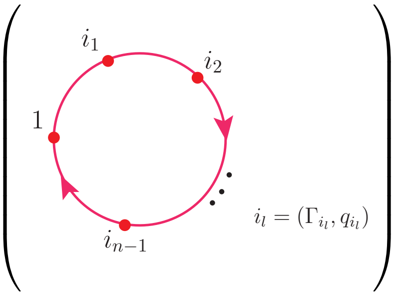
However, there is not a priori relation among arbitrariness of coordinates in one and in the other graph. If the graphs are strictly convergent this distinction is empty, since translation invariance sweeps out the very possibility of dependence on another variable but the kinematic data. Hence, we have introduced the property that translation invariance, or lack thereof is unrelated through the graphs that contribute to some correlator. Even though their differences are related, the independent summations $k_i + k_j$ and $l_i + l_j$ varies independently.

Let us pass to a wider discussion, for higher n-point functions, where this simple pair of graphs gets more aspects. For $n!$ connected cycle graphs, the n cyclic permutations of external vertices of a particular graph return the same expression due to the trace, hence, only $(n-1)!$ are independent. Moreover, the set can be grouped in $(n-1)!/2$ pairs whose only distinction is the orientation of the fermion line. Any permutation of some reference graph with one fixed vertex, say the first, will not be a cyclic one and it will have a pair. That pair come from considering the permutation

$$\sigma_r = \begin{pmatrix} 1 & i_1 & i_2 & \cdots & i_{n-1} \\ 1 & i_{n-1} & i_{n-2} & \cdots & i_1 \end{pmatrix} , \tag{2.65}$$

which reverses the order of the vertices, and is seen graphically as the original permutation, σ, but the graph has direction reversed from clockwise to anticlockwise direction. To see this more clearly, the amplitude corresponding to $T^{\Gamma}_{\mathrm{I}_n}$ with fixed first vertex has one permutation contributing as

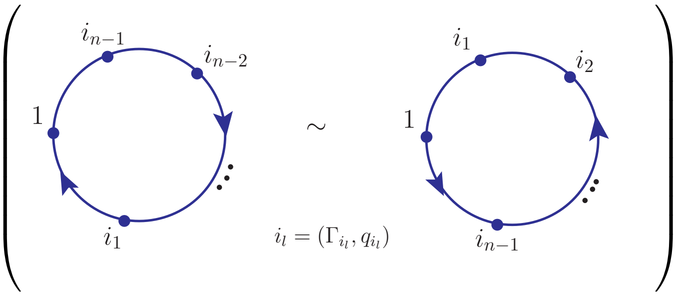
$$T^{\Gamma_1 \Gamma_{i_1} \ldots \Gamma_{i_{n-1}}}_{\mathrm{I}_1 \sigma(\mathrm{I}_{n-1})} (k_1, \cdots k_n) \quad = \tag{2.66}$$



$$\left(q_1, (q_{i_s})_{s=1}^{n-1}\right) \quad = \quad \left(p_{1,n}, (p_{j,j-1})_{j=2}^{n}\right) . \tag{2.67}$$

Whereas the reversed permutation is represented by the anticlockwise expression

$$T^{\Gamma_1 \Gamma_{i_{n-1}} \ldots \Gamma_{i_1}}_{\mathrm{I}_1 \sigma_r \sigma(\mathrm{I}_{n-1})} (l_1, \cdots l_n) \quad = \tag{2.68}$$



$$\left(q_1, (q_{i_s})_{s=n-1}^{1}\right) \quad = \quad \left(l_{1,n}, (l_{j,j-1})_{j=2}^{n}\right) . \tag{2.69}$$

These simple and standard considerations have non-trivial consequences in scenarios of non-negative power counting. For starting let us consider what happens if the amplitude is convergent. In that case we use the external momenta and the previous expression to write the following, simplified for $\sigma = \mathrm{id}$, expression

$$T_{\mathrm{I}_n}^{\Gamma_1\Gamma_2\ldots\Gamma_n}\left(q_1,\left(q_s\right)_{s=2}^{n}\right)+T_{\mathrm{I}_1\sigma_r\left(\mathrm{I}_{n-1}\right)}^{\Gamma_1\Gamma_{n-1}\ldots\Gamma_2}\left(q_1,\left(q_s\right)_{s=n}^{2}\right)=\left(1+\left(-1\right)^{\pi(\mathbf{\Gamma})}\right)T_{\mathrm{I}_n}^{\Gamma_1\Gamma_2\ldots\Gamma_n}. \tag{2.70}$$

We must now pause to make some considerations, and explain the $\pi\left(\mathbf{\Gamma}\right)$ written in the last equation. In even dimensions, modulo similarity transformation, there is only one irreducible representation of the Clifford Algebra, the Pauli theorem in four dimensions[4]. This implies specific connections among the vertexes, propagators, and graphs having a closed loop. Then the perturbative set of graphs contributing to a particular amplitude can be related and constrained. Translations of the internal momentum are essential here. That information in turn is codified in the set of surface terms, see section (3.3) for how we handle these objects.

Continuing, by defining new matrices through

$$\gamma_\mu^{\pm} := \pm\gamma_\mu^T \Rightarrow \{\gamma_\mu^{\pm},\gamma_\nu^{\pm}\} = \{\gamma_\mu,\gamma_\nu\}^T = 2g_{\mu\nu}\mathbf{1}, \tag{2.71}$$

it is seen that they satisfy the defining product of the Clifford algebra. Therefore, there are matrices connecting by similarity transformations the two representations with the starting one, namely

$$C_{\pm}\gamma_\mu C_{\pm}^{-1} = \pm\gamma_\mu^T. \tag{2.72}$$

Both $C_{\pm}$ exist for even dimensions, however only one exists for odd dimensions, C_{-} for $d = 4k+3$ and C_{+} for $d = 4k+1$, $k \in \mathbb{N}$. Focusing in the negative sign, the matrix is the charge conjugation matrix and we just call it C. In dimension even $d = 2n$ the two types of vertices, tensor and pseudo-tensor, behave under conjugation by the matrix C as

$$C\gamma_{[\mathrm{I}_l]}C^{-1} = \left(-1\right)^{l(l+1)/2}\gamma_{[\mathrm{I}_l]}^T; \quad \text{and} \quad C\gamma_{*[\mathrm{I}_l]}C^{-1} = \left(-1\right)^{l(l-1)/2+n}\left(\gamma_{*[\mathrm{I}_l]}\right)^T. \tag{2.73}$$

Hence, the charge parity of the vertices are

$$\pi\left(\Gamma_i\right) = \begin{cases} l\left(l+1\right)/2 & \text{if } \Gamma_i = \gamma_{[\mathrm{I}_l]} \\ l\left(l-1\right)/2+n & \text{if } \Gamma_i = \gamma_*\gamma_{[\mathrm{I}_l]} \end{cases} \tag{2.74}$$

For us they are written below

$$\begin{aligned} d &= 4 : C\left(S,P,V,A,T,\tilde{T}\right)C^{-1} = \left(S,P,-V,A,-T,-\tilde{T}\right), && (2.75) \\ d &= 6 : C\left(S,P,V,A,T,\tilde{T}\right)C^{-1} = \left(S,-P,-V,-A,-T,\tilde{T}\right). && (2.76) \end{aligned}$$

Returning to our considerations where we want to relate clockwise and anti-clockwise graphs. By simple operations, using eqs. (2.74) for the parities which for the propagator implies

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[4] See Günter Scharf, *Finite Quantum Electrodynamics: The Causal Approach*, 3rd ed, [64]. In section 1.3 for an interesting derivation.

$CS\left(K_i\right)C^{-1}=S^T\left(-K_i\right)$, we obtain a identity among integrands

$$\begin{aligned}\operatorname{tr}\prod_{i=1}^{n}\Gamma_i S\left(K_i\right) &= \operatorname{tr}\prod_{i=1}^{n}\left(C\Gamma_i C^{-1}\right)\left(CS\left(K_i\right)C^{-1}\right)=(-1)^{\pi(\mathbf{\Gamma})}\operatorname{tr}\prod_{i=n}^{1}S\left(-K_i\right)\Gamma_i. \quad (2.77)\\ \pi\left(\mathbf{\Gamma}\right) &: = \textstyle\sum_{i=1}^{n}\pi\left(\Gamma_i\right) \quad (2.78)\end{aligned}$$

This is the most we can reach only in integrand level. Now working out what would be for integrals. Until this point we have the pair

$$\begin{aligned} t^{\mathbf{\Gamma}}_{\circlearrowright} &= (-1)^{\pi(\mathbf{\Gamma})}\operatorname{tr}\prod_{i=n}^{1}S\left(-k-k_i\right)\Gamma_i, \quad t^{\mathbf{\Gamma}}_{\circlearrowleft}=\operatorname{tr}\prod_{i=n}^{1}S\left(k+l_i\right)\Gamma_i, \quad (2.79)\\ K_i &= k+k_i, \quad L_i=k+l_i. \quad (2.80)\end{aligned}$$

Then, inside the integral we reflect the integration variable $k\rightarrow-k$ and adopt the notation $K_i^*=k-k_i$. Furthermore, observe that in the reversed direction momenta conservation fixes $l_{i,i-1}=-q_i$, hence $l_{i,i-1}+k_{i,i-1}=0$. This easily can be seen as the relation

$$K_i^*=L_i-\Sigma_{i-1}, \quad \Sigma_{i-1}:=l_{i-1}+k_{i-1}. \quad (2.81)$$

Therefore, the integrated amplitudes acquire the form

$$T^{\mathbf{\Gamma}}_{\circlearrowright}=(-1)^{\pi(\mathbf{\Gamma})}\int_{\mathbb{R}^d}\mathrm{d}k\operatorname{tr}\left[\prod_{i=n}^{1}S\left(L_i-\Sigma_{i-1}\right)\Gamma_i\right], \quad T^{\mathbf{\Gamma}}_{\circlearrowleft}=\int_{\mathbb{R}^d}\mathrm{d}k\operatorname{tr}\left[\prod_{i=n}^{1}S\left(L_i\right)\Gamma_i\right]. \quad (2.82)$$

Now we come to the anticipated point. If, by hypothesis, one could transform the integration variable by $k\rightarrow k+\Sigma_{n-1}$ then, and only then, we have

$$T^{\mathbf{\Gamma}}_{\circlearrowright}=(-1)^{\pi(\mathbf{\Gamma})}T^{\mathbf{\Gamma}}_{\circlearrowleft}. \quad (2.83)$$

Observe that the relation among the graphs may change with non-abelian vertices. In abelian case and equal masses if $\pi\left(\mathbf{\Gamma}\right)$ is odd, then each pair of graphs should vanish that in turn implies the vanishing of the total amplitude. This is what should happen in all amplitudes of odd n-pt degree and just vector vertices in all dimensions, the Furry theorem. In other words, we must be able to transform the loop momenta to make this algebraic relation valid. A more subtle point is when $\pi\left(\mathbf{\Gamma}\right)$ is even in which case the result should double.

Returning to the Ward identities subject. After effecting a contraction with some $p^{\mu_i}_{i,i-1}$ and stated one IRagf as

$$q_i^{\mu_i}\left[T^{\mathbf{\Gamma}}_{\mathrm{I}}\right]=[T^{\mathbf{\Gamma}^{i-1}}_{\mathrm{I}^i}(\widehat{i-1})]-[T^{\mathbf{\Gamma}^i}_{\mathrm{I}^i}(\widehat{i})]+\left(m_i-(-m_{i-1})^{c_i}\right)[T^{\mathbf{\Gamma}_i}_{\mathrm{I}^i}], \quad (2.84)$$

then simplifying to equal masses and summing all contributions, we will get

$$q_i^{\mu_i}[T_{\mathrm{I}}^{\Gamma_1\rightarrow\Gamma_{2\cdots n}}]=\sum_{\sigma\in S_n/\mathbb{Z}_n}[\sigma(T^{\mathbf{\Gamma}^{i-1}}_{\mathrm{I}^i})-\sigma(T^{\mathbf{\Gamma}^i}_{\mathrm{I}^i})]+\left(1+(-1)^{c_i}\right)m[T^{\Gamma_1'\rightarrow\Gamma_{2\cdots n}}_{\mathrm{I}^i}]. \quad (2.85)$$

This should be a Ward identity!! The example in the next paragraph will guide the discussion better. I had to drop some arguments since expressing the aspect of having independent momenta coordinates for each graph is an unwieldy task to write. However, I must point that the differences in the r.h.s. do not cancel each other for higher than 3-pt functions. They get organized complete amplitudes of lower number of points.

The example used to finally connect both concepts (IRagfs and WIs) is a two-dimensional ($d = 2$) amplitude, single mass, but with enough features to make our statements clear. The $2D$-$AVVV$ box is finite as it is all its IRagfs. By the eqs. (2.74) and (2.78) follow $\pi\left(AV^3\right) = 1$, hence the total sum of its contributing graphs is non-zero and given by

$$\begin{aligned} T_{\mathrm{I}_4}^{A\to VVV}\left(q_2,q_3,q_4\right) &= 2\left[T_{\mathrm{I}_4}^{AVVV} + T_{\sigma_1(\mathrm{I}_4)}^{AVVV}\left(\sigma_1\left(q_i\right)\right) + T_{\sigma_2(\mathrm{I}_4)}^{AVVV}\left(\sigma_2\left(q_i\right)\right)\right] && (2.86)\\ \sigma_1\left(2,3,4\right) &= 3,2,4, \quad \sigma_2\left(2,3,4\right) = 2,4,2. && (2.87)\end{aligned}$$

Where the permutations are what is called the sums of the t and u channel for a 4-pt function, and we have chosen $\left(k_1,k_2,k_3,k_4\right) = \left(0,q_2,q_2+q_3,q_2+q_3+q_4\right)$ for the routings, since it is a finite tensor we are elaborating over. Notice the routings of the second and third graphs were taken such that their differences appear as simple permutations of the external momenta of the first one.

The contraction with $q_1^{\mu_1}$ reads

$$q_1^{\mu_1}T_{\mathrm{I}_4}^{A\to VVV} = 2[q_1^{\mu_1}T_{\mathrm{I}_4}^{AVVV}\left(q_2,q_3,q_4\right) + q_1^{\mu_1}T_{\mu_{1324}}^{AVVV}\left(q_3,q_2,q_4\right) + q_1^{\mu_1}T_{\mu_{1243}}^{AVVV}\left(q_2,q_4,q_3\right)]. \quad (2.88)$$

Applying the IRagfs for this negative power counting amplitude, it follows

$$\begin{aligned} q_1^{\mu_1}[T_{\mathrm{I}_4}^{AV^3}] &= [T_{\mu_{234}}^{AVV}\left(q_3,q_3+q_4\right)] - [T_{\mu_{423}}^{AVV}\left(q_2,q_2+q_3\right)] + 2m[T_{\mu_{234}}^{PV^3}\left(q_2,q_3,q_4\right)], && (2.89)\\ q_1^{\mu_1}[T_{\sigma_1\mathrm{I}_4}^{AV^3}] &= [T_{\mu_{324}}^{AVV}\left(q_2,q_2+q_4\right)] - [T_{\mu_{432}}^{AVV}\left(q_3,q_3+q_2\right)] + 2m[T_{\mu_{324}}^{PV^3}\left(q_3,q_2,q_4\right)], && (2.90)\\ q_1^{\mu_1}[T_{\sigma_2(\mathrm{I}_4)}^{AV^3}] &= [T_{\mu_{243}}^{AVV}\left(q_4,q_4+q_3\right)] - [T_{\mu_{324}}^{AVV}\left(q_2,q_2+q_4\right)] + 2m[T_{\mu_{243}}^{PV^3}\left(q_2,q_4,q_2\right)]. && (2.91)\end{aligned}$$

Summing them up, one pair will cancel without any action. The remaining can be written as

$$\begin{aligned} q_1^{\mu_1}T_{\mathrm{I}_4}^{A\to VVV} &= +2m\left[T_{\mathrm{I}_4}^{P\to VVV}\right] && (2.92)\\ &\quad +2\left[T_{\mu_{234}}^{AVV}\left(q_3,q_3+q_4\right) + T_{\mu_{243}}^{AVV}\left(q_4,q_4+q_3\right)\right]\\ &\quad -2\left[T_{\mu_{432}}^{AVV}\left(q_3,q_3+q_2\right) + T_{\mu_{423}}^{AVV}\left(q_2,q_2+q_3\right)\right].\end{aligned}$$

The two sums of 3-pt amplitudes are complete green functions or correlators of type $\left\langle\mathcal{J}^A\mathcal{J}^V J^V\right\rangle$, the sums of direct and crossed channel. Therefore, by the two-dimensional parities $\pi\left(A\right) = \pi\left(V\right) = -1$ (2.74), and the previous considerations applying, the result is zero. *Before this step we have a set of IRagfs and then their combinations, if everything holds, becomes a WI.* Explicitly

$$\begin{aligned} q_1^{\mu_1}T_{\mathrm{I}_4}^{A\to VVV} &= +2m\left[T_{\mathrm{I}_4}^{P\to VVV}\right] + 2\left[T_{\mu_{234}}^{A\to VV}\left(q_3,q_4\right)\right] - 2\left[T_{\mu_{432}}^{A\to VV}\left(q_3,q_2\right)\right] && (2.93)\\ &= 2m\left[T_{\mathrm{I}_4}^{P\to VVV}\right].\end{aligned}$$

The process repeats for the other vertices, from which we must get

$$q_i^{\mu_i}T_{\mathrm{I}_4}^{A\to VVV} = 0, \quad 2\leq i\leq 4. \quad (2.94)$$

The summary and rationale about the connection of algebraic relations, translational symmetry, and kinematical properties can be made as:

I All translations must be equivalent, no dependence in individual routings must be present.

II All algebraic relations must hold simultaneously to item **I**. Hitherto we see that all iRagfs must becomes IRagfs, i.e., integration linearity must be everywhere satisfied.

III Even if there is no individual routing dependence all algebraic relations must hold.

In even dimensions ($d = 2n$) the odd amplitudes of vertices $\mathbf{\Gamma} = (\Gamma_i)_{i=1}^{2n+1}$, with $\Gamma_i \in \{A, V\}$, can not satisfy these conditions and the reason is a *low energy theorem* (LET) which we shall elaborate in the next section. Such theorem requires quantification just over finite amplitudes related to correlators with scalar and pseudoscalar, and depends solely in the analytical structure of a tensor representing the type of amplitudes called anomalous.

2.3 Low-Energy Analytical Behavior

To entertain in the matter of kinematics and symmetry, where we will transform WIs together with an ancillary hypothesis in a predicate about kinematics. We must start by defining kinematic invariants as follows

$$s_{ij} := (q_i + q_j)^2 . \tag{2.95}$$

The range of indices I will establish in a moment. These Mandelstan variables are not all independent: their independent set varies with the relation between dimension and r-pt degree, with the type of external particles, and so on. In the case we are considering, just the r-pt degree in d dimensions, we have for the number invariants:

$$|\{s_{ij}\}| = \frac{r\,(r-1)}{2}, \quad r < d \quad \text{and} \quad |\{s_{ij}\}| = \frac{d\,(2r-d-1)}{2}, \quad r \geq d. \tag{2.96}$$

If $d > r$, then the number of independent invariants is just the number of $s_{ij} = s_{ji}$ discounting its symmetry. Whereas for $r \geq d$ linear dependence of $d + 1$, or more vectors in d dimensions must be taken into account. That has the result described above. This fairly common consideration appear in references where a selection of variables that reduces redundancy in the kinematic setup are sought for, e.g., spinor helicity variables [78, 75]. For another context-based reference see Davydychev and Delbourgo [93]. They show how to interpret 1-loop Feynman integrals as volumes of simplices formed by masses and the kinematic setup.

We shall focus rather precisely in one point of the kinematic data, the point where all invariants vanish, i.e.

$$s_{ij} = 0, \quad \forall i, j \in [1, n+1] = \{1, \cdots, n+1\}, \tag{2.97}$$

where the range will eventually corresponds to a $(n+1)$-pt amplitudes in $d = 2n$ dimensions. Here, the important aspect, for now, is the number of variables. Let it be the scalar functions of the kinematics

$$\left\{F_a\left(\{s_{ij}\}\right), G_{ba}\left(\{s_{ij}\}\right) : a \in [1,n]; \quad b \in [1, n+1]\right\}, \tag{2.98}$$

and the sequences of free μ indices and contracted indices ν given by

$$\begin{aligned} \mathrm{I}_{n+1} &= \mu_{1\cdots n+1} = \left(\mu_a\right)_{a=1}^{n+1}, \quad \mathrm{I}_n^i = \left(\mu_a\right)_{a=1,a\neq i}^{n+1}, & (2.99) \\ \mathrm{C}_n &= \nu_{2\cdots n+1} = \left(\nu_a\right)_{a=2}^{n+1}, \quad \mathrm{C}_{n-1}^i = \left(\nu_a\right)_{a=2}^{n+1}{}_{,a\neq i}. & (2.100) \end{aligned}$$

With their help we define the sequence of products of momenta q_i as

$$q^{\mathrm{C}_n} := \prod_{s=2}^{n+1} q_s^{\nu_s}, \quad q^{\mathrm{C}_{n-1}^i} := \prod_{s=2,s\neq i}^{n+1} q_s^{\nu_s} \Rightarrow q_i^{\nu_i} q^{\mathrm{C}_{n-1}^i} = q^{\mathrm{C}_n}. \tag{2.101}$$

The role of second definition is to omit the $i \geq 2$ index and momenta, basically, the structure $q^{\mathrm{C}_{n-1}^i}$ is an ascending subsequence of q^{C_n} excluding $q_i^{\nu_i}$. Observe that we are singling out $q_1 = -q_2 - \cdots - q_{n+1}$ as dependent vector.

Aiming the amplitudes of type $T_{\mathrm{I}_{n+1}}^{\mathbf{\Gamma}}$, with $\mathbf{\Gamma}$ containing a odd number of γ_*-matrix, we define the most ample form of a pseudo-tensor constituted by n vectors, $q_{2\leq i\leq n+1}$, such as

$$F_{\mathrm{I}_{n+1}} = \sum_{a=2}^{n+1} \left[\varepsilon_{\mathrm{I}_{n+1}\mathrm{C}_{n-1}^a} q^{\mathrm{C}_{n-1}^a}\right] \left[F_{n-a+1}\right] + \sum_{a=1}^{n+1}\sum_{b=2}^{n+1} \left[\varepsilon_{\mathrm{I}_n^a \mathrm{C}_n} q^{\mathrm{C}_n}\right] \left[q_{b,\mu_a} G_{ab}\right]. \tag{2.102}$$

Examples will appear after we deduce the general case.

Then, for the expression to be valid in each dimension, without adaptation, we separate the contractions with the variables q_i. First contracting $F_{\mathrm{I}_{n+1}}$ with $q_i^{\mu_i} (i = 2, \ldots n+1)$ and analyzing the first sum. We have the following

$$q_i^{\mu_i} \varepsilon_{\mathrm{I}_{n+1}\mathrm{C}_{n-1}^a} q^{\mathrm{C}_{n-1}^a} = \left(-1\right)^{n+1} \delta_i^a \varepsilon_{\mathrm{I}_n^i \nu_a \mathrm{C}_{n-1}^a} q_i^{\nu_a} q^{\mathrm{C}_{n-1}^a} = \left(-1\right)^{n+1} \delta_i^a \varepsilon_{\mathrm{I}_n^i \mathrm{C}_n} q^{\mathrm{C}_n}. \tag{2.103}$$

The result follows from the fact that to bring an i^{th} index to the right next position of μ_{n+1} one needs $n - i + 1$ permutations; moreover, if the vector q_i is equal to some vector in $q^{\mathrm{C}_{n-1}^a}$ the contraction in null. Therefore, it must be the case that $i = a$ for a non-zero result. Then, the now dummy index μ_i is written as ν_a and inserting it in the string C_{n-1}^a we have more $i - 2$ permutations to perform, hence the $(-1)^{n+1}$ factor.

For the double sum we have

$$q_i^{\mu_i} \varepsilon_{\mathrm{I}_n^a \mathrm{C}_n} q^{\mathrm{C}_n} q_{b,\mu_a} = \delta_a^i q_i^{\mu_a} \varepsilon_{\mathrm{I}_n^a \mathrm{C}_n} q^{\mathrm{C}_n} q_{b,\mu_a} = \delta_a^i \varepsilon_{\mathrm{I}_n^a \mathrm{C}_n} q^{\mathrm{C}_n} \left(q_i \cdot q_b\right). \tag{2.104}$$

The result follows because the q^{C_n} contains all vectors from q_2 to q_{n+1}; hence, if $i = a$, we are adding one more contraction and repeating one vector, then the result is null. What remains is a contraction with q_{b,μ_a}, i.e., the term $(q_i \cdot q_b)$.

Lumping everything together, one may express the contractions until now as

$$q_i^{\mu_i} F_{\mathrm{I}_{n+1}} = \varepsilon_{\mathrm{I}_n^i \mathrm{C}_n} q^{\mathrm{C}_n} \left\{ (-1)^{n+1} \left[F_{n-i+1}\right] + \sum_{j=2}^{n+1} \left(q_i \cdot q_j\right) \left[G_{ij}\right] \right\}, i \in [2, n+1]\,. \tag{2.105}$$

Now the contraction with $q_1^{\mu_1} = -q_2^{\mu_1} - \cdots - q_{n+1}^{\mu_1}$. For this one observe the following sequence of derivations

$$\begin{aligned} q_1^{\mu_1} \varepsilon_{\mathrm{I}_{n+1} \mathrm{C}^a_{n-1}} q^{\mathrm{C}^a_{n-1}} &= -\sum_{i=2}^{n+1} q_i^{\mu_1} \varepsilon_{\mathrm{I}_{n+1} \mathrm{C}^a_{n-1}} q^{\mathrm{C}^a_{n-1}} \\ &= -\sum_{i=2}^{n+1} (-1)^n \varepsilon_{\mathrm{I}_n^1 \nu_i \mathrm{C}^a_{n-1}} q_i^{\nu_i} q^{\mathrm{C}^a_{n-1}} \\ &= -\sum_{i=2}^{n+1} (-1)^{n+i-2} \delta_i^a \varepsilon_{\mathrm{I}_n^1 \mathrm{C}_n} q^{\mathrm{C}_n} = (-1)^{n+a-1} \varepsilon_{\mathrm{I}_n^1 \mathrm{C}_n} q^{\mathrm{C}_n}\,. \end{aligned} \tag{2.106}$$

In the derivation, the second line comes from observing that the index contracted is in the same position for all the momenta. Then n permutations are necessary to bring it to the end of the string of free indices. The third line comes from the permutations needed to insert the new dummy index in its ascending position inside C^a_{n-1}. The delta δ_i^a arises because if $i \neq a$, then $q_{2 \leq i \leq n}$ is present in the contracted-indices string $q^{\mathrm{C}^a_{n-1}}$, therefore the result is null. In the opposite case the vector gets inserted into q^{C_n}.

The next step is to show that

$$+q_1^{\mu_1} \varepsilon_{\mathrm{I}_n^a \mathrm{C}_n} q^{\mathrm{C}_n} q_{b,\mu_a} = \delta_1^a \varepsilon_{\mathrm{I}_n^a \mathrm{C}_n} q^{\mathrm{C}_n} \left(q_1 \cdot q_b\right).$$

This happens since if $a \neq 1$ we are contracting with the ε-indices and have zero because q^{C_n} contain all the vectors and $q_1^{\mu_1} = -q_2^{\mu_1} - \cdots - q_{n+1}^{\mu_1}$. We could have expanded everything in terms of what we have chosen to be the independent set, but as we shall see, for our objectives, this is enough.

Gathering these facts, we have for contractions with $q_i^{\mu_i}$ the following structure

$$q_1^{\mu_1} F_{\mathrm{I}_{n+1}} = \varepsilon_{\mathrm{I}_n^1 \mathrm{C}_n} q^{\mathrm{C}_n} \left\{ \sum_{a=2}^{n+1} \left[(-1)^{n+a+1} F_{n-a+1}\right] + \sum_{b=2}^{n+1} \left(q_1 \cdot q_b\right) \left[G_{1,b}\right] \right\}, \tag{2.107}$$

$$q_i^{\mu_i} F_{\mathrm{I}_{n+1}} = \varepsilon_{\mathrm{I}_n^i \mathrm{C}_n} q^{\mathrm{C}_n} \left\{ (-1)^{n+1} \left[F_{n-i+1}\right] + \sum_{j=2}^{n+1} \left(q_i \cdot q_j\right) \left[G_{ij}\right] \right\}, \quad i \in [2, n+1] \tag{2.108}$$

Motivated by this we offer a definition that will be used to analyze the interrelation of kinematics and symmetry.

Definition 2.3.1 *In $d = 2n$ dimensions, consider a pseudo-tensor of rank $n+1$ formed by n linearly independent vectors from a set $\left\{q_i \in \mathbb{M}^d : i \in [1, n+1]\right\}$, which satisfy $\sum_{j=1}^{n+1} q_j = 0$, its Lorentz indices coded in $\mathrm{I}_{n+1} = \mu_1 \mu_2 \cdots \mu_{n+1}$, and the product $q^{\mathrm{C}_n} = \prod_{s=2}^{n+1} q_s^{\nu_s}$. With these ingredients we identify, by definition, the invariant functions $V_i^L\left(\{s_{ab}\}\right)$*

$$q_i^{\mu_i} F_{\mathrm{I}_{n+1}} =: \varepsilon_{\mathrm{I}_n^i \mathrm{C}_n} q^{\mathrm{C}_n} V_i^L\left(\{s_{ab}\}\right), \quad \forall i \in [1, n+1]\,. \tag{2.109}$$

They are a way to store the informations characterizing the tensor. The superscript L in V_i^L is a tag for the scalar coming from the contractions of a general tensor with its external momenta which we commonly put in the left hand side of our equations. Nothing more is said.

Now, for the most general form assumed in the beginning of the deduction, we have

$$V_1^L = (-1)^{n+1}\sum_{a=2}^{n+1}(-1)^a\left[F_{n-a+1}\right]+\sum_{j=2}^{n+1}(q_1\cdot q_j)\left[G_{1,j}\right] \tag{2.110}$$

$$V_i^L = (-1)^{n+1}\left[F_{n-i+1}\right]+\sum_{j=2}^{n+1}(q_i\cdot q_j)\left[G_{ij}\right]. \tag{2.111}$$

Note the alternation of signs, $(-1)^a$, in the first term of V_1^L. Then, one may linearly combine the functions above to fully eliminate the form factors F_i. Just multiplying by $(-1)^{i+1}V_i^L$, and summing over $i\in[1,n+1]$. Explicitly

$$\sum_{i=1}^{n+1}(-1)^{i+1}V_i^L=\sum_{j=2}^{n+1}(q_1\cdot q_j)\left[G_{1,j}\right]+\sum_{i=2}^{n+1}\sum_{j=2}^{n+1}(-1)^{i+1}(q_i\cdot q_j)\left[G_{ij}\right], \tag{2.112}$$

which can be further reduced to

$$\sum_{i=1}^{n+1}(-1)^{i+1}V_i^L=-\sum_{i=2}^{n+1}\sum_{j=2}^{n+1}(q_i\cdot q_j)\left[G_{1,i}+(-1)^i G_{ij}\right]. \tag{2.113}$$

The only place where we will use the environment of theorem is now, because the proposition can be stated reasonably precisely and if all hypotheses hold it is independent of any well or ill definiteness question or even if we are treating a problem in QFT.

Theorem 2.3.2 (Low energy theorem) *Consider a Lorentz pseudotensor of rank $n+1$ in dimension $d=2n$, function of* ***only*** *$|\{q_i\}|=n$ independent vector variables, as described for $F_{\mathrm{I}_{n+1}}$. In its most general form it has a set of $n(n+2)$ Lorentz invariant functions, $\{F_a(\{s_{ij}\}),G_{ba}(\{s_{ij}\})\}$. If those functions are analytic in a neighborhood of the point $s_{ij}=(q_i+q_j)^2=0,\forall i,j\in[1,n+1]$, or at least bounded there, then the combinations of the definition (2.3.1) satisfies*

$$\sum_{i=1}^{n+1}(-1)^{i+1}V_i^L(0)=0. \tag{2.114}$$

In other words, the structure of the tensor, and a very specific hypothesis, implies in the vanishing of the combination. Because the point is in the momentum space, we call it Low Energy Theorem (LET).

We will not furnish a formal proof, since it is very simple statement which require just to observe that in the combinations we have created, without assumption of symmetry, even bose symmetry, there appear sums of the type

$$(q_i\cdot q_j)\left[G_{1,i}+(-1)^i G_{ij}\right]. \tag{2.115}$$

Therefore, if the G's are bounded there, the zero point, the limit vanishes. In the real world, there are sets of parameters for which this does not hold, all massless fermion amplitudes do have poles. However, the residue of the pole can be understood from the massive situation. We will not delve into that question anyway.

Now, returning to perturbative amplitudes. In the moment one identifies a general tensor F_{I} carrying Lorentz indexes it is possible to study or anticipate, under some set of hypothesis, properties of any computation of that tensor. We select the particular class corresponding to the first nontrivial pseudo-tensor that are named anomalous amplitudes. In that case we will produce

$$q_i^{\mu_i} T^{\mathbf{\Gamma}}_{\mathrm{I}_{n+1}} = \varepsilon_{\mathrm{I}^i_n \mathrm{C}_n} q^{\mathrm{C}_n} \left[V_i^L \right], \tag{2.116}$$

and does not matter if arbitrary masses are involved, in zero the combination vanish. Note that the combination 2.114 interrelates all the vertices.

The point is that if those amplitudes of $(n+1)$-pt coming from the right hand side[5] of WI or IRafgs, which are finite and can be written in the form

$$\left(m_i - (-1)^{c_i} m_{i-1} \right) T^{\mathbf{\Gamma}_i}_{\mathrm{I}^i_n} = \varepsilon_{\mathrm{I}^i_n \mathrm{C}_n} q^{\mathrm{C}_n} V_i^R \left(\{ s_{i,j} \} \right), \tag{2.117}$$

can be unrestricted linked to the set $\left\{ q_i^{\mu_i} T^{\mathbf{\Gamma}}_{\mathrm{I}_{n+1}} : i \in [1, n+1] \right\}$, then they must satisfy

$$\sum_{i=1}^{n+1} (-1)^{i+1} V_i^R (0) = 0 = \sum_{i=1}^{n+1} (-1)^{i+1} V_i^L (0). \tag{2.118}$$

Then without much thinking one may just go after them, which do not suffer from technical definitions, and test the property. After realizing they do not have the property can one settle for a anomaly? Only if the left hand side functions V_i^L do not have poles. (Remember that $c_i = 1$ if $\Gamma_i = A$ or $c_i = 0$ if $\Gamma_i = V$ and $\mathbf{\Gamma}_i$ represents the vertex content associated to the contraction with $q_i^{\mu_i}$, i.e., containing one scalar or pseudo-scalar density)

In other words, if $s_{ij} = 0$ is not a pole of $\left(V_1^L, \cdots, V_{n+1}^L \right)$ neither a zero of $\left(V_1^R, \cdots, V_{n+1}^R \right)$, then we cannot have the set $\left\{ q_i^{\mu_i} T^{\mathbf{\Gamma}}_{\mathrm{I}_{n+1}} : i \in [1, n+1] \right\}$ identical with the set $\left\{ \left(m_i - (-1)^{c_i} m_{i-1} \right) T^{\mathbf{\Gamma}_i}_{\mathrm{I}^i_n} : i \in [1, n+1] \right\}$. This observation is interesting since the second set is outside the realm of indeterminacy or ill-definiteness typical of perturbative definitions. We can analyze it without ever thinking something problematic exists. The program we endeavour to study is the relations the equations below have with the symmetry and algebraic features of anomalous amplitudes

$$\begin{aligned} \text{General tensor} \quad &: \quad \left(V_1^L, \cdots, V_{n+1}^L \right) \rightarrow \sum_{i=1}^{n+1} (-1)^i V_i^L (0) = 0 \\ \text{Known r.h.s.} \quad &: \quad \left(V_1^R, \cdots, V_{n+1}^R \right) \rightarrow \sum_{i=1}^{n+1} (-1)^i V_i^R (0) = ? \end{aligned}$$

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[5] An example in $6D$

$$\begin{aligned} q_1^{\mu_1} T^{VAAA}_{\mathrm{I}_4} \quad &\rightarrow \quad \left(m_1 - m_4 \right) T^{SAAA}_{\mathrm{I}^1_3} = \varepsilon_{\mathrm{I}^1_3 \mathrm{C}_3} q^{\mathrm{C}_n} V_1^{R,SAAA} \\ V_1^{R;SAAA} \quad &= \quad \left(m_1 - m_4 \right) \left[+ \left(m_4 + m_1 \right) Z^{(-1)}_{001} + \left(m_1 - m_3 \right) Z^{(-1)}_{010} + \left(m_2 + m_1 \right) Z^{(-1)}_{100} - m_1 Z^{(-1)}_{000} \right] \end{aligned}$$

The interrogation stands for what we will set in the course of the dimension specific chapters.

Chapter 3

Handling Divergent and the Finite Integrals

This chapter will organize a series of notations and considerations about conceptual and practical uses of the definitional recipes we employ in the work. Before presenting the strategy to define non-negative power counting amplitudes, let us digress into the divergent-integrals issue in QFT. It is well-known that products of propagators (that are not regular distributions) are ill-defined in general. A good example is the following equation

$$\int \frac{\mathrm{d}^4 k}{(2\pi)^4} \mathrm{tr}[S_F\,(k)\,S_F\,(k-p)] = \int \mathrm{d}^4 x \mathrm{tr}[\hat{S}_F\,(x)\,\hat{S}_F\,(-x)]\mathrm{e}^{ip\cdot x}. \tag{3.1}$$

The l.h.s. displays a divergent integral that should represent (in generalized-function notation) the convolution of two Feynman propagators in momentum space. As for the r.h.s., it is the Fourier transform of a product of propagators in position space. Both sides do not define distributions because, when the point-wise product of distributions does not exist, the convolution product of their Fourier transform either.

These short-distance UV singularities manifest in divergences of loop momentum integrals. Their origins trace back to multiplications of distributions by discontinuous step function in the chronological ordering of operators in the interaction picture. That leads, through the Wick theorem, to Feynman rules; see G. Scharf [64, 65], originally in Epstein and Glaser [66]. Although the undefined Feynman diagrams can be circumvented by carefully studying the splitting of distributions with causal support in the setting of causal perturbation theory [67, 68] (where no divergent integral appears at all), we work with Feynman rules in the context of regularizations.

We use the systematic procedure known as Implicit Regularization (IReg) to handle divergences. Its development dates back to the late 1990s in the Ph.D. thesis of O. A. Battistel [47]. To do this we have two first sections with concepts and definitions and then the third and fourth with tools which allow to express any 1-loop amplitude. Without changing their bare forms.

3.1 Counting Powers: Pure and Simple

The Power counting theorem of Weinberg [84] states that: if the superficial degree of divergence is negative and all sub-integrals also are superficially convergent, then the integral as whole is absolutely convergent. The theorem, clearly, is more complex than this lightning statement. We shall only make some simple definitions that serve the purpose of illustrate the reasoning we employ.

First, if the limit

$$\lim_{t\to\infty} \frac{t^d \left[\bar{j}\left(tz\right)\right]}{t^{\omega(j)}} = \bar{j}\left(z\right) \quad t \in \mathbb{R}_{>0} \tag{3.2}$$

is finite for fixed z, where z denote all variables: integration momentum, routings and masses. Then we call $\omega\left(j\right)$ the power counting. Note we undressed all indices and n-pt integrand. The theorem grants, in our 1-loop context, that if the power counting is negative the integral defining the correction is absolutely convergent, whereas if zero or positive the integral representation to the correction does not hold. Now an example.

Example 3.1.1 *In the general case we have*

$$\bar{j}_n^{\mu_1\cdots\mu_m}\left(z\right) = \frac{K_{\mathbf{m}}^{\mathrm{I}_m}}{\prod_{i=1}^{n} D_i} \rightarrow t^d \bar{j}_n^{\mathrm{I}_m}\left(tz\right) = t^{d+m-2n} \bar{j}_n^{\mathrm{I}_m}\left(tz\right), \tag{3.3}$$

what implies

$$\frac{t^d \bar{j}_n^{\mathrm{I}_m}\left(zx\right)}{t^{\omega(j)}} = \bar{j}_n^{\mathrm{I}_m}\left(tz\right) \Rightarrow \omega\left(\bar{j}_n^{\mathrm{I}_m}\right) = d + m - 2n. \tag{3.4}$$

Simple and pure power counting. However, one must be aware that the protocol overestimates the divergent character in some circumstances, mainly in the amplitudes because of antisymmetry in the indices engendered by the ε-tensor. But its value is point to places where care must be exercised. Another very particular examples.

Example 3.1.2 *The tadpole integrand* $\bar{j}_1\left(K_i, m_i\right) = \bar{j}_1\left(i\right) = 1/D_i$*:*

$$\frac{t^d}{t^{\omega(\bar{j}_1)}}\bar{j}_1\left(t\left(i\right)\right) = \frac{t^d t^{-\omega(\bar{j}_1)}}{D_i\left(t\right)} = \frac{t^{d-\omega(\bar{j}_1)}}{\left(tk + tk_i\right)^2 - \left(tm_i\right)^2} = t^{d-2-\omega(\bar{j}_1)}\bar{j}_1\left(i\right) \tag{3.5}$$

for a non-zero finite limit we need $\omega\left(\bar{j}_1\right) = d - 2$*. One more example, this time a* 2^{nd}*-rank tensor and* 2*-pt integrand* $\bar{j}_2^{\mathrm{I}_2}\left(i_1, i_2\right)$

$$\frac{t^d \bar{j}_2^{\mathrm{I}_2}}{t^\omega} = t^d t^{-\omega}\frac{t^2 K_1^{\mu_{12}}}{t^4 D_{i_1 i_2}} = t^{d-\omega-2}\bar{j}_2^{\mathrm{I}_2} \tag{3.6}$$

which implies $\omega\left(\bar{j}_2^{\mathrm{I}_2}\right) = d - 2$*.*

Within our particular definition we see the $\omega\left(\bar{J}_n^{\mathrm{I}_m}\right)$ also measures the mass dimension of the corresponding integral, which we write now with the integral as argument, since this is the target of the definition. This feature will cease to happen, uniformly, for identities satisfied by the integrands, which are bout to be explored by us. Basically the integrand behaves asymptotically as $\bar{j}_n^{\mathrm{I}_m} \sim \mathcal{O}\left(k^{\mathrm{I}_m}\left(k^2\right)^{-n}\right)$ for $k_\mu \rightarrow \infty$, then in d dimensions follows $\omega = d + m - 2m$. This is the proper way to talk about power counting. With this ideas in hand let us go to the subject of how we analyze the computation of amplitudes.

3.2 Routings and Implicit Regularization

We have noticed that routing coordinates can be interpreted as dual or region momenta. Indeed, for any planar diagram one can express the momentum carried by each line as the difference of the momenta between two regions of the plane separated by it, as mentioned in section (1.3). Let us choose a general n-pt integral $\bar{J}_n^{\mathrm{I}}$ or n-pt amplitude $T_{\mathrm{I}}^{\boldsymbol{\Gamma}}$, to make our considerations about momenta.

For the non kinematic-interpretable variable one may choose an arbitrary linear combination as

$$P_{\boldsymbol{\alpha}} = \sum\nolimits_{i=1}^{n} \alpha_i k_i; \quad |\boldsymbol{\alpha}| = \sum\nolimits_{i=1}^{n} \alpha_i \neq 0, \tag{3.7}$$

and as long as this condition is respected we can pick up quite arbitrarily the variable which violates Poincaré symmetry in momentum space, advancing that it is what it does. It is of responsibility of what comes thereafter to remove or not $P_{\boldsymbol{\alpha}}$. This definition is wider than the one we selected in definition (1.3.3):

$$P_{a_1 \cdots a_n} = P_A = \sum_{i \in A} k_i; \quad A \subseteq [1, n]\,. \tag{3.8}$$

However, it is no more general. First one question, maybe naive one but definitely useful: would it be possible to invert the relation $q_i = \phi_i\left(k_1, \cdots k_n\right)$, i.e. write $k_i = \phi_i^{-1}\left(q_1, \cdots, q_n\right)$, with no further artifice? Looking into the relation in matrix form:

$$\begin{pmatrix} q_1 \\ q_2 \\ \vdots \\ q_n \end{pmatrix} = \begin{pmatrix} 1 & 0 & \dots & -1 \\ -1 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & -1 & 1 \end{pmatrix} \begin{pmatrix} k_1 \\ k_2 \\ \vdots \\ k_n \end{pmatrix} \tag{3.9}$$

$$\left(q_i\right)_{i=1}^{n} = \phi\left(\left(k_i\right)_{i=1}^{n}\right), \tag{3.10}$$

we see that $\det \phi = 0$; therefore, the set of external momenta must be augmented with one more linearly independent variable, necessarily related to individual routings or a summation of them. Because of hardwired total momenta conservation we have a constraint which prevents invertibility, the constraint is the linear dependence of the set $\{q_i : i \in [1, n]\}$. The definition P_{ij} (1.3.3) do the job, as does $P_{\boldsymbol{\alpha}}$. The variable P_A is just $P_{\boldsymbol{\alpha}}$ with $\alpha_i = 1$ for $i \in A$ and $\alpha_i = 0$ if $i \notin A$. Therefore, as long as $\sum_{i=1}^{n} \alpha_i \neq 0$ the transformation that follows is invertible

$$\left(\sum\nolimits_{i=1}^{n} \alpha_i k_i, \left(p_{i,i-1}\right)_{i=2}^{n}\right) = \left(P_{\boldsymbol{\alpha}}, \left(q_i\right)_{i=2}^{n}\right) = \phi\left(k_1, \ldots, k_n\right). \tag{3.11}$$

Formally q_i are relative coordinates and $P_{\boldsymbol{\alpha}}$ is the center of mass coordinate or barycentric coordinate. The inversion is harder, but simple in a sense, it is given by

$$k_{1 \leq l \leq n} = \phi_l^{-1}\left(P_{\boldsymbol{\alpha}}, \left(q_i\right)_{i=2}^{n}\right) = \frac{1}{|\boldsymbol{\alpha}|}\left\{P_{\boldsymbol{\alpha}} - \sum_{j=l+1}^{n} \delta_{l,n}^{*}\left(\sum_{i=j}^{n} \alpha_i\right) q_j + \sum_{j=2}^{l} \delta_{1,l}^{*}\left(\sum_{i=1}^{j-1} \alpha_i\right) q_j\right\} \tag{3.12}$$

where $\delta_{ij}^{*} = 1 - \delta_{ij}$, and δ_{ij} is the Kronecker delta.

Being impossible to write dual variables as momenta variables, *the way to free the expressions from sums of those variables is through transformation of the loop momentum.* By mere coordinate changes we can only write

$$\bar{\jmath}_n^{\mathrm{I}_m}\left(k, k_1, \ldots, k_n\right) = \bar{\jmath}_n^{\mathrm{I}_m}\left(k, \phi^{-1}\left(P_{\boldsymbol{\alpha}}, q_i\right)\right)$$

and then

$$\bar{J}_n^{\mathrm{I}_m}\left(P_{\boldsymbol{\alpha}}, q_i\right) = \int_{\mathbb{R}^d} \mathrm{d}k \left[\bar{\jmath}_n^{\mathrm{I}_m}\left(k, \phi^{-1}\left(P_{\boldsymbol{\alpha}}, q_i\right)\right)\right].$$

Imposing translational invariance in position space makes amplitudes supported on the zero-sum of all momenta, and this is a direct reflection of Poincaré symmetry in position space. The constraint is automatically present in dual coordinates, however, the inversion of coordinates from dual to center of mass and relative coordinates leaves room for the appearance of the center of mass coordinate.

In distribution products where the Fourier transform is a defined operation, the role of center of mass or any individual dual variable is swept out through a change of variables, i.e. by translation in the loop momentum. In formal integrals, which maybe thought of as integrals of non-integrable rational functions, the idea of transmuting them into parametric functions, which are unbounded in the region corresponding to the original formal object, introduces in some strategies a dependence on the center of mass coordinate.

Albeit polynomially dependent, the dependence of the center of mass coordinate implicates ambiguous and consequential results if one requires that only variables employed in the Fourier transform must appear in the result. It is largely interpreted due to the presence of surface terms, but such terms are seen from inside manipulations, they are residues of some limit or symmetric integration attitude.

One of the features of the strategy we shall adopt is to promptly rewrite any rational-function integrand in two sets. In one of these sets we have only integrable functions, and the routing dependence is rational in the same vein as generalized Feynman integrals (i.e. with propagator powers). The other set contains non-integrable rational functions whose routing dependence is solely polynomial. What is consistent with the type of indeterminacy observed in general, and required by renormalization, in effective theories also.

This decomposition into two sets is universally possible. The idea, in schematic form reads:

$$r = r_1 + r_2; \qquad R_2 = \int \frac{\mathrm{d}^d k}{c\left(d\right)} r_2 < \infty; \qquad \partial_i^{\mathrm{I}_l} r_1 = 0. \tag{3.13}$$

Where $\partial_i^{\mathrm{I}_l} r_1$ denotes the derivative of order $l \in \mathbb{N}$,

$$\partial_i^{\mathrm{I}} r_1 = \frac{\partial^l r_1}{\partial k_i^{\mu_1} \cdots \partial k_i^{\mu_1}},$$

w.r.t. (with respect to) to all routings that $r\left(k_1, k_2 \cdots\right)$ may depend upon. But how to realize the idea? We need an algorithm. This main idea is realized in a series of observations starting with the possible and simple decomposition

$$D_i = D_\lambda + A_i. \tag{3.14}$$

Being the mass and the routing localized in the second polynomial, linear in the future integration variable, as the first is quadratic in that variable. Explicitly,

$$\begin{aligned} D_i &= (k+k_i)^2 - m_i^2, & (3.15)\\ D_\lambda &= \left(k^2-\lambda^2\right), & (3.16)\\ A_i &= 2k\cdot k_i + \left(k_i^2+\lambda^2-m_i^2\right). & (3.17)\end{aligned}$$

Then one can take the reciprocal of D_i and factor D_λ as

$$\frac{1}{D_i} = \left(\frac{1}{D_\lambda}\right)\frac{1}{\left[1-(-A_i/D_\lambda)\right]}. \tag{3.18}$$

In this point we may remember that the finite geometric series, or the partial sum of the geometric series, is always well defined. Thus, we can organize the expression in the form

$$\frac{1}{(x-y)} = \sum_{n=0}^{m}\frac{y^n}{x^{n+1}} + \frac{y^{m+1}}{x^{m+1}\,(x-y)}, \quad m \in \mathbb{N}. \tag{3.19}$$

With the simple identification $(x, y) = (D_\lambda, A_i)$, we have for any natural m an identity. Continuing, as a necessary task, the proper integrals identified in the course of our procedure are performed by standard methods. For us, transforming the momentum into a parametric representation, specifically, a Feynman-like parametrization. Before reaching that point, let us cast the identity in the form which is going to be used by us:

$$\frac{1}{D_i} = \sum_{r=0}^{n}(-1)^r\frac{A_i^r}{D_\lambda^{r+1}} + (-1)^{n+1}\frac{A_i^{n+1}}{D_\lambda^{n+1}D_i}, \quad n \in \mathbb{N}. \tag{3.20}$$

Fully developed, for the sake of clarity and better grasping of the mechanism, we have

$$\begin{aligned}\frac{1}{\left(K_i^2-m_i^2\right)} &= \sum_{r=0}^{n}\frac{(-1)^r\left[2k\cdot k_i+\left(k_i^2+\lambda^2-m_i^2\right)\right]^r}{\left(k^2-\lambda^2\right)^{r+1}} \\ &+\frac{(-1)^{n+1}\left[2k\cdot k_i+\left(k_i^2+\lambda^2-m_i^2\right)\right]^{n+1}}{\left(k^2-\lambda^2\right)^{n+1}\left(K_i^2-m_i^2\right)}.\end{aligned} \tag{3.21}$$

Remembering the power counting: We have introduced a definition, in the previous section, where the power counting basically equates with the mass dimension of the integral. That way of thinking is perfectly fine; however, the conclusion holds due to routings and masses entering—with the same weight— in that power-counting definition. If one have a quadratic term in k, one have a quadratic term in k_i and so on.

They turn up with the same degree in the rational-function integrand. Now, the identity above changes this feature because A_i is linear in k, the integration variable, but quadratic in the other parameters. For each term we have the same mass dimension but different power counting. This decoupling of power counting and mass dimension by means of an identity has very rich consequences.

First, by applying the identity in all propagators of some integral or amplitude we position the routings that remain in the denominators in superficially and absolutely integrable expressions. Second, the remaining part is a sequence of rational functions which are strictly polynomials in the momentum coordinates. Third, the expression is always identical to the starting point, hence it can be understood as an organization for whatever action the user may have in mind. Before proceeding to computations themselves lets make some more remarks.

This set of actions which constitutes the approach do not fall far of traditional perspectives, since the objects under investigation are naturally undefined in their original form. The difference resides in how the tensorial and kinematical dependence are handled. The strategy of analysis observes that only localized portions of the integrands are non integrable and then proceeds to explicitly exhibit those portions before additional actions. Therefore, kinematic and non-kinematic sectors can be individually investigated and their separate consequences understood. By kinematic sector we mean the part which is absolutely convergent and that will carry the information contained in the external data-momenta and physical masses.

One does not need to mix them by regularization. The will-be finite or kinematic sector that we isolate corresponds, modulo a polynomial of finite coefficients, to the constant term in a regularization parameter; if regularizing the integrals and seeking for finite parts is the attitude chosen. The finite-polynomial part is not unique in regularization, the question of it coming from a finite integral or in the expansion of a divergent integral is open[1]. On the contrary, we will always know the origin of our terms. Since the variables k_i are entrenched inside a definition of a divergent integral, the determinations of a particular set of amplitudes require a fine work to be exhibited.

In summary, we are looking to exhibit the analytic structure of Feynman integrals without introducing arbitrary interventions—or keeping this step on hold—which combines the lack of a traditional definition of the original object with what one knows and what one needs to know. One may still consider regularizing only and exactly the divergent set, if that is the goal. That set can always be determined without loosing touch with the initial rational function. Rather than modifying it too early, one may adopt the attitude of applying minimal changes later on.

It is not our aim to investigate the methodology per se, but instead, only to give a glimpse of our interests and point of view. The caveats and reasoning which we are trying to spot, within the realm of possibility (as much as possible), are not present in the literature of the method and perhaps may be useful for more clear communication, mainly with students and newcomers (those new to the idea). At least this is the sincere position of the thesis's author.

To close this section, I want to retrieve the point of the phrase "lack of a traditional definition

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[1] It does mean one can not get definite answers in particular cases.

of the original object". Some literature use of the term implicit regularization is a interpretation of the definitions and protocols proposed by O. A. Battistel in his Ph.d. [47] and its first publications [69, 70]. Here, we are also having another interpretation of the idea, similar to his own. We are dropping the position of being before an integral which diverges, or a divergent object. Hence, it is not a matter of $\infty - \infty$ type of manipulations. However, the interpretation is conservative, i.e. it is compatible with regularized integrals and some statements are borrowed from that interpretation. *Linearity of the functional that transforms rational functions into special functions, which is the meaning of Feynman integrals, is the ultimate and fundamental working hypothesis we are employing in all the thesis*[2].

A typical example we shall meet in the context of the method is a decomposition like

$$\bar{j}_2 = \frac{1}{D_{12}} = \frac{1}{D_\lambda^2} - \frac{1}{D_\lambda^2 D_1} - \frac{1}{D_\lambda D_{12}}, \tag{3.22}$$

now if the operation we apply to it were an integral then both sides, after "integration", are undefined. They do not converge absolutely, hence, even in the case of conditional convergence one must make a convention about how the integral is done. However, one could observe that a derived identity is convergent,

$$\bar{j}_2 - \frac{1}{D_\lambda^2} = -\frac{1}{D_\lambda^2 D_1} - \frac{1}{D_\lambda D_{12}}. \tag{3.23}$$

Nonetheless trying to separate the symbols of integration in the l.h.s., using the argument of linearity, to write

$$\bar{J}_2 - \int \frac{\mathrm{d}^4 k}{c\,(4)} \frac{1}{D_\lambda^2} = -\int \frac{\mathrm{d}^4 k}{c\,(4)} \frac{1}{D_\lambda^2 D_1} - \int \frac{\mathrm{d}^4 k}{c\,(4)} \frac{1}{D_\lambda D_{12}}, \tag{3.24}$$

you lose the property of linearity as a theorem when all integrals are defined. What you have now is the hypothesis or axiom of linearity to your functional. Implicitly and sometimes very explicitly, this assumption is made in the context of regularizations, and in the work of O. A. Battistel.

We will continue to use the symbol of integration with the meaning of a functional which distribute linearly over a sum of terms.

$$\bar{J}_2 = \mathsf{Fy}\left[\bar{j}_2\right] = \mathsf{Fy}\left[\frac{1}{D_\lambda^2}\right] - \mathsf{Fy}\left[\frac{1}{D_\lambda^2 D_1}\right] - \mathsf{Fy}\left[\frac{1}{D_\lambda^2 D_{12}}\right] \tag{3.25}$$

But! It can only quantifies in one way: If the decomposition constructed using the identity (3.20) reaches the form

$$\bar{j}_n^{\mathrm{I}_m} = r_1 + r_2; \quad r_1 = \sum_{i \in I} \frac{p_i^{\mathrm{I}_m}\left(k, k_1, \ldots, k_n\right)}{q\left(k\right)} \wedge \omega\left(R_2\right) < 0, \tag{3.26}$$

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[2] We are just trying to de-emphasize the idea and idea of need to literally interpret the objects of QFT perturbation theory as divergent integrals. The methodology does not obscure the analytic structure of the series coefficients, quite the opposite, it fully develops it.

where $p_i^{\mathrm{I}_m}$ and $q(k)$ are polynomials and $q(k)$ is strictly independent of routings. Plus, the second term r_2 have the property $\omega(R_2) < 0$, i.e. the integral is finite. Then the functional acts as

$$\bar{J}_n^{\mathrm{I}_m} = \mathsf{Fy}\left[\bar{j}_n^{\mathrm{I}_m}\right] = \mathsf{Fy}\left[r_1\right] + \mathsf{Fy}\left[r_2\right] = \mathsf{Fy}\left[r_1\right] + \int \frac{\mathrm{d}^d k}{c(d)}\left[r_2\right]. \tag{3.27}$$

The meaning of these statements for the example above is

$$\bar{J}_2 = \int \frac{\mathrm{d}^4 k}{c(4)}\left[\bar{j}_2\right] = I_{\log}\left(\lambda^2\right) - \int \frac{\mathrm{d}^4 k}{c(4)} \frac{1}{D_\lambda^2 D_1} - \int \frac{\mathrm{d}^4 k}{c(4)} \frac{1}{D_\lambda^2 D_{12}} = I_{\log} + J_2 \tag{3.28}$$

Notwithstanding all this elaboration, and to emphasize, we keep the use of the integration symbol, what is evident in the first equality. The other symbol, $I_{\log}$, will appear in the next section. Anticipating and seizing the opportunity to discuss some interpretation, and to talk for the first time about the role of the overbar notation, we write it down explicitly:

$$I_{\log} = \mathsf{Fy}\left[1/D_\lambda^2\right] \equiv \int \frac{\mathrm{d}^4 k}{c(4)} \frac{1}{D_\lambda^2}. \tag{3.29}$$

Note that the strictly convergent part of the decomposition (3.28) gets a notation without the overbar, J_2, and for finite integrals we do not have a distinction at all. This means that without bar the integrals are finite and represent some function to be determined.

In this point the finite integrals are standardly evaluated, what put everything else in this sector in common ground with well-known sets of Feynman integrals with propagator powers (the subset of finite ones). The non-integrable part will be almost thoroughly surveyed in the next section.

Furthermore, looking at the way in which $I_{\log}$ appear as a coefficient of a constant polynomial one could perform renormalization by imposing some condition on the whole integral and absorbing the complete object without have explicitly evaluated a divergent integral—obviously in a place where renormalization holds, such as the 4-pt function in the $4D$-$\lambda\varphi^4$ model. When in an effective model of non-renormalizable nature the symmetry analysis and definition of physical parameters can fully determine the non-integrable sector, again, without computing explicitly a divergent integral. Examples of such investigations by the conceiver of the technique in the NJL-model can be found in refs. [91, 92], with and without isospin breaking terms (different fermion masses).

What we need to answer now is: Given a rational decomposition, how to organize it in order to analyze the problem? How are the routings and their arbitrariness preserved, and where is it allocated? The next section aims at a systematization through a basis of tensor surface terms and scalar integrals, and then at how to integrate the finite ones.

3.3 Generalized Surface terms and Scalar Objects

We have been introducing the phases of computation, and the reasoning techniques to be able to account the computations. This section deals with the question of what we do with the part that is strictly polynomial in the routings and may be or not divergent, since there may exists parts which are finite and polynomial in the routings.

Let us start by one mode for splitting the sectors that we talked about in the previous section: We have to split the kinematic dependence in products of propagators, not only a single one, but each time we split one it leaves a streak of functions behind it, which do not contain the routing of that propagator. The multiplications with the remaining ones still has non-negative power counting and have routings. Nonetheless, their power countings has lowered from zero to the original power counting, therefore, we can apply systematically the separating identity with the appropriate length[3].

To be more clear, let us simplify even more the linear term in the integration momentum, A_i, separating its constant term

$$
\begin{aligned}
A_i &= 2k\cdot k_i + B_i, &(3.30)\\
B_i &= \left(k_i^2+\lambda^2-m_i^2\right). &(3.31)
\end{aligned}
$$

Now we take a power counting $\omega = n$, in the simplest nontrivial case of two propagator $\frac{1}{D_{12}}$ (it can be in a tensor integral). Then, we separate k_1 with that degree, for subsequently separate k_2 by the corresponding degree $n-r_1$. With r_1 being the index of sum in the first separation, in another words, apply sequentially the identities

$$
\begin{aligned}
\frac{1}{D_1} &= \sum_{r_1=0}^{n}(-1)^{r_1}\frac{A_1^{r_1}}{D_\lambda^{r_1+1}}+(-1)^{n+1}\frac{A_1^{n+1}}{D_\lambda^{n+1}D_1}, &(3.32)\\
\frac{1}{D_2} &= \sum_{r_2=0}^{n-r_1}(-1)^{r_2}\frac{A_2^{r_2}}{D_\lambda^{r_2+1}}+(-1)^{n-r_1+1}\frac{A_2^{n-r_1+1}}{D_\lambda^{n-r_1+1}D_2}. &(3.33)
\end{aligned}
$$

From which will follow the result

$$
\begin{aligned}
\frac{1}{D_{12}} &= \sum_{r_1=0}^{n}\sum_{r_2=0}^{n-r_1}(-1)^{r_1+r_2}\left[\frac{A_1^{r_1}A_2^{r_2}}{D_\lambda^{r_1+r_2+2}}\right] &(3.34)\\
&+(-1)^{n+1}\sum_{r_1=0}^{n}\sum_{s_1=0}^{r_1}2^{s_1}\binom{r_1}{s_1}(B_1)^{r_1-s_1}\,k_1^{\mathrm{C}_{s_1}}\left[\frac{k_{\mathrm{C}_{s_1}}A_2^{n+1-r_1}}{D_\lambda^{n+2}D_2}\right] &(3.35)\\
&+(-1)^{n+1}\left[\frac{A_1^{n+1}}{D_\lambda^{n+1}D_{12}}\right].
\end{aligned}
$$

The second summation arose in that way because we have expanded the power below

$$
A_1^{r_1}=(2k\cdot k_1+B_1)^{r_1}=\sum_{s_1=0}^{r_1}2^{s_1}\binom{r_1}{s_1}k_1^{\mathrm{C}_{s_1}}k_{\mathrm{C}_{s_1}}(B_1)^{r_1-s_1}, \qquad (3.36)
$$

where the $k_1^{\mathrm{C}_{s_1}}k_{\mathrm{C}_{s_1}} = k_1^{\nu_1}\cdots k_1^{\nu_{s_1}}k_{\nu_1}\cdots k_{\nu_{s_1}} = (k\cdot k_1)^{s_1}$, this notation will be explained in a moment. The proliferation of indices r_i, s_i occurred because the result can be extended

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[3] The result is the same as if done uniformly in all propagators, since additional terms obtained from a stage where the result is already finite can be connected to the one where it appeared finite for the first time.

systematically to higher number of propagators. The last term and the second line, after integrated, are finite integrals and for them an algorithmical attitude can be adopted as well.

At this point of derivation we can focus on the first summation, which contain terms like this

$$\frac{A_1^{r_1} A_2^{r_2}}{D_\lambda^{r_1+r_2+2}}, \tag{3.37}$$

in it, by expanding the powers $A_1^{r_1} A_2^{r_2}$, we get linear combinations of finite and will be divergent structures. This part which appear in any tensor integral (implicitly assuming that there are tensor factors, besides the scalar structure analyzed, to compose the power-counting $\omega = n$) is the one we will show how to organize.

- **Tensor and routing-independent integrals**.

For any string I follows the tensor $k_{\mathrm{I}} = \prod_{\mu_i \in \mathrm{dom(I)}} k_{\mu_i}$ in particular $k_{\mathrm{I}_{2n}} = \prod_{i=1}^{2n} k_{\mu_i}$, where k_μ is the integration momenta. To remind, when we exclude an index μ_i we are saying that $\mu_i \notin \mathrm{dom}\left(\mathrm{I}_{2n-1}^i\right)$, this detail is only technical since the strings are not set, however due to their objects symmetry they behave as set. Nonetheless, expressions as the following are easy to grasp

$$k_{\mu_i} k^{\mathrm{I}_{2n-1}^i} = \prod_{i=1}^{2n} k_{\mu_j} = k_{\mathrm{I}_{2n}} \tag{3.38}$$

and its permutation symmetry

$$\mathrm{I}_{2n} = (\mu_i)_{i=1}^{2n}, \quad \mathrm{I}_{\sigma(2n)} = \left(\mu_{\sigma(i)}\right)_{i=1}^{2n} \quad k_{\mathrm{I}_{2n}} = k_{\mu_1} \cdots k_{\mu_{2n}}, \quad k_{\mathrm{I}_{\sigma(2n)}} = k_{\mathrm{I}_{2n}}. \tag{3.39}$$

Note that the superscript or subscript of the index set is irrelevant $k^{\mathrm{I}_{2n-1}^i} = k_{\mathrm{I}_{2n-1}^i}$. When it is necessary to translate a result to a form with explicit indices it is just a matter of position all occurrences in covariant or contravariant form, in both side of an equation, for example. Only the contractions ask for care.

Remark 3.3.1 *With the integration variable, the notation $k_{\mathrm{I}_{2n}}$, using italics and not San Serif letter, does not correspond to the one introduced in definition 1.1.8 of notation section. That notation will be used for vectors not integrated, let it be routings, or external momenta, and so on.*

Anticipating its use, we introduce a notation that can be coded by the one put forth in (1.1.6), viz.

$$k_{\mathrm{I}_{2n}}^{g^r} = \left\{[g]^r [k]^{2n-2r}\right\}_{\mathrm{I}_{2n}} = g_{\mu_{12}} \cdots g_{\mu_{2r-1,2r}} k_{\mu_{2r+1}} \cdots k_{\mu_{2n}} + \cdots$$

Now we enter the problem itself, the sub-space of tensor integrands of fixed mass dimensions, that includes finite integrals also, can be defined as

$$\mathfrak{i}_m = \left\{ \frac{k^{\mathrm{I}_{2n}}}{D_\lambda^{m+n}}, \frac{1}{D_\lambda^m} \middle| n \geq 1, m \geq 1 \right\}. \tag{3.40}$$

This set sustains the rational functions that appear in $A_1^{r_1}A_2^{r_2}/D_\lambda^{r_1+r_2+2}$, which was used as motivation and example[4]. However, it is not restricted to that motivational situation.

Example 3.3.2 *The set* $\mathfrak{i}_m$ *includes structures like*

$$\cdots\frac{k^{\mu_1}k^{\mu_2}}{\left(k^2-\lambda^2\right)^2},\frac{1}{\left(k^2-\lambda^2\right)}; \tag{3.41}$$

$$\cdots\frac{k^{\mu_1}k^{\mu_2}}{\left(k^2-\lambda^2\right)^3},\frac{1}{\left(k^2-\lambda^2\right)^2}; \tag{3.42}$$

$$\cdots\frac{k^{\mu_1}k^{\mu_2}k^{\mu_3}k^{\mu_4}}{\left(k^2-\lambda^2\right)^4},\frac{k^{\mu_1}k^{\mu_2}}{\left(k^2-\lambda^2\right)^3},\frac{1}{\left(k^2-\lambda^2\right)^2}. \tag{3.43}$$

The set of irreducible tensors integrands $\mathfrak{i}_m$ can always be recombined into another set, $\mathfrak{b}_m$, of total derivatives whose integrals comprised surface terms.

$$\mathfrak{b}_m=\left\{b_{m+n}^{\mathrm{I}_{2n}}\left(D_\lambda\right),\frac{1}{D_\lambda^m}\middle| n\geq 1, m\geq 1\right\}. \tag{3.44}$$

To precisely exhibit it let us start by analyzing a derivative. With the notation $\partial^{\mu_i}=\partial/\partial k_{\mu_i}$, we take the derivative of a rank-odd tensor, namely

$$-\partial^{\mu_i}\frac{k^{\mathrm{I}_{2n-1}^i}}{D_\lambda^{s-1}}=2\left(s-1\right)\frac{k^{\mu_i}k^{\mathrm{I}_{2n-1}^i}}{D_\lambda^s}-\frac{1}{D_\lambda^{s-1}}\partial^{\mu_i}k^{\mathrm{I}_{2n-1}^i}. \tag{3.45}$$

As μ_i is not an index in I_{2n-1}^i, then $k^{\mu_i}k^{\mathrm{I}_{2n-1}^i}=k^{\mathrm{I}_{2n}}$, hence the first term is an element if $\mathfrak{i}_{2(n-s)}$. In the second term we have

$$\partial^{\mu_i}k^{\mathrm{I}_{2n-1}^i}=\sum\nolimits_{j\neq i}\left(g_{\mu_{ij}}k^{\mathrm{I}_{2n-2}^{ij}}\right), \tag{3.46}$$

where I_{2n-2}^{ij} excludes the indices $\left\{\mu_i,\mu_j\right\}$, however, the last expression is not explicitly fully symmetric. To avoid this situation, which complicates matters for tensors with rank four or higher, we sum over i:

$$-\sum_{i=1}^{2n}\partial_k^{\mu_i}\left(\frac{k^{\mathrm{I}_{2n-1}^i}}{D_\lambda^{s-1}}\right)=4n\left(s-1\right)\frac{k^{\mathrm{I}_{2n}}}{D_\lambda^s}-\frac{1}{D_\lambda^{s-1}}\sum_{i=1}^{2n}\sum\nolimits_{j\neq i}\left(g_{\mu_{ij}}k^{\mathrm{I}_{2n-2}^{ij}}\right). \tag{3.47}$$

The first summand in r.h.s. was a constant and we got the factor $2n$ from it, whereas the second one, in the double summation, will overcount the number of terms needed to have full symmetry. In other words, it stands for $2n\left(2n-1\right)$ terms. However, with only $n\left(2n-1\right)=(2n)!/\left(2\left(2n-2\right)!\right)$ terms, by simple degeneracy counting, the result is already fully symmetric. Therefore we can cut the sums by half and only cast the linearly independent tensor monomials in the symmetric combination, which means that the following decomposition holds,

$$\sum_{i,j=1;i\neq j}^{2n}\left(g_{\mu_{ij}}k^{\mathrm{I}_{2n-2}^{ij}}\right)=\sum_{i<j}\left(g_{\mu_{ij}}k^{\mathrm{I}_{2n-2}^{ij}}\right)+\sum_{i>j}\left(g_{\mu_{ij}}k^{\mathrm{I}_{2n-2}^{ij}}\right)=2\sum_{i<j}\left(g_{\mu_{ij}}k^{\mathrm{I}_{2n-2}^{ij}}\right). \tag{3.48}$$

---

[4] When the power-counting, $\omega=n<d$, does not reaches the space-time dimension. Those cases can also be dealt, see P.h.d. thesis of L. Ebani ([89]), where in gravitational anomalies it is necessary to extend our present derivations to situations where non-rational functions will take place in the non-integrable sector.

Each of the summations give rise to the same set of terms and are fully symmetric. Based on this observation we identify

$$k^g_{\mathrm{I}_{2n}} = \sum_{i<j\in[1,n]} \left(g_{\mu_{ij}} k^{\mathrm{I}^{ij}_{2n-2}} \right) = \left\{ [g]\, [k]^{2n-2} \right\}^{\mathrm{I}_{2n}} = g_{\mu_{12}} k_{\mu_3} \dots k_{\mu_{2n}} + \text{perm.} \tag{3.49}$$

The second equality came from the section of notations in definition (1.1.6), is a notation first introduced by Passarino and Veltman [85]. To the author's knowledge.

Hitherto, we have obtained an explicitly symmetric tensor that is a total derivative. It is a linear combination of elements of $\mathfrak{i}_{2(n-s)}$. Itself it is a element of $\mathfrak{b}_{2(n-s)}$, and in final form is given by

$$-\frac{1}{2} \sum_{i=1}^{2n} \partial^{\mu_i}_{k} \left(\frac{k^{\mathrm{I}^i_{2n-1}}}{D^{s-1}_{\lambda}} \right) = \frac{2n\,(s-1)\, k^{\mathrm{I}_{2n}}}{D^s_{\lambda}} - \frac{k^g_{\mathrm{I}_{2n}}}{D^{s-1}_{\lambda}} =: b^{\mathrm{I}_{2n}}_s \left(D_{\lambda} \right). \tag{3.50}$$

Therefore, we offer the following definition.

Definition 3.3.3 *The boundary or surface term, of tensor degree* $2n$ *and explicitly symmetric in its Lorentz indices, corresponds to the expression*

$$\mathcal{B}^{(d)}_{s,\mathrm{I}_{2n}} \left(\lambda^2 \right) = -\frac{1}{2} \int_{\mathbb{R}^d} \mathrm{d}k \left\{ \sum_{i=1}^{2n} \frac{\partial}{\partial k_{\mu_i}} \left(\frac{k^{\mathrm{I}^i_{2n-1}}}{D^{s-1}_{\lambda}} \right) \right\} = \int_{\mathbb{R}^d} \mathrm{d}k \left\{ \frac{2n\,(s-1)\, k_{\mathrm{I}_{2n}}}{D^s_{\lambda}} - \frac{k^g_{\mathrm{I}_{2n}}}{D^{s-1}_{\lambda}} \right\}, \tag{3.51}$$

where $s > 1$ *and* $n \geq 1$. *Its power counting is* $d + 2n - 2s$. *As the scalar irreducible objects, they are defined by*

$$\mathcal{I}^{(d)}_{s-n} \left(\lambda^2 \right) = \int_{\mathbb{R}^d} \mathrm{d}k \frac{1}{D^{s-n}_{\lambda}}. \tag{3.52}$$

The superscript associated to the dimension will be dropped in context, as the scale λ^2 also. Our objective is not realized yet, since the scalar object still did not appear. Now we come back to eq. (3.50) above and write

$$\frac{2n\,(s-1)\, k^{\mathrm{I}_{2n}}}{D^s_{\lambda}} = b^{\mathrm{I}_{2n}}_s + \frac{1}{2\,(n-1)\,(s-2)} \frac{2\,(n-1)\,(s-2)\, k^g_{\mathrm{I}_{2n}}}{D^{s-1}_{\lambda}}. \tag{3.53}$$

At this moment additional exclusion of indices will occur, they will be represented by adding that information to I. Employing the definition of $k^g_{\mathrm{I}_{2n}}$ and summing and subtracting what is necessary to identify another surface term of lower tensor rank, we obtain

$$\begin{aligned} \frac{2\,(n-1)\,(s-2)\, k^g_{\mathrm{I}_{2n}}}{D^{s-1}_{\lambda}} &= \sum_{i_1<j_1\in[1,n]} g_{\mu_{i_1 j_1}} \left\{ \frac{2\,(n-1)\,(s-2)\, k^{\mathrm{I}^{i_1 j_1}_{2n-2}}}{D^{s-1}_{\lambda}} \right\} \\ &= \sum_{i_1<j_1\in[1,n]} g_{\mu_{i_1 j_1}} \left\{ \frac{2\,(n-1)\,(s-2)\, k_{\mathrm{I}^{i_1 j_1}_{2n-2}}}{D^{s-1}_{\lambda}} - \frac{k^g_{\mathrm{I}^{i_1 j_1}_{2n-2}}}{D^{s-2}_{\lambda}} \right\} + \\ &\quad -\frac{1}{D^{s-2}_{\lambda}} \sum_{i_1<j_1\in[1,n]} g_{\mu_{i_1 j_1}} k^g_{\mathrm{I}^{i_1 j_1}_{2n-2}}. \end{aligned} \tag{3.54}$$

The first term in the last passage is a symmetric combination of $b_{s-1}^{\mathrm{I}_{2n-2}}$ total derivatives and metric tensors. For the last term, the recursive use of eq. (3.49), now with μ_{i_1} and μ_{j_1}, deleted from $\mathrm{I}_{2n-2}^{i_1j_1}$, produces

$$\begin{aligned} k^{g}_{\mathrm{I}_{2n-2}^{i_1j_1}} &= \sum_{i_2<j_2\in[1,n]\backslash\{i_1,j_1\}} g_{\mu_{i_2j_2}} k_{\mathrm{I}_{2n-4}^{i_1i_2j_1j_2}} && (3.55) \\ \sum_{\substack{i_1<j_1\\ \in[1,n]}} g_{\mu_{i_1j_1}} k^{g}_{\mathrm{I}_{2n-2}^{i_1j_1}} &= \sum_{i_1<j_1\in[1,n]} \sum_{i_2<j_2\in[1,n]\backslash\{i_1,j_1\}} g_{\mu_{i_1j_1}} g_{\mu_{i_2j_2}} k_{\mathrm{I}_{2n-4}^{i_1i_2j_1j_2}} = 2k^{g^2}_{\mathrm{I}_{2n}}. && (3.56) \end{aligned}$$

The second line represents a fully symmetric tensor by construction; however, the number of terms is $n\,(n-1)\,(2n-1)\,(2n-3)$. It means that as we drag more and more indices from I_{2n}, the symmetry of the tensor power of metric tensors produces a factor of degeneracy. The combinations with two metrics, viz. $k^{g^2}_{\mathrm{I}_{2n}}$, has $n\,(n-1)\,(2n-1)\,(2n-3)\,/2$ terms, hence the factor of two in the formula. The tensor $k^{g^2}_{\mathrm{I}_{2n}}$ is part of the general definition

$$k^{g^r}_{\mathrm{I}_{2n}} = \left\{[g]^r\,[k]^{2n-2r}\right\}_{\mathrm{I}_{2n}}, \tag{3.57}$$

see definition (1.1.6) of full symmetric tensors. It has a number of independent monomials given by

$$\left|k^{g^r}_{\mathrm{I}_{2n}}\right| = \frac{(2n-1)!!}{[2\,(n-r)-1]!!}\binom{n}{r}. \tag{3.58}$$

Recapitulating: We have in a first iteration of the reduction

$$\frac{2n\,(s-1)\,k^{\mathrm{I}_{2n}}}{D_{\lambda}^{s}} = b_{s}^{\mathrm{I}_{2n}} + \frac{1}{2\,(n-1)\,(s-2)}\left\{\sum_{i_1<j_1\in[1,n]} g_{\mu_{i_1j_1}} b_{s-1}^{\mathrm{I}_{2n-2}^{i_1j_1}} - \frac{2k^{g^2}_{\mathrm{I}_{2n}}}{D_{\lambda}^{s-2}}\right\}, \tag{3.59}$$

where the first term is simply a fully symmetric combination of metric tensor and other symmetric tensor–surface term one–and as usual we define it either with curly bracket notation, or the power one,

$$b^{g}_{s-1,\mathrm{I}_{2n}} \equiv \left\{[g]\left[b_{s-1,\mathrm{J}_{2n-2}}\right]\right\}_{\mathrm{I}_{2n}} = \sum_{i_1<j_1\in[1,n]} g_{\mu_{i_1j_1}} b_{s-1}^{\mathrm{I}_{2n-2}^{i_1j_1}}.$$

Now, after $1 \leq l \leq n$ steps, there will be a factor of degeneracy $l!$ arising. The factor arise because the summations under the restriction $i_r < j_r$ in an each time smaller set of indices will overcount the number of terms necessary for full symmetry of the object, viz., we have the formula

$$\prod_{r=1}^{l} \sum_{i_r<j_r,\in[1,2n]\backslash R_r} g_{i_rj_r}\left(k^{\mathrm{I}_{2(n-r)}^{R_{r+1}}}\right) = l!\left(k^{g^r}_{\mathrm{I}_{2n}}\right). \tag{3.60}$$

The set of excluded indices in each iteration is $R_r = \{i_s, j_s : 1 \leq s \leq r-1\}$ with $R_1 = \varnothing$. In particular for $l = n$, which produces the irreducible terms since in that case we would have eliminated all the factors of k_μ, we have

$$\prod_{r=1}^{n} \sum_{i_r<j_r,\in[1,2n]\backslash R_r} g_{i_rj_r}\left(k^{\mathrm{I}_{2(n-r)}^{R_{r+1}}}\right) = n!\left(g_{\mathrm{I}_{2n}}\right). \tag{3.61}$$

One may see then, the first tensor defined in the thesis $g_{\mathrm{I}_{2n}}$. It lasts to us to introduce the notation for when the will-be surface terms are combined with ever increasing number of metrics, viz.

$$b^{g^r}_{s-r;\mathrm{I}_{2n}} := \left\{ [g]^r \left[b_{s-r;\mathrm{J}_{2n-2r}} \right] \right\}_{\mathrm{I}_{2n}} . \tag{3.62}$$

Finally, all factors and notations together yields a formula expressing any term of $\mathfrak{i}_{2(n-s)}$ as linear combinations of terms in $\mathfrak{b}_{2(b-s)}$. Explicitly

$$\frac{2n\left(s-1\right)k_{\mathrm{I}_{2n}}}{D^s_\lambda} = \sum_{r=0}^{n-1} c_r\left(n,s\right) b^{g^r}_{s-r;\mathrm{I}_{2n}} + 2n\left(s-1\right) c_n\left(n+1,s+1\right) g^{\mathrm{I}_{2n}} \frac{1}{D^{s-n}_\lambda}, \tag{3.63}$$

being the coefficients $c_r\left(n,s\right)$ expressed as

$$c_r\left(n,s\right) = \frac{\Gamma\left(n-r\right)\Gamma\left(s-r-1\right)\Gamma\left(r+1\right)}{2^r\Gamma\left(n\right)\Gamma\left(s-1\right)}. \tag{3.64}$$

Therefore, we have a very pleasant formula expressing the transformation from $\mathfrak{i}_m$ to $\mathfrak{b}_m$

$$\int_{\mathbb{R}^d} \mathrm{d}k \frac{2n\left(s-1\right)k_{\mathrm{I}_{2n}}}{D^s_\lambda} = \sum_{r=0}^{n-1} c_r\left(n,s\right) \left[\mathcal{B}^{(d),g^r}_{s-r;\mathrm{I}_{2n}}\right] + 2n\left(s-1\right) c_n\left(n+1,s+1\right) g_{\mathrm{I}_{2n}} \left[\mathcal{I}^{(d)}_{s-n}\left(\lambda^2\right)\right], \tag{3.65}$$

where the tensor surface-terms $\mathcal{B}_{s-r;\mathrm{I}_{2n}}$ were defined in (3.51) and the irreducible objects in (3.52). *Now we can add greek-index labels if needed.*

For the amplitudes in the thesis we need just two very simple objects in dimension $d = 2n$ that we define by

$$\Delta^{(2n)}_{n+1;\mu_{12}}\left(\lambda^2\right) \; : = \mathcal{B}^{(d)}_{n+1;\mu_{12}}\left(\lambda^2\right) = \int_{\mathbb{R}^{2n}} \mathrm{d}k \left(\frac{2nk_{\mu_1}k_{\mu_2}}{D^{n+1}_\lambda} - \frac{g_{\mu_1\mu_2}}{D^n_\lambda} \right), \tag{3.66}$$

$$I^{(2n)}_{\log}\left(\lambda^2\right) \; : = \mathcal{I}^{(2n)}_n\left(\lambda^2\right) = \int_{\mathbb{R}^{2n}} \mathrm{d}k \left(\frac{1}{D^n_\lambda} \right). \tag{3.67}$$

In the Ph.d. thesis of L. Ebani ([89]) she and partly the present author have investigated gravitational anomalies through the present technique. In there, the enormous growth of variety and number of surface terms required an efficient keep-tracking scheme that is implicit in the derivations done so far. Basically any tensor can be combined even more compactly as

$$\int_{\mathbb{R}^d} \mathrm{d}k \frac{k_{\mathrm{I}_{2n}}}{D^s_\lambda} = \frac{1}{2n\left(s-1\right)} \left[\mathcal{W}^{(d)}_{s;\mathrm{I}_{2n}}\right] + \frac{1}{2^n\Gamma\left(s\right)} g^{\mathrm{I}_{2n}} \left[\mathcal{I}^{(d)}_{s-n}\right], \tag{3.68}$$

$\mathcal{W}^{(d)}_{s;\mathrm{I}_{2n}}$ being merely the summation of $\mathcal{B}^{(d),g^r}_{s-r;\mathrm{I}_{2n}}$. Lots of operations, but not all, may be performed only knowing the type of $\mathcal{W}^{(d)}_{s;\mathrm{I}_{2n}}$, i.e., the parameters (d,s,n) and how many free or contracted indices it carries.

We locked the open values of $\mathcal{W}$ or $\mathcal{B}$ over their integrands and do not touch its integral forms anymore. Their values are restricted by consistency of its ambient space: by tensor calculus, amplitudes, symmetries, and so on.

They preserve the possibility, or not, of shifting the integration variable since they appear multiplied by k_i in general. The map $\phi\left(k_1,\cdots\right) = \left(P_\alpha, p_{2,1},\ldots\right)$, or with less unnecessary complication, through the transformations $p_{i,j} = k_i - k_j$ and $P_{i,j} = k_i + k_j$, permits to pinpoint the dependence on the labelling of an amplitude. The surface terms are always present for linear and higher divergent or logarithmic-divergent tensor integrals. Although their coefficients depend on $P_{i,j}$ in the first case, only external momenta $p_{i,j}$ appear in the second.

The separation highlights diverging structures and organizes them without performing any analytic operation. Moreover, it makes evident that the divergent content is a local polynomial in the ambiguous and physical momenta obtained without expansions or limits.

3.4 Definitions of the Finite Functions

In this section, instead of simply listing the definitions, I will mix the text with the techniques and notation used to achieve the definitions themselves. First, from multiple forms of transforming the momentum representation to a parametric one, we choose the one where the region of integration is a standard simplex

$$\mathbb{R}^n \supset \Delta_n = \left\{ x \in \mathbb{R}^n \middle| \textstyle\sum_{i=1}^{n} x_i \leq 1; x_i \geq 0 \right\}, \tag{3.69}$$

and an integral over this auxiliary space will be denoted by forms like

$$\int_{\Delta_n} \mathrm{d}x \left[f\left(x \right) \right] \equiv \int_{\Delta} \mathrm{d}^n x \left[f\left(x \right) \right]. \tag{3.70}$$

Explicitly, for us, it is a notation representing the iterated integral that follows

$$\int_{\Delta} \mathrm{d}^n x \left[f\left(x \right) \right] = \prod_{i=1}^{n} \int_{0}^{\mathcal{X}_{i-1}} \mathrm{d}x_i \left[f\left(x \right) \right] = \int_{0}^{\mathcal{X}_0} \mathrm{d}x_1 \cdots \int_{0}^{\mathcal{X}_{n-1}} \mathrm{d}x_n \left[f\left(x \right) \right], \tag{3.71}$$

where the upper limits are given by

$$\mathcal{X}_0 = 1, \quad \mathcal{X}_l = 1 - \sum_{i=1}^{l} x_i = \mathcal{X}_{l-1} - x_l. \tag{3.72}$$

As functions of the parameters x_l, two important values must be stated, and the derivative of its powers

$$\mathcal{X}_i \left(\mathcal{X}_{i-1} \right) \equiv \mathcal{X}_i \left(x_1, \cdots, x_i = \mathcal{X}_{i-1} \right) = 0, \tag{3.73}$$

$$\mathcal{X}_i \left(0 \right) \equiv \mathcal{X}_i \left(x_1, \cdots, x_i = 0 \right) = \mathcal{X}_{i-1}, \tag{3.74}$$

$$\frac{\partial \mathcal{X}_i^s}{\partial x_l} = -s \mathcal{X}_i^{s-1}. \tag{3.75}$$

Now, let us introduce the integrals to really be solved, because any other can be written in terms of them. They have the form

$$f^{\mathrm{I}_a} \left(b, M, N \right) = \frac{k_{\mathrm{I}_a} A_i^b}{D_\lambda^N D_{[1,M]}}, \quad i \in [1, M], \quad D_{[1,M]} = \prod_{i=1}^{M} D_i, \tag{3.76}$$

$$F^{\mathrm{I}_a} \left(b, M, N \right) = \int_{\mathbb{R}^d} \mathrm{d}k \left[f^{\mathrm{I}_a} \left(b, M, N \right) \right] \tag{3.77}$$

for some non-negative integers a, b, N, M, and just to remember that $k_{\mathrm{I}_a} = k_{\mu_1} \cdots k_{\mu_a}$. Leaving parameters free of range, with the exception of corresponding to finite structures, the set $f^{\mathrm{I}_a}(b, M, N)$ account for any 1-loop integral.

One could ask why this form? Well one example can be seen in the systematic separation we presented in eq. (3.34), section (3.3). Without further elaborations, for now, we put a tensor component in that expression and ignore some terms which are trivial in this analysis. Then that expression becomes

$$\begin{aligned} \frac{k_{\mathrm{I}_t}}{D_{12}} &= (-1)^{n+1} \sum_{r=0}^{n=\omega} \sum_{s=0}^{r} 2^s \binom{r}{s} (B_1)^{r-s} k_1^{\mathrm{C}_s} f_{\mathrm{I}_c \mathrm{C}_t}(n-r+1, 1, n+2) \\ &\quad + (-1)^{n+1} f_{\mathrm{I}_t}(n+1, 2, n+1) . \end{aligned} \tag{3.78}$$

Therefore, when the solution for the integrals $f^{\mathrm{I}_a}(b, M, N)$ are found we are one step closer to resolve a general Feynman integral. One must notice two things, first

$$\omega = d + a + b - 2(M + N) < 0, \tag{3.79}$$

since the separation ought to be effective, $f^{\mathrm{I}_a}(b, M, N)$ are integrands of strictly finite integrals. Second, the mass dimension of the integrals were decoupled from power counting as discussed in the section (3.2). The mass dimension itself is given by

$$\dim_m \left[F^{\mathrm{I}_a}(b, M, N) \right] = d + a + 2b - 2(M + N),$$

it can be positive or zero meanwhile $\omega < 0$ always. For future use, we define a related quantity $c_m = \left\lfloor \dim_m \left[F^{\mathrm{I}_a}(b, M, N) \right] / 2 \right\rfloor$.

Coming back to the motivating path leading to the organization of the finite parts. As everything here is traditional we will try to be brief. The formula below can be derived through an integral representation of the Beta function and, by induction, extended to multiple factors. It reads

$$\frac{1}{D_\lambda^N D_{[1,M]}} = (N)_M \int_\Delta \mathrm{d}^M x \frac{\mathcal{X}_M^{N-1}}{\left[D_\lambda + \sum_{i=1}^M x_i (D_i - D_\lambda) \right]^{N+M}}, \tag{3.80}$$

remembering that

$$\mathcal{X}_M = 1 - x_1 - \cdots - x_M,$$

and $(N)_M = \Gamma(N + M) / \Gamma(N)$ is the Pochhammer symbol.

Now one may recognize the combination $(D_i - D_\lambda) = A_i$, based on which all the strategy began. Let us retrieve its definition and call the denominator by E, i.e.,

$$\begin{aligned} E &= D_\lambda + \sum_{i=1}^{M} x_i A_i, & (3.81) \\ A_i &= (2k \cdot k_i) + B_i & (3.82) \\ B_i &= k_i^2 + \lambda^2 - m_i^2 . & (3.83) \end{aligned}$$

In a first look, one could figure out that by taking the derivative of the denominator, E, w.r.t. the parameter x_i she or he would have identified a relation between numerator and denominator. In equation form, it is better to see this fact

$$f^{\mathrm{I}_a}\left(b,M,N\right)=\left(N\right)_M\int_{\Delta}\mathrm{d}^Mx\frac{\mathcal{X}_M^{N-1}\left(k_{\mathrm{I}_a}A_i^b\right)}{\left[D_\lambda+\sum_{i=1}^{M}x_iA_i\right]^{N+M}}. \tag{3.84}$$

Right! This identification is the one we fully exploit in explicit computations, nonetheless, only after we have integrated the result. The direct procedure suggested can be followed, but, to the author's knowledge is not as effective as the one we employ in the real computations.

To anticipate, the format of integrals $J_n^{\mathrm{I}_m}$ that will be defined are obtained by just reducing the derivatives identified in A_i. This identification occur after translation in the loop momentum, and necessary to put the integral in standard form. Therefore, let us bring the momenta integration into play.

The integral of the expression above reads

$$F^{\mathrm{I}_a}\left(b,M,N\right)=\left(N\right)_M\int_{\Delta}\mathrm{d}^Mx\mathcal{X}_M^{N-1}\int\frac{\mathrm{d}^dk}{c\left(d\right)}\left\{\frac{k_{\mathrm{I}_a}A_i^b}{\left(E_M\right)^{N+M}}\right\}, \tag{3.85}$$

then, looking in the equation above for A_i, we get

$$\begin{aligned}\sum_{i=1}^{M}x_iA_i &= 2k\cdot L_M+\sum_{i=1}^{M}x_iB_i, &(3.86)\\ E &= k^2+2k\cdot L_M+\sum_{i=1}^{M}x_iB_i-\lambda^2, &(3.87)\end{aligned}$$

where we have defined a parameter-dependent vector as

$$L_M\left(x,\{k_i\}\right)=\sum_{i=1}^{M}x_ik_i. \tag{3.88}$$

By shifting the variable k_μ to eliminate the linear term, $k\rightarrow k-L_M$, it follows

$$E\rightarrow k^2+Q_M,\quad A_i\rightarrow 2\left(k\cdot k_i\right)+\partial_iQ_M, \tag{3.89}$$

with the definition of the quadratic polynomial Q, in momenta, masses, and parameters given by

$$\begin{aligned}Q_M &= -L_M^2+\sum_{i=1}^{M}x_iB_i-\lambda^2 &(3.90)\\ \mathcal{D}_i &\equiv \partial_iQ_M:=\frac{\partial Q_M}{\partial x_i}=-2k_i\cdot L_M+B_i. &(3.91)\end{aligned}$$

And we have seized the opportunity to introduce the notation for its derivatives $\mathcal{D}_i$.

With this identifications we write the result, up to now, as

$$F^{\mathrm{I}_a}\left(b,M,N\right)=\left(N\right)_M\int_{\Delta}\mathrm{d}^Mx\mathcal{X}_M^{N-1}\int\frac{\mathrm{d}^dk}{c\left(d\right)}\left\{\frac{\left(k-L_M\right)^{\mathrm{I}_a}\left(2k\cdot k_i+\partial_iQ_M\right)^b}{\left(k^2+Q_M\right)^{N+M}}\right\}. \tag{3.92}$$

The main, longest, intricate, and prone to error part of obtaining the finite parts is the elimination of the derivative factors ($\mathcal{D}_i^n = (\partial_i Q)^n$) from the general structure above. Itself this is only a part of the task, since there is multiple such integrals that must be combined to reach a final result.

Nonetheless, it lives in the realm of standard techniques and the reader may discover a shorter path; however, the caveats being introduced may avoid dead ends for those which want to understand and obtain the results used in the thesis.

The elimination of all derivatives is plain and simple partial integration; furthermore, after that is done in all places the remaining terms will be the integral representation of the Feynman integrals $J_n^{\mathrm{I}_m}$.

Therefore, we start by noticing that after integration there appear rational functions of Q, as it must be since the integration is a finite one. That being the very first objective: to perform only such type of operation. They fall in the class

$$\begin{aligned} \int \frac{\mathrm{d}^d k}{c\left(d\right)} \frac{k_{\mathrm{I}_{2l}}}{\left(k^2+Q\right)^{N+M}} &= \frac{g_{\mathrm{I}_{2l}}}{2^l \Gamma\left(N+M\right)} \frac{\Gamma\left(N+M-l-d/2\right)}{Q^{N+M-l-d/2}}, \\ & \quad 2\left(N+M\right)-2l-d>0, \end{aligned} \tag{3.93}$$

where l runs over an appropriate range due to the tensor power $\left(k-L_M\right)^{\mathrm{I}_a} = \left(k-L_M\right)_{\mu_1} \cdots \left(k-L_M\right)_{\mu_a}$, and the powers of $(k \cdot k_i)$, which appear on the expression above. Now choosing any two $n, m > 0$, a representative term will be of the form

$$\frac{\mathcal{D}_i^n}{Q^m} = \mathcal{D}_i^{n-1} \frac{\partial Q}{\partial x_i} \frac{1}{Q^m} = -\mathcal{D}_i^{n-1} \frac{\partial}{\partial x_i} \frac{1}{Q^{m-1}}. \tag{3.94}$$

See that if $m = 1$ we get a logarithm not a power, because of this feature we introduce the important and often used definitions

$$\Xi^{(-k)}\left[Q\right] = \frac{\left(-1\right)^{k-1}\Gamma\left(k\right)}{Q^k}, \quad k>0 \tag{3.95}$$

$$\Xi^{(0)}\left[Q,\lambda^2\right] = \log\frac{Q}{-\lambda^2} \tag{3.96}$$

$$\Xi^{(k)}\left[Q,\lambda^2\right] = \frac{Q^k}{k!}\left(\Xi^{(0)}-H_k\right), \quad k>0 \tag{3.97}$$

$$H_k = \sum_{l=1}^{k}\left(\frac{1}{l}\right), \quad H_0 := 0. \tag{3.98}$$

The rationale behind the definition is that derivatives work smoothly when integrating by parts with these definitions. For example, in the process of partial integration arises sequences given schematically as

$$\cdots \frac{1}{Q^p} \to \frac{1}{Q^{p-1}} \to \cdots \to \frac{1}{Q} \to \log\frac{Q}{-\lambda^2} \to Q\left[\log\frac{Q}{-\lambda^2} - Q\right] \to \cdots . \tag{3.99}$$

Thus the reason for the coefficients adopted becomes clear, because for each power of derivative $\mathcal{D}_i$ eliminated there will be no other coefficient, nor sign, to bother with, i.e.

$$\mathcal{D}_i \Xi^{(k-1)} = \left(\partial_i Q\right)\Xi^{(k-1)} = \partial_i \Xi^{(k)}. \tag{3.100}$$

Which is valid independent of the sign of $k \in \mathbb{Z}$. We call the functions $\Xi^{(k)}$, the kernels. They appear in the definition of the finite functions and are used in the process of obtaining them.

Here we have enough motivation to introduce the collection of functions over which we project the finite parts obtained by the method of analysis adopted.

Definition 3.4.1 *Let $\mathbf{n}_r = (n_1, \cdots, n_r) \in \mathbb{N}^r$ be a multi-index, $x = (x_1, \cdots, x_r) \in \mathbb{R}^r$, the simplex $\Delta_r \subset \mathbb{R}^r$, the definition of power $x^{\mathbf{n}} = x_1^{n_1} \cdots x_r^{n_r}$, then a basis of finite integrals are given by*

$$Z^{(k)}_{\mathbf{n}_r} = \prod_{i=1}^{r} \int_0^{\mathcal{X}_{i-1}} \mathrm{d}x_i \left[x^{\mathbf{n}_r} \Xi^{(k)} \left(Q, \lambda^2 \right) \right] \equiv \int_{\Delta_r} \mathrm{d}^r x \left[x^{\mathbf{n}_r} \Xi^{(k)} \right], \tag{3.101}$$

where the kernels $\Xi^{(k)}$ are the ones in eqs. (3.95-3.98). The sequence of upper limits $(\mathcal{X}_0, \cdots \mathcal{X}_{r-1})$ satisfies $\mathcal{X}_0 = 1$, $\mathcal{X}_l = \mathcal{X}_{l-1} - x_l$.

The index in the region of integration will be dropped when it is present in the measure $\mathrm{d}^r x$. Some common examples appearing in $r+1$-pt integrals in the next sections. Here let us continue the discussion with some examples.

Example 3.4.2 *A general class of examples are*

$$Z^{(-1)}_{i_1, i_2, \cdots i_r} = \int_{\Delta} \mathrm{d}^s x \left[x_1^{i_1} \cdots x_r^{i_r} \frac{1}{Q} \right] \tag{3.102}$$

$$Z^{(0)}_{i_1, i_2, \cdots i_r} = \int_{\Delta} \mathrm{d}^s x \left[x_1^{i_1} \cdots x_r^{i_r} \log \left(\frac{Q}{-\lambda^2} \right) \right] \tag{3.103}$$

$$Z^{(1)}_{i_1, i_2, \cdots i_r} = \int_{\Delta} \mathrm{d}^s x \left\{ x_1^{i_1} \cdots x_r^{i_r} Q \left[\log \left(\frac{Q}{-\lambda^2} \right) - 1 \right] \right\}. \tag{3.104}$$

Some more specific ones

$$Z^{(-1)}_{n} = \int_0^1 \mathrm{d}x x^n \frac{1}{Q} \tag{3.105}$$

$$Z^{(0)}_{n} = \int_0^1 \mathrm{d}x x^n \log \left(\frac{Q}{-\lambda^2} \right) \tag{3.106}$$

$$Z^{(-1)}_{n,m} = \int_0^1 \mathrm{d}x_1 \int_0^{1-x_1} \mathrm{d}x_2 x_1^n x_2^m \frac{1}{Q} \tag{3.107}$$

$$Z^{(0)}_{n,m} = \int_0^1 \mathrm{d}x_1 \int_0^{1-x_1} \mathrm{d}x_2 x_1^n x_2^m \log \left(\frac{Q}{-\lambda^2} \right) \tag{3.108}$$

that appear in $2, 3$-pt amplitudes in $2D$ and $4D$, and the next one appear in....

Now we have some idea of how positive mass dimensions integrals arise even if just finite integrals were computed, partial integration lifts the index of $\Xi^{(k)}$ from negative values up to positive ones where logarithms appear. After stretching the identity to some point to get finiteness, i.e.,

$$\frac{k_{\mathrm{I}}}{D_{[1,M]}} \rightarrow \cdots \frac{k_{\mathrm{I}} A_i^N}{D_\lambda^N D_{[1,M]}}. \tag{3.109}$$

We return to a minimal stage, that is to say, the separation which has led to a complicated expression is brought to a simple form by "undoing" it through partial integrations. We lift the momenta coordinates to the road of finite integrals and from there we descend to the forms that otherwise are coefficients in asymptotic or Laurent expansions in other strategies. Ok, I have not show this yet! Two examples of interest, the vector $\bar{J}_n^{\mu}$ and tensor $\bar{J}_{n+1}^{\mu\nu}$ in $d = 2n$ dimensions will be explicitly computed later...

We have left the definition of the quadratic polynomial behind, let us retrieve it, and the super momentum L_M too, i.e.

$$Q_M = -L_M^2 + \sum_{i=1}^{M} x_i B_i - \lambda^2; \quad L_M = \sum_{i=1}^{M} k_i x_i. \tag{3.110}$$

Then we square L_M and combine it with $B_i = k_i^2 + \lambda^2 - m_i^2$, i.e.

$$L_M^2 = \sum_{i,j=1}^{M} (k_i \cdot k_j)\, x_i x_j = \sum_{i=1}^{M} k_i^2 x_i^2 + 2 \sum_{i=1,i<j}^{M} (k_i \cdot k_j)\, x_i x_j, \tag{3.111}$$

the result will be

$$Q_M \left(\{k_i\}, \{x_i\}, \lambda^2\right) = \sum_{i=1}^{M} k_i^2 x_i (1 - x_i) - 2 \sum_{i=1,i<j}^{M} (k_i \cdot k_j)\, x_i x_j + \sum_{i=1}^{M} \left(\lambda^2 - m_i^2\right) x_i - \lambda^2. \tag{3.112}$$

Right! A fairly standard result, since it is a combination of Symanzik polynomials; however, in the final result for the integrals $J_n^{\mu_1 \cdots \mu_m}$ (not barred) it is not the routings that appear, what invariably appear are the differences $p_{i,j}$. The dependence in k_i only occur in intermediary steps.

For the integrals necessary in study of the first anomalous amplitudes for $d = 2n$, AV^n type ones. We set, for instance, the expression

$$\begin{aligned} Q\left(p_{21}, \cdots, p_{M,1},; m_1^2\right) &= \sum_{i=1}^{M} p_{i+1,1}^2 x_i (1 - x_i) - 2 \sum_{i,j=1,i<j}^{M} (p_{i+1,1} \cdot p_{j+1,1})\, x_i x_j \\ &\quad + \sum_{i=1}^{M} \left(m_{i+1}^2 - m_1^2\right) x_i - m_1^2. \end{aligned} \tag{3.113}$$

Notice that we used the parameters k_1, m_1 as references, but any other could be chosen. Sometimes, in context, we will drop the index in $Q \equiv Q_M$ and if the pair (k_1, m_1) is not available we use the next (k_2, m_2), remember that in the IRagfs there is one amplitude which the propagator carrying (k_1, m_1) is removed. Moreover, the polynomial can be interpreted as function of the kinematic invariants, as it can be written as $p_{i+1,1} = p_{i+1,i} + p_{i,i-1} + \cdots + p_{2,1} = q_{i+1} + q_i + \cdots q_2$, at least in one graph, and its associates by IRagfs, thus we may use q's in manipulations.

The low-energy value for all identical masses, $m_i = m$ (scale $\lambda^2 = m^2$ also), in the point $q_i \cdot q_j = 0$ $(s_{ij} = 0)$ follows from $Q\left(0, m^2\right) = -m^2$ and

$$\Xi^{(k)}\left(0, m^2\right) = (-1)^{k+1} \left(m^2\right)^k \frac{H_k}{k!}, \quad \Xi^{(-k)}\left(0, m^2\right) = (-1)^{k-1} \frac{\Gamma(k)}{\left(-m^2\right)^k}, \quad (k \geq 0)$$

that for $n \in \mathbb{N}^r$ end up in the following results

$$\begin{aligned}
Z^{(k)}_{\mathbf{n}}\left(s_{ij}=0\right) &= \left(-1\right)^{k+1} m^{2k}\frac{H_k}{k!}\left(\int_{\Delta_r} \mathrm{d}^r x x^{\mathbf{n}_r}\right) = m^{2k}\frac{\left(-1\right)^{k+1} H_k}{k!}\frac{\mathbf{n}!}{\left(r+|\mathbf{n}|\right)!} && (3.114)\\
Z^{(-k)}_{\mathbf{n}}\left(s_{ij}=0\right) &= \left(-1\right)^{k-1}\frac{\Gamma\left(k\right)}{\left(-m^2\right)^k}\left(\int_{\Delta_r} \mathrm{d}^r x x^{\mathbf{n}_r}\right) = -\frac{1}{m^{2k}}\Gamma\left(k\right)\frac{\mathbf{n}!}{\left(r+|\mathbf{n}|\right)!} && (3.115)\\
|\mathbf{n}| &= \sum_{i=1}^{r} n_i, \quad \mathbf{n}! = \prod_{i=1}^{r} n_i!. && (3.116)
\end{aligned}$$

For distinct masses on the point $q_i \cdot q_j = 0$ $\left(s_{ij}=0\right)$ these functions may depend on logarithms and other transcendental functions, but a particular combination which will appear in the sequel is easy to get and will be treated along the chapters, e.g. eqs. (5.114) chapter 5 and (6.67) in chapter 6.

$$\begin{aligned}
&\sum_{i=1}^{n}\left(m_{i+1}^2 - m_1^2\right) Z^{(-1)}_{\mathbf{e}_i} - m_1^2 Z^{(-1)}_{\mathbf{0}} && (3.117)\\
&= \int_{\Delta} \mathrm{d}^n x \left[\sum_{i=1}^{n}\left(m_{i+1}^2 - m_1^2\right) x_i - m_1^2\right] \Xi^{(-1)} = \int_{\Delta} \mathrm{d}^n x \frac{\left[Q\left(s_{ij}=0\right)\right]}{Q}
\end{aligned}$$

at $s_{ij}=0$ this reduces to the volume of the standard simplex Δ^n

$$\left[\sum_{i=1}^{n}\left(m_{i+1}^2 - m_1^2\right) Z^{(-1)}_{\mathbf{e}_i} - m_1^2 Z^{(-1)}_{\mathbf{0}}\right]\left(s_{ij}=0\right) = \int_{\Delta} \mathrm{d}^n x = \frac{1}{n!} \qquad (3.118)$$

Everything said we define the finite parts of the tensor integrals and thereafter we illustrate some aspects of their obtainment.

Definition 3.4.3 *Let it be the multi-indices introduced in definition (3.4.1)* $\mathbf{n} = \mathbf{0}, \mathbf{e}_i \in \mathbb{N}^{n-1}$, $\left(\mathbf{e}_i\right)^j = \delta^i_j, \mathbf{0} = \left(0, \cdots, 0\right)$*, the n-pt* $d = 2n$ *dimensional scalar and vector integrals assume the forms*

$$\begin{aligned}
J^{(2n)}_{n}\left(p_{21}, \cdots, p_{n,1}\right) &= -Z^{(0)}_{\mathbf{0}}, && (3.119)\\
J^{(2n),\mu}_{n}\left(p_{21}, \cdots, p_{n,1}\right) &= \sum_{i\in[1,n-1]} p^{\mu}_{i+1,1} Z^{(0)}_{\mathbf{e}_i}, && (3.120)
\end{aligned}$$

whereas the $(n+1)$*-pt integrals, which appear with one more parameter integral, are given by*

$$\begin{aligned}
J^{(2n)}_{n+1}\left(p_{21}, \cdots, p_{n+1,1}\right) &= Z^{(-1)}_{\mathbf{0}}, && (3.121)\\
J^{(2n),\mu}_{n+1}\left(p_{21}, \cdots, p_{n+1,1}\right) &= -\sum_{i\in[1,n]} p^{\mu}_{i+1,1} Z^{(-1)}_{\mathbf{e}_i}, && (3.122)\\
J^{(2n),\mu_{12}}_{n+1}\left(p_{21}, \cdots, p_{n+1,1}\right) &= -\frac{1}{2} g^{\mu_{12}} Z^{(0)}_{\mathbf{0}} + \sum_{i,j\in[1,n]}\left[\left(p^{\mu_1}_{i+1,1} p^{\mu_2}_{j+1,1}\right) Z^{(-1)}_{\mathbf{e}_{i,j}}\right], && (3.123)
\end{aligned}$$

where the multi-indices $\mathbf{0}, \mathbf{e}_i, \mathbf{e}_{i,j} \in \mathbb{N}^n$, $\mathbf{e}_{i,j} = \mathbf{e}_i + \mathbf{e}_j$.

Remark 3.4.4 *Observe that the depth of the multi-indices can be read from the r-pt degree of the integrals they are indexing. As $\mathbf{e}_i$ is basis for $\mathbf{n} \in \mathbb{N}^n$, that is $\mathbf{n} = \sum_{i=1}^n n_i \mathbf{e}_i$, $\mathbf{e}_{i,j}$, it is just capturing the fact that for $p^{\mu_1}_{i+1,1} p^{\mu_2}_{j+1,1}$ only the i^{th} and j^{th} powers appear with unit coefficient $x_i x_j$, which is x_i^2 for $\mathbf{e}_{i,i} = 2\mathbf{e}_i$. Furthermore, the structure arises from considering the notation introduced in cap. 1, definition 1.1.8; on which the symmetric tensor $\mathsf{k}^{u^r, v^s}_{\mathrm{I}_t}$ was introduced. To show this, through a brief example, let it be $r = 0$, $s = 2$, $t = 2$, and*

$$\begin{aligned} L &= \sum_{i=[1,n]} p_{i+1,1} x_i = p_{21} x_1 + \cdots + p_{n+1} x_n, \textit{ then} \\ \mathsf{k}^{L^2}_{\mathrm{I}_2} &= L^{\mu_1} L^{\mu_2} = \sum_{i,j \in [1,n]} \left(p^{\mu_1}_{i+1,1} p^{\mu_2}_{j+1,1} \right) x_i x_j . \end{aligned}$$

As for $r = 0$, $s = 0$, and $t = 2$ follows trivially that $\mathsf{k}_{\mathrm{I}_2} = g_{\mathrm{I}_2} = g^{\mu_{12}}$ (the position of the $\mathrm{I}_2 = \mu_{12}$ is irrelevant as commented elsewhere). The number of metrics such notation ($\mathsf{k}^{u^r, v^s}_{\mathrm{I}_t}$) follows from $\lfloor (t - r - s)/2 \rfloor$, but in applications the number comes naturally as an integer. Now it is possible to express the tensor $J^{(2n),\mu_{12}}_{n+1}$ as

$$J^{(2n),\mathrm{I}_2}_{n+1} = \int_\Delta \mathrm{d}^n x \left[-\frac{1}{2} \mathsf{k}_{\mathrm{I}_2} \Xi^{(0)} + \mathsf{k}^{LL}_{\mathrm{I}_2} \Xi^{(-1)} \right].$$

This will be useful when contractions with the momenta and metric will be worked out later.

Examples: expanding the most the expression of a tensor type of integral, we have for $2D$ and 2-pt tensor integral

$$J^{(2)\mu_{12}}_2 = -\frac{1}{2} g^{\mu_{12}} Z^{(0)}_0 + p^{\mu_1}_{21} p^{\mu_2}_{21} Z^{(-1)}_2 , \tag{3.124}$$

for a $4D$ and 3-pt tensor integral

$$J^{(4)\mu_{12}}_3 = -\frac{1}{2} g^{\mu_{12}} Z^{(0)}_{00} + p^{\mu_1}_{21} p^{\mu_2}_{21} Z^{(-1)}_{20} + p^{\mu_1}_{31} p^{\mu_2}_{31} Z^{(-1)}_{02} + 2 p^{(\mu_1}_{21} p^{\mu_2)}_{31} Z^{(-1)}_{11} \tag{3.125}$$

and a $6D$ and 4-pt tensor integral

$$\begin{aligned} J^{(6)\mu_{12}}_4 = & -\frac{1}{2} g^{\mu_{12}} Z^{(0)}_{000} + p^{\mu_1}_{21} p^{\mu_2}_{21} Z^{(-1)}_{200} + p^{\mu_1}_{31} p^{\mu_2}_{31} Z^{(-1)}_{020} + p^{\mu_1}_{41} p^{\mu_2}_{41} Z^{(-1)}_{002} + \\ & + 2 p^{(\mu_1}_{21} p^{\mu_2)}_{31} Z^{(-1)}_{110} + 2 p^{(\mu_1}_{21} p^{\mu_2)}_{41} Z^{(-1)}_{101} + 2 p^{(\mu_1}_{31} p^{\mu_2)}_{41} Z^{(-1)}_{011} \end{aligned} \tag{3.126}$$

being $p^{(\mu} q^{\nu)} = \left(p^\mu q^\nu + p^\nu q^\mu \right)/2$.

3.5 Obtaining the Finite Functions

The integrals necessary to evaluate the AV^n-type amplitudes, following our conventions, are listed by

$$\left(\bar{J}^{(2n)}_n ; \bar{J}^{(2n)\mu_1}_n \right) = \int_{\mathbb{R}^{2n}} \mathrm{d}k \frac{\left(1; K^{\mu_1}_{i_1}\right)}{D_{i_1 i_2 \cdots i_n}}, \tag{3.127}$$

$$i_l < i_{l+1} \in [1, n+1], \text{ and} \tag{3.128}$$

$$\left(\bar{J}^{(2n)}_{n+1} ; \bar{J}^{(2n),\mu_1}_{n+1} ; \bar{J}^{(2n),\mu_{12}}_{n+1} \right) = \int_{\mathbb{R}^{2n}} \mathrm{d}k \frac{\left(1; K^{\mu_1}_1 ; K^{\mu_{12}}_1\right)}{D_{[1,n+1]}} . \tag{3.129}$$

The power countings reveal finite and non finite integral, the more complex case is the vector n-pt one ($\bar{J}_n^{(2n)\mu_1}$) they arise due to cancellation of denominators since factors as $K_i \cdot K_j$ appear after Dirac traces are performed and in 1-loop they always can expand as inverse denominator D_i and D_j. Explicitly, the power countings are

$$\begin{aligned} \omega\left(\bar{J}_n^{(2n)},\ \bar{J}_n^{(2n)\mu_1}\right) &= (0,1)\,, \\ \omega\left(\bar{J}_{n+1}^{(2n)},\ \bar{J}_{n+1}^{(2n),\mu_1},\ \bar{J}_{n+1}^{(2n),\mu_{12}}\right) &= (-2,-1,0)\,. \end{aligned}$$

The reference routing (mass as well) can be any (k_i, m_i) for $i \in [1, n+1] = \{1, 2, \cdots, n+1\}$, but in integrals where all denominators are present we use (k_1, m_1). However, other cases may appear in contractions with $p_{i,j}$, hence for completeness and keep in mind that the general dependence may come in the form

$$\begin{aligned} & Q_{n-1}\left(p_{i_2,i_1}, p_{i_3,i_1}, \cdots, p_{i_n,i_1}; m_{i_1}^2\right) \\ = {} & \sum_{j=1}^{n-1} p_{i_j,i_1}^2 x_j\left(1-x_j\right) - 2\sum_{j<l}^{n-1}\left(p_{i_j,i_1}\cdot p_{i_l,i_1}\right) x_j x_l + \sum_{j=1}^{n-1}\left(m_{i_j}^2 - m_{i_1}^2\right) x_j - m_{i_1}^2. \end{aligned}$$

3.5.1 Computing Finite Integrals

I will divert from the convention to illustrate a different choice of reference routing to serve as argument in the section of sign tensors (3.7), and to show some notational manipulations we do for reducing expressions in non-finite integrals, next subsection.

Selecting a tensor integral of rank $2l$ (to facilitate the considerations), dimension $d = 2k$, and $k + l + 1$ propagators to stay in the edge of finiteness. For which the integrand can be stated as

$$\bar{j}_{k+l+1}^{\mathrm{I}_{2l}} = j_{k+l+1}^{\mathrm{I}_{2l}} = \frac{K_c^{\mathrm{I}_{2l}}}{D_{[1,k+l+1]}}. \tag{3.130}$$

The chosen routing k_c^μ, in $K_c^\mu = k^\mu + k_c^\mu$, has its index $c \in [1, k+l+1]$ not necessarily the lowest one, i.e. k_1. The tensor power $K_c^{\mathrm{I}_{2l}}$ is simply $K_c^{\mu_1} \cdots K_c^{\mu_{2l}}$. The power counting is trivially negative, since we arrange it in that way, $\omega = 2k + 2l - 2(k + l + 1) = -1$. Fact that was implicit above when we equated the barred and unbarred definition.

The parametrization for the finite integral is a little different from the one presented before. That parametrization was tailored to be used when the factor D_λ appear. Notwithstanding this fact, it is a simple matter of renaming some variables to express

$$j_{k+l+1}^{\mathrm{I}_{2l}} = \Gamma(k+l+1)\int_\Delta \mathrm{d}^{k+l}x \frac{K_c^{\mathrm{I}_{2l}}}{\left[D_c + \sum_{i=1}^{k+l} x_i\left(D_{b_i} - D_c\right)\right]^{k+l+1}}. \tag{3.131}$$

Besides, we cleverly arranged them in order $b_i < b_{i+1} \in [1, k+l+1] \setminus \{c\}$ (for matters of organization), i.e. we select k_c as reference; moreover, we tagged the $x_i D_{b_i}$ in such a way the parameters "say" which vector will appear in the i^{th} integration.

The point now is that $(D_{b_i} - D_c) \neq A_{b_i}$, instead we have

$$(D_{b_i} - D_c) = 2k \cdot p_{b_i,c} - k_c^2 + \left(k_{b_i}^2 + m_c^2 - m_{b_i}^2\right) = 2k \cdot p_{b_i,c} + \left(B_{b_i} - k_c^2\right), \tag{3.132}$$

and thus the denominator acquires the form below

$$\begin{aligned} E \;=\; & \left[D_c + \sum_{i=1}^{k+l} x_i \left(D_{b_i} - D_c\right)\right] = k^2 + 2k \cdot \left[k_c + \sum_{i=1}^{k+l} p_{b_i,c}\, x_i\right] \\ & + k_c^2 + \sum_{i=1}^{k+l} x_i \left(B_{b_i} - k_c^2\right) - m_c^2. \end{aligned} \tag{3.133}$$

Then we identify two momenta-parameter dependent vectors by

$$\tilde{L} = k_c + L \quad \text{and} \quad L = p_{b_1,c} x_1 + \cdots + p_{b_{k+l+1},c}\, x_{k+l}. \tag{3.134}$$

The result is integrated over the loop-momenta $\int_{\mathbb{R}^{2k}} \mathrm{d}k$. After that we shift the integration variable as $k \to k - \tilde{L}$, the effect on the denominator is

$$E \to k^2 - \tilde{L}^2 + k_c^2 + \sum_{i=1}^{k+l} x_i \left(B_{b_i} - k_c^2\right) - m_c^2 = k^2 + Q. \tag{3.135}$$

One more consideration, by means of $\tilde{L}^2 = L^2 + 2k_c \cdot L + k_c^2$ the Q-polynomial becomes

$$Q = -L^2 + \sum_{i=1}^{k+l} x_i \left(p_{b_i,c}^2 + m_c^2 - m_{b_i}^2\right) - m_c^2. \tag{3.136}$$

Here we could set $B_i = p_{b_i,c}^2 + m_c^2 - m_{b_i}^2$, which will be used in the section about reductions (3.6).

Over the numerator the effect will be $K_c^\mu = k^\mu + k_c^\mu \to k^\mu - \tilde{L}^\mu + k_c^\mu = k^\mu - L^\mu$. This means that the expression is function solely of the difference $p_{b_i,c}$, because L^μ is function of $p_{b_i,c}$. What it is expected since a very early shift $k \to k - k_c$, before parametrization, would make evident the property (it is a valid operation since the integral is finite).

At this point, we must keep only the even term in the integration variable, in other words, we must select them in the tensor power below

$$(k - L)^{\mathrm{I}_{2l}} = (k - L)^{\mu_1} \cdots (k - L)^{\mu_{2l}}.$$

To this aim a notational device is employed for integrands, i.e.

$$(k - L)^{\mathrm{I}_{2l}} = \sum_{r=0}^{2l} (-1)^r\, k_{\mathrm{I}_{2l}}^{L^r} = \sum_{r=0}^{l} k_{\mathrm{I}_{2l}}^{L^{2r}} - \sum_{r=1}^{l} k_{\mathrm{I}_{2l}}^{L^{2r-1}}. \tag{3.137}$$

Reminding the definition of $k_{\mathrm{I}_{2l}}^{L^r} = \{[k]^{2l-r}\,[L]^r\}_{\mathrm{I}_{2l}}$. For example, in a rank-two tensor we obtain

$$(k - L)_{\mu_1\mu_2} = \left(k_{\mu_{12}} - k_{\mu_{12}}^{L} + k_{\mu_{12}}^{LL}\right) = \left[k_{\mu_1} k_{\mu_2} - \left(L_{\mu_1} k_{\mu_2} + L_{\mu_2} k_{\mu_1}\right) + L_{\mu_1} L_{\mu_2}\right]. \tag{3.138}$$

Caveat: before the integration, the "super" tensor $k_{\mathrm{I}_{2l}}^{L^r}$ doesn't carry metric tensors. I know it can get confusing, but the stages before and after integration will free us of this problem. In context and by the definition with italics or San serif font we may distinguish them.

Now we are prepared to explicitly integrate the loop-momentum, i.e.

$$J_{k+l+1}^{\mathrm{I}_{2l}} = \int_{\mathbb{R}^{2k}} \mathrm{d}k \frac{K_c^{\mathrm{I}_{2l}}}{D_{[1,k+l+1]}} = \sum_{r=0}^{l} \Gamma\left(k+l+1\right) \int_\Delta \mathrm{d}^{k+l}x \int_{\mathbb{R}^{2k}} \mathrm{d}k \frac{k_{\mathrm{I}_{2l}}^{L^{2r}}}{\left(k^2+Q\right)^{k+l+1}}. \tag{3.139}$$

Notice that we have kept only the even tensor $k_{\mathrm{I}_{2l}}^{L^{2r}}$ in the loop momentum ($2\left(l-r\right)$ is the number of k_μ in the tensor).

Let us repeat the formula of eq. (3.93) for the integral of a pure tensor. Adapting that formula with the choices $d=2k$, $N+M=k+l+1$, and $2l=2a$ for the tensor rank (more about this in a moment). Thus, conveniently organized, it is expressed as

$$\int_{\mathbb{R}^{2k}} \mathrm{d}k \frac{\Gamma\left(k+l+1\right) k_{\mathrm{I}_{2a}}}{\left(k^2+Q\right)^{k+l+1}} = \frac{g_{\mathrm{I}_{2a}}}{2^a} \frac{\Gamma\left(l+1-a\right)}{Q^{l+1-a}} = g_{\mathrm{I}_{2a}} \frac{\left(-1\right)^{l-a} \Xi^{\left(-\left(l+1-a\right)\right)}\left[Q\right]}{2^a}. \tag{3.140}$$

Where it was identified the kernels Ξ; the definitions (3.95-3.98).

An important point is the connection between the definitions $k_{\mathrm{I}_{2l}}^{L^{2r}}$ and $\mathsf{k}_{\mathrm{I}_{2l}}^{L^{2r}}$; the one with italics belong to the integrand level and the one with Serif letter, denoting $\mathsf{k}_{\mathrm{I}_{2l}}^{L^{2r}} = \{\left[g\right]^{l-r}\left[L\right]^{2r}\}_{\mathrm{I}_{2l}}$, arises from integration. The number of monomials for $k_{\mathrm{I}_{2l}}^{L^{2r}}$ is

$$\left|k_{\mathrm{I}_{2l}}^{L^{2r}}\right| = \binom{2l}{2r},$$

akin of the coefficient in a binomial expansion of $\left(k-L\right)^{\mathrm{I}_{2l}}$, if it was not a tensor power. Meanwhile, the after-integration structure $\mathsf{k}_{\mathrm{I}_{2l}}^{L^{2r}}$ has

$$\left|\mathsf{k}_{\mathrm{I}_{2l}}^{L^{2r}}\right| = \frac{\left(2l\right)!}{2^{l-r}\left(l-r\right)!\left(2r\right)!}$$

monomials. A short scheme illustrates how integration transforms one form into another (no factor arises)

$$\begin{aligned} k_{\mathrm{I}_6}^{L^2} &= L_{\mu_{12}} k_{\mu_{3456}} + L_{\mu_{13}} k_{\mu_{2456}} + L_{\mu_{14}} k_{\mu_{2356}} + \cdots \\ &\xrightarrow{\int} L_{\mu_{12}} \left(g_{\mu_{34}} g_{\mu_{56}} + g_{\mu_{35}} g_{\mu_{46}} + g_{\mu_{36}} g_{\mu_{45}}\right) + \cdots \\ &= L_{\mu_{12}} g_{\mathrm{I}_4} + \cdots = \mathsf{k}_{\mathrm{I}_6}^{L^2}. \end{aligned}$$

The fifteen ($\binom{6}{2}$) terms in $k_{\mathrm{I}_6}^{L^2}$ has turned into forty five ($\frac{6!}{2^2 2! 2!}$) terms of $\mathsf{k}_{\mathrm{I}_6}^{L^2}$. The metrics, resulting from integration, get adjoined in the definition of $\mathsf{k}_{\mathrm{I}_{2l}}^{L^{2r}}$.

When we are integrating the expression $\int_{\mathbb{R}^{2k}} \mathrm{d}k [k_{\mathrm{I}_{2l}}^{L^{2r}}] f\left(k^2, Q\right)$, the tensor power $2a$ in $k_{\mathrm{I}_{2a}}$ from the formula (3.140) must be read as $a=l-r$. Therefore, with due care, the final result turn up as

$$J_{k+l+1}^{\mathrm{I}_{2l}} = \int_\Delta \mathrm{d}^{k+l}x \left\{ \sum_{r=0}^{l} \frac{\left(-1\right)^r}{2^{l-r}} \mathsf{k}_{\mathrm{I}_{2l}}^{L^{2r}} \Xi^{\left(-1-r\right)}\left[Q\right] \right\}. \tag{3.141}$$

Where L and Q in the expression mean

$$\begin{aligned} L &= p_{b_1,c}x_1+\cdots+p_{b_{k+l+1},c}x_{k+l}, & (3.142)\\ Q &= -L^2+\sum_{i=1}^{k+l}x_iB_i-m_c^2=Q\left(p_{b_1,c},\cdots,p_{b_{k+l},c},m_c^2\right). & (3.143)\end{aligned}$$

We can add the length of L if needed.

Remark 3.5.1 *If the $J_{k+l+1}^{\mathrm{I}_{2l}}$ integral carries all the momenta, in some context, we omit its dependence on the momenta and masses, even Q, and generically adopt $(k_c,m_c)=(k_1,m_1)$ as reference.*

3.5.2 Computing Finite Integrals Associated to Non-Finite Ones

The role of the derivations done for a class of finite integrals in the previous subsection stems from the considerations about and practical use of the notations.

The value of the practice resides in the necessary task of extracting derivatives from integrals defined in eq. (3.76), section 3.4. That means, in the integrals produced by the strategy to bring all kinematic content into finite integrals. Let us bring them here to better develop the next steps,

$$F^{\mathrm{I}_a}\left(b,M,N\right)=\int_{\mathbb{R}^d}\mathrm{d}k\left[\frac{k_{\mathrm{I}_a}A_i^b}{D_\lambda^N D_{[1,M]}}\right]. \tag{3.144}$$

To keep the matters simple, I will just point the places where care must be exercised. The routing k_i that appeared in A_i can be any from the ones present in $D_{[1,M]}=D_1D_2\cdots D_M$. We have to tag it with the last parameter of integration, besides better organization it has no larger significance. We may rename the variables in the simplex at will, or rename momenta variables if necessary. Continuing, I remind us that in a certain step in the section where we have defined the finite functions (3.4), we have arrived at an expression, eq. (3.92), which is reproduced here

$$F^{\mathrm{I}_a}\left(b,M,N\right)=\frac{\Gamma\left(N+M\right)}{\Gamma\left(N\right)}\int_\Delta\mathrm{d}^Mx\mathcal{X}_M^{N-1}\int_{\mathbb{R}^d}\mathrm{d}k\left\{\frac{\left(k-L\right)^{\mathrm{I}_a}\left(2k\cdot k_i+\mathcal{D}_i\right)^b}{\left(k^2+Q\right)^{N+M}}\right\}. \tag{3.145}$$

In there, it was said that one must eliminate the derivatives $\mathcal{D}_i=\partial_iQ$ from the expression. Now, we will use the tools introduced to at least show the expression which one must operate to systematically achieve that goal.

First, we must separate by cases: for tensor degree even/odd and power of derivative even/odd ($A_{i,k-L}=2k\cdot k_i+\mathcal{D}_i$). There are four cases, only one we illustrate. For the next expressions we made convenient choices $d=2l$ and lowercase letters, $m=M,n=N$.

For even tensor and even derivative power, i.e. $k_{\mathrm{I}_{2a}}A_i^{2b}$, the even parts, in the loop-momentum, can be adjusted by

$$\begin{aligned}
&(k-L)_{\mathrm{I}_{2a}}\left(2k\cdot k_i+\mathcal{D}_i\right)^{2b} \\
=\;& \sum_{s=0}^{b}\sum_{r=0}^{a} 2^{2b-2s}\binom{2b}{2s}\mathcal{D}_i^{2s}k_i^{\mathrm{C}_{2b-2s}}\left[k_{\mathrm{C}_{2b-2s}}k_{\mathrm{I}_{2a}}^{L^{2r}}\right] \\
&-\sum_{s=1}^{b}\sum_{r=1}^{a} 2^{2b-2s+1}\binom{2b}{2s-1}\mathcal{D}_i^{2s-1}k_i^{\mathrm{C}_{2b-2s+1}}\left[k_{\mathrm{C}_{2b-2s+1}}k_{\mathrm{I}_{2a}}^{L^{2r-1}}\right].
\end{aligned} \tag{3.146}$$

This formula comes from separating both summations in their even and odd parts

$$\sum_{n=0}^{2a} f(n)=\sum_{n=0}^{a} f(2n)+\sum_{n=1}^{a} f(2n-1), \quad \sum_{n=0}^{2a+1} f(n)=\sum_{n=0}^{a} f(2n)+\sum_{n=0}^{a} f(2n+1), \tag{3.147}$$

which was already use in eq. (3.137) to expand $(k-L)_{\mathrm{I}_{2a}}$. The other ingredient was to write the powers of $(k\cdot k_i)$ as

$$(k\cdot k_i)^s=k_i^{\mathrm{C}_s}k_{\mathrm{C}_s}=\prod_{j=1}^{s}k_i^{\nu_j}k_{\nu_j}.$$

Just the even powers of k_μ were showed above, by example $k_{\mathrm{C}_{2b-2s+1}}k_{\mathrm{I}_{2a}}^{L^{2r-1}}$ has $2b-2s+1$ factors from the contracted set and $2a-2r+1$ from the free index part.

Now, when we integrate, a novel and crucial feature emerges. To be detailed as possible, we have to work by means of a piece of the expression, namely

$$k_i^{\mathrm{C}_{2b-2s}}\int_{\mathbb{R}^{2l}}\mathrm{d}k\frac{k_{\mathrm{C}_{2b-2s}}k_{\mathrm{I}_{2a}}^{L^{2r}}}{\left(k^2+Q\right)^{n+m}}. \tag{3.148}$$

Roughly, the numerator under the integral sign is given by

$$\begin{aligned}
k_{\mathrm{C}_{2b-2s}}k_{\mathrm{I}_{2a}}^{L^{2r}} &= k_{\nu_1}\cdots k_{\nu_{2b-2s}}\{[k]^{2a-2r}[L]^{2r}\}_{\mu_1\cdots\mu_{2a}} \\
&= k_{\nu_1}\cdots k_{\nu_{2b-2s}}\underbrace{k_{\mu_1}\cdots k_{\mu_{2a-2r}}L_{\mu_{2a-2r}}\cdots L_{\mu_{2a}}}_{\text{plus } (2a)!/[(2r)!(2(a-r))!] \text{ permutations}}.
\end{aligned} \tag{3.149}$$

Let it be clear that the symmetrization in this level occurs only in the free indices. However, after integration we get a certain mixing between ν and μ indices, let us see it

$$k_{\mathrm{C}_{2b-2s}}k_{\mathrm{I}_{2a}}^{L^{2r}}\xrightarrow{\int}\overbrace{k_{\nu_1}\cdots k_{\nu_{2b-2s}}k_{\mu_1}\cdots k_{\mu_{2a-2r}}}^{g_{\nu_1\cdots\nu_{2b-2s}\mu_1\cdots\mu_{2a-2r}}}L_{\mu_{2a-2r+1}}\cdots L_{\mu_{2a}}+\text{perm. in the } \mu \text{ indices.} \tag{3.150}$$

Which means that after integration all monomials in the loop momentum will become fully symmetric tensors $g_{\mathrm{J}_{2a+2b-2r}}=g_{\nu_1\cdots\nu_{2b-2s}\mu_1\cdots\mu_{2a-2r}}$, but the vectors L will not receive contracted indices; nonetheless, their symmetry (in the free indices) will be intact. In total we have the scheme ()

$$k_{\mathrm{C}_{2b-2s}}k_{\mathrm{I}_{2a}}^{L^{2r}}\underset{\int}{\rightarrow}g_{\nu_1\cdots\nu_{2b-2s}(\mu_1\cdots\mu_{2a-2r}}L_{\mu_{2a-2r+1}}\cdots L_{\mu_{2a})}.^{5} \tag{3.151}$$

The next step is the contraction with the labels k_i. As the symmetry of results is unaltered through this process, the question is combinatorial in nature. Before resolving that issue, let us see one example to make clear what is going on.

Example 3.5.2 *For two contracted indices, two vectors L, and four free indices, we have in total six indices named by $\{\nu_1, \nu_2, \mu_1, \mu_2, \mu_3, \mu_4\}$. The $k_{\mathrm{C}_2}$and $k_{\mathrm{I}_{2a}}^{L^{2r}}$ factors read*

$$\begin{aligned} k_{\mathrm{I}_4}^{L^2} &= k_{\mu_{12}} L_{\mu_{34}} + k_{\mu_{13}} L_{\mu_{24}} + k_{\mu_{14}} L_{\mu_{23}} + k_{\mu_{23}} L_{\mu_{14}} + k_{\mu_{24}} L_{\mu_{13}} + k_{\mu_{34}} L_{\mu_{12}} && (3.152)\\ k_{\mathrm{C}_2} &= k_{\nu_{12}}, && (3.153)\end{aligned}$$

their product, therefore, is given by

$$k_{\mathrm{C}_2} k_{\mathrm{I}_4}^{L^2} = k_{\nu_{12}\mu_{12}} L_{\mu_{34}} + k_{\nu_{12}\mu_{13}} L_{\mu_{24}} + k_{\nu_{12}\mu_{14}} L_{\mu_{23}} + k_{\nu_{12}\mu_{23}} L_{\mu_{14}} + k_{\nu_{12}\mu_{24}} L_{\mu_{13}} + k_{\nu_{12}\mu_{34}} L_{\mu_{12}}. \quad (3.154)$$

Notice the intense merging of indices for identical vectors that we have done. Now, each factor of type $k_{\nu_{12}\mu_{12}}$ gets transformed into a sum of products of metric tensors, under integration!. Explicitly

$$\begin{aligned} k_{\mathrm{C}_2} k_{\mathrm{I}_4}^{L^2} \;\to\;& g_{\nu_{12}\mu_{12}} L_{\mu_{34}} + g_{\nu_{12}\mu_{13}} L_{\mu_{24}} + g_{\nu_{12}\mu_{14}} L_{\mu_{23}} + g_{\nu_{12}\mu_{23}} L_{\mu_{14}} + k_{\nu_{12}\mu_{24}} L_{\mu_{13}} + k_{\nu_{12}\mu_{34}} L_{\mu_{12}}\\ =\;& \left(g_{\nu_{12}} g_{\mu_{12}} + g_{\nu_1\mu_1} g_{\nu_2\mu_2} + g_{\nu_1\mu_2} g_{\nu_2\mu_1}\right) L_{\mu_{34}}\\ &+ \left(g_{\nu_{12}} g_{\mu_{13}} + g_{\nu_1\mu_1} g_{\nu_2\mu_3} + g_{\nu_1\mu_3} g_{\nu_2\mu_1}\right) L_{\mu_{24}}\\ &+ \left(g_{\nu_{12}} g_{\mu_{14}} + g_{\nu_1\mu_1} g_{\nu_2\mu_4} + g_{\nu_1\mu_4} g_{\nu_2\mu_1}\right) L_{\mu_{23}}\\ &+ \left(g_{\nu_{12}} g_{\mu_{23}} + g_{\nu_1\mu_2} g_{\nu_2\mu_3} + g_{\nu_1\mu_3} g_{\nu_2\mu_2}\right) L_{\mu_{14}}\\ &+ \left(g_{\nu_{12}} g_{\mu_{24}} + g_{\nu_1\mu_2} g_{\nu_2\mu_4} + g_{\nu_1\mu_4} g_{\nu_2\mu_2}\right) L_{\mu_{13}}\\ &+ \left(g_{\nu_{12}} g_{\mu_{34}} + g_{\nu_1\mu_3} g_{\nu_2\mu_4} + g_{\nu_1\mu_4} g_{\nu_2\mu_3}\right) L_{\mu_{12}}, \quad (3.155)\end{aligned}$$

observe there is no degeneracy here, there is no two factors that sum to produce a numerical coefficient and reduce the expression. Moreover, the vector L is a spectator and do not assume ν indices. The symmetry in the indices $\{\mu_1, \mu_2, \mu_3, \mu_4\}$ remained untouched. However, for each fixed $L_{\mu_{ij}}$ the tensor coefficient is fully symmetric in a mixed set of ν and μ indices.

Continuing, we must return to the eq. (3.148). Our considerations about this transition serves the purpose to introduce a notation and a result, i.e., the piece of the integral we are exploiting until now is given by

$$\begin{aligned} & k_i^{\mathrm{C}_{2b-2s}} \int_{\mathbb{R}^{2l}} \mathrm{d}k \frac{k_{\mathrm{C}_{2b-2s}} k_{\mathrm{I}_{2a}}^{L^{2r}}}{\left(k^2+Q\right)^{n+m}} \quad (3.156)\\ =\;& \frac{1}{\Gamma\left(n+m\right)} \frac{1}{2^{b+a-s-r}} \mathcal{C}_i^{2b-2s}(\mathsf{k}_{\mathrm{I}_{2a}}^{L^{2r}}) \left(-1\right)^{r+s+m+n-a-b-l-1} \Xi^{(-(r+s+m+n-a-b-l))}. \end{aligned}$$

Some clarifications are naturally necessary. First, the tensor power of the integration variable k is hidden in $k_{\mathrm{C}_{2b-2s}} k_{\mathrm{I}_{2a}}^{L^{2r}}$, which is $2\left(a+b-r-s\right)$; second, we just used the definition of kernels (3.95). This type of identification follows from the formula in eq.(3.93), it was already used in the discussion of finite integrals, e.g. see eq.(3.140).

Third, and most importantly, the meaning of $\mathcal{C}_i^{2b-2s}(\mathsf{k}_{\mathrm{I}_{2a}}^{L^{2r}})$. In finite integrals the integrand tensor $k_{\mathrm{I}_{2a}}^{L^{2r}}$ simply loses its loop-momentum parts (which become metric tensors), with the net result that it gets transformed into $\mathsf{k}_{\mathrm{I}_{2a}}^{L^{2r}}$ (with Serif font). For the present situation we get an additional task to perform; this additional task that is to contract with the $2b-2s$ routings k_i awaiting the aftermath of integration is represented by $\mathcal{C}_i^{2b-2s}(\mathsf{k}_{\mathrm{I}_{2a}}^{L^{2r}})$. Let us write its formula before discussing its derivation,

$$\mathcal{C}_i^{2(b-s)}(\mathsf{k}_{\mathrm{I}_{2a}}^{J^{2r}}) = (2\,(b-s)-1)!! \sum_{j=0}^{\min(a-r,b-s)} \frac{2^j\,(b-s)!}{(b-s-j)!} \left(k_i^2\right)^{b-s-j} \mathsf{k}_{\mathrm{I}_{2a}}^{i^{2j},L^{2r}}. \tag{3.157}$$

To express it, we have to consider the combinatorial problem of the following contraction

$$k_i^{\nu_1} \cdots k_i^{\nu_{2b-2s}} g_{\nu_1 \cdots \nu_{2b-2s} \mu_1 \cdots \mu_{2a-2r}}. \tag{3.158}$$

To this aim we have dropped the factor $L_{\mu_{2a-2r+1}} \cdots L_{\mu_{2a}}$ and the symmetrization in μ, whose role is to ensure the appearance of the symmetric tensor $\mathsf{k}_{\mathrm{I}_{2a}}^{i^{2j},L^{2r}}$. Then, some terms of a particular organization of the tensor $g_{\nu_1 \cdots \nu_{2b-2s} \mu_1 \cdots \mu_{2a-2r}}$ are presented:

$$\begin{aligned} g_{\nu_1 \cdots \nu_{2b-2s} \mu_1 \cdots \mu_{2a-2r}} &= g_{\nu_1 \cdots \nu_{2b-2s}} g_{\mu_1 \cdots \mu_{2a-2r}} \\ &+ g_{\nu_3 \cdots \nu_{2b-2s}} \left(g_{\nu_1\mu_1} g_{\nu_2\mu_2} + g_{\nu_1\mu_2} g_{\nu_2\mu_1}\right) g_{\mu_3 \cdots \mu_{2a-2r}} + \cdots \\ &+ g_{\nu_5 \cdots \nu_{2b-2s}} \left(g_{\nu_1\mu_1} g_{\nu_2\mu_2} g_{\nu_3\mu_3} g_{\nu_4\mu_4} + \cdots\right) g_{\mu_5 \cdots \mu_{2a-2r}} + \cdots \end{aligned} \tag{3.159}$$

In this highly schematic representation[6], we must know how many ways to select ν-type indices from the set $\{\nu_1, \cdots, \nu_{2b-2s}\}$ with all of them in the metrics $g_{\nu_1 \cdots \nu_{2b-2s}}$, for which there only one

---

[6] Suppose there is $2n$ contracted indices and $2m$ free indices, $n > m$, and in oposite case we exchange their roles. The tensor $g_{\nu_1 \cdots \nu_{2n} \mu_1 \cdots \mu_{2m}} = g_{\mathrm{C}_{2n} \mathrm{I}_{2m}}$ has $(2n+2m-1)!!$ monomials, we can dissect its terms as the schematic equation present in the text. Considering permutation is each group we always have full symmetry,

$$g_{\nu_1 \cdots \nu_{2n} \mu_1 \mu_2} = g_{\nu_1 \cdots \nu_{2n}} g_{\mu_1 \mu_2} + \underbrace{g_{\widehat{\nu_{12}} \nu_3 \cdots \nu_{2n}} \left(g_{\nu_1\mu_1} g_{\nu_2\mu_2} + g_{\nu_1\mu_2} g_{\nu_2\mu_1}\right) + g_{\nu_{12} \widehat{\nu_{34}} \cdots \nu_{2n}} \left(g_{\nu_3\mu_1} g_{\nu_4\mu_2} + g_{\nu_3\mu_2} g_{\nu_4\mu_1}\right) + \cdots}_{n(2n-1)\text{terms}}$$

all monomials are distinct in the expansion above, considering the total number we have $(2n-1)!! + n\,(2n-1)\,(2n-3)!!2 = (2n+1)!!$, which is the number in $g_{\nu_1 \cdots \nu_{2n} \mu_1 \mu_2}$. If one exchanges indices that dismantle the organization one gets the same tensor, nothing is affected. The decomposition is just to deduce the result of contraction with k_i.

As the decomposition do not reapeat terms, the task is to verify the number of term is the same. The number of terms fully symmetric in μ (some mixed with ν) in the decomposition of $g_{\nu_1 \cdots \nu_{2n} \mu_1 \cdots \mu_{2m}}$ is $(2m)!/2^{m-j}\,(m-j)!$ and we must have the validity of the equation below

$$(2n+2m-1)!! = \sum_{j=0}^{m} \binom{2n}{2j} (2n-2j-1)!! \frac{(2m)!}{2^{m-j}\,(m-j)!}.$$

To prove it, let us work the binomials and double factorial to write

$$\left(\prod_{j=0}^{m-1} (2n+2m-2j-1) \right) (2n-1)!! = (2n-1)!! \sum_{j=0}^{m} \frac{n!}{(n-j)!} \left[\frac{2^{2j-m}\,(2m)!}{(2j)!\,(m-j)!} \right].$$

Then cancelling the factor $(2n-1)!!$ and reversing the index of summation in the productorium ($j \to m-1-j$), one has to prove

$$\left(\prod_{j=0}^{m-1} (2n+2j+1) \right) = \sum_{j=0}^{m} \frac{n!}{(n-j)!} \left[\frac{2^{2j-m}\,(2m)!}{(2j)!\,(m-j)!} \right],$$

that is easy job to do by mathematical induction in m.

possibility. Then how many ways to select $2b-2s-2$ indices occupying the metrics without mixing with a μ one and so on. Therefore, the number of possibilities to adjust $2\left(b-s\right)-2j$ indices, all in contracted mode, is

$$\binom{2\left(b-s\right)}{2j}. \tag{3.160}$$

They will produce a scalar power of k_i given by $\left(k_i^2\right)^{b-s-j}$, and $2j$ denotes the number of indices combined with μ indices. However, there is a degeneracy since in each of the $\binom{2(b-s)}{2j}$ modes of grouping ν-type indices, the corresponding tensor $g_{\mathrm{C}_{2b-2s-2j}}$ has $\left(2\left(b-s-j\right)-1\right)!!$ terms that produce the same result, e.g.,

$$k_i^{\nu_1}k_i^{\nu_2}k_i^{\nu_3}\cdots k_i^{\nu_{2b-2s}}g_{\nu_3\cdots\nu_{2b-2s}}=\left(2\left(b-s-1\right)-1\right)!!\left[k_i^{\nu_1}k_i^{\nu_2}\right]\left[\left(k_1^2\right)^{b-s-1}\right]. \tag{3.161}$$

The last factor will come from the number of ways the remaining indices (in $k_i^{\nu_1}k_i^{\nu_2}$) can acquire μ indices. Let us see this progressing from the last equation, now with $g_{\mu_3\cdots\mu_{2a-2r}}$ included, that is

$$\begin{aligned}
&k_i^{\nu_1}k_i^{\nu_2}k_i^{\nu_3}\cdots k_i^{\nu_{2b-2s}}g_{\nu_3\cdots\nu_{2b-2s}}\left(g_{\nu_1\mu_1}g_{\nu_2\mu_2}+g_{\nu_1\mu_2}g_{\nu_2\mu_1}\right)g_{\mu_3\cdots\mu_{2a-2r}} \\
=\;&\left(2\left(b-s-1\right)-1\right)!!\left[k_i^{\nu_1}k_i^{\nu_2}\right]\left(g_{\nu_1\mu_1}g_{\nu_2\mu_2}+g_{\nu_1\mu_2}g_{\nu_2\mu_1}\right)\left[\left(k_1^2\right)^{b-s-1}\right]g_{\mu_3\cdots\mu_{2a-2r}} \\
=\;&\left(2\left(b-s-1\right)-1\right)!!\left(2k_{i\mu_1}k_{i\mu_2}\right)\left[\left(k_1^2\right)^{b-s-1}\right]g_{\mu_3\cdots\mu_{2a-2r}}.
\end{aligned} \tag{3.162}$$

We got $2!k_{i\mu_1}k_{i\mu_2}$ and for $2j$ contracted indices, selected to be mixed with μ indices, there will be $(2j)!$ identical results. The symmetrization in μ at this level guarantees that $k_{i\mu_1}k_{i\mu_2}$ will get all the indices from $\{\mu_1,\cdots,\mu_{2a-2r}\}$. What about the remaining indices $\{\mu_{2a-2r+1},\cdots,\mu_{2r}\}$? They also will be symmetrized with the indices in $L_{\mu_{2a-2r+1}}\cdots L_{\mu_{2a}}$, which results in a tensor fully symmetric in $\{g,L,k_i\}$. For our working example, gathering all factors, we have

$$\left[\binom{2\left(b-s\right)}{2}\left(2\left(b-s-1\right)-1\right)!!\left(2!\right)\right]\left(k_1^2\right)^{b-s-1}\mathsf{k}_{\mathrm{I}_{2a}}^{k_i^2,L^{2r}}, \tag{3.163}$$

the k_i vector is suppressed in $k_{\mathrm{I}_{2a}}^{k_i^2,L^{2r}}\equiv k_{\mathrm{I}_{2a}}^{i^2,L^{2r}}$, only the index is kept.

Summarizing, we have three degeneracy factors: partition of the contracted index set, number of ways to form the corresponding power of k_i^2, and the number of ways to fit in the label (k_i^μ) in the super tensor. That means we have the total factor

$$\binom{2\left(b-s\right)}{2j}\left(2\left(b-s-j\right)-1\right)!!\left(2j\right)!=\left(2b-2s-1\right)!!\frac{2^j\left(b-s\right)!}{\left(b-s-j\right)!}, \tag{3.164}$$

which addresses the numerical coefficient in the eq. (3.157) for $\mathcal{C}_i^{2(b-s)}(\mathsf{k}_{\mathrm{I}_{2a}}^{J^{2r}})$.

The upper limit ($\min\left(a-r,b-s\right)$) in that formula stems from two facts: if $a-r<b-s$, then there will be no space in $\mathsf{k}_{\mathrm{I}_{2a}}^{i^{2j},L^{2r}}$ to accommodate more tensor powers of k_i since the maximum number is $2a-2r$, occurring when there is no metric present in the tensor. But if $2a$ is large enough to accommodate $2\left(b-s\right)$ tensor powers of k_i and $2r$ of L, i.e. $a-r>b-s$, the summation stops in $b-s$, since there is no more k_i to "upload" in $\mathsf{k}_{\mathrm{I}_{2a}}^{i^{2j},L^{2r}}$.

Recapitulating: we have eq. (3.145) for $F^{\mathrm{I}_a}(b,M,N)$ that is adapted to $F^{\mathrm{I}_{2a}}(2b,m,n)$, the collection of the even terms in the loop-momentum occurred in eq. (3.146). The analysis of a piece of the integral started in eq. (3.148), in the next eq. (3.156) we reached the transition from the integrand $k_i^{\mathrm{C}_{2b-2s}} k_{\mathrm{C}_{2b-2s}} k_{\mathrm{I}_{2a}}^{L^{2r}}$ to the operation of contracting with routings denoted by $\mathcal{C}_i^{2(b-s)}(\mathsf{k}_{\mathrm{I}_{2a}}^{J^{2r}})$[7], which we have guided the derivation.

The complete result for this type of integrals finally follows.

- **Even-Even (Numerator $k_{\mathrm{I}_{2a}} A_i^{2b}$)**

First, the mass dimension and power counting (dimension $d = 2l$)

$$\begin{aligned} \dim_{mass}\left[F^{\mathrm{I}_{2a}}(2b,m,n)\right] &= 2(a+2b+l-m-n) = 2c \\ \omega\left[F^{\mathrm{I}_{2a}}(2b,m,n)\right] &= 2(c-b)\,. \end{aligned} \tag{3.165}$$

The organized result reads

$$\begin{aligned} F^{\mathrm{I}_{2a}}(2b,m,n) &= \frac{-(-1)^{b-c}}{\Gamma(n)} \int_\Delta \mathrm{d}^m x \mathcal{X}^{n-1} \times \\ &+ \sum_{s=0}^{b}\sum_{r=0}^{a} (-1)^{r+s} \frac{2^{b-s}}{2^{a-r}} \binom{2b}{2s} \mathcal{C}_i^{2b-2s}(\mathsf{k}_{\mathrm{I}_{2a}}^{L^{2r}}) \mathcal{D}_i^{2s} \Xi^{-(r+s+b-c)} \\ &+ \sum_{s=1}^{b}\sum_{r=1}^{a} (-1)^{r+s} \frac{2^{b-s}}{2^{a-r}} \binom{2b}{2s-1} \mathcal{C}_i^{2b-2s+1}(\mathsf{k}_{\mathrm{I}_{2a}}^{L^{2r-1}}) \mathcal{D}_i^{2s-1} \Xi^{-(r+s+b-c-1)} \end{aligned} \tag{3.166}$$

- **Odd-Odd (Numerator $k_{\mathrm{I}_{2a+1}} A_i^{2b+1}$)**

The parameters

$$\begin{aligned} \dim_{mass}\left[F^{\mathrm{I}_{2a+1}}(2b+1,m,n)\right] &= 2(a+2b+1+l-m-n)+1 = 2c+1 \\ \omega\left[F^{\mathrm{I}_{2a+1}}(2b+1,m,n)\right] &= 2(c-b) \end{aligned}$$

The integral

$$\begin{aligned} F^{\mathrm{I}_{2a+1}}(2b+1,m,n) &= \frac{-(-1)^{b-c}}{\Gamma(n)} \int_\Delta \mathrm{d}^m x \mathcal{X}^{n-1} \\ &+ \sum_{s=0}^{b}\sum_{r=0}^{a} (-1)^{r+s} \frac{2^{b-s}}{2^{a-r}} \binom{2b+1}{2s} \mathcal{C}_i^{2b-2s+1} \left(\mathsf{k}_{\mathrm{I}_{2a+1}}^{L^{2r}}\right) \mathcal{D}_i^{2s} \Xi^{-(r+s+b-c)} \\ &+ \sum_{s=0}^{b}\sum_{r=0}^{a} (-1)^{r+s} \frac{2^{b-s}}{2^{a-r}} \binom{2b+1}{2s+1} \mathcal{C}_i^{2b-2s} \left(\mathsf{k}_{\mathrm{I}_{2a+1}}^{L^{2r+1}}\right) \mathcal{D}_i^{2s+1} \Xi^{-(r+s+b-c+1)} \end{aligned} \tag{3.167}$$

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[7] If you are more comfortable with the G. Passarino notation the expression may be reads as

$$\mathcal{C}_i^{2t}(\mathsf{k}_{\mathrm{I}_{2a}}^{J^{2r}}) = (2t-1)!! \sum_{j=0}^{\min(a-r,t)} \frac{2^j t!}{(t-j)!} \left(k_i^2\right)^{t-j} \left\{[k_i]^{2j}\,[L]^{2r}\,[g]^{a-j-r}\right\}_{\mu_1\cdots\mu_{2a}}$$

- **Odd-Even (Numerator $k_{\mathrm{I}_{2a+1}} A_i^{2b}$)**

The parameters

$$\begin{aligned} \mathrm{dim}_{mass}\left[F^{\mathrm{I}_{2a+1}}\left(2b,m,n\right)\right] &= 2\left(a+2b+l-n-m\right)+1=2c+1 \\ \omega\left[F^{\mathrm{I}_{2a+1}}\left(2b,m,n\right)\right] &= 2\left(c-b\right)+1 \end{aligned}$$

The organized result

$$\begin{aligned} F^{\mathrm{I}_{2a+1}}\left(2b,m,n\right) &= \frac{\left(-1\right)^{b-c}}{\Gamma\left(n\right)}\int \mathrm{d}^m x\mathcal{X}^{n-1}\times \\ &+\sum_{s=0}^{b}\sum_{r=0}^{a}\left(-1\right)^{r+s}\binom{2b}{2s}\frac{2^{b-s}}{2^{a-r}}\left[\mathcal{C}_i^{2b-2s}\left(\mathsf{k}_{\mathrm{I}_{2a+1}}^{L^{2r+1}}\right)\right]\mathcal{D}_i^{2s}\Xi^{-(r+s+b-c)} \\ &+\sum_{s=1}^{b}\sum_{r=0}^{a}\left(-1\right)^{r+s}\binom{2b}{2s-1}\frac{2^{b-s}}{2^{a-r}}\left[\mathcal{C}_i^{2(b-s)+1}\left(\mathsf{k}_{\mathrm{I}_{2a+1}}^{L^{2r}}\right)\right]\mathcal{D}_i^{2s-1}\Xi^{-(r+s+b-c-1)} \end{aligned} \tag{3.168}$$

- **Even-Odd (Numerator $k_{\mathrm{I}_{2a}} A_i^{2b+1}$)**

The parameters

$$\begin{aligned} \mathrm{dim}_{mass}\left[F^{\mathrm{I}_{2a}}\left(2b+1,m,n\right)\right] &= 2\left(a+2b+1-m-n+l\right)=2c \\ \omega\left[F^{\mathrm{I}_{2a}}\left(2b+1,m,n\right)\right] &= 2\left(c-b\right)-1 \end{aligned}$$

The integral

$$\begin{aligned} F^{\mathrm{I}_{2a}}\left(2b+1,m,n\right) &= \frac{\left(-1\right)^{b-c}}{\Gamma\left(n\right)}\int_{\Delta} \mathrm{d}^m x\mathcal{X}^{n-1}\times \\ &+\sum_{s=0}^{b}\sum_{r=1}^{a}\left[\left(-1\right)^{r+s}\frac{2^{b-a}}{2^{s-r}}\binom{2b+1}{2s}\right]\left[\mathcal{C}_i^{2b-2s+1}\left(\mathsf{k}_{\mathrm{I}_{2a}}^{L^{2r-1}}\right)\right]\mathcal{D}_i^{2s}\Xi^{-(r+s+b-c)} \\ &+\sum_{s=0}^{b}\sum_{r=0}^{a}\left[\left(-1\right)^{r+s}\frac{2^{b-a}}{2^{s-r}}\binom{2b+1}{2s+1}\right]\left[\mathcal{C}_i^{2b-2s}\left(\mathsf{k}_{\mathrm{I}_{2a}}^{L^{2r}}\right)\right]\mathcal{D}_i^{2s+1}\Xi^{-(r+s+b-c+1)} \end{aligned} \tag{3.169}$$

Owing to this systematization, the user just needs to reduce the derivatives $\mathcal{D}_i$ from the expressions. Once this is done one has finite parts of any Feynman integral. Together with the non integrable part that was left intact and projected in integral representations for surface terms and scalar objects (3.51,3.52), virtually[8] any one-loop problem can be handled by IReg. The heightened focus on these facts is rooted in the lack of detailed instructions on operating with IReg in the literature. Our exposition still does not fit all the needs. However, the essential ingredients, besides the simple idea of isolating routings and masses in finite integrals, to operate with IReg can be extracted from our derivations. The next section applies this toolkit to an integral relevant to our aims.

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[8] Virtually because we could not expand more about surface terms with logarithms in the integrand, which in fact arise in some gravitational-anomalies/effective models.

3.5.3 The $d = 2n$ Tensor Integral $\bar{J}_{n+1}^{(2n)\mu_{12}}$

The choice for exemplify the computation of this integral is the importance it has in the discussion of anomalies. Furthermore, in arbitrary even dimension it owns enough features, in the most simple form, to worth some detailed derivation.

The definition is

$$\bar{J}_{n+1}^{(2n)\mu_{12}} = \int_{\mathbb{R}^{2n}} \mathrm{d}k \left[\bar{j}_{n+1}^{(2n)\mu_{12}} \right], \quad \bar{j}_{n+1}^{(2n)\mu_{12}} = \frac{K_1^{\mu_{12}}}{D_{[1,n+1]}}. \tag{3.170}$$

It is a $(n+1)$-pt integral of tensor rank two from which we may adopt $\mathrm{I}_2 = \mu_{12}$, thus the power counting is $\omega = 2n + 2 - 2(n+1) = 0$. We will drop the superscript for dimension.

The next step is the isolation of routings into finite integrals. Then we exercise the separation: If one chooses to separate $A_{n+1}, A_n, \cdots A_1$ in the respective sequence, there follows the first iteration

$$\bar{j}_{n+1}^{\mathrm{I}_2} = \frac{K_1^{\mathrm{I}_2}}{D_{[1,n+1]}} = \frac{K_1^{\mathrm{I}_2}}{D_{[1,n]}} \left(\frac{1}{D_\lambda} - \frac{A_{n+1}}{D_\lambda D_{n+1}} \right). \tag{3.171}$$

The second integral is finite, whereas the first maintains $\omega = 0$; thus we separate the next propagator—by an identity—and obtain

$$\bar{j}_{n+1}^{\mathrm{I}_2} = \frac{K_1^{\mathrm{I}_2}}{D_\lambda^2 D_{[1,n-1]}} - \frac{K_1^{\mathrm{I}_2} A_n}{D_\lambda^2 D_{[1,n]}} - \frac{K_1^{\mathrm{I}_2} A_{n+1}}{D_\lambda D_{[1,n+1]}}. \tag{3.172}$$

So, the result where there is no more routing in a $\omega = 0$ integral, follows as

$$\bar{j}_{n+1}^{\mathrm{I}_2} = \frac{K_1^{\mathrm{I}_2}}{D_\lambda^{n+1}} - \sum_{i=1}^{n+1} \frac{K_1^{\mathrm{I}_2} A_i}{D_\lambda^{n+2-i} D_{[1,i]}}. \tag{3.173}$$

As we are going to parametrize a sequence of more and more propagators, $D_{[1,i]}$, we will have the L vector of the last sections indexed such that

$$L_i = \sum_{j=1}^{i} k_j x_j = k_1 x_1 + \cdots + k_i x_i, \tag{3.174}$$

in ascending order. The important thing is the parameter's index that links the collection of integrals. We make the convention that the power i^r in $\mathsf{k}_{\mathrm{I}_a}^{i^r, L_i^s}$ represents the tensor power of k_i, when in the ascending order for L_i, explicitly

$$\mathsf{k}_{\mathrm{I}_a}^{i^r, L_i^s} = \left\{ [k_i]^r [L_i]^s [g]^{(a-r-s)/2} \right\}_{\mathrm{I}_a}. \tag{3.175}$$

Now, each time we evaluate the last parameter in zero $x_i = 0$ we transform L_i into L_{i-1}, or $L_i(0) = L_{i-1}$.The only situation where we will have non-zero results by evaluating in the upper limit $\mathcal{X}_i$ will be for $i = n+1$, in $x_{n+1} = \mathcal{X}_n = 1 - x_1 - \cdots x_n$.

$$L_{n+1}(\mathcal{X}_n) = \sum_{j=1}^{n} k_j x_j + k_{n+1} \left(1 - \sum_{j=1}^{n} x_j \right) = \sum_{j=1}^{n} (p_{j,n+1}) x_j + k_{n+1}. \tag{3.176}$$

The same occur for the Q polynomial, for ascending order $i \in [1, n+1]$ we identify

$$\begin{aligned} Q_i\left(x_i=0\right) &= Q_{i-1}\left(\left\{k_j\right\}, \left\{x_j\right\}; \lambda^2\right), & (3.177) \\ Q_i\left(x_i=\mathcal{X}_{i-1}\right) &= Q_{i-1}\left(\left\{p_{j,i}\right\}, \left\{x_i\right\}; m_i^2\right). & (3.178) \end{aligned}$$

In what follows we suppress subscripts for $Q_i = Q$.

Let us start the computation by the numerator. Under shift used to eliminate crossed terms in the denominator, the factors K_1^{μ} in the numerator will become

$$K_1 = k + k_1 \to k - \left(L_i - k_1\right) = \left(k - \tilde{L}_i\right).$$

We are choosing to instead of integrate a pure-loop momentum factor $k^{\mathrm{I}_{2a}}$, integrate a combination with a reference routing $K_1^{\mathrm{I}_{2a}}$. That leads more directly to the result of the integral. The form of the integrals from the previous section does not change, in the tensor part instead of L_i we put $\tilde{L}_i = L_i - k_1$, not even the reference routing is important.

Returning to (3.173), in integral form it reads

$$\bar{J}_{n+1}^{\mathrm{I}_2} = \int_{\mathbb{R}^{2n}} \mathrm{d}k \frac{K_1^{\mathrm{I}_2}}{D_\lambda^{n+1}} - \sum_{i=1}^{n+1} \int_{\mathbb{R}^{2n}} \mathrm{d}k \frac{K_1^{\mathrm{I}_2} A_i}{D_\lambda^{n+2-i} D_{[1,i]}}. \tag{3.179}$$

The first integral has a finite and a divergent part, one which we organize by our definition of surface terms and scalar objects (3.66,3.67). Explicitly

$$\begin{aligned} \int_{\mathbb{R}^{2n}} \mathrm{d}k \frac{K_1^{\mu_{12}}}{D_\lambda^{n+1}} &= \int_{\mathbb{R}^{2n}} \mathrm{d}k \frac{k^{\mu_{12}}}{D_\lambda^{n+1}} + k_1^{\mu_{12}} \int_{\mathbb{R}^{2n}} \mathrm{d}k \frac{1}{D_\lambda^{n+1}} & (3.180) \\ &= \frac{1}{2n}\left(\Delta_{n+1}^{\mu_{12}} + g^{\mu_{12}} I_{\log}\right) + \frac{k_1^{\mu_1} k_1^{\mu_2}}{\Gamma\left(n+1\right)\left(-\lambda^2\right)}, \end{aligned}$$

where the term $k_1^{\mu_1} k_1^{\mu_2} = \mathsf{k}_{\mathrm{I}_2}^{11}$ can be written in various forms. Once more, in a simple integral like this, low power counting and low rank, it may be confusing and inadequate to write like this. However, as the complexity of computation grows they start to be very effective.

Right! Now, we call the general form of the integral with $k_{\mathrm{I}_{2a}} A_i^{2b+1}$ as numerator, adapted to our scenario, with the parameters

$$\begin{aligned} \dim_{mass}\left[F^{\mathrm{I}_2}\left(1, i, n+2-i\right)\right] &= 0, & (3.181) \\ \omega\left[F^{\mathrm{I}_2}\left(1, i, n+2-i\right)\right] &= -1. & (3.182) \end{aligned}$$

The integrals, where we must extract the derivative $\mathcal{D}_i$, are given by

$$\begin{aligned} & F^{\mathrm{I}_2}\left(1, i, n+2-i\right) & (3.183) \\ = \; & \frac{1}{\Gamma\left(n+2-1\right)} \int_\Delta \mathrm{d}^i x \mathcal{X}_i^{n+1-i} \left\{ -\mathcal{C}_i^1(\mathsf{k}_{\mathrm{I}_2}^{\tilde{L}_i}) \Xi^{(-1)} + \frac{1}{2} g_{\mathrm{I}_2} \mathcal{D}_i \Xi^{(-1)} - \mathsf{k}_{\mathrm{I}_2}^{\tilde{L}_i^2} \mathcal{D}_i \Xi^{(-2)} \right\}. \end{aligned}$$

The only contraction term is $\mathcal{C}_i^1(\mathsf{k}_{\mathrm{I}_2}^{\tilde{L}_i}) = \mathsf{k}_{\mathrm{I}_2}^{i,\tilde{L}_i}$, because for one contraction the routing is just inserted in the tensor $\mathsf{k}_{\mathrm{I}_2}^{\tilde{L}_i}$. Hence, the integral can be worked out in the form

$$\begin{aligned} & F^{\mathrm{I}_2}\left(1, i, n+2-i\right) & (3.184) \\ = \; & \frac{1}{\Gamma\left(n+2-1\right)} \int_\Delta \mathrm{d}^i x \mathcal{X}_i^{n+1-i} \left\{ -\mathsf{k}_{\mathrm{I}_2}^{i,\tilde{L}_i} \Xi^{(-1)} + \frac{1}{2} g_{\mathrm{I}_2} \mathcal{D}_i \Xi^{(-1)} - \mathsf{k}_{\mathrm{I}_2}^{\tilde{L}_i^2} \mathcal{D}_i \Xi^{(-2)} \right\}. \end{aligned}$$

In more complex[9] cases you may keep the contraction in the way they appear for longer, because the partial integration will produce a series of terms which starts to match exactly the contractions, and it is only necessary to have a table of results to compare and cancels terms. One more comment, we have put an index in $\mathcal{X}_i$ to indicate the number of parameter being summed.

Therefore, at this point we have

$$\bar{J}^{\mathrm{I}_2}_{n+1} = \int_{\mathbb{R}^{2n}} \mathrm{d}k \frac{K^{\mathrm{I}_2}_1}{D^{n+1}_\lambda} - \sum_{i=1}^{n+1} F^{\mathrm{I}_2}\left(1, i, n+2-i\right). \tag{3.185}$$

To free us from $\mathcal{D}_i$ we start from the highest power, unit in this case, and the most negative index in Ξ, that is $\Xi^{(-2)}$. Then use $\mathcal{D}_i \Xi^{(-2)} = \partial_i \Xi^{(-1)}$ in the integrand of $F^{\mathrm{I}_2}\left(1, i, n+2-i\right)$, and "partial integrate" it. That means, construct the result

$$\begin{aligned} \mathcal{X}_i^{n+1-i} \mathsf{k}_{\mathrm{I}_2}^{\tilde{L}_i^2} \mathcal{D}_i \Xi^{(-2)} &= \mathcal{X}_i^{n+1-i} \mathsf{k}_{\mathrm{I}_2}^{\tilde{L}_i^2} \partial_i \Xi^{(-1)} \\ &= \left[\left(n+1-i\right) \mathcal{X}_i^{n-i} \mathsf{k}_{\mathrm{I}_2}^{\tilde{L}_i^2} - \mathcal{X}_i^{n+1-i} \mathsf{k}_{\mathrm{I}_2}^{i, \tilde{L}_i}\right] \Xi^{(-1)} \\ &\quad + \partial_i \left[\mathcal{X}_i^{n+1-i} \mathsf{k}_{\mathrm{I}_2}^{\tilde{L}_i^2} \Xi^{(-1)}\right], \end{aligned} \tag{3.186}$$

where we employed two basic results

$$\partial_i \mathcal{X}_i^{n+1-i} = -\left(n+1-i\right) \mathcal{X}_i^{n-i}, \quad \partial_i \mathsf{k}_{\mathrm{I}_a}^{i^r, \tilde{L}_i^s} = \left(r+1\right) \mathsf{k}_{\mathrm{I}_a}^{i^{r+1}, \tilde{L}_i^{s-1}}. \tag{3.187}$$

Gathering all of this in the expression for $F^{\mathrm{I}_2}\left(1, i, n+2-i\right)$, cancelling some terms, we must get

$$\begin{aligned} &\Gamma\left(n+2-i\right) F^{\mathrm{I}_2}\left(1, i, n+2-i\right) \\ = &\int_\Delta \mathrm{d}^i x \left\{-\left(n+1-i\right) \mathcal{X}_i^{n-i} \mathsf{k}_{\mathrm{I}_2}^{\tilde{L}_i^2} \Xi^{(-1)} + \frac{1}{2} \mathcal{X}_i^{n+1-i} g_{\mathrm{I}_2} \mathcal{D}_i \Xi^{(-1)}\right\} \\ &- \int_\Delta \mathrm{d}^i x \partial_i \left[\mathcal{X}_i^{n+1-i} \mathsf{k}_{\mathrm{I}_2}^{\tilde{L}_i^2} \Xi^{(-1)}\right]. \end{aligned} \tag{3.188}$$

We use $\mathcal{D}_i \Xi^{(-1)} = \partial_i \Xi^{(0)}$ in the second term of the r.h.s., plus product rule, to obtain

$$\begin{aligned} \mathcal{X}_i^{n+1-i} g_{\mathrm{I}_2} \mathcal{D}_i \Xi^{(-1)} &= \mathcal{X}_i^{n+1-i} g_{\mathrm{I}_2} \partial_i \Xi^{(0)} \\ &= \left(n+1-i\right) \mathcal{X}_i^{n-i} g_{\mathrm{I}_2} \Xi^{(0)} + \partial_i \left[\mathcal{X}_i^{n+1-i} g_{\mathrm{I}_2} \Xi^{(0)}\right]. \end{aligned} \tag{3.189}$$

For $i = n+1$ the power in $\mathcal{X}_i^{n+1-i}$ is zero and there is no parameter dependence, hence the formula is simpler,

$$g_{\mathrm{I}_2} \mathcal{D}_{n+1} \Xi^{(-1)} = g_{\mathrm{I}_2} \partial_i \Xi^{(0)} = \partial_i \left[g_{\mathrm{I}_2} \Xi^{(0)}\right]. \tag{3.190}$$

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[9] Writing everything in a more traditional form we would have something like

$$\begin{aligned} &\Gamma\left(n+2-1\right) F_{\mu_{12}}\left(1, i, n+2-i\right) \\ = &\int_\Delta \mathrm{d}^i x \left(1 - \textstyle\sum_{r=1}^i x_r\right)^{n+1-i} \left\{-\left(k_{i,\mu_1} \tilde{L}_{i,\mu_2} + k_{i,\mu_2} \tilde{L}_{i,\mu_1}\right) \frac{1}{Q} + \frac{1}{2} g_{\mu_{12}} \frac{\partial Q}{\partial x_i} \frac{1}{Q} - \tilde{L}_{i,\mu_1} \tilde{L}_{i,\mu_2} \frac{\partial Q}{\partial x_i} \frac{1}{Q^2}\right\}. \end{aligned}$$

Which is good, however it is for higher power couting and tensor rank the compaction really starts to help.

Observe $g_{\mathrm{I}_2} = \mathsf{k}_{\mathrm{I}_2}$ is a constant for the parametric derivative ∂_i. Therefore, we have eliminated the derivative $\mathcal{D}_i = (\partial_i Q)$; the organized result comes as follows

$$\begin{aligned} F^{\mathrm{I}_2}(1,i,n+2-i) &= \frac{(n+1-i)}{\Gamma(n+2-i)} \int_\Delta \mathrm{d}^i x \mathcal{X}_i^{n-i} \left[\frac{1}{2}\mathsf{k}_{\mathrm{I}_2}\Xi^{(0)} - \mathsf{k}_{\mathrm{I}_2}^{\tilde{L}_i^2}\Xi^{(-1)}\right] \\ &\quad + \frac{1}{\Gamma(n+2-i)} \int_\Delta \mathrm{d}^i x \partial_i \left\{\mathcal{X}_i^{n+1-i}\left[\frac{1}{2}\mathsf{k}_{\mathrm{I}_2}\Xi^{(0)} - \mathsf{k}_{\mathrm{I}_2}^{\tilde{L}_i^2}\Xi^{(-1)}\right]\right\}. \end{aligned} \tag{3.191}$$

Now we evaluate the integration limits in the second line, and the result is

$$\begin{aligned} &\int_\Delta \mathrm{d}^{i-1}x \int_0^{\mathcal{X}_{i-1}} \mathrm{d}x_i \partial_i \left\{\mathcal{X}_i^{n+1-i}\left[\frac{1}{2}\mathsf{k}_{\mathrm{I}_2}\Xi^{(0)} - \mathsf{k}_{\mathrm{I}_2}^{\tilde{L}_i^2}\Xi^{(-1)}\right]\right\} \\ &= -\int_\Delta \mathrm{d}^{i-1}x \left\{\mathcal{X}_{i-1}^{n+1-i}\left[\frac{1}{2}\mathsf{k}_{\mathrm{I}_2}\Xi^{(0)} - \mathsf{k}_{\mathrm{I}_2}^{\tilde{L}_{i-1}^2}\Xi^{(-1)}\right]\right\}. \end{aligned} \tag{3.192}$$

Notice that $\tilde{L}_i = k_1x_1 + \cdots + x_ik_i - k_1$ has become $\tilde{L}_i(x_i = 0) = \tilde{L}_{i-1}$, and $\mathcal{X}_i = 1 - x_1 - \cdots - x_i$ has turned into $\mathcal{X}_{i-1}$ in the lower limit. The Q-polynomial suffer from the same effect, everything related to the parameter x_i drops off. The upper limit does not remain, since $\mathcal{X}_i(\mathcal{X}_{i-1}) = 0$; nonetheless, care must be exercised because the very last integral with $i = n+1$ (i.e. the largest number of propagators present) does not have that factor, i.e. $\mathcal{X}_{n+1}^{n+1-(n+1)} = 1$. Thus, the upper limit survives for $F^{\mathrm{I}_2}(1, n+1, 1)$.

Moreover, the last integral is solely a total derivative, for it we have

$$\begin{aligned} F^{\mathrm{I}_2}(1,n+1,1) &= \int_\Delta \mathrm{d}^{n+1}x \partial_{n+1} \left\{\frac{1}{2}\mathsf{k}_{\mathrm{I}_2}\Xi^{(0)} - \mathsf{k}_{\mathrm{I}_2}^{\tilde{L}_{n+1}^2}\Xi^{(-1)}\right\} \\ &= \int_\Delta \mathrm{d}^n x \left\{\frac{1}{2}\mathsf{k}_{\mathrm{I}_2}\Xi^{(0)} - \mathsf{k}_{\mathrm{I}_2}^{\tilde{L}_{n+1}^2}\Xi^{(-1)}\right\}(x_{n+1} = \mathcal{X}_n) \\ &\quad - \int_\Delta \mathrm{d}^n x \left\{\frac{1}{2}\mathsf{k}_{\mathrm{I}_2}\Xi^{(0)} - \mathsf{k}_{\mathrm{I}_2}^{\tilde{L}_n^2}\Xi^{(-1)}\right\}(x_{n+1} = 0). \end{aligned} \tag{3.193}$$

Before addressing the upper limit, which will be the result of the integral, the other part about limits is the integral with $i = 1$, only one parametric integration. For that integral we have

$$\begin{aligned} &\int_0^1 \mathrm{d}x_1 \partial_1 \left\{(1-x_1)^n \left[\frac{1}{2}\mathsf{k}_{\mathrm{I}_2}\Xi^{(0)} - \mathsf{k}_{\mathrm{I}_2}^{\tilde{L}_1^2}\Xi^{(-1)}\right]\right\} \\ &= -\left[\frac{1}{2}\mathsf{k}_{\mathrm{I}_2}\Xi^{(0)} - \mathsf{k}_{\mathrm{I}_2}^{\tilde{L}_1^2}\Xi^{(-1)}\right](x_1 = 0) = \frac{\mathsf{k}_{\mathrm{I}_2}^{11}}{(-\lambda^2)}. \end{aligned} \tag{3.194}$$

The kernel $\Xi^{(0)}$ for a polynomial Q in one variable, vanishes in the lower limit, i.e. $\Xi^{(0)}[Q(x_1 = 0)] = \log\left((-\lambda^2)/(-\lambda^2)\right) = 0$. Additionally, the reason for the last equality is that at $x_1 = 0$ we have $\tilde{L}_1(0) = -k_1$ and $Q(0) = -\lambda^2$ what implies $\mathsf{k}_{\mathrm{I}_2}^{\tilde{L}_1^2(0)}\Xi^{(-1)}[Q(0)] = \mathsf{k}_{\mathrm{I}_2}^{11}/(-\lambda^2)$.

These observations point us to separate the first and last summation. As it was done in the expression below

$$\bar{J}_{n+1}^{\mathrm{I}_2} = \int_{\mathbb{R}^{2n}} \mathrm{d}k \frac{K_1^{\mathrm{I}_2}}{D_\lambda^{n+1}} - F^{\mathrm{I}_2}(1,1,n+1) - \sum_{i=2}^{n} F^{\mathrm{I}_2}(1,i,n+2-i) - F^{\mathrm{I}_2}(1,n+1,1). \tag{3.195}$$

The intermediary integrals, with their limits already applied, will cancel as a whole. Explicitly,

$$\begin{aligned}
\bar{J}^{\mathrm{I}_2}_{n+1} = & \int_{\mathbb{R}^{2n}} \mathrm{d}k \frac{K_1^{\mathrm{I}_2}}{D_\lambda^{n+1}} - \frac{1}{\Gamma(n+1)} \frac{\mathsf{k}^{11}_{\mathrm{I}_2}}{\left(-\lambda^2\right)} - \int_\Delta \mathrm{d}^n x \left\{ \frac{1}{2} \mathsf{k}_{\mathrm{I}_2} \Xi^{(0)} - \mathsf{k}_{\mathrm{I}_2}^{\tilde{L}^2_{n+1}} \Xi^{(-1)} \right\} (\mathcal{X}_n) \\
& - \frac{1}{\Gamma(n)} \int_\Delta \mathrm{d}x \mathcal{X}_1^{n-1} \left\{ \frac{1}{2} \mathsf{k}_{\mathrm{I}_2} \Xi^{(0)} - \mathsf{k}_{\mathrm{I}_2}^{\tilde{L}^2_1} \Xi^{(-1)} \right\} \\
& + \sum_{i=2}^{n} \frac{1}{\Gamma(n+2-i)} \int_\Delta \mathrm{d}^{i-1} x \mathcal{X}_{i-1}^{n+1-i} \left\{ \frac{1}{2} \mathsf{k}_{\mathrm{I}_2} \Xi^{(0)} - \mathsf{k}_{\mathrm{I}_2}^{\tilde{L}^2_{i-1}} \Xi^{(-1)} \right\} \\
& - \sum_{i=2}^{n} \frac{1}{\Gamma(n+1-i)} \int_\Delta \mathrm{d}^{i} x \mathcal{X}_{i}^{n-i} \left\{ \frac{1}{2} \mathsf{k}_{\mathrm{I}_2} \Xi^{(0)} - \mathsf{k}_{\mathrm{I}_2}^{\tilde{L}^2_{i}} \Xi^{(-1)} \right\} \\
& + \int_\Delta \mathrm{d}^n x \left\{ \frac{1}{2} \mathsf{k}_{\mathrm{I}_2} \Xi^{(0)} - \mathsf{k}_{\mathrm{I}_2}^{\tilde{L}^2_{n}} \Xi^{(-1)} \right\}. \qquad (3.196)
\end{aligned}$$

The first term of the summation in the third line cancels the second line, and the last term of the summation in the fourth line cancels the last line. After that, it is just a matter of re-indexing the summations, and the remaining terms then cancel completely. For the first line which still remains we simplify it even more, by the previous result for $\int_{\mathbb{R}^{2n}} \mathrm{d}k K_1^{\mathrm{I}_2}/D_\lambda^{n+1}$, eq. (3.180). Observe that only the upper limit of the integral with all the propagators and the lower limit of the integral with one propagator remain; this happens in general.

Thus, we have a full application of the technique for computing a whole class of integrals; simple, however, rich enough to illustrate how to proceed. The result is

$$\begin{aligned}
\bar{J}^{\mathrm{I}_2}_{n+1} &= \frac{1}{2n} \left(\Delta_{n+1,\mathrm{I}_2} + g_{\mathrm{I}_2} I_{\log} \right) + J^{\mathrm{I}_2}_{n+1} \\
J^{\mathrm{I}_2}_{n+1} &= \int_\Delta \mathrm{d}^n x \left\{ -\frac{1}{2} g_{\mathrm{I}_2} \Xi^{(0)} + \mathsf{k}_{\mathrm{I}_2}^{\tilde{L}^2_{n+1}} \Xi^{(-1)} \right\} (\mathcal{X}_n).
\end{aligned}$$

It is a little preposterous to use the string of indices $\mathrm{I}_2 = \mu_{12}$ to represent the integral, so the choice was made to employ the notation in practice. Its power and convenience appear in higher-tensor ranks, large numbers of propagators, and mainly in large power countings.

Remark 3.5.3 *Careful attention in the limit* $x_{n+1} = \mathcal{X}_n = \sum_{j=1}^{n} x_j$. *It reveals that the super vector* $\tilde{L}$,

$$\tilde{L}_{n+1} = L_{n+1} - k_1 = \sum_{j=1}^{n+1} k_j x_j - k_1, \qquad (3.197)$$

when computed in that limit becomes

$$\tilde{L}_{n+1} = \sum_{j=1}^{n} p_{j,n+1} x_j + p_{n+1,1}. \qquad (3.198)$$

Which is not in the format we have presented, $J^{\mathrm{I}_2}_{n+1}$, *for the form we use it* (k_1, m_1) *appears as reference. Well, we could instead say that let us start with* $K_{n+1} = k + k_{n+1}$ *to define the integrals and the additional piece* $p_{n+1,1}$ *would not be present. Thereafter we would change* $(1 \leftrightarrow n+1)$ *in the labels, giving us*

$$\begin{aligned}
\tilde{L}_{n+1} &= p_{1,n+1} x_1 + p_{2,n+1} x_2 + \cdots + p_{n,n+1} x_n \qquad (3.199) \\
&\to p_{n+1,1} x_1 + p_{2,1} x_2 + \cdots + p_{n,1} x_n.
\end{aligned}$$

But the order of integration does not correspond to the definition we gave for $J^{\mathrm{I}_2}_{n+1}$*, then you could invoke the fact that you can permute the order of integrations, over the same function, as long as they cover the simplex* Δ^n*. Right! However, this detour can be easily avoided by just remembering that the integration over the simplex is a choice of an integral over the hyper-octant* $\mathbb{R}^{n+1}_{\geq 0}$ *cut off by a hyperplane* $\mathcal{X}_{n+1} = 1 - \sum_{j=1}^{n+1} x_j = 0$*. That means, we have*

$$\int_{x_i \geq 0} \mathrm{d}^{n+1} x \delta\left(\mathcal{X}_{n+1}\right) f\left(\tilde{L}_{n+1}, Q_{n+1}\right). \tag{3.200}$$

Now it is easy to deploy any reference routing in the desired place and manner. Suppose we have started with k_c *as reference in* K_c *for the last integral, then in its upper limit we will obtain* $\tilde{L}_{n+1}\left(\mathcal{X}_n\right) = \sum_{j=1}^{n} p_{1,n+1} x_j + p_{n+1,c}$*. After that we go to the equation*

$$\int_\Delta \mathrm{d}^n x f\left(\tilde{L}_{n+1}\left(\mathcal{X}_n\right), Q\right) = \int_{x_i \geq 0} \mathrm{d}^{n+1} x \delta\left(\mathcal{X}_{n+1}\right) f\left(\tilde{L}_{n+1}, Q\right), \tag{3.201}$$

and choose the coordinate x_c *corresponding to* k_c*, in* L_{n+1}*, instead of* x_{n+1}*. Thus, we can shuffle the parameters around as in the next equation*

$$\mathcal{X}_{n+1} = \left(1 - \sum_{j=1, b_j \neq c}^{n} x_{b_j}\right) - x_c,$$

then integrate in x_c*. The rest is organized with the order* $b_j < b_{j+1}$*, and the consequence is*

$$\int_{x_i \geq 0} \mathrm{d}^{n+1} x \delta\left(1 - \sum_{j=1, b_j \neq c}^{n} x_{b_j} - x_c\right) f\left(\tilde{L}_{n+1}, Q_{n+1}\right) = \int_0^1 \mathrm{d}x_{b_1} \cdots \int_0^{\mathcal{X}_{b_n}} \mathrm{d}x_{b_n} f\left(L, Q\right). \tag{3.202}$$

What amounts to L *and* Q *having the desired configuration*

$$L = \sum_{j=1}^{n} p_{b_j,c} x_j, \quad Q = -L^2 + \sum_{j=1}^{n}\left(p^2_{b_j,c} + m^2_c - m^2_{b_j}\right) x_j - m^2_c. \tag{3.203}$$

After this long remark about permutations in the integration region, and by using the reference (k_1, m_1) we have completely computed the finite part of $\bar{J}^{\mathrm{I}_2}_{n+1}$ presented in definition (3.4.3), section (3.4),

$$\begin{aligned} J^{(2n)\mathrm{I}_2}_{n+1}\left(p_{21}, \cdots, p_{n+1,1}\right) &= \int_\Delta \mathrm{d}^n x \left\{-\frac{1}{2} g_{\mathrm{I}_2} \Xi^{(0)} + \mathsf{k}^{L^2}_{\mathrm{I}_2} \Xi^{(-1)}\right\} \\ &= -\frac{1}{2} g^{\mu_{12}} Z^{(0)}_{\mathbf{0}} + \sum_{i,j \in [1,n]} \left[\left(p^{\mu_1}_{i+1,1} p^{\mu_2}_{j+1,1}\right) Z^{(-1)}_{\mathbf{e}_{i,j}}\right]. \end{aligned} \tag{3.204}$$

The method and manipulations thoroughly applied in this subsection are the same for other integrals used in the work, the more difficult one is the vector n-pt integral $\bar{J}^{(2n)\mu}_n$ due to the linear power counting, but mainly because the number of propagators grow with the dimension. There are more features to handle that is not possible to present without become to far from illustration[10]. When more than two derivatives are present the work must be broken in stages, the simply act of extract the derivative $\mathcal{D}_i$ and summing all pieces with everything straightforwardly cancelling is a feature of $\omega = 0$ integrals only.

It is the moment to fully write the expression for all integrals used in the AV^n type amplitudes. The n-pt ones

$$\begin{aligned} \bar{J}^{(2n)}_{n} &= J^{(2n)}_{n} + I^{(2n)}_{\log}, & (3.205)\\ \bar{J}^{(2n)}_{n,\mu_1} &= J^{(2n)}_{n,\mu_1} - \frac{1}{n} P^{\nu}_{[1,n]} \Delta^{(2n)}_{n+1,\mu_1} - \frac{1}{n} I^{(2n)}_{\log} \sum_{i=1}^{n-1} \left(p_{i+1,1}\right)_{\mu_1}, & (3.206) \end{aligned}$$

reminding that $P^{\nu}_{[1,n]} = \sum_{i=1}^{n} k_i^{\nu}$, and the $(n+1)$-pt integrals

$$\begin{aligned} \bar{J}^{(2n)}_{n+1} &= J^{(2n)}_{n+1}, & (3.207)\\ \bar{J}^{(2n)}_{n+1,\mu_1} &= J^{(2n)}_{n+1,\mu_1}, & (3.208)\\ \bar{J}^{(2n)}_{n+1,\mu_{12}} &= J^{(2n)}_{n+1,\mu_{12}} + \frac{1}{2n}\left(\Delta^{(2n)}_{n+1,\mu_{12}} + g_{\mu_{12}} I^{(2n)}_{\log}\right), & (3.209) \end{aligned}$$

their unbarred parts (finite parts) will be immediately coped with in the next section, for consultations you can also go to the definition (3.4.3).

The next step is to establish the consequence of contracting with the external momenta and tracing with the metric.

3.6 Reductions by Momentum and Metric Contraction

In this section, we shall obtain the contractions of the integrals established on the definition (3.4.3), that are comprised of Z functions on the definition (3.4.1). Contractions with the external momenta, or simple the variable $p_{i,j}$, and with the metric. They arise in the explicit verification of IRagfs and symmetry properties. In a general context they relate the tensors $J_n^{\mathrm{I}_m}$ among themselves, i.e. with a lower number of integration parameters (lower-point integrals) and lower tensor rank, therefore, we call them reductions. Another reason is: in the context of the $Z^{(k)}_{\mathbf{n}}$ functions (3.101) we have relations among functions with the length ($|\mathbf{n}|$) reduced by one unit.

For the reductions then, we will be using the notation[11] hinted in the previous sections, where we can express the lower n-pt integrals as

$$J^{(2n),\mathrm{I}_0}_{n} = \int_{\Delta} \mathrm{d}^{n-1}x \left[-\Xi^{(0)}\right]; \quad J^{(2n),\mathrm{I}_1}_{n} = \int_{\Delta} \mathrm{d}^{n-1}x \left[\mathsf{k}^{L}_{\mathrm{I}_1} \Xi^{(0)}\right]. \qquad (3.210)$$

---

[10] In the thesis [89] L. Ebani dealt with a two point, rank four tensor integral, in two dimensions with quadratic power counting, and not present in the thesis, in four dimensions where the power counting is four. This takes months to be done with the care we illustrate. Moreover, the computation of a sixth order tensor three point integral in four dimensions takes half a year to superficially complete. With all the tools furnished.

[11] I must point that the real usefulness of these notations comes in higher-rank tensor integrals for which taylored manipulations can be devised (we did not expresses them until now). Those integrals arise not in the first anomalies of AV^n type but in AV^{n+s} $(s > 0)$ type (considering the dimension high enough); moreover, they appear in gravitational anomalies. In our philosophy of analysis an effective notation must be available to such endeavours. However, they are already useful enough in this context to employed.

Being $\mathsf{k}_{\mathrm{I}_1}^{L} = L_\mu$, the L's can be read from their integrals, for n-pt integrals above, using the least i_1 as reference (will be k_1 or k_2) and in ascending order $i_j < i_{j+1}$, we use

$$L = \sum_{j=1}^{n-1} p_{i_j,i_1} x_j, \quad Q = \left(p_{i_2,i_1}, \cdots, p_{i_{n-1},i_1}, m_{i_1}^2\right), \tag{3.211}$$

and the bigger $n+1$-point ones can be expressed as

$$\begin{aligned} J_{n+1}^{(2n),\mathrm{I}_0} &= \int_\Delta \mathrm{d}^n x \left[\Xi^{(-1)}\right], & (3.212)\\ J_{n+1}^{(2n),\mathrm{I}_1} &= \int_\Delta \mathrm{d}^n x \left[-\mathsf{k}_{\mathrm{I}_1}^{L} \Xi^{(-1)}\right], & (3.213)\\ J_{n+1}^{(2n),\mathrm{I}_2} &= \int_\Delta \mathrm{d}^n x \left[-\frac{1}{2}\mathsf{k}_{\mathrm{I}_2} \Xi^{(0)} + \mathsf{k}_{\mathrm{I}_2}^{LL} \Xi^{(-1)}\right], & (3.214)\end{aligned}$$

$\mathsf{k}_{\mathrm{I}_1}^{L} = L_\mu$ and $\mathsf{k}_{\mathrm{I}_2}^{L^2} = L_\mu L_\nu$, with

$$L = \sum_{i=1}^{n} p_{i+1,1} x_i, \quad Q = \left(p_{21}, \cdots, p_{n+1,1}, m_1^2\right), \tag{3.215}$$

Reduction of the Vector Integral $J_{n+1,\mu}$: Instead of present the reductions of Z functions (3.101) which can be made, are useful, and present in the appendix, we will show how to do it in the integrals themselves. The trick is to identify the expression $p_{i+1,1}^{\mu} J_{n+1,\mu}$ in the parametric integrand and use the property of the kernel $\partial_i \Xi^{(k)} = \mathcal{D}_i \Xi^{(k-1)}$. Therefore, we take the derivative of $\Xi^{(0)}$ w.r.t. x_1 and obtain, for example,

$$\partial_1 \left(\Xi^{(0)}\right) = \mathcal{D}_1 \Xi^{(-1)} = 2 p_{21}^{\nu} \left(-L_\nu \Xi^{(-1)}\right) + B_1 \Xi^{(-1)}. \tag{3.216}$$

Remember that $\mathcal{D}_1 = \partial_1 Q = -2 p_{21} \cdot L + B_1$ (by derivative of eq.(3.110) for the Q-polynomial, in convenient form). Then identify your target $(-L_\nu \Xi^{(-1)})$ which is the integrand of $J_{n+1,\nu}$, the same for $\Xi^{(-1)}$, integrand of J_{n+1}. Now integrate the equation in the parametric space to obtain

$$2 p_{21}^{\nu} J_{n+1,\nu} + B_1 J_{n+1} = \int_\Delta \mathrm{d}^n x \partial_1 \left(\Xi^{(0)}\right) = \int_\Delta \mathrm{d}^{n-1} x \Xi^{(0)} \left(x_n = \mathcal{X}_{n-1}\right) - \int_\Delta \mathrm{d}^{n-1} x \Xi^{(0)} \left(x_i = 0\right). \tag{3.217}$$

Explanation: the limits that survive are the upper limit of the last integration $\int_0^{\mathcal{X}_{n-1}} \mathrm{d}x_n$ and the lower limit of $\int_0^1 \mathrm{d}x_1$ the reason we show later in a remark. In general, the lower limit has the simple implication of eliminating that dependency of momenta and masses attached to that parameter, schematically

$$\begin{aligned} L_n &= \sum_{i=[1,n]} p_{i+1,1} x_i \overset{x_j=0}{\rightarrow} \sum_{i=[1,n]\setminus\{j\}} p_{i+1,1} x_i, & (3.218)\\ Q_n \left(p_{21}, \cdots, p_{n+1,1}; m_1^2\right) &= -L_n^2 + \sum_{i=[1,n]} B_i x_i - m_1^2 & (3.219)\\ &\overset{x_j=0}{\rightarrow} Q_{n-1} \left(p_{21}, \cdots \widehat{p_{j+1,1}} \cdots, p_{n+1,1}; m_1^2\right). \end{aligned}$$

Thus, by renaming variables $x_i \to x_{i-1}$ for $i > j$ we may continue to write $\int_\Delta \mathrm{d}^{n-1}x$ without altering anything else. In other words, you have lowered the n-pt degree and excluded one momentum variable from the integral. The upper limit (which remains only in the last integration) lower the n-pt degree and reset the momenta upon which the integral depends, since we have

$$\begin{aligned} L_n &= \sum_{i=[1,n]} p_{i+1,1}x_i \qquad (3.220) \\ &\overset{x_n=(1-x_1\cdots-x_{n-1})}{\longrightarrow} \sum_{i=[1,n-1]} p_{i+1,n+1}x_i + \left(p_{n+1,1}\right), \\ Q_n\left(p_{21},\cdots,p_{n+1,1};m_1^2\right) &= -L_n^2 + \sum_{i=[1,n]} B_i x_i - m_1^2 \qquad (3.221) \\ &\overset{x_n=(1-x_1\cdots-x_{n-1})}{\longrightarrow} Q_{n-1}\left(p_{2,n+1},\cdots p_{n,n+1},m_{n+1}^2\right). \end{aligned}$$

Therefore the simplest contraction, where $B_1 = p_{21}^2 + m_2^2 - m_1^2$, delivers

$$2p_{21}^{\nu}J_{n+1,\nu} = -\left(p_{21}^2 + m_2^2 - m_1^2\right)J_{n+1} + J_n\left(p_{31},\cdots,p_{n+1,1}\right) - J_n\left(p_{2,n+1},\cdots,p_{n,n+1}\right). \qquad (3.222)$$

Before presenting all the contractions some points must be remarked.

Remark 3.6.1 *The formula for exchanging partials and integrals (Leibnitz rule), under appropriate continuity conditions, states that*

$$\int_{a(x)}^{b(x)} \mathrm{d}z\frac{\partial}{\partial z}f(x,z) = \frac{\partial}{\partial z}\int_{a(x)}^{b(x)} \mathrm{d}z f(x,z) - b'(x)f(x,b) + a'(x)f(x,a). \qquad (3.223)$$

For us $b'(x) = -1$ and $a'(x) = 0$, therefore we use

$$\int_{0}^{b(x)} \mathrm{d}z\frac{\partial}{\partial z}f(x,z) = \frac{\partial}{\partial z}\int_{a(x)}^{b(x)} \mathrm{d}z f(x,z) + f(x,b(x)). \qquad (3.224)$$

Our functions Ξ and $\mathcal{X}_i$ are continuous functions of the parameters (x_i) as long as the kinematic data stays away from the thresholds for imaginary parts. That is a matter of dependence on the kinematic data and we may choose it (stay away from thresholds) and in the aftermath analytically continue the result, and with the help of $\mathrm{i}0^+$ prescription recover the result in all regions. As the relations developed relate analytic functions in some region they are true for the extended common region. The prescription $m^2 - i0^+$ determines the branch of the logarithm, and in the $Q \equiv Q + i0^+$ polynomial the prescription avoids its vanishing in the simplex Δ^n where we integrate, and that is compact. Hence, for $\Xi^{(-k)}$ $(k > 0)$, where everything starts, we have an analytic function of the kinematic data away of certain values on the real axis. In any case the formulas used are present throughout the literature of the method.

Example 3.6.2 *Pulling back the partial ∂_2 towards the integral $\int \mathrm{d}x_2$. Consider the expression*

$$\int_0^1 \mathrm{d}x_1 \int_0^{1-x_1} \mathrm{d}x_2 \int_0^{1-x_1-x_2} \mathrm{d}x_3 \int_0^{1-x_1-x_2-x_3} \mathrm{d}x_4 \partial_2 f(x_1,x_2,x_3,x_4), \qquad (3.225)$$

for which in the first commutation we get

$$\begin{aligned}
& \partial_2 \int_0^{1-x_1-x_2-x_3} \mathrm{d}x_4 f\left(x_1,x_2,x_3,x_4\right) - \partial_2\left(\mathcal{X}_3\right) f\left(x_1,x_2,x_3,\mathcal{X}_3\right) \\
= \; & \partial_2 \int_0^{1-x_1-x_2-x_3} \mathrm{d}x_4 f\left(x_1,x_2,x_3,x_4\right) + f\left(x_1,x_2,x_3,\mathcal{X}_3\right),
\end{aligned} \tag{3.226}$$

but for any other permutation where variable of integration is not x_2 we will have

$$\begin{aligned}
& \int_0^{1-x_1-x_2} \mathrm{d}x_3 \partial_2 \int_0^{1-x_1-x_2-x_3} \mathrm{d}x_4 f\left(x\right) \\
= \; & \partial_2 \int_0^{1-x_1-x_2} \mathrm{d}x_3 \int_0^{1-x_1-x_2-x_3} \mathrm{d}x_4 f\left(x\right) + \int_0^{0} \mathrm{d}x_4 f\left(x_1,x_2,\mathcal{X}_2,x_4\right) \\
= \; & \partial_2 \int_0^{1-x_1-x_2} \mathrm{d}x_3 \int_0^{1-x_1-x_2-x_3} \mathrm{d}x_4 f\left(x\right).
\end{aligned} \tag{3.227}$$

The upper limits combine as $\mathcal{X}_i\left(\mathcal{X}_{i-1}\right)=0$. As of the term coming from the derivative of the variable limit of integration, it corresponds to an integral evaluated in one point, and hence it vanishes. Then we can commute the partials in intermediary integrations until it hits the corresponding integral, in there only the lower limit of integration will remain, again, due to $\mathcal{X}_i\left(\mathcal{X}_{i-1}\right)=0$. Summary: we can write

$$\int_\Delta \mathrm{d}^n x \partial_j f = -\int_\Delta \mathrm{d}^{n-1}x f\left(x_j=0\right) + \int_\Delta \mathrm{d}^{n-1}x f\left(x_n=\mathcal{X}_{n-1}\right). \tag{3.228}$$

Thus contractions with the external momenta $p^\nu_{i+1,1}$ ($i\in[1,n]$) of the vector integrals read

$$2p^\nu_{i+1,1}\left[J_{n+1,\nu}\right] = -\left(p^2_{i+1,1}+m_i^2-m_1^2\right)\left[J_{n+1}\right] \tag{3.229}$$

$$+\left[J_n\left(\widehat{p_{i+1,1}}\right)\right]-\left[J_n\left(p_{2,n+1},\cdots,p_{n,n+1}\right)\right]. \tag{3.230}$$

Reductions of Tensor integrals $J^{\mu_{12}}_{n+1}$:

(I)-Momenta contraction. The same procedure, however using $L_\mu\Xi^{(0)}$ as starting point. By differentiating it w.r.t. x_1 follows

$$\begin{aligned}
\partial_1\left(L_\mu\Xi^{(0)}\right) &= \left(\partial_1 L_\mu\right)\Xi^{(0)} + L_\mu\left(\mathcal{D}_1\Xi^{(-1)}\right) \\
&= p_{21\mu}\Xi^{(0)} - 2p_{21}\cdot LL_\mu\Xi^{(-1)} + L_\mu B_1\Xi^{(-1)} \\
&= -2p^\nu_{21}\left[-\frac{1}{2}g_{\mu\nu}\Xi^{(0)}+L_\nu L_\mu\Xi^{(-1)}\right] - B_1\left(-L_\mu\Xi^{(-1)}\right).
\end{aligned} \tag{3.231}$$

Now by making the identifications we need, that was already done in the equation above, and integrating one obtains

$$-2p^\nu_{21}J_{n+1;\mu\nu} - B_1 J_{n+1;\mu} = \int_\Delta \mathrm{d}^n x \partial_1\left(L_\mu\Xi^{(0)}\right). \tag{3.232}$$

The r.h.s. will become

$$\begin{aligned}
\int_\Delta \mathrm{d}^n x \partial_1\left(L_\mu\Xi^{(0)}\right) &= -\int_\Delta \mathrm{d}^{n-1}x\left(L_\mu\Xi^{(0)}\right)\left(x_1=0\right)+\int_\Delta \mathrm{d}^{n-1}x\left(L_\mu\Xi^{(0)}\right)\left(x_n=\mathcal{X}_{n-1}\right) \\
&= -J_{n,\mu}\left(p_{31},\cdots,p_{n+1,1}\right)+J_{n,\mu}\left(p_{2,n+1},\cdots,p_{n,n+1}\right)+ \\
&\quad -\left(p_{n+1,1}\right)_\mu J_n\left(p_{2,n+1},\cdots,p_{n,n+1}\right).
\end{aligned} \tag{3.233}$$

The last term $\left(p_{n+1,1}\right)_\mu J_n$ arose from eq. (3.220), i.e. $L\left(\mathcal{X}_{n-1}\right)=p_{n+1,1}+\sum_{i=[1,n-1]}p_{i+1,n+1}x_i$.

Hence, the contraction with p_{21} can be stated in the form

$$\begin{aligned} 2p_{21}^\nu J_{n+1;\mu\nu} &= -\left(p_{21}^2+m_2^2-m_1^2\right)J_{n+1;\mu} \\ &\quad +J_{n,\mu}\left(p_{31},\cdots,p_{n+1,1}\right)-J_{n,\mu}\left(p_{2,n+1},\cdots,p_{n,n+1}\right) \\ &\quad +\left(p_{n+1,1}\right)_\mu J_n\left(p_{2,n+1},\cdots,p_{n,n+1}\right). \end{aligned} \tag{3.234}$$

They all can be encompassed in a single formula, for $i\in[1,n]$ they are

$$2p_{i+1,1}^\nu\left[J_{n+1;\nu}^\mu\right] = -\left(p_{i+1,1}^2+m_i^2-m_1^2\right)\left[J_{n+1}^\mu\right] \tag{3.235}$$

$$+\left[J_n^\mu\left(\widehat{p_{i+1,1}}\right)\right]-\left[J_n^\mu\left(p_{2,n+1},\cdots,p_{n,n+1}\right)\right]$$

$$+p_{n+1,1}^\mu\left[J_n\left(p_{2,n+1},\cdots,p_{n,n+1}\right)\right]. \tag{3.236}$$

(II)-Metric contraction. For the trace we use the following facts

$$x_i\partial_i\Xi^{(0)} = x_i\mathcal{D}_i\Xi^{(-1)} \tag{3.237}$$

$$\mathcal{D}_i = -2p_{i+1,1}\cdot L+B_i, \tag{3.238}$$

$$\sum_{i=1}^n x_i\partial_i\Xi^{(0)} = \left(\sum_{i=1}^n x_i\mathcal{D}_i\right)\Xi^{(-1)}, \tag{3.239}$$

from which it is obtained

$$\left(\sum_{i=1}^n x_i\mathcal{D}_i\right)=-2\underbrace{\sum_{i=1}^n\left(x_ip_{i+1,1}\right)}_{=L}\cdot L+\sum_{i=1}^n x_iB_i=-2L^2+\sum_{i=1}^n x_iB_i. \tag{3.240}$$

On the other hand, the "partial integration" produces

$$\sum_{i=1}^n x_i\partial_i\Xi^{(0)}=\sum_{i=1}^n\partial_i\left(x_i\Xi^{(0)}\right)-n\Xi^{(0)}. \tag{3.241}$$

Now let us adopt a notation for the parametric integrands similar to the momentum integrals, that is to say, the parametric integrand of $J_{n+1}^{\mu_{12}}$ is given by

$$\jmath_{n+1,\mu_{12}}=-\frac{1}{2}g_{\mu_{12}}\Xi^{(0)}+\mathsf{k}_{\mu_{12}}^{LL}\Xi^{(-1)}, \tag{3.242}$$

hence its trace acquires the form

$$g^{\mu_{12}}\jmath_{n+1,\mu_{12}}=-n\Xi^{(0)}+L^2\Xi^{(-1)}. \tag{3.243}$$

Putting eqs. (3.240,3.241) together enable us to write

$$-n\Xi^{(0)}=-\sum_{i=1}^n\partial_i\left(x_i\Xi^{(0)}\right)+\left(-2L^2+\sum_{i=1}^n x_iB_i\right)\Xi^{(-1)}, \tag{3.244}$$

that together with the previous eq. for the integrand trace $g^{\mu_{12}}j_{n+1,\mu_{12}}$ furnishes

$$g^{\mu_{12}}j_{n+1,\mu_{12}} = -\sum_{i=1}^{n}\partial_i\left(x_i\Xi^{(0)}\right) + \left(-L^2+\sum_{i=1}^{n}x_iB_i\right)\Xi^{(-1)}. \tag{3.245}$$

The almost final consideration is the identification of the polynomial Q and the definition of $\Xi^{(-1)} = 1/Q$, they follow by

$$\begin{aligned} Q+m_1^2 &= -L^2+\sum_{i=1}^{n}x_iB_i, && (3.246)\\ Q\Xi^{(-1)} &= 1, && (3.247)\end{aligned}$$

therefore, the result below holds

$$g^{\mu_{12}}j_{n+1,\mu_{12}} = -\sum_{i=1}^{n}\partial_i\left(x_i\Xi^{(0)}\right) + \left(1+m_1^2\Xi^{(-1)}\right). \tag{3.248}$$

The final argument is just consider the integration of the partial derivatives, for which follows

$$\begin{aligned} \int_\Delta \mathrm{d}^nx\partial_i\left(x_i\Xi^{(0)}\right) &= \int_\Delta \mathrm{d}^{n-1}x\left(x_i\Xi^{(0)}\left(x_n=\mathcal{X}_{n-1}\right)\right),\ \text{for } i<n, && (3.249)\\ \int_\Delta \mathrm{d}^nx\partial_n\left(x_n\Xi^{(0)}\right) &= \int_\Delta \mathrm{d}^{n-1}x\left(\mathcal{X}_{n-1}\Xi^{(0)}\left(x_n=\mathcal{X}_{n-1}\right)\right), && (3.250)\end{aligned}$$

note that because we have an unit power of x_i, for integrals with $i<n$, only the upper limit of the last integral over x_n contributes. Putting them all together, we get

$$\begin{aligned} \sum_{i=1}^{n}\int_\Delta \mathrm{d}^nx\partial_i\left(x_i\Xi^{(0)}\right) &= \int_\Delta \mathrm{d}^{n-1}x\left(\sum_{i=1}^{n-1}x_i+\mathcal{X}_{n-1}\right)\Xi^{(0)}\left(\mathcal{X}_{n-1}\right) && (3.251)\\ &= \int_\Delta \mathrm{d}^{n-1}x\Xi^{(0)}\left(\mathcal{X}_{n-1}\right) = -J_n\left(p_{2,n+1},\cdots,p_{n,n+1}\right).\end{aligned}$$

Integration of the whole expression for the trace will result in

$$g_{\mu\nu}J_{n+1}^{\mu\nu} = J_n\left(p_{2,n+1},\cdots,p_{n,n+1}\right)+m_1^2J_{n+1}+\int_\Delta \mathrm{d}^nx, \tag{3.252}$$

using the simplex volume, $\int_\Delta \mathrm{d}^nx = 1/n!$, the final result is achieved

$$g_{\mu\nu}J_{n+1}^{\mu\nu} = J_n\left(p_{2,n+1},\cdots,p_{n,n+1}\right)+m_1^2J_{n+1}+\frac{1}{n!}. \tag{3.253}$$

For better consultation we list the formulae again.

$$\begin{aligned} 2p_{i+1,1}^{\nu}\left[J_{n+1,\nu}\right] &= -\left(p_{i+1,1}^2+m_i^2-m_1^2\right)\left[J_{n+1}\right] && (3.254)\\ &\quad+\left[J_n\left(\widehat{p_{i+1,1}}\right)\right]-\left[J_n\left(p_{2,n+1},\cdots,p_{n,n+1}\right)\right],\end{aligned}$$

$$\begin{aligned} 2p_{i+1,1}^{\nu}\left[J_{n+1;\nu}^{\mu}\right] &= -\left(p_{i+1,1}^2+m_i^2-m_1^2\right)\left[J_{n+1}^{\mu}\right] && (3.255)\\ &\quad+\left[J_n^{\mu}\left(\widehat{p_{i+1,1}}\right)\right]-\left[J_n^{\mu}\left(p_{2,n+1},\cdots,p_{n,n+1}\right)\right]\\ &\quad+p_{n+1,1}^{\mu}\left[J_n\left(p_{2,n+1},\cdots,p_{n,n+1}\right)\right]\end{aligned}$$

$$g_{\mu\nu}\left[J_{n+1}^{\mu\nu}\right] = J_n\left(p_{2,n+1},\cdots,p_{n,n+1}\right)+m_1^2J_{n+1}+\frac{1}{n!}. \tag{3.256}$$

The next step is systematization for a class of tensors that appear just after taking Dirac traces, leading traces for the AV^n type amplitudes.

3.7 The Sign Tensors and Traces

When we are analyzing amplitudes like $t^{AV^n}_{\mathrm{I}_{n+1}}$ $(d = 2n)$, two types of trace are non-zero: with $2n+2$ matrices and $2n$ matrices. The ones with $2n$ matrices have one monomial, an epsilon tensor. The ones with $2n+2$ matrices may exhibit a multitude of monomials, feature to be discussed in due time. Nonetheless, they are linear combinations of monomials with one metric tensor and one epsilon tensor. In this last set arises, in the amplitudes, a characteristic type of tensor that can be essentially determined by a set of signs, plus or minus one, for this reason we call them sign tensors. To approach them let us write our considerations for $t^{AV^n}_{\mathrm{I}_{n+1}}$ itself

$$\begin{aligned} t^{AV^n}_{\mathrm{I}_{n+1}} &= \frac{1}{D_{[1,n+1]}} K_1^{\nu_1} \cdots K_{n+1}^{\nu_{n+1}} \mathrm{tr}\left(\gamma_* \gamma_{\mu_1\nu_1\cdots\mu_{n+1}\nu_{n+1}}\right) \\ &+ \frac{1}{D_{[1,n+1]}} \underbrace{m_n m_{n+1} K_1^{\nu_1} \cdots K_{n-1}^{\nu_{n-1}} \mathrm{tr}\left(\gamma_* \gamma_{\mu_1\nu_1\cdots\nu_{n-1}\mu_n\mu_{n+1}}\right) + \cdots}_{n(n+1)/2 \text{ terms}}. \end{aligned} \tag{3.257}$$

The second line has only one type of term, e.g. $\mathrm{tr}\left(\gamma_* \gamma_{\mu_1\nu_1\cdots\nu_{n-1}\mu_n\mu_{n+1}}\right) = 2^n \left(-\mathrm{i}\right)^{n+1} \varepsilon_{\mu_1\nu_1\cdots\nu_{n-1}\mu_n\mu_{n+1}}$, and nothing more; however, the first line will be more complex. If all indices appear in the monomials, then $\mathrm{tr}\left(\gamma_* \gamma_{\mu_1\nu_1\cdots\mu_{n+1}\nu_{n+1}}\right)$ has $(n+1)(2n+1)$ such monomials. As always, one examples in better.

Example 3.7.1 *In $d = 6$ dimensions, we may utilize the formula (1.37) from the notation's section (1), to write the trace. The trace for study is $\mathrm{tr}\left(\gamma_* \gamma_{\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3\mu_4\nu_4}\right)$. If we adopt that formula with $I_8 = \mu_1\nu_1\mu_2\nu_2\mu_3\nu_3\mu_4\nu_4$ and organize the monomials by how the metric tensors acquires indices of type ν and μ, and nothing more, we have*

$$\begin{aligned} &\frac{1}{2^3} \mathrm{tr}\left(\gamma_* \gamma_{\mathrm{I}_8}\right) \\ = &\Big[+g_{\mu_1\nu_1}\varepsilon_{\mu_2\nu_2\mu_3\nu_3\mu_4\nu_4} + g_{\mu_1\nu_2}\varepsilon_{\nu_1\mu_2\mu_3\nu_3\mu_4\nu_4} + g_{\mu_1\nu_3}\varepsilon_{\nu_1\mu_2\nu_2\mu_3\mu_4\nu_4} + g_{\mu_1\nu_4}\varepsilon_{\nu_1\mu_2\nu_2\mu_3\nu_3\mu_4} \\ &+g_{\nu_1\mu_2}\varepsilon_{\mu_1\nu_2\mu_3\nu_3\mu_4\nu_4} + g_{\mu_2\nu_2}\varepsilon_{\mu_1\nu_1\mu_3\nu_3\mu_4\nu_4} + g_{\mu_2\nu_3}\varepsilon_{\mu_1\nu_1\nu_2\mu_3\mu_4\nu_4} + g_{\mu_2\nu_4}\varepsilon_{\mu_1\nu_1\nu_2\mu_3\nu_3\mu_4} \\ &+g_{\nu_1\mu_3}\varepsilon_{\mu_1\mu_2\nu_2\nu_3\mu_4\nu_4} + g_{\nu_2\mu_3}\varepsilon_{\mu_1\nu_1\mu_2\nu_3\mu_4\nu_4} + g_{\mu_3\nu_3}\varepsilon_{\mu_1\nu_1\mu_2\nu_2\mu_4\nu_4} + g_{\mu_3\nu_4}\varepsilon_{\mu_1\nu_1\mu_2\nu_2\nu_3\mu_4} \\ &+g_{\nu_1\mu_4}\varepsilon_{\mu_1\mu_2\nu_2\mu_3\nu_3\nu_4} + g_{\nu_2\mu_4}\varepsilon_{\mu_1\nu_1\mu_2\mu_3\nu_3\nu_4} + g_{\nu_3\mu_4}\varepsilon_{\mu_1\nu_1\mu_2\nu_2\mu_3\nu_4} + g_{\mu_4\nu_4}\varepsilon_{\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3}\Big] \\ &\Big[- g_{\nu_1\nu_2}\varepsilon_{\mu_1\mu_2\mu_3\nu_3\mu_4\nu_4} - g_{\nu_2\nu_3}\varepsilon_{\mu_1\nu_1\mu_2\mu_3\mu_4\nu_4} - g_{\nu_3\nu_4}\varepsilon_{\mu_1\nu_1\mu_2\nu_2\mu_3\mu_4} \\ &-g_{\nu_1\nu_3}\varepsilon_{\mu_1\mu_2\nu_2\mu_3\mu_4\nu_4} - g_{\nu_2\nu_4}\varepsilon_{\mu_1\nu_1\mu_2\mu_3\nu_3\mu_4} - g_{\nu_1\nu_4}\varepsilon_{\mu_1\mu_2\nu_2\mu_3\nu_3\mu_4}\Big] \\ &\Big[-g_{\mu_1\mu_2}\varepsilon_{\nu_1\nu_2\mu_3\nu_3\mu_4\nu_4} - g_{\mu_2\mu_3}\varepsilon_{\mu_1\nu_1\nu_2\nu_3\mu_4\nu_4} - g_{\mu_3\mu_4}\varepsilon_{\mu_1\nu_1\mu_2\nu_2\nu_3\nu_4} \\ &-g_{\mu_1\mu_3}\varepsilon_{\nu_1\mu_2\nu_2\nu_3\mu_4\nu_4} - g_{\mu_2\mu_4}\varepsilon_{\mu_1\nu_1\nu_2\mu_3\nu_3\nu_4} - g_{\mu_1\mu_4}\varepsilon_{\nu_1\mu_2\nu_2\mu_3\nu_3\nu_4}\Big]. \end{aligned} \tag{3.258}$$

The first four lines have the structure of $g_{\mu_{i_1}\nu_{i_1}}\varepsilon_{\mu_{i_2 i_3 i_4}\nu_{i_2 i_3 i_4}}$ which by contraction with $K^{\nu_{1234}}_{1234}$ produces, for example, $K^{\nu_{1234}}_{1234} g_{\mu_1\nu_1}\varepsilon_{\mu_{234}\nu_{234}} = K_{1\mu_1} K^{\nu_{123}}_{123}\varepsilon_{\mu_{234}\nu_{123}}$. They will give rise to the sign *tensors. We call this specific expression for the trace the first version, such nomenclature will be discussed in due time. In a more complete picture, the four first lines are organized as*

$$\begin{aligned}
&K_{1234}^{\nu_{1234}}\mathrm{tr}\left(\gamma_{*}\gamma_{\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3\mu_4\nu_4}\right) \\
= \;& +\varepsilon_{\mu_{234}\nu_{123}}\left[-K_{1\mu_1}K_{234}^{\nu_{123}}+K_{2\mu_1}K_{134}^{\nu_{123}}-K_{3\mu_1}K_{124}^{\nu_{123}}+K_{4\mu_1}K_{123}^{\nu_{123}}\right] \\
& +\varepsilon_{\mu_{134}\nu_{123}}\left[-K_{1\mu_2}K_{234}^{\nu_{123}}-K_{2\mu_2}K_{134}^{\nu_{123}}+K_{3\mu_2}K_{124}^{\nu_{123}}-K_{4\mu_2}K_{123}^{\nu_{123}}\right] \\
& +\varepsilon_{\mu_{124}\nu_{123}}\left[+K_{1\mu_3}K_{234}^{\nu_{123}}-K_{2\mu_3}K_{134}^{\nu_{123}}-K_{3\mu_3}K_{124}^{\nu_{123}}+K_{4\mu_3}K_{123}^{\nu_{123}}\right] \\
& +\varepsilon_{\mu_{123}\nu_{123}}\left[-K_{1\mu_4}K_{234}^{\nu_{123}}+K_{2\mu_4}K_{134}^{\nu_{123}}-K_{3\mu_4}K_{124}^{\nu_{123}}-K_{4\mu_4}K_{123}^{\nu_{123}}\right].
\end{aligned} \tag{3.259}$$

The fifth and sixth line have the structure $g_{\nu_1\nu_2}\varepsilon_{\mu_{1234}\nu_{34}}$*, they generate bilinears that go with the mass terms and are perfectly identified with even sub-amplitudes. The last two lines correspond to rank two and odd amplitudes. Due to being* 4*-pt ones, and finite, with just three independent external momenta will vanish identically, when in integral form. Besides the free index they carry, the sign tensors differ from each other only by their signs.*

After this, almost complete, discussion of the expression that we will investigate in the sequel, we have enough motivation to fix attention on the general structure the above expression fits in.

The object we are interested in have a set of indices in contraction, represented by $\mathrm{C}_n = (\nu_i)_{i=1}^{n}$, and a set of free indices, I_n, carried by the epsilon tensor. We will not specify this set for our discussion, since it is determined by context (what version of the amplitude to be defined latter and what type of amplitude). Inspired by the example above, we known it has the form

$$\varepsilon_{\mathrm{I}_n\mathrm{C}_n}t_{\mu}^{\mathbf{s},\mathrm{C}_n}=\frac{1}{D_{[1,n+1]}}\sum_{a=1}^{n+1}s_aK_{a,\mu}K_{\widehat{a}}^{\mathrm{C}_n}, \quad \varepsilon_{\mathrm{I}_n\mathrm{C}_n}T_{\mu}^{\mathbf{s},\mathrm{C}_n}=\varepsilon_{\mathrm{I}_n\mathrm{C}_n}\int_{\mathbb{R}^{2n}}\mathrm{d}k\left[t_{\mu}^{\mathbf{s},\mathrm{C}_n}\right]. \tag{3.260}$$

See that a unspecified free index μ is carried by one factor $K_{a,\mu}$, as in the example, and each time a routing k_a is in $K_{a,\mu}$ it is not present in the sequence $K_{\widehat{a}}^{\mathrm{C}_n}$, symbol that will be precisely defined below. Nothing more than the structure of the example. Now the signs, in principle all combinations of signs s_a give a distinct tensor, we have represented them in the notation by a vector of signs

$$\mathbf{s}=(s_1,\cdots s_{n+1}), \quad s_a\in\{-1,1\}. \tag{3.261}$$

The real number of independent structures would be 2^{n+1}, but it is lesser than that since they can be just proportional; therefore, only n independent signs are necessary. However, our derivations are facilitated by keeping them independent for now.

The notation for exclusion of a specific routing reads

$$K_{\widehat{a}}^{\mathrm{C}_n}=K_{b_1\cdots b_n}^{\nu_1\cdots\nu_n}=\prod_{i=1}^{n}K_{b_i}^{\nu_i}; \quad b_i\in[1,n+1]\setminus\{a\}, \quad b_i<b_{i+1} \tag{3.262}$$

Before delving into any deduction let us state some things: as it stands it sounds that the signs tensors (and amplitudes as whole) are projected, first in integrands of rank $n+1$ ($\bar{j}^{\mathrm{I}_{n+1}}_{n+1}$), and then in integrals of that rank. Obvious this is not the case, due to the high number of contraction with the epsilon tensor. Our objective is to reduce the expressions into functions of just tensor and vector integrals, which codify quite precisely any expression possible to obtain from anomalous amplitudes. Furthermore, they will show us connections between the versions of the amplitudes and what type of sign tensor appear.

Therefore, to start the process of systematizing the sign tensors we must first chose a reference index to their component integrals. Let us say $c \in [1, n+1]$, then we split the summations in the definition as

$$\sum_{a=1}^{n+1} s_a K_{a,\mu} K_{\widehat{a}}^{\mathrm{C}_n} = \sum_{a<c} s_a K_{a,\mu} K_{\widehat{a}}^{\mathrm{C}_n} + s_c K_{c,\mu} K_{\widehat{c}}^{\mathrm{C}_n} + \sum_{a>c} s_a K_{a,\mu} K_{\widehat{a}}^{\mathrm{C}_n}. \tag{3.263}$$

Let us start by the last term where the indices excluded, $a > c$, are larger than the reference one c. Keeping in mind the C_n string are in $\varepsilon_{\mathrm{I}_n \mathrm{C}_n}$, we manipulate as the tensor was antisymmetrized in those indices. Therefore we can write

$$K_{b_i}^{\nu_i} = K_c^{\nu_i} + p_{b_i,c}^{\nu_i}, \quad b_i \neq c, \tag{3.264}$$

as the excluded index is above c the index corresponding to K_c is ν_c, then we drop all $K_c^{\nu_i}$ by antisymmetry and get

$$K_{\widehat{a}}^{\mathrm{C}_n} = p_{b_1,c}^{\nu_1} \cdots p_{c-1,c}^{\nu_{c-1}} p_{b_c,c}^{\nu_{c+1}} \cdots p_{b_{n-1}}^{\nu_n} K_c^{\nu_c} = (-1)^{n-c} \left(\prod_{i=1}^{n-1} p_{b_i,c}^{\nu_i} \right) K_c^{\nu_n}, \quad b_i \neq a, c \wedge a > c. \tag{3.265}$$

For the case of $a < c$, as we exclude one routing in ascending order below c, we have and additional permutation to bring not only the b_i is ascending order but the ν_i as well. With this understanding we denote the products of external momenta by

$$p_{\widehat{a},c}^{\mathrm{C}_{n-1}} = \prod_{\substack{i=1; b_i \neq a,c \\ b_i < b_{i+1}}}^{n-1} p_{b_i,c}^{\nu_i}, \tag{3.266}$$

and write for all cases, except for $\widehat{a} = \widehat{c}$, the formula

$$K_{\widehat{a}}^{\mathrm{C}_n} = (-1)^{n-c+1+\theta(a-c)} \, p_{\widehat{a},c}^{\mathrm{C}_{n-1}} K_c^{\nu_n}, \quad a \neq c. \tag{3.267}$$

The last case is when K_c carries a free index, thereby, not present in $K_{\widehat{c}}^{\mathrm{C}_n}$, for which we select K_{c+1} as reference. It is similar to the first case since $c+1 > c$. The answer follows as

$$K_{\widehat{c}}^{\mathrm{C}_n} = (-1)^{n-c \bmod n} \, p_{\widehat{c},c+1}^{\mathrm{C}_{n-1}} K_{c+1}^{\nu_n}. \tag{3.268}$$

The $\bmod n$ operation only has effect when $c = n+1$, for which the next routing will be k_1 and associate index is ν_1, i.e.,

$$(-1)^{n-(n+1) \bmod n} \, p_{\widehat{n+1},n+2}^{\mathrm{C}_{n-1}} K_{n+2}^{\nu_n} = (-1)^{n-1} \, p_{\widehat{n+1},1}^{\mathrm{C}_{n-1}} K_1^{\nu_n},$$

by convention of cyclic labelling. We shall see in a moment that this fact will have no effect on the result.

Therefore, by gathering the manipulation so far we get

$$\sum_{a=1}^{n+1} s_a K_{a,\mu} K_{\widehat{a}}^{\mathrm{C}_n} = \sum_{a \neq c} s_a \left(-1\right)^{n-c+1+\theta(a-c)} p_{\widehat{a},c}^{\mathrm{C}_{n-1}} K_c^{\nu_n} K_{a,\mu} + s_c \left(-1\right)^{n-c \bmod n} p_{\widehat{c},c+1}^{\mathrm{C}_{n-1}} K_{c+1}^{\nu_n} K_{c,\mu}. \tag{3.269}$$

Now we drag all the reference to k_c, by means of $K_{a,\mu} = (K_{c,\mu} + p_{ac,\mu})$ and $K_{c+1}^{\nu_n} = \left(K_c^{\nu_n} + p_{c+1,c}^{\nu_n}\right)$, to obtain

$$\begin{aligned} \sum_{a=1}^{n+1} s_a K_{a,\mu} K_{\widehat{a}}^{\mathrm{C}_n} \quad = \quad & \left\{ \sum_{a \neq c} s_a \left(-1\right)^{n-c+1+\theta(a-c)} p_{\widehat{a},c}^{\mathrm{C}_{n-1}} + s_c \left(-1\right)^{n-c \bmod n} p_{\widehat{c},c+1}^{\mathrm{C}_{n-1}} \right\} K_c^{\nu_n} K_{c,\mu} \\ & + \left\{ \sum_{a \neq c} s_a \left(-1\right)^{n-c+1+\theta(a-c)} p_{\widehat{a},c}^{\mathrm{C}_{n-1}} p_{ac,\mu} \right\} K_c^{\nu_n} \\ & + \left\{ s_c \left(-1\right)^{n-c \bmod n} p_{\widehat{c},c+1}^{\mathrm{C}_{n-1}} p_{c+1,c}^{\nu_n} \right\} K_{c,\mu}. \end{aligned} \tag{3.270}$$

Now let us use $p_{b_i,c+1}^{\nu_i} = p_{b_i,c}^{\nu_i} - p_{c+1,c}^{\nu_i}$ in the product $p_{\widehat{c},c+1}^{\mathrm{C}_{n-1}} = \prod_{i=1}^{n-1} p_{b_i,c+1}^{\nu_i}$, keeping only the non-repeating terms or just one copy of $p_{c+1,c}^{\nu_i}$. What we observe is that the one which does not have the vector $p_{c+1,c}^{\nu_i}$ as a factor is always positive and the remaining ones are negative. However, in order to bring the following ordering $(\nu_{j_i<j_i+1}, b_i < b_{i+1})$ in the expression permutations must be made. The result is basically represented by

$$\cdots - + - \leftarrow \left(p_{b_1,c}^{\nu_1} \cdots \widehat{p_{c+1,c}} \cdots p_{b_{n-1},c}^{\nu_{n-1}}\right) \rightarrow - + - \cdots ,$$

the only care which must be exercised is with $c = n+1$, in that case the routing corresponds to $k_{n+1+1} \equiv k_1$ and it carries index ν_1 (in returns to one). In summary, we can cleverly codify the sign by the routings being omitted as follows

$$p_{\widehat{c},c+1}^{\mathrm{C}_n} = \sum_{a \neq c} \left(-1\right)^{c \bmod n - a + \theta(a-c)} p_{\widehat{a},c}^{\mathrm{C}_{n-1}}. \tag{3.271}$$

Thereby, the reason why, in the final result, the $\bmod n$ will not matter is that the term we are decomposing is accompanied by the same artifice. Recovering the epsilon tensor to remind us the justification of our deletions along the way, we have

$$\begin{aligned} s_c \left(-1\right)^{n-c \bmod n} \varepsilon_{\mathrm{I}_n \mathrm{C}_n} p_{\widehat{c},c+1}^{\mathrm{C}_n} \quad & = \quad s_c \left(-1\right)^{n-c \bmod n} \varepsilon_{\mathrm{I}_n \mathrm{C}_n} \sum_{a \neq c} \left(-1\right)^{c \bmod n - a + \theta(a-c)} p_{\widehat{a},c}^{\mathrm{C}_{n-1}} \qquad (3.272) \\ & = \quad \varepsilon_{\mathrm{I}_n \mathrm{C}_n} \sum_{a \neq c} s_c \left(-1\right)^{n-a+\theta(a-c)} p_{\widehat{a},c}^{\mathrm{C}_{n-1}}. \end{aligned}$$

As of the term $(-1)^{n-c \bmod n} p_{\widehat{c},c+1}^{\mathrm{C}_{n-1}} p_{c+1,c}^{\nu_n}$, we take $p_{b_i,c+1} = p_{b_i,c} - p_{c+1,c}$ what cancels any copy of $p_{c+1,c}$ since its is multiplied by $p_{c+1,c}^{\nu_n}$, and the reordering of positions just consumes the sign which arose from the same type of permutation, in equations:

$$\left(-1\right)^{n-c \bmod n} p_{\widehat{c},c+1}^{\mathrm{C}_{n-1}} p_{c+1,c}^{\nu_n} = \left(-1\right)^{n-c \bmod n} p_{\widehat{c},c}^{\mathrm{C}_{n-1}} p_{c+1,c}^{\nu_n} = p_{\widehat{c},c}^{\mathrm{C}_n} = \prod_{i=1, b_i \neq c}^{n} p_{b_i,c}^{\nu_i}. \tag{3.273}$$

What implies the following result, until now and with some arrangement,

$$\begin{aligned}\frac{\varepsilon_{\mathrm{I}_n\mathrm{C}_n}}{D_{[1,n+1]}}\sum_{a=1}^{n+1} s_a K_{a,\mu} K_{\widehat{a}}^{\mathrm{C}_n} &= \frac{\varepsilon_{\mathrm{I}_n\mathrm{C}_n}\left(-1\right)^{n-c}}{D_{[1,n+1]}}\left\{\sum_{a\neq c}\left(-1\right)^{\theta(a-c)}\left[s_c\left(-1\right)^{c-a}-s_a\right] p_{\widehat{a},c}^{\mathrm{C}_{n-1}} K_c^{\nu_n} K_{c,\mu}\right\} \\ &+\frac{\varepsilon_{\mathrm{I}_n\mathrm{C}_n}}{D_{[1,n+1]}}\left\{\left(-1\right)^{n-c}\sum_{a\neq c} s_a\left(-1\right)^{\theta(a-c)+1} p_{ac,\mu} p_{\widehat{a},c}^{\mathrm{C}_{n-1}} K_c^{\nu_n}+s_c p_{\widehat{c},c}^{\mathrm{C}_n} K_{c,\mu}\right\}.\end{aligned}\tag{3.274}$$

At this moment, we must point out a rather important fact: if the signs of the terms from a specific representation for the trace of Dirac matrices are such that they lend us a sign tensor, in the amplitudes, for which the relation below holds

$$s_a = s_c\left(-1\right)^{c-a}, \tag{3.275}$$

then the term which should represent a tensor integral $\bar{J}_{n+1,\mu}^{\nu_n}$ does not appear. Hence, that component is finite. The next feature is the completely vanishing of it as well, because the same coefficient will appear to the remaining terms. To show this we must appeal to the representation of the vector integrals as below

$$\int_{\mathbb{R}^{2n}} \mathrm{d}k \frac{K_{c,\mu}}{D_{[1,n+1]}} = J_{n+1,\mu} = -\sum_{i=1}^{n}\left(p_{b_i,c}\right)_\mu Z_{\mathbf{e}_i}^{(-1)}, \tag{3.276}$$

$$\int_{\mathbb{R}^{2n}} \mathrm{d}k \frac{K_c^{\nu_n}}{D_{[1,n+1]}} = J_{n+1}^{\nu_n} = -\sum_{i=1}^{n} p_{b_i,c}^{\nu_n} Z_{\mathbf{e}_i}^{(-1)}. \tag{3.277}$$

Then we write

$$\varepsilon_{\mathrm{I}_n\mathrm{C}_n} p_{\widehat{c},c}^{\mathrm{C}_n} J_{n+1,\mu} = -\varepsilon_{\mathrm{I}_n\mathrm{C}_n}\sum_{i=1}^{n}\left(p_{b_i,c}\right)_\mu p_{\widehat{c},c}^{\mathrm{C}_n} Z_{\mathbf{e}_i}^{(-1)} = -\varepsilon_{\mathrm{I}_n\mathrm{C}_n}\sum_{i=1}^{n}\left(p_{b_i,c}\right)_\mu \prod_{j=1}^{n} p_{b_j,c}^{\nu_j} Z_{\mathbf{e}_i}^{(-1)}, \tag{3.278}$$

thereafter we extract the momenta $p_{b_i,c}$, which corresponds to $Z_{\mathbf{e}_i}^{(-1)}$ in the vector integral, and re-index it to keep a constant index on $p_{b_i,c}$, in equations:

$$\begin{aligned}\varepsilon_{\mathrm{I}_n\mathrm{C}_n}\prod_{j=1}^{n} p_{b_j,c}^{\nu_j} &= \varepsilon_{\mathrm{I}_n\mathrm{C}_n}\left(\prod_{j=1,j\neq i}^{n} p_{b_j,c}^{\nu_j}\right) p_{b_i,c}^{\nu_i} \\ &= \left(-1\right)^{n-i}\varepsilon_{\mathrm{I}_n\mathrm{C}_n}\left(\prod_{j=1,b_j\neq b_i}^{n-1} p_{b_j,c}^{\nu_j}\right) p_{b_i,c}^{\nu_n} = \left(-1\right)^{n-i}\varepsilon_{\mathrm{I}_n\mathrm{C}_n} p_{\widehat{b_i},c}^{\mathrm{C}_{n-1}} p_{b_i,c}^{\nu_n}.\end{aligned}\tag{3.279}$$

Continuing from the previous equation. We have to complete the remaining terms of the vector integral, they will vanish in the result, since they are already present in $p_{\widehat{b_i},c}^{\mathrm{C}_{n-1}}$; again, in equations:

$$\begin{aligned}\varepsilon_{\mathrm{I}_n\mathrm{C}_n} p_{\widehat{c},c}^{\mathrm{C}_n} J_{n+1,\mu} &= \varepsilon_{\mathrm{I}_n\mathrm{C}_n}\sum_{i=1}^{n}\left(p_{b_i,c}\right)_\mu\left(-1\right)^{n-i} p_{\widehat{b_i},c}^{\mathrm{C}_{n-1}}\left[-p_{b_i,c}^{\nu_n} Z_{\mathbf{e}_i}^{(-1)}\right] \\ &= \varepsilon_{\mathrm{I}_n\mathrm{C}_n}\sum_{i=1}^{n}\left(p_{b_i,c}\right)_\mu\left(-1\right)^{n-i} p_{\widehat{b_i},c}^{\mathrm{C}_{n-1}}\left[J_{n+1}^{\nu_n}\right] \\ &= \varepsilon_{\mathrm{I}_n\mathrm{C}_n}\sum_{a\neq c}\left(p_{a,c}\right)_\mu\left(-1\right)^{n-a+\theta(a-c)} p_{\widehat{a},c}^{\mathrm{C}_{n-1}}\left[J_{n+1}^{\nu_n}\right].\end{aligned}\tag{3.280}$$

The last passage is a way to write the summation as function of the routing labels.

Finally, after gathering everything, we write

$$\begin{aligned}
&\varepsilon_{\mathrm{I}_n\mathrm{C}_n}\left\{(-1)^{n-c}\sum_{a\neq c} s_a\,(-1)^{\theta(a-c)+1}\,p_{ac,\mu}p_{\widehat{a},c}^{\mathrm{C}_{n-1}}\left[J_{n+1}^{\nu_n}\right]+s_c p_{\widehat{c},c}^{\mathrm{C}_n}\left[J_{n+1,\mu}\right]\right\} \qquad (3.281)\\
&= (-1)^{n-c}\,\varepsilon_{\mathrm{I}_n\mathrm{C}_n}\sum_{a\neq c}(-1)^{\theta(a-c)}\left[s_c\,(-1)^{c-a}-s_a\right](p_{a,c})_\mu\,p_{\widehat{a},c}^{\mathrm{C}_{n-1}}\left[J_{n+1}^{\nu_n}\right],
\end{aligned}$$

and the complete tensor now can be stated as

$$\varepsilon_{\mathrm{I}_n\mathrm{C}_n}T_\mu^{\mathbf{s},\mathrm{C}_n}=(-1)^{n-c}\,\varepsilon_{\mathrm{I}_n\mathrm{C}_n}\sum_{a\neq c}(-1)^{\theta(a-c)}\left[s_c\,(-1)^{c-a}-s_a\right]p_{\widehat{a},c}^{\mathrm{C}_{n-1}}\left\{\left[\bar{J}_{n+1,\mu}^{\nu_n}\right]+(p_{a,c})_\mu\left[J_{n+1}^{\nu_n}\right]\right\}. \qquad (3.282)$$

Ok, now we have learned how to use any routing as reference, which is extremely useful if one is studying the AV^{n+1} amplitudes, since in their IRagfs the type AV^n appear, and some AV^n do not having the least routing k_1 as argument.

Notwithstanding we have slightly adapt the definition to our uses. First, the reference routing will be k_1 that is the one used in the sections about the J integrals and their contractions. Second the integral carry index ν_1 instead of ν_n what, in tandem with the first point, free us from the sign $(-1)^{n-c}$. Third and most important, we fix the sign $s_1 = 1$ and then the vector of signs shorten by a unit,

$$\mathbf{s}'=(s_1',s_2',\cdots s_n'),\quad s_i'\in\{-1,+1\},$$

what implies that the number of independent sign tensors is 2^n for odd amplitude of vertex length $|\mathbf{\Gamma}| = n+1$, in $d = 2n$ dimensions. For these reasons, we adopt the definition.

Definition 3.7.2 *Consider $i, j \in [1, n+1]$, the strings of indices $\mathrm{I}_n^j = (\mu_i)_{i=1,i\neq j}^{n+1}$, $\mathrm{C}_n = (\nu_i)_{i=1}^n$, and $\mathrm{C}_{n-1} = (\nu_i)_{i=2}^n$. Then, with the signs coded by $\mathbf{s} = (s_1, s_2, \cdots s_n)$, the expression used for the sign tensors is*

$$\varepsilon_{\mathrm{I}_n^j\mathrm{C}_n}T_{\mu_j}^{\mathbf{s},\mathrm{C}_n}=\varepsilon_{\mathrm{I}_n^j\mathrm{C}_n}\sum_{a=1}^{n}\left[s_a+(-1)^{a+1}\right]p_{\widehat{a+1},1}^{\mathrm{C}_{n-1}}\left\{\left[\bar{J}_{n+1,\mu_j}^{\nu_1}\right]+(p_{a+1,1})_{\mu_j}\left[J_{n+1}^{\nu_1}\right]\right\}. \qquad (3.283)$$

The indices in I_n^j are in ascending order, e.g. $\mathrm{I}_4^3 = \mu_1\mu_2\mu_4$. The remaining index μ_j, which completes the sequence of indices ($\mathrm{I}_{n+1} = (\mu_i)_{i=1}^{n+1}$) for some amplitude, is carried by the tensor under definition (it is not in ε-tensor). The integrals are referenced by (k_1, m_1).

Owing to this formula, the point we want to retrieve is: there is one and only one sign tensor whose component $K_1^{\nu_1}K_{1\mu_j}$, which would be divergent, however, its coefficient is identically null. It occurs for the sign-tensor with the signs

$$s_a=(-1)^a\rightarrow s_a+(-1)^{a+1}=0,\forall a\in[1,n],$$

hence, in this conventions for the signs the number of independent non-vanishing tensors is $2^n - 1$, e.g. in four dimensions is three. Furthermore, this property stems from the signs only, not from our convenient convention. By retrieving the equation $s_a = s_c (-1)^{c-a}$ (3.275) and choosing $c = 1$, we get $s_a = (-1)^{1-a}$, and remembering we pulled down the indexing of signs, then everything fits.

Let us re-take the example in the beginning of this part and discuss some points.

Example 3.7.3 *In the previous example we presented the trace* $\mathrm{tr}\left(\gamma_* \gamma_{\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3\mu_4\nu_4}\right)$ *(six dimensions), which we call first version. In there, we extracted the part from where originates the sign tensors; here, we bring that piece back, and already arrange the signs ($K_{1\mu}$ always positive), i.e.,*

$$\begin{aligned}
& K_{1234}^{\nu_{1234}} \mathrm{tr}\left(\gamma_* \gamma_{\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3\mu_4\nu_4}\right) \\
= \;& -\varepsilon_{\mu_{234}\nu_{123}} \left[K_{1\mu_1} K_{234}^{\nu_{123}} - K_{2\mu_1} K_{134}^{\nu_{123}} + K_{3\mu_1} K_{124}^{\nu_{123}} - K_{4\mu_1} K_{123}^{\nu_{123}}\right] \\
& -\varepsilon_{\mu_{134}\nu_{123}} \left[K_{1\mu_2} K_{234}^{\nu_{123}} + K_{2\mu_2} K_{134}^{\nu_{123}} - K_{3\mu_2} K_{124}^{\nu_{123}} + K_{4\mu_2} K_{123}^{\nu_{123}}\right] \\
& +\varepsilon_{\mu_{124}\nu_{123}} \left[K_{1\mu_3} K_{234}^{\nu_{123}} - K_{2\mu_3} K_{134}^{\nu_{123}} - K_{3\mu_3} K_{124}^{\nu_{123}} + K_{4\mu_3} K_{123}^{\nu_{123}}\right] \\
& -\varepsilon_{\mu_{123}\nu_{123}} \left[K_{1\mu_4} K_{234}^{\nu_{123}} - K_{2\mu_4} K_{134}^{\nu_{123}} + K_{3\mu_4} K_{124}^{\nu_{123}} + K_{4\mu_4} K_{123}^{\nu_{123}}\right].
\end{aligned}$$

As this is for a four-point function, we divide by D_{1234} and use the formulas above to express

$$\bar{C}_{1;\mu_{1234}} = -\varepsilon_{\mu_{234}\nu_{123}} T_{\mu_1}^{(-,+,-)\nu_{123}} - \varepsilon_{\mu_{134}\nu_{123}} T_{\mu_2}^{(+,-,+)\nu_{123}} + \varepsilon_{\mu_{124}\nu_{123}} T_{\mu_3}^{(-,-,+)\nu_{123}} - \varepsilon_{\mu_{123}\nu_{123}} T_{\mu_4}^{(-,+,+)\nu_{123}}.$$

The $\bar{C}_{1;\mu_{1234}}$ notation will be used in the amplitudes, it stands for "common tensor". The reason stems from the property of it to only depend on the trace $\mathrm{tr}\left(\gamma_* \gamma_{\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3\mu_4\nu_4}\right)$ *that is common to all amplitudes $\left\{T_{\mathrm{I}_4}^{AVVV}, T_{\mathrm{I}_4}^{VAVV}, T_{\mathrm{I}_4}^{VVAV}, T_{\mathrm{I}_4}^{VVVA}, A \leftrightarrow V\right\}$, analogously for higher dimensions (and lower). About the sign-tensors we have: the one which carries the index μ_1 (corresponding to the vertex Γ_1), in this expression to the trace, is identically vanishing because $(s_1, s_2, s_3) = (-1, +1, -1)$, $\varepsilon_{\mu_{234}\nu_{123}} T_{\mu_1}^{(-,+,-)\nu_{123}} \equiv 0$. The other ones have some components that vanish. In general*

$$\begin{aligned}
\varepsilon_{\mu_{ijk}\nu_{123}} T_{\mu_j}^{(s_1,s_2,s_3)\nu_{123}} = \varepsilon_{\mu_{ijk}\nu_{123}} \quad & \Big[(s_1+1)\, p_{31}^{\nu_2} p_{41}^{\nu_3} \left(\bar{J}_{4,\mu_l}^{\nu_1} + p_{21\mu_l} J_4^{\nu_1}\right) \\
& + (s_2-1)\, p_{21}^{\nu_2} p_{41}^{\nu_3} \left(\bar{J}_{4,\mu_l}^{\nu_1} + p_{31\mu_l} J_4^{\nu_1}\right) \\
& + (s_3+1)\, p_{21}^{\nu_2} p_{31}^{\nu_3} \left(\bar{J}_{4,\mu_l}^{\nu_1} + p_{41\mu_l} J_4^{\nu_1}\right)\Big],
\end{aligned} \tag{3.284}$$

in particular

$$\begin{aligned}
\varepsilon_{\mu_{134}\nu_{123}} T_{\mu_2}^{(+,-,+)\nu_{123}} &= 2\varepsilon_{\mu_{134}\nu_{123}} \Big[p_{31}^{\nu_2} p_{41}^{\nu_3} \left(\bar{J}_{4,\mu_2}^{\nu_1} + p_{21\mu_2} J_4^{\nu_1}\right) && (3.285) \\
& \quad - p_{21}^{\nu_2} p_{41}^{\nu_3} \left(\bar{J}_{4,\mu_2}^{\nu_1} + p_{31\mu_2} J_4^{\nu_1}\right) + p_{21}^{\nu_2} p_{31}^{\nu_3} \left(\bar{J}_{4,\mu_2}^{\nu_1} + p_{41\mu_2} J_4^{\nu_1}\right)\Big], \\
\varepsilon_{\mu_{124}\nu_{123}} T_{\mu_3}^{(-,-,+)\nu_{123}} &= 2\varepsilon_{\mu_{124}\nu_{123}} \Big[-p_{21}^{\nu_2} p_{41}^{\nu_3} \left(\bar{J}_{4,\mu_3}^{\nu_1} + p_{31\mu_3} J_4^{\nu_1}\right) && (3.286) \\
& \quad + p_{21}^{\nu_2} p_{31}^{\nu_3} \left(\bar{J}_{4,\mu_3}^{\nu_1} + p_{41\mu_3} J_4^{\nu_1}\right)\Big], \\
\varepsilon_{\mu_{123}\nu_{123}} T_{\mu_4}^{(-,+,+)\nu_{123}} &= 2\varepsilon_{\mu_{123}\nu_{123}} \Big[p_{21}^{\nu_2} p_{31}^{\nu_3} \left(\bar{J}_{4,\mu_4}^{\nu_1} + p_{41\mu_4} J_4^{\nu_1}\right)\Big]. && (3.287)
\end{aligned}$$

Remark 3.7.4 *Our example showed how uniform is our handling of the structures which will appear. Besides that, we can start to talk about in the "basic/basis-versions" of the amplitudes. In them the vanishing sign-tensor carries a free-index where the IRagfs are constrained. Notice that if a IRafg or WI were being investigated which is not in the first vertex, for example* $p_{21}^{\mu_2} T_{\mu_{1234}}^{VAVV}$*, the contractions will hit directly the integrals in the tensor* $T_{\mu_2}^{(+,-,+)\nu_{123}}$*, that are then reduced to vector integrals by the formula (3.255), deduced in §§ (3.6). In the other tensors, it hits the epsilon one with one momentum* $p_{21}^{\nu_2}$ *already contracted, resulting in zero. On the other hand, if the analyzed relation is* $p_{14}^{\mu_1} T_{\mu_{1234}}^{VAVV}$*. Then the index only hits the epsilon tensor (the component* $T_{\mu_1}^{(-,+,-)\nu_{123}} = 0$ *is lacking), and to find reductions we have to reorganize first by the Schouten identity (1.38) in § (1.1). The reshuffling of indices will bring the trace* $g_{\nu_{12}} J_4^{\nu_{12}}$ *to the scene, whose formula of reduction (3.255) brings up a constant term, such term go hand in hand with the surface-terms sector (worked out separately), thus making the symmetry and linearity analysis constrained in that vertex. This is so for every amplitude using the first version of the trace.*

I intentionally proposed ${}^{6D}T_{\mu_{1234}}^{VAVV}$ as an example to raise the point that the vertex content is unrelated to what occur in different expressions of the trace. The anomalous amplitudes can always be expressed as

$$\left(T_{\mathrm{I}_{n+1}}^{\mathbf{\Gamma}}\right)_i = \bar{C}_{i,\mathrm{I}_{n+1}} + \varepsilon_{\mathrm{I}_{n+1}}{}^{\mathrm{C}_{n-1}} T_{\mathrm{C}_{n-1}}^{\mathrm{sub}_i(\mathbf{\Gamma})},$$

where the subscript i denotes the version of the trace, for which corresponds an even parity subamplitude whose vertices are $\mathrm{sub}_i\left(\mathbf{\Gamma}\right)$. The $\bar{C}_{i,\mathrm{I}_{n+1}}$ tensor is common for all amplitudes, $T_{\mathrm{I}_{n+1}}^{\mathbf{\Gamma}}$, in that version.

The last important ingredient, before the discussion of traces themselves: In $d = 2n$ a $(n+1)$-gon graph, which is a $(n-1)$-rank tensor and odd one, has its integrand proportional to

$$t_{\mathrm{I}_{n-1}}^{\mathbf{\Gamma}} = \frac{\varepsilon_{\mathrm{I}_{n-1}\mathrm{C}_{n+1}} K_{1\cdots n+1}^{\mathrm{C}_{n+1}}}{D_{[1,n+1]}} = \varepsilon_{\mathrm{I}_{n-1}\mathrm{C}_{n+1}} p_{\hat{1},1}^{\mathrm{C}_n} \frac{K_1^{\nu_1}}{D_{[1,n+1]}}. \tag{3.288}$$

Thereby, its integral being finite and proportional to the vector one will vanish. The prefactor $p_{\hat{1},1}^{\mathrm{C}_n}$ contains all the momenta that the integral can depend upon; therefore,

$$T_{\mathrm{I}_{n-1}}^{\mathbf{\Gamma}} = \varepsilon_{\mathrm{I}_{n-1}\mathrm{C}_{n+1}} p_{\hat{1},1}^{\mathrm{C}_n} J_{n+1}^{\nu_1} = 0. \tag{3.289}$$

Particular cases that will appear in two dimensions are $T^{PS} = T^{SP} = 0$, in four dimensions $T_{\mu}^{ASS} = T_{\mu}^{PVS} \cdots = 0$, and in six dimensions $T_{\mu_{12}}^{AAPS} = T_{\mu_{12}}^{AVSS} = T_{\mu_{12}}^{TPSS} = T_{\mu_{12}}^{\tilde{T}SSS} \cdots = 0$.

3.8 Traces of $2n+2$ Dirac Matrices and γ_* in $d = 2n$

For a long, we have been talking about the traces of $d+2$ matrices and attributing some importance to them. This section aims to present some arguments to motivate the adoption of a special class of expressions for our tensors. Expressions we have been calling basis/basic versions. *To advance, the name and justification of the expressions adopted is a posteriori conclusion that is tied to the LETs.*

The necessity of knowing trace's formulae for strings of chiral matrix and $d+2$ many γ-matrices must regard the variety of tensor expressions in such objects. Let us clarify this affirmation: If one computes the trace of $\gamma_{\alpha_1}\gamma_{\alpha_2}$, one may appeal to the trace of anticommutator and trace cyclicity. Fine, one obtains $\mathrm{tr}\left(\gamma_{\alpha_1}\gamma_{\alpha_2}\right) = \mathrm{tr}\left(\mathbf{1}_{2^n\times 2^n}\right) g_{\alpha_1\alpha_2}$. When one computes the trace of three matrices, and for what matter one matrix, one appeal for the chiral matrix, which is defined for even dimensions, and get $\mathrm{tr}\left(\gamma_{\alpha_{123}}\right) = -\mathrm{tr}\left(\gamma_{\alpha_{123}}\right) \Rightarrow \mathrm{tr}\left(\gamma_{\alpha_{123}}\right) = 0$. Then, for any odd number of matrices $\mathrm{tr}\left(\gamma_{\mathrm{I}_{2n+1}}\right) = 0$. Heuristically, the reason is that the answer should be a combination of metric and Levi-Civita tensor, both of even rank, hence the vanishing result.

In this path we have to compute the trace of four matrices, for which we use the recursion of the traces (1.11) and get

$$\begin{aligned}
\mathrm{tr}\left(\gamma_{\alpha_{1234}}\right) &= g_{\alpha_{12}}\mathrm{tr}\left(\gamma_{\alpha_{34}}\right) - g_{\alpha_{13}}\mathrm{tr}\left(\gamma_{\alpha_{24}}\right) + g_{\alpha_{14}}\mathrm{tr}\left(\gamma_{\alpha_{23}}\right) \\
&= 2^n\left(g_{\alpha_{12}}g_{\alpha_{34}} - g_{\alpha_{13}}g_{\alpha_{24}} + g_{\alpha_{14}}g_{\alpha_{23}}\right),
\end{aligned}$$

Some commentary: for this section, with opportunity, we will absorb factors like 2^n.

Now, for every dimension $d \geq 2$ (odd included) this specific trace can not be expressed by lesser number of monomials, and the signs of the terms are fixed as well. The same expression codes the trace of other strings, e.g., the reversed string $\gamma_{\alpha_{4321}}$. However, for each string of four matrices we have only one disposition of monomials for tensor products of the metric tensor carrying the string's indices.

A priori, the number of monomials in an even parity trace $\mathrm{tr}\left(\gamma_{\mathrm{I}_{2l}}\right)$ is $(2l-1)!!$. Nevertheless, by noticing there are oscillations of sign (different from $g_{\mathrm{I}_{2l}}$), when the rank is high enough, one can intuitively anticipate the raising of fully antisymmetric components. If these components can have rank larger than the dimension, then they are identically vanishing and can removed from the expression. Therefore, the expressions are not unique in the sense there are null tensors among the components.

Let us continue with the traces. For six matrices we have

$$\begin{aligned}
\frac{\mathrm{tr}\left(\gamma_{\alpha_{123456}}\right)}{2^n} &= +g_{\alpha_1\alpha_2}g_{\alpha_3\alpha_4}g_{\alpha_5\alpha_6} - g_{\alpha_1\alpha_2}g_{\alpha_3\alpha_5}g_{\alpha_4\alpha_6} + g_{\alpha_1\alpha_2}g_{\alpha_3\alpha_6}g_{\alpha_4\alpha_5} \\
&\quad -g_{\alpha_1\alpha_3}g_{\alpha_2\alpha_6}g_{\alpha_4\alpha_5} + g_{\alpha_1\alpha_6}g_{\alpha_2\alpha_3}g_{\alpha_4\alpha_5} - g_{\alpha_1\alpha_6}g_{\alpha_2\alpha_4}g_{\alpha_3\alpha_5} \\
&\quad -g_{\alpha_1\alpha_5}g_{\alpha_2\alpha_6}g_{\alpha_3\alpha_4} + g_{\alpha_1\alpha_6}g_{\alpha_2\alpha_5}g_{\alpha_3\alpha_4} + g_{\alpha_1\alpha_4}g_{\alpha_2\alpha_6}g_{\alpha_3\alpha_5} \\
&\quad \left[-g_{\alpha_1\alpha_3}g_{\alpha_2\alpha_4}g_{\alpha_5\alpha_6} - g_{\alpha_1\alpha_4}g_{\alpha_2\alpha_5}g_{\alpha_3\alpha_6} - g_{\alpha_1\alpha_5}g_{\alpha_2\alpha_3}g_{\alpha_4\alpha_6}\right. \\
&\quad \left.+\, g_{\alpha_1\alpha_5}g_{\alpha_2\alpha_4}g_{\alpha_3\alpha_6} + g_{\alpha_1\alpha_3}g_{\alpha_2\alpha_5}g_{\alpha_4\alpha_6} + g_{\alpha_1\alpha_4}g_{\alpha_2\alpha_3}g_{\alpha_5\alpha_6}\right]
\end{aligned} \tag{3.290}$$

The last two rows are a determinant, specifically

$$\begin{aligned}
& \det\left(\alpha_3\alpha_4\alpha_5;\alpha_6\alpha_2\alpha_1\right) \qquad (3.291)\\
= & \det\begin{pmatrix} g_{\alpha_3\alpha_6} & g_{\alpha_4\alpha_6} & g_{\alpha_5\alpha_6}\\ g_{\alpha_3\alpha_2} & g_{\alpha_4\alpha_2} & g_{\alpha_5\alpha_2}\\ g_{\alpha_3\alpha_1} & g_{\alpha_4\alpha_1} & g_{\alpha_5\alpha_1}\end{pmatrix}\\
= & +g_{\alpha_1\alpha_5}g_{\alpha_2\alpha_4}g_{\alpha_3\alpha_6}+g_{\alpha_1\alpha_3}g_{\alpha_2\alpha_5}g_{\alpha_4\alpha_6}+g_{\alpha_1\alpha_4}g_{\alpha_2\alpha_3}g_{\alpha_5\alpha_6}\\
& -g_{\alpha_1\alpha_3}g_{\alpha_2\alpha_4}g_{\alpha_5\alpha_6}-g_{\alpha_1\alpha_4}g_{\alpha_2\alpha_5}g_{\alpha_3\alpha_6}-g_{\alpha_1\alpha_5}g_{\alpha_2\alpha_3}g_{\alpha_4\alpha_6}.
\end{aligned}$$

However, the fragment identified is not the only determinant expression that can be identified, another example is $\det\left(\alpha_1\alpha_2\alpha_3;\alpha_6\alpha_5\alpha_4\right)$. Therefore, in an adequate dimension, the column or row vectors become linearly dependent, and those terms vanish. That means, in two dimensions there are shorter expressions for the traces starting by $2.2+2=6$ matrices. There are algorithms developed to simplify high-rank traces, see Kahane [81], Caianiello [80] and Chilsom [82] for references on the subject.

In general, the trace with $2d+2$ or more matrices in d dimensions have fragments that can be removed since they are fully antisymmetric tensors of rank $d+1$ (note that we can add terms in this way under the same condition some can be removed). One question can be raised, are these simplifications affecting something? For parity even amplitudes of vertices with rank at most one, vector vertices, the answer is no. Even if one disregard the aspect of renormalizability, the pure vector amplitudes where this ingredient will appear occur in $d+1$-pt ones, each propagator contributes with one matrix and each vertex with the other pair. Hence, if no derivative interaction is assumed, the power counting is negative. Furthermore, they are the first fermionic loop of the type considered in this thesis that are finite, and exactly when parity even traces could be casted in more than one form. Examples are the 2D triangle, 4D pentagon, 6D heptagon.

Nevertheless, if in four dimensions the tensor coupling $\gamma_{[\mu\nu]}$ were included and analyzed on equal footing, despite being non-renormalizable coupling, it would entertain us into questions, at least, similar to the anomalies, and in even amplitudes. This is not what we shall do anywhere in the following, but I let it here as a note since the fully consideration of all even vertices has its own subtleties.

Having observed that even amplitudes can show in its ingredients a set of options to follow in a computation, but, only are available when the amplitudes are already finite (care here, because their IRagfs connect them to non-finite ones). We start now by describing the odd traces, the ones where the epsilon tensor appear. They have the ingredients in the level of non-negative power counting amplitudes.

The first non-zero trace with chiral matrix in $d=2n$ is[12]

$$\operatorname{tr}\left(\gamma_*\gamma_{\alpha_1\ldots\alpha_{2n}}\right)=2^n\left(-\mathrm{i}\right)^{n+1}\left(\varepsilon_{\alpha_1\ldots\alpha_{2n}}\right). \tag{3.292}$$

Only one monomial. Observe that one way to obtain it is inserting the string $\gamma_{0,1,\cdots d-1}$, and seeing that only for all distinct α values the result is no-zero. However, my point is that we have only $2d$ matrices in the string $\gamma_{0,1,\cdots d-1}\gamma_{\alpha_1\ldots\alpha_{2n}}$. Better, we could have analyzed the expression $\varepsilon^{\beta_1\ldots\beta_{2n}}\operatorname{tr}\left(\gamma_{\beta_1\ldots\beta_{2n}}\gamma_{\alpha_1\ldots\alpha_{2n}}\right)$, which obviously has at most $2d$ matrices; hence, the trace has a unique monomial expansion. The role of the contraction with epsilon is to project everything down to only one term, due to its full antisymmetry. For less than d γ_{α_i} matrices we would saturate the combinations $g_{\beta_i\alpha_i}$, and some β indices would have to be allocated in one metric, for all monomials; hence, the result is always zero is this case.

This is unavoidable because the artifice of selecting a different expression is not available. The indices of $\operatorname{tr}\left(\gamma_{\beta_1\ldots\beta_{2n}}\gamma_{\alpha_1\ldots\alpha_{i\leq 2n}}\right)$ are locked and without options in their monomials. The opportunity for diverse expressions solely occurs at the level of $2d+2$ matrices, or more (in an even trace, not counting the chiral matrix).

Being the ideas introduced, let us bring the problem of computing the following trace

$$\operatorname{tr}\left(\gamma_*\gamma_{\alpha_1\cdots\alpha_{2n+2}}\right).$$

The first thing to note is: if the definition of γ_* is called here, we immediately see that there will be a trace of $4n+2=2d+2$ gamma matrices to cope with. Moreover, this is exactly the threshold for an even trace to show up with multiple options for its expansion. Contrary to the even amplitudes, where the possibility of having multiple expressions to the same tensor occur only in finite amplitudes (with only vector vertices), this trace starts to occur at a lower number of points in $d/2+1=n+1$ in the odd amplitudes.

Below we will make a thoroughly determination of a qualified general form for this tensor. Now, the paths to compute these traces are numerous, all of the following formulas can be "massaged" to make a trade between the chiral matrix and an even string of gamma matrices,

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[12] Here we deduce $\operatorname{tr}\left(\gamma_*\gamma_{\alpha_1\ldots\alpha_{2n}}\right)=2^n\left(-\mathrm{i}\right)^{n+1}\left(\varepsilon_{\alpha_1\ldots\alpha_{2n}}\right)$. Take $\mathrm{I}_{2n}=\alpha_1\ldots\alpha_{2n}$ in $\operatorname{tr}\left(\gamma_*\gamma_{\mathrm{I}_{2n}}\right)=c_{2n}\varepsilon_{\alpha_1\ldots\alpha_{2n}}$, there is only one independent component $\gamma_*=\mathrm{i}^{n-1}\gamma_0\cdots\gamma_{2n-1}$. Thus, $\mathrm{i}^{n-1}\operatorname{tr}\left(\gamma_0\cdots\gamma_{2n-1}\gamma_0\cdots\gamma_{2n-1}\right)=c_{2n}\varepsilon_{0,1,\cdots 2n-1}$.To determine the constant c_{2n} we start by the normalization $\varepsilon_{0,1,\cdots 2n-1}=-1$; then, the number of permutations to reverse the product of matrices $\gamma_0\cdots\gamma_{2n-1}$ is $n\left(2n-1\right)$, all get a minus sign since they are distinct matrices. Thereafter $2n-1$ squares of Dirac matrices get a minus sign because they are spatial components. Summing up: $\mathrm{i}^{n-1}\left(-1\right)^{n(2n-1)}\left(-1\right)^{2n-1}\operatorname{tr}\left(\mathbf{1}_{2^n\times 2^n}\right)=-c_{2n}\Rightarrow c_{2n}=2^n\left(-\mathrm{i}\right)^{n+1}$. Thereby

$$\operatorname{tr}\left(\gamma_*\gamma_{\alpha_1\ldots\alpha_{2n}}\right)=2^n\left(-\mathrm{i}\right)^{n+1}\left(\varepsilon_{\alpha_1\ldots\alpha_{2n}}\right).$$

We could alternatively express by $\gamma_*\gamma_{[\alpha_1\ldots\alpha_{2n}]}=\left(-\mathrm{i}\right)^{n+1}\varepsilon_{\alpha_1\ldots\alpha_{2n}}$, and observe that the difference of $\gamma_{[\alpha_1\ldots\alpha_{2n}]}$ from $\gamma_{\alpha_1\ldots\alpha_{2n}}$ are lower grade skewsymmetric products whose trace vanish. Since there is no tensor to accomodate their antisymmetry, therefore the previous formula follows.

plus a contraction with the Levi-Civita tensor.

$$\gamma_* = \frac{\mathrm{i}^{n-1}}{(2n)!}\varepsilon_{\mathrm{C}_{2n}}\gamma^{[\mathrm{C}_{2n}]} = \mathrm{i}^{n-1}\gamma_0\cdots\gamma_{2n-1}, \tag{3.293}$$

$$\gamma_*\gamma_{[\mathrm{I}_1]} = \frac{\mathrm{i}^{n-1}}{(2n-1)!}\varepsilon_{\mathrm{I}_1\mathrm{C}_{2n-1}}\gamma^{[\mathrm{C}_{2n-1}]}, \tag{3.294}$$

$$\vdots$$

$$\gamma_*\gamma_{[\mathrm{I}_r]} = \frac{\mathrm{i}^{r(r-1)+n-1}}{(2n-r)!}\varepsilon_{\mathrm{I}_r\mathrm{C}_{2n-r}}\gamma^{[\mathrm{C}_{2n-r}]}, \tag{3.295}$$

$$\vdots$$

$$\gamma_*\gamma_{[\mathrm{I}_{2n}]} = (-\mathrm{i})^{n+1}\,\varepsilon_{\mathrm{I}_{2n}}. \tag{3.296}$$

We will call the list the "tower" of identities, for no reason besides a name. One path in particular is the definition, i.e. the first identity above, in multiple ways

$$\frac{\mathrm{i}^{n-1}}{(2n)!}\varepsilon^{\beta_1\ldots\beta_{2n}}\mathrm{tr}\left(\gamma_{\beta_1\ldots\beta_{2n}}\gamma_{\alpha_1\cdots\alpha_{2n+2}}\right),\ \text{or}\ \frac{\mathrm{i}^{n-1}}{(2n)!}\varepsilon^{\beta_1\ldots\beta_{2n}}\mathrm{tr}\left(\gamma_{\alpha_1\alpha_2}\gamma_{\beta_1\ldots\beta_{2n}}\gamma_{\alpha_1\cdots\alpha_{2n+2}}\right). \tag{3.297}$$

The second identity would produce, by a particular application, the following

$$\frac{\mathrm{i}^{n-1}}{(2n)!}\varepsilon_{\alpha_1}{}^{\beta_1\ldots\beta_{2n-1}}\mathrm{tr}\left(\gamma_{\beta_1\ldots\beta_{2n-1}}\gamma_{\alpha_2\cdots\alpha_{2n+2}}\right). \tag{3.298}$$

The place to use substitutions is not constrained, these are finite dimensional matrices. In the first examples, all monomials of the type $g_{\alpha_{i_1}\alpha_{i_2}}\varepsilon_{\alpha_{i_3\ldots i_{2n}}}$ appear, whereas in the last example the index α_1 will not appear in the metric tensor.

We will argue in the sequel that only one place of substitution is enough to our purposes. Since the alpha indices are arbitrary, we can rename them to get any other expression. To advance, we will choose a set of expressions arising from the use of the definition of chiral matrix as basis versions, e.g. by using (3.297). However, a priori, no reason could be offered to such choice, if not by some "symmetry" that those expressions have due to containing all monomials available. The results derived would be identical if the second procedure is employed, i.e. using (3.298) . The reason is that there are expressions whose differences are perfectly finite and vanishing integrals, combinations of the vanishing sign tensor and a finite and null amplitude, see section (3.7), eqs. (3.275,3.289), and definition (3.7.2).

- **Computing explicitly a set of traces (for each dimension).**

Using the γ_* definition, or brute force approach: Start by the provisional expression $\mathrm{tr}\left(\gamma_{\beta_1\ldots\beta_{2n}}\gamma_{\alpha_1\cdots\alpha_{2n+2}}\right)$, then use the recursion obeyed by traces, eq.(1.11). The expanded trace looks like

$$\begin{aligned}\mathrm{tr}\left(\gamma_{\beta_1\ldots\beta_{2n}}\gamma_{\alpha_1\cdots\alpha_{2n+2}}\right) &= \sum_{i=1}^{2n-1}(-1)^{i+1}g_{\beta_1\beta_{i+1}}\mathrm{tr}\left(\gamma_{\beta_2\ldots\widehat{\beta_i}\ldots\beta_{2n}}\gamma_{\alpha_1\cdots\alpha_{2n+2}}\right) \\ &+\sum_{i=1}^{2n+2}(-1)^{i}g_{\beta_1\alpha_i}\mathrm{tr}\left(\gamma_{\beta_2\ldots\beta_{2n}}\gamma_{\alpha_1\ldots\widehat{\alpha_i}\ldots\alpha_{2n+2}}\right).\end{aligned} \tag{3.299}$$

Now, the first summation gives zero by multiplication with $\varepsilon^{\beta_1 \ldots \beta_{2n}}$. Thus, it follows that

$$\varepsilon^{\beta_1 \ldots \beta_{2n}} \operatorname{tr}\left(\gamma_{\beta_1 \ldots \beta_{2n}} \gamma_{\alpha_1 \cdots \alpha_{2n+2}}\right) = \varepsilon^{\beta_1 \ldots \beta_{2n}} \sum_{i_1=1}^{2n+2} (-1)^{i_1} g_{\beta_1 \alpha_{i_1}} \operatorname{tr}\left(\gamma_{\beta_2 \ldots \beta_{2n}} \gamma_{\ldots \widehat{\alpha_{i_1}} \ldots}\right). \tag{3.300}$$

Next, let us bring β_2 out of the trace through the recursion. The process engender the expression

$$\begin{aligned} & \varepsilon^{\beta_1 \ldots \beta_{2n}} \operatorname{tr}\left(\gamma_{\beta_1 \ldots \beta_{2n}} \gamma_{\alpha_1 \cdots \alpha_{2n+2}}\right) \\ = \; & \varepsilon^{\beta_1 \ldots \beta_{2n}} \sum_{i_1 \in [1,2n+2]} (-1)^{i_1} g_{\beta_1 \alpha_{i_1}} \sum_{i_2 \in \{i_1\}^c} (-1)^{1+i_2+\theta(i_2-i_1)} g_{\beta_2 \alpha_{i_2}} \operatorname{tr}\left(\gamma_{\beta_3 \ldots \beta_{2n}} \gamma_{\ldots \widehat{\alpha_{i_1}} \ldots \widehat{\alpha_{i_2}} \ldots}\right). \end{aligned} \tag{3.301}$$

The factor $(-1)^{1+i_2+\theta(i_2-i_1)}$ secures us to have the right signs. First, since for β_2 to pick the first α_{i_2} to pair with in the metric, the number of β's it passes is one less than for β_1 and we added one. Second, if $i_1 > i_2$ is removed from the string and paired with β_1, the signs will oscillate with the index i_2. On the other hand, if $i_1 < i_2$, then the index i_2 misreads the position in the string, for example, if $i_1 = 2$ and $i_2 = 4$ the α part is $\alpha_1 \alpha_3 \alpha_4 \cdots$ (before pairing with β_2) the position of α_4 is the third one not the fourth, then we compensate this with the theta function $\theta(i_2 - i_1)$. The i_2 in $(-1)^{\cdots}$ does the rest, as in all the other pairings.

We then pair all the β indices, and there will remain only two α ones, i.e.

$$\begin{aligned} & \varepsilon^{\beta_1 \ldots \beta_{2n}} \operatorname{tr}\left(\gamma_{\beta_1 \ldots \beta_{2n}} \gamma_{\alpha_1 \cdots \alpha_{2n+2}}\right) \\ = \; & \sum_{i_1 \in [1,2n+2]} \sum_{i_2 \in \{i_1\}^c} \cdots \sum_{i_{2n} \in \{i_1, \cdots i_{2n-1}\}^c} (-1)^{\sum_{j=1}^{2n-1} j + \sum_{j=1}^{2n} i_j + \sum_{k=2}^{2n} \sum_{j=1}^{k-1} \theta(i_k - i_j)} \\ & \times \left\{ \left[\varepsilon^{\beta_1 \ldots \beta_{2n}} g_{\beta_1 \alpha_{i_1}} \cdots g_{\beta_{2n} \alpha_{i_{2n}}}\right] \operatorname{tr}\left(\gamma_{\alpha_{i_{2n+1}} \alpha_{i_{2n+2}}}\right)\right\}. \end{aligned} \tag{3.302}$$

At each new round we have introduced thetas, e.g., $\theta(i_3 - i_1) + \theta(i_3 - i_2)$ appear in the pairing with β_3, and so on.

Then we have the term $\varepsilon^{\beta_1 \ldots \beta_{2n}} g_{\beta_1 \alpha_{i_1}} \cdots g_{\beta_{2n} \alpha_{i_{2n}}} = \varepsilon_{\alpha_{i_1} \cdots \alpha_{i_{2n}}}$, for which an ordering of the summations is called for. Given an specific term, with the factor $\varepsilon_{\alpha_{i_1} \cdots \alpha_{i_{2n}}}$ carrying a particular set of indices $\{i_1, \cdots, i_{2n}\} \subset [1, 2n+2]$. There appears $(2n)!$ terms due to the order of picking the indices, but the sign coming from thetas always cancels the permutation's sign needed to bring the indices in order $\sigma(i_1) < \cdots < \sigma(i_{2n})$, that is $\varepsilon_{\alpha_{\sigma(i_1)} \cdots \alpha_{\sigma(i_{2n})}}$. A short piece with three indices illustrates the matter better

$$\begin{aligned} & (-1)^{\theta(i_2-i_1)+\theta(i_3-i_1)+\theta(i_3-i_2)+\cdots} \varepsilon_{\alpha_{i_1} \alpha_{i_2} \alpha_{i_3} \cdots}, \\ i_1 & < i_2 < i_3 \Rightarrow (-1)^{3+\cdots} \varepsilon_{\alpha_{i_1} \alpha_{i_2} \alpha_{i_3} \cdots} \\ i_2 & < i_1 < i_3 \Rightarrow (-1)^{2+\cdots} \varepsilon_{\alpha_{i_1} \alpha_{i_2} \alpha_{i_3} \cdots} = (-1)^{3+\cdots} \varepsilon_{\alpha_{i_2} \alpha_{i_1} \alpha_{i_3} \cdots} \\ i_3 & < i_2 < i_1 \Rightarrow (-1)^{0+\cdots} \varepsilon_{\alpha_{i_1} \alpha_{i_2} \alpha_{i_3} \cdots} = (-1)^{3+\cdots} \varepsilon_{\alpha_{i_3} \alpha_{i_2} \alpha_{i_1} \cdots}. \end{aligned} \tag{3.303}$$

So we can take only the ascending summations $i_1 < \cdots < i_{2n}$, and stays with the sign coming from the sum of thetas, which in this setup all give one. Thus, by factorizing $(2n)!$ identical terms from the $(2n+2)!/2$ terms in the formula above, we get

$$\varepsilon^{\beta_1 \ldots \beta_{2n}} \operatorname{tr}\left(\gamma_{\beta_1 \cdots \beta_{2n}} \gamma_{\alpha_1 \cdots \alpha_{2n+2}}\right) = (2n)! \operatorname{tr}\left(\mathbf{1}_{2^n \times 2^n}\right) =$$

$$= \left\{ \sum_{1\leq i_1<i_2\cdots<i_{2n}\leq 2n+2} (-1)^{\sum_{j=1}^{2n-1} j+\sum_{j=1}^{2n} i_j+\sum_{k=2}^{2n}\sum_{j=1}^{k-1} 1} \varepsilon_{\alpha_{i_1}\cdots\alpha_{i_{2n}}} g_{\alpha_{i_{2n+1}}\alpha_{i_{2n+2}}} \right\} \tag{3.304}$$

Now, we apply some summation formulas to obtain

$$(-1)^{\sum_{j=1}^{2n-1} j+\sum_{k=2}^{2n}\sum_{j=1}^{k-1} 1} = (-1)^{2n^2+2n(n-1)} = 1,$$

and arrive at

$$\begin{aligned} &\frac{1}{(2n)!}\varepsilon^{\beta_1\ldots\beta_{2n}}\mathrm{tr}\left(\gamma_{\beta_1\cdots\beta_{2n}}\gamma_{\alpha_1\cdots\alpha_{2n+2}}\right) \\ &= 2^n \left\{ \sum_{1\leq i_1<i_2\cdots<i_{2n}\leq 2n+2} (-1)^{\sum_{j=1}^{2n} i_j} \varepsilon_{\alpha_{i_1}\cdots\alpha_{i_{2n}}} g_{\alpha_{i_{2n+1}}\alpha_{i_{2n+2}}} \right\}. \end{aligned} \tag{3.305}$$

To wrap up, the signs and summations can re-written in term of the indices i_{2n+1}, i_{2n+2}. Notice that for $i_j \in [1, 2n+2]$ the formulae below holds

$$\begin{aligned} \sum_{j=1}^{2n+2} i_j &= (n+1)(2n+3) = \sum_{j=1}^{2n} i_j + i_{2n+1} + i_{2n+2} \\ \Rightarrow (-1)^{\sum_{j=1}^{2n} i_j} &= (-1)^{(n+1)(2n+3)-i_{2n+1}-i_{2n+2}} = (-1)^{n+1+i_{2n+1}+i_{2n+2}}. \end{aligned} \tag{3.306}$$

Thereby, multiplication by i^{n-1} and recognition of $\gamma_* = \mathrm{i}^{n-1}\varepsilon^{\beta_1\ldots\beta_{2n}}\gamma_{\beta_1\ldots\beta_{2n}}/\,(2n)!$ enable us to write

$$\begin{aligned} &\mathrm{tr}\left(\gamma_*\gamma_{\alpha_1\cdots\alpha_{2n+2}}\right) \\ &= 2^n(-\mathrm{i})^{n+1}\left\{ \sum_{1\leq i_{2n+1}<i_{2n+2}\leq 2n+2} (-1)^{i_{2n+1}+i_{2n+2}+1} g_{\alpha_{i_{2n+1}}\alpha_{i_{2n+2}}}\varepsilon_{\alpha_{i_1<\cdots<i_{2n}}} \right\}, \\ i_j &\in [1, 2n+2]. \end{aligned} \tag{3.307}$$

All indices are written in ascending order, and the result follows since the expressions have the same terms and signs. The kept dummy indices are irrelevant; their intent is to keep the formulas connected somehow.

Having at least one formula, we can discuss the value of starting with alpha indices. It stems from the possibility of choosing $(\alpha_{2i-1}, \alpha_{2i}) = (\mu_i, \nu_i)$ for $i \in [1, n+1]$ and write down the expression for

$$\mathrm{tr}(\gamma_* \prod_{i=1}^{n+1}\gamma_{\mu_i\nu_i}) = \mathrm{tr}\left(\gamma_*\gamma_{\mu_1\nu_1\mu_2\nu_2\cdots\mu_{n+1}\nu_{n+1}}\right). \tag{3.308}$$

Where the use of $(\alpha_{2i-1}, \alpha_{2i}) = (\mu_i, \nu_i)$, as we put it, corresponds to replace the definition of the chiral matrix in exactly the position it appeared, left to the γ_{μ_1} matrix. On the other hand, if we have chosen $(\alpha_{2i-1}, \alpha_{2i}) = \left(\mu_{i+1}, \nu_{i+1}\right)$, for $i \in [1, n]$ and $(\alpha_{2n-1}, \alpha_{2n+2}) = (\mu_1, \nu_1)$, then the expression for $\mathrm{tr}\left(\gamma_*\gamma_{\alpha_1\cdots\alpha_{2n+2}}\right)$ will corresponds to

$$\mathrm{tr}(\gamma_*(\prod_{i=1}^{n}\gamma_{\mu_{i+1}\nu_{i+1}})\gamma_{\mu_1\nu_1}) = \mathrm{tr}\left(\gamma_*\gamma_{\mu_2\nu_2\cdots\mu_{n+1}\nu_{n+1}\mu_1\nu_1}\right), \tag{3.309}$$

and the idea proceeds for the other vertices.

To express some trace where we deploy the definition of the chiral matrix in an arbitrary position, it is only a matter of using the appropriate interpretation for the α indices. However, what do these last two expressions have in common? Obviously, by their construction, they contain all monomials possible to form using $2n+2$ Lorentz indices, viz. $(n+1)(2n+1)$ monomials. Furthermore, they are equal since anticommuting the chiral matrix two times to the left, and by trace cyclicity we can write the identity

$$\mathrm{tr}\left(\gamma_*\gamma_{\mu_1\nu_1\mu_2\nu_2\cdots\mu_{n+1}\nu_{n+1}}\right)=\mathrm{tr}\left(\gamma_{\mu_1\nu_1}\gamma_*\gamma_{\mu_2\nu_2\cdots\mu_{n+1}\nu_{n+1}}\right)=\mathrm{tr}\left(\gamma_*\gamma_{\mu_2\nu_2\cdots\mu_{n+1}\nu_{n+1}\mu_1\nu_1}\right). \tag{3.310}$$

Such identity is consequence of finite dimensionality of the matrices, but in a scenario where undefined objects appear nontrivial things may happen.

As said, they contain all Lorentz structures possible to such a tensor. Therefore, in a first look, if not absolutely identical, they should be equivalent. In fact, their difference are linear combinations of fully antisymmetric tensors in $2n+1$ indices, thereby, vanishing in $d=2n$.

The previous expressions based on the definition of chiral matrix are not special. So, if in the computation of $\mathrm{tr}\left(\gamma_*\gamma_{\alpha_1\cdots\alpha_{2n+2}}\right)$ one uses the second identity

$$\gamma_*\gamma_{\alpha_1}=\frac{\mathrm{i}^{n-1}}{(2n-1)!}\varepsilon_{\alpha_1}{}^{\beta_1\cdots\beta_{2n-1}}\gamma_{\beta_1\cdots\beta_{2n-1}}, \tag{3.311}$$

in the "tower" we talked above, then one obtains a shorter formula. That means, this formula

$$\mathrm{tr}\left(\gamma_*\gamma_{\alpha_1\cdots\alpha_{2n+2}}\right)=2^n(-\mathrm{i})^{n+1}\left\{\sum_{2\leq a<b\leq 2n+2}(-1)^{a+b+1}g_{\alpha_a\alpha_b}\varepsilon_{\alpha_1\cdots\widehat{\alpha_a}\cdots\widehat{\alpha_b}\cdots\alpha_{2n+2}}\right\}. \tag{3.312}$$

Note the lower limit of the summation starts in two, not one. The method is the same, however, the index α_1 is fixed in the epsilon tensor, thus not appearing in the metric and the length is $n(2n+1)$ instead of $(n+1)(2n+1)$ monomials.

Now, this expression is not "symmetric", in the sense that some monomials do not appear; however, they result in the same amplitudes. *In reality, the configuration of indices acquired by the tensors that we call basic versions of some amplitude is the fruit of them satisfying the largest number of IRagfs automatically, i.e., without choice or intervention.* This feature is a posteriori observation, which can be understood by the play of linearity, momentum-space homogeneity, and LETs in the study of explicit and general forms for amplitudes, and the constraints we establish for them.

Other expressions, which do not contain the same monomials, will furnish the same integrated expression. Therefore, this clarification for what we call basic version was felt necessary. Let us work with some example.

Example 3.8.1 *As a general instance we apply the identity with $r=0$, i.e. definition of γ_*,* *to the left of the γ_{α_1}. The $\underline{d=4}$ specialization of the formula above gives us*

$$\begin{aligned}
&\overline{\mathrm{tr}}_1^{(0)}\left(\gamma_*\gamma_{\alpha_1\cdots\alpha_6}\right)\\
=\;&+g_{\alpha_1\alpha_2}\varepsilon_{\alpha_3\alpha_4\alpha_5\alpha_6}-g_{\alpha_1\alpha_3}\varepsilon_{\alpha_2\alpha_4\alpha_5\alpha_6}+g_{\alpha_1\alpha_4}\varepsilon_{\alpha_2\alpha_3\alpha_5\alpha_6}-g_{\alpha_1\alpha_5}\varepsilon_{\alpha_2\alpha_3\alpha_4\alpha_6}+g_{\alpha_1\alpha_6}\varepsilon_{\alpha_2\alpha_3\alpha_4\alpha_5}\\
&+g_{\alpha_2\alpha_3}\varepsilon_{\alpha_1\alpha_4\alpha_5\alpha_6}-g_{\alpha_2\alpha_4}\varepsilon_{\alpha_1\alpha_3\alpha_5\alpha_6}+g_{\alpha_2\alpha_5}\varepsilon_{\alpha_1\alpha_3\alpha_4\alpha_6}-g_{\alpha_2\alpha_6}\varepsilon_{\alpha_1\alpha_3\alpha_4\alpha_5}\\
&+g_{\alpha_3\alpha_4}\varepsilon_{\alpha_1\alpha_2\alpha_5\alpha_6}-g_{\alpha_3\alpha_5}\varepsilon_{\alpha_1\alpha_2\alpha_4\alpha_6}+g_{\alpha_3\alpha_6}\varepsilon_{\alpha_1\alpha_2\alpha_4\alpha_5}\\
&+g_{\alpha_4\alpha_5}\varepsilon_{\alpha_1\alpha_2\alpha_3\alpha_6}-g_{\alpha_4\alpha_6}\varepsilon_{\alpha_1\alpha_2\alpha_3\alpha_5}\\
&+g_{\alpha_5\alpha_6}\varepsilon_{\alpha_1\alpha_2\alpha_3\alpha_4}
\end{aligned}$$

The bar is a provisional notation to absorb the factor $2^n(-\mathrm{i})^{n+1}$. Applying the identity with $r=1$ to the left of γ_{α_1} will immediately implies the tensor

$$\begin{aligned}
&\overline{\mathrm{tr}}_1^{(1)}\left(\gamma_*\gamma_{\alpha_1\cdots\alpha_6}\right)\\
=\;&+g_{\alpha_2\alpha_3}\varepsilon_{\alpha_1\alpha_4\alpha_5\alpha_6}-g_{\alpha_2\alpha_4}\varepsilon_{\alpha_1\alpha_3\alpha_5\alpha_6}+g_{\alpha_2\alpha_5}\varepsilon_{\alpha_1\alpha_3\alpha_4\alpha_6}-g_{\alpha_2\alpha_6}\varepsilon_{\alpha_1\alpha_3\alpha_4\alpha_5}\\
&+g_{\alpha_3\alpha_4}\varepsilon_{\alpha_1\alpha_2\alpha_5\alpha_6}-g_{\alpha_3\alpha_5}\varepsilon_{\alpha_1\alpha_2\alpha_4\alpha_6}+g_{\alpha_3\alpha_6}\varepsilon_{\alpha_1\alpha_2\alpha_4\alpha_5}\\
&+g_{\alpha_4\alpha_5}\varepsilon_{\alpha_1\alpha_2\alpha_3\alpha_6}-g_{\alpha_4\alpha_6}\varepsilon_{\alpha_1\alpha_2\alpha_3\alpha_5}\\
&+g_{\alpha_5\alpha_6}\varepsilon_{\alpha_1\alpha_2\alpha_3\alpha_4}.
\end{aligned}$$

Note the first row of the previous expression is not present. The additional provisional notation $\overline{\mathrm{tr}}_1^{(0)}$ and $\overline{\mathrm{tr}}_1^{(1)}$ will be elaborated in a minute.

Let it be the true assertion for any two ways of express the trace

$$\forall\mu_i^0\;:\;\mu_i^0=0,\ldots,d-1:\overline{\mathrm{tr}}_1\left(\gamma_{*\mathrm{I}^0}\right)-\overline{\mathrm{tr}}_2\left(\gamma_{*\mathrm{I}^0}\right)=0$$

For all evaluations, the expression **is** the zero integer .

However, we want analyze the terms that each modality of computing the trace offer. For that, let us introduce the definition

$$\overline{\mathrm{tr}}_1\left(\gamma_{*\mathrm{I}}\right)-\overline{\mathrm{tr}}_2\left(\gamma_{*\mathrm{I}}\right)\equiv 0\Leftrightarrow\text{They have exactly the same monomials (including signs)}$$

They are indiscernible numerically, but their effect when not identical is nontrivial over a class of identities. Two expression are said redundant if their integral forms differ by a finite and zero integrals. This is the case for expressions where $\overline{\mathrm{tr}}_{2l+1}^{(0)}\left(\gamma_*\gamma_{\mathrm{I}}\right)$ and $\overline{\mathrm{tr}}_{2l+1}^{(1)}\left(\gamma_{*\mathrm{I}}\right)$ are used. Better, there are non-identical expressions in the sense above that result in the same integral, despite existing divergent integrals around.

To expresses any tensor coming from the "tower" we may devise some notation:

$$\overline{\mathrm{tr}}_l^{(k)}\left(\gamma_{*\mathrm{I}}\right)=(-1)^{l+1}\,\overline{\mathrm{tr}}\left(\gamma_{\alpha_1}\ldots\gamma_*\gamma_{\alpha_l}\ldots\gamma_{\alpha_{2n+2}}\right)\tag{3.313}$$

where the l index refers to the position the γ_* is position in the trace, hence the $(-1)^{l+1}$ factor. As of the superscript (k) point to how to compute it in term of g-ε monomials. The rule is to identify a string of length k after the chiral matrix and use

$$\gamma_*\gamma_{\alpha_l}\cdots\alpha_{l+k-1}\to\gamma_*\gamma_{[\alpha_l\ldots\alpha_{l+k-1}]}+\gamma_*\left[\text{lower order skew-symmetric products}\right].\tag{3.314}$$

The first term gets replace by

$$\gamma_*\gamma_{[\alpha_l\ldots\alpha_{l+k-1}]}=\frac{\mathrm{i}^{k(k-1)+n-1}}{(2n-k)!}\varepsilon_{\alpha_l\ldots\alpha_{l+k-1}\beta_1\cdots\beta_{2n-k}}\gamma^{\beta_1\cdots\beta_{2n-k}}. \tag{3.315}$$

This part yields monomials that do not carry any of $\{\alpha_l,\ldots,\alpha_{l+k-1}\}$ indices in the metric tensor. Meanwhile, the lower grade terms have lost two indices (carried by γ matrices), and thus their traces are of $2n$ matrices accompanying γ_*; therefore, are one monomial for each term. Now take the traces of the example above and you will have

$$\begin{aligned}
0 &= \left\{\overline{\mathrm{tr}}_1^{(0)}\left(\gamma_*\gamma_{\alpha_1\cdots\alpha_6}\right)-\overline{\mathrm{tr}}_1^{(1)}\left(\gamma_*\gamma_{\alpha_1\cdots\alpha_6}\right)\right\} \qquad (3.316)\\
&= \left[g_{\alpha_1\alpha_2}\varepsilon_{\alpha_3\alpha_4\alpha_5\alpha_6}-g_{\alpha_1\alpha_3}\varepsilon_{\alpha_2\alpha_4\alpha_5\alpha_6}+g_{\alpha_1\alpha_4}\varepsilon_{\alpha_2\alpha_3\alpha_5\alpha_6}-g_{\alpha_1\alpha_5}\varepsilon_{\alpha_2\alpha_3\alpha_4\alpha_6}+g_{\alpha_1\alpha_6}\varepsilon_{\alpha_2\alpha_3\alpha_4\alpha_5}\right]\\
&= g_{\alpha_1[\alpha_2}\varepsilon_{\alpha_3\alpha_4\alpha_5\alpha_6]}\\
&\neq 0
\end{aligned}$$

This is an illustration, we will not exploit all these tensor expressions. Besides, we are being cavalier here, the computations following the instruction must concern with the imaginary units and factors. The bar in $\overline{\mathrm{tr}}_l^{(k)}$ indicate the protocol of collecting the common factors for all terms after computing the subtraces, it is every time $2^n\left(-\mathrm{i}\right)^{n+1}$.

Now a specific example to be exploited by us in four dimensions. By using the sequence $\alpha_{123456}=\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3$ and $\alpha_{123456}=\mu_2\nu_2\mu_3\nu_3\mu_1\nu_1$, one gets the formulas for the first and the second version, those obtained with γ_* definition. Their difference is casted below

$$\begin{aligned}
\overline{\mathrm{tr}}_1^{(0)}\left(\gamma_{*\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3}\right)-\overline{\mathrm{tr}}_3^{(0)}\left(\gamma_{*\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3}\right) &= \\
\overline{\mathrm{tr}}\left(\gamma_{*\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3}\right)-\overline{\mathrm{tr}}\left(\gamma_{*\mu_2\nu_2\mu_3\nu_3\mu_1\nu_1}\right) &= -2g_{\mu_1[\mu_2}\varepsilon_{\mu_3\nu_1\nu_2\nu_3]}+2g_{\nu_1[\nu_2}\varepsilon_{\nu_3\mu_1\mu_2\mu_3]}.
\end{aligned}$$

We dropped the label for which identity, and what position of use that the expressions are computed with. They can be read from the sequence of indices in $\mathrm{tr}\left(\gamma_{*\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3}\right)$ and $\mathrm{tr}\left(\gamma_{*\mu_2\nu_2\mu_3\nu_3\mu_1\nu_1}\right)$. The difference, which is a sum of two fully antisymmetric tensors in five indices, hence vanishing, is not inert. Those two traces deliver non-automatically identical expressions after integration of some amplitude, see chapter 5.

Whereas the quite similar expressions

$$\begin{aligned}
&\overline{\mathrm{tr}}_1^{(0)}\left(\gamma_{*\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3}\right)-\overline{\mathrm{tr}}_2^{(0)}\left(\gamma_{*\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3}\right)=\mathrm{tr}\left(\gamma_{*\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3}\right)-\left[-\mathrm{tr}\left(\gamma_{*\nu_1\mu_2\nu_2\mu_3\nu_3\mu_1}\right)\right]\\
=\;& -2\left(g_{\mu_1\nu_1}\varepsilon_{\mu_2\mu_3\nu_2\nu_3}-g_{\mu_1\nu_2}\varepsilon_{\mu_2\mu_3\nu_1\nu_3}+g_{\mu_1\nu_3}\varepsilon_{\mu_2\mu_3\nu_1\nu_2}\right)-2g_{\mu_1\mu_2}\varepsilon_{\mu_3\nu_1\nu_2\nu_3}+2g_{\mu_1\mu_3}\varepsilon_{\mu_2\nu_1\nu_2\nu_3},
\end{aligned}$$

obtained with the γ_* definition, are not perfectly identical, but have all possible monomials, and do not lead to different integrated expressions. They are the "skeleton" of the vanishing sign tensor $\varepsilon_{\mu_{23}\nu_{12}}T_{\mu_1}^{(-+)\nu_{12}}$ and the null amplitude T_{μ}^{ASS} (one has to multiply by $K_{123}^{\nu_{123}}/D_{123}$ and work the expression a while to observe our assertions).

The same occur in the equation above for $[\overline{\mathrm{tr}}_1^{(0)}\left(\gamma_*\gamma_{\alpha_1\cdots\alpha_6}\right)-\overline{\mathrm{tr}}_1^{(1)}\left(\gamma_*\gamma_{\alpha_1\cdots\alpha_6}\right)]$. Choosing $\alpha_{123456}=\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3$ for them, the fully antisymmetric tensor corresponding to their difference projects down a combination of $T_{\mu}^{(-+)\nu_{12}}$ and T_{μ}^{ASS}. Therefore, the expressions[13] that follows

$$\overline{\mathrm{tr}}_{1}^{(0)}\left(\gamma_{*\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3}\right), \quad \overline{\mathrm{tr}}_{2}^{(0)}\left(\gamma_{*\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3}\right), \text{ and } \overline{\mathrm{tr}}_{1}^{(1)}\left(\gamma_{*}\gamma_{\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3}\right),$$

yield the same version of the amplitude. One in which the IRagfs is conditioned in the vertex carrying μ_1 index. For more details see the Appendix (A.6) or our letter [2].

Now, we abandon our provisional notations and this tiresome elaboration to fixate the term basic/basis versions.

Definition 3.8.2 *In the amplitude's integrand with vertices $\Gamma_i \in \{A, V\}$*

$$t_{\mathrm{I}_{n+1}}^{\mathbf{\Gamma}} = \frac{1}{D_{[1,n+1]}} K_1^{\nu_1} \cdots K_{n+1}^{\nu_{n+1}} \mathrm{tr}\left(\gamma_* \gamma_{\mu_1\nu_1\cdots\mu_{n+1}\nu_{n+1}}\right) + \text{Mass terms}, \tag{3.317}$$

the leading trace confer the amplitude (integrand and integrated) a label denoting the first, second, and so on version of the amplitude. This amounts to expressing the trace

$$\mathrm{tr}\left(\gamma_*\gamma_{\mu_1\nu_1\mu_2\nu_2\cdots\mu_{n+1}\nu_{n+1}}\right) = \mathrm{tr}\left(\gamma_*\gamma_{\mu_l\nu_l\cdots\mu_{n+1}\nu_{n+1}\mu_1\nu_1\cdots\mu_{l-1}\nu_{l-1}}\right), \tag{3.318}$$

where μ_l is the index of the l^{th} vertex (Γ_l), by the formula (3.307) above, with $\alpha_{1\cdots 2n+2} = \mu_l\nu_l \cdots \mu_{n+1}\nu_{n+1}\ \mu_1\nu_1 \cdots \mu_{l-1}\nu_{l-1}$. In other words, computing by the definition in the left of the μ_l index. There are $n+1$ independent versions.

These traces have a meaning, they allow for expressions that comply with all but one IRagf. Additionally, their linear combination with weights

$$\mathrm{tr}_{\mathbf{w}} = w_1 \mathrm{tr}_1 + \cdots w_{n+1}\mathrm{tr}_{n+1},$$

where tr_i is to indicate the version, is able to encode any information possibly coming from traces. The vector of weights satisfies $|\mathbf{w}| := w_1 + \cdots + w_{n+1} = 1$, what means if all possibilities lead in fact to the same tensor, then the integrated result would collapse onto a unique form. As will see, this is impossible simultaneously to translation invariance. The chapter (5) will do a careful analysis of this last statement.

Hitherto, we have discussed enough material to apply to AV^n-type amplitudes. I hope the application and conclusions in two, four, and six dimensions will be reasonable to convince the reader of their validity in $d = 2n$. Besides, our results and methodology enable a foray into other types of amplitudes, like tensor vertices and gravitational ones.

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[13] Notice that to study the possible difference of two versions of a amplitude, only the leading trace of $d+2$ matrices need to be take into account.

Chapter 4

$2D$-Two-Point Amplitudes

(Disclaimer: This chapter uses the normalization $\int \mathrm{d}^2k/c\,(2) = \int \mathrm{d}^2k/\,(2\pi)^2$)

In this chapter, we compute axial amplitudes of two Lorentz indexes (AV and VA) to establish the connection between the linearity of integration and symmetries, which materializes through IRagfs and Ward identities (WIs). We also initiate discussions about low-energy implications and uniqueness, which will be fundamental topics in the four-dimensional analysis.

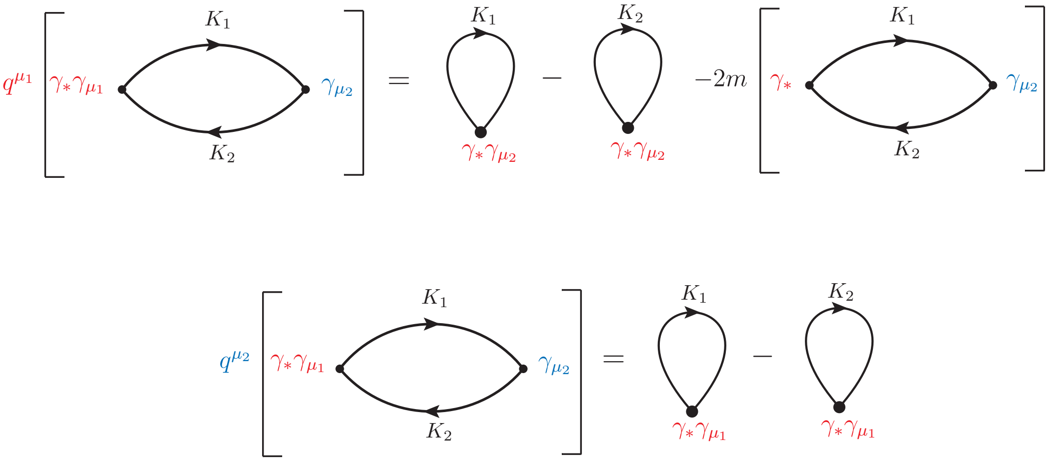

Figure 4.1: Graphic representation of the relations among Green functions arising from momenta contractions over the axial-vector bubble.

The mentioned connection manifests in contractions with the external momentum $q = p_{21} = k_2 - k_1$. After introducing the model (1), we derived identities involving integrands of amplitudes at the section (2.1). For the cases in analysis, the integration should produce RAGFs for the vector vertex

$$q^{\mu_2}T^{AV}_{\mu_{12}} = T^{A}_{\mu_1}(1) - T^{A}_{\mu_1}(2)\,, \tag{4.1}$$

$$q^{\mu_1}T^{VA}_{\mu_{12}} = T^{A}_{\mu_2}(1) - T^{A}_{\mu_2}(2)\,, \tag{4.2}$$

and for the axial vertex

$$q^{\mu_1}T^{AV}_{\mu_{12}} = T^{A}_{\mu_2}(1) - T^{A}_{\mu_2}(2) - 2mT^{PV}_{\mu_2}, \tag{4.3}$$

$$q^{\mu_2}T^{VA}_{\mu_{12}} = T^{A}_{\mu_1}(1) - T^{A}_{\mu_1}(2) + 2mT^{VP}_{\mu_1}. \tag{4.4}$$

Their satisfaction is necessary to maintain the linearity of integration. Figure 4.1 uses the AV amplitude to illustrate these relations. Meanwhile, WIs imply vanishing the one-point functions above as required by the current-conservation equations (2.24)-(2.23). Part of our objective consists of verifying these expectations explicitly, even if they are not entirely contemplated since we deal with anomalous amplitudes.

On the other hand, if symmetry constraints were valid, the general structure of these amplitudes as odd tensors implies kinematic properties to invariants. Let us take the AV structure as an example

$$T^{AV}_{\mu_{12}} \to F_{\mu_{12}} = \varepsilon_{\mu_{12}}F_1 + \varepsilon_{\mu_1\nu}q^{\nu}q_{\mu_2}F_2 + \varepsilon_{\mu_2\nu}q^{\nu}q_{\mu_1}F_3, \tag{4.5}$$

where F_i are scalar invariants. Since two-point amplitudes exhibit logarithmic power counting in a two-dimensional setting, we only considered dependence on the external momentum. Then, performing momenta contractions yields

$$\begin{aligned} q^{\mu_2}T^{AV}_{\mu_{12}} &= \varepsilon_{\mu_1\nu}q^{\nu}(q^2F_2 + F_1), & (4.6)\\ q^{\mu_1}T^{AV}_{\mu_{12}} &= \varepsilon_{\mu_2\nu}q^{\nu}(q^2F_3 - F_1). & (4.7)\end{aligned}$$

Vector conservation in the first equation implies $F_1 = -q^2F_2$, whose replacement in the second equation produces

$$q^{\mu_1}T^{AV}_{\mu_{12}} = \varepsilon_{\mu_2\nu}q^{\nu}q^2(F_3 + F_2). \tag{4.8}$$

Hence, if the invariants do not have poles in $q^2 = 0$, we have a low-energy implication for the axial contraction. This falls on the PV amplitude if the axial WI is satisfied

$$0 = q^{\mu_1}T^{AV}_{\mu_{12}}|_{q^2=0} = -2mT^{PV}_{\mu_2}\Big|_{q^2=0} =: \varepsilon_{\mu_2\nu}q^{\nu}V_1^{PV}(q^2=0), \tag{4.9}$$

with Ω^{PV} being the form factor associated with PV. As the deduction of this last behavior requires the validity of both WIs, it has the same status as a symmetry property.

The reciprocal form of this statement appears by exchanging the order of the arguments. If the axial WI is selected first, it implies $F_1 = q^2F_3 - V_1^{PV}$ in (4.7). Its replacement in the vector contraction (4.6) gives the low-energy implication for the contraction with the vector index

$$q^{\mu_2}T^{AV}_{\mu_{12}}\Big|_{q^2=0} = -\varepsilon_{\mu_1\nu}q^{\nu}V_1^{PV}(q^2=0). \tag{4.10}$$

With this scenario in hands, our objective is the analysis in the light of explicit integration (2.19). Consulting the definition (2.18), we write the general integrand of two-point amplitudes

$$\begin{aligned} t^{\Gamma_1\Gamma_2} = {}& K^{\nu_{12}}_{12}\mathrm{tr}(\Gamma_1\gamma_{\nu_1}\Gamma_2\gamma_{\nu_2})/D_{12} + m^2\mathrm{tr}(\Gamma_1\Gamma_2)/D_{12}\\ &+mK^{\nu}_1\mathrm{tr}(\Gamma_1\gamma_{\nu}\Gamma_2)/D_{12} + mK^{\nu}_2\mathrm{tr}(\Gamma_1\Gamma_2\gamma_{\nu})/D_{12}; \end{aligned} \tag{4.11}$$

thus, specific versions emerge after choosing vertices and keeping the nonzero traces:

$$\begin{aligned} t^{AV}_{\mu_{12}} &= K^{\nu_{12}}_{12}\mathrm{tr}(\gamma_*\gamma_{\mu_1\nu_1\mu_2\nu_2})/D_{12} + m^2\mathrm{tr}(\gamma_*\gamma_{\mu_{12}})/D_{12}, & (4.12)\\ t^{VA}_{\mu_{12}} &= K^{\nu_{12}}_{12}\mathrm{tr}(\gamma_*\gamma_{\mu_1\nu_1\mu_2\nu_2})/D_{12} - m^2\mathrm{tr}(\gamma_*\gamma_{\mu_{12}})/D_{12}. & (4.13)\end{aligned}$$

The next step consists of taking Dirac traces, with the lower-rank one resulting in $\mathrm{tr}(\gamma_{*}\gamma_{\mu_{12}}) = -2\varepsilon_{\mu_{12}}$. The trace of four gamma matrices is a linear combination of the metric tensor and the Levi-Civita tensor, so various expressions emerge through substitutions involving the identities (3.293)-(3.296), restricted to two dimensions:

$$2\gamma_{*} = \varepsilon_{\nu_{12}}\gamma^{\nu_{12}}; \quad \gamma_{*}\gamma_{\mu} = -\varepsilon_{\mu\nu}\gamma^{\nu}; \quad \gamma_{*}\gamma_{[\mu\nu]} = -\varepsilon_{\mu\nu}.$$

They lead to expressions that are not automatically equal after integration. To unfold this rationale, let us apply the definition of the chiral matrix as we did in eq. (3.307), by restricted to two dimensions

$$\begin{aligned}\mathrm{tr}(\gamma_{*}\gamma_{\alpha_1\alpha_2\alpha_3\alpha_4}) &= 2\left(-g_{\alpha_1\alpha_2}\varepsilon_{\alpha_3\alpha_4} + g_{\alpha_1\alpha_3}\varepsilon_{\alpha_2\alpha_4} - g_{\alpha_1\alpha_4}\varepsilon_{\alpha_2\alpha_3} + \right.\\ &\quad \left. -g_{\alpha_2\alpha_3}\varepsilon_{\alpha_1\alpha_4} + g_{\alpha_2\alpha_4}\varepsilon_{\alpha_1\alpha_3} - g_{\alpha_3\alpha_4}\varepsilon_{\alpha_1\alpha_2}\right). \qquad (4.14)\end{aligned}$$

Next, we explore two sorting of indexes corresponding to replacing the chiral matrix definition around the first and second vertices. Albeit equivalent, these traces differ through signs of some terms. As a result we get these two expressions

$$\begin{aligned}K_{12}^{\nu_{12}}\mathrm{tr}(\gamma_{*}\gamma_{\mu_1\nu_1\mu_2\nu_2}) &= 2\varepsilon_{\mu_{12}}(K_1\cdot K_2) + 2g_{\mu_{12}}\varepsilon_{\nu_{12}}K_{12}^{\nu_{12}}\\ &\quad -2\varepsilon_{\mu_1\nu_1}\left(K_{1\mu_2}K_2^{\nu_1} + K_{2\mu_2}K_1^{\nu_1}\right)\\ &\quad -2\varepsilon_{\mu_2\nu_1}\left(K_{1\mu_1}K_2^{\nu_1} - K_{2\mu_1}K_1^{\nu_1}\right) \qquad (4.15)\\ K_{12}^{\nu_{12}}\mathrm{tr}(\gamma_{*}\gamma_{\mu_2\nu_2\mu_1\nu_1}) &= -2\varepsilon_{\mu_{12}}(K_1\cdot K_2) - 2g_{\mu_{12}}\varepsilon_{\nu_{12}}K_{12}^{\nu_{12}}\\ &\quad +2\varepsilon_{\mu_1\nu_1}\left(K_{1\mu_2}K_2^{\nu_1} - K_{2\mu_2}K_1^{\nu_1}\right)\\ &\quad -2\varepsilon_{\mu_2\nu_1}\left(K_{1\mu_1}K_2^{\nu_1} + K_{2\mu_1}K_1^{\nu_1}\right). \qquad (4.16)\end{aligned}$$

It is often possible to examine the tensor structure of one amplitude to find less complex ones inside it. Despite an $\varepsilon_{\mu_1\mu_2}$ factor, using the general form (4.11) leads to scalar two-point subamplitudes below when combining the bilinears above with squared mass terms:

$$t^{PP} = t^{SS} - 4m^2\frac{1}{D_{12}} = q^2\frac{1}{D_{12}} - \frac{1}{D_1} - \frac{1}{D_2}. \qquad (4.17)$$

The following reduction was used to simplify their integrands

$$2(K_i\cdot K_j - m^2) = D_i + D_j - p_{ij}^2. \qquad (4.18)$$

All other contributions receive an organization in terms of the same object, a sign tensor present similarly in all explored dimensions

$$t_{\mu}^{(\pm)\nu} = \left(K_{1\mu}K_2^{\nu} \pm K_{2\mu}K_1^{\nu}\right)/D_{12}. \qquad (4.19)$$

Nevertheless, anticipating a connection with higher dimensions, we opt to write the last term as a pseudoscalar function $t^{SP} = -t^{PS} = \varepsilon^{\nu_{12}}t_{\nu_{12}}^{(-)}$. Therefore, given both versions for the four-matrix trace, we have the corresponding versions for the AV amplitude:

$$\begin{aligned}(t_{\mu_{12}}^{AV})_1 &= -2\varepsilon_{\mu_1\nu}t_{\mu_2}^{(+)\nu} - \varepsilon_{\mu_{12}}t^{PP} - 2\varepsilon_{\mu_2\nu}t_{\mu_1}^{(-)\nu} + g_{\mu_{12}}t^{SP} \qquad (4.20)\\ (t_{\mu_{12}}^{AV})_2 &= -2\varepsilon_{\mu_2\nu}t_{\mu_1}^{(+)\nu} - \varepsilon_{\mu_{12}}t^{SS} + 2\varepsilon_{\mu_1\nu}t_{\mu_2}^{(-)\nu} - g_{\mu_{12}}t^{SP}. \qquad (4.21)\end{aligned}$$

As mentioned at the beginning of the section, integrated amplitudes depend exclusively on the external momentum q. That precludes the construction of 2^{nd}-order antisymmetric tensors, which cancels out terms like $t^{(-)}$ and SP. The sign tensor $t^{(-)}$ and the odd subamplitude are those structures that vanishing and finite, as happens for all even dimensions, see the end of section (3.7), eq. (3.289).

Further examination of the general form (4.11) allows the identification of even amplitudes

$$t^{VV}_{\mu_{12}} = 2t^{(+)}_{\mu_{12}} + g_{\mu_{12}} t^{PP} \quad \text{and} \quad t^{AA}_{\mu_{12}} = 2t^{(+)}_{\mu_{12}} - g_{\mu_{12}} t^{SS}. \tag{4.22}$$

Hence, the integration provides relations among odd and even amplitudes

$$\begin{aligned} (T^{AV}_{\mu_{12}})_1 &= -\varepsilon_{\mu_1}{}^{\nu} T^{VV}_{\nu\mu_2}; \qquad (T^{AV}_{\mu_{12}})_2 = -\varepsilon_{\mu_2}{}^{\nu} T^{AA}_{\mu_1\nu}; && (4.23)\\ (T^{VA}_{\mu_{12}})_1 &= -\varepsilon_{\mu_1}{}^{\nu} T^{AA}_{\nu\mu_2}; \qquad (T^{VA}_{\mu_{12}})_2 = -\varepsilon_{\mu_2}{}^{\nu} T^{VV}_{\mu_1\nu}. && (4.24) \end{aligned}$$

Although we did not detail, following the same steps produced both VA versions. It is direct to achieve these associations at the integrand level using the second identity for Dirac matrices $\gamma_*\gamma_\mu = -\varepsilon_\mu{}^\nu \gamma_\nu$ in the adequate position. Even so, we need a clear distinction among versions since their comparison is not automatic for integrated amplitudes due to their diverging character.

Lastly, the third identity $\gamma_*\gamma_{\mu\nu} = -\varepsilon_{\mu\nu} + g_{\mu\nu}\gamma_*$ allows introducing another version for the discussed amplitudes. Disregarding terms on the antisymmetric tensor $t^{(-)}$, the integrated amplitude links to previous versions as follows:

$$\begin{aligned} 2(T^{AV}_{\mu_{12}})_3 &= -\varepsilon_{\mu_1}{}^{\nu} T^{VV}_{\nu\mu_2} - \varepsilon_{\mu_2}{}^{\nu} T^{AA}_{\mu_1\nu} = (T^{AV}_{\mu_{12}})_1 + (T^{AV}_{\mu_{12}})_2 && (4.25)\\ 2(T^{VA}_{\mu_{12}})_3 &= -\varepsilon_{\mu_1}{}^{\nu} T^{AA}_{\nu\mu_2} - \varepsilon_{\mu_2}{}^{\nu} T^{VV}_{\mu_1\nu} = (T^{VA}_{\mu_{12}})_1 + (T^{VA}_{\mu_{12}})_2. && (4.26) \end{aligned}$$

This particular aspect receives further attention in the four-dimensional setting, having the sole purpose of illustrating how any amplitude version follows from versions one and two here. The investigation from reference [71] uses the third version in Eq. (85).

Obtaining explicit results occurs by replacing the results from A.1 inside integrated expressions of structures derived above. Scalar two-point functions assume the forms

$$T^{PP} = T^{SS} - 4m^2 J_2 = q^2 J_2 - 2I_{\log}, \tag{4.27}$$

and the symmetric sign tensor is

$$\begin{aligned} T^{(+)}_{\mu_{12}} &= 2(\bar{J}_{2\mu_{12}} + q_{\mu_1} J_{2\mu_2}) \\ &= \Delta_{2\mu_{12}} + g_{\mu_{12}} I_{\log} + 2\theta_{\mu_{12}}(m^2 J_2 + i/4\pi) - \frac{1}{2} q^2 g_{\mu_{12}} J_2, && (4.28) \end{aligned}$$

where $\theta_{\mu\nu}(q) = (g_{\mu\nu} q^2 - q_\mu q_\nu)/q^2$ is the transversal projector. We combine these pieces into odd tensors[1]

$$(T^{AV}_{\mu_{12}})_1 = -\varepsilon_{\mu_1}{}^{\nu}[2\Delta_{2\mu_2\nu} + 4\theta_{\mu_2\nu}\left(m^2 J_2 + i/4\pi\right)], \tag{4.29}$$

$$(T^{AV}_{\mu_{12}})_2 = -\varepsilon_{\mu_2}{}^{\nu}[2\Delta_{2\mu_1\nu} + 4\theta_{\mu_1\nu}\left(m^2 J_2 + i/4\pi\right)] + \varepsilon_{\mu_2}{}^{\nu} g_{\mu_1\nu}\left(4m^2 J_2\right), \tag{4.30}$$

with objects between squared brackets being even tensors.

We also use this opportunity to introduce amplitudes that emerge through momenta contractions. They follow a strong pattern acknowledged in all IRagfs seen in this investigation. Whereas additional functions arising in axial relations are finite

$$T^{PV}_{\mu} = -T^{VP}_{\mu} = \varepsilon_{\mu\nu} q^{\nu}\left[-2mJ_2\left(q\right)\right], \tag{4.31}$$

other functions are pure surface terms proportional to the arbitrary routings k_i as follows

$$T^{A}_{\mu}\left(i\right) = -\varepsilon_{\mu}{}^{\nu_1} T^{V}_{\nu_1}\left(i\right) = 2\varepsilon_{\mu}{}^{\nu_1} k_i^{\nu_2}\Delta_{2\nu_{12}}. \tag{4.32}$$

The last structure is consistent with the linear power counting of one-point amplitudes in a two-dimensional setting.

Even though integrands of amplitudes are equivalent, the same does not apply to their integrated form. In the case of even and odd tensor amplitudes, expressions depend on the prescription adopted to evaluate divergences because they contain surface terms Δ_2. Additionally, odd amplitudes depend on the trace version since using the definition of the chiral matrix around the first or the second vertices brings implications for the index arrangement in finite and divergent parts. This perspective produced identities originally, but now the connection is not automatic. That becomes clear when we subtract one AV version from the other

$$\begin{aligned}(T^{AV}_{\mu_{12}})_1 - (T^{AV}_{\mu_{12}})_2 &= -2(\varepsilon_{\mu_1\nu}\Delta^{\nu}_{2\mu_2} - \varepsilon_{\mu_2\nu}\Delta^{\nu}_{2\mu_1}) + 4\varepsilon_{\mu_{12}} m^2 J_2 \\ &\quad -4(\varepsilon_{\mu_1\nu}\theta^{\nu}_{\mu_2} - \varepsilon_{\mu_2\nu}\theta^{\nu}_{\mu_1})\left(m^2 J_2 + i/4\pi\right).\end{aligned}$$

We use Schouten identities[2] in two dimensions to rearrange indexes in the transversal projector and surface terms; therefore, the difference reduces to

$$(T^{AV}_{\mu_{12}})_1 - (T^{AV}_{\mu_{12}})_2 = -\varepsilon_{\mu_{12}}[2\Delta^{\alpha}_{2\alpha} + i/\pi]. \tag{4.35}$$

The linearity of integration requires this difference to vanish identically, which would constrain the value of the object $\Delta^{\alpha}_{2\alpha}$. That represents a link between linearity and the uniqueness of perturbative solutions. We consider these concepts while investigating the original expectation in the subsections.

---

[1] It is possible to obtain VA versions by redefining indexes through $\mu_1 \longleftrightarrow \mu_2$.

[2] The antisymmetry of the Levi-Civita tensor establishes:

$$\varepsilon_{[\mu_1\nu}\Delta^{\nu}_{2\mu_2]} = \varepsilon_{\mu_1\nu}\Delta^{\nu}_{2\mu_2} + \varepsilon_{\mu_2\mu_1}\Delta^{\nu}_{2\nu} + \varepsilon_{\nu\mu_2}\Delta^{\nu}_{2\mu_1} = 0, \tag{4.33}$$

$$\varepsilon_{[\mu_1\nu}\theta^{\nu}_{\mu_2]} = \varepsilon_{\mu_1\nu}\theta^{\nu}_{\mu_2} + \varepsilon_{\mu_2\mu_1}\theta^{\nu}_{\nu} + \varepsilon_{\nu\mu_2}\theta^{\nu}_{\mu_1} = 0. \tag{4.34}$$

4.1 Relations Among Green Functions (IRagfs)

This subsection aims to perform momenta contractions with odd amplitudes to test the validity of IRagfs. Firstly, let us comment on even amplitudes because they appear inside odd ones in Equations (4.29)-(4.30). They also follow relations, whose proof only requires algebraic operations:

$$q^{\mu_1}T^{VV}_{\mu_{12}} = 2q^{\nu}\Delta_{2\mu_2\nu} = [T^{V}_{\mu_2}(1) - T^{V}_{\mu_2}(2)] \tag{4.36}$$

$$q^{\mu_1}T^{AA}_{\mu_{12}} + 2mT^{PA}_{\mu_2} = 2q^{\nu}\Delta_{2\mu_2\nu} = [T^{V}_{\mu_2}(1) - T^{V}_{\mu_2}(2)]. \tag{4.37}$$

Furthermore, they are automatic because they apply identically; observe the vector one-point amplitudes (4.32).

Such a feature differs from odd amplitudes although they contain the same elements, i.e., finite contributions and the surface term Δ_2. Let us perform the corresponding contractions to test relations (4.1)-(4.4). Starting with the first AV version (4.29), its vector contraction yields

$$q^{\mu_2}(T^{AV}_{\mu_{12}})_1 = -2\varepsilon_{\mu_1\nu_1}q^{\nu_2}\Delta^{\nu_1}_{2\nu_2} = [T^{A}_{\mu_1}(1) - T^{A}_{\mu_1}(2)]. \tag{4.38}$$

Analogously to the case of even amplitudes, finite terms vanish because $q^{\mu_2}\theta^{\nu}_{\mu_2} = 0$ while it is straightforward to identify the axial amplitude (4.32).

In another way, the axial contraction exhibits an inadequate tensor arranging since the momentum couples to the Levi-Civita symbol:

$$q^{\mu_1}(T^{AV}_{\mu_{12}})_1 = -q^{\mu_1}\varepsilon_{\mu_1}{}^{\nu}[2\Delta_{2\mu_2\nu} + 4\theta_{\mu_2\nu}\left(m^2J_2 + i/4\pi\right)]. \tag{4.39}$$

This feature demands index permutations through Schouten identities (4.33)-(4.34) for the surface term and the projector. Then, reminding the trace $\theta^{\nu}_{\nu} = 1$, we identify the PV amplitude (4.31) and the axials

$$q^{\mu_1}(T^{AV}_{\mu_{12}})_1 = [T^{A}_{\mu_2}(1) - T^{A}_{\mu_2}(2)] - 2mT^{PV}_{\mu_2} + \varepsilon_{\mu_2\nu}q^{\nu}[2\Delta^{\alpha}_{2\alpha} + i/\pi]. \tag{4.40}$$

The last term prevents the automatic satisfaction of this relation, depending on the value assumed by the surface term.

We observed the same situation for the second AV version (4.30); however, the additional term appears on its vector contraction[3]

$$q^{\mu_2}(T^{AV}_{\mu_{12}})_2 = [T^{A}_{\mu_1}(1) - T^{A}_{\mu_1}(2)] + \varepsilon_{\mu_1\nu}q^{\nu}[2\Delta^{\alpha}_{2\alpha} + i/\pi] \tag{4.41}$$

$$q^{\mu_1}(T^{AV}_{\mu_{12}})_2 = [T^{A}_{\mu_2}(1) - T^{A}_{\mu_2}(2)] - 2mT^{PV}_{\mu_2}. \tag{4.42}$$

This same pattern repeats for the VA amplitude regardless of the vertex arrangement. Additional terms arise for the first contraction (vector) of the first version and the second contraction

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[3] Since the third version is a combination of the others (4.25), both vertices have additional terms.

(axial) of the second version:

$$q^{\mu_1}(T^{VA}_{\mu_{12}})_1 = [T^{A}_{\mu_2}(1) - T^{A}_{\mu_2}(2)] + \varepsilon_{\mu_2\nu}q^{\nu}[2\Delta^{\alpha}_{2\alpha} + i/\pi] \tag{4.43}$$

$$q^{\mu_2}(T^{VA}_{\mu_{12}})_2 = [T^{A}_{\mu_1}(1) - T^{A}_{\mu_1}(2)] + 2mT^{VP}_{\mu_1} + \varepsilon_{\mu_1\nu}q^{\nu}[2\Delta^{\alpha}_{2\alpha} + i/\pi]. \tag{4.44}$$

The iRagfs, deduced as identities for integrands, represent the linearity of integration within this context. Even amplitudes automatically satisfy their relations as there is no dependence on the surface term value. On the other hand, odd amplitudes exhibit a potentially-violating term, so linearity would require the condition

$$\Delta^{\alpha}_{2\alpha} = -i\,(2\pi)^{-1}. \tag{4.45}$$

This contribution emerges for the contraction with the vertex that defines the amplitude version, definition (3.8.2). Choosing this finite value for the surface term ensures that all versions are equal (4.35), elucidating the relation between linearity and uniqueness. Any formula to the Dirac traces leads to one unique answer that respects the linearity of integration.

Nevertheless, this condition sets nonzero values for the one-point functions (4.32), affecting symmetry implications through WIs. That occurs for all relations in this section since amplitudes depend on the surface term. This subject receives attention in the sequence.

4.2 Ward Identities (WIs) and Low-Energy Implications

We discussed the divergence of axial and vector currents (2.24)-(2.23), indicating implications through WIs for perturbative amplitudes. The adopted strategy translates these implications into restrictions over IRagfs, linking linearity and symmetries. This subsection analyses such a connection focusing on anomalous amplitudes (AV and VA), known for the impossibility of satisfying all WIs simultaneously.

Adopting a prescription that eliminates surface terms reduces all IRagfs for even amplitudes (VV and AA) to the corresponding WIs. Regarding odd amplitudes, this condition satisfies those WIs corresponding to automatic relations while violating others. Observe the first version of the AV to elucidate this statement. Identifying the vector relation was automatic; however, the axial relation has an additional term. Hence, the zero value for the surface term satisfies the vector WI while violating the axial WI through one anomalous contribution. We see the opposite for the second version of the amplitude, which violates the vector WI. Both identities are violated for the third version since it is a composition of the first two. Table 4.1 shows the mentioned results for the AV and some examples of even amplitudes.

This argumentation applies to the VA without changes regarding vertex arrangement. Under this perspective, selecting an amplitude version would set the vertex (or vertices) with one anomalous contribution. Furthermore, this perspective breaks the linearity of integration in anomalous amplitudes for violating non-automatic RAGFs.

Table 4.1: Ward identities using the zero value for the surface term.

$q^{\nu}(T^{AV}_{\nu\mu})_1 = -2mT^{PV}_{\mu} + (i/\pi)\,\varepsilon_{\mu\nu}q^{\nu}$	$q^{\nu}(T^{AV}_{\mu\nu})_1 = 0$
$q^{\nu}(T^{AV}_{\nu\mu})_2 = -2mT^{PV}_{\mu}$	$q^{\nu}(T^{AV}_{\mu\nu})_2 = (i/\pi)\,\varepsilon_{\mu\nu}q^{\nu}$
$q^{\nu}(T^{AV}_{\nu\mu})_3 = -2mT^{PV}_{\mu} + (i/2\pi)\,\varepsilon_{\mu\nu}q^{\nu}$	$q^{\nu}(T^{AV}_{\mu\nu})_3 = (i/2\pi)\,\varepsilon_{\mu\nu}q^{\nu}$
$q^{\nu}T^{AA}_{\nu\mu} = -2mT^{PA}_{\mu}$	$q^{\nu}T^{VV}_{\mu\nu} = 0$

On the other hand, choosing the value that preserves linearity (4.45) collapses different amplitude versions into one unique form[4] (4.35). Nevertheless, that violates all WIs for odd and even amplitudes as they depend on the surface term value; see Table 4.2.

Table 4.2: Ward identities using the non-zero value for the surface term.

$q^{\nu}T^{AV}_{\nu\mu} = -2mT^{PV}_{\mu} + (i/2\pi)\,\varepsilon_{\mu\nu}q^{\nu}$	$q^{\nu}T^{AV}_{\mu\nu} = (i/2\pi)\,\varepsilon_{\mu\nu}q^{\nu}$
$q^{\nu}T^{AA}_{\nu\mu} = -2mT^{PA}_{\mu} - (i/2\pi)\,q_{\mu}$	$q^{\nu}T^{VV}_{\nu\mu} = -\,(i/2\pi)\,q_{\mu}$

Low-energy properties of finite functions are crucial to deepen this analysis. Under the hypothesis that both WIs for the AV amplitude apply, we established the kinematical behavior at zero of V_1^{PV} as being zero (4.9). Nonetheless, employing the PV expression (4.31) together with the limit $m^2 Z_0^{(-1)}(0) = -1$ (3.115) yields a non-zero outcome:

$$V_1^{PV}(0) = 4m^2 J_2\big|_0 = \frac{i}{\pi} m^2 Z_0^{(-1)}(0) = -\frac{i}{\pi}. \tag{4.46}$$

That means the hypothesis is false. Hence, when satisfying the vector WI, the axial WI violation is the value corresponding to the negative of $V_1^{PV}(0)$. The other expectation (4.10) leads to the reciprocal; thus, satisfying the axial WI implies violating the vector WI.

Let us extend these ideas using the general structure of a 2nd-order odd tensor (4.5). In both AV and VA cases, momenta contractions lead to a set of functions written in terms of form factors

$$\begin{aligned} q^{\mu_1} F_{\mu_{12}} &= \varepsilon_{\mu_2\nu} q^{\nu} V_1^L\left(q^2\right) = \varepsilon_{\mu_2\nu} q^{\nu}\left(q^2 F_3 - F_1\right), & (4.47)\\ q^{\mu_2} F_{\mu_{12}} &= \varepsilon_{\mu_1\nu} q^{\nu} V_2^L\left(q^2\right) = \varepsilon_{\mu_1\nu} q^{\nu}\left(q^2 F_2 + F_1\right). & (4.48)\end{aligned}$$

If form factors are free of kinematic singularities observed in the explicit forms of amplitudes, the implication at zero follows[5]

$$V_1^L(0) + V_2^L(0) = 0. \tag{4.49}$$

Thereby, if one term vanishes, the other must do so. Otherwise, if one term relates to a finite function (PV or VP), an additional constant must appear as compensation within the last equation. These statements are inconsistent with the satisfaction of both WIs, which only occurs if linearity of integration holds with null surface terms. Thus, the low-energy behavior of

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[4] The third AV version is independent of the surface term value. Parametrizing $\Delta_{2\mu\nu} = a g_{\mu\nu}$ in its equation, we get an expression independent of the coefficient a and equal to the unique form.

[5] The conventions for the invariants V_1^L used in this chapter differ from the constructions of section (2.3) related to general LETs. However, the conclusion is not affected.

these finite functions is the source of anomalous terms in amplitudes and not their perturbative ambiguity.

Nevertheless, ambiguities relate to these low-energy implications. Under the condition of linearity and considering surface terms in the general tensor, this limit implies the constraint $2\Delta_{2\alpha}^{\alpha} = V_1^{PV}(0)$. Such an aspect will be fully explored in the following section considering axial triangles in the physical dimension. Conclusions similar to those drawn here anticipate the presence of anomalies and linearity breaking in this new context.

Chapter 5

$4D$ Three-Point Amplitudes

(Disclaimer: This chapter uses the normalization $\int \mathrm{d}^4k/c\,(4) = \int \mathrm{d}^4k/\,(2\pi)^4)$

The analysis developed in the physical dimension focuses on odd amplitudes that are rank-3 tensors, namely AVV, VAV, VVA, and AAA. They depend on the trace involving six Dirac matrices plus the chiral one, whose computation yields products between the Levi-Civita (LC) and metric tensor. After the integration, that generates expressions that differ in their dependence on surface terms and finite parts. We want to verify these prospects by evaluating the triangles' basic versions[1]. Once these resources are clear, we study how symmetries, linearity of integration, and uniqueness manifest.

5.1 Obtaining Explicit Expressions for $T_{\mathrm{I}}^{\Gamma_{123}}$

The integrand of the 3-pt amplitudes is in general expressed by

$$t_{\mathrm{I}}^{\Gamma_{123}} = \mathrm{tr}\left[\Gamma_1 S\left(1\right)\Gamma_2 S\left(2\right)\Gamma_3 S\left(3\right)\right]. \tag{5.1}$$

Thus, after replacing vertex operators and disregarding vanishing traces, 3^{rd}-order amplitudes assume the forms

$$\begin{aligned}
t^{AVV}_{\mu_{123}} &= [K^{\nu_{123}}_{123}\mathrm{tr}(\gamma_{*\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3}) + m^2\mathrm{tr}(\gamma_{*\mu_1\mu_2\mu_3\nu_1})(K^{\nu_1}_1 - K^{\nu_1}_2 + K^{\nu_1}_3)]\frac{1}{D_{123}}, && (5.2)\\
t^{VAV}_{\mu_{123}} &= [K^{\nu_{123}}_{123}\mathrm{tr}(\gamma_{*\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3}) + m^2\mathrm{tr}(\gamma_{*\mu_1\mu_2\mu_3\nu_1})(K^{\nu_1}_1 + K^{\nu_1}_2 - K^{\nu_1}_3)]\frac{1}{D_{123}}, && (5.3)\\
t^{VVA}_{\mu_{123}} &= [K^{\nu_{123}}_{123}\mathrm{tr}(\gamma_{*\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3}) - m^2\mathrm{tr}(\gamma_{*\mu_1\mu_2\mu_3\nu_1})(K^{\nu_1}_1 - K^{\nu_1}_2 - K^{\nu_1}_3)]\frac{1}{D_{123}}, && (5.4)\\
t^{AAA}_{\mu_{123}} &= [K^{\nu_{123}}_{123}\mathrm{tr}(\gamma_{*\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3}) - m^2\mathrm{tr}(\gamma_{*\mu_1\mu_2\mu_3\nu_1})(K^{\nu_1}_1 + K^{\nu_1}_2 + K^{\nu_1}_3)]\frac{1}{D_{123}}. && (5.5)
\end{aligned}$$

Although the trace involving four Dirac matrices plus the chiral one is univocal, the leading trace features various possibilities as discussed in subsection (3.8). It is possible to consider any of the identities (3.293)-(3.296) to start a computation, all of them can be understood through definition (3.8.2). For this see the Appendix A.6 or preprint [1].

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[1] To this aim, we compute twenty-four triangles of rank-one. Twelve parity-even triangles: VPP, ASP, VSS, and their permutations. Twelve parity-odd tensors: ASS, APP, VPS, and their permutations, all trivially vanishing.

There are three basic versions, each corresponding to replacing the chiral matrix near a specific vertex operator. We introduce a numeric label to distinguish them:

$$[\mathrm{tr}(\gamma_{*\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3})]_1 = [\mathrm{tr}(\gamma_{*\mu_2\nu_2\mu_3\nu_3\mu_1\nu_1})]_2 = [\mathrm{tr}(\gamma_{*\mu_3\nu_3\mu_1\nu_1\mu_2\nu_2})]_3. \tag{5.6}$$

They differ in the signs of terms; however, these three are not automatically identical. We cast their contraction with $K_{123}^{\nu_{123}}$ in the sequence. As an illustrative example we list the result for the first term in the above equation, namely

$$\begin{aligned}[K_{123}^{\nu_{123}}\mathrm{tr}(\gamma_{*\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3})]_1 \;=\; & -4\mathrm{i}\varepsilon_{\mu_{23}\nu_{12}}[K_{1\mu_1}K_{23}^{\nu_{12}} - K_{2\mu_1}K_{13}^{\nu_{12}} + K_{3\mu_1}K_{12}^{\nu_{12}}] \\ & -4\mathrm{i}\varepsilon_{\mu_{13}\nu_{12}}[K_{1\mu_2}K_{23}^{\nu_{12}} + K_{2\mu_2}K_{13}^{\nu_{12}} - K_{3\mu_2}K_{12}^{\nu_{12}}] \\ & +4\mathrm{i}\varepsilon_{\mu_{12}\nu_{12}}[K_{1\mu_3}K_{23}^{\nu_{12}} - K_{2\mu_3}K_{13}^{\nu_{12}} - K_{3\mu_3}K_{12}^{\nu_{12}}] \\ & -4\mathrm{i}\varepsilon_{\mu_{123}\nu_1}[K_1^{\nu_1}(K_2\cdot K_3) - K_2^{\nu_1}(K_1\cdot K_3) + K_3^{\nu_1}(K_1\cdot K_2)] \\ & +4\mathrm{i}[-g_{\mu_{12}}\varepsilon_{\mu_3\nu_{123}} - g_{\mu_{23}}\varepsilon_{\mu_1\nu_{123}} + g_{\mu_{13}}\varepsilon_{\mu_2\nu_{123}}]K_{123}^{\nu_{123}}\end{aligned} \tag{5.7}$$

Our next task consists of organizing and integrating the complete expressions. From section the section of sign tensors (3.7), the three first lines are directly identifiable. The last line is written as an odd subamplitude as discussed around the eq. (3.289). Whereas the third line gather with the mass terms and gives specific even subamplitudes, which depend on version and the original amplitude. Let us exemplify by the AVV: together with masses we have

$$\begin{aligned}(t_{\mu_{123}}^{AVV})_1 \;=\; & 4\mathrm{i}\left[-\varepsilon_{\mu_{23}\nu_{12}}t_{\mu_1}^{(-+)} - \varepsilon_{\mu_{13}\nu_{12}}t_{\mu_2}^{(+-)} + \varepsilon_{\mu_{12}\nu_{12}}t_{\mu_3}^{(--)}\right] \\ & -4\mathrm{i}\varepsilon_{\mu_{123}\nu_1}\left[\mathfrak{s}_{23}^{(-)}K_1^{\nu_1} - \mathfrak{s}_{13}^{(-)}K_2^{\nu_1} + \mathfrak{s}_{12}^{(-)}K_3^{\nu_1}\right]\frac{1}{D_{123}} \\ & +[-g_{\mu_{12}}t_{\mu_3}^{ASS} - g_{\mu_{23}}t_{\mu_1}^{ASS} + g_{\mu_{13}}t_{\mu_2}^{ASS}].\end{aligned} \tag{5.8}$$

Where

$$2\mathfrak{s}_{ij}^{(\pm)} = 2\left(K_i\cdot K_j \pm m_i m_j\right) = D_i + D_j + \left[(m_i - m_j)^2 - p_{ij}^2\right], \tag{5.9}$$

is an abbreviation that comes handy if different masses appear. Obvious here $m_i = m_j$, but the expression at this point would be identical for distinct masses. The second term is easily identified with

$$(t^{VPP})^{\nu_1} = 4(-\mathfrak{s}_{23}^{(-)}K_1^{\nu_1} + \mathfrak{s}_{13}^{(-)}K_2^{\nu_1} - \mathfrak{s}_{12}^{(-)}K_3^{\nu_1})\frac{1}{D_{123}}, \tag{5.10}$$

what gives us

$$\begin{aligned}(t_{\mu_{123}}^{AVV})_1 \;=\; & 4\mathrm{i}\left[-\varepsilon_{\mu_{23}\nu_{12}}t_{\mu_1}^{(-+)} - \varepsilon_{\mu_{13}\nu_{12}}t_{\mu_2}^{(+-)} + \varepsilon_{\mu_{12}\nu_{12}}t_{\mu_3}^{(--)}\right] + \mathrm{i}\varepsilon_{\mu_{123}\nu_1}(t^{VPP})^{\nu_1} \\ & +[-g_{\mu_{12}}t_{\mu_3}^{ASS} - g_{\mu_{23}}t_{\mu_1}^{ASS} + g_{\mu_{13}}t_{\mu_2}^{ASS}].\end{aligned} \tag{5.11}$$

This is the way to reach all versions of all amplitudes. Some comments are in order: all amplitudes computed with the same trace's version share the first term in square brackets, we call it common tensor.

The integrated amplitudes: As extensively discussed in the sign-tensors section, $\varepsilon_{\mu_{ab}\nu_{12}}T_{\mu_c}^{(-,+)\nu_{12}}$ is finite and zero. The others we need are listed here:

$$\begin{aligned}
2\varepsilon_{\mu_{ab}\nu_{12}}T_{\mu_c}^{(-,+)\nu_{12}} &= 2\varepsilon_{\mu_{ab}\nu_{12}}[p_{21}^{\nu_1}p_{32}^{\nu_2}J_{3\mu_c}+(-p_{21\mu_c}p_{31}^{\nu_2}+p_{31\mu_c}p_{21}^{\nu_2})J_3^{\nu_1}]\equiv 0, && (5.12)\\
2\varepsilon_{\mu_{ab}\nu_{12}}T_{\mu_c}^{(+,-)\nu_{12}} &= 4\varepsilon_{\mu_{ab}\nu_{12}}[p_{31}^{\nu_2}(J_{3\mu_c}^{\nu_1}+p_{21\mu_c}J_3^{\nu_1})-p_{21}^{\nu_2}(J_{3\mu_c}^{\nu_1}+p_{31\mu_c}J_3^{\nu_1})] && (5.13)\\
&\quad +(\varepsilon_{\mu_{ab}\nu_{12}}p_{32}^{\nu_2}\Delta_{3\mu_c}^{\nu_1}+\varepsilon_{\mu_{abc}\nu_1}p_{32}^{\nu_1}I_{\log}),\\
2\varepsilon_{\mu_{ab}\nu_{12}}T_{\mu_c}^{(-,-)\nu_{12}} &= -4\varepsilon_{\mu_{ab}\nu_{12}}p_{21}^{\nu_2}(J_{3\mu_c}^{\nu_1}+p_{31\mu_c}J_3^{\nu_1}) && (5.14)\\
&\quad -(\varepsilon_{\mu_{ab}\nu_{12}}p_{21}^{\nu_2}\Delta_{3\mu_c}^{\nu_1}+\varepsilon_{\mu_{abc}\nu_1}p_{21}^{\nu_1}I_{\log}),\\
2\varepsilon_{\mu_{ab}\nu_{12}}T_{\mu_c}^{(+,+)\nu_{12}} &= +4\varepsilon_{\mu_{ab}\nu_{12}}p_{31}^{\nu_2}(J_{3\mu_c}^{\nu_1}+p_{21\mu_c}J_3^{\nu_1}) && (5.15)\\
&\quad +(\varepsilon_{\mu_{ab}\nu_{12}}p_{31}^{\nu_2}\Delta_{3\mu_c}^{\nu_1}+\varepsilon_{\mu_{abc}\nu_1}p_{31}^{\nu_1}I_{\log}).
\end{aligned}$$

For each trace version from (5.6) we have a common tensor and in them sign tensor vanishes. Thus, after disregarding the vanishing contribution, we identify the corresponding combinations

$$\begin{aligned}
\bar{C}_{1,\mu_{123}} &= -\varepsilon_{\mu_{13}\nu_{12}}T_{\mu_2}^{\nu_{12}(+-)}+\varepsilon_{\mu_{12}\nu_{12}}T_{\mu_3}^{\nu_{12}(--)} && (5.16)\\
\bar{C}_{2,\mu_{123}} &= -\varepsilon_{\mu_{12}\nu_{12}}T_{\mu_3}^{\nu_{12}(++)}-\varepsilon_{\mu_{23}\nu_{12}}T_{\mu_1}^{\nu_{12}(+-)} && (5.17)\\
\bar{C}_{3,\mu_{123}} &= -\varepsilon_{\mu_{23}\nu_{12}}T_{\mu_1}^{\nu_{12}(--)}-\varepsilon_{\mu_{13}\nu_{12}}T_{\mu_2}^{\nu_{12}(++)}. && (5.18)
\end{aligned}$$

The sampling of indexes reflects the absence of the index μ_i of the vertex Γ_i in the sign tensors of the $C_{i,\mu_{123}}$, enabling the anticipation of violations of either WIs or IRagfs. That occurs because the procedure of reducing the finite parts of the J tensor integrals asks for reshuffling of indices via Schouten identity. As a collateral effect the trace of that integral appear and produce a constant that goes with the trace of surface term. We will see this in detail.

For the last, the odd subamplitudes vanish. Hence for the first version of AVV we need only the result for VPP:

$$\begin{aligned}
(T^{VPP})^{\nu_1} &= 2[P_{31}^{\nu_2}\Delta_{3\nu_2}^{\nu_1}+(p_{21}^{\nu_1}-p_{32}^{\nu_1})I_{\log}]-4\,(p_{21}\cdot p_{32})\,J_3^{\nu_1} && (5.19)\\
&\quad +2[(p_{31}^{\nu_1}p_{21}^2-p_{21}^{\nu_1}p_{31}^2)J_3+p_{21}^{\nu_1}J_2\,(p_{21})-p_{32}^{\nu_1}J_2\,(p_{32})].
\end{aligned}$$

Therefore, presenting all versions, we have

$$\begin{aligned}
(T_{\mu_{123}}^{AVV})_1 &= 4\mathrm{i}\bar{C}_{1\mu_{123}}+\mathrm{i}\varepsilon_{\mu_{123}\nu_1}(T^{VPP})^{\nu_1}, && (5.20)\\
(T_{\mu_{123}}^{AVV})_2 &= 4\mathrm{i}\bar{C}_{2\mu_{123}}-\mathrm{i}\varepsilon_{\mu_{123}\nu_1}(T^{SAP})^{\nu_1}, && (5.21)\\
(T_{\mu_{123}}^{AVV})_3 &= 4\mathrm{i}\bar{C}_{3\mu_{123}}+\mathrm{i}\varepsilon_{\mu_{123}\nu_1}(T^{SPA})^{\nu_1}. && (5.22)
\end{aligned}$$

Each investigated case leads to a subamplitude identified by vertex arrangements in (5.1). This result is general: besides $C_{i,\mu_{123}}$ tensors, different rank-1 even subamplitudes appear in each version of rank-3 odd amplitudes. Table 5.2 accounts for all of these possibilities, while Appendix (A.4) presents explicit expressions for subamplitudes.

Table 5.1:

Version/Type	AVV	VAV	VVA	AAA
1	$+VPP$	$+ASP$	$-APS$	$-VSS$
2	$-SAP$	$+PVP$	$+PAS$	$-SVS$
3	$+SPA$	$-PSA$	$+PPV$	$-SSV$

Table 5.2: Even sub-amplitudes related to each version of 3rd order odd amplitudes.

Since all pieces are known, compounding triangle amplitudes is a direct task. For instance, the i^{th} version of the AVV arises as a combination involving the C_i-tensor and the corresponding vector subamplitude. Thus, consulting Table 5.2 leads to the following associations The generalization for VAV, VVA, and AAA is straightforward:

$$(T^{\Gamma_{123}}_{\mu_{123}})_i = 4\mathrm{i}\bar{C}_{i,\mu_{123}} \pm \mathrm{i}\varepsilon_{\mu_{123}\nu_1} \left(\text{Corresponding sub-amplitude}\right)^{\nu_1}. \tag{5.23}$$

We still want to detail some important points about these amplitudes. To illustrate this subject, we use the tools developed in this section to build up the first version of AVV

$$\begin{aligned}(T^{AVV}_{\mu_{123}})_1 \;=\;& \mathcal{S}_{1,\mu_{123}} && (5.24)\\ & -8\mathrm{i}\varepsilon_{\mu_{12}\nu_{12}}p_{21}^{\nu_2}(J^{\nu_1}_{3\mu_3} + p_{31\mu_3}J_3^{\nu_1}) && (5.25)\\ & -8\mathrm{i}\varepsilon_{\mu_{13}\nu_{12}}[p_{31}^{\nu_2}(J^{\nu_1}_{3\mu_2} + p_{21\mu_2}J_3^{\nu_1}) - p_{21}^{\nu_2}(J^{\nu_1}_{3\mu_2} + p_{31\mu_2}J_3^{\nu_1})]\\ & -4\mathrm{i}\varepsilon_{\mu_{123}\nu_1}(p_{21}\cdot p_{32})J_3^{\nu_1} + 2i\varepsilon_{\mu_{123}\nu_1}[(p_{31}^{\nu_1}p_{21}^2 - p_{21}^{\nu_1}p_{31}^2)]J_3\\ & +2\mathrm{i}\varepsilon_{\mu_{123}\nu_1}[p_{21}^{\nu_1}J_2\left(p_{21}\right) - p_{32}^{\nu_1}J_2\left(p_{32}\right)].\end{aligned}$$

The divergent part of the tensor (5.16) comes from eqs. (5.13) and (5.14) as

$$4\mathrm{i}\bar{C}_{1,\mu_{123}} = -2\mathrm{i}[\varepsilon_{\mu_{13}\nu_{12}}p_{32}^{\nu_2}\Delta^{\nu_1}_{3\mu_2} + \varepsilon_{\mu_{12}\nu_{12}}p_{21}^{\nu_2}\Delta^{\nu_1}_{3\mu_3} + \varepsilon_{\mu_{123}\nu_1}(p_{21}^{\nu_1} - p_{32}^{\nu_1})I_{\text{log}}].$$

When combined with the VPP subamplitude, we acknowledge the exact cancellation of the object I_{log} as it occurs for all investigated versions of all amplitudes. Thus, surface terms compound the whole structure of divergences

$$\mathcal{S}_{1,\mu_{123}} = -2\mathrm{i}(\varepsilon_{\mu_{13}\nu_{12}}p_{32}^{\nu_2}\Delta^{\nu_1}_{3\mu_2} + \varepsilon_{\mu_{12}\nu_{12}}p_{21}^{\nu_2}\Delta^{\nu_1}_{3\mu_3}) + 2\mathrm{i}\varepsilon_{\mu_{123}\nu_1}P_{31}^{\nu_2}\Delta^{\nu_1}_{3\nu_2}. \tag{5.26}$$

Moreover, contributions from vector subamplitudes exhibit arbitrary momenta $P_{ij} = k_i + k_j$ as coefficients. We stress that the divergent content is shared; the first version of each amplitude among AVV, VAV, VVA, and AAA contains the same structure (5.26). That is a feature of the specific version and not on the vertex content of the diagram. For later use, we define the other sets of surface terms

$$\begin{aligned}\mathcal{S}_{2,\mu_{123}} &= -2\mathrm{i}(\varepsilon_{\mu_{12}\nu_{12}}p_{31}^{\nu_2}\Delta^{\nu_1}_{3\mu_3} + \varepsilon_{\mu_{23}\nu_{12}}p_{32}^{\nu_2}\Delta^{\nu_1}_{3\mu_1}) + 2\mathrm{i}\varepsilon_{\mu_{123}\nu_1}P_{21}^{\nu_2}\Delta^{\nu_1}_{3\nu_2}, && (5.27)\\ \mathcal{S}_{3,\mu_{123}} &= -2\mathrm{i}(\varepsilon_{\mu_{13}\nu_{12}}p_{31}^{\nu_2}\Delta^{\nu_1}_{3\mu_2} - \varepsilon_{\mu_{23}\nu_{12}}p_{21}^{\nu_2}\Delta^{\nu_1}_{3\mu_1}) + 2\mathrm{i}\varepsilon_{\mu_{123}\nu_1}P_{32}^{\nu_2}\Delta^{\nu_1}_{3\nu_2}. && (5.28)\end{aligned}$$

That concludes the preliminary discussion on rank-3 triangles, so investigating RAGFs is possible. That is the subject of the following subsections.

5.2 Ragfs and Uniqueness

The next step is to perform momenta contractions that lead to iRagfs following the recipes of section (2.1). Although they are algebraic identities at the integrand level, their satisfaction is not automatic after integration. Possibilities for Dirac traces and values of surface terms have implications for this analysis.

$$\begin{aligned} p_{31}^{\mu_1} t_{\mu_{123}}^{AVV} &= t_{\mu_{32}}^{AV}(1,2) - t_{\mu_{23}}^{AV}(2,3) - 2m t_{\mu_{23}}^{PVV} \\ p_{21}^{\mu_2} t_{\mu_{123}}^{AVV} &= t_{\mu_{13}}^{AV}(1,3) - t_{\mu_{13}}^{AV}(2,3) \\ p_{32}^{\mu_3} t_{\mu_{123}}^{AVV} &= t_{\mu_{12}}^{AV}(1,2) - t_{\mu_{12}}^{AV}(1,3) \end{aligned} \tag{5.29}$$

$$\begin{aligned} p_{31}^{\mu_1} t_{\mu_{123}}^{VAV} &= t_{\mu_{23}}^{AV}(2,1) - t_{\mu_{23}}^{AV}(2,3) \\ p_{21}^{\mu_2} t_{\mu_{123}}^{VAV} &= t_{\mu_{31}}^{AV}(3,1) - t_{\mu_{13}}^{AV}(2,3) + 2m t_{\mu_{13}}^{VPV} \\ p_{32}^{\mu_3} t_{\mu_{123}}^{VAV} &= t_{\mu_{21}}^{AV}(2,1) - t_{\mu_{21}}^{AV}(3,1) \end{aligned} \tag{5.30}$$

$$\begin{aligned} p_{31}^{\mu_1} t_{\mu_{123}}^{VVA} &= t_{\mu_{32}}^{AV}(1,2) - t_{\mu_{32}}^{AV}(3,2) \\ p_{21}^{\mu_2} t_{\mu_{123}}^{VVA} &= t_{\mu_{31}}^{AV}(3,1) - t_{\mu_{31}}^{AV}(3,2) \\ p_{32}^{\mu_3} t_{\mu_{123}}^{VVA} &= t_{\mu_{12}}^{AV}(1,2) - t_{\mu_{21}}^{AV}(3,1) + 2m t_{\mu_{12}}^{VVP} \end{aligned} \tag{5.31}$$

$$\begin{aligned} p_{31}^{\mu_1} t_{\mu_{123}}^{AAA} &= t_{\mu_{23}}^{AV}(2,1) - t_{\mu_{32}}^{AV}(3,2) - 2m t_{\mu_{23}}^{PAA} \\ p_{21}^{\mu_2} t_{\mu_{123}}^{AAA} &= t_{\mu_{13}}^{AV}(1,3) - t_{\mu_{31}}^{AV}(3,2) + 2m t_{\mu_{13}}^{APA} \\ p_{32}^{\mu_3} t_{\mu_{123}}^{AAA} &= t_{\mu_{21}}^{AV}(2,1) - t_{\mu_{12}}^{AV}(1,3) + 2m t_{\mu_{12}}^{AAP} \end{aligned} \tag{5.32}$$

Let us introduce the structures that emerged within the relations above. First, the r.h.s.'s three-point functions are finite tensors, and external-momenta dependent. That is transparent due to their connection with finite Feynman integrals introduced in Appendix (A.2), so we only remove the overbar notation from corresponding tensors $\bar{J}_3^{\nu_1} = J_3^{\nu_1}$ and $\bar{J}_3 = J_3$. We have for single axial triangles

$$2mT_{\mu_{23}}^{PVV} = \varepsilon_{\mu_{23}\nu_{12}} p_{21}^{\nu_1} p_{32}^{\nu_2} (-8\mathrm{i}m^2 J_3), \tag{5.33}$$

$$2mT_{\mu_{13}}^{VPV} = \varepsilon_{\mu_{13}\nu_{12}} p_{21}^{\nu_1} p_{32}^{\nu_2} (8\mathrm{i}m^2 J_3), \tag{5.34}$$

$$2mT_{\mu_{12}}^{VVP} = \varepsilon_{\mu_{12}\nu_{12}} p_{21}^{\nu_1} p_{32}^{\nu_2} (-8\mathrm{i}m^2 J_3), \tag{5.35}$$

while momenta contractions for the triple axial triangle lead to

$$2mT_{\mu_{23}}^{PAA} = \varepsilon_{\mu_{23}\nu_{12}} p_{31}^{\nu_2} [-8\mathrm{i}m^2 (2J_3^{\nu_1} + p_{21}^{\nu_1} J_3)], \tag{5.36}$$

$$2mT_{\mu_{13}}^{APA} = \varepsilon_{\mu_{13}\nu_{12}} p_{21}^{\nu_2} [-8\mathrm{i}m^2 (2J_3^{\nu_1} + p_{31}^{\nu_1} J_3)], \tag{5.37}$$

$$2mT_{\mu_{12}}^{AAP} = \varepsilon_{\mu_{12}\nu_{12}} p_{32}^{\nu_2} [8\mathrm{i}m^2 (2J_3^{\nu_1} + p_{21}^{\nu_1} J_3)]. \tag{5.38}$$

These amplitudes have a low-energy behavior that we aim to explore in connection with the IRagfs in Subsections (5.3) and (5.4). Since they depend on $Z^{(-1)}_{n_1 n_2}$ functions (3.101) through the scalar and the vector three-point integrals (J_3, J_3^{ν}), definition 3.4.3 ; we use (3.115) to determine the behavior of these tensors when all bilinears in their momenta are zero:

$$2mT^{PVV}_{\mu_{23}}\Big|_0 = -\frac{1}{(2\pi)^2}; \quad 2mT^{VPV}_{\mu_{13}}\Big|_0 = \frac{1}{(2\pi)^2}; \quad 2mT^{VVP}_{\mu_{12}}\Big|_0 = -\frac{1}{(2\pi)^2}; \tag{5.39}$$

$$2mT^{PAA}_{\mu_{23}}\Big|_0 = -\frac{1}{3(2\pi)^2}; \quad 2mT^{APA}_{\mu_{13}}\Big|_0 = \frac{1}{3(2\pi)^2}; \quad 2mT^{AAP}_{\mu_{12}}\Big|_0 = -\frac{1}{3(2\pi)^2}. \tag{5.40}$$

Each term above is multiplied by the corresponding tensor $\varepsilon_{\mu_{kl}\nu_{12}} p_{21}^{\nu_1} p_{32}^{\nu_2}$ with $k < l$.

Second, the other structures that appeared in the RAGFs are AV functions, which are proportional to two-point vector integrals. Using the result (A.12), we achieve

$$T^{AV}_{\mu_{ij}}(a,b) = -4\mathrm{i}\varepsilon_{\mu_i \mu_j \nu_1 \nu_2} p_{ba}^{\nu_2} \left[\bar{J}_2^{\nu_1}(a,b)\right] = 2\mathrm{i}\varepsilon_{\mu_i \mu_j \nu_1 \nu_2} p_{ba}^{\nu_2} P_{ab}^{\nu_3} \Delta_{3\nu_3}^{\nu_1}. \tag{5.41}$$

As contributions (exclusively) on the external momentum cancel out in the contraction, they are pure surface terms proportional to arbitrary label combinations. After replacing the adequate labels (k_a and k_b), combinations seen in the IRagfs above arise:

$$\begin{aligned}
T^{AV}_{\mu_{32}}(1,2) - T^{AV}_{\mu_{23}}(2,3) &= -2\mathrm{i}\varepsilon_{\mu_{23}\nu_{12}} \left(p_{21}^{\nu_2} P_{12}^{\nu_3} + p_{32}^{\nu_2} P_{32}^{\nu_3}\right) \Delta_{3\nu_3}^{\nu_1} && (5.42)\\
T^{AV}_{\mu_{13}}(1,3) - T^{AV}_{\mu_{13}}(2,3) &= -2\mathrm{i}\varepsilon_{\mu_{13}\nu_{12}} \left(p_{32}^{\nu_2} P_{32}^{\nu_3} - p_{31}^{\nu_2} P_{31}^{\nu_3}\right) \Delta_{3\nu_3}^{\nu_1} && (5.43)\\
T^{AV}_{\mu_{12}}(1,2) - T^{AV}_{\mu_{12}}(1,3) &= -2\mathrm{i}\varepsilon_{\mu_{12}\nu_{12}} \left(p_{31}^{\nu_2} P_{31}^{\nu_3} - p_{21}^{\nu_2} P_{21}^{\nu_3}\right) \Delta_{3\nu_3}^{\nu_1}. && (5.44)
\end{aligned}$$

We stress that these forms depend only on the vertex contraction and not the specific amplitude (AVV, VAV, VVA, and AAA). That occurs because there is a sign change in the AV when permuting the position of free indexes (see $\varepsilon_{\mu_i \mu_j \nu_1 \nu_2}$) or changing the role of routings (see $p_{ba}^{\nu_2} P_{ab}^{\nu_3}$). *This fact holds in any dimensions for the difference of n-point and odd amplitudes that are pure surface terms.*

To verify IRagfs, we must contract external momenta with the explicit forms of amplitudes. Observe the finite contributions displayed in the example (5.24) to clarify operations involving finite contributions. These results use well-defined relations involving finite quantities. After contracting with momenta, some terms vanish due to the Levi-Civita symbol. Then, we manipulate the remaining terms using tools developed in Appendix (A.2). The procedure involves reducing J-tensors to identify finite 2nd-order amplitudes or achieve some cancellations. The referred reductions are for tensor integrals

$$\begin{aligned}
2p_{21}^{\nu_2} J_{3\nu_2}^{\nu_1} &= -p_{21}^2 J_3^{\nu_1} + J_2^{\nu_1}(p_{31}) + J_2^{\nu_1}(p_{32}) + p_{31}^{\nu_1} J_2(p_{32}), && (5.45)\\
2p_{31}^{\nu_2} J_{3\nu_2}^{\nu_1} &= -p_{31}^2 J_3^{\nu_1} + J_2^{\nu_1}(p_{21}) + J_2^{\nu_1}(p_{32}) + p_{31}^{\nu_1} J_2(p_{32}), && (5.46)\\
2J_{3\nu_1}^{\nu_1} &= 2m^2 J_3 + 2J_2(p_{32}) + i(4\pi)^{-2}, && (5.47)
\end{aligned}$$

and vector integrals

$$\begin{aligned}
2p_{21\nu_1} J_3^{\nu_1} &= -p_{21}^2 J_3 + J_2(p_{31}) - J_2(p_{32}), && (5.48)\\
2p_{31\nu_1} J_3^{\nu_1} &= -p_{31}^2 J_3 + J_2(p_{21}) - J_2(p_{32}). && (5.49)
\end{aligned}$$

Although some reductions arise directly, other occurrences require further algebraic manipulations. This circumstance manifests in cases where a J-tensor couples to the LC tensor so that rearranging indexes is necessary to find momenta contractions. For vector integrals, we consider the identity $\varepsilon_{[\mu_a \mu_b \nu_1 \nu_2 p_{\nu_3]}} J_3^{\nu_1} = 0$ to achieve the formula[2]

$$2\varepsilon_{\mu_{ab}\nu_{12}} \left[p_{21}^{\nu_2} \left(p_{ij} \cdot p_{31}\right) - p_{31}^{\nu_2} \left(p_{ij} \cdot p_{21}\right)\right] J_3^{\nu_1} = -\varepsilon_{\mu_{ab}\nu_{23}} p_{21}^{\nu_2} p_{31}^{\nu_3} \left[2p_{ij\nu_1} J_3^{\nu_1}\right]. \tag{5.50}$$

Similarly, we use $\varepsilon_{[\mu_a \nu_1 \nu_2 \nu_3} J_{3\mu_c]}^{\nu_1} = 0$ to reorganize terms involving the tensor integral

$$\begin{aligned} & 2\varepsilon_{\mu_b \nu_{123}} p_{21}^{\nu_2} p_{31}^{\nu_3} J_{3\mu_a}^{\nu_1} - 2\varepsilon_{\mu_a \nu_{123}} p_{21}^{\nu_2} p_{31}^{\nu_3} J_{3\mu_b}^{\nu_1} \\ = \ & \varepsilon_{\mu_{ab}\nu_{13}} p_{31}^{\nu_3} \left[2p_{21}^{\nu_2} J_{3\nu_2}^{\nu_1}\right] - \varepsilon_{\mu_{ab}\nu_{12}} p_{21}^{\nu_2} \left[2p_{31}^{\nu_3} J_{3\nu_3}^{\nu_1}\right] - \varepsilon_{\mu_{ab}\nu_{23}} p_{21}^{\nu_2} p_{31}^{\nu_3} \left[2J_{3\nu_1}^{\nu_1}\right]. \end{aligned} \tag{5.51}$$

Once more the procedure extends to all even dimensions with larger expressions.

In the amplitudes, we have two structures: standard tensors $C_{i\mu_{123}}$ (5.16)-(5.18) and sub-amplitudes. The tensors are common to the amplitudes versions and are comprised of the sign tensors (5.12)-(5.15). To illustrate operations necessary for IRagfs, let us take the first case

$$C_{1,\mu_{123}} = -2\varepsilon_{\mu_{13}\nu_{12}}[p_{31}^{\nu_2}(J_{3\mu_2}^{\nu_1} + p_{21\mu_2} J_3^{\nu_1}) - p_{21}^{\nu_2}(J_{3\mu_2}^{\nu_1} + p_{31\mu_2} J_3^{\nu_1})] - 2\varepsilon_{\mu_{12}\nu_{12}} p_{21}^{\nu_2}(J_{3\mu_3}^{\nu_1} + p_{31\mu_3} J_3^{\nu_1}). \tag{5.52}$$

The first term in curvy brackets cancels when performing the contraction with $p_{31}^{\mu_1}$, the remaining terms are

$$p_{31}^{\mu_1}\left[C_{1,\mu_{123}}\right] = -2[\varepsilon_{\mu_3 \nu_{123}} p_{21}^{\nu_2} p_{31}^{\nu_3} J_{3\mu_2}^{\nu_1} - \varepsilon_{\mu_2 \nu_{123}} p_{21}^{\nu_2} p_{31}^{\nu_3} J_{3\mu_3}^{\nu_1}]. \tag{5.53}$$

Then, we employ the identity above (5.51) to permute indices and perform reductions. That accomplishes our objective; furthermore, this rearrangement implies the presence of eq. (5.47) and that brings two additional contributions: one proportional to squared mass (J_3) and a constant term. That differs from contractions with $p_{21}^{\mu_2}$ and $p_{32}^{\mu_3}$, where reductions of the tensor integrals are immediate, and it is only necessary to use (5.50). We cast all possible contractions of tensors below, emphasizing that their behavior is a feature of the amplitude version.

$$p_{31}^{\mu_1}\left[C_{1,\mu_{123}}\right] = \varepsilon_{\mu_{23}\nu_1\nu_2}\{\left(p_{31}^{\nu_2} p_{21}^2 - p_{21}^{\nu_2} p_{31}^2\right) J_3^{\nu_1} + p_{21}^{\nu_1} p_{31}^{\nu_2}[2m^2 J_3 + i(4\pi)^{-2} + J_2\left(p_{32}\right)]\} \tag{5.54}$$

$$p_{21}^{\mu_2}\left[C_{1,\mu_{123}}\right] = \frac{1}{2}\varepsilon_{\mu_{13}\nu_{12}} p_{32}^{\nu_2}\left[2p_{21}^2\left(J_3^{\nu_1} + p_{21}^{\nu_1} J_3\right) - p_{21}^{\nu_1} J_2\left(p_{31}\right)\right] \tag{5.55}$$

$$p_{32}^{\mu_3}\left[C_{1,\mu_{123}}\right] = \frac{1}{2}\varepsilon_{\mu_{12}\nu_1\nu_2} p_{21}^{\nu_2}[-2p_{32}^2 J_3^{\nu_1} - p_{31}^{\nu_1} J_2\left(p_{31}\right)] \tag{5.56}$$

$$p_{31}^{\mu_1}\left[C_{2,\mu_{123}}\right] = \frac{1}{2}\varepsilon_{\mu_{23}\nu_{12}} p_{32}^{\nu_2}[2p_{31}^2\left(J_3^{\nu_1} + p_{21}^{\nu_1} J_3\right) - p_{21}^{\nu_1} J_2\left(p_{21}\right)] \tag{5.57}$$

$$p_{21}^{\mu_2}\left[C_{2,\mu_{123}}\right] = \varepsilon_{\mu_{13}\nu_{12}}\left\{(p_{31}^{\nu_2} p_{21}^2 - p_{21}^{\nu_2} p_{31}^2) J_3^{\nu_1} + p_{21}^{\nu_1} p_{31}^{\nu_2}[2m^2 J_3 + i(4\pi)^{-2} + J_2(p_{32})]\right\} \tag{5.58}$$

$$p_{32}^{\mu_3}\left[C_{2,\mu_{123}}\right] = \frac{1}{2}\varepsilon_{\mu_{12}\nu_{12}}[2p_{31}^{\nu_2} p_{32}^2 J_3^{\nu_1} + p_{21}^{\nu_1} p_{31}^{\nu_2} J_2\left(p_{21}\right)] \tag{5.59}$$

---

[2] Two terms like $p_a \varepsilon_{b\nu_{123}} p_{21}^{\nu_2} p_{31}^{\nu_3} J_3^{\nu_1}$ cancel due to triple contraction.

$$\begin{aligned}
p_{31}^{\mu_1}\left[C_{3,\mu_{123}}\right] &= \frac{1}{2}\varepsilon_{\mu_{23}\nu_{12}}p_{21}^{\nu_2}[2p_{31}^2 J_3^{\nu_1} + p_{31}^{\nu_1}J_2(p_{32})] && (5.60)\\
p_{21}^{\mu_2}\left[C_{3,\mu_{123}}\right] &= \frac{1}{2}\varepsilon_{\mu_{13}\nu_{12}}p_{31}^{\nu_2}[-2p_{21}^2 J_3^{\nu_1} - p_{21}^{\nu_1}J_2(p_{32})] && (5.61)\\
p_{32}^{\mu_3}\left[C_{3,\mu_{123}}\right] &= \varepsilon_{\mu_{12}\nu_{12}}\{\left(p_{21}^{\nu_2}p_{31}^2 - p_{31}^{\nu_2}p_{21}^2\right)J_3^{\nu_1} - p_{21}^{\nu_1}p_{31}^{\nu_2}[2m^2 J_3 + i(4\pi)^{-2} + J_2(p_{32})]\}. && (5.62)
\end{aligned}$$

We have to sum contributions from the subamplitudes to complete the finite sector. That requires the same resources discussed above, but only vector integrals remain, and again we use Eq. (5.50) to reduce these integrals to scalar ones. Terms proportional to the squared mass arise from some of the common tensors and subamplitudes. They cancel in all vector-vertex contractions and combine into the expected finite functions for all axial-vertex contractions (5.33)-(5.38). Lastly, regardless of the specific amplitude, the additional term $i\,(4\pi)^{-2}$ arises when the contracted index μ_i matches the i^{th} version.

To complete the analysis of IRagfs, we recall that divergent structures from eqs. (5.26)-(5.28) are pure surface terms $\mathcal{S}_{i,\mu_{123}}$ with the index μ_i appearing exclusively within the Levi-Civita tensor and not in $\Delta_{3\mu\nu}$. Our task is to perform their momenta contractions to identify differences between AVs (5.42)-(5.44). For all triangle amplitudes, that is automatic whenever these operations consider another index μ_j with $i \neq j$. Nevertheless, contracting the version-defining index $(i = j)$ does not produce momenta contractions with surface terms required to recognize AV functions. Thus, in parallel to the procedure for 2nd-order J-tensors, indices are reorganized through the identity

$$\varepsilon_{\mu_1\mu_3\nu_1\nu_2}\Delta_{3\mu_2}^{\nu_1} - \varepsilon_{\mu_1\mu_2\nu_1\nu_2}\Delta_{3\mu_3}^{\nu_1} = \varepsilon_{\mu_2\mu_3\nu_1\nu_2}\Delta_{3\mu_1}^{\nu_1} + \varepsilon_{\mu_1\mu_2\mu_3\nu_1}\Delta_{3\nu_2}^{\nu_1} - \varepsilon_{\mu_1\mu_2\mu_3\nu_2}\Delta_{3\nu_1}^{\nu_1}. \qquad (5.63)$$

Taking the first version $\mathcal{S}_{1,\mu_{123}}$ to illustrate. While relations involving indices μ_2 and μ_3 are automatic, contracting μ_1 demands the permutation introduced above. These operations yield (5.64) after organizing the momenta by $p_{ij} = P_{ir} - P_{jr}$. Besides the expected contributions, note the presence of an additional term, the trace $\Delta_{3\nu}^{\nu}$, resembling what occurred for the finite part. We also cast results for other contractions in the sequence, so that the understanding of this pattern becomes clearer.

$$\begin{aligned}
p_{31}^{\mu_1}\mathcal{S}_{1,\mu_{123}} &= -2\mathrm{i}\varepsilon_{\mu_{23}\nu_{12}}\left(p_{21}^{\nu_2}P_{12}^{\nu_3} + p_{32}^{\nu_2}P_{32}^{\nu_3}\right)\Delta_{3\nu_3}^{\nu_1} + 2\mathrm{i}\varepsilon_{\mu_2\mu_3\nu_2\nu_3}p_{21}^{\nu_2}p_{31}^{\nu_3}\Delta_{3\nu_1}^{\nu_1} && (5.64)\\
p_{21}^{\mu_2}\mathcal{S}_{1,\mu_{123}} &= -2\mathrm{i}\varepsilon_{\mu_{13}\nu_{12}}(p_{32}^{\nu_2}P_{32}^{\nu_3} - p_{31}^{\nu_2}P_{31}^{\nu_3})\Delta_{3\nu_3}^{\nu_1} && (5.65)\\
p_{32}^{\mu_3}\mathcal{S}_{1,\mu_{123}} &= -2\mathrm{i}\varepsilon_{\mu_{12}\nu_{12}}\left(p_{31}^{\nu_2}P_{31}^{\nu_3} - p_{21}^{\nu_2}P_{21}^{\nu_3}\right)\Delta_{3\nu_3}^{\nu_1} && (5.66)
\end{aligned}$$

$$\begin{aligned}
p_{31}^{\mu_1}\mathcal{S}_{2,\mu_{123}} &= -2\mathrm{i}\varepsilon_{\mu_{23}\nu_{12}}\left(p_{21}^{\nu_2}P_{12}^{\nu_3} + p_{32}^{\nu_2}P_{32}^{\nu_3}\right)\Delta_{3\nu_3}^{\nu_1} && (5.67)\\
p_{21}^{\mu_2}\mathcal{S}_{2,\mu_{123}} &= -2\mathrm{i}\varepsilon_{\mu_{13}\nu_{12}}\left(p_{32}^{\nu_2}P_{32}^{\nu_3} - p_{31}^{\nu_2}P_{31}^{\nu_3}\right)\Delta_{3\nu_3}^{\nu_1} + 2\mathrm{i}\varepsilon_{\mu_1\mu_3\nu_2\nu_3}p_{21}^{\nu_2}p_{31}^{\nu_3}\Delta_{3\nu_1}^{\nu_1} && (5.68)\\
p_{32}^{\mu_3}\mathcal{S}_{2,\mu_{123}} &= -2\mathrm{i}\varepsilon_{\mu_{12}\nu_{12}}\left(p_{31}^{\nu_2}P_{31}^{\nu_3} - p_{21}^{\nu_2}P_{21}^{\nu_3}\right)\Delta_{3\nu_3}^{\nu_1} && (5.69)
\end{aligned}$$

$$\begin{aligned}
p_{31}^{\mu_1}\mathcal{S}_{3,\mu_{123}} &= -2\mathrm{i}\varepsilon_{\mu_{23}\nu_{12}}\left(p_{21}^{\nu_2}P_{12}^{\nu_3} + p_{32}^{\nu_2}P_{32}^{\nu_3}\right)\Delta_{3\nu_3}^{\nu_1} && (5.70)\\
p_{21}^{\mu_2}\mathcal{S}_{3,\mu_{123}} &= -2\mathrm{i}\varepsilon_{\mu_{13}\nu_{12}}\left(p_{32}^{\nu_2}P_{32}^{\nu_3} - p_{31}^{\nu_2}P_{31}^{\nu_3}\right)\Delta_{3\nu_3}^{\nu_1} && (5.71)\\
p_{32}^{\mu_3}\mathcal{S}_{3,\mu_{123}} &= -2\mathrm{i}\varepsilon_{\mu_{12}\nu_{12}}\left(p_{31}^{\nu_2}P_{31}^{\nu_3} - p_{21}^{\nu_2}P_{21}^{\nu_3}\right)\Delta_{3\nu_3}^{\nu_1} - 2\mathrm{i}\varepsilon_{\mu_1\mu_2\nu_2\nu_3}p_{21}^{\nu_2}p_{31}^{\nu_3}\Delta_{3\nu_1}^{\nu_1} && (5.72)
\end{aligned}$$

With these properties in hands, we establish IRagfs for the explicit $(T^{AVV}_{\mu_{123}})_1$ (5.20) to illustrate how to proceed in any case. The axial contraction comes from reducing the common tensor in eq. (5.54) plus the nonzero terms from subamplitude (5.19)

$$\mathrm{i}\varepsilon_{\mu_{123}\nu_1}p_{31}^{\mu_1}(T^{VPP})^{\nu_1} = 2\mathrm{i}\varepsilon_{\mu_{23}\nu_{12}}p_{31}^{\nu_2}\left\{2(p_{21}\cdot p_{32})J_3^{\nu_1} + p_{21}^{\nu_1}[p_{31}^2 J_3 - J_2(p_{21}) - J_2(p_{32})]\right\}.$$

At this stage, by summing both contributions (common tensor and subamplitude), we get

$$\begin{aligned} p_{31}^{\mu_1}(T^{AVV}_{\mu_{123}})_1 &= \left[p_{31}^{\mu_1}\mathcal{S}_{1,\mu_{123}}\right] + 4\mathrm{i}\varepsilon_{\mu_{23}\nu_{12}}p_{21}^{\nu_1}p_{31}^{\nu_2}[2m^2 J_3 + \mathrm{i}\,(4\pi)^{-2}] \\ &\quad +4\mathrm{i}\varepsilon_{\mu_{23}\nu_{12}}\left[p_{31}^{\nu_2}\,(p_{21}\cdot p_{31}) - p_{21}^{\nu_2}p_{31}^2\right]J_3^{\nu_1} \\ &\quad +2\mathrm{i}\varepsilon_{\mu_{23}\nu_{12}}p_{21}^{\nu_1}p_{31}^{\nu_2}[p_{31}^2 J_3 + J_2\,(p_{32}) - J_2\,(p_{21})]. \end{aligned} \tag{5.73}$$

To find reductions in terms like the second row, we use (5.50) to identify the needed contraction and obtain a cancellation

$$p_{31}^{\mu_1}(T^{AVV}_{\mu_{123}})_1 = p_{31}^{\mu_1}S_{1\mu_{123}} + 4\mathrm{i}\varepsilon_{\mu_{23}\nu_{12}}p_{21}^{\nu_1}p_{31}^{\nu_2}[2m^2 J_3 + \mathrm{i}\,(4\pi)^{-2}]. \tag{5.74}$$

After contracting surface terms using (5.64) and identifying the PVV function (5.33), we write

$$p_{31}^{\mu_1}(T^{AVV}_{\mu_{123}})_1 = T^{AV}_{\mu_{32}}\,(1,2) - T^{AV}_{\mu_{23}}\,(2,3) - 2mT^{PVV}_{\mu_{23}} + \underline{2\mathrm{i}\varepsilon_{\mu_{23}\nu_{12}}p_{21}^{\nu_1}p_{31}^{\nu_2}[\Delta^{\alpha}_{3\alpha} + 2\mathrm{i}\,(4\pi)^{-2}]}. \tag{5.75}$$

Similarly, IRagfs coming from vector vertices need (5.55)-(5.56) for the common tensor, and the identity (5.50). They imply the vanishing of finite parts, while the remaining parts correspond to AV differences:

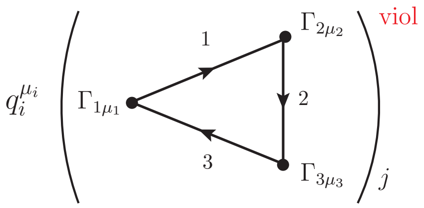
$$p_{21}^{\mu_2}(T^{AVV}_{\mu_{123}})_1 = p_{21}^{\mu_2}\mathcal{S}_{1\mu_{123}} = T^{AV}_{\mu_{13}}\,(1,3) - T^{AV}_{\mu_{13}}\,(2,3) \tag{5.76}$$

$$p_{32}^{\mu_3}(T^{AVV}_{\mu_{123}})_1 = p_{32}^{\mu_3}\mathcal{S}_{1\mu_{123}} = T^{AV}_{\mu_{12}}\,(1,2) - T^{AV}_{\mu_{12}}\,(1,3)\,. \tag{5.77}$$

This pattern repeats for the first version of the other amplitudes (VAV, VVA, and AAA). Whereas the contraction with first vertex exhibits the additional term, the other IRagfs are satisfied without conditions. The pattern changes to the second and third versions, for they show the violating term in the second and third vertex independent of its nature: axial or vector vertex.

Following the developed steps the equations below subsume all potentially offending terms, which emerge in momentum contractions where the version is defined. Adopting the notation of external momenta $q_1 = p_{13}$, $q_2 = p_{21}$, and $q_3 = p_{32}$, it will be simple to code all information about IRagfs. For all combinations of vertices $\Gamma_i \in \{A, V\}$ we are investigating, the version and place where there is a condition for satisfaction of a IRagfs is given by

$$\begin{aligned} q_1^{\mu_1}(T^{\Gamma_{123}}_{\mu_{123}})_1^{\mathrm{viol}} &= -2\mathrm{i}\varepsilon_{\mu_{23}\nu_{12}}q_2^{\nu_1}q_3^{\nu_2}[\Delta^{\alpha}_{3\alpha} + 2i\,(4\pi)^{-2}] \\ q_2^{\mu_2}(T^{\Gamma_{123}}_{\mu_{123}})_2^{\mathrm{viol}} &= +2\mathrm{i}\varepsilon_{\mu_{13}\nu_{12}}q_2^{\nu_1}q_3^{\nu_2}[\Delta^{\alpha}_{3\alpha} + 2i\,(4\pi)^{-2}] \\ q_3^{\mu_3}(T^{\Gamma_{123}}_{\mu_{123}})_3^{\mathrm{viol}} &= -2\mathrm{i}\varepsilon_{\mu_{12}\nu_{12}}q_2^{\nu_1}q_3^{\nu_2}[\Delta^{\alpha}_{3\alpha} + 2i\,(4\pi)^{-2}]. \end{aligned} \tag{5.78}$$



$$= 2i\delta_{ij}\varepsilon_{\mu_a\mu_b\nu_1\nu_2}p_{21}^{\nu_1}p_{31}^{\nu_2}\left[\Delta_{3\alpha}^{\alpha} + 2i\,(4\pi)^{-2}\right]$$

Figure 5.1: The violation factor of the RAGF established for the contraction with momenta $q_i^{\mu_1}$.

The other vertices (for each version) have their IRagfs identically satisfied. To visualize this violation pattern, we offer the schematic graph in Figure 5.1.

IRagfs are not automatic, they require further assessment. Meaning, they only apply under the constraint

$$\Delta_{3\alpha}^{\alpha} = -\frac{2\mathrm{i}}{(4\pi)^2}. \tag{5.79}$$

From another perspective, if these relations apply identically we could satisfy all Ward identities by making surface terms null (this works channel by channel). That is not the case, because it requires conflicting interpretations of surface terms: zero for the momentum-space translational invariance and nonzero for linearity of integration. Thence, these properties do not hold simultaneously. The reason is found in general tensor properties and the low-energy behavior of *PVV-PAA*. In section (5.4) it is shown that these conclusions are inescapable. Fact that is independent of any conceivable trace.

Once the computational aspects of IRagfs are clear, we would like to deepen the discussion about different versions of amplitudes. Their integrands are well-defined tensors, and obey $(t_{\mu_{123}}^{\Gamma_{123}})_i = (t_{\mu_{123}}^{\Gamma_{123}})_j$. Even if we separate expressions in finite and divergent sectors without commitment to the divergences, as we did, after integration, the sampling of indexes makes results of finite parts and tensor surface terms different. We highlight differences among the three basis versions to elucidate this point:

$$(T_{\mu_{123}}^{\Gamma_{123}})_1 - (T_{\mu_{123}}^{\Gamma_{123}})_2 = +2\mathrm{i}\varepsilon_{\mu_{123}\nu_1}p_{32}^{\nu_1}[\Delta_{3\alpha}^{\alpha} + 2i\,(4\pi)^{-2}], \tag{5.80}$$

$$(T_{\mu_{123}}^{\Gamma_{123}})_1 - (T_{\mu_{123}}^{\Gamma_{123}})_3 = -2\mathrm{i}\varepsilon_{\mu_{123}\nu_1}p_{21}^{\nu_1}[\Delta_{3\alpha}^{\alpha} + 2i\,(4\pi)^{-2}], \tag{5.81}$$

$$(T_{\mu_{123}}^{\Gamma_{123}})_2 - (T_{\mu_{123}}^{\Gamma_{123}})_3 = +2\mathrm{i}\varepsilon_{\mu_{123}\nu_1}p_{13}^{\nu_1}[\Delta_{3\alpha}^{\alpha} + 2i\,(4\pi)^{-2}]. \tag{5.82}$$

After subtracting two versions, we reorganized indexes to identify reductions of finite functions and recognize the same potentially violating term acknowledged in (5.78). At this point, we define the meaning of uniqueness adopted within this investigation: any possible form to compute the same expression returns the same result. Canceling the r.h.s. of these equations would be required to achieve this property. That only happens when adopting the same prescription seen above $\Delta_{3\alpha}^{\alpha} = -2i\,(4\pi)^{-2}$. This notion of uniqueness implies that an amplitude does not depend on Dirac traces. Nevertheless, unlike in the two-dimensional context, the nonzero surface terms required by this notion allow dependence on ambiguous combinations of arbitrary internal momenta. In this sense, there is not one unique expression in the external momenta.

The trace of six matrices is the unique place where the amplitude versions differ. Other traces can be obtained through identities involving the chiral matrix, eqs. (3.293)-(3.296). Nonetheless, as detailed in appendix of our preprint [1] or Appendix (A.6), all possibilities become linear combinations of the basis ones.

If we define the linear combination of two basis versions by

$$[T^{\Gamma_{123}}_{\mu_{123}}]_{i;j} = [(T^{\Gamma_{123}}_{\mu_{123}})_i + (T^{\Gamma_{123}}_{\mu_{123}})_j]/2, \tag{5.83}$$

which manifests potentially violating terms in IRagfs for both vertices Γ_i and Γ_j. The three independent combinations are enough to reproduce any expressions achieved through the referred identities[3]. That justifies taking $(T^{\Gamma_{123}}_{\mu_{123}})_i$ as the basic versions; moreover, *they have the maximum number of RAGFs identically satisfied*, see section (5.4).

For instance, the expression associated with the common substitution

$$\gamma_* \gamma_{\mu_i \nu_i \mu_{i+1}} = \mathrm{i}\varepsilon_{\mu_i \nu_i \mu_{i+1} \nu} \gamma^{\nu} + \gamma_* (g_{\nu_i \mu_{i+1}} \gamma_{\mu_i} - g_{\mu_i \mu_{i+1}} \gamma_{\nu_i} + g_{\mu_i \nu_i} \gamma_{\mu_{i+1}}) \tag{5.84}$$

has an integrand differing from $[T^{\Gamma_{123}}_{\mu_{123}}]_{i;i+1}$ ($i \bmod 3$) in terms that have finite and identically vanishing integrals ($T^{(-,+)}$, 5.12 and T^{ASS}). Using this last identity or combining traces of basic versions before integration makes expressions exhibit the same terms when integrated, divergent and finite parts.

With these facts in mind, we define linear combinations that reproduce any possible expression with the basis versions (see the discussion just after definition 3.8.2, section 3.8).

$$[t^{\Gamma_{123}}_{\mu_{123}}]_{\mathbf{w}} = \sum_{i=1}^{3} w_i (t^{\Gamma_{123}}_{\mu_{123}})_i \rightarrow [T^{\Gamma_{123}}_{\mu_{123}}]_{\mathbf{w}} = \sum_{i=1}^{3} w_i (T^{\Gamma_{123}}_{\mu_{123}})_i \tag{5.85}$$

where $\mathbf{w} = (w_1, w_2, w_3)$ is a vector of weights and $|\mathbf{w}| = w_1 + w_2 + w_3 = 1$.

They have equivalent integrands as it occurs for combinations above (5.83). This general form compiles all involved arbitrariness, accounting for any choices regarding routings or Dirac traces. From this formula, by nullifying the surface terms[4], we identify an infinity set of amplitudes that violate IRagfs by arbitrary amounts, which are only constrained by LETs. That is useful for obtaining different violation values found in the literature, e.g., [72]. However, if the surface term has the value (5.79), then all components are identical and by $|\mathbf{w}| = 1$ collapse to a unique form.

We have shown how traces and surface terms interfere with linearity of integration and uniqueness of the investigated tensors. In the subsequent sections, we demonstrate these properties are unavoidable since conditions for IRagfs arise without explicit computations of the primary amplitudes.

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[3] In four dimensions there no way to use just the "tower" and get a linear combination of all three basis versions.

[4] Or taking any other value which is not $\Delta^{\alpha}_{3\alpha} = -\frac{2i}{(4\pi)^2}$.

5.3 Low-Energy Theorems I

This section proposes a structure depending only on external momenta to formulate a low-energy implication for a tensor that represents three-point amplitudes, a detailed version of what we did in section (2.3). That does not mean we ignore the possible presence of ambiguous routing combinations because these terms can be transformed into linear covariant combinations of physical momenta. The referred structure is a general 3rd-order tensor having odd parity:

$$\begin{aligned} F_{\mu_{123}} &= \varepsilon_{\mu_{123}\nu}(q_2^{\nu}F_1 + q_3^{\nu}F_2) + \varepsilon_{\mu_{12}\nu_{12}}q_2^{\nu_1}q_3^{\nu_2}(q_{2\mu_3}G_1 + q_{3\mu_3}G_2) \\ &\quad +\varepsilon_{\mu_{13}\nu_{12}}q_2^{\nu_1}q_3^{\nu_2}\left(q_{2\mu_2}G_3 + q_{3\mu_2}G_4\right) + \varepsilon_{\mu_{23}\nu_{12}}q_2^{\nu_1}q_3^{\nu_2}(q_{2\mu_1}G_5 + q_{3\mu_1}G_6). \end{aligned} \tag{5.86}$$

That is a function of two variables; namely, the incoming external momenta q_2 and q_3 associated with vertices Γ_2 and Γ_3. Conservation sets the relation $q_1 + q_2 + q_3 = 0$.

After performing momenta contractions, one identifies the arrangements $q_i^{\mu_i}F_{\mu_{123}} = \varepsilon_{\mu_{kl}\nu_{12}}q_2^{\nu_1}q_3^{\nu_2}V_i^L$ with $k < l \neq i$. These operations lead to a set of three functions written in terms of form factors belonging to the general tensor

$$\begin{aligned} V_1^L &= F_1 - F_2 - (q_1 \cdot q_2)\, G_5 - (q_1 \cdot q_3)\, G_6, \\ V_2^L &= -F_2 + q_2^2 G_3 + (q_2 \cdot q_3)\, G_4, \\ V_3^L &= -F_1 + q_3^2 G_2 + (q_2 \cdot q_3)\, G_1. \end{aligned} \tag{5.87}$$

Without any sort of hypothesis about eventual symmetries nor restrictions over the value of any of quantities above, we construct an identity, namely

$$V_1^L - V_2^L + V_3^L = -q_2^2 G_3 + q_3^2 G_2 + (q_2 \cdot q_3)\,(G_1 - G_4) - (q_1 \cdot q_2)\, G_5 - (q_1 \cdot q_3)\, G_6.$$

At the kinematic point where all bilinears are zero $q_i \cdot q_j = 0$, if G_i are regular or at most bounded there, we derive a structural identity among invariants

$$V_1^L(0) - V_2^L(0) + V_3^L(0) = 0. \tag{5.88}$$

This relation contains information about symmetries or violations thereof, at the zero limit, even if no particular symmetry is needed for its deduction. That occurs because it represents a constraint over three-point structures arising on the r.h.s. of proposed WIs.

To illustrate this resource, suppose that the AVV axial contraction connects to the amplitude coming from the pseudoscalar density in the following fashion

$$\varepsilon_{\mu_{23}\nu_{12}}q_2^{\nu_1}q_3^{\nu_2}V_1^L(0) = 2mT_{\mu_{23}}^{PVV}(0) =: \varepsilon_{\mu_{23}\nu_{12}}q_2^{\nu_1}q_3^{\nu_2}V_1^{PVV}(0), \tag{5.89}$$

with the behavior (5.39) leading to the value for the first invariant $V_1^L(0) = -1/(2\pi)^2$. Since the constraint above prevents the simultaneous vanishing of both other invariants $V_2^L(0) = V_3^L(0) = 0$, at least one vector WI is violated. On the other hand, supposing that both vector WIs apply, implies violating the axial one. That occurs because parameters defining the considered tensor and regularity require the existence of an additional term $V_1(0) = -1/(2\pi)^2 + \mathcal{A}$, the anomaly. Thus, $\mathcal{A} = -V_1^{PVV}(0)$, relating a property of a finite amplitude and the symmetry content of a rank-3 amplitude. Satisfying the symmetry at this point does not guarantee invariance for all points; however, its violation at zero implies symmetry violation.

That is the starting point of the violation pattern in anomalous amplitudes. Numerical values presented above for invariants V_i^L at zero represent the preservation of corresponding WIs. Nevertheless, their co-occurrence imply a violation of the linear-algebra type solution (5.88). No tensor, independent of its origin, can connect to the PVV and have vanishing contractions with momenta $q_2^{\mu_2}$ and $q_3^{\mu_3}$ at the same time. Whenever an axial-vertex contraction is connected to an amplitude coming from the pseudoscalar density (anomalously or not), there will be an anomaly in at least one of the vertices; the same conclusion stands for other diagrams. These facts are known; however, the form we raise is general. The low-energy theorem invoking vector WIs is only one of the solutions, as in Section (4.2) of [41]. The built equation is an exclusive and inviolable consequence of properties assumed to the 3rd-order tensor, and symmetry violations occur when the r.h.s. terms of WIs do not behave accordingly.

The explicit computation of perturbative expressions corroborates these assertions. Moreover, the IRagfs furnish an exact connection among ultraviolet and infrared features of amplitudes, namely $V_1^{PVV}(0) = -2i\Delta_{3\alpha}^{\alpha}$. That is the requirement for linearity seen after evaluating the IRagfs, and it will be derived in the next subsection. There, we assume the form $V_i^L = V_i^R + \mathcal{A}_i$ and demonstrate the implication

$$V_1^R(0) - V_2^R(0) + V_3^R(0) = -(2\pi)^{-2}, \tag{5.90}$$

where we suppress upper-indexes in V_i^R coming from finite functions (e.g., PVV-PAA), see (5.93). The equation above holds even to classically non-conserved vector currents or amplitudes with three arbitrary masses running in the loop. Although rank-2 amplitudes of multiple masses are complicated functions of these masses, the relation at the point zero is ever the finite constant above.

Independently of divergent aspects, the last equation is incompatible with (5.88); therefore, characterizing violations for rank-3 triangles under the form (5.86). Hence, anomalous terms coming from different vertices $\mathcal{A}_i$ obey the general constraint

$$\mathcal{A}_1 - \mathcal{A}_2 + \mathcal{A}_3 = (2\pi)^{-2}, \tag{5.91}$$

This equation shows that by preserving two vector WIs (in AVV), the value of the axial anomaly is unique. Likewise, any explicit tensor[5] having WIs violated by any quantity obeys this equation if the $\mathcal{A}_i$ relate to finite amplitudes coming from Feynman rules. The crossed channel of finite amplitudes just brings a multiplicative factor 2 in the last couple of equations.

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[5] This tensor can be obtained via regularization or not. See the approach of G. Scharf ([65]) in Section 5.1, using causal perturbation theory. The analogous to PVV is not computed until the very end. Instead, the authors study analogous differences between the contraction of AVV and the PVV without Feynman diagrams.

It is possible to anticipate restrictions over surface terms based on the general dependence that 3rd-order tensors have on such terms and preserving the independence and arbitrariness of internal momenta sums. That is achieved through the connection with AV functions via integration linearity. In the next section, this reasoning leads to the proposition $V_1^{PVV}(0) = -2i\Delta_{3\alpha}^{\alpha}$ and eq. (5.90).

5.4 Low-Energy Theorems II

In section (5.2), we performed explicit calculations related to different amplitude versions. A condition connecting the surface term with a finite contribution emerged for the satisfaction of at least one IRagf (5.79), without manipulating the integral expression of the surface term. This phenomenon happens in the IRagf coming from the vertex defining the version (5.78).

Meanwhile, the previous section established a low energy implication that is a result of assuming a general tensor of the external momenta (5.86). In the eventual presence of routing sums (that are ambiguous), one may project them as covariant combinations of the external momenta (P_{ij} onto q_i), and then the tensor acquires the form of the equation (5.86), the same as if it were finite. The zero mass-dimension form factors (F_1 and F_2) carry the whole arbitrariness for this projection. Thus, our statements about low-energy behavior are general, as they can be under the circumstances at work.

Nonetheless, the involved expressions contain integrals of linear and logarithmic power counting, i.e., the vector $J_{2\mu}$ and the tensor $J_{3\mu\nu}$, see (A.12) and (A.16). They depend on surface terms having physical and ambiguous momenta as coefficients. In the absence of automatic translational invariance, the three routings parametrizing the propagators are independent and cannot be reduced to the physical momenta, see section (3.2) or (1.3) for this discussion. Therefore, respecting this arbitrary feature when investigating kinematic limits (intrinsic to the subject of anomalies), one must account for the fact that the general tensor has not the form of the previous section, or for what matter of the theorem 2.3.2 in section (2.3).

Considering this change of perspective, we will show that the low-energy behavior of finite amplitudes precludes the simultaneous maintenance of integration linearity and translational invariance. Ultimately, this situation leads to anomalies since both these properties are requirements for satisfying all WIs. This discussion will finally show the basic versions as those that automatically satisfy the maximum number of IRagfs, albeit not all. We advance that there is no need for computing anomalous amplitudes, so these derivations are independent of specific trace versions.

Thereby, besides contributions on the external momenta (5.86), the general tensor must also consider the following terms

$$\begin{aligned} F^{\Delta}_{\mu_{123}} &= +\varepsilon_{\mu_{23}\nu_{12}}\left(a_{11}P_{21}^{\nu_2}+a_{12}P_{31}^{\nu_2}+a_{13}P_{32}^{\nu_2}\right)\Delta_{3\mu_1}^{\nu_1} \\ &\quad +\varepsilon_{\mu_{13}\nu_{12}}\left(a_{21}P_{21}^{\nu_2}+a_{22}P_{31}^{\nu_2}+a_{23}P_{32}^{\nu_2}\right)\Delta_{3\mu_2}^{\nu_1} \\ &\quad +\varepsilon_{\mu_{12}\nu_{12}}\left(a_{31}P_{21}^{\nu_2}+a_{32}P_{31}^{\nu_2}+a_{33}P_{32}^{\nu_2}\right)\Delta_{3\mu_3}^{\nu_1} \\ &\quad +\varepsilon_{\mu_{123}\nu_1}\left(b_1P_{21}^{\nu_2}+b_2P_{31}^{\nu_2}+b_3P_{32}^{\nu_2}\right)\Delta_{3\nu_2}^{\nu_1}. \end{aligned} \tag{5.92}$$

with $P_{ij}=k_i+k_j$. The arbitrary constants b_j and a_{ij} summarize all degrees of freedom[6]; thus, it is convenient to compact them into the vectors $\boldsymbol{b}=(b_1,b_2,b_3)$ and $\boldsymbol{a}_i=(a_{i1},a_{i2},a_{i3})$. The subindex i links to the index μ_i associated with the vertex of amplitudes $T^{\Gamma_{123}}_{\mu_{123}}$.

Contracting amplitudes with external momenta shows how finite amplitudes determine surface terms. To clarify this idea, we propose one general equation representing the satisfaction of all RAGFs

$$q_i^{\mu_i}T^{\Gamma_{123}}_{\mu_{123}}=T^{AV}_{i,(-)\mu_{kl}}+\varepsilon_{\mu_{kl}\nu_{12}}q_2^{\nu_1}q_3^{\nu_2}V_i^R\left(q_1,q_2,q_3\right); \quad k<l\neq i\in\{1,2,3\}. \tag{5.93}$$

The first term on the r.h.s. stands for the differences (5.42)-(5.44) from section (5.2). The second one corresponds to the invariants of the 2nd-rank amplitudes in the r.h.s. of IRagfs, where there is one. Now, we express the three independent AV differences in terms of P_{ij} which are given by

$$\begin{aligned} T^{AV}_{1,(-)\mu_{23}} &= 2\mathrm{i}\varepsilon_{\mu_{23}\nu_{12}}\left[P_{21}^{\nu_2}P_{32}^{\nu_3}-P_{31}^{\nu_2}\left(P_{32}^{\nu_3}-P_{21}^{\nu_3}\right)-P_{32}^{\nu_2}P_{21}^{\nu_3}\right]\Delta_{3\nu_3}^{\nu_1} \qquad (5.94) \\ T^{AV}_{2,(-)\mu_{13}} &= 2\mathrm{i}\varepsilon_{\mu_{13}\nu_{12}}\left[-P_{21}^{\nu_2}\left(P_{31}^{\nu_3}-P_{32}^{\nu_3}\right)-P_{31}^{\nu_2}P_{32}^{\nu_3}+P_{32}^{\nu_2}P_{31}^{\nu_3}\right]\Delta_{3\nu_3}^{\nu_1} \qquad (5.95) \\ T^{AV}_{3,(-)\mu_{12}} &= 2\mathrm{i}\varepsilon_{\mu_{12}\nu_{12}}\left[P_{21}^{\nu_2}P_{31}^{\nu_3}-P_{31}^{\nu_2}P_{21}^{\nu_3}-P_{32}^{\nu_2}\left(P_{31}^{\nu_3}-P_{21}^{\nu_3}\right)\right]\Delta_{3\nu_3}^{\nu_1}. \qquad (5.96) \end{aligned}$$

The notation $T^{AV}_{i,(-)}$ indicates it came from the IRagf where we contracted with $q_i^{\mu_i}$ in the i^{th} vertex. These equations preserve the arbitrary label for the internal lines and the value of the surface term and do not depend on the traces used because there is no ambiguity in expressing the trace of four Dirac matrices and a chiral one.

Performing contractions of the general structure $F^{\Delta}_{\mu_{123}}$ (5.92) is necessary to verify the possibility of identifying the two-point functions above in RAGFs. If that occurs without additional conditions, they would be simultaneously valid for any surface term values. Let us test this possibility in the sequence.

We start by taking the first contraction and writing the result in terms of the appropriate P_{ij} combinations:

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[6] We have used the algebraic identity $\varepsilon_{[\mu_{123}\nu_1}\Delta_{3\nu_2]}^{\nu_2}=0$ when expressing the tensor to simplify the study of relations with AV tensors. All our results can also be obtained without this simplification, it reduces the number of parameters without losing information.

$$\begin{aligned}
p_{31}^{\mu_1}F_{\mu_{123}}^{\Delta} \;=\;& +\varepsilon_{\mu_3\nu_{123}}\Delta_{3\mu_2}^{\nu_1}[-(a_{21}+a_{23})P_{21}^{\nu_2}P_{32}^{\nu_3}+a_{22}P_{31}^{\nu_2}(P_{21}^{\nu_3}-P_{32}^{\nu_3})] \\
& +\varepsilon_{\mu_2\nu_{123}}\Delta_{3\mu_3}^{\nu_1}[-(a_{31}+a_{33})P_{21}^{\nu_2}P_{32}^{\nu_3}+a_{32}P_{31}^{\nu_2}(P_{21}^{\nu_3}-P_{32}^{\nu_3})] \\
& +\varepsilon_{\mu_{23}\nu_{12}}\Delta_{3\nu_3}^{\nu_1}[-(a_{11}-b_1)P_{21}^{\nu_2}P_{21}^{\nu_3}+(a_{13}-b_3)P_{32}^{\nu_2}P_{32}^{\nu_3}+b_2(P_{21}^{\nu_2}-P_{32}^{\nu_2})P_{31}^{\nu_3}] \\
& +\varepsilon_{\mu_{23}\nu_{12}}\Delta_{3\nu_3}^{\nu_1}[(a_{11}+b_3)P_{21}^{\nu_2}P_{32}^{\nu_3}+a_{12}P_{31}^{\nu_2}(P_{32}^{\nu_3}-P_{21}^{\nu_3})-(a_{13}+b_1)P_{32}^{\nu_2}P_{21}^{\nu_3}].
\end{aligned} \tag{5.97}$$

After comparing this result with AV amplitudes (5.94), we organized non-zero terms in the first row. Vanishing the other rows sets most coefficients directly, so one has to solve the remaining linear equations to find $b_3 = 2\mathrm{i} - b_1$ and $a_{12} = -2\mathrm{i}$. By requiring the satisfaction of the first RAGF, the original twelve parameters reduce to just three. Hence, adopting a subindex corresponding to the considered contraction (with $q_1 = -p_{31}$), we organize this solution as follows:

$$(F_{\mu_{123}}^{\Delta})_1 : \begin{pmatrix} \boldsymbol{b} \\ \boldsymbol{a}_1 \\ \boldsymbol{a}_2 \\ \boldsymbol{a}_3 \end{pmatrix}_1 = \begin{pmatrix} b_1 & 0 & 2\mathrm{i}-b_1 \\ b_1 & -2\mathrm{i} & 2\mathrm{i}-b_1 \\ -a_{23} & 0 & a_{23} \\ -a_{33} & 0 & a_{33} \end{pmatrix}_1 . \tag{5.98}$$

Extending this analysis to contractions with q_2 and q_3, we infer the requirements to satisfy the corresponding relations. A comparison with the differences between AVs establishes a system of linear equations whose solutions follow

$$(F_{\mu_{123}}^{\Delta})_2 \;:\; \begin{pmatrix} \boldsymbol{b} \\ \boldsymbol{a}_1 \\ \boldsymbol{a}_2 \\ \boldsymbol{a}_3 \end{pmatrix}_2 = \begin{pmatrix} 0 & b_2 & 2\mathrm{i}-b_2 \\ 0 & -a_{13} & a_{13} \\ 2\mathrm{i} & -b_2 & b_2-2\mathrm{i} \\ 0 & -a_{33} & a_{33} \end{pmatrix}_2 ; \tag{5.99}$$

$$(F_{\mu_{123}}^{\Delta})_3 \;:\; \begin{pmatrix} \boldsymbol{b} \\ \boldsymbol{a}_1 \\ \boldsymbol{a}_2 \\ \boldsymbol{a}_3 \end{pmatrix}_3 = \begin{pmatrix} b_1 & 2\mathrm{i}-b_1 & 0 \\ a_{11} & -a_{11} & 0 \\ a_{21} & -a_{21} & 0 \\ b_1 & 2\mathrm{i}-b_1 & -2\mathrm{i} \end{pmatrix}_3 . \tag{5.100}$$

Let us study the simultaneous satisfaction of two relations by combining solutions $(F_{\mu_{123}}^{\Delta})_{ij} = (F_{\mu_{123}}^{\Delta})_i \cap (F_{\mu_{123}}^{\Delta})_j$. The intersection of the first two sets determines all coefficients without recurring to further conditions regarding surface terms. In other words, the hypothesis of satisfaction of the first and second RAGFs constrains the general tensor to

$$\begin{aligned}
(F_{\mu_{123}}^{\Delta})_{12} \;=\;& 2\mathrm{i}[\varepsilon_{\mu_{13}\nu_{12}}(P_{21}^{\nu_2}-P_{32}^{\nu_2})\Delta_{3\mu_2}^{\nu_1}+\varepsilon_{\mu_{123}\nu_1}P_{32}^{\nu_2}\Delta_{3\nu_2}^{\nu_1} \\
& +\varepsilon_{\mu_{23}\nu_{12}}(P_{32}^{\nu_2}-P_{31}^{\nu_2})\Delta_{3\mu_1}^{\nu_1}],
\end{aligned} \tag{5.101}$$

which is incompatible with the coefficients of the third set. We observe the same circumstances when combining other solutions. Single-solutions depend on three independent parameters and are compatible in pairs, which means that coefficients are unique once one pair of RAGFs is determined. Therefore, the complementary contraction always leads to an incompatible solution.

Now, identifying $p_{ij} = P_{il} - P_{jl}$, the achieved tensors correspond to the divergent sector of amplitude versions computed explicitly (5.26)-(5.28):

$$\begin{aligned} (F^{\Delta}_{\mu_{123}})_{23} &= \mathcal{S}_{1,\mu_{123}}; & (5.102)\\ (F^{\Delta}_{\mu_{123}})_{13} &= \mathcal{S}_{2,\mu_{123}}; & (5.103)\\ (F^{\Delta}_{\mu_{123}})_{12} &= \mathcal{S}_{3,\mu_{123}}. & (5.104) \end{aligned}$$

As a consequence, their contractions also follow the properties (5.64)-(5.72). We stress that these results come from the analysis of the divergent structure of a general rank-3 tensor of mass-dimension one $F^{\Delta}_{\mu_{123}}$(5.92), independently of the explicit approach developed at the outset of this chapter.

Let us resume the discussion about low-energy implications by considering this new information. For instance, in the hypothesis of satisfying both vector RAGFs, the complete tensor structure of any *anomalous* amplitude (AVV, VAV, VVA, AAA) assumes the form:

$$T^{\Gamma_{123}}_{\mu_{123}} = (F^{\Delta}_{\mu_{123}})_{23} + \hat{F}_{\mu_{123}} = \mathcal{S}_{1,\mu_{123}} + \hat{F}_{\mu_{123}}. \qquad (5.105)$$

Differently from the original context (5.86), the term $\hat{F}_{\mu_{123}}$ represents strictly finite parts this time, justifying the adoption of the "hats" notation. In that sense, note that all considerations from the previous subsection extend to this analysis. Momenta contractions of these finite contributions lead to $q_i^{\mu_i}\hat{F}_{\mu_{123}} = \varepsilon_{\mu_{kl}\nu_{12}} q_2^{\nu_1} q_3^{\nu_2} \hat{V}_i^L$ as in (5.87), linking to invariants $\hat{V}_i^L$ that are functions of form factors belonging to the general tensor. We cast these results in the sequence (the first equation has a minus sign because the derivations used $p_{31} = -q_1$, the same reason for the sign of $-V_1^L$, $-V_1^R$, in the paper [2] we used q_1 outgoing)

$$\begin{aligned} -q_1^{\mu_1} T^{\Gamma_{123}}_{\mu_{123}} - T^{AV}_{1(-)\mu_{23}} &= \varepsilon_{\mu_{23}\nu_{12}} q_2^{\nu_1} q_3^{\nu_2} \left(-V_1^R\right) = \varepsilon_{\mu_{23}\nu_{12}} q_2^{\nu_1} q_3^{\nu_2} (-V_1^L + 2i\Delta^{\alpha}_{3\alpha}) & (5.106)\\ q_2^{\mu_2} T^{\Gamma_{123}}_{\mu_{123}} - T^{AV}_{2(-)\mu_{13}} &= \varepsilon_{\mu_{13}\nu_{12}} q_2^{\nu_1} q_3^{\nu_2} V_2^R = \varepsilon_{\mu_{13}\nu_{12}} q_2^{\nu_1} q_3^{\nu_2} V_2^L & (5.107)\\ q_3^{\mu_3} T^{\Gamma_{123}}_{\mu_{123}} - T^{AV}_{3(-)\mu_{12}} &= \varepsilon_{\mu_{12}\nu_{12}} q_2^{\nu_1} q_3^{\nu_2} V_3^R = \varepsilon_{\mu_{12}\nu_{12}} q_2^{\nu_1} q_3^{\nu_2} V_3^L. & (5.108) \end{aligned}$$

where V_i^R are the invariants associated with finite amplitudes (5.93). The additional object $\Delta^{\alpha}_{3\alpha}$ emerged from index permutations within the $\mathcal{S}_{1,\mu_{123}}$-contraction, as acknowledged in the analysis of IRagfs (5.64). Its presence characterizes the corresponding relation as non-automatic since the equality between $\hat{V}_1^L$ and V_1^R is not direct.

$$-q_1^{\mu_1}\mathcal{S}_{1,\mu_{123}} = T^{AV}_{1(-)\mu_{23}} + \varepsilon_{\mu_{23}\nu_{23}} q_2^{\nu_2} q_3^{\nu_3} (2\mathrm{i}\Delta^{\nu_1}_{3\nu_1}).$$

Considering the V_i^L expressions in terms of invariants (5.87), the mentioned connections materializes through the formulas

$$\begin{aligned} -\hat{V}_1^L &= -\hat{F}_1 + \hat{F}_2 + (q_1\cdot q_2)\,G_5 + (q_1\cdot q_3)\,G_6 = -V_1^R - 2i\Delta^{\alpha}_{3\alpha}\\ \hat{V}_2^L &= -\hat{F}_2 + q_2^2 G_3 + (q_2\cdot q_3)\,G_4 = V_2^R\\ \hat{V}_3^L &= -\hat{F}_1 + q_3^2 G_2 + (q_2\cdot q_3)\,G_1 = V_3^R. \end{aligned}$$

Eliminating the $\hat{F}_i$ form factors gives

$$-V_1^R + V_2^R - V_3^R - 2\mathrm{i}\Delta_{3\alpha}^{\alpha} = -q_3^2 G_2 - (q_2 \cdot q_3)\, G_1 + q_2^2 G_3 + (q_2 \cdot q_3)\, G_4 + (q_1 \cdot q_2)\, G_5 + (q_1 \cdot q_3)\, G_6$$

Under the condition that G_i functions are regular at zero[7], it follows

$$2\mathrm{i}\Delta_{3\alpha}^{\alpha} = V_1^R(0) - V_2^R(0) + V_3^R(0) \tag{5.109}$$

This equation is true irrespective of which relation is satisfied without restriction. Suppose one starts with the second version of one amplitude, characterized by the structure $\mathcal{S}_{2,\mu_{123}}$. Then, it satisfies IRagfs for the first and third vertexes. Although the position of $\Delta_{3\alpha}^{\alpha}$ changed to the second contraction, see eq. (5.68), the result above still applies. We obtained a proper relation connecting surface terms with a kinematical property of finite functions, generalizing the occurrence $V_1^{PVV}(0) = 2i\Delta_{3\alpha}^{\alpha}$ (5.3). The outset was a tensor with two RAGFs satisfied without restriction, connected to AV differences and finite amplitudes. Explicit computations from section (5.2) corroborate the result above.

The r.h.s. of the expression above is the low-energy behavior of finite amplitudes (5.39)-(5.40). In the case of AVV-type amplitudes, two V_i^R are zero, and the third has the value $-V_1^{R,PVV} = V_2^{R,VPV} = -V_3^{R,VVP} = (2\pi)^{-2}$. The same result happens for AAA, which has three non-zero contributions. Combining the constants cast in eq. (5.40), we have

$$-V_1^{R,PAA}(0) + V_2^{R,APA}(0) - V_3^{R,AAP}(0) = (2\pi)^{-2}. \tag{5.110}$$

The AV differences depend only on the contraction with the momenta, but the correlators with the P density are distinct, it could be that distinct diagrams would require different numerical values to the surface term. Despite that, one always find

$$\text{IRagfs} \Leftrightarrow 2\Delta_{3\alpha}^{\alpha} = -\mathrm{i}(2\pi)^{-2}. \tag{5.111}$$

This answer the question posed in the beginning of section of Ragfs (2.1). iRagfs are identities, however IRagfs are equations. Equations for what variable? For the indefinite piece Δ, determined by the zeros of V_i^R finite invariants.

The Case of Distinct Masses

This analysis remains valid for three distinct masses running in the internal lines. We comment on the AVV case in the sequence. In this scenario, the vector currents are not classically conserved. Now, vector contractions admit an additional term proportional to the difference between masses $(m_i - m_j)$; namely, the three contractions $q_i^{\mu_i} T_{\mathrm{I}_3}^{\Gamma_{123}}$ are associated with

$$\begin{aligned} q_1^{\mu_1} T_{\mu_{123}}^{AVV} &\rightarrow (m_1 + m_3) T_{\mu_{23}}^{PVV}, \\ q_2^{\mu_2} T_{\mu_{123}}^{AVV} &\rightarrow (m_2 - m_1) T_{\mu_{13}}^{ASV}, \\ q_3^{\mu_3} T_{\mu_{123}}^{AVV} &\rightarrow (m_3 - m_2) T_{\mu_{12}}^{AVS}. \end{aligned}$$

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[7] The functions $Z_{nm}^{(0)}$, $Z_{nm}^{(-1)}$, $Z_n^{(0)}$ that comprise the finite part of any of these amplitudes do not have kinematical singularities at the point $q_i \cdot q_j = 0$.

The ASV, AVS, and PVV do not comply with eq. (5.88) for V_i^L invariants. They will satisfy a type relation (5.110), i.e. $V_1^R(0) - V_2^R(0) + V_3^R(0) = -(2\pi)^{-2}$.

It is very direct to check that

$$\begin{aligned} T^{PVV}_{\mu_{23}} &= -(2\pi)^{-2}\varepsilon_{\mu_{23}\nu_{12}}p_{21}^{\nu_1}p_{32}^{\nu_2}[(m_1-m_2)\,Z_{10}^{(-1)}+(m_1-m_3)\,Z_{01}^{(-1)}-m_1Z_{00}^{(-1)}] \\ T^{ASV}_{\mu_{13}} &= -(2\pi)^{-2}\varepsilon_{\mu_{13}\nu_{12}}p_{21}^{\nu_1}p_{32}^{\nu_2}[(m_1+m_2)\,Z_{10}^{(-1)}+(m_1+m_3)\,Z_{01}^{(-1)}-m_1Z_{00}^{(-1)}] \\ T^{AVS}_{\mu_{12}} &= -(2\pi)^{-2}\varepsilon_{\mu_{12}\nu_{12}}p_{21}^{\nu_1}p_{32}^{\nu_2}[(m_2-m_1)\,Z_{10}^{(-1)}-(m_3+m_1)\,Z_{01}^{(-1)}+m_1Z_{00}^{(-1)}]. \end{aligned} \tag{5.112}$$

Identifying the form factors

$$\begin{aligned} \varepsilon_{\mu_{23}\nu_{12}}p_{21}^{\nu_1}p_{32}^{\nu_2}V_1^{PVV} &= +(m_1+m_3)T^{PVV}_{\mu_{23}} \\ \varepsilon_{\mu_{13}\nu_{12}}p_{21}^{\nu_1}p_{32}^{\nu_2}V_2^{ASV} &= +(m_2-m_1)T^{ASV}_{\mu_{13}} \\ \varepsilon_{\mu_{12}\nu_{12}}p_{21}^{\nu_1}p_{32}^{\nu_2}V_3^{AVS} &= +(m_3-m_2)T^{AVS}_{\mu_{12}}. \end{aligned} \tag{5.113}$$

These functions as in the previous cases combine into the following form

$$V_1^{R,PVV} - V_2^{R,ASV} + V_3^{R,AVS} = -\frac{2}{(2\pi)^2}[(m_1^2-m_2^2)Z_{10}^{(-1)}+(m_1^2-m_3^2)Z_{01}^{(-1)}-m_1^2Z_{00}^{(-1)}], \tag{5.114}$$

with the combination in squared brackets equals to one-half in the zero limit, see (3.118). Therefore, we found again

$$(V_1^{R,PVV} - V_2^{R,ASV} + V_3^{R,AVS})\,(0) = -(2\pi)^{-2}. \tag{5.115}$$

In summary: In the previous section, we deduced a *structural identity* from scalar invariants V_i^L of a general third-rank tensor (5.88). This equation applies as long as there are no poles at zero, and it does not require any hypothesis about symmetries. When identifying the result of momenta contractions with amplitudes coming from WIs, we get a device to anticipate the impossibility of realizing all WIs. As these amplitudes are finite and immune to ambiguities, this analysis does not depend on the scheme used to compute divergences. This competition involving symmetries materializes into the invariants $\hat{V}_i^L = V_i^R + \mathcal{A}_i$, which produce anomalous factors $\mathcal{A}_i$ to maintain the *structural identity*:

$$(\hat{V}_1^L - \hat{V}_2^L + \hat{V}_3^L)|_0 = \left(V_1^R - V_2^R + V_3^R\right)|_0 + \mathcal{A}_1 - \mathcal{A}_2 + \mathcal{A}_3 = 0. \tag{5.116}$$

Meanwhile, by preserving the arbitrariness of internal momenta and surface terms, we observed that the low-energy behavior of these finite amplitudes links to the numerical value of surface terms. This value is the same that guarantees the uniqueness of axial amplitudes while satisfying all IRagfs. *For these perturbative amplitudes, shift-invariance is lost when the linearity of integration is obeyed and vice-versa.* Hence, kinematical limits of finite amplitudes are incompatible with the whole set of WIs, as already established in two dimensions. The main counterpoint of this work is that anomalies originate in finite functions, differing from the literature and its focus on regularization properties.

However, in the following section we will explore the role of surface terms in this discussion. Not now as determinants of anomalies, but how they interact with the various ambiguities in this discussion. For this, we will adopt all parameters as arbitrary: linearity breaking, traces, and routings.

We extend this argumentation to extra dimensions in the chapter 6.

5.5 General Parameters to the Violations and Discussions

Throughout this section, we factored out 3-pt rank-2 finite amplitudes from the discussion.

Summarizing the last sections, we have: (i) Integration linearity holds if and only if the surface terms are nonzero, eq. (5.111). Simultaneously the results are independent of the Dirac traces for the same value, which saves linearity. (ii) Since some surface-terms coefficients are ambiguous combinations of the routings, we must make choices for them. (iii) From (ii) if a procedure nullifies that terms, the linearity is violated by $q_i^{\mu_i}(T^{\Gamma_{123}}_{\mu_{123}})^{\text{viol}}_i \sim \pm(2\pi)^{-2}$, see these results in (5.78). In this way, there is an equilibrium between the routing ambiguities and trace ambiguities organized by the value of the surface term. Let us see the parameter space for this competition.

First, combine the versions that save the maximum number of IRagfs, condition-less on the surface term[8], to obtain

$$[t^{\Gamma_{123}}_{\mu_{123}}]_{\mathbf{w}} = [w_1(t^{\Gamma_{123}}_{\mu_{123}})_1 + w_2(t^{\Gamma_{123}}_{\mu_{123}})_2 + w_3(t^{\Gamma_{123}}_{\mu_{123}})_3], \tag{5.117}$$

where $|\mathbf{w}| = w_1+w_2+w_3 = 1$. As discussed at the end of section (5.2), they are identical before integration. However, when $\Delta_{3\mu\nu} = 0$, they become distinct and dependent on the weights $\mathbf{w}$. For zero surface terms their symmetry violations occur in all vertices. In the i^{th} vertex they get a factor of w_i.

Now, taking the surface term as an arbitrary parameter, i.e, given by a constant c_1 as

$$\Delta_{3\mu_{12}} = -\mathrm{i}\frac{c_1}{2(4\pi)^2} g_{\mu_{12}}. \tag{5.118}$$

It is $c_1 = 1$ for the satisfaction of IRagfs, or $c_1 = 0$ for the momentum-space translational invariance.

Then we handle the routings, at least in a covariant scenario, parametrizing the internal lines by the external momenta. We do this by choosing one of the sums P_{ij} as linear combination of two independent momenta q_i.

$$P_{31} = c_2 q_2 + c_3 q_3 \rightarrow P_{21} = c_2 q_2 + (c_3 - 1)\, q_3, \quad P_{32} = (c_2 + 1)\, q_2 + c_3 q_3.$$

---

[8] This claim is independent of explicit computations performed in the previous Section.

Therefore, the AV functions from the previous sections become functions of c_1, c_2,and c_3

$$\begin{aligned} T^{AV}_{1,(-)\mu_{23}} &= c_1 \varepsilon_{\mu_{23}\nu_{12}} \left[2P^{\nu_1}_{21} P^{\nu_2}_{32} + P^{\nu_1}_{31} \left(P^{\nu_2}_{21} - P^{\nu_2}_{32}\right)\right] / (4\pi)^2 && (5.119) \\ T^{AV}_{2,(-)\mu_{13}} &= c_1 \varepsilon_{\mu_{13}\nu_{12}} \left[+2P^{\nu_1}_{31} P^{\nu_2}_{32} - P^{\nu_1}_{21} \left(P^{\nu_2}_{32} - P^{\nu_2}_{31}\right)\right] / (4\pi)^2 && (5.120) \\ T^{AV}_{3,(-)\mu_{12}} &= c_1 \varepsilon_{\mu_{12}\nu_{12}} \left[-2P^{\nu_1}_{21} P^{\nu_2}_{31} - P^{\nu_1}_{32} \left(P^{\nu_2}_{21} - P^{\nu_2}_{31}\right)\right] / (4\pi)^2, && (5.121) \end{aligned}$$

Now we borrow the eqs. (5.78) for the potential IRagfs violations.

$$q_i^{\mu_i} (T^{\Gamma_{123}}_{\mu_{123}})^{\text{viol}}_j = T^{AV}_{i,(-)\mu_{23}} + (-1)^i \delta_{ij} \left(2\pi\right)^{-2} \varepsilon_{\mu_{kl}\nu_{12}} q_2^{\nu_1} q_3^{\nu_2} \left(c_1 - 1\right); \quad k < l \neq i \in \{1,2,3\}.$$

Thereby, the parameters c_1, c_2, and c_3 will express any possible values for the contractions of basic versions. They will interact with the parameters from arbitrary traces $\mathbf{w} = (w_1, w_2, w_3)$.

Following these considerations, the linear combinations $[T^{\Gamma_{123}}_{\mu_{123}}]_{\mathbf{w}}$ defined in eq. (5.117) obey

$$\begin{aligned} q_1^{\mu_1} [T^{\Gamma_{123}}_{\mu_{123}}]_{\mathbf{w}} &= \varepsilon_{\mu_{23}\nu_{12}} q_2^{\nu_1} q_3^{\nu_2} \mathcal{A}_1 = \varepsilon_{\mu_{23}\nu_{12}} q_2^{\nu_1} q_3^{\nu_2} \left[\frac{-4w_1 \left(c_1 - 1\right) + c_1 \left(c_2 - c_3 + 2\right)}{4(2\pi)^2}\right] \\ q_2^{\mu_2} [T^{\Gamma_{123}}_{\mu_{123}}]_{\mathbf{w}} &= \varepsilon_{\mu_{13}\nu_{12}} q_2^{\nu_1} q_3^{\nu_2} \mathcal{A}_2 = \varepsilon_{\mu_{13}\nu_{12}} q_2^{\nu_1} q_3^{\nu_2} \left[\frac{4w_2 \left(c_1 - 1\right) - c_1 \left(c_3 + 1\right)}{4\left(2\pi\right)^2}\right] && (5.122) \\ q_3^{\mu_3} [T^{\Gamma_{123}}_{\mu_{123}}]_{\mathbf{w}} &= \varepsilon_{\mu_{12}\nu_{12}} q_2^{\nu_1} q_3^{\nu_2} \mathcal{A}_3 = \varepsilon_{\mu_{12}\nu_{12}} q_2^{\nu_1} q_3^{\nu_2} \left[\frac{4w_3 \left(1 - c_1\right) - c_1 \left(c_2 - 1\right)}{4\left(2\pi\right)^2}\right]. \end{aligned}$$

What implies

$$\mathcal{A}_1 - \mathcal{A}_2 + \mathcal{A}_3 = \frac{1}{(2\pi)^2}. \tag{5.123}$$

Hence, when we have numerical amounts of two violations, no matter the path leading them, the third arises without ambiguity. The only concession we did was write the internal momenta as covariant functions of the external ones.

Let us analyze the consequences: If $c_1 = 1$, there is no dependence in w_i, we have the unique solution that satisfies linearity but is not a function of the external momenta. However, if $c_1 = 0$, there is no dependence on c_2 and c_3, the tensors are functions of the external momenta but not unique, they depend on w_i. The crossed diagrams work in the same way, adding more parameters but having the same behavior. The break of linearity, ambiguities, and violation of symmetries in the way we described is independent of this feature.

From our point of view, the crucial factor is the kinematic behavior of finite functions. In the massless limit, this aspect appears in the residue of poles for the form factors, which are regular in the massive case. Breaking linearity has a function in the divergent amplitudes, though. It corroborates with the low-energy value of PV^n amplitudes in dimension $d = 2n$. If it does not occur and shifts are allowed by getting rid of the AV functions, then through the equations relating V_i^L and V_i^R, the finite amplitudes would have to be zero at $q_i \cdot q_j = 0$, which is not true.

Linearity Breaking and Algebraic Identities

The linearity breaking occurs when the surface term is put to zero and also its trace, as a minimum consistency requirement. In other words, the Schouten identity $\varepsilon_{[\mu_{1234}}\Delta^{\mu_5}_{3,\mu_5]} = 0$

dictates that. The identity holds for other values as well; actually, any value such that $\Delta_{3\mu\nu} = [g^{\alpha\beta}\Delta_{3\alpha\beta}]/4$. It also holds for the finite tensors $J_{3\mu\nu}$, and notice that we employed and analyzed them separately.

From another perspective, the breaking happens when integrating a zero tensor and obtaining a nonzero result. Individually we did not observe this. Nonetheless, considering all possible Schouten identities, what encompass the one for the complete integral $\bar{J}_{3\mu\nu}$, we see that linearity breaking is the breaking of a Schouten identity.

Let us clarify our statements, first consider the identity for the integrand of the Feynman integral $\bar{J}_{3\mu\nu}$,

$$\varepsilon_{[\mu_{1234}}\bar{j}^{\mu_5}_{3\mu_5]} = \frac{1}{D_{123}}\left[\left(\varepsilon_{\mu_{5123}}K_{1\mu_4} + \varepsilon_{\mu_{4512}}K_{1\mu_3} + \varepsilon_{\mu_{3451}}K_{1\mu_2} + \varepsilon_{\mu_{2345}}K_{1\mu_1}\right)K_1^{\mu_5} + \varepsilon_{\mu_{1234}}K_1^2\right] = 0. \tag{5.124}$$

Using $K_1^2 = D_1 + m^2$, we obtain

$$\frac{1}{D_{123}}\left(\varepsilon_{\mu_{5123}}K_{1\mu_4} + \varepsilon_{\mu_{4512}}K_{1\mu_3} + \varepsilon_{\mu_{3451}}K_{1\mu_2} + \varepsilon_{\mu_{2345}}K_{1\mu_1}\right)K_1^{\mu_5} + \varepsilon_{\mu_{1234}}\left(\frac{m^2}{D_{123}} + \frac{1}{D_{23}}\right) = 0. \tag{5.125}$$

Now, using the identities (3.20) to split $\bar{j}_3^{\mu\nu} = K_1^{\mu}K_1^{\nu}/D_{123}$ and $\bar{j}_2 = 1/D_{12}$, we can rewrite them as

$$\bar{j}_3^{\mu\nu} = \frac{1}{4}\left[-\frac{\partial}{\partial k_\nu}\left(\frac{k^\mu}{D_\lambda^3}\right) + g^{\mu\nu}\frac{1}{D_\lambda^2}\right] + j_3^{\mu\nu} \tag{5.126}$$

$$\bar{j}_2 = \frac{1}{D_\lambda^2} + j_2\left(k_2, k_3\right). \tag{5.127}$$

The integral of $j_3 = 1/D_{123}$ is finite, whereas the integrals of $j_3^{\mu\nu}$ and $j_2\left(k_2, k_3\right)$ are present in the Appendix (A.2).

Then, without disregarding any term the identity has become

$$\begin{aligned} 0 = {} & +\varepsilon_{\mu_{5123}}\left[-\partial_{\mu_4}\left(k^{\mu_5}D_\lambda^{-2}\right) + 4j^{\mu_5}_{3\mu_4}\right] + \varepsilon_{\mu_{4512}}\left[-\partial_{\mu_3}\left(k^{\mu_5}D_\lambda^{-2}\right) + 4j^{\mu_5}_{3\mu_3}\right] \\ & +\varepsilon_{\mu_{3451}}\left[-\partial_{\mu_2}\left(k^{\mu_5}D_\lambda^{-2}\right) + 4j^{\mu_5}_{3\mu_2}\right] + \varepsilon_{\mu_{2345}}\left[-\partial_{\mu_1}\left(k^{\mu_5}D_\lambda^{-2}\right) + 4j^{\mu_5}_{3\mu_1}\right] \\ & +\varepsilon_{\mu_{1234}}\left[4j_2\left(k_2, k_3\right) + 4m^2 j_3\right] + \left(\varepsilon_{\mu_{4123}} + \varepsilon_{\mu_{4312}} + \varepsilon_{\mu_{3421}} + \varepsilon_{\mu_{2341}} + 4\varepsilon_{\mu_{1234}}\right)D_\lambda^{-2}, \end{aligned} \tag{5.128}$$

and it is rather direct that the scalar integrand, $1/D_\lambda^2$, drop from the expression. What remains are convergent integrands and total derivatives formally defining the tensor surface term.

At this point, all the quantities are well-defined. The next step is to integrate this expression; however, in such a way we do not manipulate the integral representation for the surface term. The above equation then turns into

$$\begin{aligned} 0 = {} & \varepsilon_{\mu_{1234}}\left[4m^2 J_3 + 4J_2\left(p_{32}\right)\right] + \left[\varepsilon_{\mu_{5123}}\Delta^{\mu_5}_{3\mu_4} + \varepsilon_{\mu_{4512}}\Delta^{\mu_5}_{3\mu_3} + \varepsilon_{\mu_{3451}}\Delta^{\mu_5}_{3\mu_2} + \varepsilon_{\mu_{2345}}\Delta^{\mu_5}_{3\mu_1}\right] \\ & +4\left[\varepsilon_{\mu_{5123}}J^{\mu_5}_{3\mu_4} + \varepsilon_{\mu_{4512}}J^{\mu_5}_{3\mu_3} + \varepsilon_{\mu_{3451}}J^{\mu_5}_{3\mu_2} + \varepsilon_{\mu_{2345}}J^{\mu_5}_{3\mu_1}\right]. \end{aligned} \tag{5.129}$$

Now we use the following equations

$$\varepsilon_{\mu_{5123}}\Delta^{\mu_5}_{3\mu_4} + \varepsilon_{\mu_{4512}}\Delta^{\mu_5}_{3\mu_3} + \varepsilon_{\mu_{3451}}\Delta^{\mu_5}_{3\mu_2} + \varepsilon_{\mu_{2345}}\Delta^{\mu_5}_{3\mu_1} = -\varepsilon_{\mu_{1234}}\Delta^{\mu_5}_{3\mu_5}, \tag{5.130}$$

$$\varepsilon_{\mu_{5123}}J^{\mu_5}_{3\mu_4} + \varepsilon_{\mu_{4512}}J^{\mu_5}_{3\mu_3} + \varepsilon_{\mu_{3451}}J^{\mu_5}_{3\mu_2} + \varepsilon_{\mu_{2345}}J^{\mu_5}_{3\mu_1} = -\varepsilon_{\mu_{1234}}J^{\mu_5}_{3\mu_5}. \tag{5.131}$$

The validity of the first only constrains $\Delta_{3,\mu\nu} = [g^{\alpha\beta}\Delta_{3,\alpha\beta}]/4$, and the finite parts undeniably satisfy it. Reinserting these equations in the previous one, we get

$$\varepsilon_{\mu_{1234}}[-\Delta_{3\mu_5}^{\mu_5} - 4J_{3\mu_5}^{\mu_5} + 4m^2 J_3 + 4J_2\left(p_{32}\right)] = 0. \tag{5.132}$$

The last ingredient is the trace of the tensor integral (A.19)

$$J_{3\mu_5}^{\mu_5} = m^2 J_3 + J_2\left(p_{32}\right) + \mathrm{i}[2\left(4\pi\right)^2]^{-1}, \tag{5.133}$$

what leads to

$$\left[\Delta_{3\mu_5}^{\mu_5} + \frac{2\mathrm{i}}{(4\pi)^2}\right]\varepsilon_{\mu_{1234}} = 0. \tag{5.134}$$

This shows that despite the identity $\varepsilon_{[\mu_{1234}}\Delta_{3,\mu_5]}^{\mu_5} = 0$ be consistent with any value of Δ_3, the same is not true to the bare integral $\varepsilon_{[\mu_{1234}}\bar{J}_{3,\mu_5]}^{\mu_5} = 0$. The identity is respected generally only if $\Delta_{3\alpha}^{\alpha} = -2\mathrm{i}/(4\pi)^2$, which is derived without manipulating divergent integrals (we just encapsulate them). Only convergent quantities deliver the constraint on the surface term. As a part of the Feynman integrals, the satisfaction of the Schouten identity for surface terms and finite parts individually is not enough to make it valid for entire integrals. Therefore, the initial identity transforms into a condition to the break, or not, of integration linearity.

Let us see this more precisely; take (5.8) for $(t_{\mu_{123}}^{AVV})_1$ in the beginning of this chapter, including the fragments that vanish under integration ($t^{(-+)}$ and t^{ASS}). Then, write $p_{31}^{\mu_1} = K_3^{\mu_1} - K_1^{\mu_1}$, as in a derivation of an iRagf and carefully arranging the terms, by summing and subtracting what is necessary to find t^{PVV} and t^{AV}, one finds

$$\begin{aligned} p_{31}^{\mu_1}(t_{\mu_{123}}^{AVV})_1 \;\; = \;\; & \left[-2mt_{\mu_{23}}^{PVV} + t_{\mu_{32}}^{AV}\left(1,2\right) - t_{\mu_{23}}^{AV}\left(2,3\right)\right] \\ & -8\mathrm{i}p_{21}^{\nu_2}p_{31}^{\nu_3}\left[\varepsilon_{[\mu_{23}\nu_{12}}\bar{j}_{3,\nu_3]}^{\nu_1} + \varepsilon_{[\mu_{23}\nu_{12}}p_{31\nu_3]}j_3^{\nu_1}\right]. \end{aligned} \tag{5.135}$$

The last two terms are identically zero; there is no question here. The other two contractions (with $p_{21}^{\mu_2}$ and $p_{32}^{\mu_3}$) fall directly into t^{AV} tensors for this version of the trace. This means that for linearity of integration (IRagfs) to hold, we must have $\varepsilon_{[\mu_{23}\nu_{12}}\bar{J}_{3,\nu_3]}^{\nu_1} = 0$. Which repeats the condition above $\Delta_{3\alpha}^{\alpha} = -2\mathrm{i}/(4\pi)^2$, and the eq. (5.78) for the linearity constraint on $q_1^{\mu_1}(T_{\mu_{123}}^{\Gamma_{123}})_1$. In the analysis of the section (5.2), all amplitudes were individually and explicitly computed. Here, we are observing an amplitude-independent algebraic aspect behind the breaking of linearity.

Comments:

To end I want to point out that low-energy theorems and our careful considerations about AV^n-type anomalous amplitudes, extends to gravitational amplitudes. I will do some brief comments only: the investigation on some of the aspects discussed here was performed by L. Ebani in her Ph.d. thesis [89] and partially by the author. In a simple $2D$ model (which can be found in [30, 31]), a 2-pt correlator of the energy-momentum tensor for a Weyl fermion is considered. What we found is that all IRagfs and anomalies are in one way or another related to the constraints posed by the identities below

$$2D : \left\{\varepsilon_{[\mu\nu}\bar{J}_{2\alpha\beta\gamma]}^{\nu} = 0; \quad \varepsilon_{[\mu\nu}\bar{J}_{2\alpha\beta]}^{\nu} = 0; \quad \varepsilon_{[\mu\nu}\bar{J}_{2\alpha]}^{\nu} = 0\right\}. \tag{5.136}$$

The phenomenon also holds for diagrams where we do not have a low-energy theorem. For example, a four dimensional triangle with scalar, vector, and pseudo-tensor vertex $\widetilde{T} = \gamma_{*}\gamma_{[\alpha_{12}]}$, may be written by the same ingredients of the amplitudes discussed in the thesis. Explicitly

$$4D : T^{\tilde{T}SV}_{\alpha_{12}\mu} = 4i\varepsilon_{\alpha_{2}\mu\nu_{12}} T^{(--)\nu_{12}}_{\alpha_{1}} - 4i\varepsilon_{\alpha_{1}\mu\nu_{12}} T^{(--)\nu_{12}}_{\alpha_{2}} + i\varepsilon_{\alpha_{12}\mu\nu_{1}} \left(T^{PSA}\right)^{\nu_{1}} , \tag{5.137}$$

and it exhibits the break of linearity even if we could not yet establish a kinematical theorem to it. Moreover, the scalar object, $I_{\log}$, does not cancel among the sign tensors and the sub-amplitude; therefore, the structure of the divergent part is not solely built up on surface terms. There are other amplitudes, with tensor vertices, e.g., $T^{TAV}_{\alpha_{12}\mu_{12}}$, whose IRagfs are not automatic and have kinematic theorems for them, more intricate ones. Finally, these other scenarios were mentioned because the methodology and criteria we constructed enable us to investigate fine features of perturbative anomalies with secure conclusions, beyond particular prescriptions, and common scenarios. At least, we hope for that.

Let us see six dimensions now!

Chapter 6

$6D$-Box Amplitudes $AVVV$ and $VAAA$

(Disclaimer: This chapter uses the normalization $\int \mathrm{d}^6k/c\,(6) = \int \mathrm{d}^6k/\,(i\pi^3)$.)

The main ingredients of the four dimensional scenario arise in the boxes in six dimensions with the interesting appearance of tensor ($T = \gamma_{[\mu_{12}]}$) and pseudotensor ($\widetilde{T} = \gamma_*\gamma_{[\mu_{12}]}$) vertices in subamplitudes. The second case was already noted in [53]. This is a particular phenomenon in the basic versions of the $n+1$ point anomalous amplitudes in $2n$ dimensions where the $n-1$ rank tensor or pseudotensor appear in and even combination with the P and S vertexes.

Let us start by the non-zero traces in the definitions of our amplitudes

$$\begin{aligned} t^{AVVV}_{\mu_{1234}} &= K^{\nu_{1234}}_{1234}\mathrm{tr}(\gamma_{*\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3\mu_4\nu_4})\frac{1}{D_{1234}} \\ &+\varepsilon_{\mu_{1234}\nu_{12}}\left(-m^2K^{\nu_{12}}_{12} + m^2K^{\nu_{12}}_{13} - m^2K^{\nu_{12}}_{14} - m^2K^{\nu_{12}}_{23} + m^2K^{\nu_{12}}_{24} - m^2K^{\nu_{12}}_{34}\right)\frac{1}{D_{1234}} \end{aligned} \tag{6.1}$$

and

$$\begin{aligned} t^{VAAA}_{\mu_{1234}} &= K^{\nu_{1234}}_{1234}\mathrm{tr}(\gamma_{*\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3\mu_4\nu_4})\frac{1}{D_{1234}} \\ &+\varepsilon_{\mu_{1234}\nu_{12}}\left(m^2K^{\nu_{12}}_{12} + m^2K^{\nu_{12}}_{13} + m^2K^{\nu_{12}}_{14} + m^2K^{\nu_{12}}_{23} + m^2K^{\nu_{12}}_{24} + m^2K^{\nu_{12}}_{34}\right)\frac{1}{D_{1234}} \end{aligned} \tag{6.2}$$

Before integrate we find definite relations with even subamplitudes exhibiting tensor vertexes. They are compiled in the table below.

Version/Type	$AVVV$	$VAAA$
1	$\tilde{T}PPP$	$TSSS$
2	$STPP$	$P\tilde{T}SS$
3	$SPTP$	$PS\tilde{T}S$
4	$SPPT$	$PSS\tilde{T}$

Table 6.1: Sub-amplitudes with tensor and pseudotensor vertexes related to each version of the boxes AVVV and VAAA

A very important comment is worth here, the factor $2^n\,(-\mathrm{i})^{n+1} = 8$ for 6D traces will be omitted to make the presentation more clean, of course they are retrievable without much effort. All the subamplitudes that appear in the basic versions are in the formulae below. In them we employed the definitions

$$\begin{aligned} \mathfrak{s}_{ab}^{(\pm)} &= K_a \cdot K_b \pm m^2 = \frac{1}{2}\left(D_a + D_b - p_{ab}^2 + (m_a \pm m_b)^2\right), && (6.3) \\ K_{ab}^{[\nu_{12}]} &= \frac{1}{2}\left(K_a^{\nu_1} K_b^{\nu_2} - K_a^{\nu_2} K_b^{\nu_1}\right), && (6.4) \end{aligned}$$

to simplify the writing of the expressions. For the triple vector boxes

$$\begin{aligned} \left[t^{\tilde{T}PPP}\right]^{\nu_{12}} &= -2\left[+K_{34}^{[\nu_{12}]}\mathfrak{s}_{12}^{(-)} - K_{24}^{[\nu_{12}]}\mathfrak{s}_{13}^{(-)} + K_{23}^{[\nu_{12}]}\mathfrak{s}_{14}^{(-)}\right. && (6.5) \\ &\quad \left. + K_{14}^{[\nu_{12}]}\mathfrak{s}_{23}^{(-)} - K_{13}^{[\nu_{12}]}\mathfrak{s}_{24}^{(-)} + K_{12}^{[\nu_{12}]}\mathfrak{s}_{34}^{(-)}\right]\frac{1}{D_{1234}}, \\ \left[t^{STPP}\right]^{\nu_{12}} &= -2\left[-K_{34}^{[\nu_{12}]}\mathfrak{s}_{12}^{(+)} + K_{24}^{[\nu_{12}]}\mathfrak{s}_{13}^{(+)} - K_{23}^{[\nu_{12}]}\mathfrak{s}_{14}^{(+)}\right. && (6.6) \\ &\quad \left. +K_{14}^{[\nu_{12}]}\mathfrak{s}_{23}^{(-)} - K_{13}^{[\nu_{12}]}\mathfrak{s}_{24}^{(-)} + K_{12}^{[\nu_{12}]}\mathfrak{s}_{34}^{(-)}\right]\frac{1}{D_{1234}}, \\ \left[t^{SPTP}\right]^{\nu_{12}} &= -2\left[\mathfrak{s}_{12}^{(-)}K_{34}^{[\nu_{12}]} + \mathfrak{s}_{13}^{(+)}K_{24}^{[\nu_{12}]} - \mathfrak{s}_{14}^{(+)}K_{23}^{[\nu_{12}]}\right. && (6.7) \\ &\quad \left. -\mathfrak{s}_{23}^{(+)}K_{14}^{[\nu_{12}]} + \mathfrak{s}_{24}^{(+)}K_{13}^{[\nu_{12}]} + \mathfrak{s}_{34}^{(-)}K_{12}^{[\nu_{12}]}\right]\frac{1}{D_{1234}}, \\ \left[t^{SPPT}\right]^{\nu_{12}} &= -2\left[\mathfrak{s}_{12}^{(-)}K_{34}^{[\nu_{12}]} - \mathfrak{s}_{13}^{(-)}K_{24}^{[\nu_{12}]} - \mathfrak{s}_{14}^{(+)}K_{23}^{[\nu_{12}]}\right. && (6.8) \\ &\quad \left. +\mathfrak{s}_{23}^{(-)}K_{14}^{[\nu_{12}]} + \mathfrak{s}_{24}^{(+)}K_{13}^{[\nu_{12}]} - \mathfrak{s}_{34}^{(+)}K_{12}^{[\nu_{12}]}\right]\frac{1}{D_{1234}}. \end{aligned}$$

For the triple axial box, we have

$$\begin{aligned} \left[t^{TSSS}\right]^{\nu_{12}} &= -2\left[\mathfrak{s}_{12}^{(+)}K_{34}^{[\nu_{12}]} - \mathfrak{s}_{13}^{(-)}K_{24}^{[\nu_{12}]} + \mathfrak{s}_{14}^{(+)}K_{23}^{[\nu_{12}]}\right. && (6.9) \\ &\quad \left. +\mathfrak{s}_{23}^{(+)}K_{14}^{[\nu_{12}]} - \mathfrak{s}_{24}^{(-)}K_{13}^{[\nu_{12}]} + \mathfrak{s}_{34}^{(+)}K_{12}^{[\nu_{12}]}\right]\frac{1}{D_{1234}}, \\ \left[t^{P\tilde{T}SS}\right]^{\nu_{12}} &= -2\left[-\mathfrak{s}_{12}^{(-)}K_{34}^{[\nu_{12}]} + \mathfrak{s}_{13}^{(+)}K_{24}^{[\nu_{12}]} - \mathfrak{s}_{14}^{(-)}K_{23}^{[\nu_{12}]}\right. && (6.10) \\ &\quad \left. +\mathfrak{s}_{23}^{(+)}K_{14}^{[\nu_{12}]} - \mathfrak{s}_{24}^{(-)}K_{13}^{[\nu_{12}]} + \mathfrak{s}_{34}^{(+)}K_{12}^{[\nu_{12}]}\right]\frac{1}{D_{1234}}, \\ \left[t^{PS\tilde{T}S}\right]^{\nu_{12}} &= -2\left[\mathfrak{s}_{12}^{(+)}K_{34}^{[\nu_{12}]} + \mathfrak{s}_{13}^{(+)}K_{24}^{[\nu_{12}]} - \mathfrak{s}_{14}^{(-)}K_{23}^{[\nu_{12}]}\right. && (6.11) \\ &\quad \left. -\mathfrak{s}_{23}^{(-)}K_{14}^{[\nu_{12}]} + \mathfrak{s}_{24}^{(+)}K_{13}^{[\nu_{12}]} + \mathfrak{s}_{34}^{(+)}K_{12}^{[\nu_{12}]}\right]\frac{1}{D_{1234}}, \\ \left[t^{PSS\tilde{T}}\right]^{\nu_{12}} &= -2\left[\mathfrak{s}_{12}^{(+)}K_{34}^{[\nu_{12}]} - \mathfrak{s}_{13}^{(-)}K_{24}^{[\nu_{12}]} - \mathfrak{s}_{14}^{(-)}K_{23}^{[\nu_{12}]}\right. && (6.12) \\ &\quad \left. +\mathfrak{s}_{23}^{(+)}K_{14}^{[\nu_{12}]} + \mathfrak{s}_{24}^{(+)}K_{13}^{[\nu_{12}]} - \mathfrak{s}_{34}^{(-)}K_{12}^{[\nu_{12}]}\right]\frac{1}{D_{1234}}. \end{aligned}$$

The common tensors, which are combinations of the sign tensors in the appropriate trace of eight matrices indicated by the lower index in C_i, namely

$$\begin{aligned} C_{1\mu_{1234}} &= -\varepsilon_{\mu_{134}\nu_{123}}T_{\mu_2}^{(+-+)\nu_{123}} + \varepsilon_{\mu_{124}\nu_{123}}T_{\mu_3}^{(--+)\nu_{123}} - \varepsilon_{\mu_{123}\nu_{123}}T_{\mu_4}^{(-++)\nu_{123}} && (6.13) \\ C_{2\mu_{1234}} &= -\varepsilon_{\mu_{234}\nu_{123}}T_{\mu_1}^{(+-+)\nu_{123}} - \varepsilon_{\mu_{124}\nu_{123}}T_{\mu_3}^{(++-)\nu_{123}} + \varepsilon_{\mu_{123}\nu_{123}}T_{\mu_4}^{(+--)\nu_{123}} && (6.14) \\ C_{3\mu_{1234}} &= -\varepsilon_{\mu_{234}\nu_{123}}T_{\mu_1}^{(--+)\nu_{123}} - \varepsilon_{\mu_{134}\nu_{123}}T_{\mu_2}^{(++-)\nu_{123}} + \varepsilon_{\mu_{123}\nu_{123}}T_{\mu_4}^{(---)\nu_{123}} && (6.15) \\ C_{4\mu_{1234}} &= -\varepsilon_{\mu_{234}\nu_{123}}T_{\mu_1}^{(-++)\nu_{123}} - \varepsilon_{\mu_{134}\nu_{123}}T_{\mu_2}^{(+--)\nu_{123}} + \varepsilon_{\mu_{124}\nu_{123}}T_{\mu_3}^{(---)\nu_{123}}. && (6.16) \end{aligned}$$

The sign-tensors, whose general definition (3.7.2) can be found in section (3.7), are given by

$$\begin{aligned}
\varepsilon_{\mu_{abc}\nu_{123}} T^{(s_1 s_2 s_3)\nu_{123}}_{\mu_d} &= (1+s_1)\,\varepsilon_{\mu_{abc}\nu_{123}} p_{31}^{\nu_2} p_{41}^{\nu_3} \left(J^{\nu_1}_{4\mu_d} + p_{21\mu_d} J^{\nu_1}_{4} \right) \qquad (6.17)\\
&- (1-s_2)\,\varepsilon_{\mu_{abc}\nu_{123}} p_{21}^{\nu_2} p_{41}^{\nu_3} \left(J^{\nu_1}_{4\mu_d} + p_{31\mu_d} J^{\nu_1}_{4} \right)\\
&+ (1+s_3)\,\varepsilon_{\mu_{abc}\nu_{123}} p_{21}^{\nu_2} p_{31}^{\nu_3} \left(J^{\nu_1}_{4\mu_d} + p_{41\mu_d} J^{\nu_1}_{4} \right)\\
&+ \frac{1}{6} \varepsilon_{\mu_{abc}\nu_{123}} \left[(1+s_1)\, p_{31}^{\nu_2} p_{41}^{\nu_3} - (1-s_2)\, p_{21}^{\nu_2} p_{41}^{\nu_3} + (1+s_3)\, p_{21}^{\nu_2} p_{31}^{\nu_3} \right] \Delta^{\nu_1}_{4\mu_d}\\
&+ \frac{1}{6} \varepsilon_{\mu_{abcd}\nu_{12}} \left[(1+s_1)\, p_{31}^{\nu_1} p_{41}^{\nu_2} - (1-s_2)\, p_{21}^{\nu_1} p_{41}^{\nu_2} + (1+s_3)\, p_{21}^{\nu_1} p_{31}^{\nu_2} \right] I_{\log},
\end{aligned}$$

The amplitude versions will be

$$\begin{aligned}
(T^{AVVV}_{\mu_{1234}})_i &= (C_i)_{\mu_{1234}} - \frac{1}{2} \varepsilon_{\mu_{1234}}{}^{\nu_{12}} \left(\text{Corresponding subamplitude} \right)_{\nu_{12}} \qquad (6.18)\\
(T^{VAAA}_{\mu_{1234}})_i &= (C_i)_{\mu_{1234}} - \frac{1}{2} \varepsilon_{\mu_{1234}}{}^{\nu_{12}} \left(\text{Corresponding subamplitude} \right)_{\nu_{12}}, \qquad (6.19)
\end{aligned}$$

where the corresponding subamplitude can be checked in the table (6.1).

The surface terms of the i^{th} version (that depend only on the versions of trace) are given by

$$\begin{aligned}
\mathcal{S}_{1,\mu_{1234}} &= \frac{1}{3} \left[\varepsilon_{\mu_{134}\nu_{123}} p_{42}^{\nu_2} p_{32}^{\nu_3} \Delta^{\nu_1}_{4\mu_2} - \varepsilon_{\mu_{124}\nu_{123}} p_{21}^{\nu_2} p_{43}^{\nu_3} \Delta^{\nu_1}_{4\mu_3} - \varepsilon_{\mu_{123}\nu_{123}} p_{21}^{\nu_2} p_{31}^{\nu_3} \Delta^{\nu_1}_{4\mu_4} \right] \qquad (6.20)\\
&- \frac{1}{3} \varepsilon_{\mu_{1234}\nu_{12}} \left(p_{43}^{\nu_2} P_{134}^{\nu_3} + p_{21}^{\nu_2} P_{124}^{\nu_3} \right) \Delta^{\nu_1}_{4\nu_3}\\
\mathcal{S}_{2,\mu_{1234}} &= \frac{1}{3} \left[-\varepsilon_{\mu_{234}\nu_{123}} p_{32}^{\nu_2} p_{42}^{\nu_3} \Delta^{\nu_1}_{4\mu_1} - \varepsilon_{\mu_{124}\nu_{123}} p_{31}^{\nu_2} p_{41}^{\nu_3} \Delta^{\nu_1}_{4\mu_3} + \varepsilon_{\mu_{123}\nu_{123}} p_{32}^{\nu_2} p_{41}^{\nu_3} \Delta^{\nu_1}_{4\mu_4} \right] \qquad (6.21)\\
&+ \frac{1}{3} \varepsilon_{\mu_{1234}\nu_{12}} \left(p_{32}^{\nu_2} P_{123}^{\nu_3} - p_{41}^{\nu_2} P_{124}^{\nu_3} \right) \Delta^{\nu_1}_{4\nu_3}\\
\mathcal{S}_{3,\mu_{1234}} &= \frac{1}{3} \left[\varepsilon_{\mu_{234}\nu_{123}} p_{21}^{\nu_2} p_{43}^{\nu_3} \Delta^{\nu_1}_{4\mu_1} - \varepsilon_{\mu_{134}\nu_{123}} p_{31}^{\nu_2} p_{41}^{\nu_3} \Delta^{\nu_1}_{4\mu_2} - \varepsilon_{\mu_{123}\nu_{123}} p_{21}^{\nu_2} p_{41}^{\nu_3} \Delta^{\nu_1}_{4\mu_4} \right] \qquad (6.22)\\
&- \frac{1}{3} \varepsilon_{\mu_{1234}\nu_{12}} \left(p_{43}^{\nu_2} P_{234}^{\nu_3} + p_{21}^{\nu_2} P_{123}^{\nu_3} \right) \Delta^{\nu_1}_{4\nu_3}\\
\mathcal{S}_{4,\mu_{1234}} &= \frac{1}{3} \left[-\varepsilon_{\mu_{234}\nu_{123}} p_{21}^{\nu_2} p_{31}^{\nu_3} \Delta^{\nu_1}_{4\mu_1} - \varepsilon_{\mu_{134}\nu_{123}} p_{32}^{\nu_2} p_{41}^{\nu_3} \Delta^{\nu_1}_{4\mu_2} - \varepsilon_{\mu_{124}\nu_{123}} p_{21}^{\nu_2} p_{41}^{\nu_3} \Delta^{\nu_1}_{4\mu_3} \right] \qquad (6.23)\\
&+ \frac{1}{3} \varepsilon_{\mu_{1234}\nu_{12}} \left(p_{32}^{\nu_2} P_{234}^{\nu_3} - p_{41}^{\nu_2} P_{134}^{\nu_3} \right) \Delta^{\nu_1}_{4\nu_3}
\end{aligned}$$

Adding to this the reductions of section (3.6) eqs. (3.254),(3.255), and (3.256) it is a direct task to verify and analyze anything we propose to analyze about these boxes.

6.1 iRagfs and IRagfs

Following the steps delineated in the specific subsection (2.1) and the previous development in four dimensions, the iRagfs reads

$$\begin{aligned}
p_{14}^{\mu_1} \left[t^{AVVV}_{\mu_{1234}} \right] &= \left[t^{AVV}_{\mu_{234}} (2,3,4) \right] - \left[t^{AVV}_{\mu_{423}} (1,2,3) \right] + 2m \left[t^{PVVV}_{\mu_{234}} \right], \qquad (6.24)\\
p_{21}^{\mu_2} \left[t^{AVVV}_{\mu_{1234}} \right] &= \left[t^{AVV}_{\mu_{134}} (1,3,4) \right] - \left[t^{AVV}_{\mu_{134}} (2,3,4) \right], \qquad (6.25)\\
p_{32}^{\mu_3} \left[t^{AVVV}_{\mu_{1234}} \right] &= \left[t^{AVV}_{\mu_{124}} (1,2,4) \right] - \left[t^{AVV}_{\mu_{124}} (1,3,4) \right], \qquad (6.26)\\
p_{43}^{\mu_4} \left[t^{AVVV}_{\mu_{1234}} \right] &= \left[t^{AVV}_{\mu_{123}} (1,2,3) \right] - \left[t^{AVV}_{\mu_{123}} (1,2,4) \right], \qquad (6.27)
\end{aligned}$$

and the triple axial one

$$\begin{aligned}
p_{14}^{\mu_1}\left[t_{\mu_{1234}}^{VAAA}\right] &= \left[t_{\mu_{234}}^{AAA}\left(2,3,4\right)\right]-\left[t_{\mu_{423}}^{AAA}\left(1,2,3\right)\right], && (6.28)\\
p_{21}^{\mu_2}\left[t_{\mu_{1234}}^{VAAA}\right] &= \left[t_{\mu_{134}}^{VVA}\left(1,3,4\right)\right]-\left[t_{\mu_{134}}^{AAA}\left(2,3,4\right)\right]+2m\left[t_{\mu_{134}}^{VPAA}\right], && (6.29)\\
p_{32}^{\mu_3}\left[t_{\mu_{1234}}^{VAAA}\right] &= \left[t_{\mu_{124}}^{VAV}\left(1,2,4\right)\right]-\left[t_{\mu_{124}}^{VVA}\left(1,3,4\right)\right]+2m\left[t_{\mu_{124}}^{VAPA}\right], && (6.30)\\
p_{43}^{\mu_4}\left[t_{\mu_{1234}}^{VAAA}\right] &= \left[t_{\mu_{123}}^{AAA}\left(1,2,3\right)\right]-\left[t_{\mu_{123}}^{VAV}\left(1,2,4\right)\right]+2m\left[t_{\mu_{123}}^{VAPA}\right]. && (6.31)
\end{aligned}$$

Because their expression, before and after integrated, are independent of the vertex content, the only difference is located in what contraction they come from. We will repeat the four dimensions scenario and adopt a simplified notation. To justify this feature, which hold in all even dimensions, it is enough to observe that all the surface terms amplitudes are equal in any vertex setup. Their integrand can be written as

$$t_{\mathrm{I}_n}^{\mathbf{\Gamma}}\left(i_1,\cdots i_n\right)=\left(-1\right)^{n(n-1)/2}\left(\frac{1}{n}\right)\varepsilon_{\mathrm{I}_n\mathrm{C}_n}p_{i_2,i_1}^{\nu_2}\cdots p_{i_n,i_1}^{\nu_n}\frac{1}{D_{i_1i_2\cdots i_n}}\sum_{j=1}^{n}K_{i_j}^{\nu_1},\quad \mathrm{C}_n=\left(\nu_i\right)_1^n \tag{6.32}$$

Due to the antisymmetry of the expression only the surface term of the vector n-pt integrals (3.206) remains. Thence

$$T_{\mathrm{I}_n}^{\mathbf{\Gamma}}\left(i_1,\cdots i_n\right)=-\left(-1\right)^{n(n-1)/2}\left(\frac{1}{n}\right)\varepsilon_{\mathrm{I}_n\mathrm{C}_n}p_{i_2i_1}^{\nu_2}\cdots p_{i_ni_1}^{\nu_n}\left(P_{i_1\cdots i_n}\right)^{\nu_{n+1}}\Delta_{(n+1),\nu_{n+1}}^{\nu_1} \tag{6.33}$$

Choosing a configuration with $i_1<i_2\cdots<i_n$ and we will have for six dimensions our prototype

$$T_{\mu_{123}}^{\Gamma_{123}}\left(i,j,l\right)=\frac{1}{3}\varepsilon_{\mu_{123}\nu_{123}}\left(p_{ji}^{\nu_2}p_{li}^{\nu_3}P_{ijl}^{\nu_4}\right)\Delta_{4,\nu_4}^{\nu_1}. \tag{6.34}$$

Example: the differences that appear in the IRagf for the first vertex are subsumed by

$$\begin{aligned}
T_{1,(-)\mu_{234}}^{A} &= T_{\mu_{423}}^{AAA}\left(1,2,3\right)-T_{\mu_{234}}^{AAA}\left(2,3,4\right)=T_{\mu_{423}}^{\Gamma_{123}}\left(1,2,3\right)-T_{\mu_{234}}^{\Gamma_{123}}\left(2,3,4\right) && (6.35)\\
&= -\frac{1}{3}\varepsilon_{\mu_{234}\nu_{123}}p_{32}^{\nu_2}\left(p_{21}^{\nu_3}P_{123}^{\nu_4}+p_{42}^{\nu_3}P_{234}^{\nu_4}\right)\Delta_{4\nu_4}^{\nu_1}.
\end{aligned}$$

Again, this is the result for the 3-pt functions coming from the first contraction for both boxes. We collect all the final forms in

$$\begin{aligned}
T_{1(-)\mu_{234}}^{A} &= -\frac{1}{3}\varepsilon_{\mu_{234}\nu_{123}}p_{32}^{\nu_2}\left(p_{21}^{\nu_3}P_{123}^{\nu_4}+p_{42}^{\nu_3}P_{234}^{\nu_4}\right)\Delta_{4\nu_4}^{\nu_1}, && (6.36)\\
T_{2(-)\mu_{134}}^{A} &= -\frac{1}{3}\varepsilon_{\mu_{134}\nu_{123}}p_{43}^{\nu_2}\left(p_{31}^{\nu_3}P_{134}^{\nu_4}-p_{32}^{\nu_3}P_{234}^{\nu_4}\right)\Delta_{4\nu_4}^{\nu_1}, && (6.37)\\
T_{3(-)\mu_{124}}^{A} &= -\frac{1}{3}\varepsilon_{\mu_{124}\nu_{123}}p_{41}^{\nu_2}\left(p_{21}^{\nu_3}P_{124}^{\nu_4}-p_{31}^{\nu_3}P_{134}^{\nu_4}\right)\Delta_{4\nu_4}^{\nu_1}, && (6.38)\\
T_{4(-)\mu_{123}}^{A} &= -\frac{1}{3}\varepsilon_{\mu_{123}\nu_{123}}p_{21}^{\nu_2}\left(p_{41}^{\nu_3}P_{124}^{\nu_4}-p_{31}^{\nu_3}P_{123}^{\nu_4}\right)\Delta_{4\nu_4}^{\nu_1}. && (6.39)
\end{aligned}$$

Summary for the explicit results: with the identification $q_i=p_{i,i-1}$ it is possible to express all the contractions relating to the non-finite sector by following formula

$$q_i^{\mu_i}\mathcal{S}_{j,\mu_{1234}}=T_{i(-)\mu_{abc}}^{A}+\left(-1\right)^{i}\delta_{i,j}\varepsilon_{\mu_{abc}\nu_{123}}q_2^{\nu_1}q_3^{\nu_2}q_4^{\nu_3}\Delta_{4\alpha}^{\alpha},\ \text{with } a<b<c\neq i. \tag{6.40}$$

Notice that the conditioning term $\Delta^{\alpha}_{4\alpha}$ appear in the contraction with the index related to the substitution of the chiral-matrix definition–or the simplified substitution $\gamma_{*}\gamma_{\mu_i} = \varepsilon_{\mu_1\nu_{12345}}\gamma^{\nu_{12345}}/5!$. This property hold in all dimensions, since the difference among these modes of computation resides in finite and vanishing integrals.

To compute the contraction in the finite part we may express everything in terms of the Z functions, or use two identities: One is $p_{21}^{\nu_2}p_{31}^{\nu_3}p_{41}^{\nu_4}\varepsilon_{[\mu_{ijk}\nu_{123}}J_{4\nu_4]}^{\nu_1} = 0$, leading to

$$\begin{aligned} & +\varepsilon_{\mu_{ij}\nu_{1234}}p_{21}^{\nu_2}p_{31}^{\nu_3}p_{41}^{\nu_4}J_{4\mu_k}^{\nu_1} - \varepsilon_{\mu_{ik}\nu_{1234}}p_{21}^{\nu_2}p_{31}^{\nu_3}p_{41}^{\nu_4}J_{4\mu_j}^{\nu_1} + \varepsilon_{\mu_{jk}\nu_{1234}}p_{21}^{\nu_2}p_{31}^{\nu_3}p_{41}^{\nu_4}J_{4\mu_i}^{\nu_1} \qquad (6.41) \\ = & -\varepsilon_{\mu_{ijk}\nu_{123}}p_{21}^{\nu_2}p_{31}^{\nu_3}\left(p_{41}^{\nu_4}J_{4\nu_4}^{\nu_1}\right) + \varepsilon_{\mu_{ijk}\nu_{124}}p_{21}^{\nu_2}p_{41}^{\nu_4}\left(p_{31}^{\nu_3}J_{4\nu_3}^{\nu_1}\right) \\ & -\varepsilon_{\mu_{ijk}\nu_{134}}p_{31}^{\nu_3}p_{41}^{\nu_4}\left(p_{21}^{\nu_2}J_{4\nu_2}^{\nu_1}\right) + \varepsilon_{\mu_{ijk}\nu_{234}}p_{21}^{\nu_2}p_{31}^{\nu_3}p_{41}^{\nu_4}\left(J_{4\nu_1}^{\nu_1}\right). \qquad (6.42) \end{aligned}$$

The term within brackets are expressed as lower rank integrals by the formulas of the Appendix (A.3). The other one is $p_{21}^{\nu_2}p_{31}^{\nu_3}p_{41}^{\nu_4}\varepsilon_{[\mu_{ijk}\nu_{123}}p_{ab\nu_4]}J_4^{\nu_1}$, explicitly

$$\begin{aligned} & \varepsilon_{\mu_{ijk}\nu_{123}}\left[p_{21}^{\nu_2}p_{31}^{\nu_3}\left(p_{ab}\cdot p_{41}\right) - p_{21}^{\nu_2}p_{41}^{\nu_3}\left(p_{ab}\cdot p_{31}\right) + p_{31}^{\nu_2}p_{41}^{\nu_3}\left(p_{ab}\cdot p_{21}\right)\right]J_4^{\nu_1} \qquad (6.43) \\ = & \ \varepsilon_{\mu_{ijk}\nu_{234}}p_{21}^{\nu_2}p_{31}^{\nu_3}p_{41}^{\nu_4}\left(p_{a,b\nu_1}J_4^{\nu_1}\right). \end{aligned}$$

To get a tensor reduction, e.g. $p_{a,b} = p_{21}$, use $2p_{21\nu_1}J_4^{\nu_1} = -p_{21}^2 J_4 + J_3\left(p_{31},p_{41}\right) - J_3\left(p_{42},p_{43}\right)$, and so on.

Following strictly the steps of four dimensions we can show that all IRagfs hold automatically, with the exception of those defined on the vertices chosen to define the idea of version. Examples where a determination of the surface term is necessary are

$$\begin{aligned} p_{14}^{\mu_1}(T_{\mu_{1234}}^{AVVV})_1 &= \left[T_{\mu_{234}}^{AVV}\left(2,3,4\right)\right] - \left[T_{\mu_{423}}^{AVV}\left(1,2,3\right)\right] + 2m\left[T_{\mu_{234}}^{PVVV}\right] \qquad (6.44) \\ & -\frac{1}{3}\varepsilon_{\mu_{234}\nu_{123}}p_{21}^{\nu_1}p_{32}^{\nu_2}p_{43}^{\nu_3}[\Delta_{4\alpha}^{\alpha}+1]. \\ p_{21}^{\mu_2}(T_{\mu_{1234}}^{AVVV})_2 &= \left[T_{\mu_{134}}^{AVV}\left(1,3,4\right)\right] - \left[T_{\mu_{134}}^{AVV}\left(2,3,4\right)\right] + \frac{1}{3}\varepsilon_{\mu_{134}\nu_{123}}p_{21}^{\nu_1}p_{32}^{\nu_2}p_{43}^{\nu_3}[\Delta_{4\alpha}^{\alpha}+1]. (6.45) \end{aligned}$$

and

$$\begin{aligned} p_{32}^{\mu_3}(T_{\mu_{1234}}^{VAAA})_3 &= \left[T_{\mu_{124}}^{VAV}\left(1,2,4\right)\right] - \left[T_{\mu_{124}}^{VVA}\left(1,3,4\right)\right] + 2m\left[T_{\mu_{124}}^{VAPA}\right] \qquad (6.46) \\ & -\varepsilon_{\mu_{124}\nu_{123}}p_{21}^{\nu_1}p_{32}^{\nu_2}p_{43}^{\nu_3}\left[\Delta_{4\alpha}^{\alpha}+1\right], \\ p_{43}^{\mu_4}(T_{\mu_{1234}}^{VAAA})_4 &= \left[T_{\mu_{123}}^{AAA}\left(1,2,3\right)\right] - \left[T_{\mu_{123}}^{VAV}\left(1,2,4\right)\right] + 2m\left[T_{\mu_{123}}^{VAPA}\right] \qquad (6.47) \\ & +\varepsilon_{\mu_{123}\nu_{123}}p_{21}^{\nu_1}p_{32}^{\nu_2}p_{43}^{\nu_3}\left[\Delta_{4\alpha}^{\alpha}+1\right]. \end{aligned}$$

By linearity we get $\Delta_{4,\alpha}^{\alpha} = -1$, which immediately activate the routing dependence coded by the three-point amplitudes like $\left[T_{\mu_{234}}^{AVV}\left(2,3,4\right)\right]$. We will not pursue a detailed discussion of these features, since they are essentially an extension of chapter 5. However, at least a difference of the versions we show below

$$(T_{\mu_{1234}}^{AVVV})_1 - (T_{\mu_{1234}}^{AVVV})_2 = -\frac{1}{3}\varepsilon_{\mu_{1234}\nu_{12}}p_{32}^{\nu_1}p_{43}^{\nu_2}\left[\Delta_{4\alpha}^{\alpha}+1\right].$$

As a final comment in these matters, this difference above does not really depend on the vertex content, only the version.

6.2 Finite Amplitudes in the r.h.s. of IRagfs and LETs

The 4-pt amplitudes in the left-hand-side of the IRagfs are finite and their integrals are listed here,

$$\begin{aligned}
2mT^{PVVV}_{\mu_{234}} &= \varepsilon_{\mu_{234}\nu_{123}} p^{\nu_1}_{21} p^{\nu_2}_{32} p^{\nu_3}_{43} V^{R,PVVV}_{1} = 2m^2 \varepsilon_{\mu_{234}\nu_{123}} p^{\nu_1}_{21} p^{\nu_2}_{32} p^{\nu_3}_{43} \left[-J_4\right] && (6.48)\\
2mT^{VPAA}_{\mu_{134}} &= \varepsilon_{\mu_{134}\nu_{123}} p^{\nu_1}_{21} p^{\nu_2}_{32} p^{\nu_3}_{43} V^{R,VPAA}_{2} = 2m^2 \varepsilon_{\mu_{134}\nu_{123}} \left[-2p^{\nu_2}_{21} p^{\nu_3}_{41} J^{\nu_1}_4 + p^{\nu_1}_{21} p^{\nu_2}_{32} p^{\nu_3}_{43} J_4\right] && (6.49)\\
2mT^{VAPA}_{\mu_{124}} &= \varepsilon_{\mu_{124}\nu_{123}} p^{\nu_1}_{21} p^{\nu_2}_{32} p^{\nu_3}_{43} V^{R,VAPA}_{3} = 2m^2 \varepsilon_{\mu_{124}\nu_{123}} \left[+2p^{\nu_2}_{32} p^{\nu_3}_{41} J^{\nu_1}_4 + p^{\nu_1}_{21} p^{\nu_2}_{32} p^{\nu_3}_{43} J_4\right] && (6.50)\\
2mT^{VAAP}_{\mu_{123}} &= \varepsilon_{\mu_{123}\nu_{123}} p^{\nu_1}_{21} p^{\nu_2}_{32} p^{\nu_3}_{43} V^{R,VAAP}_{4} = 2m^2 \varepsilon_{\mu_{123}\nu_{123}} \left[+2p^{\nu_2}_{31} p^{\nu_3}_{41} J^{\nu_1}_4 + p^{\nu_1}_{21} p^{\nu_2}_{32} p^{\nu_3}_{43} J_4\right] && (6.51)
\end{aligned}$$

Recalling the vector and scalar four-point integrals,

$$J^{\nu_1}_4 = \left(-p^{\nu_1}_{21} Z^{(-1)}_{100} - p^{\nu_1}_{31} Z^{(-1)}_{010} - p^{\nu_1}_{41} Z^{(-1)}_{001}\right), \quad J_4 = Z^{(-1)}_{000}, \tag{6.52}$$

we may write everything in terms of Z's functions. Their low-energy limits, the formulas (3.115) of the section (3.4)

$$m^2 Z^{(-1)}_{100}(0) = -\frac{1}{4!}, \quad m^2 Z^{(-1)}_{010}(0) = -\frac{1}{4!}, \quad m^2 Z^{(-1)}_{001}(0) = -\frac{1}{4!}, \quad m^2 Z^{(-1)}_{000}(0) = -\frac{1}{3!}, \tag{6.53}$$

lead to

$$V^{R,PVVV}_1(0) = \frac{2}{3!}; \quad V^{R,VPAA}_2(0) = -\frac{1}{3!}; \quad V^{R,VAPA}_3(0) = 0; \quad V^{R,VAAP}_4(0) = -\frac{1}{3!}. \tag{6.54}$$

Distinct from four dimensions, there are diagrams that vanish in zero. However, the combination of these invariants is again incompatible with all WIs satisfaction. First, by the general low-energy theorem presented is section (2.3) theorem (2.3.2) and analogously to the four dimensional result (5.88), we have

$$V^L_1 - V^L_2 + V^L_3 - V^L_4 = 0. \tag{6.55}$$

These are the form factors of a arbitrary tensor for the l.h.s., the rank-four boxes. Nonetheless, for T^{VAAA} the right hand side of the first contraction, with $q^{\mu_1}_1$, does not have 4-pt functions in the equal masses scenario. Hence $V^R_1 \equiv 0$, and the other three satisfies

$$\left(V^R_1 - V^R_2 + V^R_3 - V^R_4\right)(0) = \left(-V^{R,VPAA}_2 + V^{R,VAPA}_3 - V^{R,VAAP}_4\right)(0) = \frac{2}{3!}. \tag{6.56}$$

Whereas the case of T^{AVVV} is opposite, and follows that

$$\left(V^R_1 - V^R_2 + V^R_3 - V^R_4\right)(0) = V^{R,PVVV}_1(0) = \frac{2}{3!}. \tag{6.57}$$

Therefore our situation is this: if the point where all bilinears $s_{ij} = (q_i + q_j)^2$ vanish is not a pole of the functions V^L neither a zero of the functions V^R, then there can be no full connection among the r.h.s. and l.h.s. of four-point amplitudes, simultaneously respecting all WIs in these amplitudes is ruled out. This statement follows independently of the nature of the tensor (in this case, a positive power-counting amplitude). Moreover, it does not use any particular symmetry in its derivation, only studying what happens at one point. That is to say, we do not impose some WIs and deduce a LET for the divergence of another current. The analysis is solely on tensor structure and kinematic behavior.

Actually, the matter of linearly divergent integrals, shifts, violation of trace identities, and so on is a derived phenomenon. It is a way to avoid, in a certain manner, a contradiction. If there was no obstruction, the scalar form factors were regular (for the main amplitude), and the amplitude was dependent only on the relative coordinates (external momenta), we would prove that the combination of known and determined functions on the r.h.s. is vanishing. This contradicts an observation—that the combination does not vanish—which can be checked before any computational endeavor into the l.h.s., the anomalous amplitudes, be attempted.

When the masses are distinct it becomes clear, once more, how the whole set of functions in the r.h.s. of the equations works to obstruct the simultaneous compliance of linearity and translation invariance.

- **Four Distinct Masses**

Let us verify our claim when four arbitrary masses are allowed in the edges of the diagram for $T^{VAAA}_{\mu_{1234}}$. The finite functions in the IRagfs are

$$\begin{aligned}
\left[(m_1-m_4)\,T^{SAAA}_{\mu_{234}}\right] &= \varepsilon_{\mu_{234}\nu_{123}}q_2^{\nu_1}q_3^{\nu_2}q_4^{\nu_3}\left[V_1^{R;SAAA}\right] && (6.58)\\
\left[(m_1+m_2)\,T^{VPAA}_{\mu_{134}}\right] &= \varepsilon_{\mu_{134}\nu_{123}}q_2^{\nu_1}q_3^{\nu_2}q_4^{\nu_3}\left[V_2^{R;VPAA}\right] && (6.59)\\
\left[(m_2+m_3)\,T^{VAPA}_{\mu_{124}}\right] &= \varepsilon_{\mu_{124}\nu_{123}}q_2^{\nu_1}q_3^{\nu_2}q_4^{\nu_3}\left[V_3^{R;VAPA}\right] && (6.60)\\
\left[(m_3+m_4)\,T^{VAPA}_{\mu_{123}}\right] &= \varepsilon_{\mu_{123}\nu_{123}}q_2^{\nu_1}q_3^{\nu_2}q_4^{\nu_3}\left[V_4^{R;VAAP}\right]. && (6.61)
\end{aligned}$$

They depend on the vector and scalar integrals for multiple masses; by using them, we systematize the results by their invariant amplitudes below.

$$\begin{aligned}
& V_1^{R;SAAA} && (6.62)\\
= \;& (m_1-m_4)\left[+(m_4+m_1)\,Z_{001}^{(-1)}+(m_1-m_3)\,Z_{010}^{(-1)}+(m_2+m_1)\,Z_{100}^{(-1)}-m_1Z_{000}^{(-1)}\right],
\end{aligned}$$

$$\begin{aligned}
& V_2^{R;VPAA} && (6.63)\\
= \;& (m_2+m_1)\left[-(m_1-m_4)\,Z_{001}^{(-1)}-(m_3+m_1)\,Z_{010}^{(-1)}-(m_1-m_2)\,Z_{100}^{(-1)}+m_1Z_{000}^{(-1)}\right],
\end{aligned}$$

$$\begin{aligned}
& V_3^{R;VAPA} && (6.64)\\
= \;& (m_2+m_3)\left[-(m_1-m_4)\,Z_{001}^{(-1)}-(m_3+m_1)\,Z_{010}^{(-1)}-(m_2+m_1)\,Z_{100}^{(-1)}+m_1Z_{000}^{(-1)}\right],
\end{aligned}$$

$$\begin{aligned}
& V_4^{R;VAAP} && (6.65)\\
= \;& (m_3+m_4)\left[-(m_1-m_4)\,Z_{001}^{(-1)}+(m_3-m_1)\,Z_{010}^{(-1)}-(m_2+m_1)\,Z_{100}^{(-1)}+m_1Z_{000}^{(-1)}\right].
\end{aligned}$$

Combining these functions as they should appear in the low energy limit,

$$\begin{aligned}
&\left[V_1^{R,SAAA} - V_2^{R,VPAA} + V_3^{R,VAPA} - V_4^{R,VAAP}\right] \\
= \;& 2\left[\left(m_1^2 - m_2^2\right) Z_{100}^{(-1)} + \left(m_1^2 - m_3^2\right) Z_{010}^{(-1)} + \left(m_1^2 - m_4^2\right) Z_{001}^{(-1)} - m_1^2 Z_{000}^{(-1)}\right]. \qquad (6.66)
\end{aligned}$$

Then invoking the result (3.118) of the section (3.4) to evaluate the above expression in $q_i{\cdot}q_j = 0$, we obtain

$$\left(V_1^{R,SAAA} - V_2^{R,VPAA} + V_3^{R,VAPA} - V_4^{R,VAAP}\right)(0) = \frac{1}{3}. \qquad (6.67)$$

In other words, we repeat the conclusion for triangles in four dimensions. The r.h.s. amplitudes for a set of IRagfs is studied independently. In the result it is found the need to add a compensation in each vertex for comparison with the l.h.s. that is under a LET.

$$\begin{aligned}
&\left(V_1^L - V_2^L + V_3^L - V_4^L\right)(0) \qquad (6.68)\\
= \;& \left(V_1^R - V_2^R + V_3^R - V_4^R\right)(0) + \mathcal{A}_1 - \mathcal{A}_2 + \mathcal{A}_3 - \mathcal{A}_4 = 0 \\
& \rightarrow \mathcal{A}_1 - \mathcal{A}_2 + \mathcal{A}_3 - \mathcal{A}_4 = \frac{1}{3}
\end{aligned}$$

This sets the known unavoidable anomaly for this type of amplitude, since they are a kinematic aspect of the finite boxes. Any computation of the T^{VAAA} results in the equations

$$\begin{aligned}
q_1^{\mu_1}\left[T_{\mu_{1234}}^{VAAA}\right] &= \varepsilon_{\mu_{234}\nu_{123}} q_2^{\nu_1} q_3^{\nu_2} q_4^{\nu_3}\left[V_1^{R;SAAA} + \mathcal{A}_1\right] \qquad (6.69)\\
q_2^{\mu_2}\left[T_{\mu_{1234}}^{VAAA}\right] &= \varepsilon_{\mu_{134}\nu_{123}} q_2^{\nu_1} q_3^{\nu_2} q_4^{\nu_3}\left[V_2^{R;VPAA} + \mathcal{A}_2\right] \qquad (6.70)\\
q_3^{\mu_3}\left[T_{\mu_{1234}}^{VAAA}\right] &= \varepsilon_{\mu_{124}\nu_{123}} q_2^{\nu_1} q_3^{\nu_2} q_4^{\nu_3}\left[V_3^{R;VAPA} + \mathcal{A}_3\right] \qquad (6.71)\\
q_4^{\mu_4}\left[T_{\mu_{1234}}^{VAAA}\right] &= \varepsilon_{\mu_{123}\nu_{123}} q_2^{\nu_1} q_3^{\nu_2} q_4^{\nu_3}\left[V_4^{R;VAAP} + \mathcal{A}_4\right]. \qquad (6.72)
\end{aligned}$$

Then, in turn, this aspect interferes with the algebraic properties. Why? Because differences of 3-pt functions $T_{\mu_{abc}}^{\Gamma_{123}}(i,j,l) \sim \varepsilon_{\mu_{abc}\nu_{123}} p_{ji}^{\nu_2} p_{li}^{\nu_3} P_{ijl}^{\nu_4} \Delta_{4\nu_4}^{\nu_1}$, which are part of the IRagfs representing integration linearity, carry arbitrary internal momenta $P_{ijl} = k_i + k_j + k_l$ that may violate translation invariance. To then assess the WIs we must eliminate such terms, but we need to keep the IRagfs to be certain that no other term can appear. If all of this were possible, then a contradiction arises with what we just observed, that the 4-pt functions form factors V_i^R can not be matched with the contractions V_i^L ($q_i^{\mu_i} F_{\mu_{1234}} = \varepsilon_{\mu_{abc}\nu_{123}} q_2^{\nu_1} q_3^{\nu_2} q_4^{\nu_3} V_i^L$) of any tensor representing the boxes $T_{\mu_{1234}}^{AVVV}$ and $T_{\mu_{1234}}^{VAAA}$. The same reasons that were presented in chapter 5.

This argumentation also shed light why the conditions for translation invariance and integration linearity, restrictions on $\Delta_{4,\alpha\beta}$ values, are incompatible ($\Delta_{4\alpha}^{\alpha} = 0$ and $\Delta_{4\alpha}^{\alpha} = -1$).

6.3 Max Number of Automatic IRagfs

The section (3.8), dedicated to discuss traces such as tr $\left(\gamma_*\gamma_{\mathrm{I}_{2n+2}}\right)$ $(d = 2n)$, saw an insistence on calling the expressions constructed through specific formulae as basic/basis versions for yielding the largest number of IRagfs automatically. But not by just using the definition of the chiral matrix, since other ways to begin a computation at the trace level lead to the same integrals. Then in section (5.4), we demonstrated how fixing IRagfs, one by one, leaves no freedom to fix all of them without constraining the surface term value. At this point it is clear that the LET is behind all of this.

Here, we will adopt some simplifications to derive one example is six dimensions. We start by simplifying the sum of three routings which appear in the triangles, $T^{\Gamma_{123}}_{\mu_{abc}}(i,j,l)$,

$$P_{i_1} = P_{i_2 i_3 i_4} = k_{i_2} + k_{i_3} + k_{i_4} : \ i_1 \neq i_2 \neq i_3 \neq i_4 \in \{1,2,3,4\} \to p_{ij} = P_j - P_i. \tag{6.73}$$

With the four variables above we can write every difference of 3-pt functions in these variables, $T^{A}_{i(-)\mu_{abc}}$ (6.36-6.39). We will not present explicit expressions since its is a straightforward task. Next simplification is to adopt a parametrization for $\Delta_{4\mu\nu} = c g_{\mu\nu}$. This attitude enable us to write a shorter expression to the surface-terms sector of the general tensor representing the amplitudes studied, namely

$$F^{\Delta}_{\mu_{1234}} = c\varepsilon_{\mu_{1234}\nu_{12}} \left(a_1 P_1^{\nu_1} P_2^{\nu_2} + a_2 P_1^{\nu_1} P_3^{\nu_2} + a_3 P_1^{\nu_1} P_4^{\nu_2} + a_4 P_2^{\nu_1} P_3^{\nu_2} + a_5 P_2^{\nu_1} P_4^{\nu_2} + a_6 P_3^{\nu_1} P_4^{\nu_2}\right). \tag{6.74}$$

Now the task is to contract with the external momenta and fix the coefficient a_i through the known differences $T^{A}_{i(-)\mu_{abc}}$.

The notation $\mathbf{a} = (a_1, a_2, a_3, a_4, a_5, a_6)$ helps expressing the results more compactly. Adopting vectors $\mathbf{a}_i (i = 1,2,3,4)$ for each IRagf automatically satisfied, where the index refers to the vertex the relation comes from. Then, for each vertex individually selected, we have

$$\begin{aligned}
\mathbf{a}_1 &= (a_1, a_2, a_3, 1/3, a_1 - 2/3, a_2 + 2/3) && (6.75)\\
\mathbf{a}_2 &= (a_1, a_2, a_3, -a_2 - 2/3, -a_3 + 2/3, -1/3) && (6.76)\\
\mathbf{a}_3 &= (a_1, -a_1 + 2/3, -/3, a_4, a_5, -a_5 + 2/3) && (6.77)\\
\mathbf{a}_4 &= (1, a_2, -a_2 + 2/3, a_4, -a_4 - 2/3, a_6)\,. && (6.78)
\end{aligned}$$

The freedom dropped to three coefficients. The next step is to equate two of these vectors, $\mathbf{a}_{ij} = \mathbf{a}_i = \mathbf{a}_j$ $(i \neq j)$. This means two IRagfs are simultaneously guaranteed. Now we descend to a specific example: the IRagfs for the first and second vertex contraction ($p_{14}^{\mu_1}$ and $p_{21}^{\mu_2}$) result in

$$\mathbf{a}_{12} = (a_1, -1, -a_1 + 4/3, 1/3, a_1 - 2/3, -1/3)\,. \tag{6.79}$$

Finally, three relations satisfied fix the result completely, and we chose the third vertex (carrying index μ_3) to present,

$$\mathbf{a}_{123} = (5/3, -1, -1/3, 1/3, 1, -1/3)\,.$$

It is tedious, but direct task to verify that the object obtained is identical with the set of surface terms for the boxes 4th version, $\mathcal{S}_{4,\mu_{1234}}$ (6.23). Namely,

$$\begin{aligned}\left(F^{\Delta}_{\mu_{1234}}\right)_{123} &= \mathcal{S}_{4,\mu_{1234}} &(6.80)\\ &= -\frac{1}{3}c\varepsilon_{\mu_{1234}\nu_{12}}\left(-5P_1^{\nu_1}P_2^{\nu_2}+3P_1^{\nu_1}P_3^{\nu_2}+P_1^{\nu_1}P_4^{\nu_2}-3P_2^{\nu_1}P_4^{\nu_2}-P_2^{\nu_1}P_3^{\nu_2}+P_3^{\nu_1}P_4^{\nu_2}\right).\end{aligned}$$

Which by the results exposed above does not satisfies the 4th IRagf, since by eq. (6.40) we have

$$q_i^{\mu_i}\mathcal{S}_{4,\mu_{1234}} = T^A_{i,(-)\mu_{abc}} + (-1)^i\,\delta_{i,4}\varepsilon_{\mu_{abc}\nu_{123}}q_2^{\nu_1}q_3^{\nu_2}q_4^{\nu_3}\Delta^{\alpha}_{4,\alpha}. \tag{6.81}$$

specifically

$$q_4^{\mu_4}\mathcal{S}_{4,\mu_{1234}} = T^A_{4,(-)\mu_{123}} + \varepsilon_{\mu_{123}\nu_{123}}q_2^{\nu_1}q_3^{\nu_2}q_4^{\nu_3}\Delta^{\alpha}_{4,\alpha}. \tag{6.82}$$

This means the preservation of the last IRagf will involve the remaining part of the tensor, i.e., the finite parts ($F_{\mu_{1234}} = F^{\Delta}_{\mu_{1234}} + \hat{F}_{\mu_{1234}}$). They will connect with the invariants V_i^R furnishing a constraint to $\Delta^{\alpha}_{4,\alpha}$, similar to the derivations performed in four dimensions . However, we will not repeat the derivation because it is clear now how to extend it. Our objective here was to illustrate a conclusion valid for all even dimensions.

Comments:

The differences of 3-pt amplitudes, $T^A_{i,(-)\mu_{abc}}$, are not simply surface terms that we can turn on or off. They encode information in the coefficients p_{ij} and P_{ijl}. Information that manifests when the mere admittance of general surface terms for $F^{\Delta}_{\mu_{1234}}$ is constrained by $T^A_{i,(-)\mu_{abc}}$ to the point of uniqueness, through just three IRagfs. The fourth one forces $\Delta^{\alpha}_{4,\alpha}$ to go with the finite amplitudes (like $T^{PVVV}_{\mu_{234}}$) to be determined by them. Let us imagine that the finite amplitudes had determined a non-zero value for the surface term, and we met all IRagfs; however, still with the hope of obtaining all WIs, we go after choices for routings as functions of the kinematic data, q_i, because maybe the parameters available could allow us to simultaneously fix $T^A_{i,(-)\mu_{abc}} = 0$. The hope ends when the equations obtained cannot be solved, and the reason for this obstruction is the coefficients of p_{ij} and P_{ijl} present in the differences $T^A_{i,(-)\mu_{abc}}$. The same coefficients that prohibit $F^{\Delta}_{\mu_{1234}}$ to be contracted with the external momenta and straightforwardly linked to $T^A_{i,(-)\mu_{abc}}$. It is as if the set of differences, $T^A_{i,(-)\mu_{abc}}$, "know" they are part of a system of identities with finite amplitudes having a particular LET.

I want to contrast these assertions with the comments done at the end of the last chapter about the $4D$ tensor amplitude ${}^{4D}T^{\widetilde{T}SV}_{\alpha_{12}\mu}$. This amplitude has essentially the structure of the AV^n-type ones and it exhibits breaking of linearity too. Still, an anomaly in the vector or tensor current cannot be unavoidably stated. If we put aside the conceptual matter of breaking translational symmetry and preserve linearity we can route the diagram for the amplitude such that the violating terms disappear. The lower-point functions in its IRagfs have coefficients that make this possible.

The analytical tools developed along the three dimensions chosen and the AV^n-type amplitudes permits to localize algebraic and kinematic behavior of symmetry violations, which is to be used for investigating amplitudes with antisymmetric tensor vertices and gravitational amplitudes.

Chapter 7

Final Remarks and Perspectives

This investigation looks for a better understanding of algebraic and kinematic features of odd tensors described by Feynman integrals. It was applied in the subject of anomalies by approaching $(n+1)$-point perturbative amplitudes in a $2n$-dimensional setting. These amplitudes combine axial and vector vertices to form odd tensors, whose Dirac trace of the highest order contains two gamma matrices beyond the space-time dimension. This structure allows different expressions that are considered identities at the integrand level. Nevertheless, connecting them is not automatic after loop integration since the divergent character of amplitudes implies the presence of surface terms.

The IReg strategy was crucial to this exploration because it avoids evaluating divergent objects initially. That maintains the connection among all expressions attributed to the same object, allowing a clear view of the consequences of trace choices. As results are analogous in different dimensions, the reader may consult the two-dimensional case for a simpler view (4.23)-(4.24). By replacing the chiral matrix definition adjacent to one vertex, we limit the occurrence of this *version-defining* index solely to the Levi-Civita tensor because the sign tensor and subamplitude carrying the index are zero. We stress that this tensor structure is unrelated to the nature of the vertex as axial or vector.

Such a feature affects momenta contractions embodied in Relations Among Green Functions (Ragfs). Notwithstanding these constraints originate from algebraic operations; potentially-violating terms arise after integration for contractions with the version-defining index (5.78). These terms also distinguish amplitude versions achieved through different trace choices (5.80)-(5.82). From these results, it is possible to obtain unique perturbative solutions that satisfy all IRagfs by choosing specific finite values for surface terms (5.79). That preserves the linearity of integration in this context; however, it breaks all symmetry expectations for odd and even correlators.

At the same time, symmetry implications arise from momenta contractions through Ward identities (WIs). Under the hypothesis that IRagfs apply, translational invariance would be sufficient to ensure the validity of both axial and vector WIs. This invariance imposes the vanishing of lower-point amplitudes inside these relations, leading to the cancellation of surface terms (individually). Nevertheless, that is not enough to maintain the IRagf with potentially-violating terms. Even by imposing translational invariance, one anomalous contribution emerges from the finite sector of the amplitudes.

The result above agrees with the recognized competition between gauge and chiral symmetries; however, we propose a broader perspective. By investigating strategies to take Dirac traces, we derived distinct expressions for an amplitude (5.85). These expressions are combinations of the most fundamental ones (called version-defining) and carry violations in more contractions. Under this reasoning, preserving the vector symmetry is only one possibility. That is the case of the first AVV version, which prioritizes the index of the axial vertex and violates the corresponding WI. Table 4.1 casts the two-dimensional cases, emphasizing the version-defining occurrences and one of their combinations (third version).

Here, we also proposed a general tensor form for amplitudes to investigate low-energy theorems, clarifying the opposition between translational invariance and linearity of integration. First, supposing coefficients on external momenta, *structural identities* involving invariants arise in different dimensions: (4.49), (5.88), and (6.68). They contain kinematical limits of finite functions that should be zero but assume another value instead. Hence, the finite content V^R demands anomalous contributions $\mathcal{A}$ to satisfy these identities

$$\sum_{i=1}^{n+1} (-1)^{i+1} V_i^L (0) = 0 = \sum_{i=1}^{n+1} (-1)^{i+1} \left[V_i^R (0) + \mathcal{A}_i \right], \tag{7.1}$$

showing that violations are unavoidable and have a fixed value[1].

$$\sum_{i=1}^{n+1} (-1)^{i+1} V_i^R (0) = \frac{2^{n+1} (-\mathrm{i})^n}{(4\pi)^n} [\sum_{i=1}^{n} \left(m_1^2 - m_{i+1}^2 \right) Z_{\mathbf{e}_i}^{(-1)} - m_1^2 Z_{\mathbf{0}}^{(-1)}]_0 = \frac{2^{n+1} (-\mathrm{i})^n}{(4\pi)^n n!} \tag{7.2}$$

Nonetheless, the distribution of anomalous contributions still depends on trace choices.

Second, we admit the dependence on arbitrary routings that break translational invariance. That allows deriving the structure of surface terms without computing amplitudes, emphasizing the impossibility of automatic satisfaction of all IRagfs. Meanwhile, *structural identities* still apply and associate the surface term value with the kinematical limits of finite functions

$$\frac{2^n (-\mathrm{i})^{n-1}}{n} \Delta_{n+1;\alpha}^{\alpha} = \sum_{i=1}^{n+1} (-1)^{i+1} V_i^R (0), \tag{7.3}$$

reproducing the condition for linearity maintenance

$$\Delta_{n+1;\alpha}^{\alpha} = -\frac{2}{(n-1)!} \frac{\mathrm{i}}{(4\pi)^n}. \tag{7.4}$$

All explored facets apply to amplitudes in other even dimensions, with the final equations being general. Valid for propagators featuring arbitrary masses.

---

[1] Normalization of integrals $\int \frac{\mathrm{d}^{2n}k}{(2\pi)^{2n}}$ and trace $\mathrm{tr}\left(\gamma_* \gamma_{\mathrm{I}_{2n}}\right) = 2^n (-\mathrm{i})^{n+1} \varepsilon_{\mathrm{I}_{2n}}$ are used in this value.

Following our perspective on ambiguities and exclusive manipulation of finite integrals, we intend to study trace anomalies for Weyl fermions in four-dimensional three-point correlators. Such a theme appears in recent debates about the contributions of Pontryagin density to these anomalies [24]-[28], [94], and [95, 96]-[99]. The key point in the controversy (our reading of it) is whether a particular diagram summed with its permutation of external edges vanishes or not. In section (2.2) we glanced over the fact that the doubling or vanishing of pairs of diagrams for a full 1-loop correlator is a non-trivial matter (even without routing ambiguity, but with non-negative power counting). A pair of diagrams that should cancel each other requires translation invariance for that to happen because the sum of the pair is a collection (a complex one for gravitational amplitudes) of surface terms. However, as extensively demonstrated, if one turns off the surface terms the expressions lose their algebraic properties and the cancellation occurs only for specific versions of the tensor being studied (both translation invariance and permanence of algebraic forms need to be valid). Here enter the tools developed in the thesis as means to systematize the problem conceptually and mechanically.

Appendix A

Feynman Integrals, Subamplitudes and Traces

A.1 Two-Dimensional Feynman Integrals

One-propagator integrals with denominator D_i

$$\bar{J}_1 = I_{\log}^{(2)} \tag{A.1}$$

$$\bar{J}_1^{\mu} = -k_i^{\nu} \Delta_{2\nu}^{(2)\mu} \tag{A.2}$$

Two-propagator integrals with denominator D_{12}

$$\overline{J}_2 = J_2 = i\,(4\pi)^{-1}\, Z_0^{(-1)} \tag{A.3}$$

$$\overline{J}_2^{\mu_1} = J_2^{\mu_1} = -i\,(4\pi)^{-1}\, p^{\mu_1} Z_1^{(-1)} \tag{A.4}$$

$$\bar{J}_2^{\mu_{12}} = J_2^{\mu_{12}} + (\Delta_2^{(2)\mu_{12}} + g^{\mu_{12}} I_{\log}^{(2)})/2 \tag{A.5}$$

$$J_2^{\mu_{12}} = i\,(4\pi)^{-1}\, [p^{\mu_{12}} Z_2^{(-1)} - g^{\mu_{12}} Z_0^{(0)}/2] \tag{A.6}$$

Reductions of finite functions - $n = 0, 1, 2, ...$

$$Z_0^{(0)} = 2p^2 Z_2^{(-1)} - p^2 Z_1^{(-1)} \text{ and } 2Z_1^{(-1)} = Z_0^{(-1)} \tag{A.7}$$

$$p^2 Z_{n+2}^{(-1)} = p^2 Z_{n+1}^{(-1)} - m^2 Z_n^{(-1)} - (n+1)^{-1} \tag{A.8}$$

Reductions of tensors

$$2J_2^{\mu_1} = -p^{\mu_1} J_2 \text{ and } 2p_{\mu_1} J_2^{\mu_1} = -p^2 J_2 \tag{A.9}$$

$$2p_{\mu_1} J_2^{\mu_{12}} = -p^2 J_2^{\mu_2} \text{ and } g_{\mu_{12}} J_2^{\mu_{12}} = m^2 J_2 + \frac{i}{4\pi} \tag{A.10}$$

A.2 Four-Dimensional Feynman Integrals

Two-propagator integrals

$$\bar{J}_2 = J_2\,(p_{ij}) + I_{\log}^{(4)} \text{ with } J_2\,(p_{ij}) = i\,(4\pi)^{-2}\, [-Z_0^{(0)}\left(p_{ij}^2, m^2\right)] \tag{A.11}$$

$$\bar{J}_{2\mu} = J_{2\mu}\,(p_{ij}) - \frac{1}{2}(P_{ij}^{\nu} \Delta_{3\mu\nu}^{(4)} + p_{ji\mu} I_{\log}^{(4)}) \tag{A.12}$$

$$J_{2\mu}\,(p_{ij}) = i\,(4\pi)^{-2}\, [p_{ij\mu} Z_1^{(0)}\left(p_{ij}^2, m^2\right)] \tag{A.13}$$

Three-propagator integrals using general variables p and q

$$\bar{J}_3 = J_3 = i\,(4\pi)^{-2}\,[Z_{00}^{(-1)}\,(p,q)] \tag{A.14}$$

$$\bar{J}_{3\mu} = J_{3\mu} = i\,(4\pi)^{-2}\,[-p_\mu Z_{10}^{(-1)} - q_\mu Z_{01}^{(-1)}] \tag{A.15}$$

$$\bar{J}_{3\mu_1\mu_2} = J_{3\mu_1\mu_2} + 4^{-1}(\Delta^{(4)}_{3\mu_1\mu_2} + g_{\mu_1\mu_2} I^{(4)}_{\log}) \tag{A.16}$$

$$J_{3\mu_1\mu_2} = i\,(4\pi)^{-2}\,[p_{\mu_1}p_{\mu_2}Z_{20}^{(-1)} + q_{\mu_1}q_{\mu_2}Z_{02}^{(-1)} + \left(p_{\mu_1}q_{\mu_2} + p_{\mu_2}q_{\mu_1}\right)Z_{11}^{(-1)} - \frac{1}{2}g_{\mu_1\mu_2}Z_{00}^{(0)}] \tag{A.17}$$

Reductions of finite functions using $2Z_1^{(0)} = Z_0^{(0)}$ and the Kronecker symbol δ_{n0}

$$\begin{aligned}
&2[p^2 Z_{n+1;m}^{(-1)} + (p\cdot q)\,Z_{n;m+1}^{(-1)}] \\
= \;& p^2 Z_{n;m}^{(-1)} + (1-\delta_{n0})\,nZ_{n-1,m}^{(0)} + \delta_{n0}Z_m^{(0)}\,(p_{31}) - \sum_{s=0}^{m}(-1)^s\binom{m}{s}Z_{n+s}^{(0)}\,(p_{32}) \\
&2[q^2 Z_{n;m+1}^{(-1)} + (p\cdot q)\,Z_{n+1;m}^{(-1)}] \\
= \;& q^2 Z_{n;m}^{(-1)} + (1-\delta_{m0})\,mZ_{n;m-1}^{(0)} + \delta_{m0}Z_n^{(0)}\,(p_{21}) - \sum_{s=0}^{m}(-1)^s\binom{m}{s}Z_{n+s}^{(0)}\,(p_{32})
\end{aligned}$$

$$2Z_{00}^{(0)} = [p^2 Z_{10}^{(-1)} + q^2 Z_{01}^{(-1)}] - 2m^2 Z_{00}^{(-1)} + 2Z_1^{(0)}\,(q-p) - 1 \tag{A.18}$$

Reductions of tensors

$$\begin{aligned}
2p^{\mu_1}J_{3\mu_1} &= -p^2 J_3 + [J_2\,(q) - J_2\,(q-p)] \\
2q^{\mu_1}J_{3\mu_1} &= -q^2 J_3 + [J_2\,(p) - J_2\,(q-p)]
\end{aligned}$$

$$\begin{aligned}
2p^{\mu_1}J_{3\mu_1\mu_2} &= -p^2 J_{3\mu_2} + \left[J_{2\mu_2}\,(q) + J_{2\mu_2}\,(q-p) + q_{\mu_2}J_2\,(q-p)\right] \\
2q^{\mu_1}J_{3\mu_1\mu_2} &= -q^2 J_{3\mu_2} + \left[J_{2\mu_2}\,(p) + J_{2\mu_2}\,(q-p) + q_{\mu_2}J_2\,(q-p)\right]
\end{aligned}$$

$$g^{\mu_1\mu_2}J_{3\mu_1\mu_2} = m^2 J_3 + J_2\,(q-p) + i\left[2\,(4\pi)^2\right]^{-1} \tag{A.19}$$

A.3 Six-Dimensional Feynman Integrals

Three-propagator integrals

$$\begin{aligned}
\bar{J}_3 &= J_3 + I^{(6)}_{\log} \text{ with } J_3\,(p,q) = [-Z_{00}^{(0)}\,(p,q)] \\
\bar{J}_3^{\mu_1}\,(k_1,k_2,k_3) &= J_3^{\mu_1}\,(k_1,k_2,k_3) - \frac{1}{3}\,(k_1^{\nu_1} + k_2^{\nu_1} + k_3^{\nu_1})\,\Delta^{(6)\mu_1}_{4\nu_1} - \frac{1}{3}\,(p_{21}^{\mu_1} + p_{31}^{\mu_1})\,I^{(6)}_{\log} \\
J_3^{\mu_1}\,(k_1,k_2,k_3) &= [p_{21}^{\mu_1}Z_{10}^{(0)} + p_{31}^{\mu_1}Z_{01}^{(0)}]
\end{aligned}$$

Four-propagator integrals

$$\overline{J}_4 = J_4 = [Z_{000}^{(-1)}\,(p,q,r)] \tag{A.20}$$

$$\overline{J}_{4\mu_1} = J_{4\mu_1} = [-p_{\mu_1}Z_{100}^{(-1)} - q_{\mu_1}Z_{010}^{(-1)} - r_{\mu_1}Z_{001}^{(-1)}] \tag{A.21}$$

$$\bar{J}_{4\mu_1\mu_2} = J_{4\mu_1\mu_2} + (\Delta^{(6)}_{4\mu_1\mu_2} + g_{\mu_1\mu_2}I^{(6)}_{\log})/6 \tag{A.22}$$

$$J_{4\mu_1\mu_2} = [-\frac{1}{2}g_{\mu_1\mu_2}Z_{000}^{(0)} + p_{\mu_1}p_{\mu_2}Z_{200}^{(-1)} + q_{\mu_1}q_{\mu_2}Z_{020}^{(-1)} + r_{\mu_1}r_{\mu_2}Z_{002}^{(-1)} \tag{A.23}$$

$$+ \left(p_{\mu_1}q_{\mu_2} + p_{\mu_2}q_{\mu_1}\right)Z_{110}^{(-1)} + \left(p_{\mu_1}r_{\mu_2} + p_{\mu_2}r_{\mu_1}\right)Z_{101}^{(-1)} + \left(q_{\mu_1}r_{\mu_2} + r_{\mu_1}q_{\mu_2}\right)Z_{011}^{(-1)}] \tag{A.24}$$

Reductions of finite functions using the binomial coefficient $C_s^k = \binom{k}{s}$

$$\begin{aligned} & 2[p^2 Z^{(-1)}_{n+1;m;k} + (p\cdot q)\, Z^{(-1)}_{n;m+1;k} + (p\cdot r)\, Z^{(-1)}_{n;m;k+1}] && \text{(A.25)} \\ = \; & p^2 Z^{(-1)}_{n;m;k} + (1-\delta_{n0})\, n Z^{(0)}_{n-1;m;k} + \delta_{n0} Z^{(0)}_{m;k}\,(q,r) - \sum_{s_1=0}^{k}\sum_{s_2=0}^{s_1} (-1)^{s_1}\, C^k_{s_1} C^{s_1}_{s_2} Z^{(0)}_{n+s_1-s_2;m+s_2}\,(p_{42},p_{43}) \end{aligned}$$

$$\begin{aligned} & 2[q^2 Z^{(-1)}_{n;m+1;k} + (p\cdot q)\, Z^{(-1)}_{n+1;m;k} + (q\cdot r)\, Z^{(-1)}_{n;m;k+1}] && \text{(A.26)} \\ = \; & q^2 Z^{(-1)}_{n;m;k} + (1-\delta_{m0})\, m Z^{(0)}_{n;m-1;k} + \delta_{m0} Z^{(0)}_{n;k}\,(p,r) - \sum_{s_1=0}^{k}\sum_{s_2=0}^{s_1} (-1)^{s_1}\, C^k_{s_1} C^{s_1}_{s_2} Z^{(0)}_{n+s_1-s_2;m+s_2}\,(p_{42},p_{43}) \end{aligned}$$

$$\begin{aligned} & 2[r^2 Z^{(-1)}_{n;m;k+1} + (p\cdot r)\, Z^{(-1)}_{n+1;m;k} + (q\cdot r)\, Z^{(-1)}_{n;m+1;k}] && \text{(A.27)} \\ = \; & r^2 Z^{(-1)}_{n;m;k} + (1-\delta_{k0})\, k Z^{(0)}_{n;m;k-1} + \delta_{k0} Z^{(0)}_{n;m}\,(p,q) - \sum_{s_1=0}^{k}\sum_{s_2=0}^{s_1} (-1)^{s_1}\, C^k_{s_1} C^{s_1}_{s_2} Z^{(0)}_{n+s_1-s_2;m+s_2}\,(p_{42},p_{43}) \end{aligned}$$

$$-3Z^{(0)}_{000} = 2m^2 Z^{(-1)}_{000} + \frac{1}{3} - [p^2 Z^{(-1)}_{100} + q^2 Z^{(-1)}_{010} + r^2 Z^{(-1)}_{001}] - Z^{(0)}_{00}\,(p_{42},p_{43})$$

Reductions of tensors using $p = p_{21}$, $q = p_{31}$, and $r = p_{41}$

$$\begin{aligned} 2p^{\mu_1} J_{4\mu_1} &= -p^2 J_4 + J_3\,(q,r) - J_3\,(r-p,r-q) && \text{(A.28)} \\ 2q^{\mu_1} J_{4\mu_1} &= -q^2 J_4 + J_3\,(p,r) - J_3\,(r-p,r-q) && \text{(A.29)} \\ 2r^{\mu_1} J_{4\mu_1} &= -r^2 J_4 + J_3\,(p,q) - J_3\,(r-p,r-q) && \text{(A.30)} \end{aligned}$$

$$\begin{aligned} 2p^{\mu_1} J_{4\mu_1\mu_2} &= -p^2 J_{4\mu_2} + J_{3\mu_2}\,(p_{42},p_{43}) + J_{3\mu_2}\,(p_{31},p_{41}) + p_{41\mu_2} J_3\,(p_{42},p_{43}) && \text{(A.31)} \\ 2q^{\mu_1} J_{4\mu_1\mu_2} &= -q^2 J_{4\mu_2} + J_{3\mu_2}\,(p_{42},p_{43}) + J_{3\mu_2}\,(p_{21},p_{41}) + p_{41\mu_2} J_3\,(p_{42},p_{43}) && \text{(A.32)} \\ 2r^{\mu_1} J_{4\mu_1\mu_2} &= -r^2 J_{4\mu_2} + J_{3\mu_2}\,(p_{42},p_{43}) + J_{3\mu_2}\,(p_{21},p_{31}) + p_{41\mu_2} J_3\,(p_{42},p_{43}) && \text{(A.33)} \end{aligned}$$

$$2g^{\mu_{12}} J_{4\mu_1\mu_2} = \frac{1}{3} + 2m^2 J_4 + 2J_3\,(p_{42},p_{43})\,. \qquad \text{(A.34)}$$

A.4 $4D$ Subamplitudes

First version

$$\begin{aligned} (T^{VPP})^{\nu_1} &= 2\left[P_{31}^{\nu_2}\Delta_{3\nu_2}^{\nu_1} + (p_{21}^{\nu_1} - p_{32}^{\nu_1}) I_{\log}\right] - 4\,(p_{21}\cdot p_{32})\, J_3^{\nu_1} \\ &\quad +2\left[(p_{31}^{\nu_1} p_{21}^2 - p_{21}^{\nu_1} p_{31}^2) J_3 + p_{21}^{\nu_1} J_2\,(p_{21}) - p_{32}^{\nu_1} J_2\,(p_{32})\right] \\ \left(T^{ASP}\right)^{\nu_1} &= 2\left[P_{31}^{\nu_2}\Delta_{3\nu_2}^{\nu_1} + (p_{21}^{\nu_1} - p_{32}^{\nu_1})\, I_{\log}\right] - 4\,(p_{21}\cdot p_{32})\, J_3^{\nu_1} \\ &\quad +2\left[\left(p_{31}^{\nu_1} p_{21}^2 - p_{21}^{\nu_1} p_{31}^2 - 4m^2 p_{32}^{\nu_1}\right) J_3 + p_{21}^{\nu_1} J_2\,(p_{21}) - p_{32}^{\nu_1} J_2\,(p_{32})\right] \\ -\left(T^{APS}\right)^{\nu_1} &= 2\left[P_{31}^{\nu_2}\Delta_{3\nu_2}^{\nu_1} + (p_{21}^{\nu_1} - p_{32}^{\nu_1}) I_{\log}\right] - 4\,(p_{21}\cdot p_{32})\, J_3^{\nu_1} \\ &\quad +2\left[\left(p_{31}^{\nu_1} p_{21}^2 - p_{21}^{\nu_1} p_{31}^2 + 4m^2 p_{21}^{\nu_1}\right) J_3 + p_{21}^{\nu_1} J_2\,(p_{21}) - p_{32}^{\nu_1} J_2\,(p_{32})\right] \\ -\left(T^{VSS}\right)^{\nu_1} &= 2\left[P_{31}^{\nu_2}\Delta_{3\nu_2}^{\nu_1} + (p_{21}^{\nu_1} - p_{32}^{\nu_1}) I_{\log}\right] - 4\left(p_{21}\cdot p_{32} + 4m^2\right) J_3^{\nu_1} \\ &\quad +2\left[\left(p_{31}^{\nu_1} p_{21}^2 - p_{21}^{\nu_1} p_{31}^2 - 4m^2 p_{31}^{\nu_1}\right) J_3 + p_{21}^{\nu_1} J_2\,(p_{21}) - p_{32}^{\nu_1} J_2\,(p_{32})\right] \end{aligned}$$

Second version

$$
\begin{aligned}
-\left(T^{SAP}\right)^{\nu_1} &= 2\left[P_{21}^{\nu_2}\Delta_{3\nu_2}^{\nu_1}+\left(p_{32}^{\nu_1}+p_{31}^{\nu_1}\right)I_{\log}\right]+4\left(p_{32}\cdot p_{31}\right)J_3^{\nu_1}\\
&\quad+2\left[\left(p_{21}^{\nu_1}p_{31}^2-p_{31}^{\nu_1}p_{21}^2+4m^2p_{32}^{\nu_1}\right)J_3+p_{32}^{\nu_1}J_2\left(p_{32}\right)+p_{31}^{\nu_1}J_2\left(p_{31}\right)\right]\\
\left(T^{PVP}\right)^{\nu_1} &= 2\left[P_{21}^{\nu_2}\Delta_{3\nu_2}^{\nu_1}+\left(p_{32}^{\nu_1}+p_{31}^{\nu_1}\right)I_{\log}\right]+4\left(p_{32}\cdot p_{31}\right)J_3^{\nu_1}\\
&\quad+2\left[\left(p_{21}^{\nu_1}p_{31}^2-p_{31}^{\nu_1}p_{21}^2\right)J_3+p_{32}^{\nu_1}J_2\left(p_{32}\right)+p_{31}^{\nu_1}J_2\left(p_{31}\right)\right]\\
\left(T^{PAS}\right)^{\nu_1} &= 2\left[P_{21}^{\nu_2}\Delta_{3\nu_2}^{\nu_1}+\left(p_{32}^{\nu_1}+p_{31}^{\nu_1}\right)I_{\log}\right]+4\left(p_{32}\cdot p_{31}\right)J_3^{\nu_1}\\
&\quad+2\left[\left(p_{21}^{\nu_1}p_{31}^2-p_{31}^{\nu_1}p_{21}^2+4m^2p_{31}^{\nu_1}\right)J_3+p_{32}^{\nu_1}J_2\left(p_{32}\right)+p_{31}^{\nu_1}J_2\left(p_{31}\right)\right]\\
-\left(T^{SVS}\right)^{\nu_1} &= 2\left[P_{21}^{\nu_2}\Delta_{3\nu_2}^{\nu_1}+\left(p_{32}^{\nu_1}+p_{31}^{\nu_1}\right)I_{\log}\right]+4\left(p_{32}\cdot p_{31}-4m^2\right)J_3^{\nu_1}\\
&\quad+2\left[\left(p_{21}^{\nu_1}p_{31}^2-p_{31}^{\nu_1}p_{21}^2-4m^2p_{21}^{\nu_1}\right)J_3+p_{32}^{\nu_1}J_2\left(p_{32}\right)+p_{31}^{\nu_1}J_2\left(p_{31}\right)\right]
\end{aligned}
$$

Third version

$$
\begin{aligned}
\left(T^{SPA}\right)^{\nu_1} &= 2\left[P_{32}^{\nu_2}\Delta_{3\nu_2}^{\nu_1}-\left(p_{21}^{\nu_1}+p_{31}^{\nu_1}\right)I_{\log}\right]+4\left(p_{21}\cdot p_{31}\right)J_3^{\nu_1}\\
&\quad+2\left[\left(p_{31}^{\nu_1}p_{21}^2+p_{21}^{\nu_1}p_{31}^2-4m^2p_{21}^{\nu_1}\right)J_3-p_{21}^{\nu_1}J_2\left(p_{21}\right)-p_{31}^{\nu_1}J_2\left(p_{31}\right)\right]\\
-\left(T^{PSA}\right)^{\nu_1} &= 2\left[P_{32}^{\nu_2}\Delta_{3\nu_2}^{\nu_1}-\left(p_{21}^{\nu_1}+p_{31}^{\nu_1}\right)I_{\log}\right]+4\left(p_{21}\cdot p_{31}\right)J_3^{\nu_1}\\
&\quad+2\left[\left(p_{31}^{\nu_1}p_{21}^2+p_{21}^{\nu_1}p_{31}^2-4m^2p_{31}^{\nu_1}\right)J_3-p_{21}^{\nu_1}J_2\left(p_{21}\right)-p_{31}^{\nu_1}J_2\left(p_{31}\right)\right]\\
\left(T^{PPV}\right)^{\nu_1} &= 2\left[P_{32}^{\nu_2}\Delta_{3\nu_2}^{\nu_1}-\left(p_{21}^{\nu_1}+p_{31}^{\nu_1}\right)I_{\log}\right]+4\left(p_{21}\cdot p_{31}\right)J_3^{\nu_1}\\
&\quad+2\left[\left(p_{31}^{\nu_1}p_{21}^2+p_{21}^{\nu_1}p_{31}^2\right)J_3-p_{21}^{\nu_1}J_2\left(p_{21}\right)-p_{31}^{\nu_1}J_2\left(p_{31}\right)\right]\\
-\left(T^{SSV}\right)^{\nu_1} &= 2\left[P_{32}^{\nu_2}\Delta_{3\nu_2}^{\nu_1}-\left(p_{21}^{\nu_1}+p_{31}^{\nu_1}\right)I_{\log}\right]+4\left(p_{21}\cdot p_{31}-4m^2\right)J_3^{\nu_1}\\
&\quad+2\left[\left(p_{31}^{\nu_1}p_{21}^2+p_{21}^{\nu_1}p_{31}^2-4m^2\left(p_{21}^{\nu_1}+p_{31}^{\nu_1}\right)\right)J_3-p_{21}^{\nu_1}J_2\left(p_{21}\right)-p_{31}^{\nu_1}J_2\left(p_{31}\right)\right]
\end{aligned}
$$

A.5 $6D$ Subamplitudes

In this dimension the text of chapter 6 ignored the factor $\mathrm{tr}\left(\mathbf{1}_{8\times 8}\right)=8$. Here we included.

First version

$$
\begin{aligned}
-\varepsilon_{\mu_{1234}}{}^{\nu_{12}}T_{\nu_{12}}^{\tilde{T}PPP} &= \varepsilon_{\mu_{1234}\nu_{12}}\{16[\left(p_{31}\cdot p_{43}\right)p_{21}^{\nu_2}-\left(p_{21}\cdot p_{42}\right)p_{31}^{\nu_2}+\left(p_{21}\cdot p_{32}\right)p_{41}^{\nu_2}]J_4^{\nu_1}\\
&\quad+8(p_{21}^{\nu_1}p_{41}^{\nu_2}p_{31}^2-p_{31}^{\nu_1}p_{41}^{\nu_2}p_{21}^2-p_{21}^{\nu_1}p_{31}^{\nu_2}p_{41}^2)J_4 \qquad\qquad (\mathrm{A}.35)\\
&\quad+8[2p_{43}^{\nu_2}J_3^{\nu_1}\left(p_{31},p_{41}\right)+p_{31}^{\nu_1}p_{41}^{\nu_2}J_3\left(p_{31},p_{41}\right)]+8[2p_{21}^{\nu_2}J_3^{\nu_1}\left(p_{21},p_{41}\right)]\\
&\quad+8[-p_{21}^{\nu_1}p_{41}^{\nu_2}J_3\left(p_{21},p_{41}\right)+p_{32}^{\nu_1}p_{43}^{\nu_2}J_3\left(p_{32},p_{42}\right)+p_{21}^{\nu_1}p_{31}^{\nu_2}J_3\left(p_{21},p_{31}\right)]\}\,,
\end{aligned}
$$

Second version

$$
\begin{aligned}
-\varepsilon_{\mu_{1234}}{}^{\nu_{12}}T_{\nu_1\nu_2}^{STPP} &= \varepsilon_{\mu_{1234}\nu_{12}}\{16\left[-\left(p_{41}\cdot p_{43}\right)p_{21}^{\nu_2}+\left(p_{41}\cdot p_{42}\right)p_{31}^{\nu_2}-\left(p_{31}\cdot p_{32}\right)p_{41}^{\nu_2}\right]J_4^{\nu_1}\\
&\quad+8\left[p_{31}^{\nu_1}p_{43}^{\nu_2}p_{21}^2-p_{21}^{\nu_1}p_{42}^{\nu_2}p_{31}^2+p_{21}^{\nu_1}p_{32}^{\nu_2}p_{41}^2-4m^2p_{32}^{\nu_1}p_{42}^{\nu_2}\right]J_4 \qquad\qquad (\mathrm{A}.36)\\
&\quad+8\left[2p_{41}^{\nu_2}J_3^{\nu_1}\left(p_{21},p_{41}\right)-2p_{32}^{\nu_2}J_3^{\nu_1}\left(p_{21},p_{31}\right)-p_{32}^{\nu_1}p_{43}^{\nu_2}J_3\left(p_{42},p_{43}\right)\right]\\
&\quad+8\left[-p_{31}^{\nu_1}p_{43}^{\nu_2}J_3\left(p_{31},p_{41}\right)+p_{21}^{\nu_1}p_{42}^{\nu_2}J_3\left(p_{21},p_{41}\right)-p_{21}^{\nu_1}p_{32}^{\nu_2}J_3\left(p_{21},p_{31}\right)\right]\}\,.
\end{aligned}
$$

A.6 4D-Traces: Six Gamma Matrices and γ_*

The way to insert a Levi-Civita tensor in the traces with the chiral matrix come from the use of

$$\gamma_*\gamma_{[\mu_1\cdots\mu_r]}=\frac{i^{n-1+r(r+1)}}{(2n-r)!}\varepsilon_{\mu_1\cdots\mu_r\nu_{r+1}\cdots\nu_{2n}}\gamma^{[\nu_{r+1}\cdots\nu_{2n}]}.$$

In $2n=4$ dimensions they are the identities with $0,1,2,3,4$ antisymmetrized products, giving rise in traces of a string of six gamma matrices to $(15,10,7,6,7)$ monomials respectively.

Trace Using $\gamma_*=i\varepsilon_{\nu_1\nu_2\nu_3\nu_4}\gamma^{\nu_1\nu_2\nu_3\nu_4}/4!$ (Definition)

The three main positions to deploy the definition of the chiral matrix is around the gamma matrices present in the vertexes Γ_1,Γ_2, and Γ_3, in the left or the right they return the same integrated results.

Distinct positions of the chiral matrix in the trace. First one

$$\begin{aligned}
t_1 &= \operatorname{tr}\left(\gamma_*\gamma_{\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3}\right)=i\varepsilon^{\alpha_1\alpha_2\alpha_3\alpha_4}\operatorname{tr}\left(\gamma_{\alpha_1\alpha_2\alpha_3\alpha_4}\gamma_{\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3}\right)/4!\\
&= +g_{\mu_1\nu_1}\varepsilon_{\mu_2\nu_2\mu_3\nu_3}-g_{\mu_1\mu_2}\varepsilon_{\nu_1\nu_2\mu_3\nu_3}+g_{\mu_1\nu_2}\varepsilon_{\nu_1\mu_2\mu_3\nu_3}-g_{\mu_1\mu_3}\varepsilon_{\nu_1\mu_2\nu_2\nu_3}+g_{\mu_1\nu_3}\varepsilon_{\nu_1\mu_2\nu_2\mu_3}\\
&\quad +g_{\nu_1\mu_2}\varepsilon_{\mu_1\nu_2\mu_3\nu_3}-g_{\nu_1\nu_2}\varepsilon_{\mu_1\mu_2\mu_3\nu_3}+g_{\nu_1\mu_3}\varepsilon_{\mu_1\mu_2\nu_2\nu_3}-g_{\nu_1\nu_3}\varepsilon_{\mu_1\mu_2\nu_2\mu_3}+g_{\mu_2\nu_2}\varepsilon_{\mu_1\nu_1\mu_3\nu_3}\\
&\quad -g_{\mu_2\mu_3}\varepsilon_{\mu_1\nu_1\nu_2\nu_3}+g_{\mu_2\nu_3}\varepsilon_{\mu_1\nu_1\nu_2\mu_3}+g_{\nu_2\mu_3}\varepsilon_{\mu_1\nu_1\mu_2\nu_3}-g_{\nu_2\nu_3}\varepsilon_{\mu_1\nu_1\mu_2\mu_3}+g_{\mu_3\nu_3}\varepsilon_{\mu_1\nu_1\mu_2\nu_2}.
\end{aligned}$$

Second one

$$\begin{aligned}
t_2 &= \operatorname{tr}\left(\gamma_{\mu_1\nu_1}\gamma_*\gamma_{\mu_2\nu_2\mu_3\nu_3}\right)=i\varepsilon^{\alpha_1\alpha_2\alpha_3\alpha_4}\operatorname{tr}\left(\gamma_{\mu_1\nu_1}\gamma_{\alpha_1\alpha_2\alpha_3\alpha_4}\gamma_{\mu_2\nu_2\mu_3\nu_3}\right)/4!\\
&= +g_{\mu_1\nu_1}\varepsilon_{\mu_2\nu_2\mu_3\nu_3}+g_{\mu_1\mu_2}\varepsilon_{\nu_1\nu_2\mu_3\nu_3}-g_{\mu_1\nu_2}\varepsilon_{\mu_2\nu_2\mu_3\nu_3}+g_{\mu_1\mu_3}\varepsilon_{\nu_1\mu_2\nu_2\nu_3}-g_{\mu_1\nu_3}\varepsilon_{\nu_1\mu_2\nu_2\mu_3}\\
&\quad -g_{\nu_1\mu_2}\varepsilon_{\mu_1\nu_2\mu_3\nu_3}+g_{\nu_1\nu_2}\varepsilon_{\mu_1\mu_2\mu_3\nu_3}-g_{\nu_1\mu_3}\varepsilon_{\mu_1\mu_2\nu_2\nu_3}+g_{\nu_1\nu_3}\varepsilon_{\mu_1\mu_2\nu_2\mu_3}+g_{\mu_2\nu_2}\varepsilon_{\mu_1\nu_1\mu_3\nu_3}\\
&\quad -g_{\mu_2\mu_3}\varepsilon_{\mu_1\nu_1\nu_2\nu_3}+g_{\mu_2\nu_3}\varepsilon_{\mu_1\nu_1\nu_2\mu_3}+g_{\nu_2\mu_3}\varepsilon_{\mu_1\nu_1\mu_2\nu_3}-g_{\nu_2\nu_3}\varepsilon_{\mu_1\nu_1\mu_2\mu_3}+g_{\mu_3\nu_3}\varepsilon_{\mu_1\nu_1\mu_2\nu_2}.
\end{aligned}$$

Third one

$$\begin{aligned}
t_3 &= \operatorname{tr}\left(\gamma_{\mu_1\nu_1\mu_2\nu_2}\gamma_*\gamma_{\mu_3\nu_3}\right)=i\varepsilon^{\alpha_1\alpha_2\alpha_3\alpha_4}\operatorname{tr}\left(\gamma_{\mu_1\nu_1\mu_2\nu_2}\gamma_{\alpha_1\alpha_2\alpha_3\alpha_4}\gamma_{\mu_3\nu_3}\right)/4!\\
&= +g_{\mu_1\nu_1}\varepsilon_{\mu_2\nu_2\mu_3\nu_3}-g_{\mu_1\mu_2}\varepsilon_{\nu_1\nu_2\mu_3\nu_3}+g_{\mu_1\nu_2}\varepsilon_{\mu_2\nu_2\mu_3\nu_3}+g_{\mu_1\mu_3}\varepsilon_{\nu_1\mu_2\nu_2\nu_3}-g_{\mu_1\nu_3}\varepsilon_{\nu_1\mu_2\nu_2\mu_3}\\
&\quad +g_{\nu_1\mu_2}\varepsilon_{\mu_1\nu_2\mu_3\nu_3}-g_{\nu_1\nu_2}\varepsilon_{\mu_1\mu_2\mu_3\nu_3}-g_{\nu_1\mu_3}\varepsilon_{\mu_1\mu_2\nu_2\nu_3}+g_{\nu_1\nu_3}\varepsilon_{\mu_1\mu_2\nu_2\mu_3}+g_{\mu_2\nu_2}\varepsilon_{\mu_1\nu_1\mu_3\nu_3}\\
&\quad +g_{\mu_2\mu_3}\varepsilon_{\mu_1\nu_1\nu_2\nu_3}-g_{\mu_2\nu_3}\varepsilon_{\mu_1\nu_1\nu_2\mu_3}-g_{\nu_2\mu_3}\varepsilon_{\mu_1\nu_1\mu_2\nu_3}+g_{\nu_2\nu_3}\varepsilon_{\mu_1\nu_1\mu_2\mu_3}+g_{\mu_3\nu_3}\varepsilon_{\mu_1\nu_1\mu_2\nu_2},
\end{aligned}$$

we omit the imaginary unit $\mathrm{i}=\sqrt{-1}$ and the trace $\operatorname{tr}\left(\mathbf{1}_{4\times 4}\right)$.

Now, these three expressions cast all the indexes of the tensor and they have fifteen terms each. In a narrow sense, they could be called symmetric, and considered to be as respecting all the symmetries among the indices, for example, see the appendix of [73] or [72][57]. We do not focus on such adjectives, but on the fact they are enough to obtain any other result by a careful analysis. As chapter 5 worked out, the "privileged" status of these traces is to produce integrated results for AVV-type amplitudes that automatically satisfy two IRagfs. The maximum allowed due to LET analysis.

First things first, the sign differences in the traces above are the unique distinguishing factor among them. They effectively sample the indices among finite and surface terms in the real calculations. The aim is to demonstrate that any expression for the triangles investigated are linear combinations of the ones we have detailed in the main body of this work.

Making the combinations only using sums not Schouten identities

$$t_{ij} = \frac{1}{2}\left(t_i + t_j\right),$$

we will have

$$\begin{aligned} t_{12} &= -g_{\mu_1\nu_1}\varepsilon_{\mu_2\mu_3\nu_2\nu_3} - g_{\mu_2\nu_2}\varepsilon_{\mu_1\mu_3\nu_1\nu_3} + g_{\mu_2\nu_3}\varepsilon_{\mu_1\mu_3\nu_1\nu_2} \\ &\quad -g_{\nu_2\mu_3}\varepsilon_{\mu_1\mu_2\nu_1\nu_3} - g_{\mu_3\nu_3}\varepsilon_{\mu_1\mu_2\nu_1\nu_2} - g_{\mu_2\mu_3}\varepsilon_{\mu_1\nu_1\nu_2\nu_3} - g_{\nu_2\nu_3}\varepsilon_{\mu_1\mu_2\mu_3\nu_1}, \end{aligned}$$

$$\begin{aligned} t_{13} &= -g_{\mu_3\nu_3}\varepsilon_{\mu_1\mu_2\nu_1\nu_2} - g_{\mu_1\nu_1}\varepsilon_{\mu_2\mu_3\nu_2\nu_3} + g_{\mu_1\nu_2}\varepsilon_{\mu_2\mu_3\nu_1\nu_3} \\ &\quad -g_{\nu_1\mu_2}\varepsilon_{\mu_1\mu_3\nu_2\nu_3} - g_{\mu_2\nu_2}\varepsilon_{\mu_1\mu_3\nu_1\nu_3} - g_{\mu_1\mu_2}\varepsilon_{\mu_3\nu_1\nu_2\nu_3} - g_{\nu_1\nu_2}\varepsilon_{\mu_1\mu_2\mu_3\nu_3}, \end{aligned}$$

$$\begin{aligned} t_{23} &= -g_{\mu_2\nu_2}\varepsilon_{\mu_1\mu_3\nu_1\nu_3} - g_{\mu_1\nu_1}\varepsilon_{\mu_2\mu_3\nu_2\nu_3} + g_{\mu_1\nu_3}\varepsilon_{\mu_2\mu_3\nu_1\nu_2} \\ &\quad -g_{\nu_1\mu_3}\varepsilon_{\mu_1\mu_2\nu_2\nu_3} - g_{\mu_3\nu_3}\varepsilon_{\mu_1\mu_2\nu_1\nu_2} - g_{\mu_1\mu_3}\varepsilon_{\mu_2\nu_1\nu_2\nu_3} - g_{\nu_1\nu_3}\varepsilon_{\mu_1\mu_2\mu_3\nu_2}. \end{aligned}$$

Now, we also can employ the identities involving the antisymmetrized products to compute the same trace as well obtaining other formulas.

Trace Using $\gamma_*\gamma_a = -i\varepsilon_{a\nu_1\nu_2\nu_3}\gamma^{\nu_1\nu_2\nu_3}/3!$

The straightforward application

$$\begin{aligned} \mathrm{tr}^{(1)}\left(a\right) &= \mathrm{tr}\left(\gamma_*\gamma_{abcdef}\right) = g_{bc}\varepsilon_{adef} - g_{bd}\varepsilon_{acef} + g_{be}\varepsilon_{acdf} - g_{bf}\varepsilon_{acde} \\ &\quad +g_{cd}\varepsilon_{abef} - g_{ce}\varepsilon_{abdf} + g_{cf}\varepsilon_{abde} + g_{de}\varepsilon_{abcf} + g_{ef}\varepsilon_{abcd} - g_{df}\varepsilon_{abce} \end{aligned}$$

the notation means that it uses a product with one Dirac matrix with index a in the substitution of $\gamma_*\gamma_a$

$$\mathrm{tr}^{(1)}\left(a\right) = -i\varepsilon_{a}{}^{\nu_1\nu_2\nu_3}\mathrm{tr}\left(\gamma_{\nu_1\nu_2\nu_3}\gamma_{bcdef}\right)/6.$$

The Trace Using $\gamma_*\gamma_{[ab]} = -i\varepsilon_{ab\nu_1\nu_2}\gamma^{\nu_1\nu_2}/2!$

The application of this one requires to express the ordinary product in terms of the antisymmetrized one

$$\gamma_*\gamma_{ab} = -\frac{i}{2}\varepsilon_{ab\nu_1\nu_2}\gamma^{\nu_1\nu_1} + g_{ab}\gamma_*,$$

from which follows that

$$\begin{aligned} \mathrm{tr}^{(2)}\left(ab\right) &= \mathrm{tr}\left(\gamma_*\gamma_{abcdef}\right) = g_{ab}\varepsilon_{cdef} + g_{cd}\varepsilon_{abef} - g_{ce}\varepsilon_{abdf} + g_{cf}\varepsilon_{abde} \\ &\quad +g_{de}\varepsilon_{abcf} - g_{df}\varepsilon_{abce} + g_{ef}\varepsilon_{abcd}. \end{aligned}$$

The Trace Using $\gamma_*\gamma_{[abc]} = i\varepsilon_{abc\nu}\gamma^{\nu}$

Expressing the antisymmetric product as common products we get

$$\gamma_*\gamma_{abc} = i\varepsilon_{abc\nu}\gamma^{\nu} + \gamma_*\left(g_{bc}\gamma_a - g_{ac}\gamma_b + g_{ab}\gamma_c\right),$$

Then with arbitrary indexes we get

$$\mathrm{tr}^{(3)}\left(abc\right) = \mathrm{tr}\left(\gamma_*\gamma_{abcdef}\right) = g_{ab}\varepsilon_{cdef} - g_{ac}\varepsilon_{bdef} + g_{bc}\varepsilon_{adef} + g_{de}\varepsilon_{abcf} - g_{df}\varepsilon_{abce} + g_{ef}\varepsilon_{abcd}.$$

The notation means that we absorb the indexes a, b and c with the identity and compute the resulting trace. This can be used to apply the substitution in any place desired. The use of this identity is a common choice on computation of this type of diagrams. In them, and all other possible results, after integration, we get some of the results obtained through the linear combinations t_{12}, t_{13}, and t_{23}. One example, which applies for the other case:

$$\begin{aligned}\mathrm{tr}^{(3)}\left(\mu_2\nu_2\mu_3\right) &= \mathrm{tr}\left(\gamma_*\gamma_{\mu_2\nu_2\mu_3\nu_3\mu_1\nu_1}\right) = g_{\mu_2\nu_2}\varepsilon_{\mu_3\nu_3\mu_1\nu_1} - g_{\mu_2\mu_3}\varepsilon_{\nu_2\nu_3\mu_1\nu_1} + g_{\nu_2\mu_3}\varepsilon_{\mu_2\nu_3\mu_1\nu_1}\\ &+g_{\nu_3\mu_1}\varepsilon_{\mu_2\nu_2\mu_3\nu_1} - g_{\nu_3\nu_1}\varepsilon_{\mu_2\nu_2\mu_3\mu_1} + g_{\mu_1\nu_1}\varepsilon_{\mu_2\nu_2\mu_3\nu_3}.\end{aligned}$$

The Trace Using $\gamma_*\gamma_{[abcd]} = i\varepsilon_{abcd}$

With the help of

$$\begin{aligned}\gamma_*\gamma_{abcd} &= i\varepsilon_{abcd}\mathbf{1} + g_{ab}\gamma_*\gamma_{[cd]} - g_{ac}\gamma_*\gamma_{[bd]} + g_{ad}\gamma_*\gamma_{[bc]}\\ &+g_{bc}\gamma_*\gamma_{[ad]} - g_{bd}\gamma_*\gamma_{[ac]} + g_{cd}\gamma_*\gamma_{[ab]} + \left(g_{ab}g_{cd} - g_{ac}g_{bd} + g_{ad}g_{bc}\right)\gamma_*,\end{aligned}$$

under the trace, and with arbitrary indexes, we obtain

$$\begin{aligned}\mathrm{tr}^{(4)}\left(abcd\right) &= \mathrm{tr}\left(\gamma_*\gamma_{abcdef}\right) = +g_{ab}\varepsilon_{cdef} - g_{ac}\varepsilon_{bdef} + g_{ad}\varepsilon_{bcef}\\ &+g_{bc}\varepsilon_{adef} - g_{bd}\varepsilon_{acef} + g_{cd}\varepsilon_{abef} + g_{ef}\varepsilon_{abcd}.\end{aligned}$$

The Interconnection Among the Formulas

The difference on the integrated amplitudes, even if we segregated the divergent parts in surface terms, can occur due to how indices of the trace of six matrices are distributed in a particular expression. The mass terms are unique and finite, hence we focused in the leading trace. Our method of comparison is to take all expressions above and subtract from the basic versions t_1, t_2, t_3, and their pair-wise combinations. Then the result is multiplied by $K_{123}^{\nu_{123}}/D_{123}$, which the factor appearing in the amplitude integrands. In this manner we will discover that either they identically vanish, because in the level of traces are identical

$$\begin{aligned}\left[t_{12} - \mathrm{tr}^{(2)}\left(\mu_1\nu_1\right)\right] &= 0\\ \left[t_{13} - \mathrm{tr}^{(4)}\left(\mu_1\nu_1\mu_2\nu_2\right)\right] &= 0,\end{aligned}$$

or will vanish because the difference, when integrated, corresponds to finite null integrals.

$$\begin{aligned}
\left[t_1 - \mathrm{tr}^{(1)}\left(\mu_1\right)\right]\frac{K_{123}^{\nu_{123}}}{D_{123}} &= \varepsilon_{\mu_2\mu_3\nu_1\nu_2}t_{\mu_1}^{(-+)\nu_{12}} + g_{\mu_1\mu_2}t_{\mu_3}^{ASS} - g_{\mu_1\mu_3}t_{\mu_2}^{ASS}\\
\left[t_{12} + \mathrm{tr}^{(2)}\left(\nu_1\mu_2\right)\right]\frac{K_{123}^{\nu_{123}}}{D_{123}} &= -\varepsilon_{\mu_2\mu_3\nu_1\nu_2}t_{\mu_1}^{(-+)\nu_{12}} + \varepsilon_{\mu_1\mu_3\nu_1\nu_2}t_{\mu_2}^{(-+)\nu_{12}} - g_{\mu_2\mu_3}t_{\mu_1}^{ASS} + g_{\mu_1\mu_3}t_{\mu_2}^{ASS}\\
\left[t_{12} - \mathrm{tr}^{(3)}\left(\mu_1\nu_1\mu_2\right)\right]\frac{K_{123}^{\nu_{123}}}{D_{123}} &= +\varepsilon_{\mu_1\mu_3\nu_1\nu_2}t_{\mu_2}^{(-+)\nu_{12}} + g_{\mu_1\mu_2}t_{\mu_3}^{ASS} - g_{\mu_2\mu_3}t_{\mu_1}^{ASS}\\
\left[t_{13} + \mathrm{tr}^{(4)}\left(\nu_1\mu_2\nu_2\mu_3\right)\right]\frac{K_{123}^{\nu_{123}}}{D_{123}} &= -\varepsilon_{\mu_1\mu_2\nu_1\nu_2}t_{\mu_3}^{(-+)\nu_{12}} - \varepsilon_{\mu_2\mu_3\nu_1\nu_2}t_{\mu_1}^{(-+)\nu_{12}} + g_{\mu_2\mu_3}t_{\mu_1}^{ASS} - g_{\mu_1\mu_2}t_{\mu_3}^{ASS}
\end{aligned}$$

Observation: some relations appear as a sum since the number of permutations before the chiral matrix reaches the position to apply the specific identity is odd. The second relation above is one example.

In section (3.7) and chapter 5, it was proved that the well-defined integrals for $\varepsilon_{\alpha\beta\nu_1\nu_2}T_{\rho}^{(-+)\nu_{12}}$ (5.12), and T_{μ}^{ASS} (3.289), are null. Therefore, for example, we have

$$\int_{\mathbb{R}^4}\mathrm{d}k\left\{\left[t_{12} - \mathrm{tr}^{(3)}\left(\mu_1\nu_1\mu_2\right)\right]\frac{K_{123}^{\nu_{123}}}{D_{123}}\right\} = \varepsilon_{\mu_1\mu_3\nu_1\nu_2}T_{\mu_2}^{(-+)\nu_{12}} + g_{\mu_1\mu_2}T_{\mu_3}^{ASS} - g_{\mu_2\mu_3}T_{\mu_1}^{ASS} \equiv 0.$$

The conclusion we have arrived at is that any form of substitution or manipulation is accounted by the linear combination of version one, two, or three replacing the definition of γ_* left or right of the matrices γ_{μ_1}, γ_{μ_2}, and γ_{μ_3}. What we showed here is the forms that identically correspond, not that all differences are finite and vanishing. The form obtained from t_{12} is not identical, without conditions, to any t_i, neither is $\mathrm{tr}^{(4)}\left(\nu_1\mu_2\nu_2\mu_3\right)$ automatically related to t_{12}. Furthermore, integrals that start with the combination $t_{123} = 1/3\left(t_1 + t_2 + t_3\right)$ are not reproduced by any identity that we began with.

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