Centro Brasileiro de Pesquisas Físicas
Programa de Doutorado

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Wealth Dynamics: A Model Integrating Information and Risk in Agent-Based Systems

Rio de Janeiro/RJ

2024

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Tese de Doutorado submetida a coordenação de formação científica do Centro Brasileiro de Pesquisas Físicas como parte dos requisitos para a obtenção do título de Doutor em Física.
Orientador: Evaldo M. F. Curado

Rio de Janeiro/RJ

2024

"WEALTH DYNAMICS: A MODEL INTEGRATING INFORMATION AND RISK IN AGENT-BASED SYSTEMS"

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Tese de Doutorado em Física apresentada no Centro Brasileiro de Pesquisas Físicas do Ministério da Ciência Tecnologia e Inovação. Fazendo parte da banca examinadora os seguintes professores:

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Rio de Janeiro, 06 de junho de 2024.

Resumo

Nós propomos um novo modelo baseado em agentes para estudar a distribuição de riqueza, onde nós demonstramos que os vínculos entre riqueza, informação (conectividade e comércio entre agentes) e vantagem comercial (risco) são chave para reproduzir qualitativamente distribuições reais de riqueza, assim como sua evolução ao longo do tempo e distribuições de equilíbrio. Essas distribuições são apresentadas em quatro cenários, com dois esquemas diferentes de tributação nos quais, em cada cenário, apenas um dos esquemas de tributação é aplicado. Em geral, o estado de equilibrio final é de concentração extrema de riqueza, a qual pode ser combatida com impostos apropriados.

Palavras-chave: Modelos baseados em agentes; economia; informação; imposto; sistemas complexos; distribuição de renda, riqueza, desigualdade

Abstract

We propose a new agent-based model for studying wealth distribution. We show that a model that links wealth to information (connectivity and trade among agents) and to trade advantage (risk) is able to qualitatively reproduce real wealth distributions, as well as their evolution over time and equilibrium distributions. These distributions are shown in four scenarios, with two different taxation schemes where, in each scenario, only one of the taxation schemes is applied. In general, the evolving end state is one of extreme wealth concentration, which can be counteracted with an appropriate taxation.

Key-words: agent-based; economy; information; taxation; complex systems; income distributions, wealth, inequality

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1 Introduction

1.1 A Brief History

1.1.1 Wealth Distribution

The study of wealth distributions has its origins in the late 19th century when Vilfredo Pareto [1] examined the distribution of land in Italy - a proxy for wealth at the time. Pareto's investigation revealed that for individuals with higher[1] wealth, this distribution followed a power-law pattern characterized by an exponent α.

This significant discovery led Pareto, and other scholars, to wonder if such phenomenon was something unique to Italy or something universal. Hence, Pareto sought to extend his research to other countries in Europe, and found that this distribution pattern could be seen in every other European country where data were available. He later estimated that Europe, as a whole, had an exponent of approximately $\alpha \approx \frac{3}{2}$ [2, 3].

The probability density function, $P(x)$, associated with Pareto distribution can be written, in general form, as

$$P(x) = \begin{cases} F(x) & \textit{for } x < x_c \\ \frac{\lambda}{x^{\alpha+1}} & \textit{for } x \geq x_c \textit{ where } \lambda > 0 \textit{ is a parameter,} \end{cases} \tag{1.1}$$

where for $x < x_c$ (x can be land, money, income, etc., in general, any kind of wealth), we have a function $F(x)$, but for $x > x_c$, a power law appears, having a typical exponent α. The Pareto distribution, $\mathcal{P}(x)$, is usually associated with the complementary cumulative distribution function for the higher values of x, i.e.,

$$\mathcal{P}(x) = \int_x^\infty dx' P(x')\,. \tag{1.2}$$

This research sparked the study not only of wealth but also income distributions and prompted many economists, who continued its research, to believe that with sufficient data, a similar pattern would emerge in most, if not all, other countries worldwide.

In the following decades, however, as data became more readily available, the nature of this distribution underwent significant transformations. Notably, the 20th century witnessed the emergence of a robust middle class in certain countries, largely due to the impact of the industrial revolution, wars, and hyperinflation that reshaped the global economy [4]. Furthermore, this distribution was found to also describe not only wealth, which was Pareto's initial subject (land), but also income distributions [5, 6].

[1] At the time, data were only available for large owners.

Hence, when talking about the shape of the distribution we will use income and wealth interchangeably in this thesis.

Interestingly, despite these changes, the Pareto distribution remained the default description for the tail end of this distribution, while the exponential distribution was found to be more suitable for characterizing the broader segment of society, encompassing the poor and the middle class. This distinction, as well as the similarity between income and wealth, is evident in the income distribution for the US and the wealth distribution for the UK, as illustrated in Figure 1 - for a more current example see [7] (Brazil: 2001 to 2014).

Now, if we consider the persistent relevance of Pareto's distribution in describing the wealthier segments of society across time, we must then reconcile the discovery of the exponential portion of the distribution. To that effect, many economists often argue that it was the industrial revolution that started the creation of the middle class. However, we must also consider that when Pareto first studied this issue very little data were available for the broader population. Therefore, it might just have been a case of data scarcity. Regardless of which side of this is true, the persistence of Pareto's distribution as the best description for the tail across time speaks to its robustness and stability.

This analysis, however, often overlooks, as we will demonstrate in the following subsection 1.1.2, that the same period marked by enormous economic growth and instrumental in shaping this distribution was also witness to numerous cultural and historical revolutions which also played a role in it. Hence, simply stating that this distribution was always true or that the industrial revolution is to blame is, at least partially, incorrect.

Nonetheless, regardless of its stability or how it came to be, over the 20th century this analysis was done to multiple countries[2] and this distribution was found to be widespread, as was expected. Hence, it is reasonable to infer that it must be rooted in very fundamental trade behaviors that are common to all countries worldwide. After all, it transcends geographical regions, cultures, religions, etc.

[2] We give further evidence to that claim in subsection 1.2.1

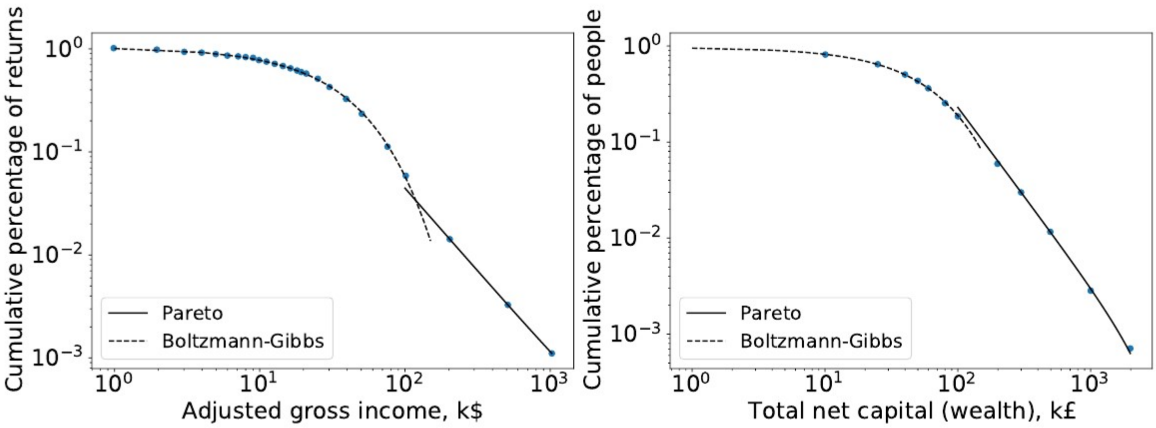

Figure 1 – The cumulative probability distribution of net wealth in the US (**left**, 1997) and UK (**right**, 1996) shown in log–log scales. Points represent data from the IRS/HMRC, and solid lines are the fitted lines to the exponential and power-law (Pareto)[2].

1.1.2 The Middle Class

The 20th century witnessed a multitude of economic and cultural shifts that have shaped the contemporary world. Amid these transformations, several stand out, particularly the heightened industrialization experienced by individual nations. Alongside a more seamlessly interconnected global economy, this synergy started a rapid and expansive worldwide growth. Frequently, this economic surge is cited as the principal drive behind the rise — and often the inception — of the middle class. However, it is crucial to recognize that both the 20th and 19th centuries were also arenas of numerous political and cultural revolutions that often remain marginalized in discourse. For a full comprehension of the evolution and historical trajectory of wealth distribution, these political and cultural shifts carry paramount significance alongside the industrial revolution itself.

1.1.3 Nineteenth Century: Growth and Wage Stagnation

In the early 19th century, many countries undergoing the industrial revolution also experienced a significant population boom due to rising agricultural productivity. These two factors combined resulted in a unprecedented population growth as well as a surplus of rural labor. Consequently, as the demand for industrial work increased, a large number of rural workers migrated to urban centers with their families in search of employment, sparking one of history's most substantial rural-to-urban migrations.

However, life within the city frequently unfolded in a harsh and demanding manner for the workers, even though employment opportunities were numerous. The routine was marked by extended work hours, the grim specter of child labor, unsanitary working conditions, and the stark reality of overcrowded apartments, often shared by multiple generations of families. All of these distressing conditions persisted, regardless of the rapid economic growth. To provide some context, if we look at the period from 1700 to 1820, we find that global production, when adjusted for inflation, experienced a rather modest annual growth rate of approximately 0.5%. However, a significant shift occurred from 1820 to 1913, where this global growth rate surged to approximately 1.5% per year – a threefold increase compared to the preceding century[8]. It is important to note that this statistic pertains to global production, predominantly driven by the waves of industrialization during that era. Consequently, it is evident that individual countries at the center of the industrial revolution witnessed considerably more substantial economic growth.

Despite the remarkable pace of economic expansion during this period, wages for the majority of workers largely remained the same. Nonetheless, the late 19th century marked a crucial turning point after decades of intensive industrialization and a substantial increase in the factory workforce. The sheer size of these factory populations made them increasingly challenging to control, which allowed for the emergence of social movements advocating for improved wages and working conditions.

It was only at this juncture, often in the wake of significant labor unrest and riots, that workers began to secure wage increases, despite still inadequate for addressing the prevailing economic inequality of the era [8]. Additionally, it is worth highlighting that this wage growth only marginally kept pace with the overall economic expansion that characterized this period. Hence, these movements marked the initial signs that economic growth alone could not effectively diminish inequality or foster the growth of the middle class.

1.1.4 Capital Returns and Inflation

Inflation was nearly non-existent prior to the last decades of the 19th century. While sporadic instances of inflation occurred, often due to a bad harvest, they were often followed by periods of deflation once supply and demand equilibrium were restored. This meant that, on average, inflation was absent for almost two centuries, spanning from the early 18th century to the late 19th century [8]. Throughout this extended timeframe, global economic growth was modest, typically below 0.5% annually, and wages remained static. In contrast, government bonds consistently yielded interest rates between 2% and 5%, while investments in foreign enterprises and debt returned around 5% on average, based on the most reliable estimates [8].

These returns, combined with the prolonged absence of inflation, enabled the upper class to lead a leisurely life without reliance on work [8]. This trend reached its pinnacle during the Belle Époque (France, 1870-1913), a period of conspicuous abundance, ceaseless festivities, and minimal worries, often depicted in literature and plays. Remarkably, this era boasted the highest recorded levels of inequality prior to the 21st century.

It was not until the early 20th century that inflation became commonplace, eroding the wealth of the upper class. Rising inflation rendered interest rates inadequate for sustaining the affluent lifestyle, consequently narrowing inequality as political movements and unions advocated for wage adjustments. For instance, France endured a consistent annual inflation rate of 13% for over 45 years, spanning from 1913 to 1950 [8]. This, combined with a substantial decrease in capital returns[3], steadily diminished the elites' amassed wealth. This transformation is illustrated in Figure 2.

Once again, this underscores that economic growth, while necessary to counterbalance inflation, fell short of effectively addressing inequality and wealth redistribution. It took more than a century for workers' conditions to ameliorate, and if not for inflation's impact on capital returns, the wealth gap between the elites, the less affluent, and the emerging middle class would likely have persisted. This elucidates that the issue of inequality is not solely tied to wages, but intrinsically linked to capital returns and inflation.

1.1.5 Twentieth Century: Wealth Destruction, Wars and Taxation

The 20th century was a period marked by transformative global events. Within a short span, World War I raged for four years (1914-1918), followed swiftly by the Russian Revolution (1917-1923) and the devastating Great Depression in 1929. Merely six years after post-war Germany, struggling with hyperinflation, had begun its recovery, World War II erupted in 1939. These tumultuous times also witnessed the Chinese Communist Revolution, the Cold War, the Vietnam War, and the Soviet-Afghan War, among others. Consequently, given how many historical events happened in such a short amount of time, discussions about this era often overlook the economic aspects and the financial underpinnings of the period. For instance, how did the world afford and financed so many wars? The answer? Money printing.

During the 18th century, countries adhered to monetary policies tied to gold and silver standards. Governments assigned a fixed value in gold/silver to their currency, maintaining an unalterable relationship. However, this standard worked only when central banks could guarantee this exchange by directly converting gold/silver to currency. This required governments to hold gold reserves, in order to honor this exchange. Consequently, for governments to enact monetary policy — increasing or decreasing the money supply — they had to either accumulate or deplete gold reserves. This hindered countries from swiftly printing money, and thus creating inflation, to counter recessions, manage debt, or rescue banks [4].

Hence, given the cost of World War I, Germany departed from the gold standard

[3]Capital returns is what an investment of capital yields (interest), whether it is in government bonds, corporate debt, stocks, etc.

in 1914, which allowed them to print money without the need for a gold counterpart. Therefore, it was no longer necessary to first increase gold reserves, nor was it necessary for banks to directly exchange currency for gold.

Later in 1929, the Great Depression led many others to follow suit, with the USA, the UK, and others, also abandoning[4] the gold standard in 1931[4].

With the gold standard no longer restricting them and economic downturns stemming from the depression, most countries turned to high rates of money printing, a trend that intensified during World War II. Therefore, because inflation rates in countries are primarily influenced by factors such as the amount of money being printed, the level of debt, and the overall economic output, this period was marked by annual inflation rates consistently exceeding 10% for many years. For instance, as mentioned before, France experienced an average annual inflation rate of 13% from 1913 to 1950, while Germany averaged 17%. Although seemingly small, this led to prices in France increasing by a factor of 100, while in Germany, they surged by a factor of 300 [8].

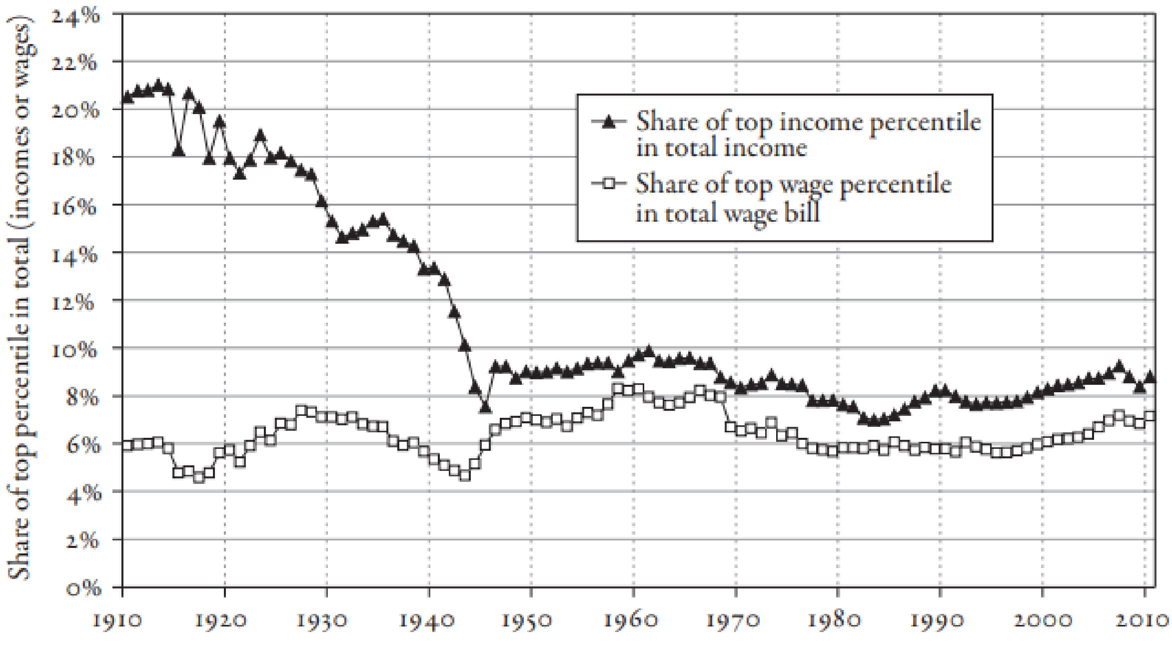

Figure 2 – France: Evolution of total salary and total income from 1910-2010 for the top 1% of the population. Reproduced from Capital in the Twentieth First Century[8]

High inflation periods, such as these, which disproportionately burden the working class, also erode capital wealth significantly. For instance, when Germany faced inflation rates exceeding 20,000% annually during the 1920s [4], the country's capital wealth disappeared. It was only in 1923, with the introduction of the Rentenmark, that inflation finally came to a halt. Additionally, during these times of war and crisis, many countries

[4]Note: The gold standard was later reinstated as part of the Bretton Woods system, which was ended by the countries over the years. The USA, for instance, ended it in 1976.

resorted to imposing steep taxation policies, which is another factor often overlooked in discussions about inequality.

In the USA, for example, during the 1940s and 1950s, income tax rates not only surged to 81% (for incomes exceeding the equivalent of about 1 million today) but the USA also enacted excess profit taxes [9] to fight war profiteering. While the purpose of these taxes is arguable—whether to fund the cost of war or redistribute wealth—this period witnessed remarkable reductions in inequality. These reductions continued until the 1980s when Ronald Reagan's presidency shifted government tax policies towards a more hands-off, free-market approach. This brought substantial cuts to income and capital gains taxes [9], fueling rapid growth in inequality. By the 1990s, the top 10% of earners, who previously represented around $32\% - 35\%$ of the country's income from the 1950s to the 1980s, claimed nearly 40% of the income. In 2010, this share had escalated to nearly 50% [10]. It is, however, worth pointing out that this change also lead one of the largest periods of sustained growth of all time. Since then, the US has grown almost 9 times, averaging a growth of about $5,3\%$ annually [11].

1.1.6 The Rise of Inequality

Similar to the Reagan era, the 1970s and 1980s marked a significant shift in economic policies across developed nations. As the memories and traumas of war receded, consensus around public welfare and the common good began to fray. Fresh leadership and ideologies emerged, advocating for a transition away from Keynesian economics and towards freer markets. This change entailed reduced government expenditure, welfare provisions, and tax rates.

By this juncture, it is unsurprising that the capital losses suffered by the wealthy elite over the past seven decades had largely been recuperated. Inflation had stabilized, and the adoption of new free-market economic models facilitated a resurgence in capital returns. Consequently, inequality once again started to rise.

For instance, in the USA, just before the Great Depression in 1929, capital returns constituted more than 80% of the income for the top 0.01% earners, see Figure 3 - which mirrored the circumstances in France during the Belle Époque.

This share, however, as mentioned in the previous section and seen in Figure 4, kept diminishing during most of the 20th century until the adoption of free-market economics in the 1980s prompted a steady escalation in inequality. By 2010, inequality had already surpassed pre-war levels, and if this trend persists, the top 10% of earners could control 60% of the nation's income by 2030 [8].

A similar trajectory unfolded in the UK with Margaret Thatcher's policies, albeit on a smaller scale. This trajectory contrasts with France, which retained many of the

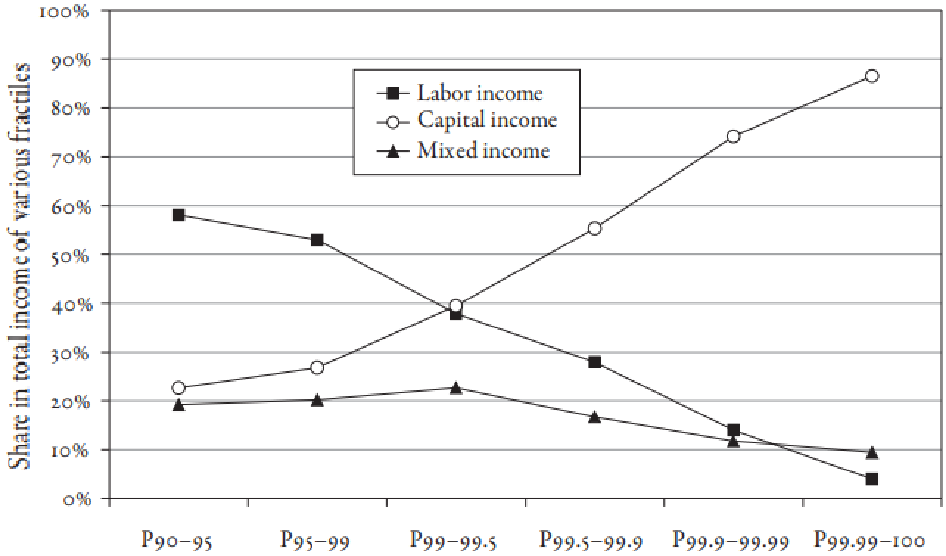

Figure 3 – USA - 1929: Composition of earnings by income percentiles. Notice how the richest get most of their income from capital. Reproduced from Capital in the Twentieth First Century [8]

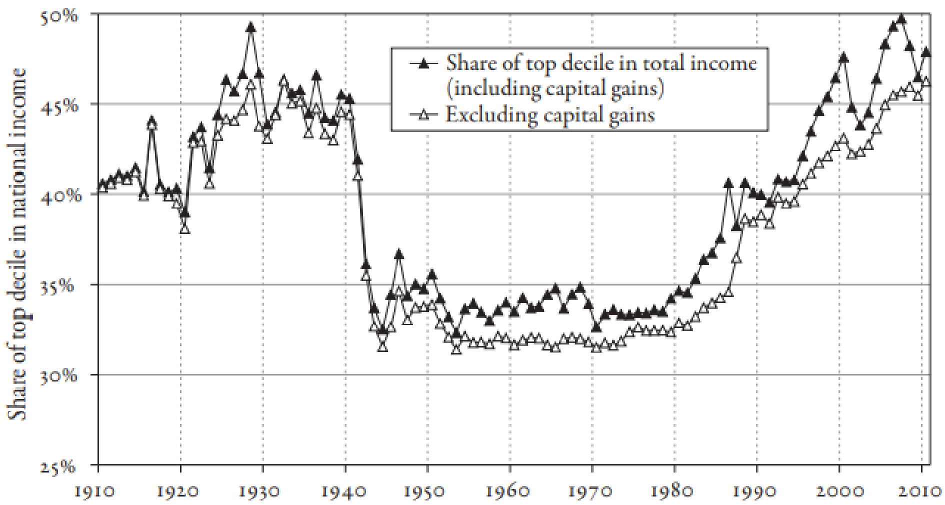

Figure 4 – USA: Evolution of total salary and total income 1910-2010 for the top 10%. Reproduced from Capital in the Twentieth First Century [8]

economic features from the first half of the century, including indexing minimum wages to inflation and economic growth, a robust welfare state, and progressive income and inheritance taxes (as depicted in Figure 5). While the USA has been consistently boosting the income share of the top 10% by nearly 5% per decade, France managed to decrease its peak from 37% in 1966 to a stable 33% throughout the century.

This divergence between countries illustrates that once inflation abated and post-war damages were repaired, and in the absence of effective inequality-controlling economic policies, these economies regressed to their previous state. It underscores that a free-market economic approach invariably leads to pronounced concentration of wealth, even though

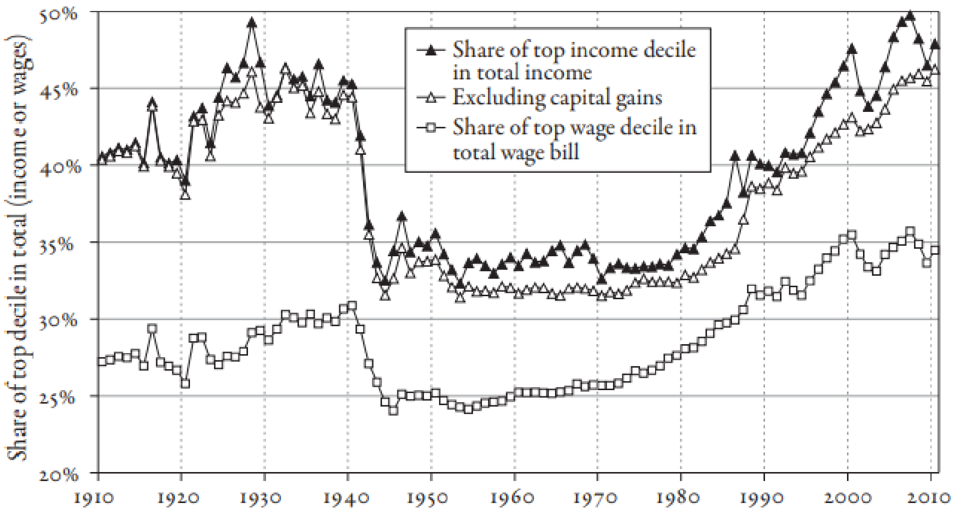

Figure 5 – France: Evolution of total salary and total income from 1910-2010 for the top 10% of the population. Reproduced from Capital in the Twentieth First Century[8]

the pace of this process might vary.

1.1.7 Conclusion

To sum up, the considerable decline in inequality throughout the 20th century was not solely the outcome of rapid economic expansion. Rather, the driving force behind inequality reduction stemmed from persistent economic upheavals driven by wars and frequent economic downturns, which reduced capital returns and fostered inflation. In these instances, due to the size of factory work forces, unions and social movements managed to get lower wages swiftly reevaluated and readjusted. This, coupled with diminished capital returns, exerted pressure on the upper classes of society, resulting in a reduction of inequality. However, this was not achieved by redistributing wealth but by eroding the affluence (and rate of return) of the elite class.

Consequently, while economic growth was essential to navigate the challenges of those tumultuous decades, it is inaccurate to attribute the reduction in inequality solely to it. Furthermore, examining countries less affected by wars, like the USA, underscores the influence of economic policies in relatively stable contexts. High upper-income and capital gains taxes successfully sustained years of economic growth without a significant rise in inequality. In contrast, the application of free-market or trickle-down economics had the opposite effect.

In essence, the intricate interplay of economic shifts, policy decisions, and external shocks—such as wars and recessions—shaped the trajectory of inequality, revealing that a nuanced combination of factors, beyond economic growth alone, played a pivotal role in reshaping the distribution of wealth and income.

1.2 The Study of Inequality

In the preceding section 1.1, we discussed the complexity of the inequality problem, which has prompted both statistical and theoretical investigations into its dynamics and origins due to its widespread and rapidly changing nature. In this section, we will review key studies to provide a current overview of the field. Our goal is to gather the essential components for constructing a model that replicates real-world wealth and income distributions.

1.2.1 Empirical Evidence

In previous sections we have mentioned that the current shape of wealth/income distributions are widespread, which poses the question: if cultures, economies and history can differ so much between countries how can its income distributions be so similar in shape? In this section we aim to provide sufficient data to validate this assertion.

In 2001, Dragulescu and Yakovenko [5] analyzed the income and wealth distributions of the UK and the income of the USA and its individual states for the years between 1994 and 1998. In their work they found that, as can be seen in Figure 6, both wealth and income for the UK follow very similar patterns and they both have the same shape. In other words, they are both described by a exponential distribution for the middle and lower class and a Pareto tail for the upper class - from this point forward we will call this a Exponential-Pareto tailed distribution. Their analysis also found a Pareto exponent α between 2.0 and 2.3 for the UK and the same shape was found for the income of US's individual states[5], as can be seen in Figure 7.

[5] With an $\alpha = [1.6, 1.8]$ for the USA

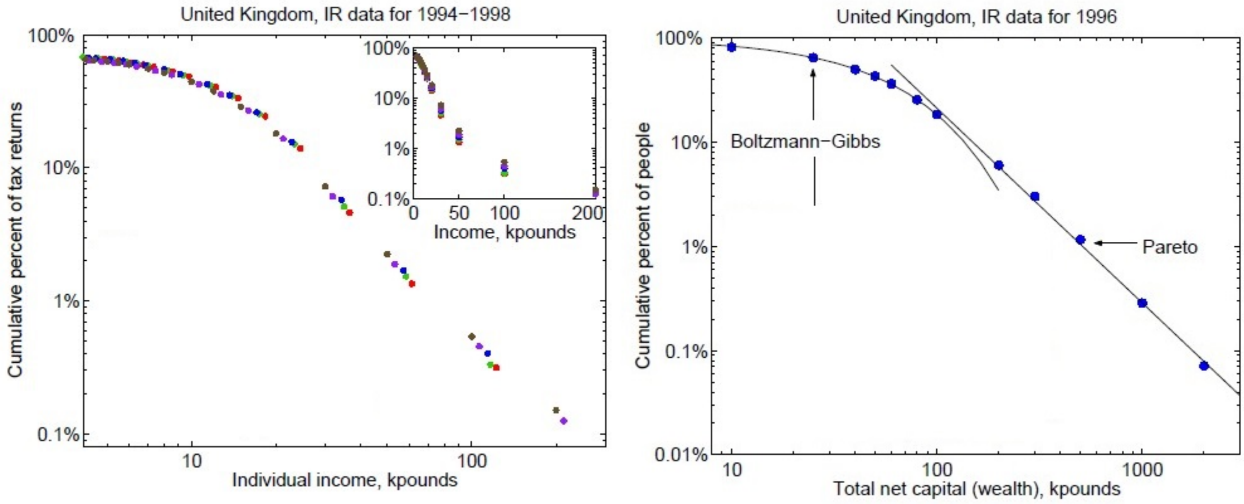

Figure 6 – Cumulative probability distribution of wealth (right panel) and cumulative percentage of tax returns by income (left panel). Both in log-log and by in thousands of pounds form for the UK. Left inset: Distribution in log-linear form. Reproduced from [5].

Clementi and Gellati [12], on the other hand, analyzed the period between 1991 and 2001 for not only the income distributions of the UK and the USA but also that of Germany. And they, once again, found the same shape pointing out that:

> Our analysis of the data for the US, the UK, and Germany shows that there are two regimes in the income distribution. For the low-middle class up to approximately 97% − 99% of the total population the incomes are well described by a two-parameter lognormal distribution, while the incomes of the top 1% − 3% are described by a power law (Pareto) distribution.

They also calculated the Pareto exponents for the period analyzed: US with $\alpha = [1.1, 3.34]$, UK with $\alpha = [3.47, 5.76]$ and Germany with $\alpha = [2.42, 3.96]$. Note how they substantially differ from the ones found by Dragulescu and Yakovenko, evidence that even though the shape might be the same, the distribution changes over time.

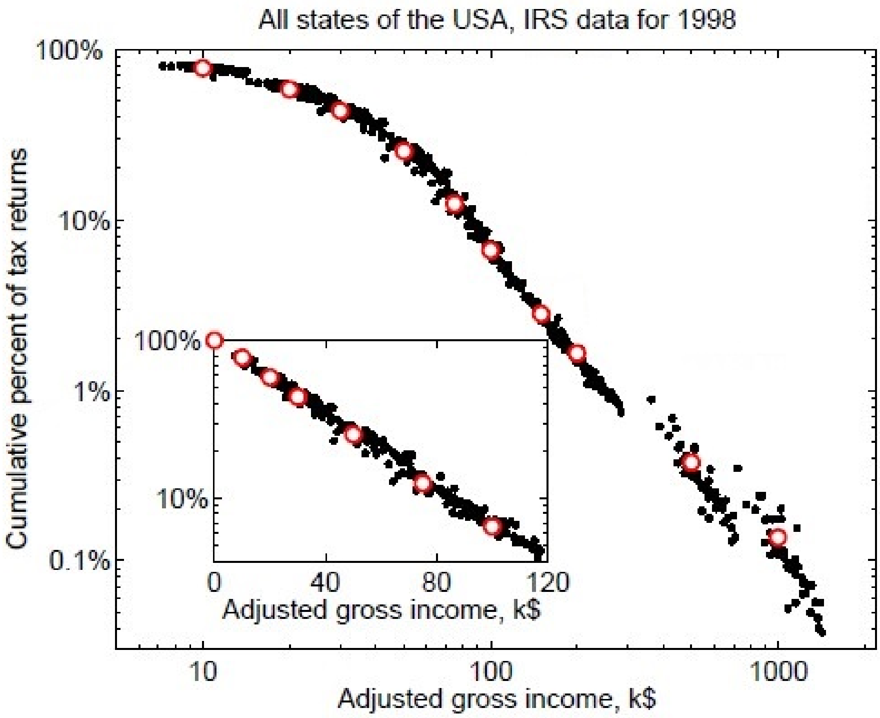

Figure 7 – Cumulative percentage of tax returns by adjusted gross income in thousands of dollars of the USA (all states) in log-log form. Inset: Log-linear for adjusted gross income between 0 and 120 thousand dollars. Reproduced from [5].

Furthermore, Tragtenberg and Siciliani [7] analyzed Brazil's income distribution between the years 2001 and 2014 and they also found the same shape, this time with an $\alpha = [2.17, 2.192]$. Similarly, Aoyama et al[13] analyzed Japan's distribution tail and, once again, found a Pareto power law with an $\alpha = [1.98, 2.06]$. Hence, with just a handful of papers we have evidence of this shape for five different countries (USA, UK, Germany, Japan and Brazil) across various years.

Nevertheless, it is important to note that there are good competing arguments about the best description of the shape of the income/wealth distribution. For example, C. Tsallis, C. Anteneodo and Sílvio M. Duarte Queiros[14] argued that, given how additive-multiplicative stochastic processes, like the economy, are at the core of nonextensive statistical mechanics [15] their naturally occurring gaps[6] could be bridged by q-exponential distributions. This idea is particularly important since, if true, it means we might be able to reproduce the distribution we are looking for with a single process that is closely linked to the q-exponential.

Later, in 2016, Abner D. Soares, N.J. Moura Jr. and Marcelo B. Ribeiro gave evidence in favor of this hypothesis by showing [16] how Brazil's income distribution, from 1978 through 2014, could be well described by a single q-exponential distribution, thus strengthening the authors' argument. In another paper [17], the authors also explored

[6] When distributions cannot be fitted by a single function and/or some of its fits are not convincing.

how taking the complex form of the q-parameter could explain periodic (over time) behavior found within these distributions, which gives further evidence of the power of the q-exponential to analyze it.

However, it is worth pointing out that the poor and middle class of this distribution (the exponential part) usually appears within just one order of magnitude, from the top 100% to the bottom 10% of the distribution. Hence, many functions are able to fit within this range. Further evidence of this can be seen in the work of Marcelo Byrro and Newton Moura in [18], where they find the Gompertz curve in the income distribution of Brazil from 1978 to 2005. And in Bernarjee, Yakovenko and Di Matteo's work in [19], where the authors also explore the Gamma and log-normal distributions.

With all that in mind, given how widespread this distribution's shape is and how economically different the countries presented here are, it is reasonable to conclude that its main drivers probably are very fundamental characteristics of trades and business interactions, since despite economic, cultural, geographic and historical differences this shape is widespread. Nonetheless, these studies also make it quite clear that the precise definition - the distributions parameters - is an ever changing phenomena and not a static fact. Hence, there is basis to think this shape will not remain the best description forever. In fact, Aoyama points out in his work[13], when analyzing the rank-size plot of the income of Japan, that in certain circumstances a single Pareto tail is not enough to model the top of the income distribution. As can be seen in Figure 8, there is, what appears like, a second Pareto tail in the upper parts of the distribution. Which hints at the fact that with enough data and time, we can differentiate between the rich too. With some much richer than others.

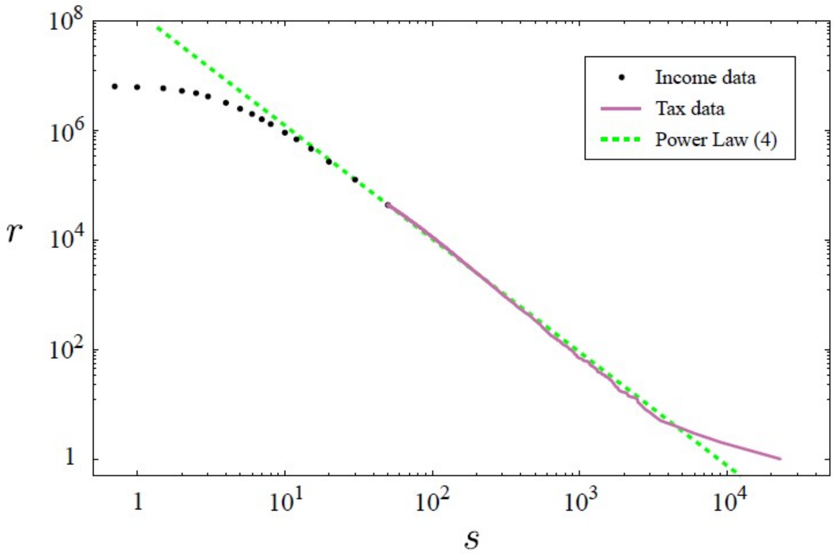

Figure 8 – The rank-size plot of the income and tax for Japan in 1998. Reproduced from Ayoama et al[5].

1.2.2 Agent-based Models

In the domain of theoretical exploration, agent-based models have arisen as a valuable instrument, providing insights that, in comparison, exhibit a closer resemblance to real-world dynamics. An illustrative case in point is the work carried out by Chatterjee and Chakrabarti, as documented in their research papers [20] and [21]. In these studies, they investigated how enhancing a basic conservative model, characterized by exchanges between agents where the total money does not change over time $(m_i(t) + m_j(t) = m_i(t+1) + m_j(t+1))$ and where the occurrence of debt is prohibited, would change its equilibrium states. Their research trajectory led them to progressively introduce higher levels of complexity, and illustrate how each modification influenced the result of the system.

In their initial study, exchanges among agents occurred randomly, characterized by a process in which a fractional amount Δm, Equation (1.3), of money was exchanged. This dynamics gave rise to a stable equilibrium state, specifically, one that aligns with the Boltzmann-Gibbs distribution, as expected. After all, this, in essence, remains a perfect gas model.

$$\Delta m = \epsilon(m_i(t) + m_j(t)) - m_i(t) \quad where \quad \epsilon \in [0, 1] \,. \tag{1.3}$$

Subsequently, the authors introduced a uniform saving parameter λ in Equation (1.3), which lead to the exchanged amount seen in Equation (1.4). This parameter represents the agent's propensity to save and, consequently, altered the amounts exchanged in transactions. This adjustment had the effect of transforming the distribution of wealth into a Gamma distribution. Notably, different values of λ resulted in distinct parameters for the Gamma distribution, highlighting the impact of this additional constraint in its shape.

$$\Delta m = (1 - \lambda)[\epsilon(m_i(t) + m_j(t)) - m_i(t)] \,. \tag{1.4}$$

To delve deeper into this added constraint, the authors extended their investigation by permitting the saving propensity to be distributed among individual agents. Consequently, each agent, denoted as i, was assigned a specific saving propensity parameter λ_i, following a distribution characterized by $\rho(\lambda)$. The results of their exploration revealed a consistent pattern: regardless of the specific shape of the $\rho(\lambda)$ distribution, the asymptotic behavior of the wealth distribution consistently conformed to a Pareto distribution. This finding aligns with the earlier work conducted by Chakraborti and Patriarca and documented in [22], which underscores the concept that a system composed of subsystems featuring different degrees of freedom, represented here by the saving propensity λ_i, tends to exhibit a Pareto power-law distribution, which is the hallmark of a multiplicative process. With this result we have the first link between varying degrees of freedom within a system and power-laws.

Building on these insights, the authors introduced an additional layer of complexity by incorporating non-consumable commodities into their model. They represented an agent's wealth as the sum of their money, denoted as m_i, and the value of the non-consumable commodities they possessed, represented by c_i. The price of these commodities was denoted as p_0, and it had a direct influence on an agent's wealth calculation ($w_i = m_i + p_0 c_i$). To account for the inherent uncertainty and the absence of bargaining power on the part of agents, the authors introduced stochastic fluctuations in commodity prices over time. Specifically, they modeled these price fluctuations as follows: $p(t) = p_0 \pm \delta$. Thus, they defined that when agents interact they exchange wealth according to equations:

$$\begin{aligned} c_i(t+1) &= c_i(t) + \frac{\Delta m_i}{p(t)} \\ c_j(t+1) &= c_j(t) - \frac{\Delta m_j}{p(t)} \,. \end{aligned} \tag{1.5}$$

Subsequently, they demonstrated that this model had the capacity to drive the steady-state wealth distribution towards a Gamma distribution when the saving propensity parameter λ was uniform, and towards a Pareto distribution when λ was distributed among agents. This observation exemplified a model that exhibited partial efficacy in bridging the gap between two contrasting approaches: one that predominantly modeled the broader segments of society, characterized by Boltzmann-Gibbs or Gamma-like features, and another that focused primarily on Pareto-like wealth concentration in the tails while disregarding the rest of the population. Therefore, by combining an stochastic process (the price fluctuation) and a system where different parts have different degrees of freedom, the authors have come closer, when compared to their initial attempts, to a real world representation, which is what we are looking for. It is also worth pointing out that the elements they explored are usually the default elements most econophysics models start from. Therefore, they are often a blueprint for new ones.

Another noteworthy contribution in this field came from Braunstein, Macri, and Iglesias, as detailed in [23]. In their work, the authors took a simple conservative model (Conservative Exchanges Market Model or CEMM), much like Chatterjee and Chakrabarti's original work, that was previously built on a nearest neighbor network and applied it to Erdos-Rényi and scale-free networks.

Scale-free networks are characterized by agents with varying degrees of connectivity that follow a power law. That is, the fraction $P(k)$ of the network that has k connections to other nodes is given by Equation (1.6).

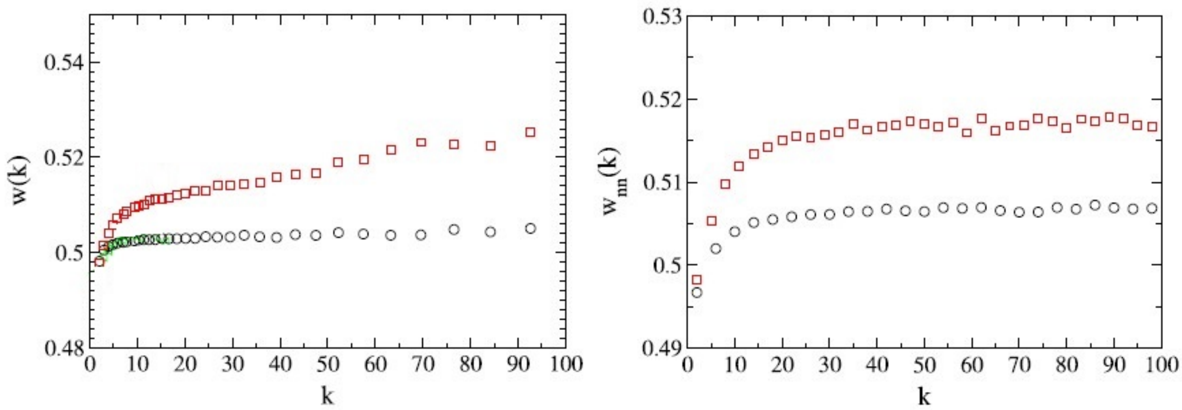

Figure 9 – Left side: Plot of the average wealth of a node of grade $k, w(k)$. Right side: Plot of the average wealth of the neighbors of a node with degree $k, w_{nn}(k)$. Both panels represent an scale free network with $\lambda = 2.5$ (black circles) $\lambda = 3.5$ (red squares). Notice that the neighbors of low connectivity nodes are in disgrace, as their average wealth is lower than the global one: 0.5. Reproduced from the original paper [23].

$$P(k) \sim k^{-\lambda}\,. \tag{1.6}$$

In this work, they found that connectivity and wealth have a very complex dynamics. Starting from a network that has average wealth $\langle w \rangle = 0.5$, the authors investigated how connectivity affected the wealth of agents/nodes across the network. In Figure 9, we reproduced one of the graphs on the paper. On the left side, the authors plot the average wealth of nodes with connectivity k, where we can see that agents with higher degrees of connectivity (higher k) are, on average, richer than agents with lower connectivity. Following suit, on the right side, they plot the average wealth of the neighboring nodes. Notice how, on average, they are also richer. This is an interesting result, since it matches common intuition: being close to rich and influential individuals increases chances of success.

What is more interesting though is that they also found that this "rich status"seems to be fleeting. Or, perhaps, very dependent on chance.

In Figure 10, they show on the y-axis the probability that an agent with connectivity k is the poorest in the network. Notice how agents with high degrees of connectivity (high values of k) are more likely to be the poorest agent in the network than the others. This, however, does not imply that a rich agent with high connectivity is constantly struggling to maintain their status, as the authors claim. A rich highly connected agent might, once rich, never fall. Meanwhile, a highly connected poor agent might suffer from the opposite but similar fate: It also never changes its status thus remaining poor. Hence, the difference between rich and poor agents with high connectivity might just be chance. The ones that gain wealth at the beginning of the simulation continue to gain and, therefore, become rich.

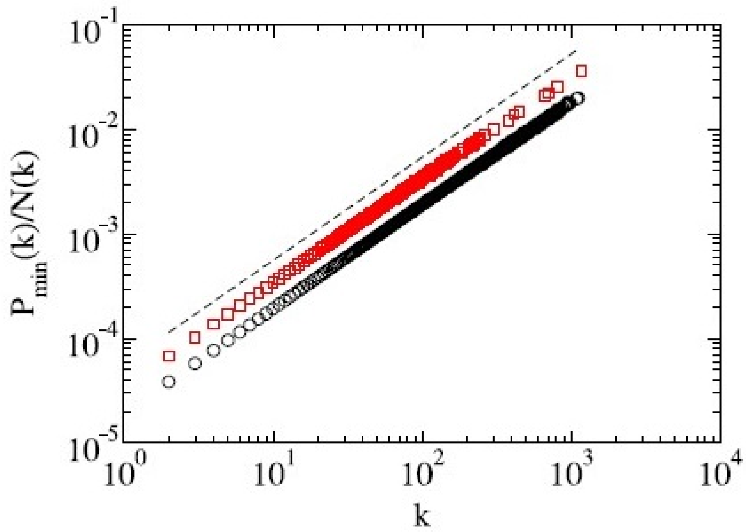

Figure 10 – Plot of the frequency any site of degree k is the poorest one divided by the number of sites of degree k and as a function of the degree. Black circles correspond to a network with exponent $\lambda = 2.5$ and red squares to $\lambda = 3.5$. The dashed line indicates the slope 1. This plot confirms that high degree nodes are the minimum wealth ones with higher frequency. Reproduced from the original paper [23].

While the ones that start losing tend to continue losing and become poor. This reinforces the previously mentioned result that chance seem to play a major role in these types of networks. However, regardless of what is precisely behind these results, what is interesting is that, even though this is quite counter intuitive, it also bares striking resemblance to the real world.

At the end, however, the different networks applied here made it so that the poverty line, here defined as the minimum wealth found in the network, was relatively healthy compared to other similar models. This result is also intuitive. Since the network is static it limits the reach of agents. Hence, once a rich agent has all or almost all of the wealth of his cluster he can grow no longer. Similarly, some agents will, simply by chance, end up in more isolated silos which will prevent them from dropping their wealth too low. Thus reducing poverty.

Furthermore, even though the model is simple and presents resulting distributions that are far from any real-world ones, as is often the case with conservative models, the effects of the network are still valid and should be kept in mind when modeling this type of problem.

Later on, in another attempt to bring conservative models closer to reality, Iglesias and de Almeida have also proposed, in another work [24], a notable modification which further helps us understand the limitations of these conservative models.

Firstly, they modified the manner in which the exchanged amount Δw was deter-

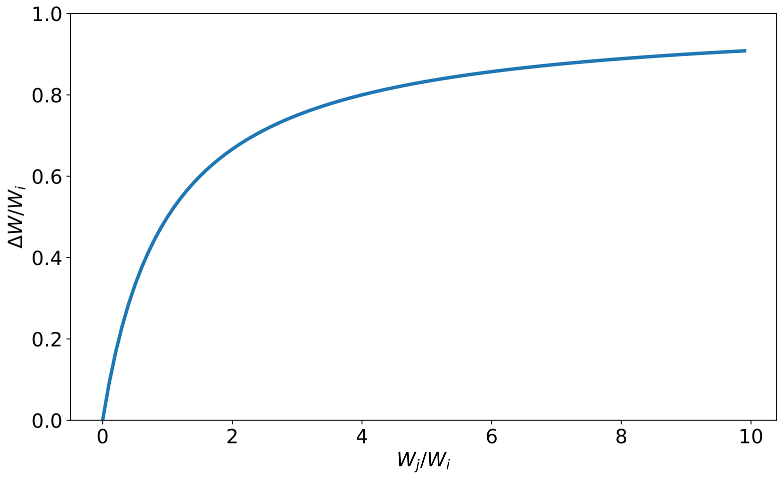

Figure 11 – Equation (1.7) rewritten as the relationship between how much larger w_j is, in reference to w_i, and how much more of w_i is available to trade.

mined. Unlike the fixed exchanges in some conservative models, Iglesias determined Δw as a combination of both agents' current wealth, as expressed in Equation (1.7). Notably, this modification meant that the wealth exchange was influenced by the relative wealth of the two agents involved in the transaction. Specifically, the richer an agent was compared to the other, the larger the portion of the poorer agent's wealth that would be exchanged. This alteration is interesting because it resembles well a very common part of trade: the richer an agent is, the higher the investment it is able to make. Hence, it is able to define the price point in which he wants to operate, thus setting the initial price. As an example, it takes a lot more money to start a car manufacturing business, which is a good with a high initial price, than to start a baking business, which has a low initial price,

$$\Delta w = \frac{w' w''}{w' + w''} \,. \tag{1.7}$$

Furthermore, they introduced an element of stochasticity into the trading process by incorporating a winning probability factor that depended on the wealth of both agents and a parameter $\gamma \in [0, 1]$, as defined in Equation (1.8) and shown in Figure 12. Notice that this probability equation exhibits symmetry, ensuring that $P(w'|w'') + P(w''|w') = 1$. Additionally, as the wealth gap between the two agents increases, their respective winning probabilities shifts to favor the poorer agent. The degree of this advantage was, however, controlled by the parameter γ. Different values of γ resulted in varying degrees of advantage for the less affluent agent, with lower values of γ indicating a smaller advantage, and higher values amplifying it. In particular, when $\gamma = 1$, the model ensured that in cases where one agent possessed significantly more wealth than the other ($w' >> w''$), the probability of the wealthier agent winning the trade approached zero:

$$P_\gamma(w'|w'') = \frac{1}{2}[1 - \gamma \tanh{(w' - w'')}] \quad where \quad \gamma \in [0, 1] \,. \tag{1.8}$$

Here we would like to show how extreme and unrealistic this advantage is by pointing to Figure 12 where we present how this probability evolves with the relative difference in wealth of both agents. Notice how steep the growth is. Very quickly wealthier agents enter a situation where they always lose, which makes no sense economically speaking. Firstly, because trade is, more often than not, a value add to both parties. And secondly, because one would never enter a trade this unfavorable in the first place - except in very extreme circumstances.

Furthermore, their exploration of this model's properties revealed a consistent trend: it converged to a state where a single agent accumulated all the wealth, except when $\gamma = 1$. This specific scenario, where the least affluent agent consistently prevails in trades, highlights three common pitfalls inherent in conservative models:

1. Extreme condensation is common and the only mechanism to avoid it is to ensure, in some form, huge advantages to the poorer agent;
2. The necessary mechanisms to stabilize these models often have no economic basis;
3. These mechanisms have no negative side-effects;

It is important to note, however, that the presence of redistribution mechanisms in economic models should not be deemed inherently wrong. In fact, such mechanisms are not only crucial from a modeling perspective but also possess significant economic and moral justifications. The underlying concern, rather, revolves around the extreme degree of redistribution required to harmonize with conservative models.

To underscore these points further, Paulo Murilo's research, as documented in [25], shows how these redistribution mechanisms end up being imperative in conservative models like this, otherwise, in essence, they all rapidly collapse. Murilo's work emphasizes that taxation serves as a vital stabilizing force, crucial for sustaining conservative models past trivial solutions.

Furthermore, these results show that, despite their utility, conservative models possess inherent limitations which hinder their ability to fully align with the complexities of reality. After all, not only do they fail to reproduce real world distributions but they also fail to capture fundamental economic principles, such as the mutual value gain that occurs when two parties engage in trade and the economic growth that leads from it. These limitations underscore the ongoing need for refinement and expansion in modeling approaches to achieve a more comprehensive understanding of this economic phenomena.

Other similar results can be found in other works by Iglesias, Cardoso and Gonçalves in [26] and [27].

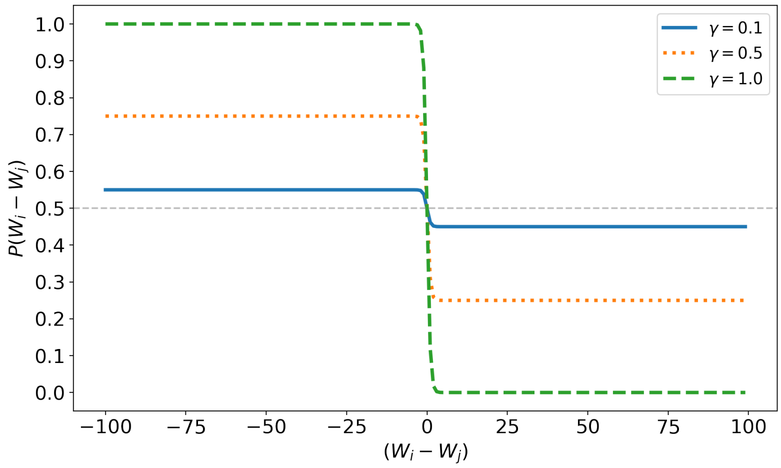

Figure 12 – Probability function (1.8) for different values of γ. Notice how quickly it grows to favor the poorest agent.

Despite these limitation, however, we can still learn from these models, much like when we analyzed the impact of networks on CEMM. And to that effect, another conservative model by the same group of researchers, Iglesias, Cardoso and Gonçalves, that is worth mentioning is [28].

In this work, the authors introduce a random network where agents interact and exchange a set amount of their wealth. The exchanged amount is given by a function of both agent's wealth (w_i and w_j) and their respective saving parameters (β_i and β_j). Here, the agents starting wealth and saving parameter is uniformly distributed in the interval $[0, 1]$ at the beginning of the simulation. Note that the saving parameter β_i does not change over the course of the simulation.

The exchanged amount is then given by Equation (1.9) and the winning probability is given by Equation (1.10). Exchanges are, obviously, conservative:

$$dw = min[(1 - \beta_i)w_i(t); (1 - \beta_j)w_j(t)] \tag{1.9}$$

$$P = \frac{1}{2} + f \times \frac{|w_i - w_j|}{w_i + w_j}, \tag{1.10}$$

note that $f \in [0, 0.5]$ controls how much the model favors the poorer agent in the exchange, with higher values of f favoring them more.

The results of this model are exactly what you would expect: $f = 0$ leads the system to a condensed state where one agent controls all the wealth. Increasing f, on the other hand, "redistributes" the wealth and reduces inequality, which they measure with a Gini index and the percentage of wealth held by the top 10% and top 1%. It also has no negative consequences. That being said, we want to draw particular attention to their introduction of a taxation and tax redistribution mechanism, which prompts interesting insights.

The authors then introduce a taxation parameter $\lambda \in [0, 1]$ that controls how much of the agents wealth is collected in taxes at the end of each Monte Carlo step. At $\lambda = 0.25$, for example, means 25% of every agent's wealth will be collected as tax. With this taxation scheme setup, they then analyze the best way to redistribute it by choosing which bottom percentage p of the population will receive taxes (targeted re-distribution). Hence, if $p = 1$ then all of the collected taxes will be equally divided among the agents and the network. On the other hand, if $p = 0.1$, the collected taxes will be equally divided among the bottom 10% of the population.

With this setup, they found that in order for one to optimize inequality reduction, different levels of taxation (λ) require different targeted re-distributions (p). This can be seem in Figure 13, where the authors plot the Gini index of the equilibrium states for different values of λ and p. This result is important firstly because it matches reality and common sense intuition: just collecting taxes is not enough, how to redistribute it is also important.

This result highlights that even in such simple models taxes are not a trivial matter. Furthermore, this is one of the few models where, simultaneously, equilibrium states depend of starting parameters (a common problem we talk about further in this section) and they are non-trivial. Hence, even in such simple models complexity can arise.

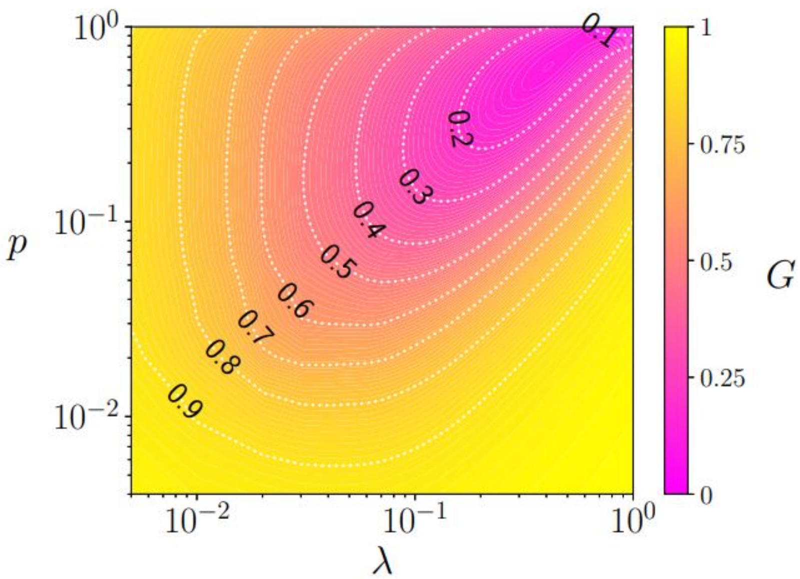

Figure 13 – Equilibrium Gini index as a function of λ and p (the bottom fraction of agents). Reproduced from [28].

To further explore and understand what we have seem so far, it is worth mentioning Dragulescu and Yakovenko's work in [29]. By looking at conservative models from a statistical mechanics point of view, the authors reverse what has been done with most of these models - which is to take gas-like models and exchange particles for agents - and evaluate, mathematically and computationally, what effects each model parameter has on its equilibrium states.

Their research indicate that within conservative systems devoid of what they term "external forces", such as robust redistribution mechanisms, equilibrium states tend to exhibit a remarkable resistance to alterations in model parameters. This insight accentuates the necessity of external interventions, such as taxation and redistribution, to steer these models toward outcomes that align more closely with empirical economic realities or that, at least, do not result in trivial equilibrium states. Which is why a new approach is necessary to study this phenomena. Furthermore, this result binds itself with Paulo Murilo's previously mentioned work [25], thus reinforcing its findings. Other noteworthy works on this subject by Scaffetta et al can be found in [30], [31] and [32].

1.2.3 Conclusion

In the preceding sections, we have emphasized the widespread prevalence of the exponential-Pareto tailed distribution. This distribution is found across nations with diverse cultures and economies, including Germany, Japan, the USA, the UK, and Brazil - with some competing evidence as to which description is best [18]. Our investigation has revealed that most attempts to model this distribution not only struggle to replicate its shape but also overlook fundamental economic principles. Notably, these models often portray economies devoid of growth, where trades are conservative, and equilibrium states seem trivial due to their parameters' inability to produce varied outcomes, thus making their features often useless. Furthermore, the redistribution mechanisms required to avoid trivial solutions are often extreme and lack any negative consequences.

On the next sections, and with these results in mind, we aim to delve deeper into not just the results shown here, but also into some of the parts we deem essential to build a model that is more realistic and capable of reproducing the distribution we are looking for.

1.3 Information Theory and Networks

In the previous section 1.2 we have explored not just empirical results but also attempts to model income distributions. However, given the pitfalls we have shown, it might be best to go back to the beginning and ask: What processes can generate this distribution? After all, ultimately, this is our goal.

To that effect, and considering we aim to model this with an agent based approach, we must first learn about information theory and how it links itself to network science and stochastic processes.

1.3.1 Information Theory

When Ludwig Boltzmann initially introduced his statistical definition of entropy, as expressed in Equation (1.11), he established a crucial connection between the number of microstates (W) and the ways in which the constituent parts of a system (atoms, in this context) can be arranged. This relationship, in turn, relates to the macroscopic properties and states of the system, such as its energy and pressure. His groundbreaking work revealed that as the number of potential microstates corresponding to a given macrostate increases, so does the entropy of that system, leading to the common analogy of entropy being associated with disorder or chaos,

$$S = k_B \ln W \, . \tag{1.11}$$

Boltzmann's definition, however, is particular to what statistical mechanics now calls microcanonical ensemble, where the number of particles and energy of the system are precisely defined and fixed. In this case, all microstates are equally probable. In other scenarios, this might not be the case.

Thankfully, we can readily generalize Equation (1.11) by incorporating the probabilities associated with each microstate (p_i), as shown in Equation (1.12). This generalized form reduces to the original Equation (1.11) when all p_i are equal. This is the case, for example, of the canonical ensemble, where the temperature is fixed and the probability of all the possible microstates is dependent on the energy of that given microstate:

$$S = -k_B \sum p_i \ln p_i \, . \tag{1.12}$$

This particular Equation, as denoted by (1.12), is often commonly referred to as Boltzmann-Gibbs' entropy. Given that it was Gibbs who first explicitly interpreted this equation in terms of the probabilities associated with microstates, in contrast to Boltzmann's original use of the symbol f to represent a phase space density.

Later, in the 1940's, Claude Shannon was working on a completely different problem: how to decode noisy messages being sent through a communication's channel [33]. Based on previous works made by Harry Nyquist [34] and Ralph Hartley [35], Shannon came to the idea that information can be measured based on how likely a new bit of data is, given prior statistical knowledge of previous messages. This idea eventually led him to a statistical problem that was very similar to Boltzmann's but without any ties to physical laws. As with Boltzmann, he attempted to maximize a function subject to a set of constraints (axioms).

Shannon would eventually solve the problem and arrive at a statistical function very similar to Boltzmann-Gibbs' entropy but with a much different interpretation. Shannon interpreted his entropy as a measure of ignorance. In other words, as the number of potential states of a system increases (resulting in higher entropy), our level of ignorance about the system also increases. Therefore, higher entropy implies that new information (or new data) becomes more valuable, as our prior state of ignorance was high.

To summarize, we can arrive at Shannon's entropy by simply reformulating Boltzmann-Gibbs' entropy, as illustrated in Equation (1.13),

$$H(p_1, ..., p_n) = -K \sum_i p_i \ln p_i \,, \tag{1.13}$$

and replacing Boltzmann's constant (k_B) with a free parameter (K), which can be customized for each particular system. This link between Boltzmann-Gibbs' entropy and Shannon's entropy was first formally discussed and formulated by E.T. Jaynes in his 1957's seminal paper called Information Theory and Statistical Mechanics [36] - it is worth pointing out, however, that his formulation and reasoning are far more in depth than this summary.

To better understand this concept let us take two examples: 1) a tossed coin; 2) a roll of dice. And let us consider $K = 1$ in order for both examples to be more readily comparable. In **1)** there are only two equally probable possibilities: heads or tails. Hence, $H_1 = \ln 2$. In **2)**, however, there are six distinct possibilities, also all equally probable. Hence, $H_2 = \ln 6$ ($H_2 > H_1$). This means that the information that an experiment with a coin has yielded a head, for example, is less valuable, contains less information, than the information that an experiment with a dice yielded a three.

These results are incredibly important in many contexts but specially in computer science and signal processing [37]. Where his work would later open many new paths to study various subjects that range from cryptography [38] to machine learning [39].

Nonetheless, it seems there is a gap in the understanding of how information theory, stochasticity and networks combine together. Something we discuss in the next subsection.

1.3.2 Networks

Networks are a ubiquitous phenomenon, observed across a wide spectrum of disciplines including physics, social sciences, biology, and many others. They are characterized by a connectivity function that governs the relationships between nodes within the network. In our context, we have shown two examples that highlight how networks can effect the equilibrium states of agent-based models (as discussed in Section 1.2). What we have not yet explored is the intriguing concept that within stochastic systems, the connectivity of a node has a direct link to information, irrespective of the nature of interactions.

To explain this it is better to start with an example. In the previous section 1.3.1, we showed how an increase in the number of possible states of a system leads to an increase in the information gained from new data. Therefore, to pull from the examples we have already explored, let us consider an agent i with number of connections k (a node denoted by $n_i(k)$), that starts with a set amount of a given quantity $w = w_0$ ($w \in R$) and then interacts and exchanges with its connections. Given our context, we can think of w as wealth and w_0 the starting wealth of the agent.

To simplify, let us say that in this interaction all it can do is gain a net amount of Δw ($\Delta w > 0$). Now let us add our stochastic element: a winning probability P_i that dictates if i will gain (add Δw to its starting w_0 quantity) with this interaction or not. Note that this is a simplified example where the agent can never lose.

As we know, the probability of winning j times in k interactions is given by the binomial distribution:

$$f(j, k, P_i) = \frac{k!}{j!(k-j)!} P_i^j (1 - P_i)^{k-j} \, . \tag{1.14}$$

Now let us consider the simplest case, where the agent will, necessarily, interact with all of his connections. Given this scenario the collection of all possible states for agent i (node $n_i(k)$) is:

$$w_0, \Delta w + w_0, 2\Delta w + w_0, 3\Delta w + w_0, ..., k\Delta w + w_0 \, , \tag{1.15}$$

notice how the number of possible states increases as the number of connections (k) increases. Now, given that every possible state has an attached probability given by Equation (1.14), we can re-write Equation (1.15) to include it:

$$w_0 \left[\frac{k!}{0!(k-0)!} P_i^0 (1 - P_i)^{k-0} \right], ..., (w_0 + k\Delta w) \left[\frac{k!}{k!(k-k)!} P_i^k (1 - P_i)^{k-k} \right] \, , \tag{1.16}$$

hence, from Equation (1.16) we can easily get how much, on average, the agent will gain. In other words, we can get the average future state of this agent, which will be given by:

$$\langle w_i(k) \rangle = w_0 + \Delta w \sum_{j=0}^{k} f(j, k, P_i) j \, . \tag{1.17}$$

Notice, that given the nature of the binomial distribution, the average state grows linearly with connectivity k, angular coefficient P_i and linear coefficient w_0. Similarly, as per Shannon's entropy, we have:

$$H_i = -K \sum_{j=0}^{k} f(j, k, P_i) \ln f(j, k, P_i) \, . \tag{1.18}$$

Thus, it is easy to see that as the connectivity of a node increases, not only does the number of possible states but also its entropy. This can be clearly seen in Figure 14 where we show $H_i(k)$ for $\Delta w = 1$, $K = 1$ and $w_0 = 0$. Notice how given the additive nature of this example an increase in the probability of winning increases the average state but decreases entropy. This highlights a fundamental physical concept: equally probable states maximize entropy. This is why we postulate that, whenever we lack information about a system, it is best to assume equal probabilities for every possible state.

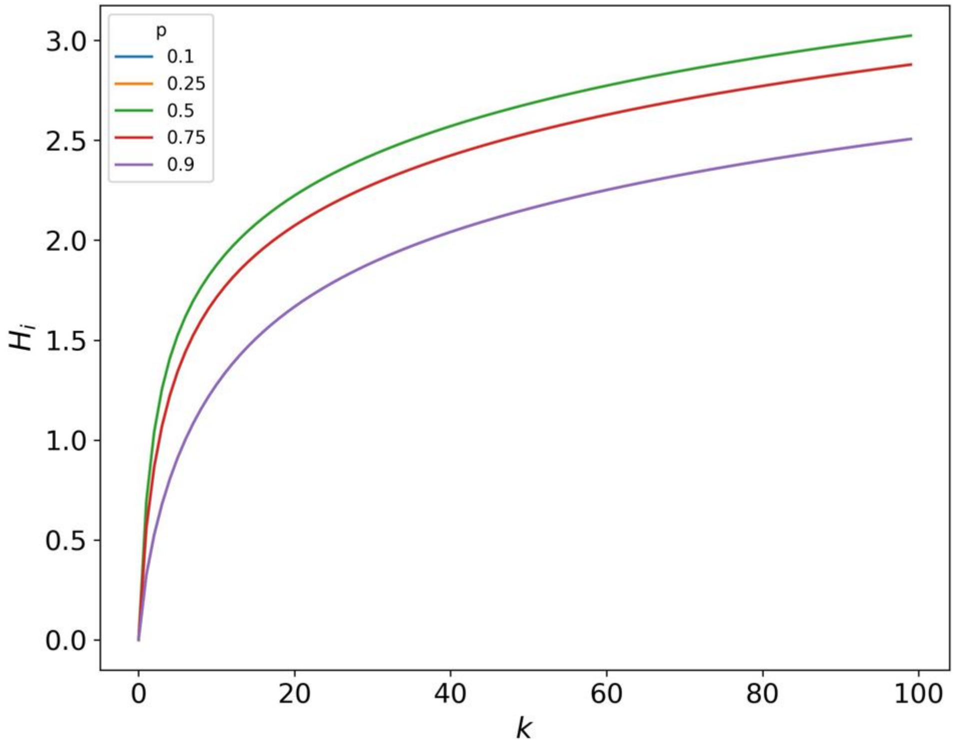

Figure 14 – Equation (1.18), the information entropy, as a function of the connectivity k. Note: Since $P \in [0, 1]$ and the distribution is binomial, distinct values of P gives an entropy that is exactly the same as its complementary value. Hence, the entropy for $P = 0.1$ and $P = 0.9$ are the same.

Furthermore, one can, obviously, write a more generalized version of this. Where P_i and Δw take much more complex forms. However, it is straightforward to see that, for most choices, the number of possible states will increase with connectivity k and so will entropy. Hence, agents/nodes with higher connectivity lead to/have more information.

To bring this concept closer to the model we are trying to build, and make it clearer, we can think about it in terms of a Monte Carlo simulation. let us consider the same example from before, but this time focus on two different agents, where their states are updated with each Monte Carlo step: An agent i, with n_i connections, and an agent j, with n_j connections. Take the case when $w_i = w_j = w_0$ but $n_i > n_j$ and, therefore, $H_i > H_j$. Even though they both start at the same place, the number of possible states the agent i has available in the next step is greater. Hence, the information about its next state, on the next Monte Carlo step, is more valuable and contains more information. This is a key concept which intrinsically ties an increase in network connectivity with an increase in information in stochastic systems.

This result also helps us better understand Iglesias' [23] result, linking higher connectivity with higher wealth: agents with more information tend to be wealthier and they also tend to make agents closer to them richer by proxy. This is a key finding because it will allow us to think about inequality as an information differential instead of more abstract, and often complex, concepts.

1.3.3 Distribution Generating Processes

In sections 1.2.2 and 1.3.2 we have given direct examples about how connectivity and wealth are intrinsically linked, we then showed how this is essentially caused by a information differential and how this allows us to better understand the wealth/income distribution process. However, there is still one element missing to build our model. We do not know which processes generate Boltzmann-Gibbs or power-law/Pareto distributions.

Boltzmann-Gibbs Distribution

The Boltzmann-Gibbs distribution arises when we maximize the entropy in Equation (1.12) in the canonical ensemble, where every microstate i of the system has an associated energy E_i, and the macrostate has average energy U. Which means, the system is at thermal equilibrium. This leads us to:

$$\Phi = -\sum_i^n p_i \ln p_i - \alpha(\sum_i^n p_1 - 1) - \beta(\sum_i^n p_i E_i - U)\,, \tag{1.19}$$

which can be solved by taking derivatives with respect to p_i:

$$0 = -\ln p_i - 1 - \alpha - \beta E_i \rightarrow p_i = e^{(-1-\alpha-\beta E_i)}\,. \tag{1.20}$$

Since $e^{(1+\alpha)}$ is simply the normalization of the distribution:

$$e^{(1+\alpha)} = \sum_i^n e^{-\beta E_i} = Z\,. \tag{1.21}$$

Therefore,

$$p_i = \frac{e^{-\beta E_i}}{Z}\,, \tag{1.22}$$

this is the Boltzmann-Gibbs distribution. Notice how this result arises from maximazing entropy in a system with no a priori preference and with fixed average energy. Hence, exchange energy by any other quantity and we realize that if the system has no preference attached to it and every exchange is conservative (the total remains constant) the result will always be this. That is why so many conservative agent-based models (see section 1.2.2) all result in Boltzmann-Gibbs/exponential distributions and it is also why when we introduce different, non-random, networks, which produce preference, this distribution changes. This is also the root of Dragulescu and Yakovenko's work in [29].

Therefore, almost by definition, agent-based random networks will produce this distribution, what we have to find is how to also generate a power-law tail.

Power-Law Distribution

In statistical mechanics and network science, additive and multiplicative processes are terms used to describe systems that evolve in time by successively adding or multiplying random numbers. Power laws, as exemplified by the Pareto tail, are a consequence of multiplicative processes, where the entities involved in the exchange typically gain or lose a percentage of their energy as a result of the interaction. A lot of non-conservative physical systems behave in this manner since drag, dampening, decay, acoustic loss, and many other losses, are exponential in nature.

Furthermore, much like non-conservative forces, economic systems are also naturally multiplicative, since compounding is an intrinsic part of it. From investments, to the growth of companies, to the growth of countries, everything is multiplicative in nature. Which is why power laws are so common in these systems. Nonetheless, from the distribution of city sizes, to particle physics, to the distribution of the mass of stars and to wealth distribution, power-laws are everywhere.

Given this nature, power-law generating processes are varied. However, given our economic interest, we will focus on just one: preference. Preference, in this context, means that if whenever something grows its chances to keep growing also increase, then we have the essential elements of a power-law generating system.

Hence, if we create a mechanism that generates some form of preference, it is very likely that this will also generate a power-law.

Combining the Two Processes

Therefore, what is left for us is to find a mechanism that can transit from broad (exponential/Boltzmann-Gibbs' distribution) to tail (Pareto's). Fortunately, based on the concepts we have elucidated, there seems to exist a straightforward method to achieve this.

In physics, a particle gains energy by colliding and interacting with other particles.

And, in essence, whenever a particle collides with a particle with higher energy it gains some. However, as the energy of this particle grows finding other particles with similar or higher energy becomes increasingly rare, thus making it much more likely that the particle will interact of other particles with lower energy and, therefore, lose some, returning closer to the mean. Similarly, in our context, what limits an agent-based system into a Boltzmann-Gibbs process is the fact that once an agent reaches a certain amount of wealth it inevitably stops growing because most other agents have such a small fraction, relative to them, of wealth to exchange that it becomes more likely for it to lose wealth and return to the mean than to keep growing. This is true regardless of the starting point of system - how much wealth is in it. As long as the total wealth of the system is fixed, whenever an agent increases his wealth away from the mean, it becomes harder and harder for it keep growing.

This constraint, however, can be solved by allowing those wealthy agents to exchange with multiple parties at once, thus effectively permitting multiple tiny exchanges to be combined into one big enough to push it further. Then, for the process to continue all we must do is to make sure that as the wealth of the agent grows so does how likely this multiple party exchange happens.

Of course, in physics, it is incredibly unlikely that an instantaneous exchange between multiple particles will happen at once. Which is why no model within physics proposes such interaction. However, in the scale of the economy, this happens all the time with large businesses, which are constantly engaging in various exchanges simultaneously. Thus, if we link the agents connectivity to its wealth or income, we will be essentially allowing them to pass the threshold and, potentially, reach the multiplicative state we need. This can also be pushed further by favoring the richer agent in each transaction, if necessary. Thus creating a system that has very strong preference towards richer agents.

1.3.4 Conclusion

So far we have highlighted the connection between a higher degree of freedom (connectivity) and both information and wealth. We have also shown how and why conservative models often produce Boltzmann-Gibbs distributions and the importance of preferential attachment to generate power-law distributions. This then allowed us to understand wealth/income differences as an information differential, and to link the growth of large companies through the lens of their connectivity. All of which, perhaps surprisingly, match common intuitions about the world and the economy.

Therefore, by combining all that we have shown, we aim to further explore the universal dynamics governing wealth and income distributions by proposing a new agent-based model that is based on fundamental ideas, intrinsic to any commercial exchange, which, we hope, will shed light on how income/wealth distributions are generated.

To that affect, we will present each element of our model in a very systematic approach, where we elucidate how each one of our assumptions contributes to the gradual transformation of wealth distribution over time. Through this process we will show that by allowing wealth to be freely generated (and destroyed) and directly linking information (connectivity) and wealth, while slightly favoring rich agents, one can achieve the results we are looking for, which is to reproduce real-world scenarios[7]. Furthermore, we will also explore different types of taxation and how these affect our results.

[7]Scenarios : Income/wealth distributions

2 Outline of the Model

Our goal with this work is to construct a simple agent-based model, rooted in very basic assumptions of business relationships. To do this, we have built our model step by step, introducing complexity along the way, while always maintaining its basic features. Hence, the description of the model, as well as its results, is presented in the same manner, as a step-by-step model built from its simplest form to its most complex, in order to help us understand what the effect of each part is and why it matters. We also separate the description of each version of the model from its results in order to discuss the impact of each parameter and to understand why we have chosen some appropriate numerical values for some of these parameters. Note that more complex models can also yield similar results, but we aim to make it as simple as possible.

2.1 Fundamental Characteristics

We consider a collection of N agents, each starting with a given value w_0 of a continuous variable w_i, representing the wealth of the agents. The evolution is probabilistic, where, in each Monte Carlo step, each agent is chosen once and trades with other agents, chosen randomly. As the system evolves, agents interact and trade according to four different sets of rules, which we call scenarios. With each one being more complex than the other. In these interactions, agents can either gain or lose a net amount of money based on the combined wealth of both agents i and j,

$$\Delta w_{i,j} = \frac{\mu(w_i w_j)}{(w_i + w_j)} \,, \tag{2.1}$$

where μ is a constant $\in [0, 1]$. This function is taken from [24] and is chosen because it reflects well the relationship of the difference in buying power among agents: the bigger the difference in buying power of one agent relative to another, the greater is its ability to set the amount of money exchanged, aka the price. This Function (2.1) can be seem in Figure 11. This also makes it impossible for agents to trade more wealth than they currently have, a basic requirement. It is important to note, however, that this function is not special and that any other function with a similar qualitative behavior would also work.

At each interaction, both agents play according to a probability distribution and wealth is not conserved, as it happens in reality. Therefore, possibilities where both agents gain (wealth creation) and where both agents lose (wealth destruction) are possible. Since wealth is not conserved in each trade exchange, the total wealth of the system is normalized at the end of each Monte Carlo step. It is important noting, however, that since our aim is to model real world economic behavior we made decisions to ensure that in most cases trades are a value add to both agents. Hence, wealth creation and wealth conservation is more likely than wealth destruction.

2.2 Basic Assumptions

Here, we will introduce the three very basic assumptions that we have adopted and that are reasonably valid anywhere in the world. Of course, each of these assumptions will be adequately mathematized.

Our first assumption, which should be self-evident, is

First basic assumption

- The number of trades an agent makes increases with his wealth.

This simple, yet powerful, mechanism encompasses all we have discussed in section 1.3.1, and introduces a direct link between how much wealth an agent has and how much information it possesses.

This link is also based in reality, after all, as businesses and individuals increase their net worth, they also increase how many business interactions they perform. For example, a store grows by selling more items and hence, having contact with a greater number of customers. Similarly, an individual usually becomes richer by taking part in multiple businesses at once, like having multiple shares in different companies. This element is what gives rise to a system with varying degrees of freedom and allows rich agents to multiply their wealth exponentially, thus shifting part of the system into a multiplicative process.

These agents will then randomly interact with others according to a given connection function, defined as $f_c(w_i)$, which gives how many connections/interactions each agent can make, based on his wealth. For example, if $f_c(w_i) = 1$, each agent in their turn, regardless of its wealth, will perform one interaction, and therefore one transaction, per Monte Carlo step. Notice that in one Monte Carlo step, a given agent may perform more than $f_c(w_i)$ interactions, as it may be chosen by other agents in their time.

Therefore, according to the assumption, the connection function $f_c(w)$ mentioned above must be a monotonically increasing function. We will assume here the simplest one, a linear function. Clearly, any other type of monotonically increasing function could be adopted, but the qualitative behavior of the evolution of the system will not change by reasonable choices of the connection function. Only the way the system evolves will change, but not the patterns of the distribution. This assumption will be used in the second, third, and fourth scenarios presented below.

Our second basic assumption, which is also self-evident, is

Second basic assumption

- The probability of making a favorable trade transaction increases with the difference in wealth between the richest and poorest.

Similarly, this aims to model the fact that the richer agent, relative to each other, in any given transaction, typically enjoys a lower risk. This also, as a reference to section 1.3.1, further reinforces the preference of the system towards richer agents, therefore increasing the likelihood that they will partake in a multiplicative process, which is necessary for the Pareto tail to appear.

To put this into perspective, and exemplify how this takes place in the real world, let us take the example of a bank performing a loan. The bank not only has a much better understanding of how risky the transaction is and hence how likely the loan-taker is to pay - thus allowing it to set the price accordingly. But it is also able to perform multiple loans with various different parties at once (first assumption) which significantly reduces its risk. Notwithstanding, it is clear to see that if both parties have similar wealth they will also have access to similar resources and their risk will be similar.

This introduces a probability of making a favorable trade exchange that depends on the difference in wealth between the two agents trading. The richer an agent is relative to each other, the higher the probability of making a good trade, as is usually the case in any negotiation. In terms of the simulation, this means that two "coins" are tossed, one for each agent; hence, they both can win or lose, and situations where one wins and the other loses are also possible. Mathematically, this means that when $w_i > w_j \rightarrow P(w_i|w_j) > P(w_j|w_i)$. This assumption will be used in the third and fourth scenarios.

Third basic assumption

- The tax is a monotonic increasing function of wealth.

We have essentially two main types of taxation: wealth taxation and income taxation (income or capital gains during 1 year, i.e., one stage).

2.3 Taxation

Taxation on Wealth

We start with taxation on wealth simply because, from the perspective of a simulation, it is the simplest form of tax. This taxation is setup as follows: After a certain number of Monte Carlo steps, here adopted as five, we have what we call a stage, which we can think of as equivalent to 1 year - the period over which income taxes are typically collected in the real world. At the end of each stage, a tax is applied on the amount of wealth the agent has at the end of the period. The standard pattern should be that the tax increases with wealth, so we have our third basic assumption. This mechanism resembles real world taxation schemes, albeit without your typical bracket and levied on wealth instead of income.

We then define as a taxation function, to be applied to the wealth of each agent at the end of a stage, the simplest one, a linear function:

$$\mathrm{Tax}(w_i) = \begin{cases} 0, & \text{if } w_i < w_o \\ \gamma(w_i - w_0) + \sigma w_0, & \text{if } w_0 \leq w_i < w^* \text{ and} \\ \tau, & \text{if } \quad w_i \geq w^* \,, \end{cases} \tag{2.2}$$

where $\gamma, \tau, \sigma \in [0, 1]$; γ indicates the growth rate of the tax according to wealth, σ the base tax rate, τ the maximum adopted tax rate, and $w^* = [\tau + (\gamma - \sigma)\, w_0]/\gamma$. It is worth noting that taxation here means taxation to be applied on the wealth each agent has. We will consider in this paper only progressive taxation, as it should be; thus, the parameter γ is considered non-negative. Of course, scenarios with negative values of γ will contribute to concentrate more wealth in the hands of fewer agents. The parameter τ, which controls how high the tax rate can reach, could be relevant in controlling inequality by preventing further concentration in the tail end of the distribution. Furthermore it is also an important political issue today in many countries - see [9]. The parameter σ, the initial tax rate, is a parameter that is not particularly important, but we keep it here for the sake of completeness.

It is also important to note that there is nothing special about the form of this function, chosen here as a linear function. It could also be another type of increasing function, such as a power law. What matters is its behavior—it grows with wealth and has an upper bound. In fact, we explored the quadratic function, and the visible difference was on how quickly the simulations reached their different patterns. We would also like to note that a similar approach but using tax brackets was previously tried and the overall behavior was unchanged hence why we defaulted to, once again, its simplest form. This, however, refers to our model only. Other authors have explored taxation and the difference between

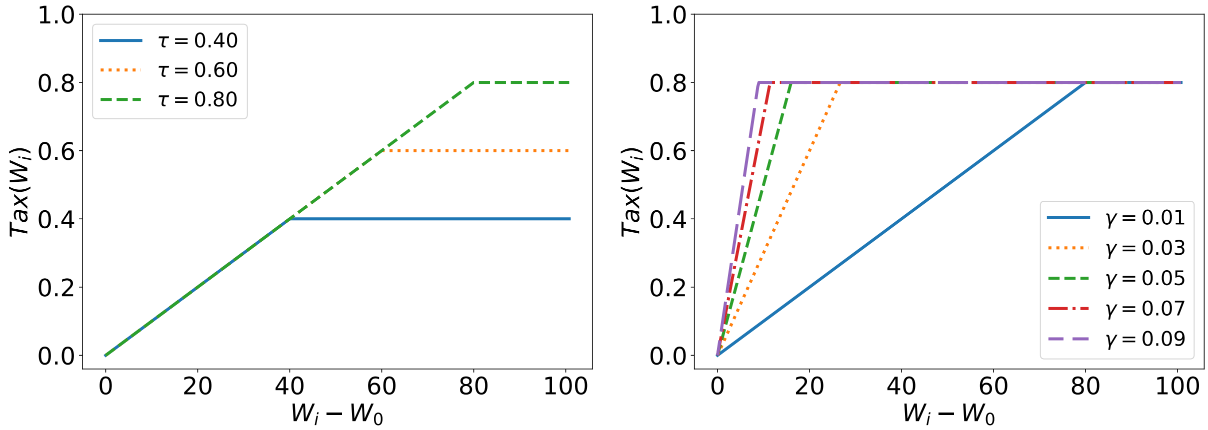

Figure 15 – Function $\text{Tax}(w_i)$ (2.2). On the left figure, the function $\text{Tax}(w)$ as function of $w_i - w_0$ is shown for $\gamma = 0.01$ and typical values of τ. On the right, the same function is represented for $\tau = 0.8$ and typical values of γ. Both figures have $\sigma = 0$

linear and non-linear taxation and found different results, see [40] and [41]. Similarly, our tax definition means taxation always starts when the wealth of an agent is greater then w_0, instead of some other point, which is an aspect that can have an effect on the results but that we do not explore - see [40].

This taxation is then applied to the share of each agent's wealth w_i above a given minimum w_0 at the end of a fixed number of Monte Carlo steps (called stages), where w_0 is each agent's wealth at the beginning of the simulation.

The total tax charged to the N agents at the end of a stage is then

$$C_T = \sum_{i=1}^{N} \text{Tax}(w_i)\, w_i \ ; \tag{2.3}$$

this total tax collected at the end of a stage is then redistributed equally among all agents (different scenarios in which, for example, the redistribution favors the poor are also possible and certainly lead to different results and conclusions). Thus, if we let C_T = total tax collected, then $C_{i,T} = C_T/N$ is the amount of tax that is returned to each agent at the end of a stage.

In the fourth scenario (see Sections 2.5.4 and 3.4), we will consider taxation on annual income, but the same third assumption applies: taxation increases with income earned during the previous year.

Taxation on Income or Capital Gain

We have discussed in Section 2.3 how to tax the wealth of agents at the end of each year (stage) rather than taxing income or capital gains, which is the much more common form of tax. Therefore, instead of taxing all agents' wealth w_i above the minimum w_0, we now tax an agent's annual capital gain whenever it is above the minimum gain ξ. Hence, regardless of how much wealth an agent has, if he has enough capital gain over a year, i.e., a stage, above this minimum (ξ), the agent will be taxed on that amount, in a monotonically increasing way. Then, we can define

$$\text{Capital Gain} \equiv G_i = w_{i,t} - w_{i,t-1}\,, \tag{2.4}$$

where t indexes the t-th year, i.e., the t-th stage—as defined in (Section 2.4). Hence, taxation to be applied to the capital gain of each agent can now be assumed as

$$\text{Tax}(G_i) = \begin{cases} 0, & \text{if } G_i < \xi \\ \gamma(G_i - \xi), & \text{if } \xi \leq G_i < G^* \text{ and} \\ \tau, & \text{if } \; G_i \geq G^*\,, \end{cases} \tag{2.5}$$

where $\xi \in [0, w_0]$ (therefore, it is never greater than the starting point of the systems) and $G^* = (\tau + \gamma\xi)/\gamma$; γ is the the growth rate of the tax according to capital gain.

Notice, however, that here, at the end of a year (a stage), we tax anyone with a capital gain above ξ, regardless of their current (wealth) condition. Therefore, a poor agent ($w_i < 1$) who has capital gains above the minimum ($G_i > \xi$) at a given time t will be taxed, even though he is poor. Note, however, that the order of magnitude of taxes in this case (earnings in a year) is very different from the case of a wealth tax. It is important to note that there are many types of taxes that can be collected during the year. There are consumption taxes, which are regressive, affecting poor agents more than rich ones, and taxes on annual income, which are progressive, hurting the rich more. All types of taxes collected during an agent's year are called here the agent's annual income tax. By annual here, we mean the earning received during a stage, of course.

Therefore, while in the wealth tax model an agent with $w_i >> w_0$ can be taxed heavily, since the tax is applied over the agent's total wealth above w_0, it also allows poorer agents to build wealth, since they are not taxed until their wealth is at least $w_i = w_0$. Here, in the case of income tax, the opposite may be true. No matter how poor an agent is, whenever he has a good year, he will be taxed, thus making it difficult for him to build wealth. Meanwhile, extremely wealthy agents could pay almost nothing—relative to their wealth—if their capital gain is not important.

As with taxation on wealth, at the end of the stage (year), the total tax collected that year, which is $\sum_i \text{Tax}(G_i)G_i$, is redistributed equally among all agents.

Note, however, that in each scenario analyzed, only one of these two types of taxation is applied, either on wealth or on annual income.

2.4 Simulation Setup

We initiate every agent with $w_i = w_0$ and define a Monte Carlo step when each agent (N) in the system finishes his turn, which means

The agent i, with income w_i and number of connections $f_c(w_i) = k$, performs all k interactions in one step. These k interactions are randomly chosen.

The system has no distance (every agent can interact with $f_c(w_i)$ other agents, chosen at random). Therefore, since the interacting agents are randomly selected, a given agent i may perform more than $f_c(w_i)$ interactions per step, since other agents may in turn randomly choose agent i.

Note that this choice of network is not random. We need the connectivity of agents to be able to grow, otherwise we will never get the multiplicative processes necessary to generate a Pareto tail.

We then define that **five Monte Carlo steps** constitute a **stage**, and every step is synchronous: the state of an agent (increase/decrease in wealth) is only updated when the Monte Carlo step is completed (all agents have been updated). Tax collection and redistribution occur only once at the end of the stage. Therefore, Monte Carlo steps can be interpreted as the passage of months, while a stage as the passage of an entire year (annual tax).

To better understand the model and how each element contributes to its' dynamics we will define the poverty line as 10% of the initial average ($w_i < \frac{w_0}{10}$) and track how this portion of the population evolves. Hence, we will separate the population in two groups:

1. Agents with $w_i \geq \frac{w_0}{10}$, which are shown in the distributions;
2. Agents with $w_i < \frac{w_0}{10}$, which are taken as the poverty rate and only appear as a percentage.

By doing so, we are essentially defining poverty as any agent with wealth inferior to 10% of the initial average. This is an arbitrary choice but one that we believe is reasonable.

In the following chapters, we explore different simulation settings (interaction rules, probability, and connection functions) and discuss some of the properties of the model. We reinforce, however, that the functions we have chosen have nothing special about them—we particularly choose the simplest functions whenever possible—it is just their qualitative behaviors that matter. In fact, in the beginning, we tested alternative functions, and the resulting patterns remained unchanged (the speed of evolution may, as mentioned earlier, change depending on the functions chosen).

2.5 Scenarios

By partially combining different elements of our model, we are able to explore different simulation settings and, therefore, evaluate the impact of each part in the end result. These different settings are what we call scenarios, which we introduce and define in the following sections.

2.5.1 First Scenario: Raw Model and Taxation on Wealth

This is the simplest, unbiased scenario. Consider a system with the trade rules defined at the beginning of Section 2, with an equal probability of winning a commercial exchange; i.e., the probability that agent i will win a commercial exchange with agent j is

$$P(w_i|w_j) = \frac{1}{2}\,. \tag{2.6}$$

We also consider the connection function—which, as it is defined in Section 2.1, means how many interactions/transactions an agent will choose at each step—as

$$f_c(w_i) = 1\,, \tag{2.7}$$

for any value of w_i, implying that at each step, each agent chooses only one other agent to trade with.

The numerical simulation results for this model can be seen in Section 3.1.

2.5.2 Second Scenario: The Wealth–Connection Model with Wealth Taxation

In this scenario, we go one step further. We present a simple connection function that links the wealth of an agent with his number of connections,

$$f_c(w_i) = \begin{cases} a\frac{(w_i - w_0)}{w_0} + 1 & \text{if} \quad w_i \geq w_0 \\ 1 & \text{if} \quad w_i < w_0\,, \end{cases} \tag{2.8}$$

where $a \in [0, 1]$. This is what links information and wealth in our model. By doing so we are adding a preference, or bias, to the system towards wealthier agents. This is the first part of the mechanism to generate a multiplicative process in our model and, therefore, generate a Pareto tail.

The probability of an agent i winning a commercial exchange with an agent j is still given by Equation (2.6).

According to the function (2.8), note that agents will always make at least one interaction and that $f_c(w_i)$ is continuous. In order to reproduce this, agents with $f_c(w_i) \in \mathbb{R}$ have an equivalent probability of having an extra interaction at each Monte Carlo step.

For example, an agent i with $f_c(w_i) = 3.14$ will have three connections plus an extra connection with a probability of 14%. A random number will be drawn, and if it is below 0.14, the agent will obtain an extra connection, while if it is above, the agent will only obtain three connections in this round. This treatment is important because, otherwise, the distribution shows discontinuity where $f_c(w_i) = 2$ given that a large portion of the population is in $f_c(w_i) \in [1, 2[$.

Furthermore, we would like to point out that the shape of this function is unimportant. What matters is that it grows with w_i, which is why we have chosen the simplest function possible - a linear one. Quadratic, exponential, etc, would also serve its purpose. In fact, earlier on, we have explored much more complex versions. One example, which is worth mentioning, was the introduction of a distance-based connectivity Function (2.9), which introduced the distance between agents ($r_{i,j}$) as part of the connectivity process by creating a probability to trade with agent j that depended on the distance and the wealth of the agent. By doing so, richer agents were able to reach and trade with agents further away than otherwise. Which, we believe, is a self-evident fact. After all, economic agents are more likely to trade with neighbors than far away businesses and as business grow so do their reach:

$$f_c(w_i) = \exp\left(-\frac{a r_{i,j}}{w_i}\right). \tag{2.9}$$

This change, however, even though much more complex than our current version, still produced very similar qualitative results. Hence why we have defaulted to its simplest form. And why we point out that it is the link between connectivity and wealth that this function generates that matters.

The numerical simulations associated with this scenario are shown in Section 3.2.

2.5.3 Third Scenario: Favoring the Rich on Transactions and Wealth Taxation

Now, according to our second basic assumption, we introduce a higher probability of winning a commercial transaction for the agent with greater wealth. Until now, each agent had an equal probability of winning a commercial exchange, but now the probability of an agent i winning a transaction with an agent j will be given by the asymmetric function

$$P(w_i|w_j) = \frac{2 + \exp(\beta \delta w_{i,j})}{5 + \exp(\beta \delta w_{i,j})}, \tag{2.10}$$

where $\delta w_{i,j} = w_i - w_j$ and $\beta \in [0, 1]$. This function aims to model the risk ratio between agents. The greater the difference between the wealths of agent i and agent j, the greater the chance that agent i will make a favorable transaction (if $w_i > w_j$), modeling the fact that the richer agent takes less risk in a trade transaction. A real world example of this can be seen when we introduced this assumption in section 2.2.

This element further reinforces preference towards richer agents, which contributes to generate the multiplicative process necessary for the Pareto tail to appear.

At each step, a random number is drawn, and an agent plays this probability with each of the other agents he trades with. Wealth, as always, is not necessarily conserved: if both agents win, wealth is created (both agents earn Δw, Equation (2.1); if only one wins, wealth is conserved (one agent loses Δw, while the other wins); and if both lose, wealth is destroyed. In Figure 16, we can see the behavior of this probability function as a function of β. Notice how the decrease in the probability of winning for the poorer agent is small, while the increase for the rich is significant. The function is not symmetric. As mentioned when we introduced the model, we make this choice with the aim to come closer to a fundamental economic principle: any given trade, more often than not, is a value add to both parties. Hence, situations where wealth is conserved or created are more likely than otherwise. After all, if the commercial negotiation is too unfavorable for an agent, in the real world, he simply would not make the trade (except in very exceptional cases, which are not considered here), something we wish to avoid.

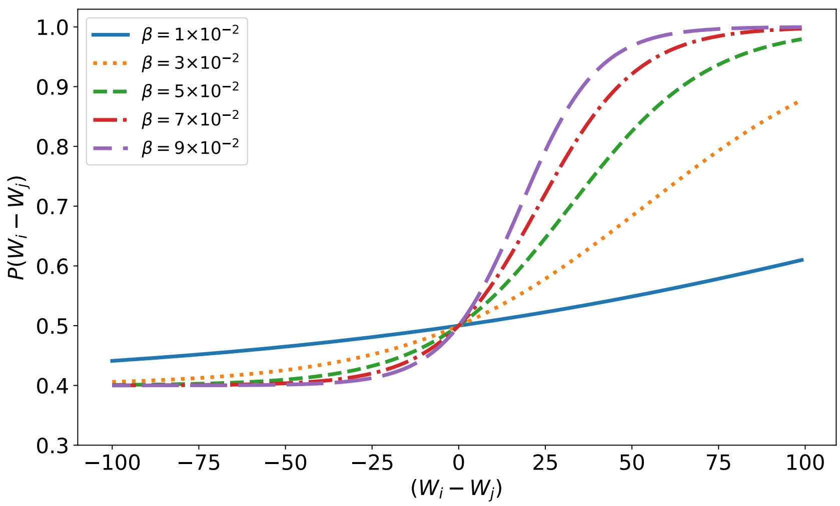

Figure 16 – Probability function, Equation (2.10).

Here, once again, the chosen function is not special, and any other function with similar behavior would work. What matters is the risk advantage that the rich agent has. After all, this mechanism simply aims to add more preference towards them.

2.5.4 Fourth Scenario: Favoring the Rich in Transactions and Taxation on Annual Income (Capital Gains)

In this scenario, we consider a connection function given by Equation (2.8), a probability to win a commercial exchange given by Equation (2.10), and a tax on income earned during a year given by Equation (2.5). This is the scenario closest to reality considered.

3 Results

When developing the model we analyzed multiple different values for its parameters and we found that, essentially, μ, β and a only control how fast the system evolves and, therefore, how quickly it goes through the different stages. Higher rates of any of these variables will mean that some of the intermediate distributions will inevitably be skipped because the system will evolve too fast. Therefore, these parameters will be kept constant in all of our simulations since we are more interested in following the different stages of evolution and seeing how the wealth/income distribution changes.

Hence, after extensive testing, we adopt from now on the values $\mu = 0.1$, $\beta = 0.01$, and $a = 1$. Similarly, σw_0, our base tax rate, will be kept at 5% of w_0 ($\sigma w_0 = 0.05$) . Furthermore, since the parameters w_0 and ξ are simply scale parameters, and therefore, their values do not affect the results, they are also kept fixed as $w_0 = 10$ and $\xi = 0$ during all numerical simulations. This means, given our previous definition, that our poverty line will be $\frac{w_0}{10} = 1$.

On the other hand, however, γ and τ could completely change both the evolution of the system and the possible equilibrium states. Therefore, our analysis will consist of varying essentially these two parameters, keeping all others previously mentioned constant.

We would also like to add that, as previously mentioned, different rules for trade have been tested and the chosen functions have nothing special about them, since, in general, what drives the distribution's shape is the link between wealth and information (connectivity) and the difference in risk[1] associated with trade. Therefore, as some of our tests have shown, we suspect that as long as these two features are maintained in some fashion, results should be qualitatively the same.

Examining the Data

To examine the evolution of the system, beyond just looking at the shape of the wealth/income distribution, we will also employ common statistical techniques for analysis. Firstly, we will assess the system's equilibrium by analyzing its standard deviation. Secondly, we will employ the Lorenz curve, as proposed by Max O. Lorenz in 1905 [42] and shown in Figure 17, to quantify inequality. This curve is essential for calculating the Gini index, which is the current standard measure of inequality introduced by Corrado Gini in 1912 [43].

[1] A richer agent having lower risk/chance of losing wealth versus the poorer agent.

We use the Lorenz curve because it provides a graphical representation of inequality, whether it is wealth or annual income. It plots the fraction of the population on the abscissa based on their income or wealth and the fraction of accumulated wealth or annual income on the vertical axis. The Gini index, when we take the x and y axis from 0 to 1, is defined as twice the area between the line representing perfect equality, i.e., the curve connecting the origin to the point $(1, 1)$, and the Lorenz curve.

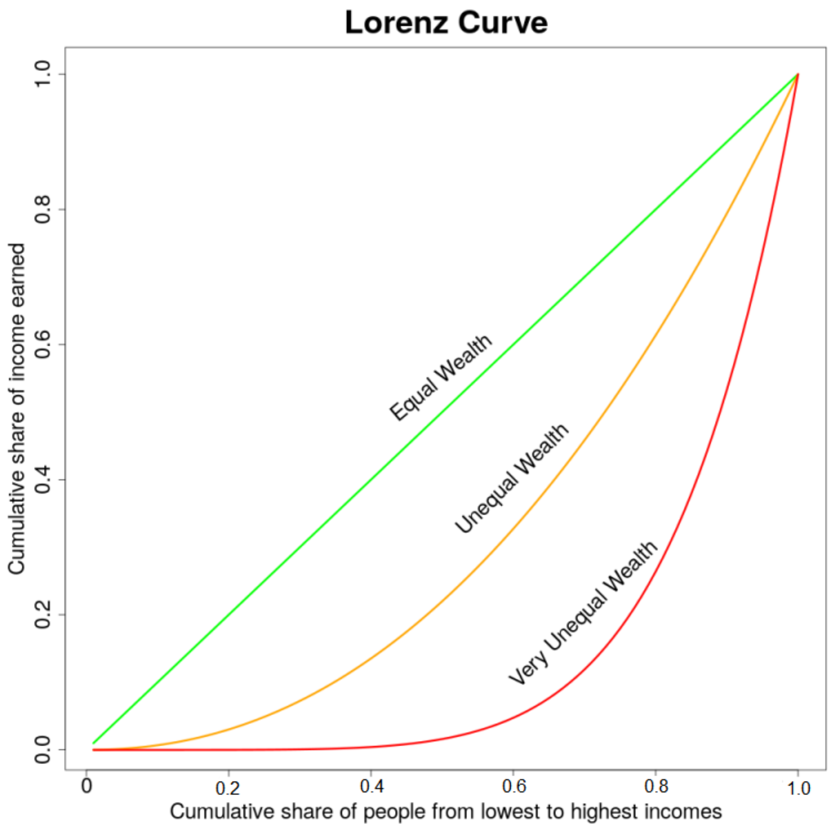

Figure 17 – A Lorenz curve depicting different inequality levels.

Finally, as we are interested in understanding the distribution of wealth across the population, we will characterize its evolution using quantiles. Quantiles are chosen because they are a standard measure[2] of inequality and represent the points that divide the population into segments. For example, the 90th quantile (or q90) separates the top 10% from the other 90%. Specifically, our focus is on the top 1% and the top 10% of the population.

Another important measure we are going to examine is the amount of wealth held by the top 10% and the 1% of the population. These quantities, often referred as the p90/p100 and p99/p100 ratios, are also commonly used to evaluate inequalities between countries. Hence, they are easy to find and reference. To that effect, whenever we reference

[2]Quantiles and percentiles can be used interchangeably.

these values we will be talking about the earliest available data from the World Inequality Database [44] (which at the time of writing this thesis was 2022 for most countries).

3.1 Raw Model

As defined in Section 2.5.1, the raw model describes a system without any assumptions that privilege any of the agents. Where the richest and the poorest have equal risks and number of connections.

The aim of this version of the model is to set a baseline for everything that follows. And it also might be taken as its most Utopian version.

Therefore, its probability function (risk) and connection function are given by Equations (2.6) and (2.7). All results presented in this section are for $N = 100{,}000$ (number of agents) averaged over 100 samples. Taxation is over wealth, given by Equation (2.2).

3.1.1 Statistics

In Figure 18a, it is evident that even in a system without any form of privilege, where no agent holds an advantage over another, and wealth is equitably redistributed among agents, there persists some level of inequality. The Gini coefficient in this scenario reaches 0.32. This observation underscores that achieving absolute equality is, in essence, unattainable, as randomness alone is capable of generating inequality.

We can also see that the 90th quantile quickly reaches about 1.6 times the average wealth (w_0). This means that in order to get into the top 10% of the population one must have at least[3] 1.6 times the average wealth (w_0). This, in turn, makes the top 10% hold about 19% of the total wealth of the population, as can be seen in Figure 18b. This measure is often called the p90/p100 ratio. To put this quantity into perspective, Sweden, one of the most equal countries on earth, has a p90/p100 ratio of 58.87% for wealth and of 32% for income.

[3]Note: this is the minimum required wealth, not the average. Hence, there are many agents within the top 10% that have higher wealth.

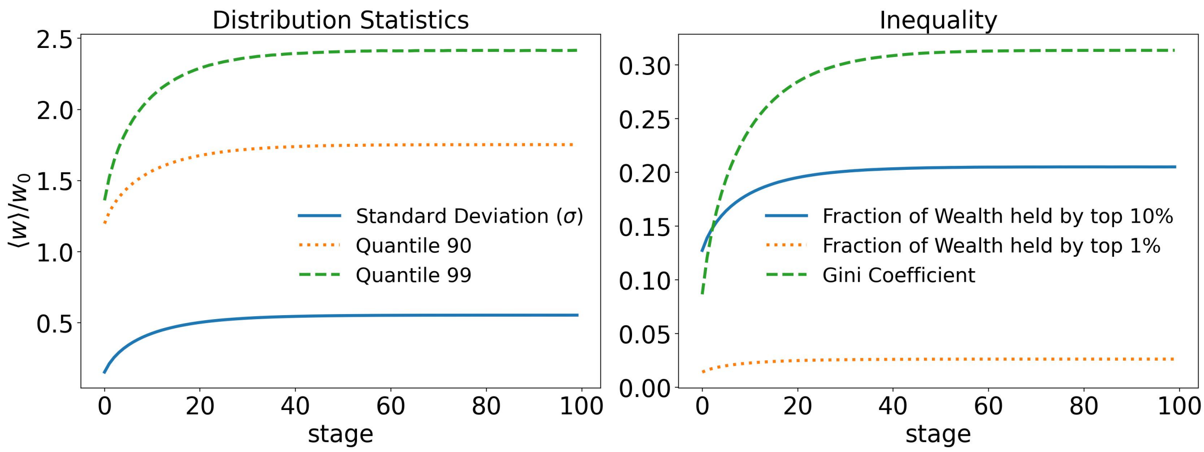

Figure 18 – Raw model with taxation on wealth: $\gamma = 10^{-3}$ and $\tau = 0.4$. In the figure on the left, we can see the average wealth held by the 90 and 99 quantiles, i.e., the 10% and 1% richest agents, respectively, compared with the standard deviation. On the right, the fraction of wealth held by the 10% and 1% richest agents is shown. The time evolution of the Gini index is also shown, stabilizing slightly above 0.3.

Going further we also analyze how the top 1% fair in this scenario. In Figure 18a, we can see that the 99 quantile has a value not greater than 2.5 times w_0, which makes the top 1% hold approximately 3% of the total wealth. This measure is commonly known as the p99/p100 ratio and for Sweden this ratio is 27.65% for wealth and $11,7\%$ for income. Therefore, it is safe to say that, as expected, this is a completely unrealistic scenario.

We have designed what essentially constitutes a perfectly egalitarian system, and yet, we have still identified the presence of inequality. This highlights the inherent difficulty in achieving absolute equality and defines the natural bounds of such a system. Consequently, it becomes evident that in ordinary circumstances where agents do not possess perfectly equal opportunities and taxation does not apply uniformly to an agent's total wealth, inequality will not only persist but increase.

Historically, we observe that significant social issues and instability tend to emerge only when wealth disparities reach extreme levels. This observation, combined with the economic rationale of maintaining some level of inequality to incentivize production, underscores the impossibility of completely eliminating inequality. Hence, our focus should be towards reducing it, rather than striving for its total eradication.

3.2 Wealth-Trade Link

As defined in Section 2.5.2, the wealth–connection model describes a system in which the richer the agent, the greater his number of connections (trade exchanges). Therefore, since his risk ratio remains at 50% at all commercial exchanges, once an agent starts to randomly gain more wealth, due to statistical fluctuations, the increase in connections makes his wealth evolution happen faster, while also providing the necessary source to further increase it. After all, if the connectivity did not increase, the rich agent's growth would be bound by the average wealth of the system. Specifically, on average, it would be limited to increase no further than μw_0 per stage. This mechanism is key to allow an effectively multiplicative process to exist within a mostly additive model.

The functions that define this scenario are given by Equations (2.6) and (2.8). All results presented in this section are for $N = 100{,}000$ averaged over 100 samples. Taxation is on wealth, given by Equation (2.2).

3.2.1 Distributions

Firstly, we focus on the model's distribution evolution in Figure 19. Where we can get an initial sense of the effects of this link. We notice that the system evolves almost equally as fast as the previous version, quickly reaching a exponential-type form. It is also noticeable that this version of the model stabilizes around stage 42, where it reaches its steady-state - this is validated in the next section. Also, see how interconnected the wealth distribution (left-hand side) and the connection distribution (right-hand side) are. We will further explore these changes in the **statistics** section.

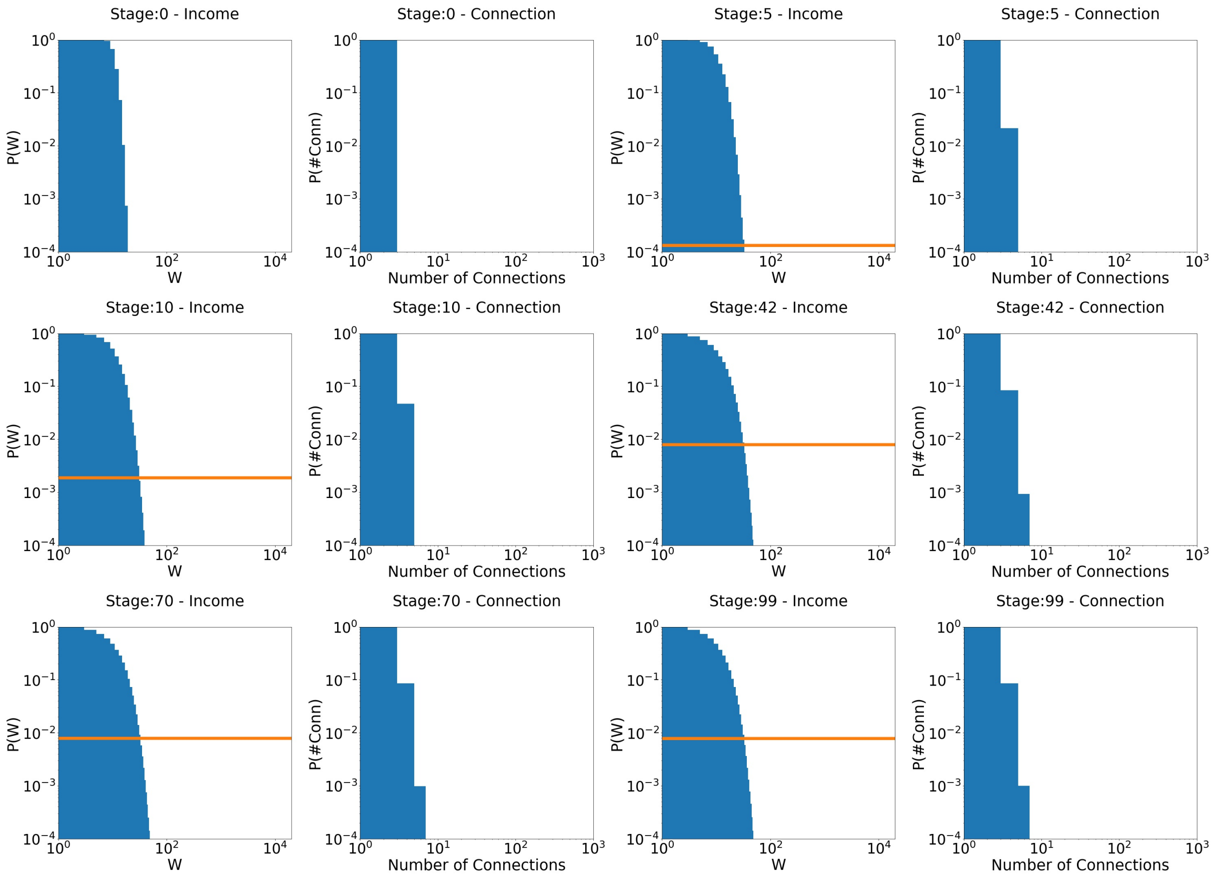

Figure 19 – Evolution of the distributions for the model that links wealth and trade: $\gamma = 10^{-4}$ and $\tau = 0.4$. At each stage, the figure on the left is the distribution of wealth, and the figure on the right is the distribution of the number of connections. The orange line is the poverty rate.

3.2.2 Statistics

In Figure 20, we can see that the link between wealth and connections allows for greater inequality. Whereas, before, the 99th quantile stabilized around 2.4 times w_0, it now stabilizes at $3.1\, w_0$, a 30% increase. Similar differences can also be seen for other statistics. The Gini index went from 0.31 to 0.37, an increase of 20%. The total wealth of the richest 10% went from 20% to 24%, an increase of 20%, and so on.

Therefore, the small advantage of allowing the agent with more wealth to have more trade opportunities (connections), and hence increasing his information, which is basically a given fact, is enough to increase inequality. Regardless of the fact that his chance of making a favorable trade is still the same as the poorer agent, ie $P(w_i|w_j) = 1/2$.

This mechanism, though apparently simple, is what allows richer agents to approach a multiplicative realm. Which means agents move away from the mainly additive nature of making small trades and slowly increasing their wealth - a Boltzmann-Gibbs like mechanism - and into a multiplicative situation which allows agents to increase their wealth by percentages - a Pareto like mechanism. This is key for achieving a distribution that resembles the real world.

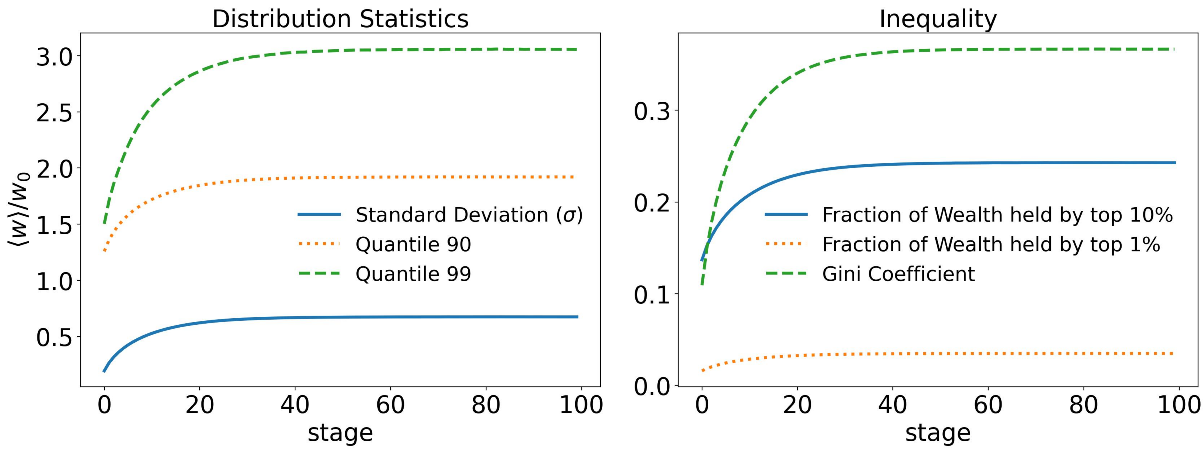

Figure 20 – Statistics for the wealth–connection linked model and taxation on wealth: $\gamma = 10^{-3}$ and $\tau = 0.4$. In the figure on the left, we can see the average wealth held by the 90 and 99 quantiles, i.e., the 10% and 1% richest agents, respectively, compared with the standard deviation. Note that these values are larger than in the raw case, Figure 21. On the right, the fraction of wealth held by the 10% and 1% richest agents is shown. The increase in wealth concentration is evident. Consequently, the Gini index also increases. The time evolution of the Gini index is also shown, stabilizing just below 0.4.

3.3 Favoring the Rich on Transactions

As defined in Section 2.5.3, this model describes a system where an agent's risk ratio, $P(w_i|w_j)$, given by Equation (2.10), and his number of connections, $f_c(w_i)$, given by Equation (2.8), are linked to his wealth. Hence, we have what is, essentially, a system where richer agents have more opportunities to do businesses while also taking on less risk, on average, than poorer agents. This mechanism, once again, further reinforces the system's preference towards richer agents and pushes it further into the multiplicative processes necessary for a Pareto tail to emerge.

The taxation here is on wealth, given by Equation (2.2), and all results presented in this section are for $N = 100{,}000$ averaged over 500 simulations.

3.3.1 Distributions

First, we start by exploring the evolution of the wealth distribution in this scenario, which is the main focus of this research: can the model reproduce real-world wealth distributions with these simple assumptions? The parameter values adopted are $\tau = 0.4$ and $\gamma = 10^{-4}$ (Figures 21 and 23), and $\gamma = 10^{-3}$ (Figure 24).

On the left-hand side of Figure 21, we can see that at the beginning of the simulation, the system quickly evolves into a exponential-type form. However, at stage 5, as the poverty rate (orange line) begins to increase, what resembles a Pareto tail begins to appear. At stage 10, when the poverty rate has passed 0.5% and continues to increase, the Pareto-shaped tail starts to become clearer. At stage 23, its shape reaches exactly the expected behavior, as can be seen in the fitted curve in Figures 22 and 23: A exponential-like middle and poor classes, with a Pareto-shaped tail for the upper 10% of the population and a poverty rate just above 1%. Interestingly, however, as the system reaches that point, the poverty rate begins to decrease due to the wealth tax, as inequality increases. At stage 38, we see that a "secondary" Pareto tail appears with a higher coefficient, much like the distribution for Japanese firms shown by Aoyama et al in [13], showing that, in practice, if given enough time, even the rich begin to differentiate themselves, some much richer than others. Then, at stage 45, poverty continues to decrease, around 0.1% (remember that taxation is levied on wealth), even though inequality is still present and evolving.

The much richer, due to the taxation on wealth, rather than annual income, help reduce poverty. The rounded part of the curve for higher values of wealth is due to finite size effects. To the right of the distributions, we can also see how the distribution of number of connections evolves.

We would like to note that, due to the not-so-large number of agents, we cannot claim that these behaviors are true power laws. For this, we would have to run simulations for at least 100 times larger number of agents, which is beyond our scope at the moment. Until the last stage presented in the image (stage 45), the system does not seem to have reached an equilibrium state yet.

This result not only shows that the model is capable of reproducing real world scenarios but it also highlights the points put forward at the end of section 3.2: The combination of reduced risk of doing business (a $P(w_i|w_j)$ that favors the rich) and increased number of opportunities/information (number of connections $f_c(w_i)$) for the rich is what finally drives the richest agents towards a multiplicative process. This, in turn, makes the Pareto tail appear. The hallmark we were looking for.

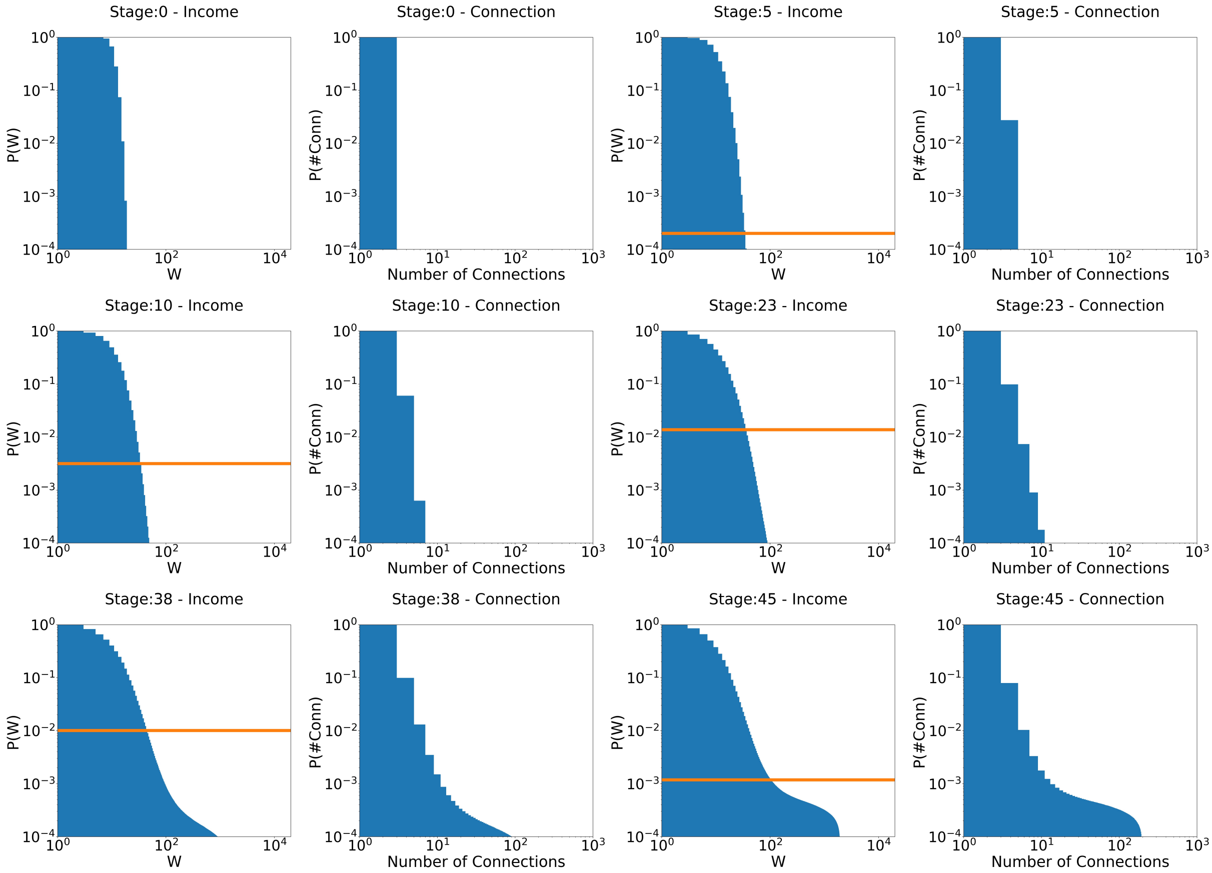

Figure 21 – Evolution of the distributions for the model that favors the rich: $\gamma = 10^{-4}$ and $\tau = 0.4$. At each stage, the figure on the left is the distribution of wealth, and the figure on the right is the distribution of the number of connections. The orange line is the poverty rate.

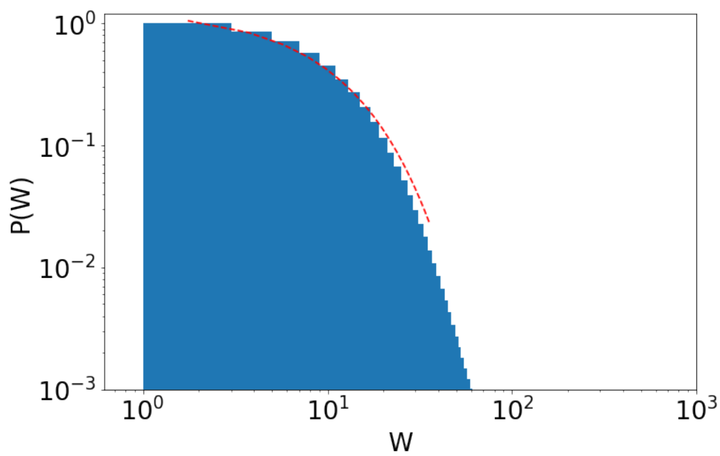

Figure 22 – Stage 23 of Figure 21. Exponential-like middle class (dotted red line) is clear, with $e^{-x/t}$ and $t = 1.18$. $\gamma = 10^{-4}$ and $\tau = 0.4$.

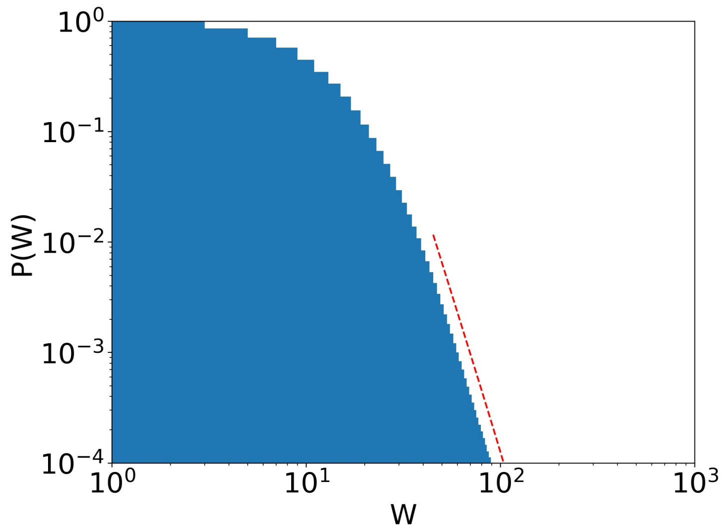

Figure 23 – Stage 23 of Figure 21. Pareto tail (dotted red line) is clear, with Pareto exponent $\alpha = 5.63$. $\gamma = 10^{-4}$ and $\tau = 0.4$.

To further explore the model, we apply a 10 times taxation increase thus taking γ from 10^{-4} to $\gamma = 10^{-3}$.

As can be seen in Figure 24, this change does not affect the model's ability to evolve to the expected/desired behavior. Hence, we can still see the exponential-Pareto tailed distribution we have been looking for.

However, even though finite-sized effects that start to appear around stage 45 for $\gamma = 10^{-4}$ (Figure 21) prevents us from being certain about these comparisons, the increase in taxation (see Figure 24), which leads to a reduction in inequality, also reduces tax revenue, and, therefore, the redistribution of wealth. This makes the system apparently reach an equilibrium state (from stage 42 to stage 99) faster and with a much more egalitarian wealth distribution. This is an important point because standard economic theory postulates that if taxes are too severe, decision making economic agents will be less encouraged to keep working and producing, thus reducing welfare. The reduction of tax revenues when taxation is increased could lead to an increase in poverty, but the limitations of our simulations do no allow us to verify this effect with certainty.

This result shows that in our model taxation is not absent of negative consequences - which is often the case with most, if not all, agent-based modeling approaches.

Therefore, to conclude, we have increased taxation which, according to most agent-based models should, theoretically, make the system more egalitarian. This, however, did not necessarily occur. Hence, given how inequality has persisted across all model versions, so far, this shows us four things about a system that favors the wealthy (note that, again, this is in the context of a model with perfectly equal tax redistribution; unequal redistribution—those that favor the poor, for example—could lead to different results):

1. The problem of poverty seems to not be simply solved with higher tax rates.

2. There might be tipping points where tax increases lead to reduction of tax revenues. A case well studied and understood within economics.

3. Indirectly, we can also infer that the way in which the tax collected is redistributed is also an essential point. In this case, the tax has been redistributed equally among the agents and the results are clear: poverty can still persist even with a perfect redistribution mechanism and high tax rates. However, at least in this case, the distribution of wealth can be stabilized, unlike what happens with lower taxes, where the distribution of wealth tends towards ever greater concentrations.

4. It is not necessary to eliminate inequality in order to end poverty. Properly equated tax policies and tax redistribution mechanisms can eliminate poverty while also allowing for a healthy elite to exist, simultaneously.

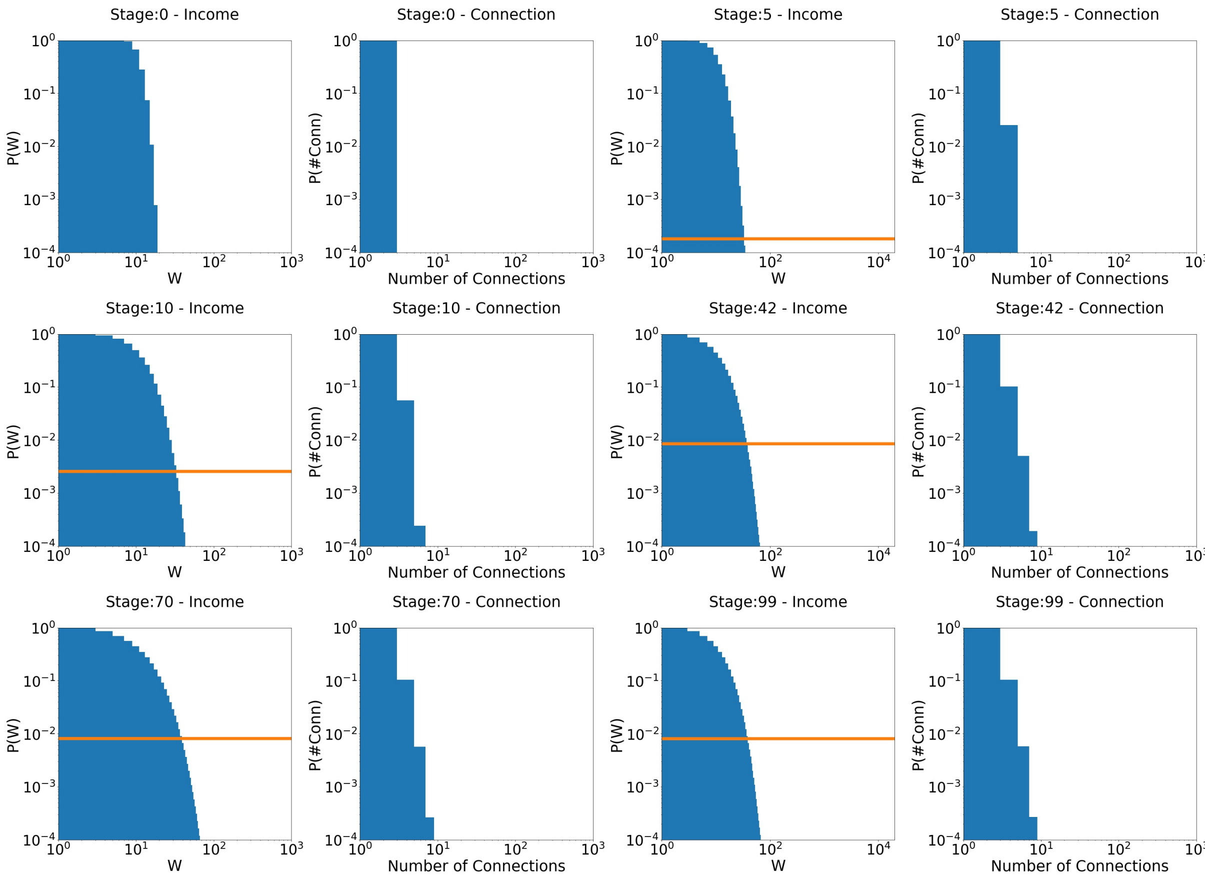

Figure 24 – Evolution of distributions for the model that favors the rich: $\gamma = 10^{-3}$ and $\tau = 0.4$. The orange line is the poverty rate.

These results show us that the model is perfectly capable of reproducing, qualitatively, the behavior of wealth distributions in the real world, from more egalitarian societies to strongly unequal ones, where even the richest end up separated into different classes.

This second observation (the double Pareto tail) is not only a remarkable result, but also something nonexistent in other approaches.

Furthermore, these results indicate that a certain level of inequality generally always exists, but that poverty can be combated with effective taxation on wealth and effective redistribution of these taxes.

3.3.2 Statistics

In order to better analyze the effects of the parameters of the model over its time evolution of the distributions, let us examine some of its statistics. The total tax revenue can be seen in Figure 25 for several values of γ (our tax rate). As the system evolves, lower tax rates lead to higher tax revenues that are applied to fewer and fewer agents. This is because lower tax rates allow a greater concentration of wealth, so fewer and fewer people can reach the minimum wealth required to pay taxes ($w_0 = 10$). This becomes clearer when we look at Figures 26 and 27, which show the evolution of the top 10% and top 1% of the population, respectively. We can see that the 90 quantile initially grows to 2 times w_0 and then suddenly falls around stage 40 for lower tax rates, although the percentage of wealth held by the top 10% continues to increase. This means that wealth is concentrated in a group of agents (much) smaller than the 10%. This quantifies the effects we saw in the last section: even among the richest agents, a differentiation starts, with some much richer than others (the second Pareto tail we saw). However, the simulation with the highest tax rate ($\gamma = 10^{-3}$) quickly reaches equilibrium (which can be better visualized in Figure 28) and inequality is greatly reduced. Figure 28b shows the Gini index. Note, however, that for $\gamma = 10^{-4}$ (tax rate), the bump around stage 40 is precisely the point at which we start to see finite size effects, which prevents us from drawing conclusions for stages higher than this value. Furthermore, in Figure 28, all the simulations eventually reach equilibrium, with a stable standard deviation.

Further analyzing the top[4] 10% of the distribution in Figure 26b (right), we can see that for $\gamma = 10^{-5}$ the fraction of wealth held by top 10% stabilizes just above 50% which is similar to Sweden, which has in 2021 registered a p90/p100 ratio of $58,9\%$ for wealth. Notice, however, that we are talking about a system with wealth taxation. Hence, a low rate like this is effectively much higher.

Similarly, analyzing the top 1% for the same value of γ in Figure 27b (right) we find that the top 1% holds just above 40% of the system's wealth. This number, on the other hand, is closer the p99/p100 ratio for Brazil in 2021, with $48,69\%$.

Therefore, this version of the model not only is capable of showing the wealth/income distribution we were looking for but it also, consequently, is capable of generating very

[4]For lower values of $\gamma = [10^{-4}, 10^{-5}]$ the same comment from before apply. Meaning, finite-size effects prevents us from making precise conclusions about wealth concentration past stage 40.

realistic values for the top 10% and 1%, respectively. Thus further establishing the validity of the model.

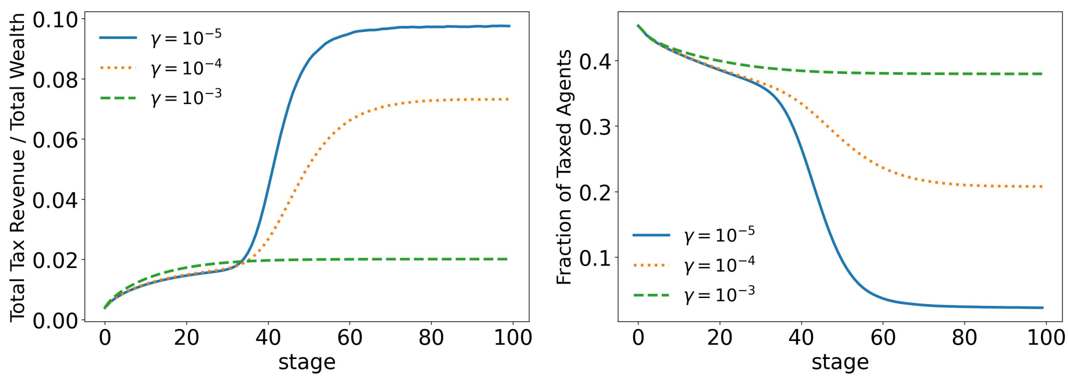

Figure 25 – Evolution of total tax revenue and total taxed agents for different values of γ (tax growth rate according to wealth).

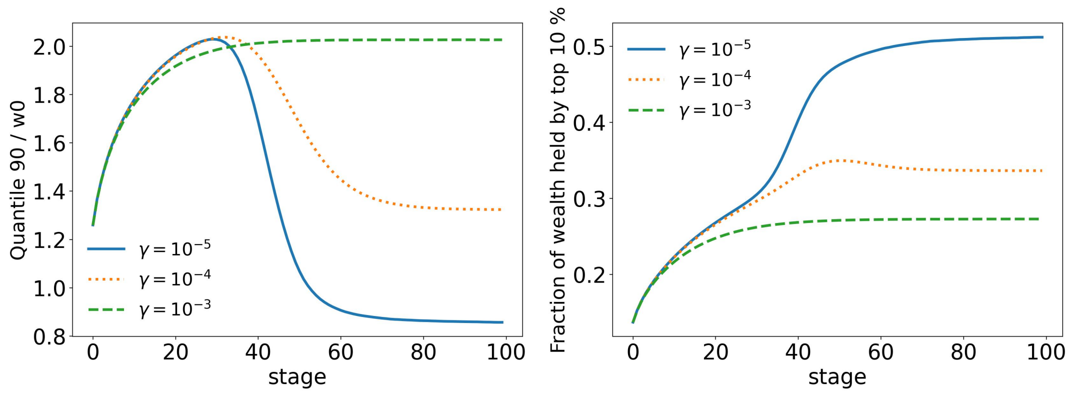

Figure 26 – Evolution of the top 10% of agents for different values of γ (tax growth rate according to wealth).

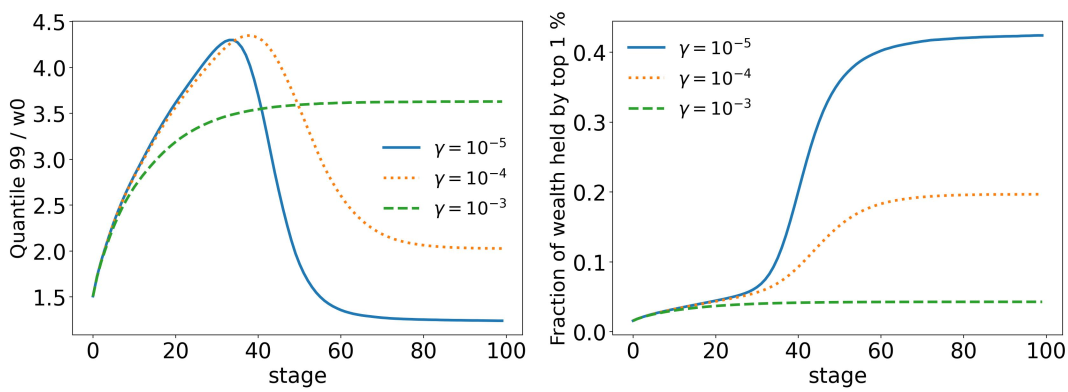

Figure 27 – Evolution of the top 1% of agents for different values of γ (tax growth rate according to wealth).

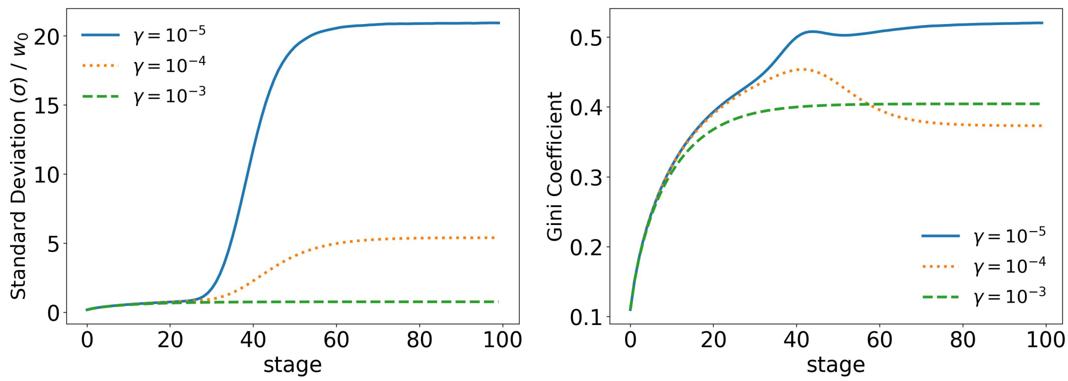

Figure 28 – Evolution of the standard deviation (σ) and the Gini coefficient for different values of γ (tax growth rate according to wealth).

3.4 Annual Income Tax Model

This scenario is similar to the model described in Section 3.3, where an agent's risk ratio ($P(w_i|w_j)$) and number of connections are linked ($f_c(w_i)$) to his wealth. The difference in this section is how taxation works. Here, instead of taxing the total wealth of an agent at the end of a stage, we tax the amount the agent earned at the last stage, which we call annual income.

This is the last scenario we are considering, as it is the closest representation to the taxation most commonly used around the world. Therefore, as a connection function, we use Equation (2.8); as risk ratio, we use Equation (2.10); and as taxation, Equation (2.5) is used.

All results presented in this section are for $N = 100{,}000$ averaged over 500 simulations.

3.4.1 Distributions

As usual, we start by exploring the wealth distribution in this scenario and try to evaluate if the change in taxation alters the qualitative characteristics we have found previously. In Figure 29 we can see the time (stage) evolution of the distribution for $\gamma = 10^{-1}$ and $\tau = 0.4$. Notice how, in accordance with what we saw in the previous section, the exponential-Pareto tailed distribution once again appear at stage 30. We also show a proper fit in Figures 30 and 31, where the exponential and Pareto tail are clearly visible.

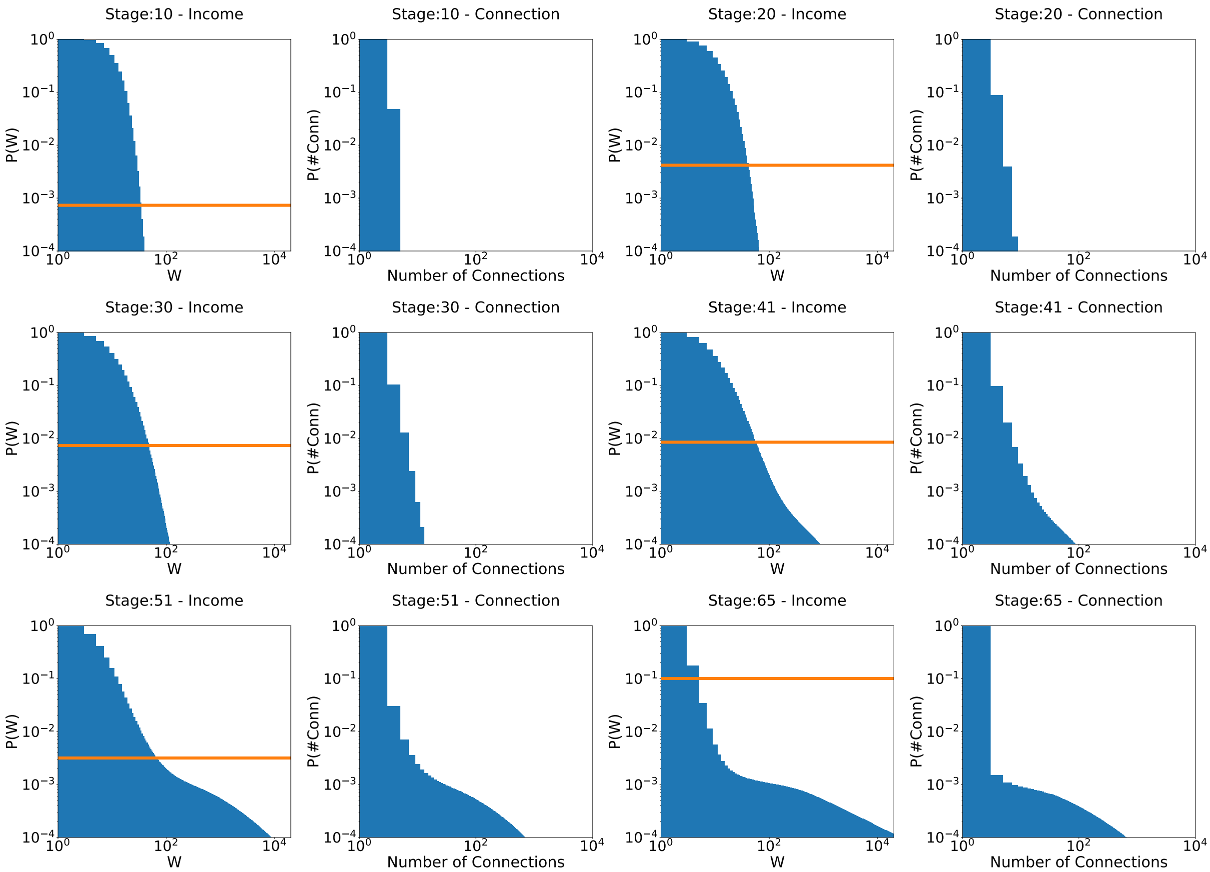

Figure 29 – Evolution of probability distributions for the model with capital gain taxation: $\gamma = 10^{-1}$ and $\tau = 0.4$. The orange line is the poverty rate.

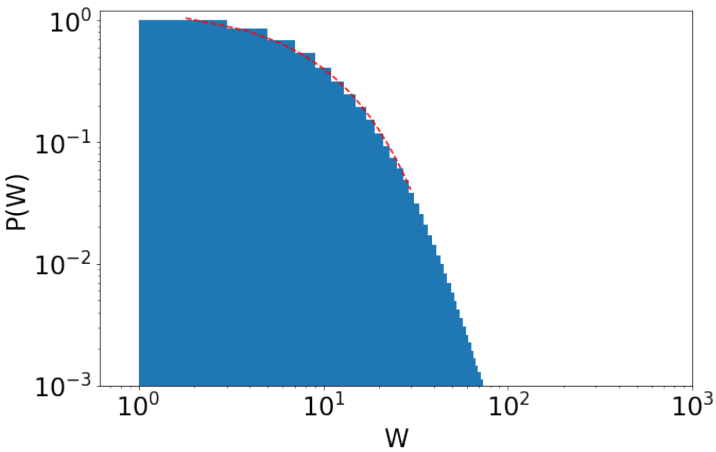

Figure 30 – Stage 30 of Figure 29. Exponential-like middle class (dotted red line) is clear, with $e^{-x/t}$ and $t = 1.17$. $\gamma = 10^{-1}$ and $\tau = 0.4$.

This time, however, we also see an even steeper second Pareto tail at stage 41, showing that inequality among the ultra rich, the top 0.1%, has increased even more. We can also see a proper fit of this phenomenon in Figure 32, where this second Tail is clearer. Interestingly, however, this is followed by a reduction in poverty rate between stages 41 and 51, although we have the highest concentration rate so far. This happens because as the income of those rich agents increases rapidly, so does the tax revenue to be redistributed. Hence, at least initially, we get a reduction in the poverty rate.

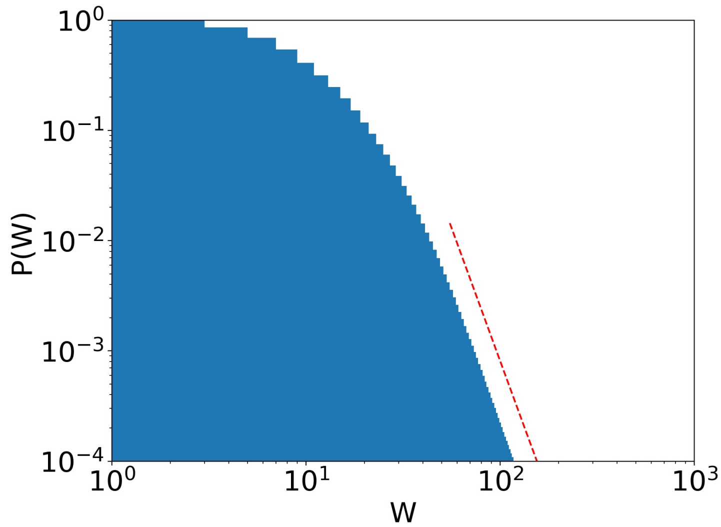

Figure 31 – Stage 30 of Figure 29. There is a Pareto tail (dotted red line) with $\alpha = 4.82$.

As the system continues to evolve, however, not only does inequality seem to keep growing, but the trend of reduced poverty dies out as the system shows no signs of reaching equilibrium. This is because, since we are taxing only annual earnings, as the ultra rich gain more and more wealth, there is less and less available income to trade among the other agents. Hence, the ultra rich can no longer keep increasing their wealth. Once this happens, tax revenues decrease and the "welfare state" collapses.

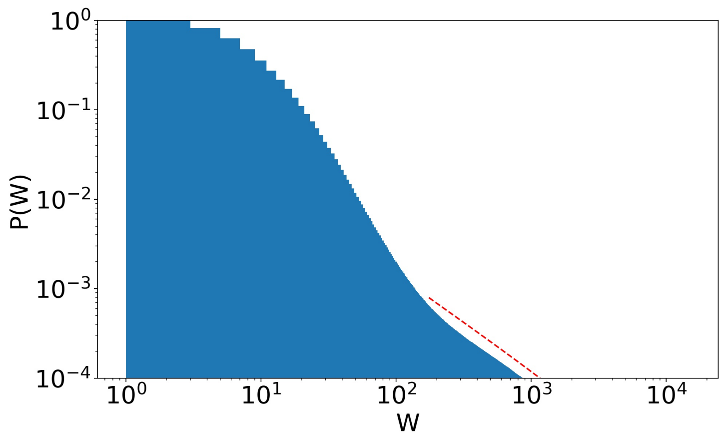

Figure 32 – Stage 41 of Figure 29, scenario of annual income taxation. A second Pareto tail appears, with $\alpha = 1.06$ (dotted red line). $\gamma = 10^{-1}$ and $\tau = 0.4$.

But perhaps the most interesting part about these results is the fact that it is, once again, similar to reality. When a country initially starts to grow its economy, the appearance of ultra rich individuals during this process is not uncommon, as the first to become rich gain a disproportional ability to take part in the broad economy and exponentially increase their wealth.

One such example is India, which went years on a trend of ever decreasing poverty which can be seen in Figure 33b [44], where we show how the poverty rate, here defined as the percentage of the population with a daily income lower than 2.25 US dollars per day, has consistently decreased from over 60%, in the 1980s to just below 12% in 2021.

This decrease in poverty is followed closely by a significant increase in income, as can be seen in the time evolution of the average Indian income in Figure 33c. Notice how from the 1950s until 1995 the average Indian income increased 250%, from 1235 Euros to 3110 Euros, with almost zero increase in inequality, as can be seen in the p90/p100 and p99/p100 ratios in Figure 33a.

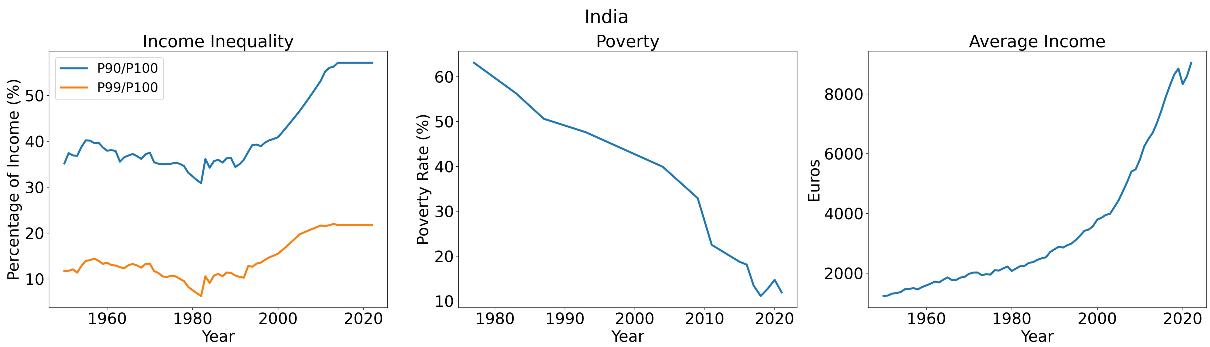

Figure 33 – Figure a (left): Time evolution of India's p90/p100 and p99/p100 ratios. Figure b (middle): Time evolution of India's poverty rate, here defined as the percentage of the population with daily income lower than 2.25 US dollars. Figure c (right): Time evolution of India's national average income in Euros. Notice how India grew its average income for decades without any significant changes to income inequality. Sources: [44] and [45].

However, from 1995 forwards India's inequality has started to increase both at the top 1% and at the top 10%, which is a similar movement from what we saw in our model - see Figure 33.

When countries grow richer quickly but fail to develop past the stage of being a manufacturer of simple goods and/or seller of commodities and into higher productivity endeavors - like high value goods and services - sometimes this improvement stagnates and the gap between richest and poorest start to grow larger. These types of development problems have been extensively studied in economics and each country has its particular problems. However, a famous case, often taught in universities, is called the Dutch disease [46], which shows how widespread such issues can be and reinforces the model's results. Nonetheless, given how complex this can be, discussing it is beyond the scope of this work.

In the context of our model, however, if we think about it in terms of an information differential, we notice that it is caused because rich agents are very "mobile", due to their large number of connections. So they are capable of finding good deals and increase their wealth even if most of the network is in disarray. On the other hand, poorer agents cannot.

Furthermore, these numerical results also confirm the widespread intuition among many economists that taxation of annual income alone fails to balance the concentration of income in a country that, with this type of taxation, always tends towards an ever more extreme concentration and does not stabilize the distribution. Consequently, according to the model, taxation of wealth and not only of annual incomes seems to be an valuable policy to avoid this trend.

To further explore this scenario, we also analyze the wealth distributions for $\gamma = 2.10^{-3}$ and $\tau = 0.4$ in Figure 34. We can see that by stage 19 the exponential-Pareto tailed distribution once again appears. This time, however, its slope (α in Equation (1.1)) seems higher which indicates a lower rate of inequality - we will verify this in the statistics section. We can also see that poverty rates are much higher than before, since it reaches 10% by stage 29. Also note that the second Pareto tail, another sign of higher inequality, appears much earlier. While before we saw its first signs at stage 41, now we see signs of it by stage 24. This shows us that there is an almost mechanical connection between tax rates, poverty and inequality. And that the model captures this relationship quite well.

Analyzing the transition from wealth taxation to taxation on annual income, the most common fiscal approach in the world today, reaffirms a central tenet of our model: the ability to qualitatively represent real-world wealth distributions based on a few simple and universally applicable assumptions. These foundational assumptions alone suffice to capture the essence of actual wealth disparities.

Of course, we can introduce additional economic variables to enhance the model's complexity, which would subsequently expand the number of parameters. Remarkably, these variables, in one way or another, are encompassed within the limited parameters employed in our model. This approach can be likened to a coarse-grained representation, where we capture the essential elements driving wealth inequality without delving into excessive intricacies.

However, when examining the fundamental principles underlying global wealth disparities, it becomes evident that advantages enjoyed by the wealthy in trade and the lower risk associated with business play a pivotal role. As our model illustrates, these factors are adequate to reproduce the qualitative aspects of wealth distributions seen across various countries.

This exploration underscores the inevitable nature of wealth concentration if effective fiscal policies and wealth redistribution, particularly targeting low-income individuals, are absent. Our findings suggest that, in the absence of such policies, the market alone is ill-equipped to address the challenge of wealth concentration. Effective wealth taxation, especially when coupled with equitable tax revenue redistribution, emerges as a plausible solution to mitigate this issue, as clearly demonstrated by our model.

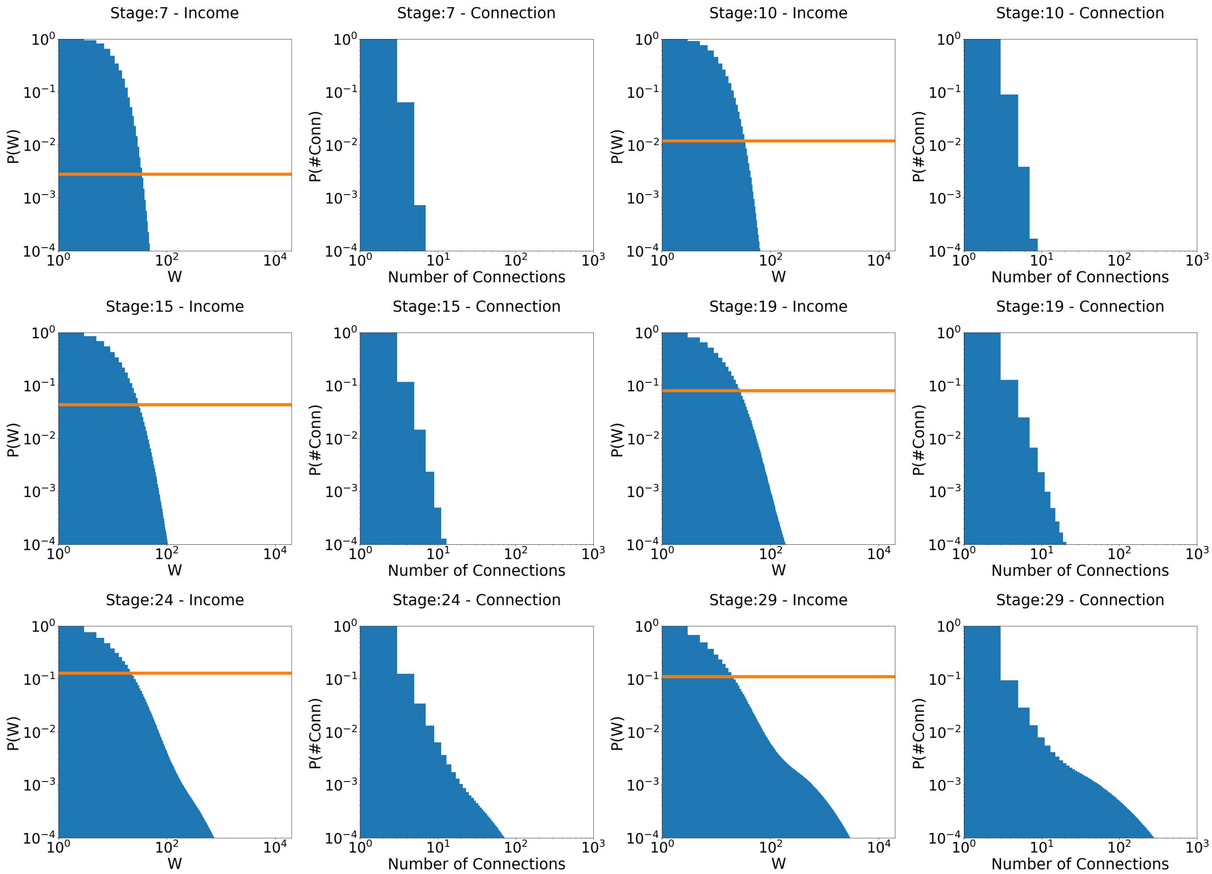

Figure 34 – Evolution of probability distributions for the model with capital gain taxation: $\gamma = 2.10^{-3}$ and $\tau = 0.4$. The orange line is the poverty rate.

3.4.2 Statistics

Examining the taxation dynamics depicted in Figure 35 and Figure 36, we gain valuable insights into the influence of the upper tax limit (τ) on our model. Notably, higher tax limits significantly affect the model's temporal progression.

As evident from both Figure 35 and Figure 36, elevating the upper tax limit extends the duration during which tax-driven wealth redistribution operates. This extension allows for the accumulation of higher total tax revenue, consequently sustaining effective wealth redistribution for a more prolonged period. Nevertheless, it is notable that, despite variations in the upper tax limit, the model ultimately converges along a similar trajectory.

Moreover, these figures indicate a noteworthy shift in tax revenue patterns due to the introduction of this novel form of taxation. Compared to previous stages in the model, tax revenues are markedly reduced, underscoring the distinctive nature of this taxation method.

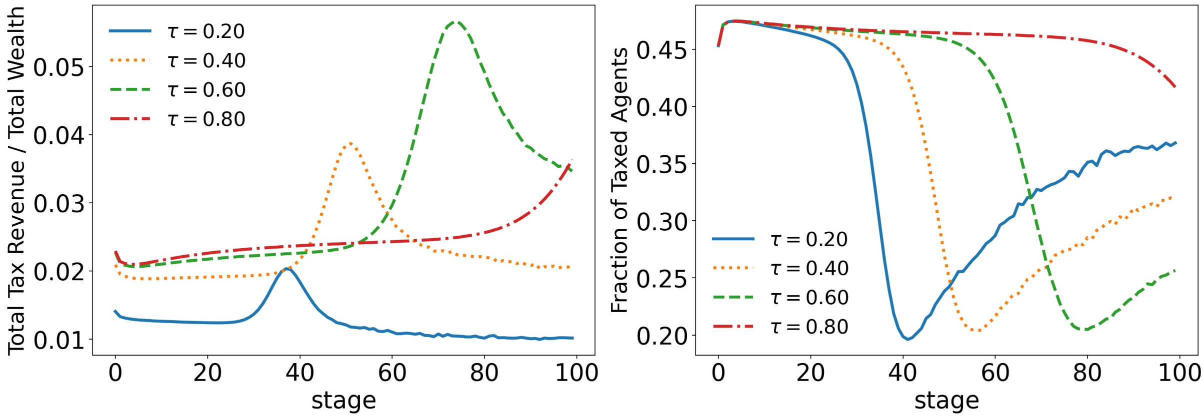

Figure 35 – Evolution of total tax revenue and total taxed agents for different values of τ (tax limit). $\gamma = 10^{-1}$.

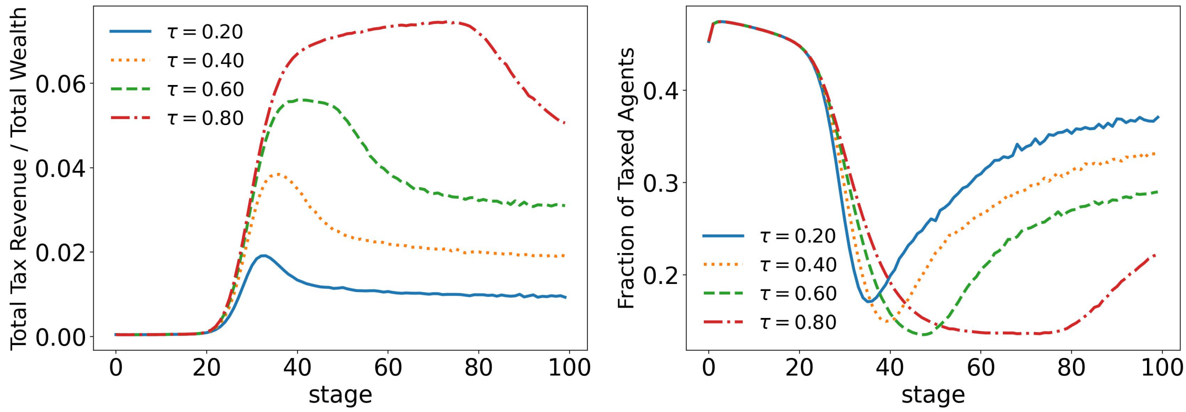

Figure 36 – Evolution of total tax revenue and total taxed agents for different values of τ (tax limit). $\gamma = 2.10^{-3}$.

When we look at the behavior of the top 10%, however, it appears that, in fact, the upper tax limit is actually able to limit how much of wealth the top 1% and top 10% hold. Specifically, if we look at Figure 38b for $\gamma = 2.10^{-3}$, we notice that the lower level of taxation, when compared to $\gamma = 10^{-1}$, allows the system to reach certain stages of evolution much faster - as we have seen before. Hence, while in Figure 37b it is not clear whether the different tax limits are capable of changing the equilibrium state of the model, in Figure 38b it seems like it indeed is. Notice how for each value of τ, a different, and lower, fraction of wealth is held by the top 10%. Of course, if we also analyze Figure 42a, we can see that besides for $\tau = 0.20$, the system has not yet reached equilibrium. Therefore, we cannot know for certain if that is the case or not. Nonetheless, the result is not unexpected.

Similar behavior can also be seen in Figures 39 and 40 for the top 1%. Meaning, the tax limit (τ) slows down the evolution of the system and, once again, $\gamma = 2.10^{-3}$ in Figure 40 suggests that different tax limits are indeed capable of changing the amount of wealth held by the elite (top 1% in this case). Here, however, based on the system's trajectory it becomes clear that for such a low taxation level, what we see in the graphs of the top 10%, Figure 38, is mainly representative of the top 1%. Hence, the top 90% to 99% actually control very little wealth.

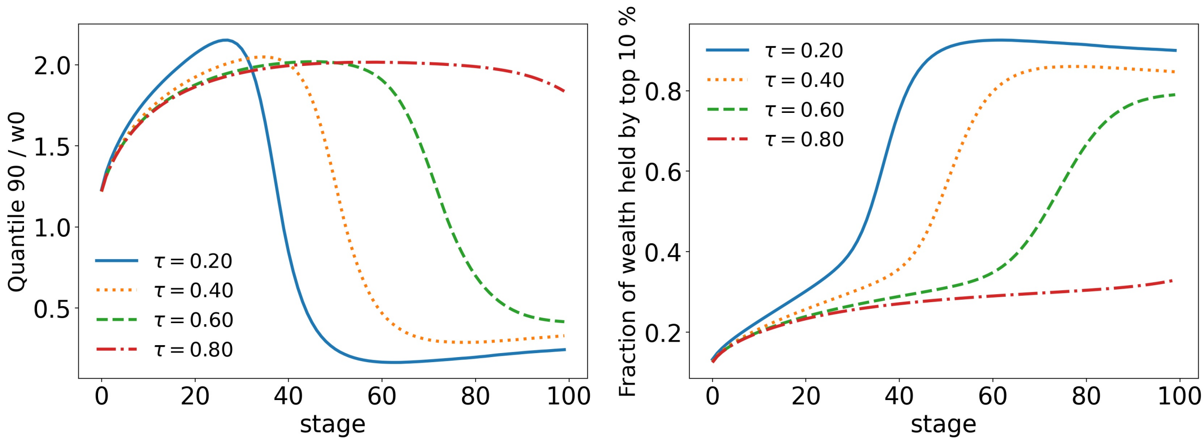

Figure 37 – Evolution of the top 10% of agents for different values of τ (tax limit) and $\gamma = 10^{-1}$.

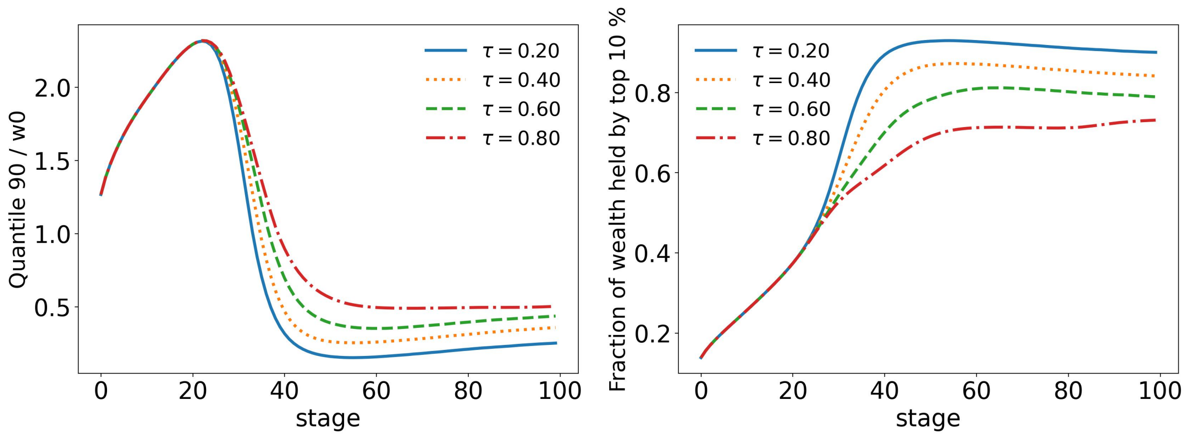

Figure 38 – Evolution of the top 10% of agents for different values of τ (tax limit) and $\gamma = 2.10^{-3}$.

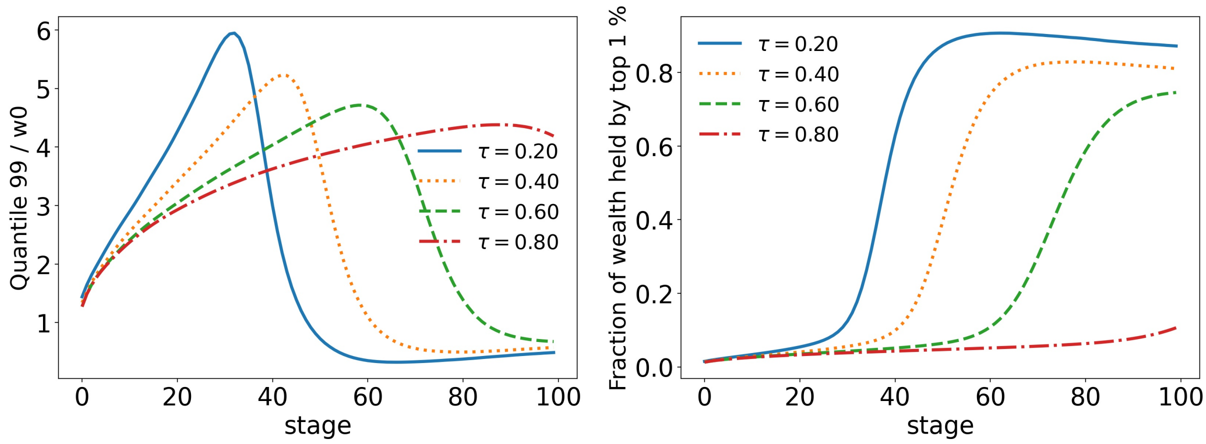

Figure 39 – Evolution of the top 1% of agents for different values of τ (tax limit) and $\gamma = 10^{-1}$.

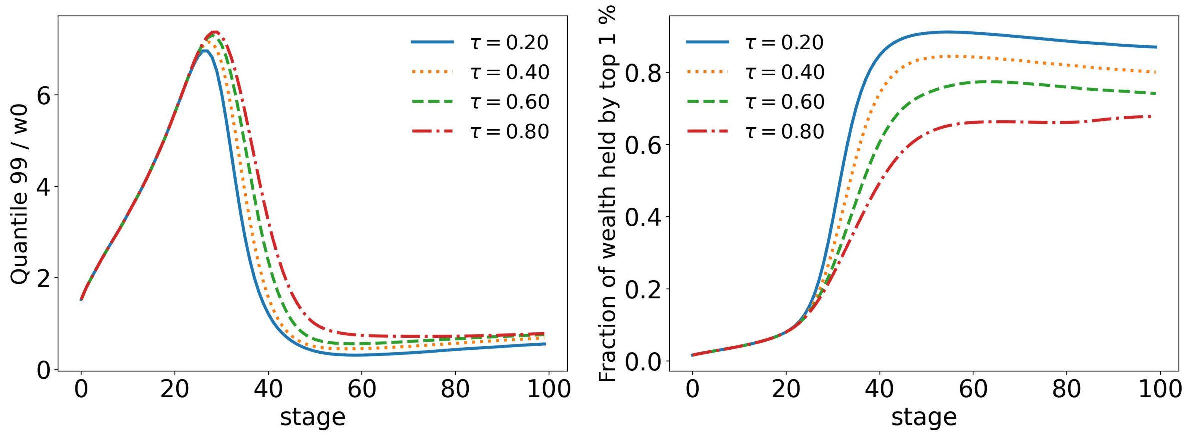

Figure 40 – Evolution of the top 1% of agents for different values of τ (tax limit) and $\gamma = 2.10^{-3}$.

Further analyzing the top of the distribution, it is clear that the concentration of wealth seen in this scenario reaches much higher levels. The richest 1% end up holding up to 90% of the wealth in both cases ($\gamma = 10^{-1}$ and $\gamma = 2.10^{-3}$), which leads Gini coefficients that also reach similar levels and goes up to 0.9.

Moreover, as we saw in the last section, the change among the rich happens in a faster and stronger way. This time, the 90th quantile reaches two times w_0 much earlier (stage 20 and stage 12 vs. stage 40 in the wealth tax model) than before and drops to much lower values: in scenario 3, the lowest value of the 90th quantile is just above $0.8w_0$ (Figure 26), whereas now it is only $0.25w_0$ (Figure 37 and 38). A similar trend can also be seen at the 99th quantile (Figure 39 and 40). This shows us that the wealth of the population is not, in fact, in the hands of the richest 10% or even the richest 1%, but, actually, in the hands of the 0.0% group (top 0.001%, 0.0001%, 0.00001%, etc.).

As high as these values might seem, they are, actually, not entirely unreasonable. let us focus on cases where it is more likely, though not certain, for the system to have reached equilibrium, which for $\gamma = 2.10^{-3}$ is the case of $\tau = 0.2$ (Figure 42):

- $\gamma = 2.10^{-3} \rightarrow p90/p100 \approx 85\%$;
- $\gamma = 2.10^{-3} \rightarrow p99/p100 \approx 80\%$;

South Africa has a p90/p100 ratio of 85, 6% and a p99/p100 ratio of 54, 92% for wealth[5]. Hence, if we take in consideration that perhaps South Africa has not reached equilibrium yet, the numbers shown by the model are actually quite possible.

Therefore, the impact of the highest level of taxation, τ, does not seem to change the qualitative behavior of the time evolution of the stages; it just delays the same pattern, and it is not a pattern-changing parameter. The parameter can, however, substantially alter the limits to how much wealth the elite is able to control. Hence, higher tax limits (τ) lead to lower wealth held by the top 1%.

Then, all the analyzed aspects of the model seem important (i) the value of the parameter γ (the tax rate), (ii) the type of taxation (on wealth or on annual income), (iii) how the total tax revenue is redistributed (to be analyzed in a future work), and (iv) the tax limit τ. All other parameters do not change the tendency of wealth concentration, according to our simulations.

Furthermore, it might be worth mentioning that the case for $\gamma = 10^{-1}$ is quite extreme. Notice how the taxation, Equation (2.5), reaches its maximum (τ) at $G_i = \frac{\tau}{\gamma} = \frac{\tau}{0.1} = 10\tau$. Hence, for $\tau = 0.2$, we have maximum taxation at $G_i = 2$, which is only $0.2w_0$. Therefore, even though the results are useful to understand and validate the model, they are a very unrealistic scenario, with very high taxation levels. And, yet still, it leads to high levels of concentration and inequality.

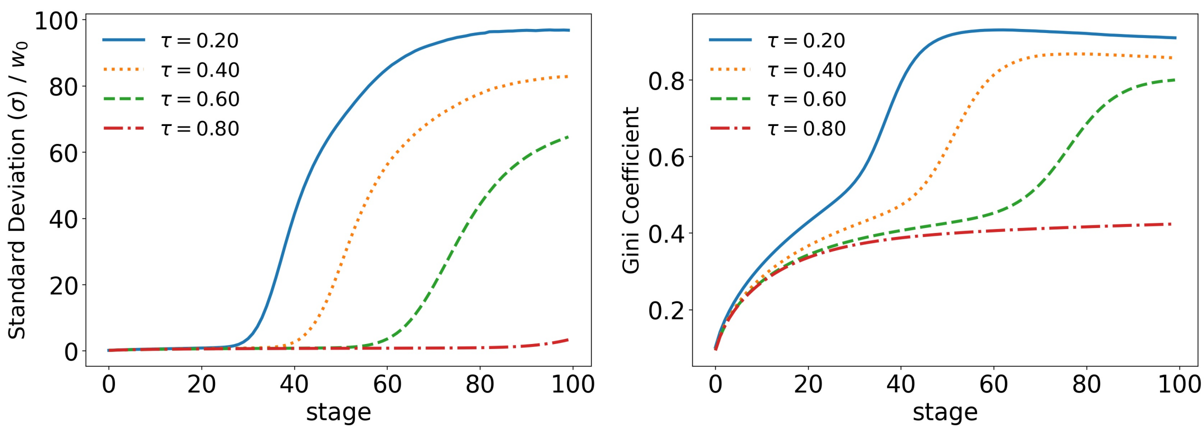

Figure 41 – Evolution of the standard deviation (σ) and the Gini coefficient for different values of τ (tax limit) and for $\gamma = 10^{-1}$.

[5]These values refer to the year of 2021.

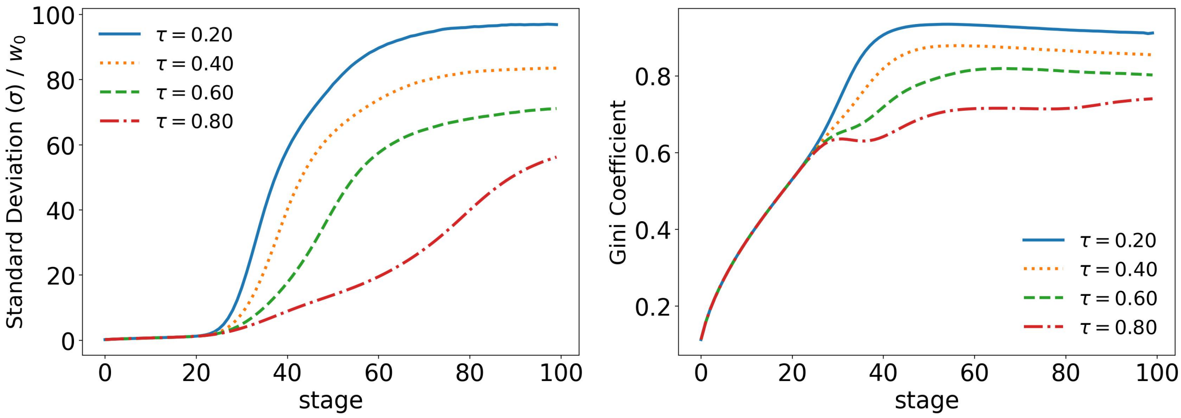

Figure 42 – Evolution of the standard deviation (σ) and the Gini coefficient for different values of τ (tax limit) and for $\gamma = 2.10^{-3}$.

4 Conclusion

Over the course of this thesis, we have adequately shown how research on the topic of income/wealth distributions has developed over time on the field of complex systems. By doing so, we saw how the problem is commonly framed as well as its typical pitfalls. In particular, how most models are based on conservative principles and how this severely hinders their ability to produce non-trivial/closer to reality solutions.

Based on this, we explored alternatives in a search for a generating process that was capable to produce the distribution we were looking for - a Boltzmann-Gibbs-Pareto tailed distribution - and through this processes showed how information, stochasticity and networks are intrinsically linked. With this we were able to frame the problem in a new light: as an information asymmetry between agents. Thus linking our model's agent connectivity to their wealth.

Following this finding, we discussed the role that preference has in these types of systems and how it is linked to power-laws - one of the reasons why we introduced the wealth-connectivity link in the first place. This, in turn, led us to further increase our model's preference towards richer agents by introducing an asymmetric risk factor that slightly favors the rich.

With all that combined we were able to build a model with very simple and universal premises and while doing so, we have showed that it can qualitatively reproduce current wealth/income distributions, with their middle and poor classes having a Exponential distribution and a Pareto tail for the richest parts of the population.

Moreover, we analyzed the model's time evolution through not only their distribution but also through statistics commonly used to study inequality. This allowed us to validate the Pareto tail we have found. Furthermore, it showed the complexities it could generate while still presenting reasonable statistics. In other words, we have showed that the model resembles the real world not only on its shape but also on how it evolves over time.

Furthermore, we have also shown that in most cases, taxation on annual income seems unable to stabilize the distribution. Hence, societies might need some form of combination of the two (wealth and income), given how hard it is to tax wealth, in order to reduce inequality in the future. After all, the model shows that a healthy equilibrium can be reached by taxing wealth even at very low rates, while simultaneously allowing a healthy economic elite to exist. Nonetheless, while it is important to note that some of these conclusions would likely change if redistribution favored the poor, particularly about how high the tax rate (γ) must be to reduce inequality and control poverty, its fundamental features are still highly likely to be correct given all the elements we have explored.

Therefore, it is reasonable to conclude that the model's principles are very likely to be correct. In other words, lower risks (greater bargaining power) and greater number of opportunities (connections/information) associated with wealthy agents are a central part of the inequality equation, so it is not simply a matter of taxing these agents more, but reducing these differences as well. A high concentration scenario does not exist when opportunities are equal and/or markets are regulated and an increase in the upper tax limit (τ) seems capable of controlling how much wealth the top 1% can control.

This means that among the main drivers of inequality, we should consider the problem of unequal opportunities and the difference in risks associated with doing business. Richer agents have much more access to business opportunities and a much lower risk rate than the rest of society, which leads to growing inequality.

Furthermore, it is worth pointing out that during the development of this model more complex elements were tested and evaluated. In particular, we explored bracket-based taxation schemes and distance-based trading advantages - all realistic mechanisms. This added complexity, however, produced similar qualitative behavior which reinforced that the basic elements, all of which remain, are what truly drives the model. In other words, it is the information differential between rich and poor agents and the reduced risk that those agents enjoy that allows the model to produce these results.

Finally, there are multiple questions yet to be explored about this model which we hope to research in future work. For example, we have yet to study the effects of unequal redistribution of taxes, hybrid taxation schemes, how taxation affects growth (remember that the model is not conservative), how an increase in the number of stages can help us better understand equilibrium states of some simulations, how with a larger set of simulations we can more readily verify the tail and some of the behaviors we have found, how stable the rich status is and further explore the robustness of the model - in other words, how much more can we change, while maintaining its core features, and still arrive at the same qualitative result.

We thank the financial support from the Brazilian scientific agencies Conselho Nacional de Desenvolvimento Científico e Tecnológico (CNPq) and Fundação de Amparo à Pesquisa do Estado do Rio de Janeiro (FAPERJ).

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