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A study of anomalies using functional integration and perturbative calculations.

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"A STUDY OF ANOMALIES USING FUNCTIONAL INTEGRATION AND PERTURBATIVE CALCULATION"

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Abstract

We present two lines of investigation involving anomalies. First, we review mechanisms behind the classical and quantum conservation of symmetries using functional integration. This discussion clarifies conditions for quantum violations, as acknowledged in chiral theories. Then, we elucidate the subject of gauge anomaly cancellation when all fields are quantized. Such an outcome requires gauge invariance of the bosonic measure, so our first object is proving this invariance within Fujikawa's approach. Second, we investigate anomalies in fermionic perturbative amplitudes using Implicit Regularization. The discussion of the single-axial triangle fundaments this analysis, bringing the elements necessary to approach the single-axial box. When organizing their mathematical structure, we highlight the role of traces involving the chiral matrix. Choosing a specific expression for them reflects on the position of symmetry violations, which has implications regarding the linearity of integration. Power counting and tensor structure imply the presence of surface terms related to momenta ambiguities. We present the results without computing these surface terms. In this neutral perspective, we explore possibilities achieved under different prescriptions.

Keywords: Gauge and Chiral Anomalies. Divergences. Implicit Regularization.

Resumo

Nós apresentamos duas linhas de investigação envolvendo anomalias. Primeiro, revisamos mecanismos por trás da conservação clássica e quântica de simetrias usando integração funcional. Essa discussão clarifica condições para a violação quântica, como reconhecido em teorias quirais. Em seguida, elucidamos o assunto de cancelamento da anomalia de calibre quando todos os campos são quantizados. Isso requer a invariância de calibre da medida bosônica, então nosso primeiro objetivo é provar essa invariância através do método de Fujikawa. Segundo, investigamos anomalias em amplitudes perturbativas fermiônicas usando Regularização Implícita. A discussão do triângulo com um vértice axial fundamenta essa análise, trazendo os elementos necessários para abordar o *box* com um vértice axial. Ao organizarmos suas estruturas matemáticas, destacamos o papel de traços envolvendo a matriz quiral. Escolher uma expressão específica para eles reflete na posição de violações de simetria, trazendo implicações quanto à linearidade da integração. Contagem de potências e estrutura tensorial implicam na presença de termos de superfície relacionados a combinações ambíguas de momenta. Apresentamos esses resultados sem calcular termos de superfície. Nesta perspectiva neutra, exploramos possibilidades encontradas em prescrições diferentes.

Palavras-chave: Anomalias de calibre e quiral. Divergências. Regularização Implícita.

Contents

Chapter 1

Introduction

When building an interacting model through a quantum field theory, one starts by constructing a functional of free fields whose interaction dynamics one aims to describe. In general, the original functional exhibits invariance under global transformations, with parameters that do not depend on the space-time position. Then, modifying this functional promotes the symmetry to a local one. The new functional emerges after introducing a set of fields (called gauge fields), with transformations chosen to ensure invariance when parameters depend on the spacetime. These local transformations of the fields are called gauge transformations, and the corresponding symmetry is called gauge invariance. The main consequence is the generation of interaction between the previously free fields. This proposed interaction generates previsions (i.e., decay rates or cross sections) capable of being compared with experimentally measured quantities.

Quantum Electrodynamics is a well-known example of this construction, corresponding to the quantum field theory for electromagnetic interaction. The first step is to build the Dirac action, describing free spin-(1/2) fermions (such as electrons and positrons). Although this functional exhibits global $U(1)$ invariance, the presence of derivatives prevents gauge invariance. The solution comes when substituting conventional derivatives with covariant ones, which induces coupling with a gauge field (interpreted as the photon field). This field arises from the only generator of the Abelian symmetry group $U(1)$. There are interacting terms involving gauge and matter fields in the modified action, so all mentioned contributions constitute a locally invariant object. One adds a gauge invariant term involving only the gauge field (the Maxwell action) to furnish dynamics for the photon.

Something analogous occurs when developing Quantum Chromodynamics to describe the strong interaction. The outset is on the Dirac action, now built with free quarks, symmetric under global $SU(3)$ symmetry. Promoting it to be local generates interaction terms involving gauge and matter fields. As $SU(3)$ has eight generators, eight gluons emerge as gauge fields. The difference from the abelian case resides in the non-commutative character of the algebra, which implies self-interacting gauge fields. The Yang-Mills functional is introduced to provide dynamics for gauge fields.

Regarding Electroweak Theory, the symmetry group is $SU(2) \times U(1)$. As this theory unifies electromagnetic and weak interactions, it adds new gauge bosons ($W^{\pm}$, Z) besides that corresponding to the photon. The main difference is that these new fields are massive, while gauge invariance does not admit this type of contribution to the action. The strategy to deal with this problem is to start from a massless theory, with the Higgs mechanism generating masses. That means quarks and leptons are seen as massless Weyl fermions (with defined chiralities) instead of Dirac fermions. Under these circumstances, the functional displays gauge invariance before spontaneous symmetry breaking. That is crucial for the renormalization of the theory. The masses are generated for all the experimentally known massive fields without spoiling the renormalizability. This mechanism is extended to the group $SU(3) \times SU(2) \times U(1)$, defining the Standard Model (SM), which unifies the three mentioned interactions.

There is another issue to be faced by the SM, the so-called *anomalies*. They are quantum violations of symmetries originally present at the action. They have a vast history, initiated by Johnson's discovery of the two-dimensional chiral anomaly [1]. A few years later, this subject received prominence due to the Adler-Bell-Jackiw anomaly [2, 3]. Both refer to the quantum breaking of the global (constant parameter) chiral symmetry, present in theories with massless fermions. It also became clear the impossibility of simultaneous maintenance of chiral and gauge symmetries at the quantum level [4]. These symmetries are mixed in the SM before spontaneous symmetry breaking, which means that gauge invariance is apparently broken at the quantum level. This phenomenon is known as gauge anomaly. In the SM, gauge invariance is only achieved through a careful adjustment of the group representation where one puts the three families of quarks and leptons so that anomalous contributions from both sectors cancel each other. Meanwhile, gauge invariance is necessary to ensure renormalizability and unitarity to the theory. Gauge anomalies modify Slavnov-Taylor identities, preventing one from relating distinct renormalization constants with each other and canceling infinities systematically to all orders of perturbation theory [5, 6]. We end with an uncomfortable situation where the SM is a superposition of apparently inconsistent theories, which result in a consistent one by a very peculiar arrangement.

This situation motivated investigations on gauge-anomalous theories. Jackiw and Rajaraman [7, 8] showed that chiral Quantum Electrodynamics in two dimensions is consistent and unitary. Furthermore, the gauge field, initially massless in the classical action, became massive after radiative corrections without needing a Higgs mechanism. Faddeev and Shatashvili clarified the quantization of this type of theory [9]. They introduced new quantum degrees of freedom that provided an equivalent gauge theory (without anomalies). In addition, Harada and Tsutsui [10] and Babelon, Schaposnik and Viallet [11] observed that these new degrees could be obtained naturally through the employment of the Faddeev-Popov procedure. That allowed them to express the effective action as a gauge scalar for any space-time dimension. These results suggested that theories with gauge anomalies could be consistent.

By taking into account gauge invariance of the gauge field measure, in the context of functional integrals, a recent investigation [12] showed the vanishing of the insertion of the anomaly operator in any correlator of gauge invariant operators. This result suggested that the anomaly vanishes in the part of the Hilbert space associated with physical states. That motivated us to investigate gauge invariance of the boson measure in more detail. We do this in Chapter (2), providing explicit proof of this fact that is, up to our knowledge, absent from the literature.

We continue to investigate symmetries in the quantum context through an approach known as *Implicit Regularization* (IReg), a procedure to identify and separate the divergent part of Feynman diagrams by manipulating the integrands before integration. The study of an amplitude associated with the neutral pion decay (the single axial triangle) establishes the foundations for this analysis. Afterward, we examine the possibility of one amplitude with an analogous mathematical structure (the single axial box) exhibiting the same characteristics. Hence, surveying features shared by these processes highlights new aspects of the anomalies. That corresponds to the second part of this thesis, whose development occurs in Chapter (3).

Conclusions will be presented separately for Chapters (2) and (3) since they use different methodologies to approach the subject of anomalies.

Chapter 2

Gauge Anomaly and Invariance of the Bosonic Measure

Investigating the consequences of gauge symmetry in classical and quantum theories is the general objective of this chapter. Starting with the classical discussion in Section (2.1), we use arguments involving action invariance to achieve current conservation. These preliminary calculations work as a guide to explorations at the quantum level, made in Section (2.2). After finding requirements for quantum invariance, the source of violations in functional integration is discussed in Section (2.3). With the mathematical structure of the anomaly in our hands, we use a simple procedure to show that its expected value vanishes when quantizing all theory fields.

The gauge invariance of the gauge field measure is central to this argumentation. This property has several usages in the literature, as in investigations involving the Faddeev-Popov method. Since there is (up to our knowledge) an absence of explicit demonstration of this invariance, our first contribution is to provide proof of it. To do so, we use general functional integral arguments to show that the Jacobian associated with the measure has to be 1 (one) when inserted in correlation functions of gauge-invariant operators. Performing the same analysis for general operators would complete this demonstration. Since this step brings complications, we employ a Fujikawa-like approach to calculate this Jacobian explicitly and show that it is 1 in general.

2.1 Classical Symmetry

This section aims for a preliminary understanding of gauge theories, emphasizing current conservation at the classical level. It is also the moment to introduce notations, which follow the material from R. Jackiw's course in reference [13] and G. L. S. Lima's works [14, 15].

Throughout the Introduction, we mentioned some aspects of theories employed to describe fundamental interactions. The starting point was the functional associated with the dynamics of free matter fields. This object is not invariant under local transformations since it contains derivatives. So, the idea was to implement this symmetry by making the derivative covariant. The price paid is inducing terms of interaction with gauge fields. In other words, gauge symmetry generates dynamics among the fields described by a theory [6]. A contribution associated with free gauge fields is also necessary. Below, we write the action with these two sectors separated, so it is clear that each part is invariant by itself:

$$S\left[\psi,\overline{\psi},A_\mu\right] = S_G\left[A_\mu\right] + S_M\left[\psi,\overline{\psi},A_\mu\right]. \tag{2.1}$$

The vector $A_\mu = A_\mu^a T_a$ represents the gauge fields with T_a being generators of the gauge group, while $(\psi, \overline{\psi})$ represent fermionic matter fields.

Saying that the action is invariant means no changes occur when fields modify through a given set of transformations. Our concern is with gauge theories, in which case these transformations belong to special unitary groups $SU(N)$. Its generators satisfy commutation relations like

$$\left[T^a, T^b\right] = i f^{abc} T_c, \tag{2.2}$$

along with the normalization

$$\operatorname{tr}\left(T^a T^b\right) = -\frac{1}{2}\delta^{ab}. \tag{2.3}$$

The symbol f^{abc} represents the structure constants, which have the property of total antisymmetry through index permutations. Indices denoted by Latin letters refer to internal degrees of freedom, ranging over the group dimension (equivalent to the number of generators). As gauge fields take values on the Lie algebra of the symmetry group, there is one field for each generator. Greek letters in the indices refer to Minkowski space-time in the chosen theory.

To analyze current conservation, let us adopt an arbitrary element $g = e^{i\theta(x)}$ to perform a transformation. The parameters depend on the space-time position $\theta(x) = \theta^a(x) T_a$, characterizing a local transformation. As mentioned, the action is invariant under simultaneous changes of boson and fermion fields

$$A_\mu \rightarrow A_\mu^g = g A_\mu g^{-1} + \frac{i}{e}\left(\partial_\mu g\right) g^{-1}, \tag{2.4}$$

$$\psi \rightarrow \psi^g = g\psi, \tag{2.5}$$

$$\overline{\psi} \rightarrow \overline{\psi}^g = \overline{\psi} g^{-1}. \tag{2.6}$$

By considering small values for the parameter, we take its first-order contribution to obtain infinitesimal transformations

$$A_\mu \rightarrow A^g_\mu = A_\mu - \frac{1}{e}\mathcal{D}_\mu\theta, \tag{2.7}$$
$$\psi \rightarrow \psi^g = \psi + i\theta\psi, \tag{2.8}$$
$$\overline{\psi} \rightarrow \overline{\psi}^g = \overline{\psi} - i\overline{\psi}\theta. \tag{2.9}$$

We define the covariant derivative of Lie algebra valued quantities through the mathematical expression

$$\mathcal{D}_\mu\theta = \partial_\mu\theta + ie\left[A_\mu, \theta\right], \tag{2.10}$$

so using the commutation relations allows specifying its components

$$\mathcal{D}_\mu\theta = T^a\left(\partial_\mu\delta^{ac} - ef^{abc}A^b_\mu\right)\theta^c \equiv T^a\mathcal{D}^{ac}_\mu\theta^c. \tag{2.11}$$

Since the action is invariant under local transformations, it is also invariant under global transformations. As the parameter is constant in the second case, the derivative $\partial_\mu\theta$ cancels out within the vector field transformation. Then, by reversing this line of reasoning, starting from global transformations and imposing dependence on the position is feasible. In such a case, the absence of the inhomogeneous term implies symmetry is lost. That means the following variation must be proportional to derivatives of the parameter

$$\delta S_M = S_M\left[\psi^g, \overline{\psi}^g, gA_\mu g^{-1}\right] - S_M\left[\psi, \overline{\psi}, A_\mu\right] = \int \mathrm{dx}\ \partial_\mu\theta^a\left(x\right)J^\mu_a\left(x\right), \tag{2.12}$$

where we introduce the vector $J^\mu_a\left(x\right)$, determined by the fields present in the model. Recalling that both sectors of the action are invariant when considered by themselves, we focus exclusively on the matter action. On the other hand, an infinitesimal transformation over the action leads to another form for the same variation:

$$\delta S_M = \int \mathrm{dx}\ \theta^a\left(x\right)\left[\frac{\delta S_M}{\delta\psi\left(x\right)}iT^a\psi\left(x\right) - i\overline{\psi}\left(x\right)T^a\frac{\delta S_M}{\delta\overline{\psi}\left(x\right)} + f^{abc}A^b_\mu\left(x\right)\frac{\delta S_M}{\delta A^c_\mu\left(x\right)}\right]. \tag{2.13}$$

Equating both expressions to produce one identity is feasible. By performing an integration by parts on the contribution from (2.12), the parameter θ^a factorizes inside the integration sign:

$$\int \mathrm{dx}\ \theta^a\left(x\right)\left[\partial_\mu J^\mu_a\left(x\right) + \frac{\delta S_M}{\delta\psi\left(x\right)}iT^a\psi\left(x\right) - i\overline{\psi}\left(x\right)T^a\frac{\delta S_M}{\delta\overline{\psi}\left(x\right)} + f^{abc}A^b_\mu\left(x\right)\frac{\delta S_M}{\delta A^c_\mu\left(x\right)}\right] = 0. \tag{2.14}$$

Hence, the arbitrariness of this object implies that the structure in squared brackets vanishes regardless of the integration

$$\partial_\mu J^\mu_a(x) + \frac{\delta S_M}{\delta \psi(x)} i T^a \psi(x) - i\overline{\psi}(x) T^a \frac{\delta S_M}{\delta \overline{\psi}(x)} + f^{abc} A^b_\mu(x) \frac{\delta S_M}{\delta A^c_\mu(x)} = 0. \tag{2.15}$$

As we have not considered local symmetry up to this point, such a result is a consequence of global invariance.

Next, observe that equations of motion associated with fermion fields fall over the matter action

$$\frac{\delta S}{\delta \psi(x)} = \frac{\delta S_M}{\delta \psi(x)} = 0, \tag{2.16}$$
$$\frac{\delta S}{\delta \overline{\psi}(x)} = \frac{\delta S_M}{\delta \overline{\psi}(x)} = 0. \tag{2.17}$$

Hence, replacing them in Equation (2.15) cancels out some contributions, which leads to the simplified version[1]

$$\partial_\mu J^\mu_a(x) + f^{abc} A^b_\mu(x) \frac{\delta S_M}{\delta A^c_\mu(x)} = 0. \tag{2.18}$$

Now, we consider gauge transformations as the final step before achieving conservation. Invariance of the action establishes the relation

$$S_M\left[\psi^g, \overline{\psi}^g, A_\mu\right] = S_M\left[\psi, \overline{\psi}, A^{g^{-1}}_\mu\right] = S_M\left[\psi, \overline{\psi}, g^{-1} A_\mu g + \frac{i}{e}\left(\partial_\mu g^{-1}\right) g\right]. \tag{2.19}$$

By adopting the configuration for the gauge field $A'_\mu = g A_\mu g^{-1}$, the variation of S_M is achievable again. To that end, rewrite the relation above by considering the infinitesimal form of the transformation

$$S_M\left[\psi^g, \overline{\psi}^g, g A_\mu g^{-1}\right] = S_M\left[\psi, \overline{\psi}, A_\mu + \frac{1}{e}\partial_\mu \theta\right]. \tag{2.20}$$

The mentioned variation emerges through an expansion over the parameter

$$\delta S_M = \frac{1}{e}\int \mathrm{dx}\, \partial_\mu \theta^a(x) \frac{\partial S_M}{\partial A^a_\mu(x)}. \tag{2.21}$$

Therefore, a comparison between this form and Equation (2.12) generates the following result

$$\int \mathrm{dx}\, \partial_\mu \theta^a(x) \left[J^\mu_a(x) - \frac{1}{e}\frac{\partial S_M}{\partial A^a_\mu(x)}\right] = 0. \tag{2.22}$$

Again, the quantity in squared brackets has to vanish by itself as transformation parame-

[1] As structure constants cancel out in the Abelian theory, the conservation of the current comes directly from this equation. That means it is unnecessary to consider gauge transformations at any point in the calculations.

ters are arbitrary. That produces the relation

$$J_a^\mu (x) = \frac{1}{e} \frac{\partial S_M}{\partial A_\mu^a (x)}, \tag{2.23}$$

whose replacement within Equation (2.18) allows recognizing the covariant derivative introduced in the gauge field transformation

$$\mathcal{D}_\mu^{ac} J_c^\mu (x) = \left(\partial_\mu \delta^{ac} - e f^{abc} A_\mu^b\right) J_c^\mu (x) = 0. \tag{2.24}$$

We identify the vector $J_a^\mu (x)$ as a current, while the last equation represents its covariant conservation. Two ingredients were necessary to achieve this outcome: local gauge invariance of the matter action and equations of motion for fermions.

2.2 Quantum Symmetry

Since we finalized exploring manifestations of gauge symmetry in classical theories, let us extend this discussion to the quantum context. To accomplish this goal, we start by introducing the effective action $W[A_\mu]$ through the functional integral

$$e^{iW[A_\mu]} = \int d\psi d\overline{\psi} \exp\left(iS\left[\psi, \overline{\psi}, A_\mu\right]\right). \tag{2.25}$$

Since gauge fields are considered external classical fields, the integration occurs exclusively over (quantized) fermion fields.

Following the same reasoning from the previous section, we consider global transformation and impose that parameters depend on the position. When applying infinitesimal transformations (2.7)-(2.9), the changed expression for the exponential follows

$$e^{iW'} = \int d\psi d\overline{\psi} \exp\left(iS_M\left[\psi + i\theta\psi, \overline{\psi} - i\overline{\psi}\theta, A_\mu - i\left[A_\mu, \theta\right]\right] + iS_G\left[A_\mu\right]\right), \tag{2.26}$$

with the gauge action invariant. Although gauge fields change through the covariant derivative, only the contribution on the commutator concerns global invariance. Recognizing the exponential argument as the action plus a variation allows detaching both parts

$$e^{iW'} = \int d\psi d\overline{\psi} \exp\left(i\delta S_M\right) \exp\left(iS\left[\psi, \overline{\psi}, A_\mu\right]\right). \tag{2.27}$$

Hence, an expansion on the infinitesimal parameter leads to the exponential variation of the effective action

$$e^{iW'} - e^{iW} = \int d\psi d\overline{\psi} \left(i\delta S_M\right) \exp\left(iS\left[\psi, \overline{\psi}, A_\mu\right]\right). \tag{2.28}$$

As this result depends on the action variation, let us recall the information obtained. On the one hand, we reasoned that it is proportional to the derivative of the parameter and the current $J_a^\mu(x)$; see Equation (2.12). In the quantum context, that leads to the expression

$$e^{iW'} - e^{iW} = -\int d\psi d\overline{\psi} \exp\left(iS\left[\psi, \overline{\psi}, A_\mu\right]\right)\left[i\int \mathrm{dx}\ \theta^a(x)\,\partial_\mu J_a^\mu(x)\right], \tag{2.29}$$

where integration by parts changes the derivative position. On the other hand, the infinitesimal transformation produced result (2.13), which reflects on the form

$$\begin{aligned} e^{iW'} - e^{iW} &= \int d\psi d\overline{\psi} \exp\left(iS\left[\psi, \overline{\psi}, A_\mu\right]\right) \times \\ &\quad \times i\int \mathrm{dx}\ \theta^a(x)\left[\frac{\delta S_M}{\delta\psi(x)} iT^a\psi(x) - i\overline{\psi}(x)\,T^a\frac{\delta S_M}{\delta\overline{\psi}(x)} + ef^{abc}A_\mu^b(x)\,J_c^\mu(x)\right]. \end{aligned} \tag{2.30}$$

We already used the association (2.23) to recognize the current within this equation.

Since there are two forms for the same object, let us equate them to produce an identity. Due to the arbitrariness of the transformation parameter, the relation applies regardless of space-time integration. We emphasize that this does not occur if the parameter is constant, as it would factor from the integration sign without further simplifications. By identifying the covariant derivative, the variation produces the result

$$\begin{aligned} &\int d\psi d\overline{\psi} \exp\left(iS\left[\psi, \overline{\psi}, A_\mu\right]\right)\left[D_\mu^{ab}J_b^\mu(x)\right] \\ &= \int d\psi d\overline{\psi} \exp\left(iS\left[\psi, \overline{\psi}, A_\mu\right]\right)\left[i\overline{\psi}(x)\,T^a\frac{\delta S_M}{\delta\overline{\psi}(x)} - \frac{\delta S_M}{\delta\psi(x)} iT^a\psi(x)\right]. \end{aligned} \tag{2.31}$$

In the classical discussion, the conservation law arose posteriorly to employing equations of motion for fermions in an analogous equation. We would expect Dyson-Schwinger equations to perform this task here, as they embody the equations of motion within this context. In that case, current conservation would result from the translational invariance of the fermion measure [16]. Nonetheless, gauge invariance emerges as a condition at the quantum level. Let us integrate an arbitrary functional and explore its transformation to understand the consequences:

$$\begin{aligned} \int d\psi d\overline{\psi}\ F\left[\psi, \overline{\psi}, A_\mu\right] &= \int d\psi^g d\overline{\psi}^g\ F\left[\psi^g, \overline{\psi}^g, A_\mu\right] \\ &= \int d\psi d\overline{\psi}\ F\left[\psi, \overline{\psi}, A_\mu\right] + \int d\psi d\overline{\psi}\int \mathrm{dx}\left[\frac{\delta F}{\delta\psi}\delta\psi + \frac{\delta F}{\delta\overline{\psi}}\delta\overline{\psi}\right]. \end{aligned} \tag{2.32}$$

Under the hypothesis of gauge-invariance of the fermion measure

$$d\psi^g d\overline{\psi}^g = d\psi d\overline{\psi}, \tag{2.33}$$

the condition applies

$$\int d\psi d\overline{\psi} \int \mathrm{dx} \left[\frac{\delta F}{\delta\psi}\delta\psi + \delta\overline{\psi}\frac{\delta F}{\delta\overline{\psi}}\right] = 0. \tag{2.34}$$

Disregarding space-time integration, observe that this object cancels out the right-hand side of Equation (2.31) when we set the functional. Hence, the referred equation turns into the quantum version of the gauge current covariant conservation:

$$\int d\psi d\overline{\psi} \left[D^{ab}_\mu J^\mu_b (x)\right] \exp\left(iS\left[\psi, \overline{\psi}, A_\mu\right]\right) = 0. \tag{2.35}$$

Such an argumentation shows that gauge invariance of the fermion measure is enough for current conservation. Invariance of the matter action does not guarantee symmetry maintenance within quantum theory, even if it guarantees classical conservation.

2.3 Gauge Anomaly

After shedding light on conditions for quantum conservation, we aim to inquire about situations characterized by violations. The literature on functional integrals recognizes non-trivial Jacobians for the fermion measure as the cause of symmetry breaking [17]. This non-invariance is typical of investigations involving chiral fermions, as in the Standard Model before spontaneous symmetry breaking.

We approach this subject by introducing the fermion measure Jacobian as follows

$$d\psi^g d\overline{\psi}^g = J\left[g, A_\mu\right] d\psi d\overline{\psi} \tag{2.36}$$

while considering the possibility of dependence on gauge fields. Although that is unreasonable for usual integration, this type of contribution might arise through regularization techniques when dealing with divergent objects associated with functional integrals [18, 19]. That means integrals and functional derivatives do not necessarily commute, requiring extra care to avoid inconsistent results.

Given the structure of calculations developed in the previous section, expressing the Jacobian as the exponential of another functional is convenient

$$J\left[g, A_\mu\right] = \exp\left(i\alpha_1\left[g, A_\mu\right]\right). \tag{2.37}$$

Thence, writing the Jacobian associated with the inverse transformation is straightforward

$$J\left[g^{-1}, A_\mu\right] = \exp\left(i\alpha_1\left[g^{-1}, A_\mu\right]\right) = \exp\left(-i\alpha_1\left[g, A_\mu\right]\right), \tag{2.38}$$

and so is the property attributed to the exponential argument

$$\alpha_1\left[g^{-1}, A_\mu\right] = -\alpha_1\left[g, A_\mu\right]. \tag{2.39}$$

Besides, we consider first-order contributions on the infinitesimal transformation parameter to build the expansion

$$\alpha_1\left[g, A_\mu\right] = \alpha_1\left[1, A_\mu\right] + \int \mathrm{dx}\ \theta^a(x) \left.\frac{\delta\alpha_1\left[g, A_\mu\right]}{\delta\theta^a(x)}\right|_{\theta=0}. \tag{2.40}$$

As the first term represents the case without transformation, the Jacobian corresponds to the identity $J\left[1, A_\mu\right] = 1$ and implies the vanishing argument $\alpha_1\left[1, A_\mu\right] = 0$.

Since we discussed how fermionic variables change, let us explore the implications for the effective action introduced in Equation (2.25). By relabeling fermion fields as $\psi \to \psi^{g^{-1}}$ and $\overline{\psi} \to \overline{\psi}^{g^{-1}}$, we get the modified expression

$$e^{iW[A_\mu]} = \int d\psi^{g^{-1}} d\overline{\psi}^{g^{-1}} \exp\left(iS\left[\psi^{g^{-1}}, \overline{\psi}^{g^{-1}}, A_\mu\right]\right). \tag{2.41}$$

After employing action invariance and inserting the Jacobian for the inverse (2.38), we achieve another form:

$$e^{iW[A_\mu]} = \exp\left(-i\alpha_1\left[g, A_\mu\right]\right) \int d\psi d\overline{\psi} \exp\left(iS\left[\psi, \overline{\psi}, A_\mu^g\right]\right). \tag{2.42}$$

The Jacobian factors out of the integral sign as it does not depend on quantized fermion fields. This integral corresponds to the effective action with modified gauge fields, so expressing the Jacobian through the effective action is feasible

$$\exp\left(i\alpha_1\left[g, A_\mu\right]\right) = \exp\left(iW\left[A_\mu^g\right] - iW\left[A_\mu\right]\right). \tag{2.43}$$

Taking the logarithm on both sides emphasizes that the effective action is not invariant under this type of transformation:

$$\alpha_1\left[g, A_\mu\right] = W\left[A_\mu^g\right] - W\left[A_\mu\right]. \tag{2.44}$$

By recalling the gauge field transformation (2.7), we expand $W\left[A_\mu^g\right]$ to the first order on the infinitesimal parameter. That allows writing the variation of the effective action through the integral

$$W\left[A_{\mu}^{g}\right]-W\left[A_{\mu}\right]=\int \mathrm{dx}\ \theta^{c}\mathcal{D}_{\mu}^{ac}\left(\frac{1}{e}\frac{\delta W\left[A_{\mu}\right]}{\delta A_{\mu}^{a}}\right). \tag{2.45}$$

But Equation (2.44) links this structure to the functional α_1, whose expansion is (2.40). Given the parameter arbitrariness, comparing both equations establishes the relation

$$\left.\frac{\delta\alpha_1\left[g,A_{\mu}\right]}{\delta\theta\left(x\right)}\right|_{\theta=0}=\mathcal{D}_{\mu}\left(\frac{1}{e}\frac{\delta W\left[A_{\mu}\right]}{\delta A_{\mu}}\right), \tag{2.46}$$

where the notation involving components is omitted.

For the last step of the current discussion, we recall that both effective action and action itself are Lorentz scalars. That means the commutation between these objects and the covariant derivative does not bring complications. Hence, multiplying the relation above and the exponential of the effective action leads to the mathematical expression

$$\begin{aligned}&\left.\frac{\delta\alpha_1\left[g,A_{\mu}\right]}{\delta\theta\left(x\right)}\right|_{\theta=0}\int d\psi d\overline{\psi}\exp\left(iS\left[\psi,\overline{\psi},A_{\mu}\right]\right)\\ =\ &\mathcal{D}_{\mu}\left\{-\frac{i}{e}\frac{\delta}{\delta A_{\mu}}\int d\psi d\overline{\psi}\exp\left(iS\left[\psi,\overline{\psi},A_{\mu}\right]\right)\right\}.\end{aligned} \tag{2.47}$$

Since the gauge action is invariant, the functional derivative acts exclusively on the matter action

$$\begin{aligned}&\left.\frac{\delta\alpha_1\left[g,A_{\mu}\right]}{\delta\theta\left(x\right)}\right|_{\theta=0}\int d\psi d\overline{\psi}\exp\left(iS\left[\psi,\overline{\psi},A_{\mu}\right]\right)\\ =\ &\int d\psi d\overline{\psi}\ \mathcal{D}_{\mu}\left(\frac{1}{e}\frac{\delta S_M\left[\psi,\overline{\psi},A_{\mu}\right]}{\delta A_{\mu}}\right)\exp\left(iS\left[\psi,\overline{\psi},A_{\mu}\right]\right).\end{aligned} \tag{2.48}$$

As the term in parenthesis is precisely the current identified in the classical discussion (2.23), the relation applies

$$\left.\frac{\delta\alpha_1\left[g,A_{\mu}\right]}{\delta\theta_a\left(x\right)}\right|_{\theta=0}=\frac{\int d\psi d\overline{\psi}\left(\mathcal{D}_{\mu}^{ab}J_b^{\mu}\right)e^{iS\left[\psi,\overline{\psi},A_{\mu}\right]}}{\int d\psi d\overline{\psi}\ e^{iS\left[\psi,\overline{\psi},A_{\mu}\right]}}. \tag{2.49}$$

We transposed the effective action to the right-hand side to identify this structure as the vacuum expectation value of the covariant divergence of the current. The non-vanishing of this expression characterizes the so-called *gauge anomaly*:

$$\mathcal{A}_a\left(A_{\mu}\right)=\left.\frac{\delta\alpha_1\left[g,A_{\mu}\right]}{\delta\theta_a\left(x\right)}\right|_{\theta=0}\neq 0. \tag{2.50}$$

This condition is what characterizes the theory as *gauge anomalous*. We stress that this happens when gauge bosons are external classical fields interacting with quantum fermion fields.

Further explorations show that the expectation value for the gauge anomaly cancels out for the fully-quantized theory. To verify that, let us define the generating functional

$$Z\left[\eta,\overline{\eta},j_a^\mu\right]=\int d\psi d\overline{\psi}dA_\mu\ \exp\left(iS\left[\psi,\overline{\psi},A_\mu\right]+i\int \mathrm{dx}\left[\overline{\eta}\psi+\overline{\psi}\eta+j_a^\mu A_\mu^a\right]\right). \tag{2.51}$$

Since our concern relates to vacuum expectation value, contributions associated with external sources are unnecessary. The notation simplifies under these circumstances, being viable to express this equation in terms of the effective action

$$Z\left[0,0,0\right]=\int dA_\mu\ e^{iW[A_\mu]}. \tag{2.52}$$

Following a strategy similar to previous cases, we start by relabeling the structure above through $A_\mu\to A_\mu^g$. The changed version for the effective action corresponds to the original plus a variation. After replacing the result from the previous section (2.45), we split the exponential argument. Then, expanding the variation part on the infinitesimal parameter produces the equation

$$Z\left[0,0,0\right]=\int dA_\mu^g\ e^{iW[A_\mu]}\left[1+i\int \mathrm{dx}\ \theta^c\mathcal{D}_\mu^{ac}\left(\frac{1}{e}\frac{\delta W\left[A_\mu\right]}{\delta A_\mu^a}\right)\right]. \tag{2.53}$$

The difference between the generating functional and the first term on the right-hand side resides in the integration variable; thus, they coincide if the bosonic measure is gauge-invariant $dA_\mu=dA_\mu^g$. The second functional integral must be zero under this condition. Since the arbitrariness of the transformation parameter allows dropping the space-time integral, the relation emerges

$$\int dA_\mu\ e^{iW[A_\mu]}\mathcal{D}_\mu^{ac}\left(\frac{1}{e}\frac{\delta W\left[A_\mu\right]}{\delta A_\mu^a}\right)=0. \tag{2.54}$$

At this point, we recall Equations (2.46) and (2.50) to recognize the anomaly. Hence, by making the dependence on the fermionic variables explicit, we showed that its vacuum expectation value is zero for the fully quantized theory:

$$\int d\psi d\overline{\psi}dA_\mu\ \mathcal{A}_a\left(A_\mu\right)\exp\left(iS\left[\psi,\overline{\psi},A_\mu\right]\right)=\langle 0|\,\mathcal{A}_a\left(A_\mu\right)|0\rangle=0. \tag{2.55}$$

In addition to its role in the demonstration above, we stress that the bosonic measure invariance has other applications in investigations in this area. Even so, we did not find proof of this property in the literature. The primary objective of this part of the thesis is to provide one, which is our next subject.

2.4 Gauge Invariance of the Bosonic Measure

This section investigates the behavior of the bosonic measure under gauge transformations. To this end, we display a preparatory argument by considering the generating functional for correlators of gauge-invariant operators $O_i\left(A_\mu^g\right) = O_i\left(A_\mu\right)$ in the pure Yang-Mills theory (without chiral fermions):

$$Z\left[\lambda^i\right] = \int dA_\mu \exp i \int \mathrm{dx}\ \mathrm{tr}\left(\frac{1}{2}F_{\mu\nu}F^{\mu\nu} + \lambda^i O_i\left[A_\mu\right]\right). \tag{2.56}$$

The quantities λ_i are currents, and functional derivatives with respect to them yield the n-point correlators

$$\frac{\delta^n}{\delta\lambda^1\left(x_1\right)...\delta\lambda^n\left(x_n\right)} Z\left[\lambda^i\right]\Bigg|_{\lambda^i=0} = \langle 0|\, T\left(O_1\left(A_\mu\right)\left(x_1\right)...O_n\left(A_\mu\right)\left(x_n\right)\right)|0\rangle\,. \tag{2.57}$$

Considering the integration over A_μ and also over its gauge transformed version A_μ^g, we develop the comparison

$$\begin{aligned} Z\left[\lambda^i\right] &= \int dA_\mu \exp i \int \mathrm{dx}\ \mathrm{tr}\left[\frac{1}{2}F_{\mu\nu}F^{\mu\nu} + \lambda^i O_i\left(A_\mu\right)\right] \\ &= \int dA_\mu^g \exp i \int \mathrm{dx}\ \mathrm{tr}\left[\frac{1}{2}\left(F_{\mu\nu}F^{\mu\nu}\right)^g + \lambda^i O_i\left(A_\mu^g\right)\right] \\ &= \int dA_\mu\ J\left[A_\mu, g\right] \exp i \int \mathrm{dx}\ \mathrm{tr}\left[\frac{1}{2}F_{\mu\nu}F^{\mu\nu} + \lambda^i O_i\left(A_\mu\right)\right], \end{aligned} \tag{2.58}$$

where the potential presence of a Jacobian $J\left[A_\mu, g\right]$ for the gauge transformation of the measure is allowed. Thus, we obtain the correlators associated with both expressions for the generating functional as follows

$$\begin{aligned} &\langle 0|\, T\left(J\left[A_\mu, g\right] O_1\left(A_\mu\right)\left(x_1\right)...O_n\left(A_\mu\right)\left(x_n\right)\right)|0\rangle \\ &= \langle 0|\, T\left(O_1\left(A_\mu\right)\left(x_1\right)...O_n\left(A_\mu\right)\left(x_n\right)\right)|0\rangle\,. \end{aligned} \tag{2.59}$$

Translated into words, that means all correlators involving the Jacobian $J\left[A_\mu, g\right]$ with gauge invariant operators are the same as those involving the identity. Thus, in the physical Hilbert space of the theory, both operators are the same.

This argument does not generalize to arbitrary operators that are not gauge-invariant, as required to recover the entire Hilbert space. However, an explicit calculation can solve this problem. Let us use the usual prescription of defining the bosonic measure through a complete set of orthonormal eigenfunctions $\{\phi_n\}$ of a hermitian operator $\bar{D}$:

$$\bar{D}\phi_n = \lambda_n \phi_n, \tag{2.60}$$

with the conditions

$$\int \mathrm{dx}\ \phi_n^\dagger \phi_m = \delta_{nm} \text{ and } \sum_n \phi_n(x)\,\phi_n^\dagger(y) = \delta(x-y)\,. \tag{2.61}$$

Posteriorly to expanding the bosonic field, we build the connection with the measure as follows

$$A_\mu^a(x) = \sum_n a_{\mu,n}^a \phi_n(x) \to dA_\mu = \prod_{a,\mu,n} da_{\mu,n}^a. \tag{2.62}$$

Next, we put the changed field into this prescription. By introducing coefficients $\bar{a}$ to the new expansion, let us rewrite the infinitesimal gauge transformation (2.7):

$$\begin{aligned} A_\mu^g &= \sum_n \bar{a}_{\mu,n}^a T_a \phi_n(x) = \sum_n a_{\mu,n}^a T_a \phi_n(x) - \frac{i}{e}\mathcal{D}_\mu \theta \\ &= \left[\sum_n \left(a_{\mu,n}^a + i a_{\mu,n}^b f_{abc}\theta^c\right)\phi_n(x) - \frac{i}{e}\partial_\mu \theta^a\right] T_a. \end{aligned} \tag{2.63}$$

Then, after decomposing parameters θ^a in terms of the same eigenfunctions of $\bar{D}$

$$-\frac{i}{e}\partial_\mu \theta^a(x) = \sum_n \tilde{a}_{\mu,n}^a \phi_n(x)\,, \tag{2.64}$$

obtaining a transformation rule to coefficients is feasible

$$\bar{a}_{\mu,n}^a = \sum_m \left(\delta_{ab}\delta_{nm} + \int \mathrm{dx}\ \phi_n^\dagger(x)\, i f_{abc}\theta^c(x)\,\phi_m(x)\right) a_{\mu,m}^b + \tilde{a}_{\mu,n}^a. \tag{2.65}$$

That reflects on the transformation linked to the bosonic measure

$$\prod_{a,\mu,n} d\bar{a}_{\mu,n}^a = \det\left[\delta_{ab}\delta_{nm} + \int \mathrm{dx}\ \phi_n^\dagger(x)\, i f_{abc}\theta^c(x)\,\phi_m(x)\right] \prod_{a,\mu,n} da_{\mu,n}^a, \tag{2.66}$$

where the term $\tilde{a}_{\mu,n}^a$ does not contribute because of the translational invariance of each measure $da_{\mu,n}^a$.

Following the steps of Fujikawa [17], we get the expression for the Jacobian:

$$J[A_\mu, \theta] = \exp\left[\sum_n \left(\mathrm{tr} \int \mathrm{dx}\ \phi_n^\dagger(x)\, i f_{abc}\theta^c(x)\,\phi_n(x)\right)\right]. \tag{2.67}$$

This trace acts over Lie algebra indices, which cancels out the total antisymmetric structure constant f_{abc}. Meanwhile, we recognize the product of fields taken at the same point $\phi_n(x)\,\phi_n^\dagger(x)$. When putting both pieces of information together, it is easy to see that the Jacobian expression is indefinite:

$$\begin{aligned}\sum_n\left(\mathrm{tr}\int \mathrm{dx}\;\phi_n^\dagger(x)\,if_{abc}\theta^c(x)\,\phi_n(x)\right) &= \mathrm{tr}\int \mathrm{dx}\;if_{abc}\theta^c(x)\sum_n\phi_n(x)\,\phi_n^\dagger(x)\\ &= \int \mathrm{dx}\;if_{aac}\theta^c(x)\,\delta(0) = 0\times\infty.\end{aligned}\tag{2.68}$$

Thus, let us regularize this object by introducing eigenvalues of the operator $\bar{D}$ as

$$\begin{aligned}J[A_\mu,\theta] &\equiv \exp\left[\lim_{M^2\to\infty}\sum_n\left(\mathrm{tr}\int \mathrm{dx}\;\phi_n^\dagger(x)\,if_{abc}\theta^c(x)\exp\left(-\frac{\lambda_n^2}{M^\alpha}\right)\phi_n(x)\right)\right]\\ &= \exp\left[\lim_{M^2\to\infty}\sum_n\left(\mathrm{tr}\int \mathrm{dx}\;\phi_n^\dagger(x)\,if_{abc}\theta^c(x)\exp\left(-\frac{\bar{D}^2}{M^\alpha}\right)\phi_n(x)\right)\right],\end{aligned}\tag{2.69}$$

where α is chosen so the exponential argument is dimensionless.

The choice of operator $\bar{D}$ usually considers the requisites of naturally appearing in the theory, being gauge invariant, and having real eigenvalues. Furthermore, our choice of coefficients $a^a_{\mu,n}$ carrying all the dependence on μ and a implies that the ϕ_n must be eigenfunctions of a scalar colorless operator; therefore, a good choice is

$$\bar{D} = \mathrm{tr}\left(\mathcal{D}_\mu\mathcal{D}^\mu\right),\tag{2.70}$$

where the trace is taken only over color indices. We see that the sum is regularized under these conditions, so proceeding with the evaluation of the Jacobian is possible. Since no additional dependence on color indices comes from the exponential argument $\bar{D}^2/M^4$, the trace can be immediately taken, yielding the unity

$$\begin{aligned}J[A_\mu,\theta] &= \exp\left[\lim_{M^2\to\infty}\sum_n\left(if_{aac}\int \mathrm{dx}\;\phi_n^\dagger(x)\,\theta^c(x)\exp\left(-\frac{\bar{D}^2}{M^4}\right)\phi_n(x)\right)\right]\\ &= \exp(0) = 1.\end{aligned}\tag{2.71}$$

Such a result accomplishes our objective of furnishing proof for the invariance of the bosonic measure. Of course, one could choose other strategies so a result different from 1 could arise. Nevertheless, the "gauge anomaly" coming from this "non-trivial" Jacobian could be removed by an adequate choice of counterterms. To say this more precisely, we can use what we know from the fact that Yang-Mills theories are renormalizable. In fact, 't Hooft's proof [20] shows that it is possible to preserve gauge invariance at every order in perturbation theory, which is crucial for demonstrating that the theory is renormalizable. Algebraic renormalization results confirm this by noticing that the cohomology of the Slavnov-Taylor operator is trivial for a Yang-Mills theory [21]. Then, even if we would regularize the theory with non-gauge invariant regulators (obtaining a non-trivial Jacobian), a change in the renormalization scheme could restore gauge invariance and set the Jacobian as the unity.

The results in this chapter are the main part of our published work [22].

2.5 Final Remarks and Conclusions

In the second chapter, we checked aspects related to gauge symmetry maintenance in gauge theories. At the classical level, current conservation arose after implementing local invariance in the theory action. Equations of motion for fermion fields were necessary to achieve this result. This part of the discussion established a route to follow in the quantum theory.

With this in mind, it would be reasonable for Dyson-Schwinger equations to play a role in the current conservation due to their analogy with classical equations of motion. It would be a consequence of the translational invariance of the fermion measure, which is a condition to obtain the mentioned equations. Nevertheless, we saw that gauge invariance of the fermion measure is the new requirement for conservation.

Once the panorama was clear, we focused on gauge-anomalous theories. For them, considering external gauge fields, the presence of a Jacobian to the fermion measure implies a non-zero result to the expectation value of the covariant derivative of the current (the anomaly). We saw that, when quantizing the gauge field, the expectation value vanishes in a simple way. This outcome is a direct consequence of considering the boson measure invariant, and the properties of the fermion measure were unnecessary. There is no gauge anomaly preventing current conservation in the fully quantized theory. That does not affect the topological interpretation of the gauge anomaly since it is present when we do not consider the integration on the gauge field.

Although our argumentation depends on gauge measure invariance, we took this property for granted. That is usual in the literature but not explicitly proved. This proof was achieved by G. de Lima e Silva, T.J. Girardi, and S. A. Dias and published in reference [22]. Such a result completes the theoretical setup for our claim that the vacuum expectation value of the gauge anomaly vanishes. The natural course of this investigation is to define a chiral theory perturbatively, aiming at a detailed analysis of its renormalizability and unitarity.

Chapter 3

Anomalies in Fermionic Amplitudes

This chapter refers to another line of investigation in this thesis, which concerns the occurrence of anomalies in fermionic amplitudes. As mentioned, the single axial triangle (AVV) establishes the foundations for this analysis. Although this process is largely explored in the literature, our perspective shows new aspects of anomalies while emphasizing patterns related to their tensor structures. The single axial box ($AVVV$) exhibits similar elements in a more complex scene, substantiating this investigation.

Both correlators depend on traces involving the chiral matrix, which lead to products between the Levi-Civita symbol and metric tensors. In addition to its manifestation in anomalous amplitudes, this type of structure is common in chiral theories and investigations developed in odd space-time dimensions. That is part of the motivation for this work and emphasizes the significance of mathematical resources developed throughout our calculations.

Integrals in perturbative calculus usually exhibit diverging content, which requires using regularization techniques in intermediate steps of calculations [23]. These prescriptions make mathematical structures finite, so manipulations problematic to the original expressions become valid. That implies modifying amplitudes through the introduction of non-physical parameters. Results independent of regularizations emerge after renormalization [24]. Then, establishing predictions to compare with experimental data becomes feasible.

Choosing a specific regularization scheme brings consequences to the interpretation of results. To clarify this aspect, we get back to the impossibility of preserving chiral and gauge symmetry simultaneously [4]. This time, however, we emphasize the issue of the maintenance of Ward identities for the single axial triangle. This amplitude unavoidably exhibits dependence on a diverging surface term [28], so choosing a prescription that eliminates this object preserves some Ward identities (but not all). Methods that allow shifts in the integration variable accomplish this task, e.g., Dimensional Regularization [25, 26, 27]. Other prescriptions do not lead to this outcome.

Even though there is an inclination towards preserving gauge symmetry, there are other possibilities. The reason for such is the presence of divergent Feynman integrals having a divergence degree higher than the logarithmic one. For them, a shift in the integration variable requires compensation through (non-zero) surface terms to maintain the connection with the original expression [6, 28, 29]. That implies the existence of different versions for perturbative contributions involving loops, which differ by these surface terms after integration. This situation is a manifestation of internal momenta arbitrariness, although they relate to external momenta through energy-momentum conservation [30]. We will illustrate that choices occur when taking Dirac traces, leading to one version with a specific behavior regarding symmetries; i.e., choosing one form sets the position of violating terms typical of anomalous amplitudes.

The mentioned aspects motivate the perspective adopted in this investigation and, therefore, the employment of Implicit Regularization (IReg) [37]. Its main feature is avoiding the evaluation of divergent structures. That means we only integrate finite contributions without modifying ill-defined objects. Our analysis falls on the accessible values for these divergences within final expressions for amplitudes. We also avoid choices for the internal momenta, adopting arbitrary routings for internal lines of the graphs. This arbitrariness is intrinsic to the perturbative calculus and received attention in recent works [31, 32, 33]. The study of schemes to compute traces involving chiral matrices also received attention from the authors [34, 35, 36]. This concept characterizes another class of possibilities for anomalous amplitudes.

The discussion is organized as follows. Section (3.1) introduces the model and the correlators that concern this investigation. We also comment on expectations about symmetries (through Ward identities) and their relation with the linearity of integration. Section (3.2) looks into integrands of amplitudes, highlighting tensor arrangements associated with structures that compound the intended organization. This feature is part of the IReg, approached in Section (3.3). We also introduce the elements used to describe diverging quantities and finite functions. With our perspective clear, Section (3.4) focus on the explicit integration of amplitudes while providing a preliminary discussion of these results. A careful analysis occurs in Section (3.5), where we inquire about relations involving amplitudes and the consequences of different prescriptions to evaluate divergent objects. Section (3.6) discusses important aspects of the investigation while presenting the conclusions.

3.1 Model and Definitions

We consider a $(1+3)$-dimensional model where massive spin $1/2$ fields interact with different types of bosons. The corresponding couplings[1] are listed in the interacting Lagrangian

$$\begin{aligned}\mathcal{L}_I &= e_S\left(\bar{\psi}\psi\right)\phi + e_P\left(\bar{\psi}\gamma_5\psi\right)\pi - e_V\left(\bar{\psi}\gamma^\mu\psi\right)V_\mu \\ &\quad - e_A\left(\bar{\psi}\gamma_5\gamma^\mu\psi\right)A_\mu + e_T\left(\bar{\psi}\gamma_5\sigma^{\mu\nu}\psi\right)H_{\mu\nu},\end{aligned} \tag{3.1}$$

where elements belonging to the set $\{\phi,\pi,V_\mu,A_\mu,H_{\mu\nu}\}$ are respectively scalar, pseudoscalar, vector, axial, and pseudotensor boson fields, while ψ corresponds to Dirac fermions. As coupling constants $\{e_S,e_P,e_V,e_A,e_T\}$ do not concern the intended discussion, we set them as the unity.

The remaining structures emerge in the context of the four-dimensional Clifford algebra. The objects γ^μ are Dirac matrices, whose commutator is denoted as $[\gamma^\mu,\gamma^\nu]=2\sigma^{\mu\nu}$. Since establishing a chiral matrix that anticommutes with all gamma matrices is feasible in even dimensions, we introduce the definition employed within this context $\gamma_5 = \frac{i}{4!}\varepsilon_{\mu\nu\alpha\beta}\gamma^\mu\gamma^\nu\gamma^\alpha\gamma^\beta$. Even though omitted, the identity $\mathbf{1}$ appears within the scalar coupling.

Those structures in parentheses within the Lagrangian correspond to Noether currents, which couple to boson fields. Current conservation establishes relations involving these quantities. Although violations are expected for anomalous amplitudes, we discuss preliminary expectations here. In a case involving fermions with different masses, the vector current divergence would be proportional to the scalar one with a coefficient depending on the difference between masses. Nevertheless, the vector current is conserved as we delimit this investigation to the equal masses context

$$\partial_\mu\left(\bar{\psi}\gamma^\mu\psi\right)=0. \tag{3.2}$$

That suggests implications at the quantum level through Ward identities for correlators involving vector vertices. The result should vanish whenever we contract an external momentum with an index corresponding to this vertex type. On the other hand, the axial current divergence is classically proportional to the pseudoscalar one

$$\partial_\mu\left(\bar{\psi}\gamma_5\gamma^\mu\psi\right)=2m\left(\bar{\psi}\gamma_5\psi\right). \tag{3.3}$$

Such relation leads to Ward identities involving similar amplitudes that differ by the corresponding vertices. Establishing an analogous association involving the pseudotensor

[1] Although some couplings do not concern this investigation at first glance, perturbative corrections bring all these possibilities.

current is not possible.

Our objective is on the next-to-leading order corrections for processes involving external bosons, which produces purely fermionic loops. We introduce them in two steps. First, we employ Feynman rules to construct graphs for a single value of the unrestricted (loop) momentum. Hence, we inspect them and survey expectations without worrying about ill-defined mathematical quantities. This problem arises when implementing the last Feynman rule, which consists of momenta integration. We only consider this operation (in the second step) after discussing a strategy to deal with the mentioned problem. Upper and lower case letters distinguish these two versions of amplitudes

$$T^{\Gamma_i \Gamma_j \cdots \Gamma_l} = \int \frac{d^4k}{(2\pi)^4} t^{\Gamma_i \Gamma_j \cdots \Gamma_l}. \tag{3.4}$$

Such notation is extended to other integrals that emerge throughout this work.

The general form of amplitudes for a single value of the loop momentum is

$$\begin{aligned} & t^{\Gamma_i \Gamma_j \cdots \Gamma_l}(k_1, k_2, \ldots, k_n) \\ = \; & \mathrm{tr}\left\{\Gamma_i \left[S_F(k+k_1;m)\right] \Gamma_j \left[S_F(k+k_2;m)\right] \cdots \Gamma_l \left[S_F(k+k_n;m)\right]\right\}, \end{aligned} \tag{3.5}$$

whose argument is omitted unless it associates with configurations different from $(k_1, k_2, \ldots, k_n)$. This structure depends on fermion propagators S_F and vertex operators Γ_l. We express the propagator of a Dirac fermion carrying momentum $K_n = k + k_n$ and mass m through the structure

$$S_F(k+k_n, m) = \frac{1}{\not{F}_n} = \frac{1}{(\not{k} + \not{k}_n) - m} = \frac{(\not{k} + \not{k}_n) + m}{D_n}. \tag{3.6}$$

Although we use the form $\not{F}_n^{-1}$ to introduce perturbative amplitudes and derive relations among them, employing the denominator $D_n = (k+k_n)^2 - m^2$ is useful to the integration. Due to the adopted simplifications, vertices have the following structures

$$\Gamma_l = \{\Gamma_S, \Gamma_P, \Gamma_V, \Gamma_A, \Gamma_{\tilde{T}}\} = \{\mathbf{1}, \gamma_5, \gamma^\mu, \gamma^\mu\gamma_5, \gamma_5\sigma^{\mu\nu}\}. \tag{3.7}$$

Capital Latin subindices denote the nature of each object. They correspond respectively to scalar, pseudoscalar, vector, axial, and pseudotensor vertices. We extend this notation to perturbative amplitudes, where these labels indicate the vertex content and the specific position of each operator.

Loop corrections to processes involving two, three, and four external bosons arise within this discussion. The Figure 3.1 shows representations through Feynman diagrams associated with these amplitudes. We have yet to specify the vertex content, but the momenta configuration is set. Although routings k_i do not have physical meaning by themselves, conservation laws on the vertices connect external (physical) momenta with differences between routings. The conventions adopted allow summarizing these relations into the object $p_i = k_1 - k_i$, whose accessible values are the following

$$p_2 = k_1 - k_2 = p, \quad p_3 = k_1 - k_3 = q, \quad p_4 = k_1 - k_4 = r.$$

In order to proceed with definitions, we must cast the processes that concern this investigation (by setting the vertex configurations). This subject is covered in the sequence.

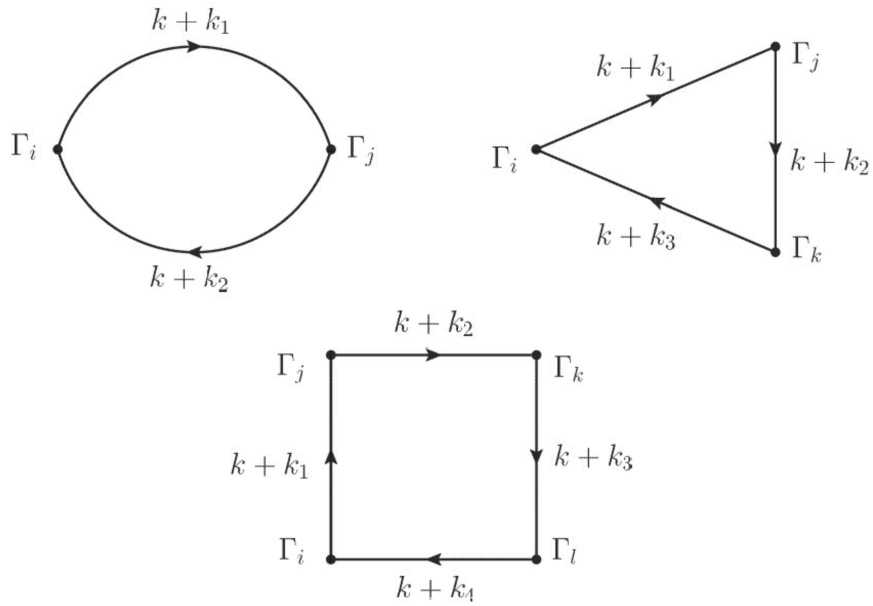

Figure 3.1: One-loop corrections to processes described by external bosons.

3.1.1 Perturbative Amplitudes

Since resources required to construct any fermionic amplitude are at our disposal, let us delimit those of interest and how they relate to each other. We consider constraints coming from their mathematical structure and symmetry implications in this process.

The neutral pion decay in two photons has a remarkable role in studies on anomalies, so we take the single axial triangle amplitude as the first laboratory. It is described by one axial and two vector vertices, assuming the form

$$t^{AVV}_{\mu\nu\alpha} = \mathrm{tr}\left\{ \gamma_\mu \gamma_5 \frac{1}{\not{D}_1} \gamma_\nu \frac{1}{\not{D}_2} \gamma_\alpha \frac{1}{\not{D}_3} \right\}. \tag{3.8}$$

From now on, we use labels to refer to a particular perturbative amplitude, i.e., we designate this one as AVV. Within the IReg perspective, the systematization of results highlights certain features concerning the mathematical structure of the AVV that relate to its anomalous character. They motivate us to pursue one higher-order amplitude exhibiting similar elements: the single axial box amplitude

$$t^{AVVV}_{\mu\nu\alpha\beta} = \mathrm{tr}\left\{\gamma_\mu\gamma_5\frac{1}{\not{\mathcal{P}}_1}\gamma_\nu\frac{1}{\not{\mathcal{P}}_2}\gamma_\alpha\frac{1}{\not{\mathcal{P}}_3}\gamma_\beta\frac{1}{\not{\mathcal{P}}_4}\right\}, \tag{3.9}$$

denominated $AVVV$. Even though its evaluation is complex, all operations involved are analogous to those performed in the triangle context. Thus, we consider the first process as a guide for analyzing the second.

These amplitudes are the central elements of this work. Nevertheless, as acknowledged in the discussion about Noether currents (3.2)-(3.3), relations among amplitudes could be derived through contractions with the external momenta. Thus, we explore this operation for the integrands above to introduce the remaining correlators while discussing potential constraints on the results.

For such purpose, let us express contractions involving physical momenta and Dirac matrices in terms of fermion propagators (3.6):

$$\not{k}_i - \not{k}_j = \not{\mathcal{P}}_i - \not{\mathcal{P}}_j. \tag{3.10}$$

Now, consider the specific contraction on the index associated with the first vector vertex of the triangle amplitude. Posteriorly to the implementation of this identity, trace linearity leads to the result

$$p^\nu t^{AVV}_{\mu\nu\alpha} = \mathrm{tr}\left\{\gamma_\mu\gamma_5\frac{1}{\not{\mathcal{P}}_2}\gamma_\alpha\frac{1}{\not{\mathcal{P}}_3}\right\} - \mathrm{tr}\left\{\gamma_\mu\gamma_5\frac{1}{\not{\mathcal{P}}_1}\gamma_\alpha\frac{1}{\not{\mathcal{P}}_3}\right\},$$

where a difference between AV two-point amplitudes is identified

$$p^\nu t^{AVV}_{\mu\nu\alpha} = t^{AV}_{\mu\alpha}\left(k_2, k_3\right) - t^{AV}_{\mu\alpha}\left(k_1, k_3\right). \tag{3.11}$$

An analogous relation arises for the second vector vertex through the same steps

$$\left(q-p\right)^\alpha t^{AVV}_{\mu\nu\alpha} = t^{AV}_{\mu\nu}\left(k_1, k_3\right) - t^{AV}_{\mu\nu}\left(k_1, k_2\right). \tag{3.12}$$

As for contractions with axial vertices, we multiply the identity (3.10) by the chiral matrix. Permuting its position is necessary to allow identifications

$$\left(\not{k}_i - \not{k}_j\right)\gamma_5 = \not{\mathcal{P}}_i\gamma_5 + \gamma_5\not{\mathcal{P}}_j + 2m\gamma_5. \tag{3.13}$$

Besides the difference between AV amplitudes, one additional term corresponds to the PVV amplitude

$$q^{\mu}t^{AVV}_{\mu\nu\alpha}=t^{AV}_{\nu\alpha}\left(k_2,k_3\right)-t^{AV}_{\alpha\nu}\left(k_1,k_2\right)-2mt^{PVV}_{\nu\alpha}. \tag{3.14}$$

Concerning the $AVVV$ box amplitude, contractions follow the same procedure and yield the results:

$$r^{\mu}t^{AVVV}_{\mu\nu\alpha\beta} = t^{AVV}_{\nu\alpha\beta}\left(k_2,k_3,k_4\right)-t^{AVV}_{\beta\nu\alpha}\left(k_1,k_2,k_3\right)-2mt^{PVVV}_{\nu\alpha\beta}, \tag{3.15}$$

$$p^{\nu}t^{AVVV}_{\mu\nu\alpha\beta} = t^{AVV}_{\mu\alpha\beta}\left(k_2,k_3,k_4\right)-t^{AVV}_{\mu\alpha\beta}\left(k_1,k_3,k_4\right), \tag{3.16}$$

$$\left(q-p\right)^{\alpha}t^{AVVV}_{\mu\nu\alpha\beta} = t^{AVV}_{\mu\nu\beta}\left(k_1,k_3,k_4\right)-t^{AVV}_{\mu\nu\beta}\left(k_1,k_2,k_4\right), \tag{3.17}$$

$$\left(r-q\right)^{\beta}t^{AVVV}_{\mu\nu\alpha\beta} = t^{AVV}_{\mu\nu\alpha}\left(k_1,k_2,k_3\right)-t^{AVV}_{\mu\nu\alpha}\left(k_1,k_2,k_4\right). \tag{3.18}$$

Although all operations lead to the difference between AVV triangles, the $PVVV$ four-point amplitude appears as the extra contribution in the axial contraction.

Obtaining these relations considers only the mathematical structure of integrands, which consist of identities at this level. Their validity after integration represents a manifestation of linearity. Nonetheless, we will see that the anomalous character of involved amplitudes might affect these prospects. Then, if their verification is successful, proper relations among Green functions (GF) are established. Expectations for contractions with the AVV triangle are the following

$$q^{\mu}T^{AVV}_{\mu\nu\alpha} \rightarrow T^{AV}_{\nu\alpha}\left(k_2,k_3\right)-T^{AV}_{\alpha\nu}\left(k_1,k_2\right)-2mT^{PVV}_{\nu\alpha}, \tag{3.19}$$

$$p^{\nu}T^{AVV}_{\mu\nu\alpha} \rightarrow T^{AV}_{\mu\alpha}\left(k_2,k_3\right)-T^{AV}_{\mu\alpha}\left(k_1,k_3\right), \tag{3.20}$$

$$\left(q-p\right)^{\alpha}T^{AVV}_{\mu\nu\alpha} \rightarrow T^{AV}_{\mu\nu}\left(k_1,k_3\right)-T^{AV}_{\mu\nu}\left(k_1,k_2\right), \tag{3.21}$$

while contractions involving the $AVVV$ box yield

$$r^{\mu}T^{AVVV}_{\mu\nu\alpha\beta} \rightarrow T^{AVV}_{\nu\alpha\beta}\left(k_2,k_3,k_4\right)-T^{AVV}_{\beta\nu\alpha}\left(k_1,k_2,k_3\right)-2mT^{PVVV}_{\nu\alpha\beta}, \tag{3.22}$$

$$p^{\nu}T^{AVVV}_{\mu\nu\alpha\beta} \rightarrow T^{AVV}_{\mu\alpha\beta}\left(k_2,k_3,k_4\right)-T^{AVV}_{\mu\alpha\beta}\left(k_1,k_3,k_4\right), \tag{3.23}$$

$$\left(q-p\right)^{\alpha}T^{AVVV}_{\mu\nu\alpha\beta} \rightarrow T^{AVV}_{\mu\nu\beta}\left(k_1,k_3,k_4\right)-T^{AVV}_{\mu\nu\beta}\left(k_1,k_2,k_4\right), \tag{3.24}$$

$$\left(r-q\right)^{\beta}T^{AVVV}_{\mu\nu\alpha\beta} \rightarrow T^{AVV}_{\mu\nu\alpha}\left(k_1,k_2,k_3\right)-T^{AVV}_{\mu\nu\alpha}\left(k_1,k_2,k_4\right). \tag{3.25}$$

Previously, we stated that current conservation (3.2)-(3.3) generates implications over quantum corrections. Ward identities (WIs) relate to momenta contractions over perturbative amplitudes. In the hypothesis that one relation among GF applies, the maintenance of the corresponding WI requires the cancellation of differences between amplitudes above (AVs in the first set and AVVs in the second). We cast these expectations in the sequence, where the required sum of channels is implicit in the notation $\mathcal{T}$. Nevertheless, we will see that our analysis applies channel by channel. The identities for the AVV amplitude are

$$q^{\mu}\mathcal{T}_{\mu\nu\alpha}^{AVV} \rightarrow -2m\mathcal{T}_{\nu\alpha}^{PVV}, \tag{3.26}$$
$$p^{\nu}\mathcal{T}_{\mu\nu\alpha}^{AVV} \rightarrow 0, \tag{3.27}$$
$$(q-p)^{\alpha}\,\mathcal{T}_{\mu\nu\alpha}^{AVV} \rightarrow 0, \tag{3.28}$$

while those for the $AVVV$ amplitude are

$$r^{\mu}\mathcal{T}_{\mu\nu\alpha\beta}^{AVVV} \rightarrow -2m\mathcal{T}_{\nu\alpha\beta}^{PVVV}, \tag{3.29}$$
$$p^{\nu}\mathcal{T}_{\mu\nu\alpha\beta}^{AVVV} \rightarrow 0, \tag{3.30}$$
$$(q-p)^{\alpha}\,\mathcal{T}_{\mu\nu\alpha\beta}^{AVVV} \rightarrow 0, \tag{3.31}$$
$$(r-p)^{\beta}\,\mathcal{T}_{\mu\nu\alpha\beta}^{AVVV} \rightarrow 0. \tag{3.32}$$

Given the impossibility of simultaneous satisfaction of gauge and axial symmetries, these are also preliminary prospects.

Through this argumentation, we connected concepts of integral linearity and symmetry implications. If relations among GF are identically satisfied, canceling those differences on their right-hand side also satisfies WIs. Nevertheless, the fact that these amplitudes exhibit diverging power counting is problematic when testing these expectations. That is particularly important in the anomalies context. We will return to this discussion after exploring the perturbative amplitudes at the integrand level.

3.2 Structure of Perturbative Amplitudes

This work implements Feynman rules in two parts, starting with obtaining perturbative amplitudes for a single value of the unrestricted (loop) momentum. Thus, organizing and examining their content without worries about the divergences that come with integration is attainable. We begin by introducing an example illustrating the elements required for this task. Subsequently, we inquire about two, three, and four-point functions concerning this investigation.

3.2.1 Two-Point Amplitudes - Preliminary Notions

This analysis uses a simple example to familiarize with calculations while producing tools for more complex scenes. Soon we will come across extensive mathematical expressions that might seem vague. Thereby, designing mechanisms to compact them and systematizing operations is part of our task.

The next-to-leading order correction to processes involving external bosons corresponds to pure fermionic loops. We denoted these amplitudes using uppercase letters (3.4), while their integrands use lowercase letters (3.5). These structures contain traces of vertex operators Γ_i and fermion propagators $\not{F}_n^{-1}$, as seen in the example of two-point functions:

$$t^{\Gamma_i\Gamma_j} = \mathrm{tr}\left(\Gamma_i \frac{1}{\not{F}_1}\Gamma_j \frac{1}{\not{F}_2}\right). \tag{3.33}$$

After rewriting the propagator (3.6), using the linearity of the trace makes its matrix content explicit

$$\begin{aligned} t^{\Gamma_i\Gamma_j} \;\; = \;\; & \mathrm{tr}\left(\Gamma_i\gamma_A\Gamma_j\gamma_B\right)\frac{K_1^A K_2^B}{D_{12}} + m^2\mathrm{tr}\left(\Gamma_i\Gamma_j\right)\frac{1}{D_{12}} \\ & + m\mathrm{tr}\left(\Gamma_i\gamma_A\Gamma_j\right)\frac{K_1^A}{D_{12}} + m\mathrm{tr}\left(\Gamma_i\Gamma_j\gamma_B\right)\frac{K_2^B}{D_{12}}. \end{aligned} \tag{3.34}$$

As several notations appear within this context, let us explain them subsequently. We introduced compact products as that of the denominator $D_{ij} = D_i D_j$ for propagator-like objects $D_i = (k + k_i)^2 - m^2$. Our goal in this section is to express integrands through combinations depending on these structures

$$\frac{1}{D_i}, \frac{[1, k_\mu, k_{\mu\nu}]}{D_{ij}}, \frac{[1, k_\mu, k_{\mu\nu}, k_{\mu\nu\alpha}]}{D_{ijk}}, \frac{[1, k_\mu, k_{\mu\nu}, k_{\mu\nu\alpha}, k_{\mu\nu\alpha\beta}]}{D_{ijkl}},$$

which leads to identifying Feynman integrals in Section (3.3). That means the usage of the symbol $K_i = k + k_i$ is limited to the current analysis, being another artifice to reduce expressions. We also introduced compact notations for products of momenta or routings:

$$k_{\mu\nu} = k_\mu k_\nu, \qquad p_{\mu\nu} = p_\mu p_\nu, \qquad k_{1\mu\nu} = k_{1\mu} k_{1\nu}.$$

The second type of notation consists of (the possibility of) adopting uppercase Latin letters for summed indices and neglecting their covariant or contravariant character. This resource facilitates the recognition of sectors with analogous index configurations inside tensor amplitudes, making substructures promptly noticeable. Hence, identifying other amplitudes inside the original only requires sign comparisons among options. Furthermore, other terms receive a suitable organization through standard tensors. We also use this notation to emphasize symmetry properties.

Since we know these tools and ideas, we implement them in the mentioned example. It consists of the double-vector function VV, which associates with the photon self-energy in the Quantum Electrodynamics context. The replacement of Dirac matrices as vertex operators ($\Gamma_i = \gamma_\mu$ and $\Gamma_j = \gamma_\nu$) on the integrand above generates the expression

$$\begin{aligned} t_{\mu\nu}^{VV} &= \mathrm{tr}\left(\gamma_\mu \gamma_A \gamma_\nu \gamma_B\right) \frac{K_1^A K_2^B}{D_{12}} + m^2 \mathrm{tr}\left(\gamma_\mu \gamma_\nu\right) \frac{1}{D_{12}} \\ &+ m \mathrm{tr}\left(\gamma_\mu \gamma_A \gamma_\nu\right) \frac{K_1^A}{D_{12}} + m \mathrm{tr}\left(\gamma_\mu \gamma_\nu \gamma_B\right) \frac{K_2^B}{D_{12}}. \end{aligned} \tag{3.35}$$

Even though Dirac traces are common ingredients, we discuss them to ground future calculations. The property of anticommutation followed by Dirac matrices is the outset

$$\gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu = 2 g_{\mu\nu}. \tag{3.36}$$

By taking the trace on both sides, linearity and invariance under cyclic permutations lead to the equation

$$\mathrm{tr}\left(\gamma_\mu \gamma_\nu\right) = g_{\mu\nu} \mathrm{tr}\left(\mathbf{1}\right) = 4 g_{\mu\nu}. \tag{3.37}$$

Any other trace involving an even number of Dirac matrices could be reduced to this one. For instance, we use the anticommutation property to express the four matrices trace as the following combination

$$\begin{aligned} \mathrm{tr}\left(\gamma_\mu \gamma_A \gamma_\nu \gamma_B\right) &= \mathrm{tr}\left(2 g_{\mu A} \gamma_\nu \gamma_B - 2 g_{\mu\nu} \gamma_A \gamma_B + 2 g_{\mu B} \gamma_A \gamma_\nu - \gamma_A \gamma_\nu \gamma_B \gamma_\mu\right) \\ &= 4 g_{\mu A} g_{\nu B} - 4 g_{\mu\nu} g_{AB} + 4 g_{\mu B} g_{A\nu}. \end{aligned} \tag{3.38}$$

As for products involving an odd number of Dirac matrices, trace operation vanishes. To prove this statement, introduce the identity $\mathbf{1} = \gamma_5^2$ inside the argument. Using (respectively) the fact that the chiral matrix anticommutes with any Dirac matrix and the cyclicity, we show that these traces are equal to their negative and, therefore, vanish.

To illustrate, take the trace of one single Dirac matrix

$$\mathrm{tr}\left(\gamma_{\mu}\right)=\mathrm{tr}\left(\gamma_{5}\gamma_{5}\gamma_{\mu}\right)=-\mathrm{tr}\left(\gamma_{5}\gamma_{\mu}\gamma_{5}\right)=-\mathrm{tr}\left(\gamma_{5}\gamma_{5}\gamma_{\mu}\right)=-\mathrm{tr}\left(\gamma_{\mu}\right).$$

When replacing these results on the VV amplitude and rearranging it, the sorting of free indices shows two sectors

$$t_{\mu\nu}^{VV}=4\frac{K_{1\mu}K_{2\nu}+K_{1\nu}K_{2\mu}}{D_{12}}+g_{\mu\nu}\left[-\mathrm{tr}\left(\gamma_{A}\gamma_{B}\right)\frac{K_{1}^{A}K_{2}^{B}}{D_{12}}+m^{2}\mathrm{tr}\left(\mathbf{1}\right)\frac{1}{D_{12}}\right]. \tag{3.39}$$

The first corresponds to the symmetric version of the following standard tensor

$$t_{2\mu\nu}^{(s)}\left(k_{i},k_{j}\right)=\frac{\left(k+k_{i}\right)_{\mu}\left(k+k_{j}\right)_{\nu}+s\left(k+k_{j}\right)_{\mu}\left(k+k_{i}\right)_{\nu}}{D_{12}}. \tag{3.40}$$

This general definition admits a numerical subindex, characterizing the number of propagator-like objects in the denominator (two in this case $D_{12}=D_1D_2$), and it allows different signs $s=\pm1$. Since this expression is a combination of structures previously mentioned, it does not require further analysis.

As for the sector proportional to the metric tensor $g_{\mu\nu}$, we recognized traces involving fewer matrices. They associate with a scalar amplitude from two possibilities: SS and PP. Thus, replace the corresponding vertices on Equation (3.34) to determine their integrands:

$$t^{SS} = \mathrm{tr}\left(\gamma_{A}\gamma_{B}\right)\frac{K_{1}^{A}K_{2}^{B}}{D_{12}}+m^{2}\mathrm{tr}\left(\mathbf{1}\right)\frac{1}{D_{12}}, \tag{3.41}$$

$$t^{PP} = -\mathrm{tr}\left(\gamma_{A}\gamma_{B}\right)\frac{K_{1}^{A}K_{2}^{B}}{D_{12}}+m^{2}\mathrm{tr}\left(\mathbf{1}\right)\frac{1}{D_{12}}. \tag{3.42}$$

Since we did not rename any index, the precise identification occurs by comparing signs, and we achieve the organization

$$t_{\mu\nu}^{VV}=4t_{2\mu\nu}^{(+)}\left(k_{1},k_{2}\right)+g_{\mu\nu}t^{PP}. \tag{3.43}$$

Exploring the PP structure is still necessary, so we draw attention to its dependence on the objects

$$2K_{ij}\rightarrow2\left(K_{i}\cdot K_{j}-m^{2}\right)=D_{i}+D_{j}-\left(k_{i}-k_{j}\right)^{2}. \tag{3.44}$$

This identity brings propagator-like objects to numerators, which reflects on reductions of denominators within the amplitude integrand

$$t^{PP}=-2\left[\frac{1}{D_{1}}+\frac{1}{D_{2}}-p^{2}\frac{1}{D_{12}}\right], \tag{3.45}$$

where we identified the external momentum $p = k_1 - k_2$. The recurrent application of this resource throughout this investigation justifies generic indices. Notice that, with the momenta integration, this identity reduces part of the Feynman integrals to those involving one less propagator.

We do not integrate these amplitudes in the future since they are not part of this work. Even so, take them as a guide to calculations performed from now on.

3.2.2 Two-Point Amplitudes - AV

Given the general expression for two-point amplitudes (3.34), we replace vertex operators to write the integrand of the axial-vector amplitude

$$\begin{aligned} t_{\mu\nu}^{AV} &= \mathrm{tr}\left(\gamma_\mu\gamma_5\gamma_A\gamma_\nu\gamma_B\right)\frac{K_1^A K_2^B}{D_{12}} + m^2\mathrm{tr}\left(\gamma_\mu\gamma_5\gamma_\nu\right)\frac{1}{D_{12}} \\ &+ m\mathrm{tr}\left(\gamma_\mu\gamma_5\gamma_A\gamma_\nu\right)\frac{K_1^A}{D_{12}} + m\mathrm{tr}\left(\gamma_\mu\gamma_5\gamma_\nu\gamma_B\right)\frac{K_2^B}{D_{12}}, \end{aligned} \tag{3.46}$$

where numerators depend on $K_i = k + k_i$ and denominators are $D_{12} = D_1 D_2$. We refer to this structure as AV, which specifies the first vertex as an axial $\Gamma_i = \gamma_\mu\gamma_5$ and the second as a vector $\Gamma_j = \gamma_\nu$. Although these traces contain the chiral matrix, replacing its definition $\gamma_5 = \frac{i}{4!}\varepsilon_{\mu\nu\alpha\beta}\gamma^\mu\gamma^\nu\gamma^\alpha\gamma^\beta$ suppresses this dependence. That adds four extra Dirac matrices to the argument while introducing a global factor through the Levi-Civita symbol. Within this perspective, we must compute even traces following steps seen in the previous subsection and then perform contractions.

Immediately, occurrences involving an odd number of Dirac matrices plus the chiral one vanish. That also happens in the case involving two Dirac matrices since it leads to contractions between symmetric and antisymmetric tensors. Hence, the only non-zero trace involves four Dirac matrices, whose computation leads to the Levi-Civita symbol

$$\mathrm{tr}\left(\gamma_5\gamma_\mu\gamma_A\gamma_\nu\gamma_B\right) = 4i\varepsilon_{\mu A\nu B}. \tag{3.47}$$

When replacing it, symmetry properties allow identifying the antisymmetric version of the standard tensor (3.40):

$$t_{\mu\nu}^{AV} = 2i\varepsilon_{\mu\nu XY} t_{2XY}^{(-)}\left(k_1, k_2\right). \tag{3.48}$$

One would expect two ingredients to compound the integrated substructure: metric tensor and external momentum $p = k_1 - k_2$. Since they combine exclusively into symmetric quantities (g_{XY} and $p_{XY} = p_X p_Y$), the contraction should cancel out. Nevertheless, two-point functions exhibit quadratic power counting in the physical dimension. Therefore, these integrals are not invariant under translations, admitting the emergence of non-physical momenta associated with surface terms. That provides another vector to build up the substructure: the sum of arbitrary routings k_1+k_2. Hence, we expect the integrated AV amplitude to have the following form

$$T^{AV}_{\mu\nu} \to \varepsilon_{\mu\nu XY} \left(k_1 - k_2\right)^X \left(k_1 + k_2\right)^Y G_0, \tag{3.49}$$

where G_0 represents a surface term that is logarithmically divergent to adjust with mass dimension.

Such dependence characterizes an ambiguity, a quantity depending on arbitrary choices. Momenta conservation sets differences between labels as external momenta; however, it does not attribute a particular meaning to routings themselves or their sum. As proposed before, this arbitrariness is preserved throughout this investigation.

3.2.3 Three-Point Amplitudes - PVV

Previously, we used lowercase letters to denote the integrand of fermionic amplitudes (3.5). They correspond to traces containing vertex operators Γ_i and fermion propagators $\not{F}_n^{-1}$, as seen for the particular case of three-point functions:

$$t^{\Gamma_i\Gamma_j\Gamma_k} = \mathrm{tr}\left(\Gamma_i \frac{1}{\not{F}_1} \Gamma_j \frac{1}{\not{F}_2} \Gamma_k \frac{1}{\not{F}_3}\right). \tag{3.50}$$

Rewriting the propagators (3.6) emphasizes the coefficients as Dirac traces

$$\begin{aligned} t^{\Gamma_i\Gamma_j\Gamma_k} = {} & \mathrm{tr}\left(\Gamma_i\gamma_A\Gamma_j\gamma_B\Gamma_k\gamma_C\right)\frac{K_1^A K_2^B K_3^C}{D_{123}} + m\mathrm{tr}\left(\Gamma_i\Gamma_j\gamma_B\Gamma_k\gamma_C\right)\frac{K_2^B K_3^C}{D_{123}} \\ & + m\mathrm{tr}\left(\Gamma_i\gamma_A\Gamma_j\Gamma_k\gamma_C\right)\frac{K_1^A K_3^C}{D_{123}} + m\mathrm{tr}\left(\Gamma_i\gamma_A\Gamma_j\gamma_B\Gamma_k\right)\frac{K_1^A K_2^B}{D_{123}} \\ & + m^2\mathrm{tr}\left(\Gamma_i\gamma_A\Gamma_j\Gamma_k\right)\frac{K_1^A}{D_{123}} + m^2\mathrm{tr}\left(\Gamma_i\Gamma_j\gamma_B\Gamma_k\right)\frac{K_2^B}{D_{123}} \\ & + m^2\mathrm{tr}\left(\Gamma_i\Gamma_j\Gamma_k\gamma_C\right)\frac{K_3^C}{D_{123}} + m^3\mathrm{tr}\left(\Gamma_i\Gamma_j\Gamma_k\right)\frac{1}{D_{123}}, \end{aligned} \tag{3.51}$$

where numerators depend on $K_i = k + k_i$ and denominators are $D_{123} = D_1 D_2 D_3$.

To study the structure of a specific amplitude, we set its vertex content and evaluate corresponding traces. For the PVV case, the first vertex indicates a pseudoscalar $\Gamma_i = \gamma_5$ while the others indicate vectors $\Gamma_j = \gamma_\nu$ and $\Gamma_k = \gamma_\alpha$. Its non-zero contributions are the following

$$\begin{aligned} t^{PVV}_{\nu\alpha} = {} & m\mathrm{tr}\left(\gamma_5\gamma_\nu\gamma_B\gamma_\alpha\gamma_C\right)\frac{K_2^B K_3^C}{D_{123}} + m\mathrm{tr}\left(\gamma_5\gamma_A\gamma_\nu\gamma_\alpha\gamma_C\right)\frac{K_1^A K_3^C}{D_{123}} \\ & + m\mathrm{tr}\left(\gamma_5\gamma_A\gamma_\nu\gamma_B\gamma_\alpha\right)\frac{K_1^A K_2^B}{D_{123}}. \end{aligned} \tag{3.52}$$

They are proportional to the Levi-Civita symbol (3.47), leading to the antisymmetric version of the standard tensor

$$t_{3\mu\nu}^{(s)}\left(k_i,k_j\right)=\frac{\left(k+k_i\right)_\mu\left(k+k_j\right)_\nu+s\left(k+k_j\right)_\mu\left(k+k_i\right)_\nu}{D_{123}}, \tag{3.53}$$

Such an object is analogous to the previous one (3.40); however, it depends on three propagators embodied in D_{123} as indicated by the numerical subindex. With these identifications, the integrand of the amplitude exhibits the form

$$t_{\nu\alpha}^{PVV}=-2im\varepsilon_{\nu\alpha XY}\left[t_{3XY}^{(-)}\left(k_2,k_3\right)+t_{3XY}^{(-)}\left(k_3,k_1\right)+t_{3XY}^{(-)}\left(k_1,k_2\right)\right]. \tag{3.54}$$

Observe the analogy between the PVV structure and that of the AV (3.48); both are 2nd-order tensors contracted with the Levi-Civita symbol. Nonetheless, expectations are different now. Even though three-point functions exhibit linear power counting, contributions involving diverging surface terms are prohibited since only finite contributions adjust to the correct mass dimension. On the other hand, after integration, two external momenta ($p=k_1-k_2$ and $q=k_1-k_3$) are available to build up the tensor structure

$$T_{\nu\alpha}^{PVV}\rightarrow\varepsilon_{\nu\alpha XY}p^X q^Y F_0. \tag{3.55}$$

The object $F_0=F_0\left(p_i\cdot p_j\right)$ represents a finite scalar function depending on momenta bilinears $p_i\cdot p_j=\{p^2,\ q^2,\ p\cdot q\}$.

3.2.4 Three-Point Amplitudes - AVV

The AVV integrand emerges by replacing the corresponding vertex operators within Equation (3.51); they are axial $\Gamma_i=\gamma_\mu\gamma_5$, vector $\Gamma_j=\gamma_\nu$ and vector $\Gamma_k=\gamma_\alpha$. Leaving null contributions aside, we cast its initial structure:

$$\begin{aligned}t_{\mu\nu\alpha}^{AVV} &= \mathrm{tr}\left(\gamma_\mu\gamma_5\gamma_A\gamma_\nu\gamma_B\gamma_\alpha\gamma_C\right)\frac{K_1^A K_2^B K_3^C}{D_{123}}+m^2\mathrm{tr}\left(\gamma_\mu\gamma_5\gamma_A\gamma_\nu\gamma_\alpha\right)\frac{K_1^A}{D_{123}}\\ &\quad+m^2\mathrm{tr}\left(\gamma_\mu\gamma_5\gamma_\nu\gamma_B\gamma_\alpha\right)\frac{K_2^B}{D_{123}}+m^2\mathrm{tr}\left(\gamma_\mu\gamma_5\gamma_\nu\gamma_\alpha\gamma_C\right)\frac{K_3^C}{D_{123}},\end{aligned} \tag{3.56}$$

where numerators depend on $K_i=k+k_i$ and denominators are $D_{123}=D_1D_2D_3$. Terms associated with the squared mass are already known, being proportional to the Levi-Civita symbol (3.47).

Our next task is to take the trace involving six Dirac matrices plus the chiral one. Nevertheless, different ways to perform this operation attribute different expressions for it. Although all forms attributed to one trace are linked through identities, the divergent character of perturbative calculations affects these relations after integration. Clarifying these aspects is essential to this investigation, so we are very detailed in this discussion.

To introduce these ideas, we use the chiral matrix anticommutation in studying two possibilities

$$\mathrm{tr}\left(\gamma_\mu\gamma_5\gamma_A\gamma_\nu\gamma_B\gamma_\alpha\gamma_C\right) = -\mathrm{tr}\left(\gamma_5\gamma_\mu\gamma_A\gamma_\nu\gamma_B\gamma_\alpha\gamma_C\right).$$

After replacing the definition $\gamma_5 = \frac{i}{4!}\varepsilon_{\mu\nu\alpha\beta}\gamma^\mu\gamma^\nu\gamma^\alpha\gamma^\beta$ in these specific places, we obtain a trace involving only Dirac matrices. Its computation yields combinations of the metric tensors, which are contracted with the Levi-Civita symbol. The expression obtained through the first path is

$$\begin{aligned} & \mathrm{tr}\left(\gamma_\mu\gamma_5\gamma_A\gamma_\nu\gamma_B\gamma_\alpha\gamma_C\right) \\ = \; & -4i\left[\varepsilon_{\mu A\nu B}g_{\alpha C} - \varepsilon_{\mu A\nu\alpha}g_{BC} + \varepsilon_{\mu A\nu C}g_{B\alpha} + \varepsilon_{\mu AB\alpha}g_{\nu C} - \varepsilon_{\mu ABC}g_{\nu\alpha}\right. \\ & +\varepsilon_{\mu A\alpha C}g_{\nu B} - \varepsilon_{\mu\nu B\alpha}g_{AC} + \varepsilon_{\mu\nu BC}g_{A\alpha} - \varepsilon_{\mu\nu\alpha C}g_{AB} + \varepsilon_{\mu B\alpha C}g_{A\nu} \\ & \left. -\varepsilon_{A\nu B\alpha}g_{\mu C} + \varepsilon_{A\nu BC}g_{\mu\alpha} - \varepsilon_{A\nu\alpha C}g_{\mu B} + \varepsilon_{AB\alpha C}g_{\mu\nu} - \varepsilon_{\nu B\alpha C}g_{\mu A}\right], \end{aligned} \tag{3.57}$$

while the other is

$$\begin{aligned} & -\mathrm{tr}\left(\gamma_5\gamma_\mu\gamma_A\gamma_\nu\gamma_B\gamma_\alpha\gamma_C\right) \\ = \; & -4i\left[\varepsilon_{\mu A\nu B}g_{\alpha C} - \varepsilon_{\mu A\nu\alpha}g_{BC} + \varepsilon_{\mu A\nu C}g_{B\alpha} + \varepsilon_{\mu AB\alpha}g_{\nu C} - \varepsilon_{\mu ABC}g_{\nu\alpha}\right. \\ & +\varepsilon_{\mu A\alpha C}g_{\nu B} - \varepsilon_{\mu\nu B\alpha}g_{AC} + \varepsilon_{\mu\nu BC}g_{A\alpha} - \varepsilon_{\mu\nu\alpha C}g_{AB} + \varepsilon_{\mu B\alpha C}g_{A\nu} \\ & \left. +\varepsilon_{A\nu B\alpha}g_{\mu C} - \varepsilon_{A\nu BC}g_{\mu\alpha} + \varepsilon_{A\nu\alpha C}g_{\mu B} - \varepsilon_{AB\alpha C}g_{\mu\nu} + \varepsilon_{\nu B\alpha C}g_{\mu A}\right]. \end{aligned} \tag{3.58}$$

Although there are other strategies to compute them, one reason to choose this path is that the results contain all contributions with non-equivalent tensor configurations. This feature is convenient for the organization developed throughout this section, which is part of IReg. Furthermore, the reason for replacing the chiral matrix definition in these specific positions (adjacent to γ_μ) is to induce a simplification.

The layout of these (equivalent) expressions highlights that they only differ by signs on the last row, characterizing one identity:

$$g_{\mu C}\varepsilon_{A\nu B\alpha} - g_{\mu\alpha}\varepsilon_{A\nu BC} + g_{\mu B}\varepsilon_{A\nu\alpha C} - g_{\mu\nu}\varepsilon_{AB\alpha C} + g_{\mu A}\varepsilon_{\nu B\alpha C} = 0. \tag{3.59}$$

From another perspective, note that this tensor is antisymmetric in five indices (μ fixed); therefore, identically zero for a four-dimensional setting. Achieving this identity is not a coincidence but a direct consequence of comparing positions adjacent to the μ-index. Finding similar identities where other free indices play this role is within reach. That is only the first example seen here of the so-called Schouten identities.

With this argumentation, we developed the know-how to find the same resources in more complex expressions from four-point amplitudes. Although that significantly reduces our efforts in these calculations, there is no damage in ignoring these identities. We verified that these contributions produce null integrals when evaluating perturbative amplitudes.

As a brief comment on this subject, suppose we achieve three trace expressions corresponding to each vertex position represented by free indices (μ, ν, and α). They are equivalent since their obtainment comes from pure algebraic manipulations. Nevertheless, due to their divergent content, their connection might not apply after integrating the amplitude. We attribute a central role to the μ-index for now, but Subsection (3.5.3) extends this notion. The author, L. Ebani, and J. F. Thuorst develop a broad investigation of the behavior of different versions of odd-tensor correlators in reference [48].

Returning to the AVV triangle, replacing traces leads to its integrand

$$\begin{aligned} t_{\mu\nu\alpha}^{AVV} &= 4i\left(g_{\nu A}\varepsilon_{\mu\alpha BC} - g_{\nu B}\varepsilon_{\mu\alpha CA} - g_{\nu C}\varepsilon_{\mu\alpha AB}\right)\frac{K_1^A K_2^B K_3^C}{D_{123}} \\ &+4i\left(-g_{\alpha A}\varepsilon_{\mu\nu BC} - g_{B\alpha}\varepsilon_{\mu\nu CA} + g_{\alpha C}\varepsilon_{\mu\nu AB}\right)\frac{K_1^A K_2^B K_3^C}{D_{123}} \\ &+4i g_{\nu\alpha}\varepsilon_{\mu ABC}\frac{K_1^A K_2^B K_3^C}{D_{123}} \\ &+\varepsilon_{\mu\nu\alpha\beta}\left[\mathrm{tr}\left(\gamma_\beta\gamma_A\gamma_B\gamma_C\right)\frac{K_1^A K_2^B K_3^C}{D_{123}} - m^2\mathrm{tr}\left(\gamma_\beta\gamma_A\right)\frac{K_1^A}{D_{123}}\right. \\ &\left.+m^2\mathrm{tr}\left(\gamma_\beta\gamma_B\right)\frac{K_2^B}{D_{123}} - m^2\mathrm{tr}\left(\gamma_\beta\gamma_C\right)\frac{K_3^C}{D_{123}}\right]. \end{aligned} \tag{3.60}$$

We already split sectors corresponding to different tensor configurations and identified less complex traces. As terms with the free index μ within the metric compound the identity (3.59), we disregarded them.

Following the reasoning established in example (3.39), trace content suggests that the last term above consists of a vector subamplitude[2]. If one maintains the notations for summed indices, comparing signs is enough to identify the VPP among all possibilities. Meanwhile, the antisymmetric character of the Levi-Civita symbol allows rewriting the remaining terms through a new standard tensor characterized by three momenta on the numerator

$$t_{3\mu;\nu\alpha}^{(s)}\left(k_l;k_i,k_j\right) = \frac{\left(k+k_l\right)_\mu\left[\left(k+k_i\right)_\nu\left(k+k_j\right)_\alpha + s\left(k+k_i\right)_\alpha\left(k+k_j\right)_\nu\right]}{D_{123}}. \tag{3.61}$$

Following previous notations, the superindex s indicates a sign choice, and the numerical subindex indicates the association with three propagators through the denominator D_{123}.

[2]The trace structure indicates this subamplitude has one Lorentz index, which links to one axial or vector vertex. Other vertices might be scalar or pseudoscalar combined to produce an even trace. That leads to amplitudes corresponding to vectors: VPP, VSS, APS, and their permutations.

Hence, we achieve the final organization

$$
\begin{aligned}
t^{AVV}_{\mu\nu\alpha} &= 2i\varepsilon_{\mu\alpha XY}\left[-t^{(-)}_{3\nu;XY}\left(k_3;k_1,k_2\right) - t^{(-)}_{3\nu;XY}\left(k_2;k_3,k_1\right) + t^{(-)}_{3\nu;XY}\left(k_1;k_2,k_3\right)\right] \\
&+2i\varepsilon_{\mu\nu XY}\left[t^{(-)}_{3\alpha;XY}\left(k_3;k_1,k_2\right) - t^{(-)}_{3\alpha;XY}\left(k_2;k_3,k_1\right) - t^{(-)}_{3\alpha;XY}\left(k_1;k_2,k_3\right)\right] \\
&+2g_{\nu\alpha}\varepsilon_{\mu XYZ}t^{(-)X;YZ}_{3}\left(k_1;k_2,k_3\right) - i\varepsilon_{\mu\nu\alpha\beta}t^{VPP}_{\beta}. \qquad (3.62)
\end{aligned}
$$

After replacing the corresponding vertices[3] in the original integrand (3.51) and taking traces, we study the vector subamplitude

$$
\begin{aligned}
t^{VPP}_{\beta} &= -4\left(g_{\beta A}g_{BC} - g_{\beta B}g_{AC} + g_{\beta C}g_{AB}\right)\frac{K_1^A K_2^B K_3^C}{D_{123}} \\
&+4m^2\left[\frac{K_{1\beta}}{D_{123}} - \frac{K_{2\beta}}{D_{123}} + \frac{K_{3\beta}}{D_{123}}\right]. \qquad (3.63)
\end{aligned}
$$

Scalar products on the momenta emerge with the contraction, which leads to reducing bilinears in analogy with scalar functions used as example (3.44). Then, some manipulations produce the structure

$$
\begin{aligned}
t^{VPP}_{\beta} &= -2p_{\beta}\frac{1}{D_{12}} - 4\frac{k_{\beta}}{D_{13}} - 2\left(k_1+k_3\right)_{\beta}\frac{1}{D_{13}} + 2\left(q-p\right)_{\beta}\frac{1}{D_{23}} \\
&+2\left(q-p\right)^2\frac{\left(k+k_1\right)_{\beta}}{D_{123}} - 2q^2\frac{\left(k+k_2\right)_{\beta}}{D_{123}} + 2p^2\frac{\left(k+k_3\right)_{\beta}}{D_{123}}. \qquad (3.64)
\end{aligned}
$$

Lastly, we recall the AV discussion to infer expectations regarding integration. The objective was to compose a 2nd-order antisymmetric tensor with available tools, namely, external and ambiguous momenta ($k_i - k_j$ and $k_i + k_j$). The only possibility was to employ them both, which necessarily implies the presence of diverging surface terms. For this to be consistent with the quadratic power counting, these surface terms must be logarithmically divergent.

We find similar circumstances for any 3rd-order tensor exhibiting the property of total antisymmetry. At least three different vectors are necessary to compound it, which requires the presence of ambiguous momenta. This structure brings diverging surface terms, which prevents obtaining the correct mass dimension. As a consequence, 3rd-order antisymmetric tensors are zero under these circumstances.

The most immediate event of this type is the (three-index) contraction between the Levi-Civita symbol and the standard tensor. For it to be non-zero, the tensor must have a total-antisymmetric component. As this leads to the argumentation above, we expect its cancellation

$$
\varepsilon_{\nu XYZ}T^{(-)X;YZ}_{3}\left(k_1;k_2,k_3\right) \to 0. \qquad (3.65)
$$

[3] There are three vertices: one vector γ_{β} followed by two pseudoscalars γ_5.

Furthermore, we combine all non-equivalent momenta configurations to produce an identity involving this tensor

$$T^{(-)}_{3\mu;\nu\alpha}\left(k_1;k_2,k_3\right)+T^{(-)}_{3\mu;\nu\alpha}\left(k_2;k_3,k_1\right)+T^{(-)}_{3\mu;\nu\alpha}\left(k_3;k_1,k_2\right)\rightarrow 0. \tag{3.66}$$

If these expectations realize, simplifications apply to the integrated amplitude, yielding the expression:

$$T^{AVV}_{\mu\nu\alpha}\rightarrow 4i\varepsilon_{\mu\alpha XY}T^{(-)}_{3\nu;XY}\left(k_1;k_2,k_3\right)+4i\varepsilon_{\mu\nu XY}T^{(-)}_{3\alpha;XY}\left(k_3;k_1,k_2\right)-i\varepsilon_{\mu\nu\alpha\beta}T^{VPP}_{\beta}. \tag{3.67}$$

We stress that the μ-index appears exclusively within the Levi-Civita symbol as a direct consequence of its prioritized role when taking the traces; simplification only made this clear.

3.2.5 Four-Point Amplitudes - $PVVV$

We still have to look into four-point amplitudes, whose integrands assume the form

$$t^{\Gamma_i\Gamma_j\Gamma_k\Gamma_l}=\mathrm{tr}\left(\Gamma_i\frac{1}{\not{K}_1}\Gamma_j\frac{1}{\not{K}_2}\Gamma_k\frac{1}{\not{K}_3}\Gamma_l\frac{1}{\not{K}_4}\right). \tag{3.68}$$

After replacing fermion propagators (3.6), linearity makes the matrix content evident within Dirac traces:

$$\begin{aligned}
&t^{\Gamma_i\Gamma_j\Gamma_k\Gamma_l}\\
=\;&\mathrm{tr}\left(\Gamma_i\gamma_A\Gamma_j\gamma_B\Gamma_k\gamma_C\Gamma_l\gamma_D\right)\frac{K_1^AK_2^BK_3^CK_4^D}{D_{1234}}+m^4\mathrm{tr}\left(\Gamma_i\Gamma_j\Gamma_k\Gamma_l\right)\frac{1}{D_{1234}}\\
&+m^2\mathrm{tr}\left(\Gamma_i\gamma_A\Gamma_j\gamma_B\Gamma_k\Gamma_l\right)\frac{K_1^AK_2^B}{D_{1234}}+m^2\mathrm{tr}\left(\Gamma_i\gamma_A\Gamma_j\Gamma_k\gamma_C\Gamma_l\right)\frac{K_1^AK_3^C}{D_{1234}}\\
&+m^2\mathrm{tr}\left(\Gamma_i\gamma_A\Gamma_j\Gamma_k\Gamma_l\gamma_D\right)\frac{K_1^AK_4^D}{D_{1234}}+m^2\mathrm{tr}\left(\Gamma_i\Gamma_j\gamma_B\Gamma_k\gamma_C\Gamma_l\right)\frac{K_2^BK_3^C}{D_{1234}}\\
&+m^2\mathrm{tr}\left(\Gamma_i\Gamma_j\gamma_B\Gamma_k\Gamma_l\gamma_D\right)\frac{K_2^BK_4^D}{D_{1234}}+m^2\mathrm{tr}\left(\Gamma_i\Gamma_j\Gamma_k\gamma_C\Gamma_l\gamma_D\right)\frac{K_3^CK_4^D}{D_{1234}}\\
&+m\mathrm{tr}\left(\Gamma_i\Gamma_j\gamma_B\Gamma_k\gamma_C\Gamma_l\gamma_D\right)\frac{K_2^BK_3^CK_4^D}{D_{1234}}+m^3\mathrm{tr}\left(\Gamma_i\gamma_A\Gamma_j\Gamma_k\Gamma_l\right)\frac{K_1^A}{D_{1234}}\\
&+m\mathrm{tr}\left(\Gamma_i\gamma_A\Gamma_j\Gamma_k\gamma_C\Gamma_l\gamma_D\right)\frac{K_1^AK_3^CK_4^D}{D_{1234}}+m^3\mathrm{tr}\left(\Gamma_i\Gamma_j\gamma_B\Gamma_k\Gamma_l\right)\frac{K_2^B}{D_{1234}}\\
&+m\mathrm{tr}\left(\Gamma_i\gamma_A\Gamma_j\gamma_B\Gamma_k\Gamma_l\gamma_D\right)\frac{K_1^AK_2^BK_4^D}{D_{1234}}+m^3\mathrm{tr}\left(\Gamma_i\Gamma_j\Gamma_k\gamma_C\Gamma_l\right)\frac{K_3^C}{D_{1234}}\\
&+m\mathrm{tr}\left(\Gamma_i\gamma_A\Gamma_j\gamma_B\Gamma_k\gamma_C\Gamma_l\right)\frac{K_1^AK_2^BK_3^C}{D_{1234}}+m^3\mathrm{tr}\left(\Gamma_i\Gamma_j\Gamma_k\Gamma_l\gamma_D\right)\frac{K_4^D}{D_{1234}},
\end{aligned} \tag{3.69}$$

where numerators depend on $K_i=k+k_i$ and denominators are $D_{1234}=D_1D_2D_3D_4$.

Obtaining a specific function requires replacing the corresponding vertex operators within this expression. For the case of $PVVV$ amplitude, we use one pseudoscalar vertex ($\Gamma_i = \gamma_5$) followed by vector ones ($\Gamma_j = \gamma_\nu$, $\Gamma_k = \gamma_\alpha$, and $\Gamma_l = \gamma_\beta$), achieving the non-zero contributions

$$\begin{aligned} t^{PVVV}_{\nu\alpha\beta} &= m\mathrm{tr}\left(\gamma_5\gamma_\nu\gamma_B\gamma_\alpha\gamma_C\gamma_\beta\gamma_D\right)\frac{K_2^B K_3^C K_4^D}{D_{1234}} + m\mathrm{tr}\left(\gamma_5\gamma_A\gamma_\nu\gamma_\alpha\gamma_C\gamma_\beta\gamma_D\right)\frac{K_1^A K_3^C K_4^D}{D_{1234}} \\ &+ m\mathrm{tr}\left(\gamma_5\gamma_A\gamma_\nu\gamma_B\gamma_\alpha\gamma_\beta\gamma_D\right)\frac{K_1^A K_2^B K_4^D}{D_{1234}} + m\mathrm{tr}\left(\gamma_5\gamma_A\gamma_\nu\gamma_B\gamma_\alpha\gamma_C\gamma_\beta\right)\frac{K_1^A K_2^B K_3^C}{D_{1234}} \\ &+ m^3\mathrm{tr}\left(\gamma_5\gamma_A\gamma_\nu\gamma_\alpha\gamma_\beta\right)\frac{K_1^A}{D_{1234}} + m^3\mathrm{tr}\left(\gamma_5\gamma_\nu\gamma_B\gamma_\alpha\gamma_\beta\right)\frac{K_2^B}{D_{1234}} \\ &+ m^3\mathrm{tr}\left(\gamma_5\gamma_\nu\gamma_\alpha\gamma_C\gamma_\beta\right)\frac{K_3^C}{D_{1234}} + m^3\mathrm{tr}\left(\gamma_5\gamma_\nu\gamma_\alpha\gamma_\beta\gamma_D\right)\frac{K_4^D}{D_{1234}}. \end{aligned} \tag{3.70}$$

All traces are known and can be consulted in Equations (3.47) and (3.58). Posteriorly to their employment, our task is to group terms that share their index configuration to recognize subamplitudes or standard tensors. We consider each of these sectors separately since their mathematical expressions are more extensive now.

Finding those terms where the metric tensor has exclusively free indices, we identify the first sector:

$$\begin{aligned} \left[t^{PVVV}_{\nu\alpha\beta}\right]_1 &= -4im\left(g_{\nu\alpha}\varepsilon_{\beta BCD} - g_{\nu\beta}\varepsilon_{\alpha BCD} + g_{\alpha\beta}\varepsilon_{\nu BCD}\right)\frac{K_2^B K_3^C K_4^D}{D_{1234}} \\ &+ 4im\left(g_{\nu\alpha}\varepsilon_{\beta ACD} - g_{\nu\beta}\varepsilon_{\alpha ACD} + g_{\alpha\beta}\varepsilon_{\nu ACD}\right)\frac{K_1^A K_3^C K_4^D}{D_{1234}} \\ &- 4im\left(g_{\nu\alpha}\varepsilon_{\beta ABD} - g_{\nu\beta}\varepsilon_{\alpha ABD} + g_{\alpha\beta}\varepsilon_{\nu ABD}\right)\frac{K_1^A K_2^B K_4^D}{D_{1234}} \\ &+ 4im\left(g_{\nu\alpha}\varepsilon_{\beta ABC} - g_{\nu\beta}\varepsilon_{\alpha ABC} + g_{\alpha\beta}\varepsilon_{\nu ABC}\right)\frac{K_1^A K_2^B K_3^C}{D_{1234}}. \end{aligned} \tag{3.71}$$

By following the same procedure from previous cases, axial vector amplitudes would be achievable. Nevertheless, since quantities in parenthesis are alike, we rename summed indices to compact them into a single object

$$\left[t^{PVVV}_{\nu\alpha\beta}\right]_1 = -4im\left(g_{\kappa\nu}g_{\alpha\beta} - g_{\kappa\alpha}g_{\nu\beta} + g_{\kappa\beta}g_{\nu\alpha}\right)f_{4\kappa}. \tag{3.72}$$

The introduced object has the following structure

$$\begin{aligned} f_{4\kappa} &= \varepsilon_{\kappa XYZ}t^{(-)}_{4X;YZ}\left(k_2;k_3,k_4\right) - \varepsilon_{\kappa XYZ}t^{(-)}_{4X;YZ}\left(k_1;k_3,k_4\right) \\ &+ \varepsilon_{\kappa XYZ}t^{(-)}_{4X;YZ}\left(k_1;k_2,k_4\right) - \varepsilon_{\kappa XYZ}t^{(-)}_{4X;YZ}\left(k_1;k_2,k_3\right), \end{aligned} \tag{3.73}$$

which depends on the new standard tensor

$$t^{(s)}_{4\mu;\nu\alpha}\left(k_l;k_i,k_j\right)=\frac{\left(k+k_l\right)_\mu\left[\left(k+k_i\right)_\nu\left(k+k_j\right)_\alpha+s\left(k+k_i\right)_\alpha\left(k+k_j\right)_\nu\right]}{D_{1234}}. \tag{3.74}$$

Although this object is analogous to that defined in Equation (3.61), the numerical subindex indicates the association with four propagators through the denominator D_{1234}.

For the second sector, let us group components where all free indices appear within the Levi-Civita symbol, including traces of four Dirac matrices. We introduce a summed index κ to isolate a global factor and recognize less complex traces

$$\begin{aligned}\left[t^{PVVV}_{\nu\alpha\beta}\right]_2 \;=\;& i\varepsilon_{\nu\alpha\beta\kappa}\Bigg[-m\mathrm{tr}\left(\gamma_B\gamma_C\gamma_D\gamma_\kappa\right)\frac{K_2^BK_3^CK_4^D}{D_{1234}}+m\mathrm{tr}\left(\gamma_A\gamma_C\gamma_D\gamma_\kappa\right)\frac{K_1^AK_3^CK_4^D}{D_{1234}}\\ &-m\mathrm{tr}\left(\gamma_A\gamma_B\gamma_D\gamma_\kappa\right)\frac{K_1^AK_2^BK_4^D}{D_{1234}}+m\mathrm{tr}\left(\gamma_A\gamma_B\gamma_C\gamma_\kappa\right)\frac{K_1^AK_2^BK_3^C}{D_{1234}}\\ &-m^3\mathrm{tr}\left(\gamma_A\gamma_\kappa\right)\frac{K_1^A}{D_{1234}}+m^3\mathrm{tr}\left(\gamma_B\gamma_\kappa\right)\frac{K_2^B}{D_{1234}}\\ &-m^3\mathrm{tr}\left(\gamma_C\gamma_\kappa\right)\frac{K_3^C}{D_{1234}}+m^3\mathrm{tr}\left(\gamma_D\gamma_\kappa\right)\frac{K_4^D}{D_{1234}}\Bigg].\end{aligned} \tag{3.75}$$

This structure associates with a vector subamplitude; thus, comparing signs among the possibilities leads to the $APPP$ function[4]:

$$\left[t^{PVVV}_{\nu\alpha\beta}\right]_2=-i\varepsilon_{\kappa\nu\alpha\beta}t^{APPP}_\kappa. \tag{3.76}$$

As bilinears arise from traces within this subamplitude, we reduce them through identity (3.44). The loop momentum from numerators cancels out with this operation. Hence, the integrand associated with this function has the final structure

$$\begin{aligned}t^{APPP}_\kappa \;=\;& 4mp_\kappa\frac{1}{D_{124}}+4m\left(r-q\right)_\kappa\frac{1}{D_{134}}\\ &-4m\left[\left(q^2-q\cdot r\right)p_\kappa-\left(p^2-p\cdot r\right)q_\kappa+\left(p^2-p\cdot q\right)r_\kappa\right]\frac{1}{D_{1234}}.\end{aligned} \tag{3.77}$$

All external momenta arose within this expression: $p=k_1-k_2$, $q=k_1-k_3$, and $r=k_1-k_4$.

[4] There are four vertices: one vector γ_κ followed by three pseudoscalars γ_5.

Lastly, consider those terms that mix free and summed indices

$$\begin{aligned}
\left[t^{PVVV}_{\nu\alpha\beta}\right]_3 &= 4im\left(g_{\nu B}\varepsilon_{\alpha C\beta D} + g_{\nu C}\varepsilon_{B\alpha\beta D} + g_{\nu D}\varepsilon_{B\alpha C\beta} + g_{B\alpha}\varepsilon_{\nu C\beta D} + g_{B\beta}\varepsilon_{\nu\alpha CD}\right.\\
&\left.+g_{\alpha C}\varepsilon_{\nu B\beta D} + g_{\alpha D}\varepsilon_{\nu BC\beta} + g_{C\beta}\varepsilon_{\nu B\alpha D} + g_{\beta D}\varepsilon_{\nu B\alpha C}\right)\frac{K_2^B K_3^C K_4^D}{D_{1234}}\\
&+4im\left(g_{A\nu}\varepsilon_{\alpha C\beta D} - g_{A\alpha}\varepsilon_{\nu C\beta D} - g_{A\beta}\varepsilon_{\nu\alpha CD} - g_{\nu C}\varepsilon_{A\alpha\beta D} - g_{\nu D}\varepsilon_{A\alpha C\beta}\right.\\
&\left.+g_{\alpha C}\varepsilon_{A\nu\beta D} + g_{\alpha D}\varepsilon_{A\nu C\beta} + g_{C\beta}\varepsilon_{A\nu\alpha D} + g_{\beta D}\varepsilon_{A\nu\alpha C}\right)\frac{K_1^A K_3^C K_4^D}{D_{1234}}\\
&+4im\left(g_{A\nu}\varepsilon_{B\alpha\beta D} + g_{A\alpha}\varepsilon_{\nu B\beta D} - g_{A\beta}\varepsilon_{\nu B\alpha D} + g_{\nu B}\varepsilon_{A\alpha\beta D} - g_{\nu D}\varepsilon_{AB\alpha\beta}\right.\\
&\left.+g_{B\alpha}\varepsilon_{A\nu\beta D} - g_{B\beta}\varepsilon_{A\nu\alpha D} - g_{\alpha D}\varepsilon_{A\nu B\beta} + g_{\beta D}\varepsilon_{A\nu B\alpha}\right)\frac{K_1^A K_2^B K_4^D}{D_{1234}}\\
&+4im\left(g_{A\nu}\varepsilon_{B\alpha C\beta} + g_{A\alpha}\varepsilon_{\nu BC\beta} + g_{A\beta}\varepsilon_{\nu B\alpha C} + g_{\nu B}\varepsilon_{A\alpha C\beta} + g_{\nu C}\varepsilon_{AB\alpha\beta}\right.\\
&\left.+g_{B\alpha}\varepsilon_{A\nu C\beta} + g_{B\beta}\varepsilon_{A\nu\alpha C} + g_{\alpha C}\varepsilon_{A\nu B\beta} + g_{C\beta}\varepsilon_{A\nu B\alpha}\right)\frac{K_1^A K_2^B K_3^C}{D_{1234}}. \qquad (3.78)
\end{aligned}$$

Once again, using the antisymmetric character of the Levi-Civita symbol, we recognize combinations of the standard tensor (3.74).Then, this sector leads to the following tensor by factorizing $2im$:

$$\begin{aligned}
f_{4\nu\alpha\beta} &= -\left(\varepsilon_{\alpha\beta XY}g_{\nu Z} - \varepsilon_{\nu\beta XY}g_{\alpha Z} + \varepsilon_{\nu\alpha XY}g_{\beta Z}\right)t^{(-)}_{4Z;XY}\left(k_1;k_3,k_4\right)\\
&+\left(\varepsilon_{\alpha\beta XY}g_{\nu Z} - \varepsilon_{\nu\beta XY}g_{\alpha Z} + \varepsilon_{\nu\alpha XY}g_{\beta Z}\right)t^{(-)}_{4Z;XY}\left(k_1;k_2,k_4\right)\\
&-\left(\varepsilon_{\alpha\beta XY}g_{\nu Z} - \varepsilon_{\nu\beta XY}g_{\alpha Z} + \varepsilon_{\nu\alpha XY}g_{\beta Z}\right)t^{(-)}_{4Z;XY}\left(k_1;k_2,k_3\right)\\
&-\left(\varepsilon_{\alpha\beta XY}g_{\nu Z} + \varepsilon_{\nu\beta XY}g_{\alpha Z} - \varepsilon_{\nu\alpha XY}g_{\beta Z}\right)t^{(-)}_{4Z;XY}\left(k_2;k_3,k_4\right)\\
&+\left(\varepsilon_{\alpha\beta XY}g_{\nu Z} + \varepsilon_{\nu\beta XY}g_{\alpha Z} - \varepsilon_{\nu\alpha XY}g_{\beta Z}\right)t^{(-)}_{4Z;XY}\left(k_2;k_1,k_4\right)\\
&-\left(\varepsilon_{\alpha\beta XY}g_{\nu Z} + \varepsilon_{\nu\beta XY}g_{\alpha Z} - \varepsilon_{\nu\alpha XY}g_{\beta Z}\right)t^{(-)}_{4Z;XY}\left(k_2;k_1,k_3\right)\\
&+\left(\varepsilon_{\alpha\beta XY}g_{\nu Z} - \varepsilon_{\nu\beta XY}g_{\alpha Z} - \varepsilon_{\nu\alpha XY}g_{\beta Z}\right)t^{(-)}_{4Z;XY}\left(k_3;k_2,k_4\right)\\
&-\left(\varepsilon_{\alpha\beta XY}g_{\nu Z} - \varepsilon_{\nu\beta XY}g_{\alpha Z} - \varepsilon_{\nu\alpha XY}g_{\beta Z}\right)t^{(-)}_{4Z;XY}\left(k_3;k_1,k_4\right)\\
&+\left(\varepsilon_{\alpha\beta XY}g_{\nu Z} - \varepsilon_{\nu\beta XY}g_{\alpha Z} - \varepsilon_{\nu\alpha XY}g_{\beta Z}\right)t^{(-)}_{4Z;XY}\left(k_3;k_1,k_2\right)\\
&-\left(\varepsilon_{\alpha\beta XY}g_{\nu Z} - \varepsilon_{\nu\beta XY}g_{\alpha Z} + \varepsilon_{\nu\alpha XY}g_{\beta Z}\right)t^{(-)}_{4Z;XY}\left(k_4;k_2,k_3\right)\\
&+\left(\varepsilon_{\alpha\beta XY}g_{\nu Z} - \varepsilon_{\nu\beta XY}g_{\alpha Z} + \varepsilon_{\nu\alpha XY}g_{\beta Z}\right)t^{(-)}_{4Z;XY}\left(k_4;k_1,k_3\right)\\
&-\left(\varepsilon_{\alpha\beta XY}g_{\nu Z} - \varepsilon_{\nu\beta XY}g_{\alpha Z} + \varepsilon_{\nu\alpha XY}g_{\beta Z}\right)t^{(-)}_{4Z;XY}\left(k_4;k_1,k_2\right). \qquad (3.79)
\end{aligned}$$

The most significant difference between both occurrences of this tensor is in the contraction. Whereas all indices were contracted with the Levi-Civita symbol in the previous case, only those that show the antisymmetry property are contracted this time.

With all sectors explored, we write the $PVVV$ final form

$$t^{PVVV}_{\nu\alpha\beta} = -4im\left(g_{\kappa\nu}g_{\alpha\beta} - g_{\kappa\alpha}g_{\nu\beta} + g_{\kappa\beta}g_{\nu\alpha}\right)f_{4\kappa} + 2imf_{4\nu\alpha\beta} - i\varepsilon_{\kappa\nu\alpha\beta}t^{APPP}_{\kappa}. \qquad (3.80)$$

It depends on two main structures: the vector subamplitude $APPP$ and the standard tensor with three momenta on the numerator. Even though four-point functions have logarithmic power counting, mass dimension analysis suggests that integrals within this particular amplitude are finite.

3.2.6 Four-Point Amplitudes - $AVVV$

The last correlator concerning this investigation is the $AVVV$ box, whose structure contains one axial vertex ($\Gamma_i = \gamma_\mu\gamma_5$) and three vector vertices ($\Gamma_j = \gamma_\nu$, $\Gamma_k = \gamma_\alpha$, and $\Gamma_l = \gamma_\beta$). We obtain its initial structure by replacing the corresponding vertices on the general integrand of four-point functions (3.69):

$$\begin{aligned}
t^{AVVV}_{\mu\nu\alpha\beta} = {} & \mathrm{tr}\left(\gamma_\mu\gamma_5\gamma_A\gamma_\nu\gamma_B\gamma_\alpha\gamma_C\gamma_\beta\gamma_D\right)\frac{K_1^A K_2^B K_3^C K_4^D}{D_{1234}} \\
& + m^2\mathrm{tr}\left(\gamma_\mu\gamma_5\gamma_A\gamma_\nu\gamma_B\gamma_\alpha\gamma_\beta\right)\frac{K_1^A K_2^B}{D_{1234}} + m^2\mathrm{tr}\left(\gamma_\mu\gamma_5\gamma_A\gamma_\nu\gamma_\alpha\gamma_C\gamma_\beta\right)\frac{K_1^A K_3^C}{D_{1234}} \\
& + m^2\mathrm{tr}\left(\gamma_\mu\gamma_5\gamma_A\gamma_\nu\gamma_\alpha\gamma_\beta\gamma_D\right)\frac{K_1^A K_4^D}{D_{1234}} + m^2\mathrm{tr}\left(\gamma_\mu\gamma_5\gamma_\nu\gamma_B\gamma_\alpha\gamma_C\gamma_\beta\right)\frac{K_2^B K_3^C}{D_{1234}} \\
& + m^2\mathrm{tr}\left(\gamma_\mu\gamma_5\gamma_\nu\gamma_B\gamma_\alpha\gamma_\beta\gamma_D\right)\frac{K_2^B K_4^D}{D_{1234}} + m^2\mathrm{tr}\left(\gamma_\mu\gamma_5\gamma_\nu\gamma_\alpha\gamma_C\gamma_\beta\gamma_D\right)\frac{K_3^C K_4^D}{D_{1234}} \\
& + m^4\mathrm{tr}\left(\gamma_\mu\gamma_5\gamma_\nu\gamma_\alpha\gamma_\beta\right)\frac{1}{D_{1234}}. \qquad (3.81)
\end{aligned}$$

This subsection deals with numerous contributions that might compromise the visualization and understanding of mathematical expressions. For this reason, we introduce a compact notation for products of gamma matrices, e.g., $\gamma_\mu\gamma_5\gamma_\nu\gamma_\alpha\gamma_\beta = \gamma_{\mu5\nu\alpha\beta}$. That is a temporary resource employed exclusively in $AVVV$ calculations.

Most traces above are known and can be consulted in Equations (3.47) and (3.58). We also identify the presence of a trace involving eight Dirac matrices plus the chiral one, which leads to products involving the Levi-Civita symbol and metric tensors. This type of structure admits equivalent expressions distinguished in their tensor structure. Nevertheless, this connection is not guaranteed for perturbative amplitudes due to their divergent character. That is analogous to the AVV case and motivated us to choose the $AVVV$ as an extension of our discussion.

Evaluating this trace follows the same procedure adopted in previous cases: replace the chiral matrix definition, take the new trace, and perform contractions with the Levi-Civita symbol. This strategy leads to a result exhibiting all non-equivalent tensor contributions, which makes the existence of identities clear. Even so, to allow a careful analysis, we chose to approach this subject after the complete organization of the amplitude. Thus, let us directly introduce the trace expression prioritizing the μ-index:

$$\begin{aligned}
& i\mathrm{tr}\left(\gamma_{5\mu A\nu B\alpha C\beta D}\right) \\
= \; & -\varepsilon_{\mu A\nu B}\mathrm{tr}\left(\gamma_{\alpha C\beta D}\right)+\varepsilon_{\mu A\nu\alpha}\mathrm{tr}\left(\gamma_{BC\beta D}\right)-\varepsilon_{\mu A\nu C}\mathrm{tr}\left(\gamma_{B\alpha\beta D}\right)+\varepsilon_{\mu A\nu\beta}\mathrm{tr}\left(\gamma_{B\alpha CD}\right) \\
& -\varepsilon_{\mu A\nu D}\mathrm{tr}\left(\gamma_{B\alpha C\beta}\right)-\varepsilon_{\mu AB\alpha}\mathrm{tr}\left(\gamma_{\nu C\beta D}\right)+\varepsilon_{\mu ABC}\mathrm{tr}\left(\gamma_{\nu\alpha\beta D}\right)-\varepsilon_{\mu AB\beta}\mathrm{tr}\left(\gamma_{\nu\alpha CD}\right) \\
& +\varepsilon_{\mu ABD}\mathrm{tr}\left(\gamma_{\nu\alpha C\beta}\right)-\varepsilon_{\mu A\alpha C}\mathrm{tr}\left(\gamma_{\nu B\beta D}\right)+\varepsilon_{\mu A\alpha\beta}\mathrm{tr}\left(\gamma_{\nu BCD}\right)-\varepsilon_{\mu A\alpha D}\mathrm{tr}\left(\gamma_{\nu BC\beta}\right) \\
& -\varepsilon_{\mu AC\beta}\mathrm{tr}\left(\gamma_{\nu B\alpha D}\right)+\varepsilon_{\mu ACD}\mathrm{tr}\left(\gamma_{\nu B\alpha\beta}\right)-\varepsilon_{\mu A\beta D}\mathrm{tr}\left(\gamma_{\nu B\alpha C}\right)+\varepsilon_{\mu\nu B\alpha}\mathrm{tr}\left(\gamma_{AC\beta D}\right) \\
& -\varepsilon_{\mu\nu BC}\mathrm{tr}\left(\gamma_{A\alpha\beta D}\right)+\varepsilon_{\mu\nu B\beta}\mathrm{tr}\left(\gamma_{A\alpha CD}\right)-\varepsilon_{\mu\nu BD}\mathrm{tr}\left(\gamma_{A\alpha C\beta}\right)+\varepsilon_{\mu\nu\alpha C}\mathrm{tr}\left(\gamma_{AB\beta D}\right) \\
& -\varepsilon_{\mu\nu\alpha\beta}\mathrm{tr}\left(\gamma_{ABCD}\right)+\varepsilon_{\mu\nu\alpha D}\mathrm{tr}\left(\gamma_{ABC\beta}\right)+\varepsilon_{\mu\nu C\beta}\mathrm{tr}\left(\gamma_{AB\alpha D}\right)-\varepsilon_{\mu\nu CD}\mathrm{tr}\left(\gamma_{AB\alpha\beta}\right) \\
& +\varepsilon_{\mu\nu\beta D}\mathrm{tr}\left(\gamma_{AB\alpha C}\right)-\varepsilon_{\mu B\alpha C}\mathrm{tr}\left(\gamma_{A\nu\beta D}\right)+\varepsilon_{\mu B\alpha\beta}\mathrm{tr}\left(\gamma_{A\nu CD}\right)-\varepsilon_{\mu B\alpha D}\mathrm{tr}\left(\gamma_{A\nu C\beta}\right) \\
& -\varepsilon_{\mu BC\beta}\mathrm{tr}\left(\gamma_{A\nu\alpha D}\right)+\varepsilon_{\mu BCD}\mathrm{tr}\left(\gamma_{A\nu\alpha\beta}\right)-\varepsilon_{\mu B\beta D}\mathrm{tr}\left(\gamma_{A\nu\alpha C}\right)+\varepsilon_{\mu\alpha C\beta}\mathrm{tr}\left(\gamma_{A\nu BD}\right) \\
& -\varepsilon_{\mu\alpha CD}\mathrm{tr}\left(\gamma_{A\nu B\beta}\right)+\varepsilon_{\mu\alpha\beta D}\mathrm{tr}\left(\gamma_{A\nu BC}\right)-\varepsilon_{\mu C\beta D}\mathrm{tr}\left(\gamma_{A\nu B\alpha}\right)-\varepsilon_{A\nu B\alpha}\mathrm{tr}\left(\gamma_{\mu C\beta D}\right) \\
& +\varepsilon_{A\nu BC}\mathrm{tr}\left(\gamma_{\mu\alpha\beta D}\right)-\varepsilon_{A\nu B\beta}\mathrm{tr}\left(\gamma_{\mu\alpha CD}\right)+\varepsilon_{A\nu BD}\mathrm{tr}\left(\gamma_{\mu\alpha C\beta}\right)-\varepsilon_{A\nu\alpha C}\mathrm{tr}\left(\gamma_{\mu B\beta D}\right) \\
& +\varepsilon_{A\nu\alpha\beta}\mathrm{tr}\left(\gamma_{\mu BCD}\right)-\varepsilon_{A\nu\alpha D}\mathrm{tr}\left(\gamma_{\mu BC\beta}\right)-\varepsilon_{A\nu C\beta}\mathrm{tr}\left(\gamma_{\mu B\alpha D}\right)+\varepsilon_{A\nu CD}\mathrm{tr}\left(\gamma_{\mu B\alpha\beta}\right) \\
& -\varepsilon_{A\nu\beta D}\mathrm{tr}\left(\gamma_{\mu B\alpha C}\right)+\varepsilon_{AB\alpha C}\mathrm{tr}\left(\gamma_{\mu\nu\beta D}\right)-\varepsilon_{AB\alpha\beta}\mathrm{tr}\left(\gamma_{\mu\nu CD}\right)+\varepsilon_{AB\alpha D}\mathrm{tr}\left(\gamma_{\mu\nu C\beta}\right) \\
& +\varepsilon_{ABC\beta}\mathrm{tr}\left(\gamma_{\mu\nu\alpha D}\right)-\varepsilon_{ABCD}\mathrm{tr}\left(\gamma_{\mu\nu\alpha\beta}\right)+\varepsilon_{AB\beta D}\mathrm{tr}\left(\gamma_{\mu\nu\alpha C}\right)-\varepsilon_{A\alpha C\beta}\mathrm{tr}\left(\gamma_{\mu\nu BD}\right) \\
& +\varepsilon_{A\alpha CD}\mathrm{tr}\left(\gamma_{\mu\nu B\beta}\right)-\varepsilon_{A\alpha\beta D}\mathrm{tr}\left(\gamma_{\mu\nu BC}\right)+\varepsilon_{AC\beta D}\mathrm{tr}\left(\gamma_{\mu\nu B\alpha}\right)-\varepsilon_{\nu B\alpha C}\mathrm{tr}\left(\gamma_{\mu A\beta D}\right) \\
& +\varepsilon_{\nu B\alpha\beta}\mathrm{tr}\left(\gamma_{\mu ACD}\right)-\varepsilon_{\nu B\alpha D}\mathrm{tr}\left(\gamma_{\mu AC\beta}\right)-\varepsilon_{\nu BC\beta}\mathrm{tr}\left(\gamma_{\mu A\alpha D}\right)+\varepsilon_{\nu BCD}\mathrm{tr}\left(\gamma_{\mu A\alpha\beta}\right) \\
& -\varepsilon_{\nu B\beta D}\mathrm{tr}\left(\gamma_{\mu A\alpha C}\right)+\varepsilon_{\nu\alpha C\beta}\mathrm{tr}\left(\gamma_{\mu ABD}\right)-\varepsilon_{\nu\alpha CD}\mathrm{tr}\left(\gamma_{\mu AB\beta}\right)+\varepsilon_{\nu\alpha\beta D}\mathrm{tr}\left(\gamma_{\mu ABC}\right) \\
& -\varepsilon_{\nu D\beta D}\mathrm{tr}\left(\gamma_{\mu AB\alpha}\right)-\varepsilon_{B\alpha C\beta}\mathrm{tr}\left(\gamma_{\mu A\nu D}\right)+\varepsilon_{B\alpha CD}\mathrm{tr}\left(\gamma_{\mu A\nu\beta}\right)-\varepsilon_{B\alpha\beta D}\mathrm{tr}\left(\gamma_{\mu A\nu C}\right) \\
& +\varepsilon_{BC\beta D}\mathrm{tr}\left(\gamma_{\mu A\nu\alpha}\right)-\varepsilon_{\alpha C\beta D}\mathrm{tr}\left(\gamma_{\mu A\nu B}\right). \qquad (3.82)
\end{aligned}$$

Since numerous components exist, we split this analysis[5] into sectors grouping terms where free indices play similar roles. This line of reasoning extends to all parts of the initial integrand (3.81). Thus, we will call upon the term proportional to $K_1^A K_2^B$ to illustrate a trace involving six Dirac matrices

$$\begin{aligned}
\mathrm{tr}\left(\gamma_{\mu 5A\nu B\alpha\beta}\right) = & -4i\left[g_{\mu A}\varepsilon_{\nu B\alpha\beta}-g_{\mu\nu}\varepsilon_{AB\alpha\beta}+g_{\mu B}\varepsilon_{A\nu\alpha\beta}-g_{\mu\alpha}\varepsilon_{A\nu B\beta}+g_{\mu\beta}\varepsilon_{A\nu B\alpha}\right. \\
& +g_{A\nu}\varepsilon_{\mu B\alpha\beta}-g_{AB}\varepsilon_{\mu\nu\alpha\beta}+g_{A\alpha}\varepsilon_{\mu\nu B\beta}-g_{A\beta}\varepsilon_{\mu\nu B\alpha}+g_{\nu B}\varepsilon_{\mu A\alpha\beta} \\
& \left.-g_{\nu\alpha}\varepsilon_{\mu AB\beta}+g_{\nu\beta}\varepsilon_{\mu AB\alpha}+g_{B\alpha}\varepsilon_{\mu A\nu\beta}-g_{B\beta}\varepsilon_{\mu A\nu\alpha}+g_{\alpha\beta}\varepsilon_{\mu A\nu B}\right]. \qquad (3.83)
\end{aligned}$$

Our first step is to find those terms depending on the metric tensor with free indices. The artifice of using uppercase Latin letters on summed indices makes this process a lot

[5] Although all vertex operators appear within this context, we only comment on cases that remain in the final form.

easier. For the equation above, the following components interest us

$$\mathrm{tr}\left(\gamma_{\mu5A\nu B\alpha\beta}\right) \rightarrow g_{\mu\nu}\varepsilon_{AB\alpha\beta} + g_{\mu\alpha}\varepsilon_{A\nu B\beta} - g_{\mu\beta}\varepsilon_{A\nu B\alpha} + g_{\nu\alpha}\varepsilon_{\mu AB\beta} - g_{\nu\beta}\varepsilon_{\mu AB\alpha} - g_{\alpha\beta}\varepsilon_{\mu A\nu B},$$

where the Levi-Civita symbols are recognized as less complex traces. Extending this idea to the complete amplitude, we have all contributions belonging to this sector:

$$\begin{aligned}
\left[t^{AVVV}_{\mu\nu\alpha\beta}\right]_1 = & -\left[g_{\alpha\beta}\mathrm{tr}\left(\gamma_{\mu5A\nu BCD}\right) + g_{\nu\beta}\mathrm{tr}\left(\gamma_{\mu5AB\alpha CD}\right) + g_{\nu\alpha}\mathrm{tr}\left(\gamma_{\mu5ABC\beta D}\right)\right. \\
& + g_{\mu\beta}\mathrm{tr}\left(\gamma_{A\nu B\alpha C5D}\right) + g_{\mu\alpha}\mathrm{tr}\left(\gamma_{A\nu B5C\beta D}\right) + g_{\mu\nu}\mathrm{tr}\left(\gamma_{A5B\alpha C\beta D}\right) \\
& \left. + \left(g_{\mu\nu}g_{\alpha\beta} - g_{\mu\alpha}g_{\nu\beta} + g_{\mu\beta}g_{\nu\alpha}\right)\mathrm{tr}\left(\gamma_{A5BCD}\right)\right] \frac{K_1^A K_2^B K_3^C K_4^D}{D_{1234}} \\
& + m^2\left[-g_{\alpha\beta}\mathrm{tr}\left(\gamma_{5\mu A\nu B}\right) - g_{\nu\beta}\mathrm{tr}\left(\gamma_{5\mu AB\alpha}\right) + g_{\nu\alpha}\mathrm{tr}\left(\gamma_{5\mu AB\beta}\right)\right. \\
& \left. - g_{\mu\beta}\mathrm{tr}\left(\gamma_{5A\nu B\alpha}\right) + g_{\mu\alpha}\mathrm{tr}\left(\gamma_{5A\nu B\beta}\right) + g_{\mu\nu}\mathrm{tr}\left(\gamma_{5AB\alpha\beta}\right)\right] \frac{K_1^A K_2^B}{D_{1234}} \\
& + m^2\left[g_{\alpha\beta}\mathrm{tr}\left(\gamma_{5\mu A\nu C}\right) - g_{\nu\beta}\mathrm{tr}\left(\gamma_{5\mu A\alpha C}\right) - g_{\nu\alpha}\mathrm{tr}\left(\gamma_{5\mu AC\beta}\right)\right. \\
& \left. - g_{\mu\beta}\mathrm{tr}\left(\gamma_{5A\nu\alpha C}\right) - g_{\mu\alpha}\mathrm{tr}\left(\gamma_{5A\nu C\beta}\right) + g_{\mu\nu}\mathrm{tr}\left(\gamma_{5A\alpha C\beta}\right)\right] \frac{K_1^A K_3^C}{D_{1234}} \\
& + m^2\left[-g_{\alpha\beta}\mathrm{tr}\left(\gamma_{5\mu A\nu D}\right) + g_{\nu\beta}\mathrm{tr}\left(\gamma_{5\mu A\alpha D}\right) - g_{\nu\alpha}\mathrm{tr}\left(\gamma_{5\mu A\beta D}\right)\right. \\
& \left. g_{\mu\beta}\mathrm{tr}\left(\gamma_{5A\nu\alpha D}\right) - g_{\mu\alpha}\mathrm{tr}\left(\gamma_{5A\nu\beta D}\right) + g_{\mu\nu}\mathrm{tr}\left(\gamma_{5A\alpha\beta D}\right)\right] \frac{K_1^A K_4^D}{D_{1234}} \\
& + m^2\left[g_{\alpha\beta}\mathrm{tr}\left(\gamma_{5\mu\nu BC}\right) + g_{\nu\beta}\mathrm{tr}\left(\gamma_{5\mu B\alpha C}\right) + g_{\nu\alpha}\mathrm{tr}\left(\gamma_{5\mu BC\beta}\right)\right. \\
& \left. - g_{\mu\beta}\mathrm{tr}\left(\gamma_{5\nu B\alpha C}\right) - g_{\mu\alpha}\mathrm{tr}\left(\gamma_{5\nu BC\beta}\right) - g_{\mu\nu}\mathrm{tr}\left(\gamma_{5B\alpha C\beta}\right)\right] \frac{K_2^B K_3^C}{D_{1234}} \\
& + m^2\left[-g_{\alpha\beta}\mathrm{tr}\left(\gamma_{5\mu\nu BD}\right) - g_{\nu\beta}\mathrm{tr}\left(\gamma_{5\mu B\alpha D}\right) + g_{\nu\alpha}\mathrm{tr}\left(\gamma_{5\mu B\beta D}\right)\right. \\
& \left. + g_{\mu\beta}\mathrm{tr}\left(\gamma_{5\nu B\alpha D}\right) - g_{\mu\alpha}\mathrm{tr}\left(\gamma_{5\nu B\beta D}\right) - g_{\mu\nu}\mathrm{tr}\left(\gamma_{5B\alpha\beta D}\right)\right] \frac{K_2^B K_4^D}{D_{1234}} \\
& + m^2\left[g_{\alpha\beta}\mathrm{tr}\left(\gamma_{5\mu\nu CD}\right) - g_{\nu\beta}\mathrm{tr}\left(\gamma_{5\mu\alpha CD}\right) - g_{\nu\alpha}\mathrm{tr}\left(\gamma_{5\mu C\beta D}\right)\right. \\
& \left. + g_{\mu\beta}\mathrm{tr}\left(\gamma_{5\nu\alpha CD}\right) + g_{\mu\alpha}\mathrm{tr}\left(\gamma_{5\nu C\beta D}\right) - g_{\mu\nu}\mathrm{tr}\left(\gamma_{5\alpha C\beta D}\right)\right] \frac{K_3^C K_4^D}{D_{1234}}.
\end{aligned} \tag{3.84}$$

The final part of this task is identifying substructures by noticing that these traces correspond to odd amplitudes that are 2nd-order tensors. Since indices are unchanged, our work reduces to replacing vertices within Equation (3.69) and comparing sign differences among all possibilities. Ultimately, this part contains exclusively odd amplitudes

$$\begin{aligned}
\left[t^{AVVV}_{\mu\nu\alpha\beta}\right]_1 = & \left[g_{\alpha\beta}t^{AVPP}_{\mu\nu} + g_{\nu\beta}t^{APVP}_{\mu\alpha} + g_{\nu\alpha}t^{APPV}_{\mu\beta}\right] \\
& - \left[g_{\mu\beta}t^{SVVP}_{\nu\alpha} + g_{\mu\alpha}t^{SVPV}_{\nu\beta} + g_{\mu\nu}t^{SPVV}_{\alpha\beta}\right] \\
& + \left(g_{\mu\nu}g_{\alpha\beta} - g_{\mu\alpha}g_{\nu\beta} + g_{\mu\beta}g_{\nu\alpha}\right)t^{SPPP}.
\end{aligned} \tag{3.85}$$

The $SPPP$ numeric factor changes because this amplitude also appears inside the others.

In the second sector, we group those terms where the Levi-Civita symbol has three or four free indices. Let us return to expression (3.83) to illustrate the analysis of these components

$$\begin{aligned}\mathrm{tr}\left(\gamma_{\mu5A\nu B\alpha\beta}\right) \quad\rightarrow\quad & \varepsilon_{\mu\nu B\alpha}g_{A\beta}+\varepsilon_{\mu A\nu\alpha}g_{B\beta}-\varepsilon_{\mu\nu B\beta}g_{A\alpha}-\varepsilon_{\mu A\nu\beta}g_{B\alpha}-\varepsilon_{\mu B\alpha\beta}g_{A\nu}\\ & -\varepsilon_{\mu A\alpha\beta}g_{\nu B}-\varepsilon_{\nu B\alpha\beta}g_{\mu A}-\varepsilon_{A\nu\alpha\beta}g_{\mu B}+\varepsilon_{\mu\nu\alpha\beta}g_{AB}.\end{aligned}$$

Our objective is finding substructures, which requires combining monomials with the same index arrangement. To do so, we introduce a new index κ to generate metric products corresponding to less complex traces

$$\begin{aligned}\mathrm{tr}\left(\gamma_{\mu5A\nu B\alpha\beta}\right) \quad\rightarrow\quad & \varepsilon_{\mu\nu\alpha\kappa}\mathrm{tr}\left(\gamma_{\kappa AB\beta}\right)-\varepsilon_{\mu\nu\beta\kappa}\mathrm{tr}\left(\gamma_{\kappa AB\alpha}\right)-\varepsilon_{\mu\alpha\beta\kappa}\mathrm{tr}\left(\gamma_{\kappa A\nu B}\right)\\ & -\varepsilon_{\nu\alpha\beta\kappa}\mathrm{tr}\left(\sigma_{\kappa\mu}\gamma_{AB}\right)-2\varepsilon_{\mu\nu\alpha\beta}\mathrm{tr}\left(\gamma_{AB}\right).\end{aligned}$$

Note that the performed manipulations changed the last numerical coefficient. The traces below are recognized when extending this discussion to the remaining cases:

$$\begin{aligned}\left[t^{AVVV}_{\mu\nu\alpha\beta}\right]_2 \;=\;& i\left[\varepsilon_{\nu\alpha\beta\kappa}\mathrm{tr}\left(\sigma_{\kappa\mu}\gamma_{ABCD}\right)-\varepsilon_{\mu\alpha\beta\kappa}\mathrm{tr}\left(\gamma_{\kappa A\nu BCD}\right)-\varepsilon_{\mu\nu\beta\kappa}\mathrm{tr}\left(\gamma_{\kappa AB\alpha CD}\right)\right.\\ & \left.-\varepsilon_{\mu\nu\alpha\kappa}\mathrm{tr}\left(\gamma_{\kappa ABC\beta D}\right)+2\varepsilon_{\mu\nu\alpha\beta}\mathrm{tr}\left(\gamma_{ABCD}\right)\right]\frac{K_1^AK_2^BK_3^CK_4^D}{D_{1234}}\\ & +im^2\left[\varepsilon_{\mu\nu\alpha\kappa}\mathrm{tr}\left(\gamma_{\kappa AB\beta}\right)-\varepsilon_{\mu\nu\beta\kappa}\mathrm{tr}\left(\gamma_{\kappa AB\alpha}\right)-\varepsilon_{\mu\alpha\beta\kappa}\mathrm{tr}\left(\gamma_{\kappa A\nu B}\right)\right.\\ & \left.-\varepsilon_{\nu\alpha\beta\kappa}\mathrm{tr}\left(\sigma_{\kappa\mu}\gamma_{AB}\right)-2\varepsilon_{\mu\nu\alpha\beta}\mathrm{tr}\left(\gamma_{AB}\right)\right]\frac{K_1^AK_2^B}{D_{1234}}\\ & +im^2\left[-\varepsilon_{\mu\nu\alpha\kappa}\mathrm{tr}\left(\gamma_{\kappa AC\beta}\right)-\varepsilon_{\mu\nu\beta\kappa}\mathrm{tr}\left(\gamma_{\kappa A\alpha C}\right)+\varepsilon_{\mu\alpha\beta\kappa}\mathrm{tr}\left(\gamma_{\kappa A\nu C}\right)\right.\\ & \left.+\varepsilon_{\nu\alpha\beta\kappa}\mathrm{tr}\left(\sigma_{\kappa\mu}\gamma_{AC}\right)+2\varepsilon_{\mu\nu\alpha\beta}\mathrm{tr}\left(\gamma_{AC}\right)\right]\frac{K_1^AK_3^C}{D_{1234}}\\ & +im^2\left[-\varepsilon_{\mu\nu\alpha\kappa}\mathrm{tr}\left(\gamma_{\kappa A\beta D}\right)+\varepsilon_{\mu\nu\beta\kappa}\mathrm{tr}\left(\gamma_{\kappa A\alpha D}\right)-\varepsilon_{\mu\alpha\beta\kappa}\mathrm{tr}\left(\gamma_{\kappa A\nu D}\right)\right.\\ & \left.-\varepsilon_{\nu\alpha\beta\kappa}\mathrm{tr}\left(\sigma_{\kappa\mu}\gamma_{AD}\right)-2\varepsilon_{\mu\nu\alpha\beta}\mathrm{tr}\left(\gamma_{AD}\right)\right]\frac{K_1^AK_4^D}{D_{1234}}\\ & +im^2\left[\varepsilon_{\mu\nu\alpha\kappa}\mathrm{tr}\left(\gamma_{\kappa BC\beta}\right)+\varepsilon_{\mu\nu\beta\kappa}\mathrm{tr}\left(\gamma_{\kappa B\alpha C}\right)+\varepsilon_{\mu\alpha\beta\kappa}\mathrm{tr}\left(\gamma_{\kappa\nu BC}\right)\right.\\ & \left.-\varepsilon_{\nu\alpha\beta\kappa}\mathrm{tr}\left(\sigma_{\kappa\mu}\gamma_{BC}\right)-2\varepsilon_{\mu\nu\alpha\beta}\mathrm{tr}\left(\gamma_{BC}\right)\right]\frac{K_2^BK_3^C}{D_{1234}}\\ & +im^2\left[\varepsilon_{\mu\nu\alpha\kappa}\mathrm{tr}\left(\gamma_{\kappa B\beta D}\right)-\varepsilon_{\mu\nu\beta\kappa}\mathrm{tr}\left(\gamma_{\kappa B\alpha D}\right)-\varepsilon_{\mu\alpha\beta\kappa}\mathrm{tr}\left(\gamma_{\kappa\nu BD}\right)\right.\\ & \left.+\varepsilon_{\nu\alpha\beta\kappa}\mathrm{tr}\left(\sigma_{\kappa\mu}\gamma_{BD}\right)+2\varepsilon_{\mu\nu\alpha\beta}\mathrm{tr}\left(\gamma_{BD}\right)\right]\frac{K_2^BK_4^D}{D_{1234}}\\ & +im^2\left[-\varepsilon_{\mu\nu\alpha\kappa}\mathrm{tr}\left(\gamma_{\kappa C\beta D}\right)-\varepsilon_{\mu\nu\beta\kappa}\mathrm{tr}\left(\gamma_{\kappa\alpha CD}\right)+\varepsilon_{\mu\alpha\beta\kappa}\mathrm{tr}\left(\gamma_{\kappa\nu CD}\right)\right.\\ & \left.-\varepsilon_{\nu\alpha\beta\kappa}\mathrm{tr}\left(\sigma_{\kappa\mu}\gamma_{CD}\right)-2\varepsilon_{\mu\nu\alpha\beta}\mathrm{tr}\left(\gamma_{CD}\right)\right]\frac{K_3^CK_4^D}{D_{1234}}\\ & -im^4\mathrm{tr}\left(\gamma_{5\mu\nu\alpha\beta}\right)\frac{1}{D_{1234}}. \qquad (3.86)\end{aligned}$$

This time, traces correspond to even amplitudes that are 2nd-order tensors. We write the ensuing organization when examining differences among all possibilities:

$$\left[t_{\mu\nu\alpha\beta}^{AVVV}\right]_2 = i\varepsilon_{\nu\alpha\beta\kappa}t_{\kappa\mu}^{\widetilde{T}PPP} - i\left[\varepsilon_{\mu\alpha\beta\kappa}t_{\kappa\nu}^{VVPP} + \varepsilon_{\mu\nu\beta\kappa}t_{\kappa\alpha}^{VPVP} + \varepsilon_{\mu\nu\alpha\kappa}t_{\kappa\beta}^{VPPV}\right] + 2i\varepsilon_{\mu\nu\alpha\beta}t^{PPPP}. \tag{3.87}$$

Observe that the commutator $\sigma_{\kappa\mu}$ appeared throughout calculations and now reflects on the emergence of the pseudo-tensor vertex $\widetilde{T}$. Since the scalar function $PPPP$ appears inside other terms, one must adjust its coefficient adequately.

The last sector comprehends all remaining contributions, which are combinations of standard tensors with four momenta in the numerator. Without performing manipulations, we group terms according to their index arrangement

$$\begin{aligned}\left[t_{\mu\nu\alpha\beta}^{AVVV}\right]_3 \;=\; 4i\Big[&\varepsilon_{\mu\nu XY}t_{XY\alpha\beta}^{(12)} + \varepsilon_{\mu\alpha XY}t_{XY\nu\beta}^{(13)} + \varepsilon_{\mu\beta XY}t_{XY\nu\alpha}^{(14)} \\ &+\varepsilon_{\nu\alpha XY}t_{XY\mu\beta}^{(23)} + \varepsilon_{\nu\beta XY}t_{XY\mu\alpha}^{(24)} + \varepsilon_{\alpha\beta XY}t_{XY\mu\nu}^{(34)}\Big].\end{aligned} \tag{3.88}$$

We will provide an adequate definition of these tensors eventually, so consider the direct associations introduced in the sequence for now.

$$\begin{aligned}&\varepsilon_{\mu\nu XY}t_{XY\alpha\beta}^{(12)} \\ \to\;& -\left[\varepsilon_{\mu A\nu B}\left(g_{\alpha C}g_{\beta D} + g_{\alpha D}g_{C\beta}\right) + \varepsilon_{\mu A\nu C}\left(g_{B\alpha}g_{\beta D} - g_{B\beta}g_{\alpha D}\right)\right. \\ &+\varepsilon_{\mu A\nu D}\left(g_{B\alpha}g_{C\beta} + g_{B\beta}g_{\alpha C}\right) + \varepsilon_{\mu\nu BC}\left(g_{A\alpha}g_{\beta D} - g_{A\beta}g_{\alpha D}\right) \\ &\left.+\varepsilon_{\mu\nu BD}\left(g_{A\alpha}g_{C\beta} + g_{A\beta}g_{\alpha C}\right) + \varepsilon_{\mu\nu CD}\left(g_{A\beta}g_{B\alpha} - g_{A\alpha}g_{B\beta}\right)\right]\frac{K_1^A K_2^B K_3^C K_4^D}{D_{1234}}\end{aligned} \tag{3.89}$$

$$\begin{aligned}&\varepsilon_{\mu\alpha XY}t_{XY\nu\beta}^{(13)} \\ \to\;& -\left[\varepsilon_{\mu AB\alpha}\left(g_{\nu C}g_{\beta D} + g_{\nu D}g_{C\beta}\right) + \varepsilon_{\mu A\alpha C}\left(g_{\nu B}g_{\beta D} + g_{\nu D}g_{B\beta}\right)\right. \\ &+\varepsilon_{\mu A\alpha D}\left(g_{\nu B}g_{C\beta} - g_{\nu C}g_{B\beta}\right) + \varepsilon_{\mu B\alpha C}\left(g_{A\nu}g_{\beta D} - g_{A\beta}g_{\nu D}\right) \\ &\left.+\varepsilon_{\mu B\alpha D}\left(g_{A\nu}g_{C\beta} + g_{A\beta}g_{\nu C}\right) + \varepsilon_{\mu\alpha CD}\left(g_{A\nu}g_{B\beta} + g_{A\beta}g_{\nu B}\right)\right]\frac{K_1^A K_2^B K_3^C K_4^D}{D_{1234}}\end{aligned} \tag{3.90}$$

$$\begin{aligned}&\varepsilon_{\mu\beta XY}t_{XY\nu\alpha}^{(14)} \\ \to\;& -\left[\varepsilon_{\mu AB\beta}\left(-g_{\nu C}g_{\alpha D} + g_{\nu D}g_{\alpha C}\right) + \varepsilon_{\mu AC\beta}\left(g_{\nu B}g_{\alpha D} + g_{\nu D}g_{B\alpha}\right)\right. \\ &+\varepsilon_{\mu A\beta D}\left(g_{\nu B}g_{\alpha C} + g_{\nu C}g_{B\alpha}\right) + \varepsilon_{\mu BC\beta}\left(g_{A\nu}g_{\alpha D} - g_{A\alpha}g_{\nu D}\right) \\ &\left.+\varepsilon_{\mu B\beta D}\left(g_{A\nu}g_{\alpha C} - g_{A\alpha}g_{\nu C}\right) + \varepsilon_{\mu C\beta D}\left(g_{A\nu}g_{B\alpha} + g_{A\alpha}g_{\nu B}\right)\right]\frac{K_1^A K_2^B K_3^C K_4^D}{D_{1234}}\end{aligned} \tag{3.91}$$

$$\begin{aligned}
&\varepsilon_{\nu\alpha XY}t^{(23)}_{XY\mu\beta}\\
\rightarrow\ & -\left[\varepsilon_{A\nu B\alpha}\left(g_{\mu C}g_{\beta D}+g_{\mu D}g_{C\beta}\right)+\varepsilon_{A\nu\alpha C}\left(g_{\mu B}g_{\beta D}+g_{\mu D}g_{B\beta}\right)\right.\\
&+\varepsilon_{A\nu\alpha D}\left(g_{\mu B}g_{C\beta}-g_{\mu C}g_{B\beta}\right)+\varepsilon_{\nu B\alpha C}\left(g_{\mu A}g_{\beta D}+g_{\mu D}g_{A\beta}\right)\\
&\left.+\varepsilon_{\nu B\alpha D}\left(g_{\mu A}g_{C\beta}-g_{\mu C}g_{A\beta}\right)+\varepsilon_{\nu\alpha CD}\left(g_{\mu A}g_{B\beta}-g_{\mu B}g_{A\beta}\right)\right]\frac{K_1^AK_2^BK_3^CK_4^D}{D_{1234}}
\end{aligned} \tag{3.92}$$

$$\begin{aligned}
&\varepsilon_{\nu\beta XY}t^{(24)}_{XY\mu\alpha}\\
\rightarrow\ & -\left[\varepsilon_{A\nu B\beta}\left(-g_{\mu C}g_{\alpha D}+g_{\mu D}g_{\alpha C}\right)+\varepsilon_{A\nu C\beta}\left(g_{\mu B}g_{\alpha D}+g_{\mu D}g_{B\alpha}\right)\right.\\
&+\varepsilon_{A\nu\beta D}\left(g_{\mu B}g_{\alpha C}+g_{\mu C}g_{B\alpha}\right)+\varepsilon_{\nu BC\beta}\left(g_{\mu A}g_{\alpha D}+g_{\mu D}g_{A\alpha}\right)\\
&\left.+\varepsilon_{\nu B\beta D}\left(g_{\mu A}g_{\alpha C}+g_{\mu C}g_{A\alpha}\right)+\varepsilon_{\nu C\beta D}\left(g_{\mu A}g_{B\alpha}-g_{\mu B}g_{A\alpha}\right)\right]\frac{K_1^AK_2^BK_3^CK_4^D}{D_{1234}}
\end{aligned} \tag{3.93}$$

$$\begin{aligned}
&\varepsilon_{\alpha\beta XY}t^{(34)}_{XY\mu\nu}\\
\rightarrow\ & -\left[\varepsilon_{AB\alpha\beta}\left(-g_{\mu C}g_{\nu D}+g_{\mu D}g_{\nu C}\right)+\varepsilon_{A\alpha C\beta}\left(-g_{\mu B}g_{\nu D}+g_{\mu D}g_{\nu B}\right)\right.\\
&+\varepsilon_{A\alpha\beta D}\left(-g_{\mu B}g_{\nu C}+g_{\mu C}g_{\nu B}\right)+\varepsilon_{B\alpha C\beta}\left(g_{\mu A}g_{\nu D}+g_{\mu D}g_{A\nu}\right)\\
&\left.+\varepsilon_{B\alpha\beta D}\left(g_{\mu A}g_{\nu C}+g_{\mu C}g_{A\nu}\right)+\varepsilon_{\alpha C\beta D}\left(g_{\mu A}g_{\nu B}+g_{\mu B}g_{A\nu}\right)\right]\frac{K_1^AK_2^BK_3^CK_4^D}{D_{1234}}
\end{aligned} \tag{3.94}$$

Once all pieces are known, the $AVVV$ integrand assumes the following form

$$\begin{aligned}
t^{AVVV}_{\mu\nu\alpha\beta} =\ & 4i\left[\varepsilon_{\mu\nu XY}t^{(12)}_{XY\alpha\beta}+\varepsilon_{\mu\alpha XY}t^{(13)}_{XY\nu\beta}+\varepsilon_{\mu\beta XY}t^{(14)}_{XY\nu\alpha}\right]\\
&+\left[g_{\alpha\beta}t^{AVPP}_{\mu\nu}+g_{\nu\beta}t^{APVP}_{\mu\alpha}+g_{\nu\alpha}t^{APPV}_{\mu\beta}\right]+2i\varepsilon_{\mu\nu\alpha\beta}t^{PPPP}\\
&-i\left[\varepsilon_{\mu\alpha\beta\kappa}t^{VVPP}_{\kappa\nu}+\varepsilon_{\mu\nu\beta\kappa}t^{VPVP}_{\kappa\alpha}+\varepsilon_{\mu\nu\alpha\kappa}t^{VPPV}_{\kappa\beta}\right]\\
&+4i\left[\varepsilon_{\nu\alpha XY}t^{(23)}_{XY\mu\beta}+\varepsilon_{\nu\beta XY}t^{(24)}_{XY\mu\alpha}+\varepsilon_{\alpha\beta XY}t^{(34)}_{XY\mu\nu}\right]\\
&+i\varepsilon_{\nu\alpha\beta\kappa}t^{\tilde{T}PPP}_{\kappa\mu}-\left[g_{\mu\beta}t^{SVVP}_{\nu\alpha}+g_{\mu\alpha}t^{SVPV}_{\nu\beta}+g_{\mu\nu}t^{SPVV}_{\alpha\beta}\right]\\
&+\left(g_{\mu\nu}g_{\alpha\beta}-g_{\mu\alpha}g_{\nu\beta}+g_{\mu\beta}g_{\nu\alpha}\right)t^{SPPP}.
\end{aligned}$$

We reiterate that expressions adopted for traces contain all non-equivalent tensor configurations, which was convenient for identifying substructures. As this task is over, let us pursue simplifications in the same fashion as the triangle discussion. There, we acknowledged the presence of a Schouten identity with the trace-defining index fixed. In other words, when replacing the chiral matrix definition adjacent to the matrix γ_μ, an identity with μ fixed arose (3.59). This feature also applies to the box amplitude; thus, let us look closer at terms having this index outside the Levi-Civita symbol to verify that each coefficient vanishes identically (last three rows of the equation above). We do not compact products involving Dirac matrices from this point on.

Following this reasoning, we check over terms proportional to the squared mass. Notwithstanding the 2nd-order tensor amplitudes count with these contributions, the following combination does not exhibit such dependence:

$$\begin{aligned}
& i\varepsilon_{\nu\alpha\beta\kappa}t_{\kappa\mu}^{\widetilde{T}PPP} - \left[g_{\mu\beta}t_{\nu\alpha}^{SVVP} + g_{\mu\alpha}t_{\nu\beta}^{SVPV} + g_{\mu\nu}t_{\alpha\beta}^{SPVV}\right] \\
= & \left[i\varepsilon_{\nu\alpha\beta\kappa}\mathrm{tr}\left(\sigma_{\kappa\mu}\gamma_A\gamma_B\gamma_C\gamma_D\right) - g_{\mu\beta}\mathrm{tr}\left(\gamma_A\gamma_\nu\gamma_B\gamma_\alpha\gamma_C\gamma_5\gamma_D\right)\right. \\
& \left.-g_{\mu\alpha}\mathrm{tr}\left(\gamma_A\gamma_\nu\gamma_B\gamma_5\gamma_C\gamma_\beta\gamma_D\right) - g_{\mu\nu}\mathrm{tr}\left(\gamma_A\gamma_5\gamma_B\gamma_\alpha\gamma_C\gamma_\beta\gamma_D\right)\right] \frac{K_1^A K_2^B K_3^C K_4^D}{D_{1234}}. \quad (3.95)
\end{aligned}$$

To prove this result, we recall the coefficient associated with $K_1^A K_2^B$ in (3.83). The first row of the referred equation are monomials having μ within the metric tensor. Since it is a tensor antisymmetric in five indices, it cancels out identically in a four-dimensional setting

$$g_{\mu A}\varepsilon_{\nu B\alpha\beta} - g_{\mu\nu}\varepsilon_{AB\alpha\beta} + g_{\mu B}\varepsilon_{A\nu\alpha\beta} - g_{\mu\alpha}\varepsilon_{A\nu B\beta} + g_{\mu\beta}\varepsilon_{A\nu B\alpha} = 0. \quad (3.96)$$

Alternatively, one generates this result by performing successive permutations of the matrix γ_μ within a more complex trace $\mathrm{tr}\left(\gamma_5\gamma_\mu\gamma_\nu\gamma_\alpha\gamma_\beta\gamma_A\gamma_B\right)$; observe the form:

$$\begin{aligned}
& g_{\mu A}\mathrm{tr}\left(\gamma_5\gamma_\nu\gamma_B\gamma_\alpha\gamma_\beta\right) - g_{\mu\nu}\mathrm{tr}\left(\gamma_5\gamma_A\gamma_B\gamma_\alpha\gamma_\beta\right) + g_{\mu B}\mathrm{tr}\left(\gamma_5\gamma_A\gamma_\nu\gamma_\alpha\gamma_\beta\right) \\
& \quad -g_{\mu\alpha}\mathrm{tr}\left(\gamma_5\gamma_A\gamma_\nu\gamma_B\gamma_\beta\right) + g_{\mu\beta}\mathrm{tr}\left(\gamma_5\gamma_A\gamma_\nu\gamma_B\gamma_\alpha\right) = 0. \quad (3.97)
\end{aligned}$$

We find the same outcome when examining other coefficients; therefore, completing this part of the demonstration.

As a primary ingredient to examine tensor contributions, we follow the ideas seen in the previous case to derive the identities:

$$\begin{aligned}
& g_{\mu A}\mathrm{tr}\left(\gamma_5\gamma_\nu\gamma_B\gamma_\alpha\gamma_C\gamma_\beta\gamma_D\right) + g_{\mu B}\mathrm{tr}\left(\gamma_5\gamma_A\gamma_\nu\gamma_\alpha\gamma_C\gamma_\beta\gamma_D\right) \\
& +g_{\mu C}\mathrm{tr}\left(\gamma_5\gamma_A\gamma_\nu\gamma_B\gamma_\alpha\gamma_\beta\gamma_D\right) + g_{\mu D}\mathrm{tr}\left(\gamma_5\gamma_A\gamma_\nu\gamma_B\gamma_\alpha\gamma_C\gamma_\beta\right) \\
= & -g_{\mu\nu}\mathrm{tr}\left(\gamma_A\gamma_5\gamma_B\gamma_\alpha\gamma_C\gamma_\beta\gamma_D\right) - g_{\mu\alpha}\mathrm{tr}\left(\gamma_A\gamma_\nu\gamma_B\gamma_5\gamma_C\gamma_\beta\gamma_D\right) \\
& -g_{\mu\beta}\mathrm{tr}\left(\gamma_A\gamma_\nu\gamma_B\gamma_\alpha\gamma_C\gamma_5\gamma_D\right) \quad (3.98)
\end{aligned}$$

and

$$\begin{aligned}
g_{\mu\nu}\mathrm{tr}\left(\gamma_5\gamma_A\gamma_B\gamma_C\gamma_D\right) = & \; g_{\mu A}\mathrm{tr}\left(\gamma_5\gamma_\nu\gamma_B\gamma_C\gamma_D\right) - g_{\mu B}\mathrm{tr}\left(\gamma_5\gamma_\nu\gamma_A\gamma_C\gamma_D\right) \\
& +g_{\mu C}\mathrm{tr}\left(\gamma_5\gamma_\nu\gamma_A\gamma_B\gamma_D\right) - g_{\mu D}\mathrm{tr}\left(\gamma_5\gamma_\nu\gamma_A\gamma_B\gamma_C\right). \quad (3.99)
\end{aligned}$$

They perform the task of permuting index positions in Equation (3.95), leading directly to the expected identifications

$$\begin{aligned}
&\left[g_{\mu\beta}t_{\nu\alpha}^{SVVP}+g_{\mu\alpha}t_{\nu\beta}^{SVPV}+g_{\mu\nu}t_{\alpha\beta}^{SPVV}\right]-\varepsilon_{\nu\alpha\beta\kappa}t_{\kappa\mu}^{\widetilde{T}PPP}\\
=\ &4\left[\varepsilon_{\nu\alpha XY}t_{XY\mu\beta}^{(23)}+\varepsilon_{\nu\beta XY}t_{XY\mu\alpha}^{(24)}+\varepsilon_{\alpha\beta XY}t_{XY\mu\nu}^{(34)}\right]\\
&+\left(g_{\mu\nu}g_{\alpha\beta}-g_{\mu\alpha}g_{\nu\beta}+g_{\mu\beta}g_{\nu\alpha}\right)t^{SPPP}. \qquad (3.100)
\end{aligned}$$

With this identity, we achieve a much simpler view of the $AVVV$ integrand

$$\begin{aligned}
t_{\mu\nu\alpha\beta}^{AVVV} =\ & f_{4\mu\nu\alpha\beta}-\left[\varepsilon_{\mu\alpha\beta\kappa}t_{\kappa\nu}^{VVPP}+\varepsilon_{\mu\nu\beta\kappa}t_{\kappa\alpha}^{VPVP}+\varepsilon_{\mu\nu\alpha\kappa}t_{\kappa\beta}^{VPPV}\right]\\
&+\left[g_{\alpha\beta}t_{\mu\nu}^{AVPP}+g_{\nu\beta}t_{\mu\alpha}^{APVP}+g_{\nu\alpha}t_{\mu\beta}^{APPV}\right]+2\varepsilon_{\mu\nu\alpha\beta}t^{PPPP}, \qquad (3.101)
\end{aligned}$$

where $f_{4\mu\nu\alpha\beta}$ represents the tensor sector (we clarify this object below).

Inquiring about each object structure is the final part of this exploration, which occurs in the subsequent topics.

Fourth-Order Tensors

First, we inspect pure tensor contributions grouped into the structure

$$f_{4\mu\nu\alpha\beta}=4\varepsilon_{\mu\nu XY}t_{XY\alpha\beta}^{(12)}+4\varepsilon_{\mu\alpha XY}t_{XY\nu\beta}^{(13)}+4\varepsilon_{\mu\beta XY}t_{XY\nu\alpha}^{(14)}. \qquad (3.102)$$

After performing index contractions in the original expressions (3.89)-(3.91), our goal is to relabel summed indices and factorize the Levi-Civita symbol. Using the antisymmetric character of tensors is recurrent throughout this process. Thus, we introduce the following organization of the parts

$$\begin{aligned}
2t_{XY\alpha\beta}^{(12)} &= t_{4XY;\alpha\beta}^{(-;+)}+t_{4X\alpha;Y\beta}^{(-;+)}-t_{4X\beta;Y\alpha}^{(-;-)}+t_{4\alpha Y;\beta X}^{(-;+)}+t_{4\beta Y;\alpha X}^{(-;-)}+t_{4\alpha\beta;XY}^{(-;-)}, \qquad (3.103)\\
2t_{XY\nu\beta}^{(13)} &= t_{4Y\beta;\nu X}^{(-;-)}-t_{4XY;\nu\beta}^{(-;+)}-t_{4\nu Y;\beta X}^{(+;+)}-t_{4\beta X;Y\nu}^{(-;-)}+t_{4\nu X;Y\beta}^{(+;+)}-t_{4\nu\beta;XY}^{(+;-)}, \qquad (3.104)\\
2t_{XY\nu\alpha}^{(14)} &= t_{4XY;\nu\alpha}^{(-;-)}+t_{4\alpha Y;\nu X}^{(-;-)}-t_{4\nu Y;\alpha X}^{(+;-)}+t_{4\alpha X;Y\nu}^{(-;-)}-t_{4\nu X;Y\alpha}^{(+;-)}+t_{4\nu\alpha;XY}^{(+;-)}, \qquad (3.105)
\end{aligned}$$

where a new standard tensor arises

$$\begin{aligned}
&t_{4\mu\nu;\alpha\beta}^{(s_1;s_2)}\left(k_i,k_j;k_l,k_m\right)\\
=\ &\left[\left(k+k_i\right)_\mu\left(k+k_j\right)_\nu+s_1\left(k+k_j\right)_\mu\left(k+k_i\right)_\nu\right]\times\\
&\times\left[\left(k+k_l\right)_\alpha\left(k+k_m\right)_\beta+s_2\left(k+k_m\right)_\alpha\left(k+k_l\right)_\beta\right]\frac{1}{D_{1234}}. \qquad (3.106)
\end{aligned}$$

This notation employs a numerical subindex to indicate dependence on four internal momenta and admits two sign choices: s_1 and s_2. We omit arguments in occurrences exhibiting the momenta hierarchy $t_{4\mu\nu;\alpha\beta}^{(s_1;s_2)}=t_{4\mu\nu;\alpha\beta}^{(s_1;s_2)}\left(k_1,k_2;k_3,k_4\right)$.

Such an organization allows reducing our efforts to computing a single object, which consists of the simplified version

$$t_{4\mu\nu\alpha\beta}\left(k_i,k_j,k_l,k_m\right)=\frac{\left(k+k_i\right)_\mu\left(k+k_j\right)_\nu\left(k+k_l\right)_\alpha\left(k+k_m\right)_\beta}{D_{1234}}. \tag{3.107}$$

Besides appearing by itself inside some amplitudes, redefining indices of this tensor to build up the standard version is attainable

$$\begin{aligned} t^{(s_1,s_2)}_{4\mu\nu;\alpha\beta}\left(k_i,k_j;k_l,k_m\right) &= t_{4\mu\nu\alpha\beta}\left(k_i,k_j,k_l,k_m\right)+s_1 t_{4\mu\nu\alpha\beta}\left(k_j,k_i,k_l,k_m\right) \\ &\quad +s_2 t_{4\mu\nu\alpha\beta}\left(k_i,k_j,k_m,k_l\right)+s_1 s_2 t_{4\mu\nu;\alpha\beta}\left(k_j,k_i,k_m,k_l\right). \end{aligned} \tag{3.108}$$

Even Amplitudes - $VVPP$**,** $VPVP$**, and** $VPPV$

Second, we inspect even amplitudes that are 2nd-order tensors: $VVPP$, $VPVP$, and $VPPV$. For convenience, we check over these possibilities together. Therefore, replacing their vertex operators[6] on the general integrand (3.69) leads to the form

$$\begin{aligned} t^{\Gamma_i\Gamma_j\Gamma_k\Gamma_l}_{\mu\nu} &= +4s_2K_{34}\left(K_{1\mu}K_{2\nu}+s_3K_{1\nu}K_{2\mu}\right)\frac{1}{D_{1234}} \\ &\quad +4s_3K_{24}\left(K_{1\mu}K_{3\nu}-s_2K_{1\nu}K_{3\mu}\right)\frac{1}{D_{1234}} \\ &\quad +4s_1K_{23}\left(K_{1\mu}K_{4\nu}+K_{1\nu}K_{4\mu}\right)\frac{1}{D_{1234}} \\ &\quad -4s_3K_{14}\left(K_{2\mu}K_{3\nu}+s_1K_{2\nu}K_{3\mu}\right)\frac{1}{D_{1234}} \\ &\quad -4s_1K_{13}\left(K_{2\mu}K_{4\nu}-s_3K_{2\nu}K_{4\mu}\right)\frac{1}{D_{1234}} \\ &\quad +4s_1K_{12}\left(K_{3\mu}K_{4\nu}+s_2K_{3\nu}K_{4\mu}\right)\frac{1}{D_{1234}} \\ &\quad -s_1 g_{\mu\nu}t^{PPPP}, \end{aligned} \tag{3.109}$$

where bilinears $K_{ij}=K_i\cdot K_j-m^2$ appear. Each vertex configuration $\Gamma_i\Gamma_j\Gamma_k\Gamma_l$ considers three signs s_i, so we obtain the $VVPP$ function by fixing $s_i=(-1,-1,+1)$, the $VPVP$ by fixing $s_i=(+1,-1,-1)$, and the $VPPV$ by fixing $s_i=(-1,+1,-1)$. The $PPPP$ scalar function appears as a subamplitude here.

When reducing bilinears with the aid of identity (3.44), identifying 2nd-order standard tensors is straightforward. Their systematization remembers the version depending on

[6] There are two vector vertices denoted by γ_μ and γ_ν; and two pseudoscalar vertices γ_5. Different configurations produce the acknowledged sign differences.

three internal momenta (3.53) and introduces the analogous involving four momenta

$$t_{4\mu\nu}^{(s)}\left(k_i,k_j\right)=\frac{\left(k+k_i\right)_\mu\left(k+k_j\right)_\nu+s\left(k+k_j\right)_\mu\left(k+k_i\right)_\nu}{D_{1234}}. \tag{3.110}$$

By performing these identifications and grouping terms with the same denominator, we achieve the structure:

$$\begin{aligned} t_{\mu\nu}^{\Gamma_i\Gamma_j\Gamma_k\Gamma_l} =\ & 2s_1\left[s_3 t_{3\mu\nu}^{(s_3)}\left(k_1,k_2\right)+s_2 t_{3\mu\nu}^{(-s_2)}\left(k_1,k_3\right)-s_2 t_{3\mu\nu}^{(s_1)}\left(k_2,k_3\right)\right] \\ & +2s_1\left[s_3 t_{3\mu\nu}^{(s_3)}\left(k_1,k_2\right)+t_{3\mu\nu}^{(+)}\left(k_1,k_4\right)-t_{3\mu\nu}^{(-s_3)}\left(k_2,k_4\right)\right]' \\ & +2s_1\left[s_2 t_{3\mu\nu}^{(-s_2)}\left(k_1,k_3\right)+t_{3\mu\nu}^{(+)}\left(k_1,k_4\right)+t_{3\mu\nu}^{(s_2)}\left(k_3,k_4\right)\right]'' \\ & +2s_1\left[-s_2 t_{3\mu\nu}^{(s_1)}\left(k_2,k_3\right)-t_{3\mu\nu}^{(-s_3)}\left(k_2,k_4\right)+t_{3\mu\nu}^{(s_2)}\left(k_3,k_4\right)\right]''' \\ & -2s_1\left[s_3\left(q-r\right)^2 t_{4\mu\nu}^{(s_3)}\left(k_1,k_2\right)+s_2\left(p-r\right)^2 t_{4\mu\nu}^{(-s_2)}\left(k_1,k_3\right)\right. \\ & +\left(p-q\right)^2 t_{4\mu\nu}^{(+)}\left(k_1,k_4\right)-s_2 r^2 t_{4\mu\nu}^{(s_1)}\left(k_2,k_3\right) \\ & \left.-q^2 t_{4\mu\nu}^{(-s_3)}\left(k_2,k_4\right)+p^2 t_{4\mu\nu}^{(s_2)}\left(k_3,k_4\right)\right]+s_1 g_{\mu\nu} t^{PPPP}. \end{aligned} \tag{3.111}$$

Objects typical of three-point amplitudes arose, bringing different momenta configurations with them. We introduced the associations below to distinguish these possibilities.

$$\begin{array}{ll} \frac{1}{D_{123}}\rightarrow[\text{structure}] & \frac{1}{D_{134}}\rightarrow[\text{structure}]'' \\ \frac{1}{D_{124}}\rightarrow[\text{structure}]' & \frac{1}{D_{234}}\rightarrow[\text{structure}]''' \end{array} \tag{3.112}$$

Odd Amplitudes - $AVPP$, $APVP$, and $APPV$

Third, we inspect odd amplitudes that are 2nd-order tensors: $AVPP$, $APVP$, and $APPV$. By replacing the corresponding vertex operators[7] on the general integrand (3.69), we approach all possibilities together

$$\begin{aligned} s_1 t_{\mu\nu}^{\Gamma_i\Gamma_j\Gamma_k\Gamma_l} =\ & 4\left(-\varepsilon_{ABCD}g_{\mu\nu}-\varepsilon_{\nu BCD}g_{\mu A}-\varepsilon_{\mu BCD}g_{\nu A}+\varepsilon_{\nu ACD}g_{\mu B}+s_2\varepsilon_{\mu ACD}g_{\nu B}\right. \\ & \left.-\varepsilon_{\nu ABD}g_{\mu C}+s_3\varepsilon_{\mu ABD}g_{\nu C}+\varepsilon_{\nu ABC}g_{\mu D}-\varepsilon_{\mu ABC}g_{\nu D}\right)\frac{K_1^A K_2^B K_3^C K_4^D}{D_{1234}} \\ & +4\varepsilon_{\mu\nu CD}K_{12}\frac{K_3^C K_4^D}{D_{1234}}-4\varepsilon_{\mu\nu BD}K_{13}\frac{K_2^B K_4^D}{D_{1234}}+4\varepsilon_{\mu\nu BC}K_{14}\frac{K_2^B K_3^C}{D_{1234}} \\ & +4\varepsilon_{\mu\nu AD}K_{23}\frac{K_1^A K_4^D}{D_{1234}}-4\varepsilon_{\mu\nu AC}K_{24}\frac{K_1^A K_3^C}{D_{1234}}+4\varepsilon_{\mu\nu AB}K_{34}\frac{K_1^A K_2^B}{D_{1234}}, \end{aligned} \tag{3.113}$$

[7] There are four vertices: one axial $\gamma_\mu\gamma_5$, one vector γ_ν, and two pseudoscalars γ_5. Different configurations produce sign differences.

where bilinears $K_{ij} = K_i \cdot K_j - m^2$ appear. Each vertex configuration $\Gamma_i\Gamma_j\Gamma_k\Gamma_l$ considers three signs s_i, so we obtain the $AVPP$ function by fixing $s_i = (-1,-1,+1)$, the $APVP$ by fixing $s_i = (+1,+1,+1)$, and the $APPV$ by fixing $s_i = (-1,+1,-1)$.

Since there is an evident distinction between both parts of these amplitudes, we rename summed indices to emphasize them

$$t_{\mu\nu}^{\Gamma_i\Gamma_j\Gamma_k\Gamma_l} = s_1\varepsilon_{\mu XYZ} f_{4\nu XYZ}^{(s_2,s_3)} + s_1\varepsilon_{\mu\nu XY} f_{4XY}. \tag{3.114}$$

The first depends on the simplified version of the 4th-order tensor (3.107). Taking a closer look at its coefficients, observe that contributions having the index μ on the metric compound the Schouten identity (3.99). Hence, these terms cancel out, and this sector assumes the form

$$\varepsilon_{\mu XYZ} f_{4\nu XYZ}^{(s_2,s_3)} = 4\left(-\varepsilon_{\mu BCD} g_{\nu A} + s_2\varepsilon_{\mu ACD} g_{\nu B} + s_3\varepsilon_{\mu ABD} g_{\nu C} - \varepsilon_{\mu ABC} g_{\nu D}\right) t_{4ABCD}. \tag{3.115}$$

Analogously to what happened with even amplitudes, bilinear reductions in the second part lead to 2nd-order standard tensors: (3.53) and (3.110). This time, however, all objects are antisymmetric tensors:

$$\begin{aligned} f_{4XY} = &\left[t_{3XY}^{(-)}(k_2,k_3) - t_{3XY}^{(-)}(k_1,k_3) + t_{3XY}^{(-)}(k_1,k_2)\right] \\ &+\left[-t_{3XY}^{(-)}(k_2,k_4) + t_{3XY}^{(-)}(k_1,k_4) + t_{3XY}^{(-)}(k_1,k_2)\right]' \\ &+\left[t_{3XY}^{(-)}(k_3,k_4) + t_{3XY}^{(-)}(k_1,k_4) - t_{3XY}^{(-)}(k_1,k_3)\right]'' \\ &+\left[t_{3XY}^{(-)}(k_3,k_4) - t_{3XY}^{(-)}(k_2,k_4) + t_{3XY}^{(-)}(k_2,k_3)\right]''' \\ &-p^2 t_{4XY}^{(-)}(k_3,k_4) + q^2 t_{4XY}^{(-)}(k_2,k_4) - r^2 t_{4XY}^{(-)}(k_2,k_3) \\ &-(p-q)^2 t_{4XY}^{(-)}(k_1,k_4) + (p-r)^2 t_{4XY}^{(-)}(k_1,k_3) \\ &-(q-r)^2 t_{4XY}^{(-)}(k_1,k_2). \end{aligned} \tag{3.116}$$

Scalar Amplitude $PPPP$

Fourth, we inspect the scalar amplitude $PPPP$. Our task is to explore the structure achieved by replacing the chiral matrix as vertices in the original integrand (3.69):

$$\begin{aligned} t^{PPPP} = &\left[K_1^A K_2^B K_3^C K_4^D \mathrm{tr}\left(\gamma_A\gamma_B\gamma_C\gamma_D\right) - m^2 K_1^A K_2^B \mathrm{tr}\left(\gamma_A\gamma_B\right)\right. \\ &+m^2 K_1^A K_3^C \mathrm{tr}\left(\gamma_A\gamma_C\right) - m^2 K_1^A K_4^D \mathrm{tr}\left(\gamma_A\gamma_D\right) - m^2 K_2^B K_3^C \mathrm{tr}\left(\gamma_B\gamma_C\right) \\ &\left.+m^2 K_2^B K_4^D \mathrm{tr}\left(\gamma_B\gamma_D\right) - m^2 K_3^C K_4^D \mathrm{tr}\left(\gamma_C\gamma_D\right) + m^4 \mathrm{tr}\left(\mathbf{1}\right)\right] \frac{1}{D_{1234}}, \end{aligned} \tag{3.117}$$

where sign changes come from permutations. All traces contain exclusively Dirac matrices, so results depend on the metric tensor in such a way that bilinears $K_{ij} = K_i \cdot K_j - m^2$ appear:

$$t^{PPPP} = 4\left(K_{12}K_{34} - K_{13}K_{24} + K_{14}K_{23}\right)\frac{1}{D_{1234}}. \tag{3.118}$$

Once again, rewriting them through identity (3.44) reduces the dependence on propagator-like objects D_i. Since there are two reductions this time, quantities typical of two and three-point functions emerge. In the end, we obtain the $PPPP$ final organization

$$\begin{aligned} t^{PPPP} &= 2\left[\frac{1}{D_{24}} + \frac{1}{D_{13}}\right] - 2\left(p^2 - p\cdot q\right)\frac{1}{D_{123}} - 2\left(p\cdot r\right)\frac{1}{D_{124}} \\ &\quad -2\left(r^2 - q\cdot r\right)\frac{1}{D_{134}} + 2\left(p-q\right)\cdot\left(q-r\right)\frac{1}{D_{234}} \\ &\quad + \left[p^2\left(r-q\right)^2 - q^2\left(p-r\right)^2 + r^2\left(p-q\right)^2\right]\frac{1}{D_{1234}}. \end{aligned} \tag{3.119}$$

3.2.7 Comments

After implementing the first part of Feynman rules, we analyzed integrands of amplitudes relevant to this investigation. The grouping of components that share similar sorting of indices allowed the identification of less complex correlators and standard tensors inside them. Ultimately, each piece corresponds to a combination of rational functions having propagator-like quantities in denominators with loop momentum products on numerators. This subsection briefly comments on them while introducing one-loop Feynman integrals.

The general integrand of two-point amplitudes (3.34) indicates they are combinations of structures having denominators $D_{ij} = D_iD_j$ and numerators $[1, k_\mu, k_{\mu\nu}]$. Nevertheless, the AV (3.48) is an antisymmetric tensor and does not admit dependence on the symmetric numerator. These objects also appear inside higher-order amplitudes due to reducing bilinears, in which cases discriminating the arguments is fundamental. That is the case of the vector VPP (3.64) and the scalar $PPPP$ (3.119). They lead to two-propagator Feynman integrals

$$[I_2, I_{2\alpha}] = \int \frac{d^4k}{(2\pi)^4}\frac{[1, k_\alpha]}{D_{ij}}. \tag{3.120}$$

Although power counting indicates quadratic divergences for two-point amplitudes, the integrals above exhibit logarithmic and linear power counting, respectively.

Extending this reasoning indicates that three-point amplitudes (3.51) are combinations of structures with denominators $D_{ijk} = D_iD_jD_k$ and numerators $[1, k_\alpha, k_{\alpha\beta}, k_{\alpha\beta\rho}]$. Again, the property of antisymmetry prohibits the emergence of the last numerator. Hence, the following three-propagator Feynman integrals manifest within this investigation:

$$[I_3, I_{3\alpha}, I_{3\alpha\beta}] = \int \frac{d^4k}{(2\pi)^4} \frac{[1, k_\alpha, k_{\alpha\beta}]}{D_{ijk}}. \tag{3.121}$$

Besides appearing within PVV (3.54) and AVV (3.62), bilinear reductions bring these structures to all subamplitudes belonging to box amplitudes. In this cases, we enforce the need for a notation to avoid confusion (3.112). Even though three-point amplitudes exhibit linear power counting, only the 2nd-order integral is (logarithmically) divergent.

As for four-point amplitudes, the general integrand indicates the need to compute the following Feynman integrals

$$[I_4, I_{4\alpha}, I_{4\alpha\beta}, I_{4\alpha\beta\rho}, I_{4\alpha\beta\rho\sigma}] = \int \frac{d^4k}{(2\pi)^4} \frac{[1, k_\alpha, k_{\alpha\beta}, k_{\alpha\beta\rho}, k_{\alpha\beta\rho\sigma}]}{D_{1234}}. \tag{3.122}$$

Only the last one indicates a logarithmic divergence in this case. Meanwhile, note that this contribution appears exclusively within 4th-order standard tensors

$$\begin{aligned} & t^{(s_1;s_2)}_{4\mu\nu;\alpha\beta} (k_i, k_j; k_l, k_m) \\ = & \left[(k+k_i)_\mu (k+k_j)_\nu + s_1 (k+k_j)_\mu (k+k_i)_\nu \right] \times \\ & \times \left[(k+k_l)_\alpha (k+k_m)_\beta + s_2 (k+k_m)_\alpha (k+k_l)_\beta \right] \frac{1}{D_{1234}}, \end{aligned} \tag{3.123}$$

which is contracted with the Levi-Civita symbol within the $AVVV$ box, see Equations (3.102) and (3.115). That simplifies some contributions, so tensors symmetric in four indices might not appear in this work.

We still want to comment on other standard tensors appearing throughout this section. They emphasize patterns followed by tensor amplitudes at the integrand level, which continues to occur after integration. This reasoning is essential to this work, particularly for 3rd-order tensors involving three and four propagators ($n = 3, 4$)

$$t^{(s)}_{n\mu;\nu\alpha} (k_l; k_i, k_j) = \frac{(k+k_l)_\mu \left[(k+k_i)_\nu (k+k_j)_\alpha + s (k+k_i)_\alpha (k+k_j)_\nu \right]}{D_{a_1 a_2 \ldots a_n}} \tag{3.124}$$

and for 2nd-order tensors involving two, three, and four propagators ($n = 2, 3, 4$)

$$t^{(s)}_{n\mu\nu} (k_i, k_j) = \frac{(k+k_i)_\mu (k+k_j)_\nu + s (k+k_j)_\mu (k+k_i)_\nu}{D_{a_1 a_2 \ldots a_n}}. \tag{3.125}$$

3.3 Strategy to Handle Divergences

As Feynman integrals are necessary to compound perturbative amplitudes, our objective becomes their explicit computation. Thus, it is crucial to adopt a strategy to deal with the divergences acknowledged above. We employ the Implicit Regularization (IReg), proposed and developed by O. A. Battistel in his Ph.D. thesis [37]. This strategy has several applications in the anomalies subject [38, 39, 40], including cases involving the single-axial triangle [41, 42]. We also draw attention to works developed in (odd and even) extra dimensions [43, 44, 45] since they relate to more complex tensor structures, as it occurs for the box amplitude.

The central ingredient of IReg is a representation of the propagator (3.6) capable of splitting ill-defined mathematical structures from finite contributions of integrals. The finite part is univocal, and its evaluation employs usual methods of perturbative calculations. Without choosing a prescription to compute diverging objects, we organize them and study properties relevant to the intended discussion. This view allows a transparent connection among mathematical expressions attributed to a perturbative amplitude in different stages of calculations.

Following this strategy, one writes the mentioned representation through an identity with the property that the power counting decreases from term to term. The performed operations are purely algebraic; therefore, this strategy has no restrictions on applicability. Besides, as such identity consists of a summation, the only requirement for its implementation is that linearity applies to Feynman integrals.

Let us use the object D_n^{-1} as a study case to illustrate the procedure. By introducing an arbitrary parameter λ, we construct the identity

$$\int \frac{d^4k}{(2\pi)^4}\frac{1}{D_n} = \int \frac{d^4k}{(2\pi)^4}\left[\frac{1}{\left(k^2-\lambda^2\right)} - \frac{2k_n\cdot k + k_n^2 + \lambda^2 - m^2}{\left(k^2-\lambda^2\right)D_n}\right]. \tag{3.126}$$

Although the power counting exhibited by the first term on the right-hand side remains the same as the original integral, this term does not depend on physical parameters. Meanwhile, the power counting of the second term decreases by one. That compels successive implementations, so finite integrals emerge eventually. Here, three iterations are enough to achieve this perspective. In the end, the separation comes as follows

$$\int \frac{d^4k}{(2\pi)^4}\frac{1}{D_n} = \int \frac{d^4k}{(2\pi)^4}\left[\frac{1}{D_\lambda} - \frac{A_n}{D_\lambda^2} + \frac{A_n^2}{D_\lambda^3} - \frac{A_n^3}{D_\lambda^3 D_n}\right], \tag{3.127}$$

where notations were introduced:

$$D_\lambda = k^2-\lambda^2 \quad \text{and} \quad A_n = 2k_n\cdot k + k_n^2 + \lambda^2 - m^2. \tag{3.128}$$

Even though we could keep repeating this process, nothing new would occur. Only redundant finite integrals would emerge, generating extra effort with their inspection. Observe that λ works as a scale connecting finite and ill-defined structures. Furthermore, the final results must not depend on it since it is an outsider to the theory.

At this point, we induce a general representation for the propagator (3.6), capable of splitting successfully any structure of interest. It assumes the form of the identity

$$\frac{1}{D_n} = \sum_{j=0}^{N} \frac{(-1)^j \, A_n^j}{D_\lambda^{j+1}} + \frac{(-1)^{N+1} \, A_n^{N+1}}{D_\lambda^{N+1} D_n}, \tag{3.129}$$

with N being equal to or higher than the superficial degree of divergence of the aimed integral. This condition guarantees that at least the last term leads to a finite structure.

By itself, the systematization proposed by the IReg is very useful as a tool in this type of calculation. The subsequent discussion brings ingredients from references [46, 47], introducing mathematical structures necessary to express the amplitudes investigated here. They are standard divergent objects and finite structure functions. Further information on the implementation of this strategy is elucidated in Section (3.4).

3.3.1 Standard Divergent Objects

For the separation to be effective, the last term of identity (3.129) must be finite. That implies any diverging object is shaped accordingly to the elements from the summation sign. Expanding the powers A_n^j shows that this sector combines structures depending on the loop momentum and the scale:

$$\frac{A_n^j}{D_\lambda^{j+1}} \to \frac{1}{D_\lambda^\alpha}, \frac{k_{\mu_1} k_{\mu_2}}{D_\lambda^{\alpha+1}}, \frac{k_{\mu_1} k_{\mu_2} k_{\mu_3} k_{\mu_4}}{D_\lambda^{\alpha+2}}, \ldots, \frac{k_{\mu_1} k_{\mu_2} k_{\mu_3} k_{\mu_4} \ldots k_{\mu_{2n-1}} k_{\mu_{2n}}}{D_\lambda^{\alpha+n}}.$$

Exclusively even terms are cast since odd ones do not generate non-zero contributions after integration.

Some configurations of parameters lead to quantities whose integration is finite. Therefore, establishing a restriction is needed since our targets are divergent quantities. Given that the investigation occurs in the physical space-time dimension, we come across the constraint $N = 2 - \alpha \geq 0$. Nevertheless, we saw that integrals with quadratic power counting cancel out, and linearly diverging objects are not allowed as they associate with odd integrands. That delimits our discussion to logarithmically divergent quantities, and we set $\alpha = 2$.

Our specific goal is to organize them into standard objects. When studying structures of amplitudes, we established expectations towards the emergence of surface terms. As we adjusted all structures above so that their integrals share the power counting, putting them together to obtain these terms is straightforward. Starting with a 2nd-order tensor, we combine the first two integrals into the object[8]

$$-\int \frac{d^4k}{(2\pi)^4}\frac{\partial}{\partial k^{\mu}}\frac{k_{\nu}}{D_{\lambda}^2}=\int \frac{d^4k}{(2\pi)^4}\left[\frac{4k_{\mu\nu}}{D_{\lambda}^3}-g_{\mu\nu}\frac{1}{D_{\lambda}^2}\right]. \tag{3.130}$$

Following the same reasoning, we have the 4th-order tensor

$$-\int \frac{d^4k}{(2\pi)^4}\frac{\partial}{\partial k^{\mu}}\frac{4k_{\nu\alpha\beta}}{D_{\lambda}^3}=\int \frac{d^4k}{(2\pi)^4}\left[\frac{24k_{\mu\nu\alpha\beta}}{D_{\lambda}^4}-g_{\mu\nu}\frac{4k_{\alpha\beta}}{D_{\lambda}^3}-g_{\mu\alpha}\frac{4k_{\nu\beta}}{D_{\lambda}^3}-g_{\mu\beta}\frac{4k_{\nu\alpha}}{D_{\lambda}^3}\right], \tag{3.131}$$

where the global numerical factor is an adjustment related to the first object. We recall the notation introduced to products involving momenta $k_{\alpha\beta}=k_{\alpha}k_{\beta}$.

These two cases comprise all elements that arise throughout this investigation, so we introduce the proper definitions. Concerning the 4th-order tensor, observe that the μ-index has a privileged role with respect to other indices. We prefer a symmetrized version, taking all different index configurations into account to introduce the surface term:

$$\begin{aligned}\Box_{\mu\nu\alpha\beta}\left(\lambda^2\right) &= \int \frac{d^4k}{(2\pi)^4}\left[\frac{24k_{\mu\nu\alpha\beta}}{D_{\lambda}^4}-\frac{1}{2}g_{\mu\nu}\frac{4k_{\alpha\beta}}{D_{\lambda}^3}-\frac{1}{2}g_{\mu\alpha}\frac{4k_{\nu\beta}}{D_{\lambda}^3}\right.\\ &\left.-\frac{1}{2}g_{\mu\beta}\frac{4k_{\nu\alpha}}{D_{\lambda}^3}-\frac{1}{2}g_{\nu\alpha}\frac{4k_{\mu\beta}}{D_{\lambda}^3}-\frac{1}{2}g_{\nu\beta}\frac{4k_{\mu\alpha}}{D_{\lambda}^3}-\frac{1}{2}g_{\alpha\beta}\frac{4k_{\mu\nu}}{D_{\lambda}^3}\right].\end{aligned} \tag{3.132}$$

Tensors that share the power counting connect, generating one irreducible object at the end of the process. That means we find 2nd-order tensors inside the expression above

$$\Delta_{\mu\nu}\left(\lambda^2\right)=\int \frac{d^4k}{(2\pi)^4}\left[\frac{4k_{\mu\nu}}{D_{\lambda}^3}-g_{\mu\nu}\frac{1}{D_{\lambda}^2}\right] \tag{3.133}$$

and ultimately the mentioned irreducible object arises

$$I_{\log}\left(\lambda^2\right)=\int \frac{d^4k}{(2\pi)^4}\frac{1}{D_{\lambda}^2}. \tag{3.134}$$

We omit the argument of divergent objects in the calculations for simplicity since variations do not appear.

The objects above represent the mathematically ill-defined part of the results. Differently from finite integrals, we do not evaluate them. In possession of amplitudes expressions, the analysis of results reflects on possibilities for these structures. From this perspective, it is feasible to investigate different prescriptions for their evaluation and the

[8] Even though we are introducing these objects beforehand, their arising is automatic when employing resources from IReg. For instance, if one uses the identity to separate the AV integrand, it will find precisely the 2nd-order surface term when performing the loop integration. Such an outcome requires only algebraic operations at the integrand level.

consequences they bring.

As an example of this reasoning, suppose our aim is computing a specific amplitude contraction. In general, most relations among Green functions (GF) arise from pure algebraic operations without further conditions. Nevertheless, for anomalous amplitudes, one or more relations might depend on the prescription for evaluating divergences. Beyond that, choosing a prescription affects the maintenance (or not) of Ward identities (WIs) corresponding to the same contractions. We aim to clarify how these constraints relate, inquiring about the role played by divergent objects.

3.3.2 Finite Structure Functions - Part I

The systematization involving finite functions is a fundamental ingredient of the IReg that makes it easier to visualize and interpret results even in the face of extensive mathematical expressions. We discuss this subject in three parts directed to structures typical of two, three, and four-point amplitudes.

Firstly, we focus on objects related to Feynman integrals depending on two internal lines (3.120). In any space-time dimension, one-loop calculations for theories involving equal masses lead to the following polynomial on the parameter z[9]:

$$Q(z) = p^2 z(1-z) - m^2. \tag{3.136}$$

Subsection (3.4.1) is very detailed in evaluating finite contributions, clarifying how this polynomial emerges after adopting a Feynman parametrization. As two-propagator integrals have divergent power counting in the physical dimension, one initially acknowledges dependence on non-physical quantities, i.e., arbitrary labels k_i and the scale λ. They cancel out in the integration, so only dependence on physical parameters ultimately remains. For this case, the polynomial carries the external momentum $p = k_1 - k_2$ and the mass.

The specific family of functions that concern four-dimensional calculations is

$$\xi_a^{(0)}\left(p^2, m^2; \lambda^2\right) = \xi_a^{(0)}(p) = \int_0^1 dz\; z^a \ln \frac{Q(z)}{-\lambda^2}. \tag{3.137}$$

Since momentum is the only parameter that changes throughout this investigation, we omit the others from the argument. We even suppress this information when the dependence is undoubtedly clear.

For our purposes, the integral representation of finite functions is enough. Neverthe-

[9] We deal with a particular form of the polynomial with different masses

$$Q(z) = p^2 z(1-z) + \left(m_1^2 - m_2^2\right) - m_1^2. \tag{3.135}$$

less, if needed, computing them is doable. The first stage of this task is integrating the function with the lowest parameter power ($a = 0$), which yields

$$\xi_0^{(0)}(p) = \ln \frac{m^2}{\lambda^2} - 2 - \frac{1}{2p^2} h\left(p^2, m^2\right). \tag{3.138}$$

The object $h(p^2, m^2)$ admits three different representations depending on the squared momentum value:

1. In the region where $p^2 < 0$

$$h\left(p^2, m^2\right) = 2\sqrt{4m^2 - p^2}\sqrt{-p^2} \ln\left[\frac{\sqrt{4m^2 - p^2} + \sqrt{-p^2}}{\sqrt{4m^2 - p^2} - \sqrt{-p^2}}\right] \tag{3.139}$$

2. In the region where $0 < p^2 < 4m^2$

$$h\left(p^2, m^2\right) = -4\sqrt{4m^2 - p^2}\sqrt{p^2} \tan^{-1}\left[\frac{\sqrt{p^2}}{\sqrt{4m^2 - p^2}}\right] \tag{3.140}$$

3. In the region where $p^2 > 4m^2$

$$\begin{aligned} h\left(p^2, m^2\right) &= 2\sqrt{p^2 - 4m^2}\sqrt{p^2} \ln\left\{\frac{\sqrt{p^2 - 4m^2} + \sqrt{p^2}}{\sqrt{p^2} - \sqrt{p^2 - 4m^2}}\right\} \\ &\quad + 2i\pi\sqrt{p^2 - 4m^2}\sqrt{p^2}. \end{aligned} \tag{3.141}$$

Instead of integrating more complex elements, the idea is to reduce them to those already known. The main ingredient for such is one identity that expresses the parameter in terms of the Q-polynomial derivative

$$z = \frac{1}{2}\left[1 - \frac{1}{p^2}\frac{\partial Q(z)}{\partial z}\right]. \tag{3.142}$$

When replacing this structure within a finite function, the first term represents another function with decreased parameter power, while the second corresponds to compensating terms evaluated posteriorly to integration by parts.

The closest example of this reasoning resides in the element defined by $a = 1$. Whereas the first term reduces the parameter power to $a = 0$, the rest is a total derivative that vanishes by considering both integration limits

$$\xi_1^{(0)}(p) = \frac{1}{2}\xi_0^{(0)}(p). \tag{3.143}$$

These instructions also lead to a general expression reducing any higher-order function

$(a \geq 2)$ to most elementary ones

$$\xi_a^{(0)}(p) = \frac{a}{a+1}\xi_{a-1}^{(0)}(p) - \frac{a-1}{a+1}\frac{m^2}{p^2}\xi_{a-2}^{(0)}(p) + \frac{1}{a+1}\frac{m^2}{p^2}\ln\frac{m^2}{\lambda^2} - \frac{1}{a}\frac{a-1}{(a+1)^2}. \tag{3.144}$$

3.3.3 Finite Structure Functions - Part II

The second part of this discussion studies structures related to Feynman integrals depending on three internal lines (3.121). Although two different families arise in this investigation

$$\xi_{ab}^{(-1)}\left(p,q,m^2\right) = \xi_{ab}^{(-1)}(p,q) = \int_0^1 dz \int_0^{1-z} dy\; y^b z^a \frac{1}{Q(y,z)}, \tag{3.145}$$

$$\xi_{ab}^{(0)}\left(p,q,m^2;\lambda^2\right) = \xi_{ab}^{(0)}(p,q) = \int_0^1 dz \int_0^{1-z} dy\; y^b z^a \ln\frac{Q(y,z)}{-\lambda^2}, \tag{3.146}$$

the second type appears exclusively for Feynman integrals that are 2nd-order tensors. Again, we suppress their argument if this is transparent throughout the discussion. Meanwhile, since different momenta configurations appear within the box exploration, we resort to the line notation wherever necessary (3.112). These functions manifest dependence on a polynomial on Feynman parameters $\{z, y\}$:

$$Q(y,z) = p^2 y(1-y) + q^2 z(1-z) - 2(p\cdot q)\, yz - m^2, \tag{3.147}$$

where $p = k_1 - k_2$ and $q = k_1 - k_3$.

Our focus is understanding how to reduce parameter powers in analogy with the ξ_a cases in Section (3.3.2). By examining both derivatives, we establish the following relations

$$2\left[p^2 y + (p\cdot q)\, z\right] = p^2 - \frac{\partial Q(y,z)}{\partial y}, \tag{3.148}$$

$$2\left[(p\cdot q)\, y + q^2 z\right] = q^2 - \frac{\partial Q(y,z)}{\partial z}. \tag{3.149}$$

Notice that both parameters appear together, which indicates reductions concern the sum of powers $a+b$. When computing Feynman integrals, we will see this is part of a pattern: finite structure functions do not emerge randomly but in packages having $a+b$ fixed.

Then, starting with the constraint $a+b=1$, let us examine how functions $\xi_{10}^{(-1)}$ and $\xi_{01}^{(-1)}$ combine. When multiplying both sides of the first relation by Q^{-1} and applying the integration, identifications are straightforward

$$2\left[p^2\xi_{10}^{(-1)} + (p\cdot q)\,\xi_{01}^{(-1)}\right] = p^2\xi_{00}^{(-1)} - \int_0^1 dz \int_0^{1-z} dy\; \frac{\partial}{\partial y}\ln\frac{Q(y,z)}{-\lambda^2}. \tag{3.150}$$

As it is the objective, parameter powers decreased and now $a+b=0$. The compensating term is a total derivative; thus, considering the integration limits allows recognizing two-point finite functions

$$2\left[p^{2}\xi_{10}^{(-1)}+(p\cdot q)\,\xi_{01}^{(-1)}\right]=p^{2}\xi_{00}^{(-1)}-\xi_{0}^{(0)}\,(p-q)+\xi_{0}^{(0)}\,(q)\,. \tag{3.151}$$

Since different momenta configurations are concomitant, their distinction is crucial.

From the second relation (3.149), using the derivative with respect to the z variable generates an analogous structure

$$2\left[(p\cdot q)\,\xi_{10}^{(-1)}+q^{2}\xi_{01}^{(-1)}\right]=q^{2}\xi_{00}^{(-1)}-\int_{0}^{1}dz\int_{0}^{1-z}dy\,\frac{\partial}{\partial z}\ln\frac{Q\,(y,z)}{-\lambda^{2}}. \tag{3.152}$$

The novelty is that exchanging positions of integral (in y) and derivative (in z) is needed before computing the last term. Nevertheless, difficulties emerge due to the z parameter presence in the integration limit. Under adequate continuity conditions, Leibniz rule for differentiation under the integral sign applies

$$\frac{d}{dz}\int_{a(z)}^{b(z)}dy\;f\,(y,z)=\int_{a(z)}^{b(z)}dy\;\frac{\partial}{\partial z}f\,(y,z)-f\,(a\,(z)\,,z)\,\frac{d}{dz}a\,(z)+f\,(b\,(z)\,,z)\,\frac{d}{dz}b\,(z)\,. \tag{3.153}$$

For the specific case from Equation (3.152), we set limits of integration and perform their derivatives. When integrating with respect to the z variable, the rule is established

$$\int_{0}^{1}dz\int_{0}^{1-z}dy\;\frac{\partial}{\partial z}f\,(y,z)=\int_{0}^{1}dz\;f\,(1-z,z)-\int_{0}^{1}dy\;f\,(y,0)\,. \tag{3.154}$$

Lastly, we establish the second reduction after replacing the corresponding integrand

$$2\left[(p\cdot q)\,\xi_{10}^{(-1)}+q^{2}\xi_{01}^{(-1)}\right]=q^{2}\xi_{00}^{(-1)}-\xi_{0}^{(0)}\,(p-q)+\xi_{0}^{(0)}\,(p)\,. \tag{3.155}$$

One might think these reductions have some redundancies since they involve the same functions, so introducing all of them would be unnecessary. In truth, they correspond to different properties attributed to the same object. That is an important feature we will address when computing the Feynman integrals. To illustrate, observe that both cases derived above consist of momenta contractions over one vector:

$$\begin{aligned}
p^{\mu}\left(p_{\mu}\xi_{10}^{(-1)}+q_{\mu}\xi_{01}^{(-1)}\right) &\quad\rightarrow\quad \left[p^{2}\xi_{10}^{(-1)}+(p\cdot q)\,\xi_{01}^{(-1)}\right],\\
q^{\mu}\left(p_{\mu}\xi_{10}^{(-1)}+q_{\mu}\xi_{01}^{(-1)}\right) &\quad\rightarrow\quad \left[(p\cdot q)\,\xi_{10}^{(-1)}+q^{2}\xi_{01}^{(-1)}\right].
\end{aligned}$$

Aiming for reductions involving $a+b=2$, multiply each relation from Equations

(3.148)-(3.149) by each Feynman parameter. Hence, following the line of reasoning employed in previous cases yields

$$2\left[p^2\xi_{20}^{(-1)} + (p\cdot q)\,\xi_{11}^{(-1)}\right] = p^2\xi_{10}^{(-1)} + \xi_{00}^{(0)} - \frac{1}{2}\xi_0^{(0)}\,(p-q)\,, \tag{3.156}$$

$$2\left[p^2\xi_{11}^{(-1)} + (p\cdot q)\,\xi_{02}^{(-1)}\right] = p^2\xi_{01}^{(-1)} - \frac{1}{2}\xi_0^{(0)}\,(p-q) + \frac{1}{2}\xi_0^{(0)}\,(q)\,, \tag{3.157}$$

$$2\left[(p\cdot q)\,\xi_{20}^{(-1)} + q^2\xi_{11}^{(-1)}\right] = q^2\xi_{10}^{(-1)} - \frac{1}{2}\xi_0^{(0)}\,(p-q) + \frac{1}{2}\xi_0^{(0)}\,(p)\,, \tag{3.158}$$

$$2\left[(p\cdot q)\,\xi_{11}^{(-1)} + q^2\xi_{02}^{(-1)}\right] = q^2\xi_{01}^{(-1)} + \xi_{00}^{(0)} - \frac{1}{2}\xi_0^{(0)}\,(p-q)\,. \tag{3.159}$$

Although the function $\xi_1^{(0)}$ appears with this procedure, we have already performed its reduction (3.143).

Observe that the $\xi_{00}^{(0)}$ emerged as compensation for integration by parts. Starting from the expression

$$\frac{1}{2} = \int_0^1 dz \int_0^{1-z} dy \frac{Q\,(y,z)}{Q\,(y,z)},$$

let us expand the numerator to identify some reductions above. Their replacement produces a similar relation

$$2\xi_{00}^{(0)} = p^2\xi_{10}^{(-1)} + q^2\xi_{01}^{(-1)} + \xi_0^{(0)}\,(p-q) - 2m^2\xi_{00}^{(-1)} - 1. \tag{3.160}$$

We stress two unusual contributions here, i.e., the term proportional to the squared mass and the constant. They will play relevant roles in this investigation, so we return to them eventually.

3.3.4 Finite Structure Functions - Part III

The last part of this discussion surveys structures related to Feynman integrals depending on four internal lines (3.122), which consist of three families of finite functions:

$$\xi_{abc}^{(-2)}\,(p,q,r) = \int_0^1 dz \int_0^{1-z} dy \int_0^{1-y-z} dx\; x^c y^b z^a \frac{1}{\left[Q\,(x,y,z)\right]^2}, \tag{3.161}$$

$$\xi_{abc}^{(-1)}\,(p,q,r) = \int_0^1 dz \int_0^{1-z} dy \int_0^{1-y-z} dx\; x^c y^b z^a \frac{1}{Q\,(x,y,z)}, \tag{3.162}$$

$$\xi_{abc}^{(0)}\,(p,q,r) = \int_0^1 dz \int_0^{1-z} dy \int_0^{1-y-z} dx\; x^c y^b z^a \ln\frac{Q\,(x,y,z)}{-\lambda^2}. \tag{3.163}$$

Although they depend on the mass and possibly the scale, our notation omits this information. These functions contain a new polynomial depending on three Feynman parameters

$\{z,\, y,\, x\}$, whose expression is

$$Q\left(x,y,z\right) = p^{2}x\left(1-x\right)+q^{2}y\left(1-y\right)+r^{2}z\left(1-z\right) -2\left(p\cdot q\right)xy-2\left(q\cdot r\right)yz-2\left(p\cdot r\right)xz-m^{2}, \tag{3.164}$$

where $p=k_1-k_2$, $q=k_1-k_3$, and $r=k_1-k_4$.

Since reducing combinations of finite functions is our primary objective, we perform derivatives of this polynomial to build up relations among parameters

$$\left[p^{2}x+\left(p\cdot q\right)y+\left(p\cdot r\right)z\right] = \frac{1}{2}p^{2}-\frac{1}{2}\frac{\partial Q\left(x,y,z\right)}{\partial x}, \tag{3.165}$$

$$\left[\left(p\cdot q\right)x+q^{2}y+\left(q\cdot r\right)z\right] = \frac{1}{2}q^{2}-\frac{1}{2}\frac{\partial Q\left(x,y,z\right)}{\partial y}, \tag{3.166}$$

$$\left[\left(p\cdot r\right)x+\left(q\cdot r\right)y+r^{2}z\right] = \frac{1}{2}r^{2}-\frac{1}{2}\frac{\partial Q\left(x,y,z\right)}{\partial z}. \tag{3.167}$$

These fundamental elements shape results when inserting the adequate multiplicative factors and performing the integration. The first term on the right-hand side represents a decrease in the parameter power. Identifications of four-point functions are straightforward in this procedure, even when they come from integration by parts.

On the other hand, evaluating the other terms might require permutations among derivatives and integrals. The Leibniz rule for differentiation under the integral sign (3.153) applies in these cases. Beforehand, we summarize these possibilities through the following set of rules:

$$\int_0^1 dz\int_0^{1-z}dy\int_0^{1-y-z}dx\;\frac{\partial}{\partial x}f\left(x,y,z\right) = \int_0^1 dz\int_0^{1-z}dy\;f\left(1-y-z,y,z\right)-\int_0^1 dz\int_0^{1-z}dy\;f\left(0,y,z\right), \tag{3.168}$$

$$\int_0^1 dz\int_0^{1-z}dy\int_0^{1-y-z}dx\;\frac{\partial}{\partial y}f\left(x,y,z\right) = \int_0^1 dz\int_0^{1-z}dy\;f\left(1-y-z,y,z\right)-\int_0^1 dz\int_0^{1-z}dx\;f\left(x,0,z\right), \tag{3.169}$$

$$\int_0^1 dz\int_0^{1-z}dy\int_0^{1-y-z}dx\;\frac{\partial}{\partial z}f\left(x,y,z\right) = \int_0^1 dz\int_0^{1-z}dy\;f\left(1-y-z,y,z\right)-\int_0^1 dy\int_0^{1-y}dx\;f\left(x,y,0\right). \tag{3.170}$$

We use them to suppress derivatives and then find quantities considered typical of calculations related to three-point amplitudes. Even though they only admit dependence on two external momenta, three are available in the box context. That means different momenta configurations appear mixed inside each reduction. Below, we reintroduce the line notation (3.112) by considering these new ingredients. All equations above exhibit the same type of object as the first term on the right-hand side, whose identifications lead to ξ'''_{ab}-like functions. In contrast, the second term varies; the associations occur respectively with ξ''_{ab}, ξ'_{ab}, and ξ_{ab}.

$$\begin{cases} \xi_{ab} \rightarrow \xi_{ab}\left(p,q\right) \\ \xi'_{ab} \rightarrow \xi_{ab}\left(p,r\right) \\ \xi''_{ab} \rightarrow \xi_{ab}\left(q,r\right) \\ \xi'''_{ab} \rightarrow \xi_{ab}\left(q-p,r-p\right) \end{cases} \tag{3.171}$$

Without further delay, we cast the required reductions in the sequence. Their presentation is divided accordingly with the sum of parameter powers, while subdivisions indicate the relation used in each calculation (3.165)-(3.167).

- Constraint $a+b+c=1$ - Functions $\xi^{(-2)}_{abc}$ are typical of four-dimensional calculations, appearing within all Feynman integrals involving four propagators. For this constraint, one considers the structure Q^{-2} in the relations.

First relation

$$2\left[p^2\xi^{(-2)}_{100}+(p\cdot q)\,\xi^{(-2)}_{010}+(p\cdot r)\,\xi^{(-2)}_{001}\right]=p^2\xi^{(-2)}_{000}+\left[\xi^{(-1)}_{00}\right]'''-\left[\xi^{(-1)}_{00}\right]'' \tag{3.172}$$

Second relation

$$2\left[(p\cdot q)\,\xi^{(-2)}_{100}+q^2\xi^{(-2)}_{010}+(q\cdot r)\,\xi^{(-2)}_{001}\right]=q^2\xi^{(-2)}_{000}+\left[\xi^{(-1)}_{00}\right]'''-\left[\xi^{(-1)}_{00}\right]' \tag{3.173}$$

Third relation

$$2\left[(p\cdot r)\,\xi^{(-2)}_{100}+(q\cdot r)\,\xi^{(-2)}_{010}+r^2\xi^{(-2)}_{001}\right]=r^2\xi^{(-2)}_{000}+\left[\xi^{(-1)}_{00}\right]'''-\left[\xi^{(-1)}_{00}\right] \tag{3.174}$$

- Constraint $a+b+c=2$ - Besides the structure Q^{-2}, multiplicative factors also consider each of the Feynman parameters $\{x,\,y,\,z\}$. We adopt the symbol $\xi^{(-1)}_{\mathbf{one}}=\xi^{(-1)}_{00}-\xi^{(-1)}_{10}-\xi^{(-1)}_{01}$ to improve the visualization.

First relation

$$2\left[p^2\xi^{(-2)}_{200}+(p\cdot q)\,\xi^{(-2)}_{110}+(p\cdot r)\,\xi^{(-2)}_{101}\right] = p^2\xi^{(-2)}_{100}-\xi^{(-1)}_{000}+\left[\xi^{(-1)}_{\mathbf{one}}\right]''' \tag{3.175}$$

$$2\left[p^2\xi^{(-2)}_{110}+(p\cdot q)\,\xi^{(-2)}_{020}+(p\cdot r)\,\xi^{(-2)}_{011}\right] = p^2\xi^{(-2)}_{010}+\left[\xi^{(-1)}_{10}\right]'''-\left[\xi^{(-1)}_{10}\right]'' \tag{3.176}$$

$$2\left[p^2\xi^{(-2)}_{101}+(p\cdot q)\,\xi^{(-2)}_{011}+(p\cdot r)\,\xi^{(-2)}_{002}\right] = p^2\xi^{(-2)}_{001}+\left[\xi^{(-1)}_{01}\right]'''-\left[\xi^{(-1)}_{01}\right]'' \tag{3.177}$$

Second relation

$$2\left[(p\cdot q)\,\xi^{(-2)}_{200}+q^2\xi^{(-2)}_{110}+(q\cdot r)\,\xi^{(-2)}_{101}\right] = q^2\xi^{(-2)}_{100}-\left[\xi^{(-1)}_{10}\right]'+\left[\xi^{(-1)}_{\mathbf{one}}\right]''' \quad (3.178)$$
$$2\left[(p\cdot q)\,\xi^{(-2)}_{110}+q^2\xi^{(-2)}_{020}+(q\cdot r)\,\xi^{(-2)}_{011}\right] = q^2\xi^{(-2)}_{010}-\xi^{(-1)}_{000}+\left[\xi^{(-1)}_{10}\right]''' \quad (3.179)$$
$$2\left[(p\cdot q)\,\xi^{(-2)}_{101}+q^2\xi^{(-2)}_{011}+(q\cdot r)\,\xi^{(-2)}_{002}\right] = q^2\xi^{(-2)}_{001}+\left[\xi^{(-1)}_{01}\right]'''-\left[\xi^{(-1)}_{01}\right]' \quad (3.180)$$

Third relation

$$2\left[(p\cdot r)\,\xi^{(-2)}_{200}+(q\cdot r)\,\xi^{(-2)}_{110}+r^2\xi^{(-2)}_{101}\right] = r^2\xi^{(-2)}_{100}+\left[\xi^{(-1)}_{\mathbf{one}}\right]'''-\left[\xi^{(-1)}_{10}\right] \quad (3.181)$$
$$2\left[(p\cdot r)\,\xi^{(-2)}_{110}+(q\cdot r)\,\xi^{(-2)}_{020}+r^2\xi^{(-2)}_{011}\right] = r^2\xi^{(-2)}_{010}+\left[\xi^{(-1)}_{10}\right]'''-\left[\xi^{(-1)}_{01}\right] \quad (3.182)$$
$$2\left[(p\cdot r)\,\xi^{(-2)}_{101}+(q\cdot r)\,\xi^{(-2)}_{011}+r^2\xi^{(-2)}_{002}\right] = r^2\xi^{(-2)}_{001}-\xi^{(-1)}_{000}+\left[\xi^{(-1)}_{01}\right]''' \quad (3.183)$$

- Constraint $a+b+c=3$ - Besides the structure Q^{-2}, multiplicative factors also consider each of the combinations $\{x^2,\,xy,\,xz,\,y^2,\,yz,\,z^2\}$. Since they appear when computing tensor integrals, $\xi^{(-1)}_{abc}$-type functions are considered here and correspond to the structure Q^{-1}. This time, we adopt the symbols $\xi^{(-1)}_{\mathbf{one}}=\xi^{(-1)}_{00}-\xi^{(-1)}_{10}-\xi^{(-1)}_{01}$ and $\xi^{(-1)}_{\mathbf{two}}=\xi^{(-1)}_{00}-2\xi^{(-1)}_{10}-2\xi^{(-1)}_{01}+2\xi^{(-1)}_{11}+\xi^{(-1)}_{20}+\xi^{(-1)}_{02}$ to improve the visualization.

First relation

$$2\left[p^2\xi^{(-2)}_{300}+(p\cdot q)\,\xi^{(-2)}_{210}+(p\cdot r)\,\xi^{(-2)}_{201}\right] = p^2\xi^{(-2)}_{200}-2\xi^{(-1)}_{100}+\left[\xi^{(-1)}_{\mathbf{two}}\right]''' \quad (3.184)$$
$$2\left[p^2\xi^{(-2)}_{210}+(p\cdot q)\,\xi^{(-2)}_{120}+(p\cdot r)\,\xi^{(-2)}_{111}\right] = p^2\xi^{(-2)}_{110}-\xi^{(-1)}_{010}+\left[\xi^{(-1)}_{\mathbf{one}}\right]''' \quad (3.185)$$
$$2\left[p^2\xi^{(-2)}_{201}+(p\cdot q)\,\xi^{(-2)}_{111}+(p\cdot r)\,\xi^{(-2)}_{102}\right] = p^2\xi^{(-2)}_{101}-\xi^{(-1)}_{001}+\left[\xi^{(-1)}_{\mathbf{one}}\right]''' \quad (3.186)$$
$$2\left[p^2\xi^{(-2)}_{120}+(p\cdot q)\,\xi^{(-2)}_{030}+(p\cdot r)\,\xi^{(-2)}_{021}\right] = p^2\xi^{(-2)}_{020}+\left[\xi^{(-1)}_{20}\right]'''-\left[\xi^{(-1)}_{20}\right]'' \quad (3.187)$$
$$2\left[p^2\xi^{(-2)}_{111}+(p\cdot q)\,\xi^{(-2)}_{021}+(p\cdot r)\,\xi^{(-2)}_{012}\right] = p^2\xi^{(-2)}_{011}+\left[\xi^{(-1)}_{11}\right]'''-\left[\xi^{(-1)}_{11}\right]'' \quad (3.188)$$
$$2\left[p^2\xi^{(-2)}_{102}+(p\cdot q)\,\xi^{(-2)}_{012}+(p\cdot r)\,\xi^{(-2)}_{003}\right] = p^2\xi^{(-2)}_{002}+\left[\xi^{(-1)}_{02}\right]'''-\left[\xi^{(-1)}_{02}\right]'' \quad (3.189)$$
$$2\left[p^2\xi^{(-1)}_{100}+(p\cdot q)\,\xi^{(-1)}_{010}+(p\cdot r)\,\xi^{(-1)}_{001}\right] = p^2\xi^{(-1)}_{000}-\left[\xi^{(0)}_{00}\right]'''+\left[\xi^{(0)}_{00}\right]'' \quad (3.190)$$

Second relation

$$2\left[(p\cdot q)\,\xi_{300}^{(-2)}+q^2\xi_{210}^{(-2)}+(r\cdot q)\,\xi_{201}^{(-2)}\right] = q^2\xi_{200}^{(-2)}+\left[\xi_{\mathbf{two}}^{(-1)}\right]'''-\left[\xi_{20}^{(-1)}\right]' \tag{3.191}$$

$$2\left[(p\cdot q)\,\xi_{210}^{(-2)}+q^2\xi_{120}^{(-2)}+(r\cdot q)\,\xi_{111}^{(-2)}\right] = q^2\xi_{110}^{(-2)}-\xi_{100}^{(-1)}+\left[\xi_{\mathbf{one}}^{(-1)}\right]''' \tag{3.192}$$

$$2\left[(p\cdot q)\,\xi_{201}^{(-2)}+q^2\xi_{111}^{(-2)}+(r\cdot q)\,\xi_{102}^{(-2)}\right] = q^2\xi_{101}^{(-2)}+\left[\xi_{\mathbf{one}}^{(-1)}\right]'''-\left[\xi_{11}^{(-1)}\right]' \tag{3.193}$$

$$2\left[(p\cdot q)\,\xi_{120}^{(-2)}+q^2\xi_{030}^{(-2)}+(r\cdot q)\,\xi_{021}^{(-2)}\right] = q^2\xi_{020}^{(-2)}-2\xi_{010}^{(-1)}+\left[\xi_{20}^{(-1)}\right]''' \tag{3.194}$$

$$2\left[(p\cdot q)\,\xi_{111}^{(-2)}+q^2\xi_{021}^{(-2)}+(r\cdot q)\,\xi_{012}^{(-2)}\right] = q^2\xi_{011}^{(-2)}-\xi_{001}^{(-1)}+\left[\xi_{11}^{(-1)}\right]''' \tag{3.195}$$

$$2\left[(p\cdot q)\,\xi_{102}^{(-2)}+q^2\xi_{012}^{(-2)}+(r\cdot q)\,\xi_{003}^{(-2)}\right] = q^2\xi_{002}^{(-2)}+\left[\xi_{02}^{(-1)}\right]'''-\left[\xi_{02}^{(-1)}\right]' \tag{3.196}$$

$$2\left[(p\cdot q)\,\xi_{100}^{(-1)}+q^2\xi_{010}^{(-1)}+(r\cdot q)\,\xi_{001}^{(-1)}\right] = q^2\xi_{000}^{(-1)}-\left[\xi_{00}^{(0)}\right]'''+\left[\xi_{00}^{(0)}\right]' \tag{3.197}$$

Third relation

$$2\left[(p\cdot r)\,\xi_{300}^{(-2)}+(q\cdot r)\,\xi_{210}^{(-2)}+r^2\xi_{201}^{(-2)}\right] = r^2\xi_{200}^{(-2)}+\left[\xi_{\mathbf{two}}^{(-1)}\right]'''-\left[\xi_{20}^{(-1)}\right] \tag{3.198}$$

$$2\left[(p\cdot r)\,\xi_{210}^{(-2)}+(q\cdot r)\,\xi_{120}^{(-2)}+r^2\xi_{111}^{(-2)}\right] = r^2\xi_{110}^{(-2)}+\left[\xi_{\mathbf{one}}^{(-1)}\right]'''-\left[\xi_{11}^{(-1)}\right] \tag{3.199}$$

$$2\left[(p\cdot r)\,\xi_{201}^{(-2)}+(q\cdot r)\,\xi_{111}^{(-2)}+r^2\xi_{102}^{(-2)}\right] = r^2\xi_{101}^{(-2)}-\xi_{100}^{(-1)}+\left[\xi_{\mathbf{one}}^{(-1)}\right]''' \tag{3.200}$$

$$2\left[(p\cdot r)\,\xi_{120}^{(-2)}+(q\cdot r)\,\xi_{030}^{(-2)}+r^2\xi_{021}^{(-2)}\right] = r^2\xi_{020}^{(-2)}+\left[\xi_{20}^{(-1)}\right]'''-\left[\xi_{02}^{(-1)}\right] \tag{3.201}$$

$$2\left[(p\cdot r)\,\xi_{111}^{(-2)}+(q\cdot r)\,\xi_{021}^{(-2)}+r^2\xi_{012}^{(-2)}\right] = r^2\xi_{011}^{(-2)}-\xi_{010}^{(-1)}+\left[\xi_{11}^{(-1)}\right]''' \tag{3.202}$$

$$2\left[(p\cdot r)\,\xi_{102}^{(-2)}+(q\cdot r)\,\xi_{012}^{(-2)}+r^2\xi_{003}^{(-2)}\right] = r^2\xi_{002}^{(-2)}-2\xi_{001}^{(-1)}+\left[\xi_{02}^{(-1)}\right]''' \tag{3.203}$$

$$2\left[(p\cdot r)\,\xi_{100}^{(-1)}+(q\cdot r)\,\xi_{010}^{(-1)}+r^2\xi_{001}^{(-1)}\right] = r^2\xi_{000}^{(-1)}-\left[\xi_{00}^{(0)}\right]'''+\left[\xi_{00}^{(0)}\right] \tag{3.204}$$

Analogously to the $\xi_{00}^{(0)}$, whose analysis was developed in Equation (3.160), expressing ξ_{abc}^{-1}-like functions in terms of ξ_{abc}^{-2}-like functions is convenient. To accomplish this task, employ $Q^{-1}=Q^{-2}Q$ as a link between both families. Following the procedure from the referred case and using $\xi_{\mathbf{one}}^{(-1)}=\xi_{00}^{(-1)}-\xi_{10}^{(-1)}-\xi_{01}^{(-1)}$, we obtain the relations that concern this investigation:

$$\xi_{000}^{(-1)} = 2m^2\xi_{000}^{(-2)}-\left[p^2\xi_{100}^{(-2)}+q^2\xi_{010}^{(-2)}+r^2\xi_{001}^{(-2)}\right]+\left[\xi_{00}^{(-1)}\right]''', \tag{3.205}$$

$$2\xi_{100}^{(-1)} = 2m^2\xi_{100}^{(-2)}-\left[p^2\xi_{200}^{(-2)}+q^2\xi_{110}^{(-2)}+r^2\xi_{101}^{(-2)}\right]+\left[\xi_{\mathbf{one}}^{(-1)}\right]''', \tag{3.206}$$

$$2\xi_{010}^{(-1)} = 2m^2\xi_{010}^{(-2)}-\left[p^2\xi_{110}^{(-2)}+q^2\xi_{020}^{(-2)}+r^2\xi_{011}^{(-2)}\right]+\left[\xi_{10}^{(-1)}\right]''', \tag{3.207}$$

$$2\xi_{001}^{(-1)} = 2m^2\xi_{001}^{(-2)}-\left[p^2\xi_{101}^{(-2)}+q^2\xi_{011}^{(-2)}+r^2\xi_{002}^{(-2)}\right]+\left[\xi_{01}^{(-1)}\right]'''. \tag{3.208}$$

3.4 Explicit Perturbative Amplitudes

After understanding the structure of correlators at the integrand level, we developed a strategy to deal with divergences associated with their integration. The objective of this section is to perform this operation explicitly. For each case, the first step is evaluating Feynman integrals since these are the fundamental pieces that build up the investigated objects. Subsequently, we obtain standard tensors and perturbative amplitudes hitherto identified.

3.4.1 Two-Point Amplitudes - Feynman Integrals and AV

Our task is to compute quantities introduced in Subsection (3.2.2), with a particular interest in the AV correlator. That is also the opportunity to elucidate elements related to the strategy. After detailing the procedure for the separation, we organize ill-defined mathematical quantities through standard divergent objects. Posteriorly, we evaluate finite contributions using common tools of perturbative calculations, such as Feynman parametrizations and finite loop integration. One might consult further information about these resources in introductory books on quantum field theories [49].

We achieved the AV structure in Equation (3.48) through a contraction with the standard tensor (3.40). Considering the antisymmetric character of the Levi-Civita symbol, the simplified integrand arises

$$t_{\mu\nu}^{AV} = 4i\varepsilon_{\mu\nu\alpha\beta}\left[k_1^{\alpha}k_2^{\beta}\frac{1}{D_{12}} + (k_1-k_2)^{\alpha}\,\frac{k^{\beta}}{D_{12}}\right]. \tag{3.209}$$

Denoted by an uppercase letter, the amplitude combines the following two-propagator Feynman integrals (3.120):

$$T_{\mu\nu}^{AV} = 4i\varepsilon_{\mu\nu\alpha\beta}\left[k_1^{\alpha}k_2^{\beta}I_2 + (k_1-k_2)^{\alpha}\,I_2^{\beta}\right]. \tag{3.210}$$

Since this expression exhibits a divergent power counting, we adopt a prescription to propagator-like objects D_n through identity (3.129). The separation is successful if the identity considers N as equal to or higher than the power counting of the integral. Thus, $N=2$ would be a logical option as two-point amplitudes have quadratic power counting in the physical dimension. Nevertheless, we acknowledged simplifications due to the antisymmetric character of the AV, which allows using the $N=1$ version. Although both routes lead to the same outcome, the first generates more finite contributions and involves more laborious calculations.

Alternatively, one might also evaluate Feynman integrals separately, adopting versions for the identity as it finds suitable. We opt for this route because these integrals also emerge within higher-order amplitudes. For instance, as the I_2 integral exhibits logarithmic power counting when integrated, employing the $N = 0$ identity version rewrites the propagator-like structure D_1 and splits its integrand as follows

$$\frac{1}{D_{12}} = \left[\frac{1}{D_\lambda} - \frac{A_1}{D_\lambda D_1}\right]\frac{1}{D_2}, \tag{3.211}$$

where denominators involve $D_n = (k + k_n)^2 - m^2$ and $D_\lambda = k^2 - \lambda^2$ and numerators exhibit the object $A_n = 2k_n \cdot k + k_n^2 + \lambda^2 - m^2$.

Power counting decreased as required, so the last contribution will generate a finite integral. Nevertheless, the first term still shows diverging power counting (when integrated). Exploring both propagator-like objects is necessary for this term, so divergent objects depend only on non-physical quantities, i.e., the loop momentum k and the scale λ^2. Such a property is intrinsic to the IReg. Therefore, by employing the identity for D_2 within this specific term, the separation assumes the form

$$\frac{1}{D_{12}} - \frac{1}{D_\lambda^2} = -\frac{A_2}{D_\lambda^2 D_2} - \frac{A_1}{D_\lambda D_{12}}. \tag{3.212}$$

This organization puts ill-defined mathematical structures on the left-hand side of equations after integration, so it is transparent that the right-hand side leads to a finite quantity. Therefore, by identifying the irreducible divergent object (3.134), we have the I_2 integral:

$$I_2(k_1, k_2) - I_{\log} = -\int \frac{d^4k}{(2\pi)^4}\left[\frac{A_2}{D_\lambda^2 D_2} + \frac{A_1}{D_\lambda D_{12}}\right]. \tag{3.213}$$

Our next task is to compute the finite part; however, dealing with products in the denominators is inconvenient. One generally rewrites these structures through Feynman parametrizations to avoid such circumstances. This resource expresses rational functions in terms of an integral representation; observe the examples:

$$\frac{1}{ab} = \int_0^1 dz \frac{1}{[(b-a)z + a]^2}, \tag{3.214}$$

$$\frac{1}{abc} = 2\int_0^1 dz \int_0^{1-z} dy \frac{1}{[(b-a)y + (c-a)z + a]^3}, \tag{3.215}$$

$$\frac{1}{abcd} = 6\int_0^1 dz \int_0^{1-z} dy \int_0^{1-y-z} dx \frac{1}{[(b-a)x + (c-a)y + (d-a)z + a]^4}, \tag{3.216}$$

where x, y, and z are parameters. Variations employed here emerge through derivatives with respect to a, which increases its power on the left-hand side.

Let us clarify this subject by adopting a variation of (3.214) to express the first finite contribution from the scalar integral (3.213). After replacing $a = D_\lambda$ and $b = D_2$, we group terms on the loop momentum by completing the square

$$-\int\frac{d^4k}{(2\pi)^4}\frac{A_2}{D_\lambda^2 D_2}=-2\int_0^1 dz\,(1-z)\int\frac{d^4k}{(2\pi)^4}\frac{A_2}{\left[(k+k_2z)^2+P_1\right]^3};\tag{3.217}$$

therefore, one polynomial dependent on the arbitrary routing arises

$$P_1(z)=k_2^2 z(1-z)+\left(\lambda^2-m^2\right)z-\lambda^2.\tag{3.218}$$

Performing a shift on the variable $k+k_2z\to k$ makes the denominator momentum-even while generating an additional term in the numerator, which allows identifying a derivative of the polynomial:

$$-\int\frac{d^4k}{(2\pi)^4}\frac{A_2}{D_\lambda^2 D_2}=-2\int_0^1 dz\,(1-z)\int\frac{d^4k}{(2\pi)^4}\left[2k_2^\rho k_\rho+\frac{\partial P_1}{\partial z}\right]\frac{1}{(k^2+P_1)^3}.\tag{3.219}$$

Any finite integral found in one-loop calculations leads to this type of structure after parametrization. Nevertheless, derivatives (and their powers) only appear if the original integral has divergent power counting. The next step consists of the loop integration, which only produces non-zero contributions for even integrands; the case above yields:

$$\int\frac{d^4k}{(2\pi)^4}\frac{1}{(k^2+P_z)^3}=\frac{i}{(4\pi)^2}\frac{1}{2P_1}.\tag{3.220}$$

Posteriorly to replacing this result, one must integrate by parts until all derivatives are eliminated. This case requires a sole operation and leads to the outcome:

$$\begin{aligned}-\int\frac{d^4k}{(2\pi)^4}\frac{A_2}{D_\lambda^2 D_2} &= -\frac{i}{(4\pi)^2}\int_0^1 dz\left[\frac{\partial}{\partial z}(1-z)\ln P_1+\ln P_1\right]\\ &= -\frac{i}{(4\pi)^2}\int_0^1 dz\ln\frac{P_1}{-\lambda^2}.\end{aligned}\tag{3.221}$$

Finite contributions follow a strong pattern since we departed from a logarithmically divergent integral. Each step described above has an analogous form in the second contribution from Equation (3.213). The fundamental difference is in the parametrization (3.216), which involves two propagators and leads to another polynomial

$$\begin{aligned}P_2(z,y) &= k_1^2 y(1-y)+k_2^2 z(1-z)-2k_1\cdot k_2 yz\\ &\quad+\left(\lambda^2-m^2\right)y+\left(\lambda^2-m^2\right)z-\lambda^2.\end{aligned}\tag{3.222}$$

Observe how this dependence reflects on the integration by parts:

$$\begin{aligned} -\int \frac{d^4k}{(2\pi)^4} \frac{A_1}{D_\lambda D_{12}} &= -\frac{i}{(4\pi)^2} \int_0^1 dz \int_0^{1-z} dy \frac{\partial}{\partial y} \ln P_2 \\ &= -\frac{i}{(4\pi)^2} \int_0^1 dz \ln \frac{Q}{P_1}. \end{aligned} \tag{3.223}$$

The lower limit of integration (in $y = 0$) returns the first polynomial; hence, this type of term disappears when summing up the entire sector. Even in more complex cases, finite contributions involving arbitrary routings k_i cancel out identically in a chain effect. Only the term achieved by applying the upper limit of integration (in $y = 1 - z$) contributes in the end. That leads to the dependence on external momentum $p = k_1 - k_2$ acknowledged in Subsection (3.3.2), embodied in the polynomial:

$$Q(z) = p^2 z (1 - z) - m^2. \tag{3.224}$$

With both contributions at our disposal, building up the scalar Feynman integral (3.213) is possible

$$I_2(k_1, k_2) - I_{\log} = -\frac{i}{(4\pi)^2} \xi_0^{(0)}(p), \tag{3.225}$$

where the finite function was identified (3.137). Such an expression clarifies that the parameter λ^2 plays the role of a scale connecting finite and ill-defined quantities. That becomes transparent by setting routings as zero $k_i = 0$ on the equation above:

$$I_{\log}\left(m^2\right) - I_{\log}\left(\lambda^2\right) = -\frac{i}{(4\pi)^2} \ln \frac{m^2}{\lambda^2}. \tag{3.226}$$

This type of scale relation is implicit whenever logarithmic functions are present in this investigation.

After detailing the first case, we directly cast one possible separation linked to the vector integral I_2^β (3.120). Since its power counting indicates linear divergence, let us set $N \leq 1$ in identity (3.129) and employ both versions to achieve the structure

$$\left[\frac{k^\beta}{D_{12}}\right]_{not\ odd} + 2(k_1 + k_2)_\rho \frac{k^\beta k^\rho}{D_\lambda^3} = \frac{A_2(A_1 + A_2) k^\beta}{D_\lambda^3 D_2} + \frac{A_1^2 k^\beta}{D_\lambda^2 D_{12}}. \tag{3.227}$$

We disregard momentum-odd terms since they vanish with the loop integration. Again, the adopted arrangement puts ill-defined structures on the left-hand side:

$$I_2^\beta(k_1, k_2) + \frac{1}{2}(k_1 + k_2)_\rho \left(\Delta^{\beta\rho} + g^{\beta\rho} I_{\log}\right) = \frac{i}{(4\pi)^2} \frac{1}{2} (k_1 + k_2)^\beta \xi_0^{(0)}(p). \tag{3.228}$$

We employed the irreducible object (3.134) and the 2nd-order surface term (3.133) to organize the divergent sector. Finite contributions lead to the family (3.137); we also employed the reduction of finite functions $\xi_1 = \frac{1}{2}\xi_0$, achieved initially in (3.143).

Lastly, let us employ the achieved integrals to build the AV amplitude (3.210). Finite contributions and irreducible divergent objects cancel out identically after using the identity $\varepsilon_{\mu\nu\alpha\beta}(k_1 - k_2)^\alpha (k_1 + k_2)^\beta = 2\varepsilon_{\mu\nu\alpha\beta}k_1^\alpha k_2^\beta$. Hence, the only non-trivial contribution is the following

$$T_{\mu\nu}^{AV} = -2i\varepsilon_{\mu\nu\alpha\beta}p^\alpha (k_1 + k_2)_\rho \Delta^{\beta\rho}. \tag{3.229}$$

That agrees with the expectation from Equation (3.49), i.e., it is a surface term proportional to an arbitrary momenta combination.

Observing this expression isolated, one might expect that restricting arbitrary labels (as in $k_2 = -k_1$) would eliminate surface terms and solve issues approached while exploring symmetry aspects. Nevertheless, that is not enough when considering the complete discussion. For this reason, we maintain the arbitrariness associated with labels, so the analysis falls over values accessible to surface terms.

3.4.2 Three-Point Amplitudes - Feynman Integrals

Our next objective is to compute quantities typical of calculations involving three-point correlators, starting with the corresponding Feynman integrals (3.121). Afterward, we evaluate standard tensors and subamplitudes necessary to build the main targets: PVV and AVV. Since some ingredients also appear when exploring four-point structures, we broaden their discussion.

As the first couple of integrals is finite, dependence on external momenta appears from the beginning. In other words, when employing the Feynman parametrization (3.215) and grouping terms on the loop momentum, denominators exhibit polynomial (3.147):

$$\frac{[1, k_\mu]}{D_{123}} = 2\int_0^1 dz \int_0^{1-z} dy \frac{[1, k_\mu]}{\left[(k + k_1 - py - qz)^2 + Q(y,z)\right]^3}. \tag{3.230}$$

Then, integrating both sides of the scalar version of this equation yields the first integral. No compensation term appears by shifting the momentum $k + k_1 - py - qz \to k$; hence, obtaining this result is straightforward

$$I_3 = \frac{i}{(4\pi)^2}\int_0^1 dz \int_0^{1-z} dy \, \frac{1}{Q(y,z)}. \tag{3.231}$$

Although this reasoning extends to the vector version, the momentum shift brings parameter powers to its numerator

$$I_{3\mu} = -\frac{i}{(4\pi)^2} \int_0^1 dz \int_0^{1-z} dy \ \frac{(k_1 - py - qz)_\mu}{Q(y,z)}. \tag{3.232}$$

Lastly, the 2nd-order tensor is the only integral exhibiting logarithmically diverging power counting here. We split its integrand by employing the $N = 0$ identity version (3.129) whenever necessary

$$\frac{k_{\mu\nu}}{D_{123}} - \frac{k_{\mu\nu}}{D_\lambda^3} = -\frac{A_3 k_{\mu\nu}}{D_\lambda^3 D_3} - \frac{A_2 k_{\mu\nu}}{D_\lambda^2 D_{23}} - \frac{A_1 k_{\mu\nu}}{D_\lambda D_{123}}. \tag{3.233}$$

Terms associated with ill-defined contributions are on the left-hand side, so we use standard objects introduced in Equations (3.133)-(3.134) to express them without additional manipulations.

Regarding finite contributions, each rational function requires a different Feynman parametrization. Although they lead to structures similar to those above, polynomials depend on non-physical parameters this time. Furthermore, momentum shifts induce derivatives of these polynomials in the numerators, requiring integrations by parts. When completing this procedure, most contributions fit perfectly, and only those depending on external momenta remain:

$$\begin{aligned} & I_{3\mu\nu} - \tfrac{1}{4}\left(\Delta_{\mu\nu} + g_{\mu\nu} I_{\log}\right) \\ = \; & \frac{i}{(4\pi)^2} \int_0^1 dz \int_0^{1-z} dy \ \ (k_1 - py - qz)_\mu (k_1 - py - qz)_\nu \frac{1}{Q(y,z)} \\ & -\frac{i}{(4\pi)^2} \frac{1}{2} g_{\mu\nu} \int_0^1 dz \int_0^{1-z} dy \ \ln \frac{Q(y,z)}{-\lambda^2}. \end{aligned} \tag{3.234}$$

The final step for evaluating these Feynman integrals is to project finite contributions in terms of structure functions from the families (3.145)-(3.146). Having two parameters highlights some patterns, which clarifies that these functions do not appear randomly but in tensors having well-defined properties. We mentioned them in Subsection (3.3.3). Then, following the identifications, we group terms depending exclusively on external momenta into what we call J-tensors. Other contributions correspond to lower-order Feynman integrals having combinations of the routing k_1 as coefficients. Such reasoning materializes in the following organization

$$I_3 = J_3, \tag{3.235}$$

$$I_{3\mu} = J_{3\mu} - [k_{1\mu} I_3], \tag{3.236}$$

$$I_{3\mu\nu} - \tfrac{1}{4}\left(\Delta_{\mu\nu} + g_{\mu\nu} I_{\log}\right) = J_{3\mu\nu} - [k_{1\mu} I_{3\nu} + k_{1\nu} I_{3\mu}] - [k_{1\mu} k_{1\nu} I_3], \tag{3.237}$$

where J-tensors are introduced

$$J_3 = \frac{i}{(4\pi)^2}\xi_{00}^{(-1)}, \tag{3.238}$$

$$J_{3\mu} = \frac{i}{(4\pi)^2}\left[p_\mu \xi_{10}^{(-1)} + q_\mu \xi_{01}^{(-1)}\right], \tag{3.239}$$

$$J_{3\mu\nu} = \frac{i}{(4\pi)^2}\left[p_\mu p_\nu \xi_{20}^{(-1)} + q_\mu q_\nu \xi_{02}^{(-1)} + (p_\mu q_\nu + q_\mu p_\nu)\,\xi_{11}^{(-1)} - \tfrac{1}{2} g_{\mu\nu}\xi_{00}^{(0)}\right]. \tag{3.240}$$

Using these tensors to express mathematical structures appearing in perturbative calculations is already very useful. They are introduced in reference [47] as part of the systematization from IReg, where they allow a compact presentation of the quadruple-vector box amplitude.

Although that is part of their purpose here, we stress their remarkable value regarding algebraic manipulations and interpretation of results. Since J-tensors concentrate all contributions on external momenta, they are enough to describe the finite part of physical amplitudes. We consider this systematization to propose a new perspective, where J-tensors are the fundamental pieces in this analysis. When computing momenta contractions, for instance, the discussion resorts to their properties as a generalization of reductions from Section (3.3). Without further delay, let us employ these ideas in the study of three-point functions subsequently.

3.4.3 Three-Point Amplitudes - PVV

Before computing the PVV amplitude, our first task is integrating the standard tensor having two momenta in the numerator. By integrating Equation (3.53), we expand products and identify the following combination of Feynman integrals:

$$\begin{aligned} T_{3\mu\nu}^{(s)}(k_i, k_j) &= (1+s)\, I_{3\mu\nu} + (k_j + s k_i)_\nu\, I_{3\mu} \\ &\quad + (k_i + s k_j)_\mu\, I_{3\nu} + (k_{i\mu}k_{j\nu} + s k_{i\nu}k_{j\mu})\, I_3. \end{aligned} \tag{3.241}$$

We adopt general structures, admitting choices for signs and routings. This expression applies to any denominator D_{klm} typical of three-point calculations and extends other cases (e.g., box) by changing the numerical subindex. The same pattern manifests in finite tensors, expressing all finite quantities below. We delimit our focus to the D_{123} case for now.

Since there is an intrinsic idea of hierarchy, we start by replacing the highest-order integral $I_{3\mu\nu}$ (3.237). Divergent objects and the 2nd-order J-tensor are ready; however, this operation brings new contributions through lower-order structures. With this, external momenta $p_i = k_1 - k_i$ appear as multiplicative coefficients of the next integral $I_{3\mu}$ (3.236). Its substitution gives continuity to a chain effect, and now the last integral I_2 (3.235) has this type of coefficient. Once this procedure is over, we achieve the general form

$$\begin{aligned} T^{(s)}_{3\mu\nu}(k_i,k_j) &= \tfrac{1}{4}(1+s)\left(\Delta_{\mu\nu}+g_{\mu\nu}I_{\log}+4J_{3\mu\nu}\right) \\ &\quad -(p_j+sp_i)_\nu J_{3\mu}-(p_i+sp_j)_\mu J_{3\nu}+(p_{i\mu}p_{j\nu}+sp_{i\nu}p_{j\mu})J_3. \end{aligned} \tag{3.242}$$

This procedure is generic, so we resort to it when examining all standard tensors. We recall that the object p_i produces three possibilities here: $p_1=0$, $p_2=k_1-k_2=p$, and $p_3=k_1-k_3=q$.

We aim to build the PVV (3.54) using this tool, so let us reintroduce its expression by using uppercase letters to characterize the integrated amplitude

$$T^{PVV}_{\mu\nu}=-2im\varepsilon_{\mu\nu XY}\left[T^{(-)}_{3XY}(k_2,k_3)+T^{(-)}_{3XY}(k_3,k_1)+T^{(-)}_{3XY}(k_1,k_2)\right]. \tag{3.243}$$

The minus sign reflects in their antisymmetry property, hence, canceling the first row of the general form (3.242). Then, setting the different momenta arrangements, we cast the required versions:

$$T^{(-)}_{3\mu\nu}(k_1,k_2)=-p_\nu J_{3\mu}+p_\mu J_{3\nu}, \tag{3.244}$$

$$T^{(-)}_{3\mu\nu}(k_3,k_1)=q_\nu J_{3\mu}-q_\mu J_{3\nu}, \tag{3.245}$$

$$T^{(-)}_{3\mu\nu}(k_2,k_3)=(p-q)_\nu J_{3\mu}-(p-q)_\mu J_{3\nu}+(p_\mu q_\nu-p_\nu q_\mu)J_3. \tag{3.246}$$

It is straightforward to sum them to find that these objects collapse into the finite function

$$T^{PVV}_{\mu\nu}=-4im\varepsilon_{\mu\nu XY}p^X q^Y J_3, \tag{3.247}$$

which agrees with the expectation from Equation (3.55).

3.4.4 Three-Point Amplitudes - AVV

Our next target is the AVV triangle (3.62), which contains a tensor sector besides the vector subamplitude VPP. Given the procedure introduced in the previous case, let us begin this discussion by writing the integrated form of the 3rd-order standard tensor (3.61) through Feynman integrals

$$\begin{aligned} T^{(-)}_{3\mu;\nu\alpha}(k_l;k_i,k_j) &= (k_j-k_i)_\alpha I_{3\mu\nu}+(k_i-k_j)_\nu I_{3\mu\alpha} \\ &\quad +(k_{j\alpha}k_{i\nu}-k_{i\alpha}k_{j\nu})I_{3\mu}+(k_j-k_i)_\alpha k_{l\mu}I_{3\nu} \\ &\quad +(k_i-k_j)_\nu k_{l\mu}I_{3\alpha}+(k_{j\alpha}k_{i\nu}-k_{i\alpha}k_{j\nu})k_{l\mu}I_3. \end{aligned} \tag{3.248}$$

We restricted this equation to the minus sign because only antisymmetric tensors appear throughout this investigation. That also comprehends the four-propagator version, achieved by changing numerical subindices.

Replacements start with the highest-order integral and follow a hierarchy until getting to the lowest. Ultimately, finite contributions depend exclusively on external momenta $p_i = k_1 - k_i$ since terms associated with k_1 combinations vanish identically:

$$\begin{aligned} T^{(-)}_{3\mu;\nu\alpha}(k_l;k_i,k_j) &= -\tfrac{1}{4}\left[(p_j-p_i)_\alpha \Delta_{\mu\nu} + (p_i-p_j)_\nu \Delta_{\mu\alpha}\right] \\ &\quad -\tfrac{1}{4}\left[(p_j-p_i)_\alpha g_{\mu\nu} + (p_i-p_j)_\nu g_{\mu\alpha}\right] I_{\log} \\ &\quad -(p_j-p_i)_\alpha J_{3\mu\nu} - (p_i-p_j)_\nu J_{3\mu\alpha} + (p_i-p_j)_\nu p_{l\mu} J_{3\alpha} \\ &\quad +(p_j-p_i)_\alpha p_{l\mu} J_{3\nu} + (p_{j\alpha}p_{i\nu} - p_{i\alpha}p_{j\nu}) J_{3\mu} \\ &\quad -(p_{j\alpha}p_{i\nu} - p_{i\alpha}p_{j\nu}) p_{l\mu} J_3. \end{aligned} \tag{3.249}$$

That becomes transparent as a consequence of J-tensors structures. Even though we introduced the scalar J_3 for generality, its coefficient vanishes here due to the unavoidable presence of $p_1 = 0$. As three routings are available, three non-equivalent configurations of this tensor are obtainable:

$$\begin{aligned} T^{(-)}_{3\mu;\nu\alpha}(k_1;k_2,k_3) &= -\tfrac{1}{4}\left[(q-p)_\alpha \Delta_{\mu\nu} + (p-q)_\nu \Delta_{\mu\alpha}\right] \\ &\quad -\tfrac{1}{4}\left[(q-p)_\alpha g_{\mu\nu} + (p-q)_\nu g_{\mu\alpha}\right] I_{\log} \\ &\quad -(q-p)_\alpha J_{3\mu\nu} - (p-q)_\nu J_{3\mu\alpha} + (q_\alpha p_\nu - p_\alpha q_\nu) J_{3\mu}, \end{aligned} \tag{3.250}$$

$$\begin{aligned} T^{(-)}_{3\mu;\nu\alpha}(k_2;k_3,k_1) &= \tfrac{1}{4}(q_\alpha \Delta_{\mu\nu} - q_\nu \Delta_{\mu\alpha}) + \frac{1}{4}(q_\alpha g_{\mu\nu} - q_\nu g_{\mu\alpha}) I_{\log} \\ &\quad +q_\alpha J_{3\mu\nu} - q_\nu J_{3\mu\alpha} + q_\nu p_\mu J_{3\alpha} - q_\alpha p_\mu J_{3\nu}, \end{aligned} \tag{3.251}$$

$$\begin{aligned} T^{(-)}_{3\mu;\nu\alpha}(k_3;k_1,k_2) &= \tfrac{1}{4}(p_\nu \Delta_{\mu\alpha} - p_\alpha \Delta_{\mu\nu}) + \frac{1}{4}(p_\nu g_{\mu\alpha} - p_\alpha g_{\mu\nu}) I_{\log} \\ &\quad -p_\alpha J_{3\mu\nu} + p_\nu J_{3\mu\alpha} - p_\nu q_\mu J_{3\alpha} + p_\alpha q_\mu J_{3\nu}. \end{aligned} \tag{3.252}$$

When looking into integrands, we made expectations regarding these structures (3.65)-(3.67). The main point is the impossibility of building a 3rd-order tensor with the property of total antisymmetry in this particular context. Having all ingredients required for the verifications, we comment on them in the sequence.

First, all terms vanish by contracting the Levi-Civita symbol with the first configuration above since they correspond to products between symmetric and antisymmetric objects. Whereas most cases are straightforward, inspecting the J-vector content (3.239) is necessary for completing this verification

$$\begin{aligned} \varepsilon^{\nu XYZ} T^{(-)}_{3X;YZ}(k_1;k_2,k_3) &= 2\varepsilon^{\nu XYZ} p_Y q_Z J_{3X} \\ &\to \varepsilon^{\nu XYZ} p_Y q_Z \left[p_X \xi^{(-1)}_{10} + q_X \xi^{(-1)}_{01}\right] = 0. \end{aligned} \tag{3.253}$$

Second, all terms cancel out identically when summing these three configurations. Again, that requires a closer look inside the J-vector

$$\begin{aligned}&T^{(-)}_{3\mu;\nu\alpha}\left(k_1;k_2,k_3\right)+T^{(-)}_{3\mu;\nu\alpha}\left(k_2;k_3,k_1\right)+T^{(-)}_{3\mu;\nu\alpha}\left(k_3;k_1,k_2\right)\\ =\;&\left(q_\alpha p_\nu-p_\alpha q_\nu\right)J_{3\mu}+\left(q_\nu p_\mu-p_\nu q_\mu\right)J_{3\alpha}+\left(p_\alpha q_\mu-q_\alpha p_\mu\right)J_{3\nu}=0.\end{aligned}\tag{3.254}$$

As these identities are indeed confirmed, the expectation over the amplitude also applies

$$T^{AVV}_{\mu\nu\alpha}=4i\varepsilon_{\mu\alpha XY}T^{(-)}_{3\nu;XY}\left(k_1;k_2,k_3\right)+4i\varepsilon_{\mu\nu XY}T^{(-)}_{3\alpha;XY}\left(k_3;k_1,k_2\right)-i\varepsilon_{\mu\nu\alpha\beta}T^{VPP}_{\beta}.\tag{3.255}$$

If compared with other free indices, μ has a distinct function in this equation. That is a direct consequence of the trace version adopted in the integrand exploration (3.58)-(3.57).

Proceeding to the last substructure, we consult Equation (3.64) to express the VPP amplitude as a combination of Feynman integrals

$$\begin{aligned}T^{VPP}_{\beta}\;=\;&-2p_\beta I_{2\beta}\left(k_1,k_2\right)-4I_{2\beta}\left(k_1,k_3\right)-2\left(k_1+k_3\right)_\beta I_2\left(k_1,k_3\right)\\ &+2\left(q-p\right)_\beta I_2\left(k_2,k_3\right)+2\left(q-p\right)^2\left(I_{3\beta}+k_{1\beta}I_3\right)\\ &-2q^2\left(I_{3\beta}+k_{2\beta}I_3\right)+2p^2\left(I_{3\beta}+k_{3\beta}I_3\right).\end{aligned}\tag{3.256}$$

Besides results obtained at the outset of the triangle discussion, two-propagator integrals (3.225)-(3.228) are also ingredients needed to build this object. Their replacement leads to the following mathematical expression:

$$\begin{aligned}T^{VPP}_{\beta}\;=\;&2\left(k_1+k_3\right)^\rho\Delta_{\beta\rho}-2\left(2p-q\right)_\beta I_{\log}\\ &+4\left(p^2-p\cdot q\right)J_{3\beta}+2\left(q^2p_\beta-p^2q_\beta\right)J_3\\ &+i\left(4\pi\right)^{-2}\left[p_\beta\xi^{(0)}_0\left(p\right)-\left(q-p\right)_\beta\xi^{(0)}_0\left(p-q\right)\right].\end{aligned}\tag{3.257}$$

Since the dependence on external momenta is not univocal for ξ_k-functions, we must specify their argument.

As we determined all substructures, renaming indices and organizing contributions is the final task before expressing the AVV amplitude:

$$\begin{aligned}T^{AVV}_{\mu\nu\alpha}\;=\;&-2i\varepsilon_{\mu\alpha\rho\sigma}\left(p-q\right)^\rho\Delta^\sigma_\nu+2i\varepsilon_{\mu\nu\rho\sigma}p^\rho\Delta^\sigma_\alpha\\ &-2i\varepsilon_{\mu\nu\alpha\rho}\left(k_1+k_3\right)_\sigma\Delta^{\rho\sigma}-8i\varepsilon_{\mu\alpha\rho\sigma}\left(p-q\right)^\rho J^\sigma_{3\nu}\\ &+8i\varepsilon_{\mu\nu\rho\sigma}p^\rho J^\sigma_{3\alpha}-8i\varepsilon_{\mu\nu\rho\sigma}p^\rho q_\alpha J_{3\sigma}+8i\varepsilon_{\mu\alpha\rho\sigma}p^\rho q^\sigma J_{3\nu}\\ &-4i\varepsilon_{\mu\nu\alpha\beta}\left(p^2-p\cdot q\right)J_{3\beta}-2i\varepsilon_{\mu\nu\alpha\beta}\left(q^2p^\beta-p^2q^\beta\right)J_3\\ &+2\left(4\pi\right)^{-2}\varepsilon_{\mu\nu\alpha\beta}\left[p^\beta\xi^{(0)}_0\left(p\right)-\left(q-p\right)^\beta\xi^{(0)}_0\left(p-q\right)\right].\end{aligned}\tag{3.258}$$

Some comments on ill-defined quantities are pertinent to conclude this analysis. Even though it appears when we survey substructures individually, the irreducible standard object $I_{\log}$ does not appear within the final expression since the corresponding coefficient vanishes. That implies all divergences concentrate on surface terms $\Delta_{\rho\sigma}$, whose coefficient unavoidably depends on a non-physical momenta combination. Interestingly, this ambiguous contribution comes from the vector function VPP; standard tensors do not manifest this type of ambiguity.

3.4.5 Four-Point Amplitudes - Feynman Integrals

The final task of this section is to compute quantities typical of calculations involving four-point amplitudes, starting with the corresponding Feynman integrals (3.122). Most are finite, therefore, polynomial (3.164) manifests after adopting the Feynman parametrization (3.216) and grouping terms on the loop momentum

$$\frac{1}{D_{1234}}=\frac{i}{(4\pi)^2}\int_0^1 dz\int_0^{1-z} dy\int_0^{1-y-z} dx\ \frac{1}{\left[(k+L)^2+Q\left(x,y,z\right)\right]^4}, \tag{3.259}$$

where x, y, and z are the parameters. The object $L=k_1-px-qy-rz$ corresponds to the quantity shifted posteriorly to applying the integration. Notations involving it are nothing more than tools to facilitate the visualization of mathematical expressions, hence suppressed later when identifying finite functions. Considering these introductions, explicit integration leads to the following results:

$$I_4 = \frac{i}{(4\pi)^2}\int_0^1 dz\int_0^{1-z} dy\int_0^{1-y-z} dx\ \frac{1}{Q^2}, \tag{3.260}$$

$$I_{4\mu} = -\frac{i}{(4\pi)^2}\int_0^1 dz\int_0^{1-z} dy\int_0^{1-y-z} dx\ L_\mu\frac{1}{Q^2}, \tag{3.261}$$

$$I_{4\mu\nu} = \frac{i}{(4\pi)^2}\int_0^1 dz\int_0^{1-z} dy\int_0^{1-y-z} dx\left[L_{\mu\nu}\frac{1}{Q^2}+\frac{1}{2}g_{\mu\nu}\frac{1}{Q}\right], \tag{3.262}$$

$$I_{4\mu\nu\alpha} = -\frac{i}{(4\pi)^2}\int_0^1 dz\int_0^{1-z} dy\int_0^{1-y-z} dx\left[L_{\mu\nu\alpha}\frac{1}{Q^2}+\frac{1}{2}L'_{\mu\nu\alpha}\frac{1}{Q}\right], \tag{3.263}$$

where we compact products involving the momentum $L_{\mu\nu}=L_\mu L_\nu$ and introduce the combination

$$L'_{\mu\nu\alpha}=L_\mu g_{\nu\alpha}+L_\nu g_{\mu\alpha}+L_\alpha g_{\mu\nu}. \tag{3.264}$$

We still have to evaluate the 4th-order Feynman integral. Since it exhibits logarithmic power counting, one form for its separation employs the $N=0$ version of identity (3.129) to write

$$\frac{k_{\mu\nu\alpha\beta}}{D_{1234}}-\frac{k_{\mu\nu\alpha\beta}}{D_\lambda^4}=-\frac{A_4k_{\mu\nu\alpha\beta}}{D_\lambda^4D_4}-\frac{A_3k_{\mu\nu\alpha\beta}}{D_\lambda^3D_{34}}-\frac{A_2k_{\mu\nu\alpha\beta}}{D_\lambda^2D_{234}}-\frac{A_1k_{\mu\nu\alpha\beta}}{D_\lambda D_{1234}}. \tag{3.265}$$

As this equation follows the developed strategy, integrating the left-hand side leads to ill-defined quantities. They receive an organization through symmetric tensors:

$$\begin{aligned} & I_{4\mu\nu\alpha\beta} - \tfrac{1}{24}A_{\mu\nu\alpha\beta} - \tfrac{1}{24}g_{\mu\nu\alpha\beta}I_{\log} \\ = & -\int \frac{d^4k}{(2\pi)^4}\left[\frac{A_4 k_{\mu\nu\alpha\beta}}{D_\lambda^4 D_4} + \frac{A_3 k_{\mu\nu\alpha\beta}}{D_\lambda^3 D_{34}} + \frac{A_2 k_{\mu\nu\alpha\beta}}{D_\lambda^2 D_{234}} + \frac{A_1 k_{\mu\nu\alpha\beta}}{D_\lambda D_{1234}}\right]. \end{aligned} \tag{3.266}$$

Here, aiming for a cleaner form, we concentrate all surface terms in the object

$$A_{\mu\nu\alpha\beta} = \Box_{\mu\nu\alpha\beta} + \tfrac{1}{2}\left(g_{\mu\nu}\Delta_{\alpha\beta} + g_{\mu\alpha}\Delta_{\nu\beta} + g_{\mu\beta}\Delta_{\nu\alpha} g_{\nu\alpha}\Delta_{\mu\beta} + g_{\nu\beta}\Delta_{\mu\alpha} + g_{\alpha\beta}\Delta_{\mu\nu}\right) \tag{3.267}$$

while products involving the metric tensor receive a compact notation

$$g_{\mu\nu\alpha\beta} = g_{\mu\nu}g_{\alpha\beta} + g_{\mu\alpha}g_{\nu\beta} + g_{\mu\beta}g_{\nu\alpha}. \tag{3.268}$$

Next, proceeding to the finite sector on the right-hand side of this integral, each rational function requires a different Feynman parametrization. They differ from the cases above because polynomials depend on non-physical parameters. This type of contribution cancels out identically after integrations by parts, which ultimately brings polynomials dependent on external momenta:

$$\begin{aligned} & -\int \frac{d^4k}{(2\pi)^4}\left[\frac{A_4 k_{\mu\nu\alpha\beta}}{D_\lambda^4 D_4} + \frac{A_3 k_{\mu\nu\alpha\beta}}{D_\lambda^3 D_{34}} + \frac{A_2 k_{\mu\nu\alpha\beta}}{D_\lambda^2 D_{234}} + \frac{A_1 k_{\mu\nu\alpha\beta}}{D_\lambda D_{1234}}\right] \\ = & \frac{i}{(4\pi)^2}\int_0^1 dz \int_0^{1-z} dy \int_0^{1-y-z} dx \left[L_{\mu\nu\alpha\beta}\frac{1}{Q^2} + \frac{1}{2}L''_{\mu\nu\alpha\beta}\frac{1}{Q} - \frac{1}{4}g_{\mu\nu\alpha\beta}\ln\frac{Q}{-\lambda^2}\right], \end{aligned} \tag{3.269}$$

where we introduce the object

$$L''_{\mu\nu\alpha\beta} = L_{\mu\nu}g_{\alpha\beta} + L_{\mu\alpha}g_{\nu\beta} + L_{\mu\beta}g_{\nu\alpha} + L_{\nu\alpha}g_{\mu\beta} + L_{\nu\beta}g_{\mu\alpha} + L_{\alpha\beta}g_{\mu\nu}. \tag{3.270}$$

That completes the expression for the last Feynman integral

$$\begin{aligned} & I_{4\mu\nu\alpha\beta} - \tfrac{1}{24}A_{\mu\nu\alpha\beta} - \tfrac{1}{24}g_{\mu\nu\alpha\beta}I_{\log} \\ = & \frac{i}{(4\pi)^2}\int_0^1 dz \int_0^{1-z} dy \int_0^{1-y-z} dx \left[L_{\mu\nu\alpha\beta}\frac{1}{Q^2} + \frac{1}{2}L''_{\mu\nu\alpha\beta}\frac{1}{Q} - \frac{1}{4}g_{\mu\nu\alpha\beta}\ln\frac{Q}{-\lambda^2}\right]. \end{aligned} \tag{3.271}$$

To complete the systematization of Feynman integrals, let us identify terms depending exclusively on external momenta and group them into J-tensors. The remaining terms are proportional to combinations of the arbitrary routing k_1, connecting to lower-order Feynman integrals. This process expands the momentum L and its combinations, so the notations introduced above are no longer necessary. Nevertheless, we recur to compact notations to products involving momenta, e.g., $k_{1\mu\nu} = k_{1\mu}k_{1\nu}$ and $p_{\mu\nu} = p_\mu p_\nu$. We cast the final forms for the integrals in the sequence:

$$I_4 = J_4, \tag{3.272}$$
$$I_{4\mu} = J_{4\mu} - k_{1\mu} I_4, \tag{3.273}$$
$$I_{4\mu\nu} = J_{4\mu\nu} - [k_{1\mu} I_{4\nu} + k_{1\nu} I_{4\mu}] - [k_{1\mu\nu} I_4], \tag{3.274}$$
$$\begin{aligned} I_{4\mu\nu\alpha} &= J_{4\mu\nu\alpha} - [k_{1\mu} I_{4\nu\alpha} + k_{1\nu} I_{4\alpha\mu} + k_{1\alpha} I_{4\mu\nu}] \\ &\quad - [k_{1\nu\alpha} I_{4\mu} + k_{1\mu\alpha} I_{4\nu} + k_{1\mu\nu} I_{4\alpha}] - [k_{1\mu\nu\alpha} I_4], \end{aligned} \tag{3.275}$$

$$\begin{aligned} & I_{4\mu\nu\alpha\beta} - \tfrac{1}{24} A_{\mu\nu\alpha\beta} - \tfrac{1}{24} g_{\mu\nu\alpha\beta} I_{\log} \\ = {} & J_{4\mu\nu\alpha\beta} - [k_{1\mu} I_{4\nu\alpha\beta} + k_{1\nu} I_{4\mu\alpha\beta} + k_{1\alpha} I_{4\mu\nu\beta} + k_{1\beta} I_{4\mu\nu\alpha}] \\ & - [k_{1\alpha\beta} I_{4\mu\nu} + k_{1\nu\beta} I_{4\mu\alpha} + k_{1\nu\alpha} I_{4\mu\beta} + k_{1\mu\beta} I_{4\nu\alpha} + k_{1\mu\alpha} I_{4\nu\beta} + k_{1\mu\nu} I_{4\alpha\beta}] \\ & - [k_{1\nu\alpha\beta} I_{4\mu} + k_{1\mu\alpha\beta} I_{4\nu} + k_{1\mu\nu\beta} I_{4\alpha} + k_{1\mu\nu\alpha} I_{4\beta}] - [k_{1\mu\nu\alpha\beta} I_4]. \end{aligned} \tag{3.276}$$

The J-tensors arise as symmetric combinations of finite functions belonging to the families (3.161)-(3.163). All non-equivalent index permutations compound these objects:

$$J_4 = i (4\pi)^{-2} \xi_{000}^{(-2)}, \tag{3.277}$$

$$J_{4\mu} = i (4\pi)^{-2} \left[p_\mu \xi_{100}^{(-2)} + q_\mu \xi_{010}^{(-2)} + r_\mu \xi_{001}^{(-2)} \right], \tag{3.278}$$

$$\begin{aligned} J_{4\mu\nu} = {} & i (4\pi)^{-2} \Big[p_{\mu\nu} \xi_{200}^{(-2)} + q_{\mu\nu} \xi_{020}^{(-2)} + r_{\mu\nu} \xi_{002}^{(-2)} + (p_\mu q_\nu + q_\mu p_\nu) \xi_{110}^{(-2)} \\ & + (p_\mu r_\nu + r_\mu p_\nu) \xi_{101}^{(-2)} + (q_\mu r_\nu + r_\mu q_\nu) \xi_{011}^{(-2)} + \tfrac{1}{2} g_{\mu\nu} \xi_{000}^{(-1)} \Big], \end{aligned} \tag{3.279}$$

$$\begin{aligned} J_{4\mu\nu\alpha} = {} & i (4\pi)^{-2} \Big[p_{\mu\nu\alpha} \xi_{300}^{(-2)} + q_{\mu\nu\alpha} \xi_{030}^{(-2)} + r_{\mu\nu\alpha} \xi_{003}^{(-2)} \\ & + (p_{\mu\nu} q_\alpha + p_{\mu\alpha} q_\nu + p_{\nu\alpha} q_\mu) \xi_{210}^{(-2)} + (q_{\mu\nu} p_\alpha + q_{\mu\alpha} p_\nu + q_{\nu\alpha} p_\mu) \xi_{120}^{(-2)} \\ & + (p_{\mu\nu} r_\alpha + p_{\mu\alpha} r_\nu + p_{\nu\alpha} r_\mu) \xi_{201}^{(-2)} + (r_{\mu\nu} p_\alpha + r_{\mu\alpha} p_\nu + r_{\nu\alpha} p_\mu) \xi_{102}^{(-2)} \\ & + (q_{\mu\nu} r_\alpha + q_{\mu\alpha} r_\nu + q_{\nu\alpha} r_\mu) \xi_{021}^{(-2)} + (r_{\mu\nu} q_\alpha + r_{\mu\alpha} q_\nu + r_{\nu\alpha} q_\mu) \xi_{012}^{(-2)} \\ & + [(p_\mu q_\nu + q_\mu p_\nu) r_\alpha + (p_\mu r_\nu + r_\mu p_\nu) q_\alpha + (q_\mu r_\nu + r_\mu q_\nu) p_\alpha] \xi_{111}^{(-2)} \\ & + \tfrac{1}{2} (g_{\mu\nu} p_\alpha + g_{\mu\alpha} p_\nu + g_{\nu\alpha} p_\mu) \xi_{100}^{(-1)} + \tfrac{1}{2} (g_{\mu\nu} q_\alpha + g_{\mu\alpha} q_\nu + g_{\nu\alpha} q_\mu) \xi_{010}^{(-1)} \\ & + \tfrac{1}{2} (g_{\mu\nu} r_\alpha + g_{\mu\alpha} r_\nu + g_{\nu\alpha} r_\mu) \xi_{001}^{(-1)} \Big], \end{aligned} \tag{3.280}$$

$$\begin{aligned}
J_{4\mu\nu\alpha\beta} &= i\left(4\pi\right)^{-2}\Big[p_{\mu\nu\alpha\beta}\xi_{400}^{(-2)}+q_{\mu\nu\alpha\beta}\xi_{040}^{(-2)}+r_{\mu\nu\alpha\beta}\xi_{004}^{(-2)}\\
&+p_{\mu\nu\alpha}q_{\beta}\xi_{310}^{(-2)}+q_{\mu\nu\alpha}p_{\beta}\xi_{130}^{(-2)}+p_{\mu\nu\alpha}r_{\beta}\xi_{301}^{(-2)}\\
&+r_{\mu\nu\alpha}p_{\beta}\xi_{103}^{(-2)}+q_{\mu\nu\alpha}r_{\beta}\xi_{031}^{(-2)}+r_{\mu\nu\alpha}q_{\beta}\xi_{013}^{(-2)}\\
&+p_{\mu\nu}q_{\alpha\beta}\xi_{220}^{(-2)}+p_{\mu\nu}r_{\alpha\beta}\xi_{202}^{(-2)}+q_{\mu\nu}r_{\alpha\beta}\xi_{022}^{(-2)}\\
&+p_{\mu}q_{\nu}r_{\alpha\beta}\xi_{112}^{(-2)}+r_{\mu}q_{\nu}p_{\alpha\beta}\xi_{211}^{(-2)}+p_{\mu}r_{\nu}q_{\alpha\beta}\xi_{121}^{(-2)}\\
&+\tfrac{1}{2}p_{\mu\nu}g_{\alpha\beta}\xi_{200}^{(-1)}+\tfrac{1}{2}q_{\mu\nu}g_{\alpha\beta}\xi_{020}^{(-1)}+\tfrac{1}{2}r_{\mu\nu}g_{\alpha\beta}\xi_{002}^{(-1)}\\
&+\tfrac{1}{2}p_{\mu}q_{\nu}g_{\alpha\beta}\xi_{110}^{(-1)}+\tfrac{1}{2}q_{\mu}r_{\nu}g_{\alpha\beta}\xi_{011}^{(-1)}+\tfrac{1}{2}p_{\mu}r_{\nu}g_{\alpha\beta}\xi_{101}^{(-1)}\\
&-\tfrac{1}{4}g_{\mu\nu}g_{\alpha\beta}\xi_{000}^{(0)}\Big]+\text{permutations}. \qquad (3.281)
\end{aligned}$$

Once the required pieces are at our disposal, the computation of perturbative amplitudes occurs in the sequence.

3.4.6 Four-Point Amplitudes - $PVVV$

The amplitude $PVVV$ emerges by integrating Equation (3.80), as symbolized through the adoption of uppercase letters:

$$\begin{aligned}
T_{\nu\alpha\beta}^{PVVV} &= -4im\left(g_{\kappa\nu}g_{\alpha\beta}-g_{\kappa\alpha}g_{\nu\beta}+g_{\kappa\beta}g_{\nu\alpha}\right)F_{4\kappa}\\
&+2imF_{4\nu\alpha\beta}-i\varepsilon_{\kappa\nu\alpha\beta}T_{\kappa}^{APPP}. \qquad (3.282)
\end{aligned}$$

Its content mirrors the AVV triangle since both are 3rd-order pseudotensors having a tensor sector and a vector subamplitude. Hence, operations performed there find their analogs here.

That is particularly evident for standard tensors with three momenta in the numerator. The four-propagator version follows the structure (3.74), whose integration resembles that with three propagators (3.248). We must only change the numerical subindex to four to establish the connection. This same reasoning applies to the result of integration (3.249); however, there are no divergent objects this time

$$\begin{aligned}
T_{4\mu;\nu\alpha}^{(-)}\left(k_l;k_i,k_j\right) &= -\left(p_j-p_i\right)_{\alpha}J_{4\mu\nu}-\left(p_i-p_j\right)_{\nu}J_{4\mu\alpha}\\
&+\left(p_{j\alpha}p_{i\nu}-p_{i\alpha}p_{j\nu}\right)J_{4\mu}+\left(p_j-p_i\right)_{\alpha}p_{l\mu}J_{4\nu}\\
&+\left(p_i-p_j\right)_{\nu}p_{l\mu}J_{4\alpha}-\left(p_{j\alpha}p_{i\nu}-p_{i\alpha}p_{j\nu}\right)p_{l\mu}J_{4}. \qquad (3.283)
\end{aligned}$$

Now, four routings k_i are available and allow twelve non-equivalent momenta configurations. That generates differences related to external momenta: $p_1=0$, $p_2=k_1-k_2=p$, $p_3=k_1-k_3=q$, and $p_4=k_1-k_4=r$. After performing these identifications, we cast the standard tensors below.

$$T^{(-)}_{4\mu;\nu\alpha}(k_1;k_2,k_3) = -(q-p)_\alpha J_{4\mu\nu} - (p-q)_\nu J_{4\mu\alpha} + (q_\alpha p_\nu - p_\alpha q_\nu) J_{4\mu} \quad (3.284)$$

$$T^{(-)}_{4\mu;\nu\alpha}(k_1;k_2,k_4) = -(r-p)_\alpha J_{4\mu\nu} - (p-r)_\nu J_{4\mu\alpha} + (r_\alpha p_\nu - p_\alpha r_\nu) J_{4\mu} \quad (3.285)$$

$$T^{(-)}_{4\mu;\nu\alpha}(k_1;k_3,k_4) = -(r-q)_\alpha J_{4\mu\nu} - (q-r)_\nu J_{4\mu\alpha} + (r_\alpha q_\nu - q_\alpha r_\nu) J_{4\mu} \quad (3.286)$$

$$T^{(-)}_{4\mu;\nu\alpha}(k_2;k_1,k_3) = -q_\alpha J_{4\mu\nu} + q_\nu J_{4\mu\alpha} + q_\alpha p_\mu J_{4\nu} - q_\nu p_\mu J_{4\alpha} \quad (3.287)$$

$$T^{(-)}_{4\mu;\nu\alpha}(k_2;k_1,k_4) = -r_\alpha J_{4\mu\nu} + r_\nu J_{4\mu\alpha} + r_\alpha p_\mu J_{4\nu} - r_\nu p_\mu J_{4\alpha} \quad (3.288)$$

$$\begin{aligned} T^{(-)}_{4\mu;\nu\alpha}(k_2;k_3,k_4) &= -(r-q)_\alpha J_{4\mu\nu} - (q-r)_\nu J_{4\mu\alpha} + (r_\alpha q_\nu - q_\alpha r_\nu) J_{4\mu} \\ &+ (r-q)_\alpha p_\mu J_{4\nu} + (q-r)_\nu p_\mu J_{4\alpha} - (r_\alpha q_\nu - q_\alpha r_\nu) p_\mu J_4 \end{aligned} \quad (3.289)$$

$$T^{(-)}_{4\mu;\nu\alpha}(k_3;k_1,k_2) = -p_\alpha J_{4\mu\nu} + p_\nu J_{4\mu\alpha} + p_\alpha q_\mu J_{4\nu} - p_\nu q_\mu J_{4\alpha} \quad (3.290)$$

$$T^{(-)}_{4\mu;\nu\alpha}(k_3;k_1,k_4) = -r_\alpha J_{4\mu\nu} + r_\nu J_{4\mu\alpha} + r_\alpha q_\mu J_{4\nu} - r_\nu q_\mu J_{4\alpha} \quad (3.291)$$

$$\begin{aligned} T^{(-)}_{4\mu;\nu\alpha}(k_3;k_2,k_4) &= -(r-p)_\alpha J_{4\mu\nu} - (p-r)_\nu J_{4\mu\alpha} + (r_\alpha p_\nu - p_\alpha r_\nu) J_{4\mu} \\ &+ (r-p)_\alpha q_\mu J_{4\nu} + (p-r)_\nu q_\mu J_{4\alpha} - (r_\alpha p_\nu - p_\alpha r_\nu) q_\mu J_4 \end{aligned} \quad (3.292)$$

$$T^{(-)}_{4\mu;\nu\alpha}(k_4;k_1,k_2) = -p_\alpha J_{4\mu\nu} + p_\nu J_{4\mu\alpha} + p_\alpha r_\mu J_{4\nu} - p_\nu r_\mu J_{4\alpha} \quad (3.293)$$

$$T^{(-)}_{4\mu;\nu\alpha}(k_4;k_1,k_3) = -q_\alpha J_{4\mu\nu} + q_\nu J_{4\mu\alpha} + q_\alpha r_\mu J_{4\nu} - q_\nu r_\mu J_{4\alpha} \quad (3.294)$$

$$\begin{aligned} T^{(-)}_{4\mu;\nu\alpha}(k_4;k_2,k_4) &= -(q-p)_\alpha J_{4\mu\nu} - (p-q)_\nu J_{4\mu\alpha} + (q_\alpha p_\nu - p_\alpha q_\nu) J_{4\mu} \\ &+ (q-p)_\alpha r_\mu J_{4\nu} + (p-q)_\nu r_\mu J_{4\alpha} - (q_\alpha p_\nu - p_\alpha q_\nu) r_\mu J_4 \end{aligned} \quad (3.295)$$

Our next step consists of building objects containing these tensors in their structure. Thus, we start by suiting the notation within the vector $F_{4\mu}$ (3.73) to write its integrated version

$$\begin{aligned} F_{4\mu} &= \varepsilon_{\mu\rho\sigma\kappa} \Big[T_4^{(-)\rho;\sigma\kappa}(k_2;k_3,k_4) - T_4^{(-)\rho;\sigma\kappa}(k_1;k_3,k_4) \\ &+ T_4^{(-)\rho;\sigma\kappa}(k_1;k_2,k_4) - T_4^{(-)\rho;\sigma\kappa}(k_1;k_2,k_3) \Big]. \end{aligned} \quad (3.296)$$

Whereas contributions on the 2nd-order J-tensor cancel out directly due to symmetry properties in the contraction, the same does not occur for other sectors

$$\begin{aligned} F_{4\mu} &= \varepsilon_{\mu\rho\sigma\kappa} \{ -(q^\kappa p^\sigma - p^\kappa q^\sigma) J_4^\rho + (r^\kappa p^\sigma - p^\kappa r^\sigma) J_4^\rho \\ &+ (r-q)^\kappa p^\rho J_4^\sigma + (q-r)^\sigma p^\rho J_4^\kappa - (r^\kappa q^\sigma - q^\kappa r^\sigma) p^\rho J_4 \}. \end{aligned} \quad (3.297)$$

A closer look at the J-vector structure (3.278) is required to verify that it also vanishes. Ultimately, all terms disappear for symmetry reasons. At the end of calculations, only the scalar sector remains

$$F_{4\mu} = -2\varepsilon_{\mu\rho\sigma\kappa} p^\rho q^\sigma r^\kappa J_4. \quad (3.298)$$

We need all momenta configurations to assemble the other tensor group (3.79) as seen in its integrated version

$$\begin{aligned}
F_{4\nu\alpha\beta} &= -\left(\varepsilon_{\alpha\beta XY}g_{\nu Z}-\varepsilon_{\nu\beta XY}g_{\alpha Z}+\varepsilon_{\nu\alpha XY}g_{\beta Z}\right)T^{(-)}_{4Z;XY}\left(k_1;k_3,k_4\right)\\
&\quad+\left(\varepsilon_{\alpha\beta XY}g_{\nu Z}-\varepsilon_{\nu\beta XY}g_{\alpha Z}+\varepsilon_{\nu\alpha XY}g_{\beta Z}\right)T^{(-)}_{4Z;XY}\left(k_1;k_2,k_4\right)\\
&\quad-\left(\varepsilon_{\alpha\beta XY}g_{\nu Z}-\varepsilon_{\nu\beta XY}g_{\alpha Z}+\varepsilon_{\nu\alpha XY}g_{\beta Z}\right)T^{(-)}_{4Z;XY}\left(k_1;k_2,k_3\right)\\
&\quad-\left(\varepsilon_{\alpha\beta XY}g_{\nu Z}+\varepsilon_{\nu\beta XY}g_{\alpha Z}-\varepsilon_{\nu\alpha XY}g_{\beta Z}\right)T^{(-)}_{4Z;XY}\left(k_2;k_3,k_4\right)\\
&\quad+\left(\varepsilon_{\alpha\beta XY}g_{\nu Z}+\varepsilon_{\nu\beta XY}g_{\alpha Z}-\varepsilon_{\nu\alpha XY}g_{\beta Z}\right)T^{(-)}_{4Z;XY}\left(k_2;k_1,k_4\right)\\
&\quad-\left(\varepsilon_{\alpha\beta XY}g_{\nu Z}+\varepsilon_{\nu\beta XY}g_{\alpha Z}-\varepsilon_{\nu\alpha XY}g_{\beta Z}\right)T^{(-)}_{4Z;XY}\left(k_2;k_1,k_3\right)\\
&\quad+\left(\varepsilon_{\alpha\beta XY}g_{\nu Z}-\varepsilon_{\nu\beta XY}g_{\alpha Z}-\varepsilon_{\nu\alpha XY}g_{\beta Z}\right)T^{(-)}_{4Z;XY}\left(k_3;k_2,k_4\right)\\
&\quad-\left(\varepsilon_{\alpha\beta XY}g_{\nu Z}-\varepsilon_{\nu\beta XY}g_{\alpha Z}-\varepsilon_{\nu\alpha XY}g_{\beta Z}\right)T^{(-)}_{4Z;XY}\left(k_3;k_1,k_4\right)\\
&\quad+\left(\varepsilon_{\alpha\beta XY}g_{\nu Z}-\varepsilon_{\nu\beta XY}g_{\alpha Z}-\varepsilon_{\nu\alpha XY}g_{\beta Z}\right)T^{(-)}_{4Z;XY}\left(k_3;k_1,k_2\right)\\
&\quad-\left(\varepsilon_{\alpha\beta XY}g_{\nu Z}-\varepsilon_{\nu\beta XY}g_{\alpha Z}+\varepsilon_{\nu\alpha XY}g_{\beta Z}\right)T^{(-)}_{4Z;XY}\left(k_4;k_2,k_3\right)\\
&\quad+\left(\varepsilon_{\alpha\beta XY}g_{\nu Z}-\varepsilon_{\nu\beta XY}g_{\alpha Z}+\varepsilon_{\nu\alpha XY}g_{\beta Z}\right)T^{(-)}_{4Z;XY}\left(k_4;k_1,k_3\right)\\
&\quad-\left(\varepsilon_{\alpha\beta XY}g_{\nu Z}-\varepsilon_{\nu\beta XY}g_{\alpha Z}+\varepsilon_{\nu\alpha XY}g_{\beta Z}\right)T^{(-)}_{4Z;XY}\left(k_4;k_1,k_2\right). \qquad (3.299)
\end{aligned}$$

Some simplifications are immediate after replacing standard tensors, yielding the expression

$$\begin{aligned}
F_{4\nu\alpha\beta} &= 4m\left(-\varepsilon_{\alpha\beta XY}g_{\nu Z}+\varepsilon_{\nu\beta XY}g_{\alpha Z}-\varepsilon_{\nu\alpha XY}g_{\beta Z}\right)\left(p_Xq_Y-p_Xr_Y+q_Xr_Y\right)J_{4Z}\\
&\quad+4m\left(-\varepsilon_{\alpha\beta XY}g_{\nu Z}-\varepsilon_{\nu\beta XY}g_{\alpha Z}+\varepsilon_{\nu\alpha XY}g_{\beta Z}\right)q_Xr_Y\left(J_{4Z}-p_ZJ_4\right)\\
&\quad+4m\left(\varepsilon_{\alpha\beta XY}g_{\nu Z}-\varepsilon_{\nu\beta XY}g_{\alpha Z}-\varepsilon_{\nu\alpha XY}g_{\beta Z}\right)p_Xr_Y\left(J_{4Z}-q_ZJ_4\right)\\
&\quad+4m\left(-\varepsilon_{\alpha\beta XY}g_{\nu Z}+\varepsilon_{\nu\beta XY}g_{\alpha Z}-\varepsilon_{\nu\alpha XY}g_{\beta Z}\right)p_Xq_Y\left(J_{4Z}-r_ZJ_4\right). \qquad (3.300)
\end{aligned}$$

As these coefficients are products between the Levi-Civita symbol and the metric tensor, rearranging indices through Schouten identities (3.59) is feasible. Nevertheless, contractions involving external momenta and J_4-vectors emerge in this process. From the explicit form (3.278), we recognize these contractions as reductions obtained in the strategy context (3.172)-(3.174). Their employment allows expressing the result as follows

$$\begin{aligned}
F_{4\nu\alpha\beta} &= 4m\varepsilon_{\nu\alpha\beta\rho}\left[(r-q)^\rho\left(J_3'''-J_3''\right)+p^\rho\left(J_3-J_3'\right)\right]\\
&\quad-8m\varepsilon_{\alpha\beta\rho\sigma}q^\rho r^\sigma J_{4\nu}-8m\varepsilon_{\nu\alpha\rho\sigma}p^\rho r^\sigma J_{4\beta}\\
&\quad+4m\varepsilon_{\nu\alpha\beta\rho}\left[\left(q^2-r^2-q\cdot r\right)p^\rho+\left(p\cdot r-p^2\right)q^\rho+p^2r^\rho\right]J_4\\
&\quad-4m\varepsilon_{\beta\alpha\rho\sigma}\left(q^\rho p_\nu-p^\rho q_\nu\right)r^\sigma J_4+4m\varepsilon_{\nu\beta\rho\sigma}\left(q^\rho p_\alpha+p^\rho q_\alpha\right)r^\sigma J_4\\
&\quad-4m\varepsilon_{\nu\alpha\rho\sigma}\left(q^\rho p_\beta-p^\rho q_\beta\right)]\,r^\sigma J_4, \qquad (3.301)
\end{aligned}$$

where we identified J_3-scalars and extended the line notation to them (3.171).

The last ingredient is the vector subamplitude $APPP$ (3.77), whose integration leads to the combination

$$\begin{aligned} T_{\sigma}^{APPP} &= 4mp_{\sigma}I_{3}\left(k_{1},k_{2},k_{4}\right)+4m\left(r-q\right)_{\sigma}I_{3}\left(k_{1},k_{3},k_{4}\right) \\ &\quad -4m\left[\left(q^{2}-q\cdot r\right)p_{\sigma}-\left(p^{2}-p\cdot r\right)q_{\sigma}+\left(p^{2}-p\cdot q\right)r_{\sigma}\right]I_{4}. \end{aligned} \tag{3.302}$$

Since these Feynman integrals are finite, see Equations (3.235) and (3.272), the link with the corresponding J-scalars is straightforward

$$\begin{aligned} T_{\sigma}^{APPP} &= 4mp_{\sigma}J_{3}'+4m\left(r-q\right)_{\sigma}J_{3}''-4m\left[\left(q^{2}-q\cdot r\right)p_{\sigma}\right. \\ &\quad \left.-\left(p^{2}-p\cdot r\right)q_{\sigma}+\left(p^{2}-p\cdot q\right)r_{\sigma}\right]J_{4}. \end{aligned} \tag{3.303}$$

Once all pieces are known, we replace them in the original form (3.282) to compound the $PVVV$ amplitude:

$$\begin{aligned} T_{\nu\alpha\beta}^{PVVV} &= 4im\left(g_{\alpha\beta}\varepsilon_{\nu\rho\sigma\kappa}-g_{\nu\beta}\varepsilon_{\alpha\rho\sigma\kappa}+g_{\nu\alpha}\varepsilon_{\beta\rho\sigma\kappa}\right)p^{\rho}q^{\sigma}r^{\kappa}J_{4} \\ &\quad +4im\varepsilon_{\nu\alpha\beta\rho}\left[\left(r-q\right)^{\rho}J_{3}'''+p^{\rho}J_{3}\right]-8im\varepsilon_{\alpha\beta\rho\sigma}q^{\rho}r^{\sigma}J_{4\nu} \\ &\quad -8im\varepsilon_{\nu\alpha\rho\sigma}p^{\rho}r^{\sigma}J_{4\beta}+4im\varepsilon_{\nu\alpha\beta\rho}\left[\left(p\cdot q\right)r^{\rho}-r^{2}p^{\rho}\right]J_{4} \\ &\quad -4im\varepsilon_{\beta\alpha\rho\sigma}\left(q^{\rho}p_{\nu}-p^{\rho}q_{\nu}\right)r^{\sigma}J_{4} \\ &\quad +4im\varepsilon_{\nu\beta\rho\sigma}\left(q^{\rho}p_{\alpha}+p^{\rho}q_{\alpha}\right)r^{\sigma}J_{4} \\ &\quad -4im\varepsilon_{\nu\alpha\rho\sigma}\left(q^{\rho}p_{\beta}-p^{\rho}q_{\beta}\right)r^{\sigma}J_{4}. \end{aligned} \tag{3.304}$$

As anticipated by the analysis of mass dimension, we found a finite structure.

3.4.7 Four-Point Amplitudes - $AVVV$

We reach the last correlator that concerns this investigation. From Equation (3.101), we write the integrated version of the $AVVV$ amplitude as

$$\begin{aligned} T_{\mu\nu\alpha\beta}^{AVVV} &= iF_{4\mu\nu\alpha\beta}-i\left[\varepsilon_{\mu\alpha\beta X}T_{X\nu}^{VVPP}+\varepsilon_{\mu\nu\beta X}T_{X\alpha}^{VPVP}+\varepsilon_{\mu\nu\alpha X}T_{X\beta}^{VPPV}\right] \\ &\quad +\left[g_{\alpha\beta}T_{\mu\nu}^{AVPP}+g_{\nu\beta}T_{\mu\alpha}^{APVP}+g_{\nu\alpha}T_{\mu\beta}^{APPV}\right]+2i\varepsilon_{\mu\nu\alpha\beta}T^{PPPP}. \end{aligned} \tag{3.305}$$

Since the involved mathematical expressions are extensive, we focus only on analyzing substructures without providing the complete object. Although this presentation follows the same steps from Section (3.2.6), we add one step to discuss 2nd-order tensors before building the corresponding subamplitudes.

Fourth-Order Standard Tensors

First, we compute all required 4th-order tensors starting with the simplified version. Besides appearing by itself within $AVPP$-like functions, this object compounds the standard version required to express the sector $F_{4\mu\nu\alpha\beta}$. These are the only places where the Feynman integral $I_{4\mu\nu\alpha\beta}$ appears; therefore, containing all contributions symmetric in four indices[10]. Since most of the involved tensors exhibit antisymmetry in some indices, we acknowledged the possibility of cancellation for these contributions. Verifying this prospect is part of our goal. If this situation indeed occurs, the surface term $\Box_{\mu\nu\alpha\beta}$ and the finite tensor $J_{4\mu\nu\alpha\beta}$ do not appear in this work.

By expanding products from the numerator of its structure (3.107) and integrating, we recognize the simplified version as a combination of four-propagator Feynman integrals:

$$\begin{aligned} T_{4\mu\nu\alpha\beta}\left(k_i,k_j,k_m,k_n\right) &= I_{4\mu\nu\alpha\beta} + \left[k_{i\mu}I_{4\nu\alpha\beta} + k_{j\nu}I_{4\mu\alpha\beta} + k_{m\alpha}I_{4\mu\nu\beta} + k_{n\beta}I_{4\mu\nu\alpha}\right] \\ &\quad + \left[k_{i\mu}k_{j\nu}I_{4\alpha\beta} + k_{i\mu}k_{m\alpha}I_{4\nu\beta} + k_{i\mu}k_{n\beta}I_{4\nu\alpha} + k_{j\nu}k_{m\alpha}I_{4\mu\beta}\right. \\ &\quad \left. + k_{j\nu}k_{n\beta}I_{4\mu\alpha} + k_{m\alpha}k_{n\beta}I_{4\mu\nu}\right] + \left[k_{j\nu}k_{m\alpha}k_{n\beta}I_{4\mu}\right. \\ &\quad \left. + k_{i\mu}k_{m\alpha}k_{n\beta}I_{4\nu} + k_{i\mu}k_{j\nu}k_{n\beta}I_{4\alpha} + k_{i\mu}k_{j\nu}k_{m\alpha}I_{4\beta}\right] \\ &\quad + k_{i\mu}k_{j\nu}k_{m\alpha}k_{n\beta}I_4. \end{aligned} \tag{3.306}$$

Next, our task consists of substituting their explicit expressions (3.272)-(3.276) while obeying the hierarchy observed in previous cases; consult Equation (3.242). This strategy allows writing all finite structures through J-tensors with external momenta $p_i = k_1 - k_i$ as coefficients. Observe that the J-scalar does not contribute due to the unavoidable dependence on $p_1 = 0$. Once these ideas are clear, we introduce the simplified version

$$\begin{aligned} T_{4\mu\nu\alpha\beta}\left(k_i,k_j,k_m,k_n\right) &= \tfrac{1}{24}A_{\mu\nu\alpha\beta} + \tfrac{1}{24}g_{\mu\nu\alpha\beta}I_{\log} + J_{4\mu\nu\alpha\beta} \\ &\quad - \left[p_{i\mu}J_{4\nu\alpha\beta} + p_{j\nu}J_{4\mu\alpha\beta} + p_{m\alpha}J_{4\mu\nu\beta} + p_{n\beta}J_{4\mu\nu\alpha}\right] \\ &\quad + \left[p_{i\mu}p_{j\nu}J_{4\alpha\beta} + p_{i\mu}p_{m\alpha}J_{4\nu\beta} + p_{i\mu}p_{n\beta}J_{4\nu\alpha} + p_{j\nu}p_{m\alpha}J_{4\mu\beta}\right. \\ &\quad \left. + p_{j\nu}p_{n\beta}J_{4\mu\alpha} + p_{m\alpha}p_{n\beta}J_{4\mu\nu}\right] - \left[p_{j\nu}p_{m\alpha}p_{n\beta}J_{4\mu}\right. \\ &\quad \left. + p_{i\mu}p_{m\alpha}p_{n\beta}J_{4\nu} + p_{i\mu}p_{j\nu}p_{n\beta}J_{4\alpha} + p_{i\mu}p_{j\nu}p_{m\alpha}J_{4\beta}\right], \end{aligned} \tag{3.307}$$

and all necessary momenta configurations

$$\begin{aligned} T_{4\mu\nu\alpha\beta}\left(k_1,k_2,k_3,k_4\right) &= \tfrac{1}{24}A_{\mu\nu\alpha\beta} + \tfrac{1}{24}g_{\mu\nu\alpha\beta}I_{\log} + J_{4\mu\nu\alpha\beta} \\ &\quad - \left[p_\nu J_{4\mu\alpha\beta} + q_\alpha J_{4\mu\nu\beta} + r_\beta J_{4\mu\nu\alpha}\right] \\ &\quad + \left[p_\nu q_\alpha J_{4\mu\beta} + p_\nu r_\beta J_{4\mu\alpha} + q_\alpha r_\beta J_{4\mu\nu}\right] - p_\nu q_\alpha r_\beta J_{4\mu}, \end{aligned} \tag{3.308}$$

[10] The mentioned structures are a combination of surface terms $A_{\mu\nu\alpha\beta}$, the irreducible divergent object $I_{\log}$, and the finite tensor $J_{\mu\nu\alpha\beta}$. Consult Equation (3.266) for further information.

$$\begin{aligned} T_{4\mu\nu\alpha\beta}\left(k_{1},k_{2},k_{4},k_{3}\right) &= \tfrac{1}{24}A_{\mu\nu\alpha\beta}+\tfrac{1}{24}g_{\mu\nu\alpha\beta}I_{\log}+J_{4\mu\nu\alpha\beta} \\ &-\left[p_{\nu}J_{4\mu\alpha\beta}+r_{\alpha}J_{4\mu\nu\beta}+q_{\beta}J_{4\mu\nu\alpha}\right] \\ &+\left[p_{\nu}r_{\alpha}J_{4\mu\beta}+p_{\nu}q_{\beta}J_{4\mu\alpha}+r_{\alpha}q_{\beta}J_{4\mu\nu}\right]-p_{\nu}r_{\alpha}q_{\beta}J_{4\mu}, \end{aligned} \tag{3.309}$$

$$\begin{aligned} T_{4\mu\nu\alpha\beta}\left(k_{2},k_{1},k_{3},k_{4}\right) &= \tfrac{1}{24}A_{\mu\nu\alpha\beta}+\tfrac{1}{24}g_{\mu\nu\alpha\beta}I_{\log}+J_{4\mu\nu\alpha\beta} \\ &-\left[p_{\mu}J_{4\nu\alpha\beta}+q_{\alpha}J_{4\mu\nu\beta}+r_{\beta}J_{4\mu\nu\alpha}\right] \\ &+\left[p_{\mu}q_{\alpha}J_{4\nu\beta}+p_{\mu}r_{\beta}J_{4\nu\alpha}+q_{\alpha}r_{\beta}J_{4\mu\nu}\right]-p_{\mu}q_{\alpha}r_{\beta}J_{4\nu}, \end{aligned} \tag{3.310}$$

$$\begin{aligned} T_{4\mu\nu\alpha\beta}\left(k_{2},k_{1},k_{4},k_{3}\right) &= \tfrac{1}{24}A_{\mu\nu\alpha\beta}+\tfrac{1}{24}g_{\mu\nu\alpha\beta}I_{\log}+J_{4\mu\nu\alpha\beta} \\ &-\left[p_{\mu}J_{4\nu\alpha\beta}+r_{\alpha}J_{4\mu\nu\beta}+q_{\beta}J_{4\mu\nu\alpha}\right] \\ &+\left[p_{\mu}r_{\alpha}J_{4\nu\beta}+p_{\mu}q_{\beta}J_{4\nu\alpha}+r_{\alpha}q_{\beta}J_{4\mu\nu}\right]-p_{\mu}r_{\alpha}q_{\beta}J_{4\nu}. \end{aligned} \tag{3.311}$$

Contributions symmetric in four indices come from the highest-order integral, appearing in the first row from the equations above. We stress that version (3.308) appears contracted to the Levi-Civita symbol with $AVPP$-type amplitudes; see Equation (3.115). That implies symmetric contributions vanish, but we return to this discussion in due time.

With these tools determined, let us obtain the standard version that admits sign choices (3.106). By integrating Equation (3.108), we write this object through the following combination:

$$\begin{aligned} T_{4\mu\nu;\alpha\beta}^{(s_1,s_2)} &= T_{4\mu\nu\alpha\beta}\left(k_{1},k_{2},k_{3},k_{4}\right)+s_{1}T_{4\mu\nu\alpha\beta}\left(k_{2},k_{1},k_{3},k_{4}\right) \\ &+s_{2}T_{4\mu\nu\alpha\beta}\left(k_{1},k_{2},k_{4},k_{3}\right)+s_{1}s_{2}T_{4\mu\nu;\alpha\beta}\left(k_{2},k_{1},k_{4},k_{3}\right). \end{aligned} \tag{3.312}$$

We omit arguments exhibiting the momenta hierarchy $T_{4\mu\nu;\alpha\beta}^{(s_1;s_2)}=T_{4\mu\nu;\alpha\beta}^{(s_1;s_2)}\left(k_1,k_2;k_3,k_4\right)$. Then, our job consists of replacing expressions attributed to different momenta configurations. This operation produces the generic form

$$\begin{aligned} T_{4\mu\nu;\alpha\beta}^{(s_1,s_2)} &= \left(1+s_{1}\right)\left(1+s_{2}\right)\left[\tfrac{1}{24}A_{\mu\nu\alpha\beta}+\tfrac{1}{24}g_{\mu\nu\alpha\beta}I_{\log}+J_{4\mu\nu\alpha\beta}\right] \\ &-\left(1+s_{2}\right)\left[s_{1}p_{\mu}J_{4\nu\alpha\beta}+p_{\nu}J_{4\mu\alpha\beta}\right] \\ &-\left(1+s_{1}\right)\left[\left(q_{\alpha}+s_{2}r_{\alpha}\right)J_{4\mu\nu\beta}+\left(r_{\beta}+s_{2}q_{\beta}\right)J_{4\mu\nu\alpha}\right] \\ &+\left(1+s_{1}\right)\left(q_{\alpha}r_{\beta}+s_{2}q_{\beta}r_{\alpha}\right)J_{4\mu\nu}+\left(r_{\beta}+s_{2}q_{\beta}\right)\left(p_{\nu}J_{4\mu\alpha}+s_{1}p_{\mu}J_{4\nu\alpha}\right) \\ &+\left(q_{\alpha}+s_{2}r_{\alpha}\right)\left(p_{\nu}J_{4\mu\beta}+s_{1}p_{\mu}J_{4\nu\beta}\right) \\ &-\left(q_{\alpha}r_{\beta}+s_{2}r_{\alpha}q_{\beta}\right)\left(p_{\nu}J_{4\mu}+s_{1}p_{\mu}J_{4\nu}\right); \end{aligned} \tag{3.313}$$

hence, setting the signs leads to four particular forms

$$\begin{aligned}
T^{(+,+)}_{4\mu\nu;\alpha\beta} &= \tfrac{1}{6}A_{\mu\nu\alpha\beta}+\tfrac{1}{6}g_{\mu\nu\alpha\beta}I_{\log}+4J_{4\mu\nu\alpha\beta}\\
&-2\left[p_{\mu}J_{4\nu\alpha\beta}+p_{\nu}J_{4\mu\alpha\beta}\right]-2\left[\left(q_{\alpha}+r_{\alpha}\right)J_{4\mu\nu\beta}+\left(r_{\beta}+q_{\beta}\right)J_{4\mu\nu\alpha}\right]\\
&+2\left(q_{\alpha}r_{\beta}+q_{\beta}r_{\alpha}\right)J_{4\mu\nu}+\left(r_{\beta}+q_{\beta}\right)\left(p_{\nu}J_{4\mu\alpha}+p_{\mu}J_{4\nu\alpha}\right)\\
&+\left(q_{\alpha}+r_{\alpha}\right)\left(p_{\nu}J_{4\mu\beta}+p_{\mu}J_{4\nu\beta}\right)-\left(q_{\alpha}r_{\beta}+r_{\alpha}q_{\beta}\right)\left(p_{\nu}J_{4\mu}+p_{\mu}J_{4\nu}\right),
\end{aligned}\tag{3.314}$$

$$\begin{aligned}
T^{(+,-)}_{4\mu\nu;\alpha\beta} &= -2\left[\left(q_{\alpha}-r_{\alpha}\right)J_{4\mu\nu\beta}+\left(r_{\beta}-q_{\beta}\right)J_{4\mu\nu\alpha}\right]+2\left(q_{\alpha}r_{\beta}-q_{\beta}r_{\alpha}\right)J_{4\mu\nu}\\
&+\left(r_{\beta}-q_{\beta}\right)\left(p_{\nu}J_{4\mu\alpha}+p_{\mu}J_{4\nu\alpha}\right)+\left(q_{\alpha}-r_{\alpha}\right)\left(p_{\nu}J_{4\mu\beta}+p_{\mu}J_{4\nu\beta}\right)\\
&-\left(q_{\alpha}r_{\beta}-r_{\alpha}q_{\beta}\right)\left(p_{\nu}J_{4\mu}+p_{\mu}J_{4\nu}\right),
\end{aligned}\tag{3.315}$$

$$\begin{aligned}
T^{(-,+)}_{4\mu\nu;\alpha\beta} &= 2\left(p_{\mu}J_{4\nu\alpha\beta}-p_{\nu}J_{4\mu\alpha\beta}\right)+\left(r_{\beta}+q_{\beta}\right)\left(p_{\nu}J_{4\mu\alpha}-p_{\mu}J_{4\nu\alpha}\right)\\
&+\left(q_{\alpha}+r_{\alpha}\right)\left(p_{\nu}J_{4\mu\beta}-p_{\mu}J_{4\nu\beta}\right)-\left(q_{\alpha}r_{\beta}+r_{\alpha}q_{\beta}\right)\left(p_{\nu}J_{4\mu}-p_{\mu}J_{4\nu}\right),
\end{aligned}\tag{3.316}$$

$$\begin{aligned}
T^{(-,-)}_{4\mu\nu;\alpha\beta} &= \left(r_{\beta}-q_{\beta}\right)\left(p_{\nu}J_{4\mu\alpha}-p_{\mu}J_{4\nu\alpha}\right)+\left(q_{\alpha}-r_{\alpha}\right)\left(p_{\nu}J_{4\mu\beta}-p_{\mu}J_{4\nu\beta}\right)\\
&-\left(q_{\alpha}r_{\beta}-r_{\alpha}q_{\beta}\right)\left(p_{\nu}J_{4\mu}-p_{\mu}J_{4\nu}\right).
\end{aligned}\tag{3.317}$$

Lastly, from Equations (3.102)-(3.105), we aim to determine the entire sector

$$F_{4\mu\nu\alpha\beta}=4\varepsilon_{\mu\nu XY}T^{(12)}_{XY\alpha\beta}+4\varepsilon_{\mu\alpha XY}T^{(13)}_{XY\nu\beta}+4\varepsilon_{\mu\beta XY}T^{(14)}_{XY\nu\alpha}.\tag{3.318}$$

Each of its pieces relates to a combination of standard tensors $T^{(s_1,s_2)}_{4\mu\nu;\alpha\beta}=T^{(s_1,s_2)}_{4\mu\nu;\alpha\beta}\left(k_1,k_2;k_3,k_4\right)$:

$$\begin{aligned}
2T^{(12)}_{XY\alpha\beta} &= T^{(-;+)}_{4XY;\alpha\beta}+T^{(-;+)}_{4X\alpha;Y\beta}-T^{(-;-)}_{4X\beta;Y\alpha}+T^{(-;+)}_{4\alpha Y;\beta X}+T^{(-;-)}_{4\beta Y;\alpha X}+T^{(-;-)}_{4\alpha\beta;XY}, &(3.319)\\
2T^{(13)}_{XY\nu\beta} &= -T^{(-;+)}_{4XY;\nu\beta}+T^{(-;-)}_{4Y\beta;\nu X}-T^{(+;+)}_{4\nu Y;\beta X}-T^{(-;-)}_{4\beta X;Y\nu}+T^{(+;+)}_{4\nu X;Y\beta}-T^{(+;-)}_{4\nu\beta;XY}, &(3.320)\\
2T^{(14)}_{XY\nu\alpha} &= T^{(-;-)}_{4XY;\nu\alpha}+T^{(-;-)}_{4\alpha Y;\nu X}-T^{(+;-)}_{4\nu Y;\alpha X}+T^{(-;-)}_{4\alpha X;Y\nu}-T^{(+;-)}_{4\nu X;Y\alpha}+T^{(+;-)}_{4\nu\alpha;XY}. &(3.321)
\end{aligned}$$

We highlight that the tensor with $s_1=s_2=+1$ is the only one containing structures symmetric in four indices; thus, it is straightforward to verify their cancellation within object $T^{(13)}_{XY\nu\beta}$. The immediate consequence is that the entire sector consists of a finite object. Considering our comment on $AVPP$-like amplitudes, this result completes the proof that the surface term $\square_{\mu\nu\alpha\beta}$ and the finite tensor $J_{4\mu\nu\alpha\beta}$ do not appear in this work.

Since all tensors exhibit the same momenta configuration, no additional ingredients are necessary for their evaluation. We only have to rename indices of the particular versions of the standard tensor (with signs set) and perform the replacements. As the adopted notations emphasize contracted indices through uppercase Latin letters, simplifications associated with symmetry properties are evident. After performing them, we present the final expressions attributed to the tensors below. Arrows indicate that only non-trivial contributions regarding contractions appear, which is compatible with Equation (3.318).

$$\begin{aligned} T^{(12)}_{XY\alpha\beta} \rightarrow\ & 4p_X J_{4Y\alpha\beta} - 2\left(p_\alpha q_X + q_\alpha p_X\right) J_{4Y\beta} \\ & -2\left[(q+r)_\beta p_X - p_\beta (q-r)_X\right] J_{4Y\alpha} + 2r_X p_Y J_{4\alpha\beta} \\ & + \left[\left(q_\alpha r_\beta + r_\alpha q_\beta\right) p_X + \left(r_\beta p_\alpha - r_\alpha p_\beta\right) q_X + \left(q_\beta p_\alpha + q_\alpha p_\beta\right) r_X\right] J_{4Y} \\ & + \left(q_\beta p_X r_Y + r_\beta p_X q_Y + p_\beta r_X q_Y\right) J_{4\alpha} \\ & + \left(q_\alpha p_X r_Y + r_\alpha q_X p_Y + p_\alpha q_X r_Y\right) J_{4\beta} \end{aligned} \tag{3.322}$$

$$\begin{aligned} T^{(13)}_{XY\nu\beta} \rightarrow\ & 4(q-p)_X J_{4Y\nu\beta} + 2\left(q_\nu p_X - p_\nu q_X\right) J_{4Y\beta} \\ & +2\left[(q+r)_\beta p_X - (p+r)_\beta q_X + (p-q)_\beta r_X\right] J_{4Y\nu} + 2(p-q)_X r_Y J_{4\nu\beta} \\ & - \left[\left(q_\nu r_\beta + r_\nu q_\beta\right) p_X - \left(r_\beta p_\nu + r_\nu p_\beta\right) q_X + \left(q_\nu p_\beta - q_\beta p_\nu\right) r_X\right] J_{4Y} \\ & + \left(q_\beta r_X p_Y + r_\beta q_X p_Y + q_X r_Y p_\beta\right) J_{4\nu} \\ & + \left(p_\nu q_X r_Y + q_\nu r_X p_Y + r_\nu p_X q_Y\right) J_{4\beta} \end{aligned} \tag{3.323}$$

$$\begin{aligned} T^{(14)}_{XY\nu\alpha} \rightarrow\ & 4(r-q)_X J_{4Y\nu\alpha} - 2\left[(q-r)_\nu p_X - p_\nu (q-r)_X\right] J_{4Y\alpha} \\ & +2\left[(q-r)_\alpha p_X + (p+r)_\alpha q_X - (p+q)_\alpha r_X\right] J_{4Y\nu} + 2q_X r_Y J_{4\nu\alpha} \\ & + \left[\left(q_\nu r_\alpha - r_\nu q_\alpha\right) p_X - \left(r_\nu p_\alpha + r_\alpha p_\nu\right) q_X + \left(q_\nu p_\alpha + q_\alpha p_\nu\right) r_X\right] J_{4Y} \\ & + \left(q_\alpha r_X p_Y + r_\alpha p_X q_Y + p_\alpha r_X q_Y\right) J_{4\nu} \\ & + \left(q_\nu p_X r_Y + r_\nu q_X p_Y + p_\nu r_X q_Y\right) J_{4\alpha} \end{aligned} \tag{3.324}$$

Second-Order Standard Tensors

Second, we compute the 2nd-order standard tensors required to build up subamplitudes. Even though we already examined those involving three propagators, we get back to this subject as the perspective is broader this time. For this purpose, recall the general form obtained succeeding the integration (3.242)

$$\begin{aligned} T^{(s)}_{3\mu\nu}(k_i,k_j) =\ & \tfrac{1}{4}(1+s)\left(\Delta_{\mu\nu} + g_{\mu\nu} I_{\log} + 4J_{3\mu\nu}\right) \\ & - \left(p_j + s p_i\right)_\nu J_{3\mu} - \left(p_i + s p_j\right)_\mu J_{3\nu} + \left(p_{i\mu} p_{j\nu} + s p_{i\nu} p_{j\mu}\right) J_3, \end{aligned} \tag{3.325}$$

where associations with external momenta occur through the relation $p_i = k_1 - k_i$.

We assigned a special role for the routing k_1 simply because it is the first to appear in the adopted ordering. This reasoning was implicit when evaluating three-point Feynman integrals in Subsection (3.4.2) and led to the external momenta p and q. The notation for the corresponding functions is $\xi_{ab}=\xi_{ab}\left(p,q\right)$ and reflects in the corresponding J_3-tensors, including the coefficients inside them.

From the first case, let us obtain the second D_{124} through the transformation $k_3\rightarrow k_4$. That changes the second external momentum $q\rightarrow r$, which reflects on the notations for functions $\xi'_{ab}=\xi_{ab}\left(p,r\right)$ and J'_3-tensors. Analogously, the third case D_{134} links to the momenta q and r seen in functions $\xi''_{ab}=\xi_{ab}\left(q,r\right)$ and J''_3-tensors.

Nevertheless, things are different for objects involving the fourth denominator D_{234}. When emphasizing the routing k_2, these particular associations come with $p'_i=k_2-k_i=p_i-p$ and lead to momenta $q-p$ and $r-p$. The notation for functions $\xi'''_{ab}=\xi_{ab}\left(q-p,r-p\right)$ and J'''_3-tensors follows previous cases; however, the differences p'_i generate more structures inside the tensors. We must consider such information when exploring reductions and other algebraic manipulations.

The generality brought by J-tensors makes extensions of the expression above direct. Besides changing the versions of these tensors, we recall that there are no ill-defined contributions for the standard tensor depending on four propagators. Therefore, the new version is the following

$$T^{(s)}_{4\mu\nu}\left(k_i,k_j\right)=\left(1+s\right)J_{4\mu\nu}-\left(p_j+sp_i\right)_\nu J_{4\mu}-\left(p_i+sp_j\right)_\mu J_{4\nu}+\left(p_{i\mu}p_{j\nu}+sp_{i\nu}p_{j\mu}\right)J_4, \tag{3.326}$$

where the original association $p_i=k_1-k_i$ applies.

Without setting signs, we cast all available momenta configurations for these objects in the sequence. The line notation (3.171) is particularly advantageous in this scene.

- Three propagators D_{123} - $\xi_{ab}=\xi_{ab}\left(p,q\right)$

$$\left[T^{(s)}_{3\mu\nu}\left(k_1,k_2\right)\right]=\tfrac{1}{4}\left(1+s\right)\left(\Delta_{\mu\nu}+g_{\mu\nu}I_{\log}+4J_{3\mu\nu}\right)-p_\nu J_{3\mu}-sp_\mu J_{3\nu} \tag{3.327}$$

$$\left[T^{(s)}_{3\mu\nu}\left(k_1,k_3\right)\right]=\tfrac{1}{4}\left(1+s\right)\left(\Delta_{\mu\nu}+g_{\mu\nu}I_{\log}+4J_{3\mu\nu}\right)-q_\nu J_{3\mu}-sq_\mu J_{3\nu} \tag{3.328}$$

$$\begin{aligned}\left[T^{(s)}_{3\mu\nu}\left(k_2,k_3\right)\right]=&\tfrac{1}{4}\left(1+s\right)\left(\Delta_{\mu\nu}+g_{\mu\nu}I_{\log}+4J_{3\mu\nu}\right)\\&-\left(q+sp\right)_\nu J_{3\mu}-\left(p+sq\right)_\mu J_{3\nu}+\left(p_\mu q_\nu+sp_\nu q_\mu\right)J_3\end{aligned} \tag{3.329}$$

- Three propagators D_{124} - $\xi'_{ab}=\xi_{ab}\left(p,r\right)$

$$\left[T^{(s)}_{3\mu\nu}\left(k_1,k_2\right)\right]'=\tfrac{1}{4}\left(1+s\right)\left(\Delta_{\mu\nu}+g_{\mu\nu}I_{\log}+4J'_{3\mu\nu}\right)-p_\nu J'_{3\mu}-sp_\mu J'_{3\nu} \tag{3.330}$$

$$\left[T^{(s)}_{3\mu\nu}\left(k_1,k_4\right)\right]'=\tfrac{1}{4}\left(1+s\right)\left(\Delta_{\mu\nu}+g_{\mu\nu}I_{\log}+4J'_{3\mu\nu}\right)-r_\nu J'_{3\mu}-sr_\mu J'_{3\nu} \tag{3.331}$$

$$\begin{aligned}\left[T^{(s)}_{3\mu\nu}\left(k_2,k_4\right)\right]'=&\tfrac{1}{4}\left(1+s\right)\left(\Delta_{\mu\nu}+g_{\mu\nu}I_{\log}+4J'_{3\mu\nu}\right)\\&-\left(r+sp\right)_\nu J'_{3\mu}-\left(p+sr\right)_\mu J'_{3\nu}+\left(p_\mu r_\nu+sp_\nu r_\mu\right)J'_3\end{aligned} \tag{3.332}$$

- Three propagators D_{134} - $\xi''_{ab} = \xi_{ab}(q,r)$

$$
\begin{aligned}
\left[T^{(s)}_{3\mu\nu}(k_1,k_3)\right]'' &= \tfrac{1}{4}(1+s)\left(\Delta_{\mu\nu}+g_{\mu\nu}I_{\log}+4J''_{3\mu\nu}\right)-q_\nu J''_{3\mu}-sq_\mu J''_{3\nu} && (3.333)\\
\left[T^{(s)}_{3\mu\nu}(k_1,k_4)\right]'' &= \tfrac{1}{4}(1+s)\left(\Delta_{\mu\nu}+g_{\mu\nu}I_{\log}+4J''_{3\mu\nu}\right)-r_\nu J''_{3\mu}-sr_\mu J''_{3\nu} && (3.334)\\
\left[T^{(s)}_{3\mu\nu}(k_3,k_4)\right]'' &= \tfrac{1}{4}(1+s)\left(\Delta_{\mu\nu}+g_{\mu\nu}I_{\log}+4J''_{3\mu\nu}\right)\\
&\quad-(r+sq)_\nu J''_{3\mu}-(q+sr)_\mu J''_{3\nu}+(q_\mu r_\nu+sq_\nu r_\mu)J''_3 && (3.335)
\end{aligned}
$$

- Three propagators D_{234} - $\xi'''_{ab} = \xi_{ab}(q-p,r-p)$

$$
\begin{aligned}
\left[T^{(s)}_{3\mu\nu}(k_2,k_3)\right]''' &= \tfrac{1}{4}(1+s)\left(\Delta_{\mu\nu}+g_{\mu\nu}I_{\log}+4J'''_{3\mu\nu}\right)\\
&\quad-(q-p)_\nu J'''_{3\mu}-s(q-p)_\mu J'''_{3\nu} && (3.336)\\
\left[T^{(s)}_{3\mu\nu}(k_2,k_4)\right]''' &= \tfrac{1}{4}(1+s)\left(\Delta_{\mu\nu}+g_{\mu\nu}I_{\log}+4J'''_{3\mu\nu}\right)\\
&\quad-(r-p)_\nu J'''_{3\mu}-s(r-p)_\mu J'''_{3\nu} && (3.337)\\
\left[T^{(s)}_{3\mu\nu}(k_3,k_4)\right]''' &= \tfrac{1}{4}(1+s)\left(\Delta_{\mu\nu}+g_{\mu\nu}I_{\log}+4J'''_{3\mu\nu}\right)\\
&\quad-\left[(r-p)+s(q-p)\right]_\nu J'''_{3\mu}-\left[(q-p)+s(r-p)\right]_\mu J'''_{3\nu}\\
&\quad+\left[(1+s)p_\mu p_\nu-(p_\nu q_\mu+sp_\mu q_\nu)\right.\\
&\quad\left.-(p_\mu r_\nu+sp_\nu r_\mu)+(q_\mu r_\nu+sq_\nu r_\mu)\right]J'''_3 && (3.338)
\end{aligned}
$$

- Four propagators D_{1234} - $\xi_{abc} = \xi_{abc}(p,q,r)$

$$
\begin{aligned}
T^{(s)}_{4\mu\nu}(k_1,k_2) &= (1+s)J_{4\mu\nu}-p_\nu J_{4\mu}-sp_\mu J_{4\nu} && (3.339)\\
T^{(s)}_{4\mu\nu}(k_1,k_3) &= (1+s)J_{4\mu\nu}-q_\nu J_{4\mu}-sq_\mu J_{4\nu} && (3.340)\\
T^{(s)}_{4\mu\nu}(k_1,k_4) &= (1+s)J_{4\mu\nu}-r_\nu J_{4\mu}-sr_\mu J_{4\nu} && (3.341)\\
T^{(s)}_{4\mu\nu}(k_2,k_3) &= (1+s)J_{4\mu\nu}-(q+sp)_\nu J_{4\mu} && (3.342)\\
&\quad-(p+sq)_\mu J_{4\nu}+(p_\mu q_\nu+sp_\nu q_\mu)J_4 && (3.343)\\
T^{(s)}_{4\mu\nu}(k_2,k_4) &= (1+s)J_{4\mu\nu}-(r+sp)_\nu J_{4\mu}\\
&\quad-(p+sr)_\mu J_{4\nu}+(p_\mu r_\nu+sp_\nu r_\mu)J_4 && (3.344)\\
T^{(s)}_{4\mu\nu}(k_3,k_4) &= (1+s)J_{4\mu\nu}-(r+sq)_\nu J_{4\mu}\\
&\quad-(q+sr)_\mu J_{4\nu}+(q_\mu r_\nu+sq_\nu r_\mu)J_4 && (3.345)
\end{aligned}
$$

Even Amplitudes - $VVPP$, $VPVP$, and $VPPV$

Third, we compute even amplitudes that are 2nd-order tensors: $VVPP$, $VPVP$, and $VPPV$. Taking their general form from Equation (3.111), integration allows writing

$$\begin{aligned}
T_{\mu\nu}^{\Gamma_i\Gamma_j\Gamma_k\Gamma_l} &= 2s_1\left[s_3T_{3\mu\nu}^{(s_3)}(k_1,k_2)+s_2T_{3\mu\nu}^{(-s_2)}(k_1,k_3)-s_2T_{3\mu\nu}^{(s_1)}(k_2,k_3)\right] \\
&+2s_1\left[s_3T_{3\mu\nu}^{(s_3)}(k_1,k_2)+T_{3\mu\nu}^{(+)}(k_1,k_4)-T_{3\mu\nu}^{(-s_3)}(k_2,k_4)\right]' \\
&+2s_1\left[s_2T_{3\mu\nu}^{(-s_2)}(k_1,k_3)+T_{3\mu\nu}^{(+)}(k_1,k_4)+T_{3\mu\nu}^{(s_2)}(k_3,k_4)\right]'' \\
&+2s_1\left[-s_2T_{3\mu\nu}^{(s_1)}(k_2,k_3)-T_{3\mu\nu}^{(-s_3)}(k_2,k_4)+T_{3\mu\nu}^{(s_2)}(k_3,k_4)\right]''' \\
&-2s_1\Big[s_3(q-r)^2T_{4\mu\nu}^{(s_3)}(k_1,k_2)+s_2(p-r)^2T_{4\mu\nu}^{(-s_2)}(k_1,k_3) \\
&+(p-q)^2T_{4\mu\nu}^{(+)}(k_1,k_4)-s_2r^2T_{4\mu\nu}^{(s_1)}(k_2,k_3) \\
&-q^2T_{4\mu\nu}^{(-s_3)}(k_2,k_4)+p^2T_{4\mu\nu}^{(s_2)}(k_3,k_4)\Big]-s_1g_{\mu\nu}T^{PPPP}, \qquad (3.346)
\end{aligned}$$

where we obtain one particular version by setting signs through the associations: the $VVPP$ function by fixing $s_i=(-1,-1,+1)$, the $VPVP$ by fixing $s_i=(+1,-1,-1)$, and the $VPPV$ by fixing $s_i=(-1,+1,-1)$. Replacing standard tensors obtained in Subsubsection (3.4.7) determines the explicit results cast in the sequence. We anticipate that these are the only substructures effectively contributing with divergent objects to the $AVVV$.

- The $VVPP$ Amplitude

$$\begin{aligned}
T_{\mu\nu}^{VVPP} &= -2\Delta_{\mu\nu}-2g_{\mu\nu}I_{\log}+g_{\mu\nu}T^{PPPP} \\
&-8J'_{3\mu\nu}+4(p-q)_\mu J_{3\nu}+4p_\nu J'_{3\mu}+4r_\mu J'_{3\nu}+4(r-q)_\nu J''_{3\mu} \\
&-2(p_\mu q_\nu-p_\nu q_\mu)J_3+2(p_\mu r_\nu-p_\nu r_\mu)J'_3-2(q_\mu r_\nu-q_\nu r_\mu)J''_3 \\
&-2\left[(p_\mu q_\nu-p_\nu q_\mu)-(p_\mu r_\nu-p_\nu r_\mu)+(q_\mu r_\nu-q_\nu r_\mu)\right]J'''_3 \\
&+8\left(q^2-p\cdot q+p\cdot r-q\cdot r\right)J_{4\mu\nu} \\
&-4\left[\left(q^2-q\cdot r\right)p_\nu+\left(p\cdot r-p^2\right)q_\nu+\left(p^2-q\cdot p\right)r_\nu\right]J_{4\mu} \\
&-4\left[\left(r^2-r\cdot q\right)p_\mu+\left(p\cdot r-r^2\right)q_\mu+\left(q^2-p\cdot q\right)r_\mu\right]J_{4\nu} \\
&+2\left[p^2(q_\mu r_\nu-q_\nu r_\mu)-q^2(p_\mu r_\nu-p_\nu r_\mu)+r^2(p_\mu q_\nu-p_\nu q_\mu)\right]J_4 \qquad (3.347)
\end{aligned}$$

- The $VPVP$ Amplitude

$$\begin{aligned}
T^{VPVP}_{\mu\alpha} &= -g_{\mu\alpha}T^{PPPP} - 4p_{\mu}J_{3\alpha} + 4p_{\alpha}J'_{3\mu} - 4\left(r-q\right)_{\alpha}J''_{3\mu} - 4\left(q-r\right)_{\mu}J'''_{3\alpha} \\
&+2\left(p_{\mu}q_{\alpha}+p_{\alpha}q_{\mu}\right)J_{3} - 2\left(p_{\mu}r_{\alpha}+p_{\alpha}r_{\mu}\right)J'_{3} + 2\left(q_{\mu}r_{\alpha}-q_{\alpha}r_{\mu}\right)J''_{3} \\
&+2\left[\left(p_{\mu}q_{\alpha}-p_{\alpha}q_{\mu}\right)-\left(p_{\mu}r_{\alpha}-p_{\alpha}r_{\mu}\right)+\left(q_{\mu}r_{\alpha}-q_{\alpha}r_{\mu}\right)\right]J'''_{3} \\
&-8\left(p\cdot r-p\cdot q\right)J_{4\mu\alpha} + 4\left[\left(r^{2}-q\cdot r\right)p_{\mu}+\left(p\cdot r\right)q_{\mu}-\left(p\cdot q\right)r_{\mu}\right]J_{4\alpha} \\
&+4\left[\left(q\cdot r-q^{2}\right)p_{\alpha}+\left(p\cdot r-p^{2}\right)q_{\alpha}+\left(p^{2}-p\cdot q\right)r_{\alpha}\right]J_{4\mu} \\
&-2\left[p^{2}\left(q_{\mu}r_{\alpha}-q_{\alpha}r_{\mu}\right)-q^{2}\left(p_{\mu}r_{\alpha}+p_{\alpha}r_{\mu}\right)+r^{2}\left(p_{\mu}q_{\alpha}+p_{\alpha}q_{\mu}\right)\right]J_{4} \qquad (3.348)
\end{aligned}$$

- The $VPPV$ Amplitude

$$\begin{aligned}
T^{VPPV}_{\mu\beta} &= -2\Delta_{\mu\beta} - 2g_{\mu\beta}I_{\log} + g_{\mu\beta}T^{PPPP} \\
&-8J''_{3\mu\beta} - 4p_{\beta}J'_{3\mu} + 4\left(r+q\right)_{\beta}J''_{3\mu} + 4r_{\mu}J''_{3\beta} + 4\left(q-p\right)_{\mu}J'''_{3\beta} \\
&+2\left(p_{\mu}q_{\beta}-p_{\beta}q_{\mu}\right)J_{3} + 2\left(p_{\mu}r_{\beta}+p_{\beta}r_{\mu}\right)J'_{3} - 2\left(q_{\mu}r_{\beta}+q_{\beta}r_{\mu}\right)J''_{3} \\
&-2\left[2p_{\mu}p_{\beta}-\left(p_{\mu}q_{\beta}+p_{\beta}q_{\mu}\right)-\left(p_{\mu}r_{\beta}+p_{\beta}r_{\mu}\right)+\left(q_{\mu}r_{\beta}+q_{\beta}r_{\mu}\right)\right]J'''_{3} \\
&+8\left(p^{2}-p\cdot q\right)J_{4\mu\beta} + 4\left[\left(q\cdot r\right)p_{\mu}-\left(p\cdot r\right)q_{\mu}+\left(p\cdot q-p^{2}\right)r_{\mu}\right]J_{4\beta} \\
&+4\left[\left(q^{2}-q\cdot r\right)p_{\beta}+\left(p\cdot r-p^{2}\right)q_{\beta}+\left(p\cdot q-p^{2}\right)r_{\beta}\right]J_{4\mu} \\
&+2\left[p^{2}\left(q_{\mu}r_{\beta}+q_{\beta}r_{\mu}\right)-q^{2}\left(p_{\mu}r_{\beta}+p_{\beta}r_{\mu}\right)-r^{2}\left(p_{\mu}q_{\beta}-p_{\beta}q_{\mu}\right)\right]J_{4} \qquad (3.349)
\end{aligned}$$

Odd Amplitudes - $AVPP$, $APVP$, and $APPV$

Forth, we compute odd amplitudes that are 2nd-order tensors: $AVPP$, $APVP$, and $APPV$. Given the general form (3.114), the integral operation characterizes two sectors corresponding to different tensor structures:

$$T^{\Gamma_i\Gamma_j\Gamma_k\Gamma_l}_{\mu\nu} = is_{1}\varepsilon_{\mu XYZ}F^{(s_2,s_3)}_{4\nu XYZ} + is_{1}\varepsilon_{\mu\nu XY}F_{4XY}. \qquad (3.350)$$

We distinguish particular functions when choosing signs through the association: the $AVPP$ function by fixing $s_i=(-1,-1,+1)$, the $APVP$ by fixing $s_i=(+1,+1,+1)$, and the $APPV$ by fixing $s_i=(-1,+1,-1)$.

The first sector is proportional to the simplified version of the 4th-order standard tensor (3.308):

$$\varepsilon_{\mu XYZ}F^{(s_2,s_3)}_{4\nu XYZ} = 4\left(-\varepsilon_{\mu BCD}g_{\nu A} + s_{2}\varepsilon_{\mu ACD}g_{\nu B} + s_{3}\varepsilon_{\mu ABD}g_{\nu C} - \varepsilon_{\mu ABC}g_{\nu D}\right)T_{4ABCD}. \qquad (3.351)$$

Following its replacement, symmetry properties bring simplifications so this product assumes the general form

$$\begin{aligned}\varepsilon_{\mu XYZ}F^{(s_2,s_3)}_{4\nu XYZ} &= -4\varepsilon_{\mu XYZ}\left[(1-s_2)\,q_Y r_Z J_{4\nu X}+(1+s_3)\,p_X r_Z J_{4\nu Y}\right.\\ &+2p_X q_Y J_{4\nu Z}-p_X q_Y r_Z J_{4\nu}+s_2 p_\nu q_Y r_Z J_{4X}\\ &\left.-s_3 q_\nu p_X r_Z J_{4Y}-r_\nu p_X q_Y J_{4Z}\right]. \qquad (3.352)\end{aligned}$$

As mentioned before, symmetric objects $J_{4\mu\nu\alpha\beta}$ and $\Box_{\mu\nu\alpha\beta}$ disappear and do not concern this investigation. Moving on to the second sector, we have another combination of 2nd-order standard tensors

$$\begin{aligned}F_{4XY} &= \left[T^{(-)}_{3XY}(k_2,k_3)-T^{(-)}_{3XY}(k_1,k_3)+T^{(-)}_{3XY}(k_1,k_2)\right]\\ &+\left[-T^{(-)}_{3XY}(k_2,k_4)+T^{(-)}_{3XY}(k_1,k_4)+T^{(-)}_{3XY}(k_1,k_2)\right]'\\ &+\left[T^{(-)}_{3XY}(k_3,k_4)+T^{(-)}_{3XY}(k_1,k_4)-T^{(-)}_{3XY}(k_1,k_3)\right]''\\ &+\left[T^{(-)}_{3XY}(k_3,k_4)-T^{(-)}_{3XY}(k_2,k_4)+T^{(-)}_{3XY}(k_2,k_3)\right]'''\\ &+\left[-p^2T^{(-)}_{4XY}(k_3,k_4)+q^2T^{(-)}_{4XY}(k_2,k_4)-r^2T^{(-)}_{4XY}(k_2,k_3)\right.\\ &-(p-q)^2\,T^{(-)}_{4XY}(k_1,k_4)+(p-r)^2\,T^{(-)}_{4XY}(k_1,k_3)\\ &\left.-(q-r)^2\,T^{(-)}_{4XY}(k_1,k_2)\right]. \qquad (3.353)\end{aligned}$$

Its structure arises after replacing results from Subsubsection (3.4.7) and performing simplifications:

$$\begin{aligned}F_{4XY} &= 4p_X J'_{3Y}+4\,(r-q)_X\,J''_{3Y}+2p_X q_Y J_3+2r_X p_Y J'_3\\ &+2q_X r_Y J''_3+2\,(q_X r_Y+r_X p_Y+p_X q_Y)\,J'''_3\\ &-4\left[\left(q^2-q\cdot r\right)p_X-\left(p^2-p\cdot r\right)q_X+\left(p^2-p\cdot q\right)r_X\right]J_{4Y}\\ &-2\left(p^2q_X r_Y+q^2 r_X p_Y+r^2 p_X q_Y\right)J_4. \qquad (3.354)\end{aligned}$$

Adjusting signs, we cast the final expressions attributed to odd perturbative amplitudes below.

• The $AVPP$ Amplitude

$$\begin{aligned}T^{AVPP}_{\mu\nu} &= 4i\varepsilon_{\mu XYZ}\left(2q_Y r_Z J_{4\nu X}+2p_X r_Z J_{4\nu Y}+2p_X q_Y J_{4\nu Z}-p_X q_Y r_Z J_{4\nu}\right)\\ &+4i\varepsilon_{\mu XYZ}\left(-p_\nu q_Y r_Z J_{4X}-q_\nu p_X r_Z J_{4Y}-r_\nu p_X q_Y J_{4Z}\right)\\ &-i\varepsilon_{\mu\nu XY}F_{4XY} \qquad (3.355)\end{aligned}$$

- The $APVP$ Amplitude

$$\begin{aligned} T_{\mu\alpha}^{APVP} &= -4i\varepsilon_{\mu XYZ}\left(2p_X r_Z J_{4\alpha Y} + 2p_X q_Y J_{4\alpha Z} - p_X q_Y r_Z J_{4\alpha}\right) \\ &-4i\varepsilon_{\mu XYZ}\left(p_\alpha q_Y r_Z J_{4X} - q_\alpha p_X r_Z J_{4Y} - r_\alpha p_X q_Y J_{4Z}\right) \\ &+i\varepsilon_{\mu\alpha XY} F_{4XY} \end{aligned} \quad (3.356)$$

- The $APPV$ Amplitude

$$\begin{aligned} T_{\mu\beta}^{APPV} &= 4i\varepsilon_{\mu XYZ}\left(2p_X q_Y J_{4\beta Z} - p_X q_Y r_Z J_{4\beta}\right) \\ &+4i\varepsilon_{\mu XYZ}\left(p_\beta q_Y r_Z J_{4X} + q_\beta p_X r_Z J_{4Y} - r_\beta p_X q_Y J_{4Z}\right) \\ &-i\varepsilon_{\mu\beta XY} F_{4XY} \end{aligned} \quad (3.357)$$

Scalar Amplitude - $PPPP$

Fifth, we compute the scalar amplitude $PPPP$. The integration of its structure (3.119) allows writing this correlator in terms of scalar Feynman integrals

$$\begin{aligned} T^{PPPP} &= 2\left[I_2(k_2,k_4) + I_2(k_1,k_3)\right] \\ &-2\left(p^2 - p\cdot q\right) I_3(k_1,k_2,k_3) - 2(p\cdot r) I_3(k_1,k_2,k_4) \\ &-2\left(r^2 - q\cdot r\right) I_3(k_1,k_3,k_4) + 2(p-q)\cdot(q-r) I_3(k_2,k_3,k_4) \\ &+\left[p^2(r-q)^2 - q^2(p-r)^2 + r^2(p-q)^2\right] I_4. \end{aligned} \quad (3.358)$$

The required tools are displayed in Equations (3.225), (3.235), and (3.272). Since structures typical of two and three-point calculations appear, specifying their momenta content is essential. After replacing them, we obtain the explicit version of the amplitude

$$\begin{aligned} T^{PPPP} &= 4I_{\log} - 2i(4\pi)^{-2}\left[\xi_0^{(0)}(r-p) + \xi_0^{(0)}(q)\right] \\ &-2\left(p^2 - p\cdot q\right) J_3 - 2(p\cdot r) J_3' \\ &-2\left(r^2 - q\cdot r\right) J_3'' + 2(p-q)\cdot(q-r) J_3''' \\ &+\left[p^2(r-q)^2 - q^2(p-r)^2 + r^2(p-q)^2\right] J_4. \end{aligned} \quad (3.359)$$

3.4.8 Comments

Before proceeding with the analysis of results, let us present a brief panorama of our calculations. In this section, we have evaluated all perturbative amplitudes needed for this investigation. Aiming to accomplish this task, we adopted a strategy to separate ill-defined mathematical structures from finite contributions of integrals.

After computing finite quantities, we projected them in terms of structure functions. They do not appear randomly but in particular arrangements named J-tensors. They stress the exclusive dependence on differences between routings, i.e., external momenta. Under this new perspective, J-tensors' properties are fundamental ingredients to the intended analysis.

On the other hand, we only organized divergent structures that arose within AV^n-type amplitudes. It is well-known that integrals exhibiting power counting equal to or higher than linear are not invariant under translations. Here, this causes the presence of divergent surface terms inside amplitudes AV and AVV. Furthermore, coefficients of these terms unavoidably carry ambiguous structures materialized into sums of arbitrary routings k_i. Interestingly, we acknowledged the same surface term in logarithmically diverging integrals corresponding to at least 2nd-order tensors, although coefficients are not ambiguous in these cases. The $AVVV$ box is an example of this type of situation.

To be more precise, only the 2nd-order surface term $\Delta_{\mu\nu}$ effectively concerns this investigation. The 4th-order surface term appears exclusively inside $AVVV$'s tensor sector but cancels out subsequently. Even if the irreducible object appears within substructures, it vanishes identically in the complete amplitudes. Taking a closer look at contributions from even subamplitudes belonging to the box, we cast its divergent sector:

$$\left[T^{AVVV}_{\mu\nu\alpha\beta}\right]_{\mathrm{div}} = 2i\left(\varepsilon_{\mu\alpha\beta\rho}\Delta^{\rho}_{\nu} + \varepsilon_{\mu\nu\alpha X}\Delta^{\rho}_{\beta}\right). \tag{3.360}$$

As no prescription was adopted to evaluate divergences, expressing them in the context of a regularization scheme is feasible. Nonetheless, by avoiding this step, our analysis inquires about the implications of different values for the surface term $\Delta_{\mu\nu}$. That occurs in the following section when investigating the connection involving linearity of integration and symmetries.

3.5 Analysis of the Results

In the model discussion, we considered the mathematical structure of perturbative amplitudes to establish identities at the integrand level. Proper relations among Green functions (GF) should emerge with the integration; however, the divergent character of calculations might affect these expectations.

Verifying these relations requires performing momenta contractions with the explicit form of AV^n-type amplitudes. Subsection (3.5.1) develops these operations for the AVV triangle while highlighting tools and patterns considered relevant to the more complex case. Afterward, Subsection (3.5.2) extends these explorations to the $AVVV$ box. Since potentially violating terms emerge in this process, Subsection (3.5.3) inquires about mathematical structures linked to them. Such analysis elucidates the roles played by different trace expressions and vertex configurations. Lastly, we study Ward identities (WIs) from their association with relations among GF in Subsection (3.5.4). All mentioned constraints depend on divergent objects materialized in surface terms; therefore, our argumentation approaches their possible values and ensuing implications.

3.5.1 Relations Among Green Functions - AVV

This subsection aims to verify relations among GF derived for AVV contractions (3.258). The corresponding expectations are cast in Equations (3.19)-(3.21), so we transcribe them here:

$$(k_1 - k_3)^\mu T^{AVV}_{\mu\nu\alpha} \rightarrow T^{AV}_{\nu\alpha}(k_2, k_3) - T^{AV}_{\alpha\nu}(k_1, k_2) - 2mT^{PVV}_{\nu\alpha}, \tag{3.361}$$

$$(k_1 - k_2)^\nu T^{AVV}_{\mu\nu\alpha} \rightarrow T^{AV}_{\mu\alpha}(k_2, k_3) - T^{AV}_{\mu\alpha}(k_1, k_3), \tag{3.362}$$

$$(k_2 - k_3)^\alpha T^{AVV}_{\mu\nu\alpha} \rightarrow T^{AV}_{\mu\nu}(k_1, k_3) - T^{AV}_{\mu\nu}(k_1, k_2). \tag{3.363}$$

Our task consists of performing operations described on the left-hand side of these equations, aiming to recognize the structures from the right. Since contractions involving finite tensors and external momenta emerge throughout this procedure, these primary ingredients are discussed in the sequence.

As anticipated in the $PVVV$ integration, connecting J-vector contractions to reductions of finite functions is straightforward. That is transparent when comparing the J-vector (3.238) with properties achieved in Equations (3.151) and (3.155). After recognizing the J-scalar (3.238), we introduce the explicit results:

$$2p^\mu J_{3\mu} = p^2 J_3 - \frac{i}{(4\pi)^2}\left[\xi_0^{(0)}(p-q) - \xi_0^{(0)}(q)\right], \tag{3.364}$$

$$2q^\mu J_{3\mu} = q^2 J_3 - \frac{i}{(4\pi)^2}\left[\xi_0^{(0)}(p-q) - \xi_0^{(0)}(p)\right]. \tag{3.365}$$

That motivates us to pursue similar cases involving higher parameter powers, following the condition $a+b=2$. From the definition of the 2nd-order tensor (3.240), contracting the external momentum p yields

$$\begin{aligned} p^{\mu}J_{3\mu\nu} &= \frac{i}{(4\pi)^{2}}p_{\nu}\left[p^{2}\xi_{20}^{(-1)}+(p\cdot q)\,\xi_{11}^{(-1)}\right] \\ &+\frac{i}{(4\pi)^{2}}q_{\nu}\left[p^{2}\xi_{11}^{(-1)}+(p\cdot q)\,\xi_{02}^{(-1)}\right]-\frac{i}{(4\pi)^{2}}\frac{1}{2}p_{\nu}\xi_{00}^{(0)}. \end{aligned} \tag{3.366}$$

Combinations between brackets are the properties established in Equations (3.156)-(3.157), whose replacement leads to the first reduction within this category

$$2p^{\mu}J_{3\mu\nu}=p^{2}J_{3\nu}-\frac{i}{(4\pi)^{2}}\frac{1}{2}\left[(p+q)_{\nu}\,\xi_{0}^{(0)}\,(p-q)-q_{\nu}\xi_{0}^{(0)}\,(q)\right]. \tag{3.367}$$

The second arises by using momentum q and repeating this process:

$$2q^{\mu}J_{3\mu\nu}=q^{2}J_{3\nu}-\frac{i}{(4\pi)^{2}}\frac{1}{2}\left[(p+q)_{\nu}\,\xi_{0}^{(0)}\,(p-q)-p_{\nu}\xi_{0}^{(0)}\,(p)\right]. \tag{3.368}$$

Returning to the relations, we start with vector vertices, whose manipulations must yield in pure surface terms since this is the structure of the AV amplitude (3.229). Let the contraction between $p=k_1-k_2$ and the first vector vertex be the outset of this discussion. Promptly, several terms cancel out for being symmetric quantities multiplied by the Levi-Civita symbol. Hence, we obtain the following expression after relabeling some indices

$$\begin{aligned} p^{\nu}T_{\mu\nu\alpha}^{AVV} &= 2i\varepsilon_{\mu\nu\alpha\beta}\left\{\left[(p-q)^{\nu}p^{\rho}-p^{\nu}(k_{1}+k_{3})^{\rho}\right]\Delta_{\rho}^{\beta}\right. \\ &+4\,(p-q)^{\nu}\,p_{\rho}J_{3}^{\rho\beta}-4p^{\nu}q^{\beta}p^{\rho}J_{3\rho}-2\left(p^{2}-p\cdot q\right)p^{\nu}J_{3}^{\beta} \\ &\left.+p^{2}p^{\nu}q^{\beta}J_{3}+i\,(4\pi)^{-2}\,p^{\nu}q^{\beta}\xi_{0}^{(0)}\,(p-q)\right\}. \end{aligned} \tag{3.369}$$

Obeying the hierarchy intrinsic to these calculations, we employ reduction (3.367) to suppress the dependence on $a+b=2$ finite functions:

$$\begin{aligned} p^{\nu}T_{\mu\nu\alpha}^{AVV} &= 2i\varepsilon_{\mu\nu\alpha\beta}\left\{\left[(p-q)^{\nu}p^{\rho}-p^{\nu}(k_{1}+k_{3})^{\rho}\right]\Delta_{\rho}^{\beta}\right. \\ &-4p^{\nu}q^{\beta}p^{\rho}J_{3\rho}+2\left[(p\cdot q)\,p^{\nu}-p^{2}q^{\nu}\right]J_{3}^{\beta}+p^{2}p^{\nu}q^{\beta}J_{3} \\ &\left.+i\,(4\pi)^{-2}\,p^{\nu}q^{\beta}\left[\xi_{0}^{(0)}\,(q)-\xi_{0}^{(0)}\,(p-q)\right]\right\}. \end{aligned} \tag{3.370}$$

Reducing J-vectors is necessary to cancel out all finite contributions; however, there is a term where the corresponding contraction is disguised. Symmetry properties allow us to uncover it through an index permutation

$$\varepsilon_{\mu\nu\alpha\beta}\left[\left(p_{ij}\cdot q\right)p^{\nu}-\left(p_{ij}\cdot p\right)q^{\nu}\right]J_{3}^{\beta}=\varepsilon_{\mu\nu\alpha\beta}q^{\nu}p^{\beta}p_{ij}^{\rho}J_{3\rho}. \tag{3.371}$$

This identity admits choices for the difference between routing $p_{ij}=k_i-k_j$, but we set the p momentum for this particular occurrence. These identifications concentrate $a+b=1$ contributions into object (3.364), reducing this sector and eliminating all finite parts.

The final step before concluding this demonstration is to recognize surface terms as a difference between AVs. Thus, we reorganize coefficients to achieve a transparent view

$$p^{\nu}T_{\mu\nu\alpha}^{AVV}=2i\varepsilon_{\mu\nu\alpha\beta}\left[\left(q-p\right)^{\nu}\left(k_2+k_3\right)^{\rho}-q^{\nu}\left(k_1+k_3\right)^{\rho}\right]\Delta_{\rho}^{\beta}. \tag{3.372}$$

Hence, a comparison with Equation (3.229) is enough to complete the proof of this relation among GF:

$$p^{\nu}T_{\mu\nu\alpha}^{AVV}=T_{\mu\nu}^{AV}\left(k_2,k_3\right)-T_{\mu\nu}^{AV}\left(k_1,k_3\right). \tag{3.373}$$

Let us briefly describe the contraction between momentum $q-p=k_2-k_3$ and the index corresponding to the second vector vertex. It deals with a difference between external momenta, which generates cancellations between reductions. We emphasize this circumstance since it will simplify the box analysis significantly. Again, only surface terms remain after contracting the amplitude and employing *all* reductions

$$\left(q-p\right)^{\alpha}T_{\mu\nu\alpha}^{AVV}=2i\varepsilon_{\mu\nu\alpha\beta}\left[p^{\alpha}\left(k_1+k_2\right)^{\rho}-q^{\alpha}\left(k_1+k_3\right)^{\rho}\right]\Delta_{\rho}^{\beta}. \tag{3.374}$$

Identifying AV functions is straightforward for this particular case:

$$\left(q-p\right)^{\alpha}T_{\mu\nu\alpha}^{AVV}=T_{\mu\nu}^{AV}\left(k_1,k_3\right)-T_{\mu\nu}^{AV}\left(k_1,k_2\right). \tag{3.375}$$

Hence, we successfully verified the vector relations associated with triangle contractions. Properties of finite tensors and algebraic operations were the only resources necessary to achieve these results.

Lastly, we aim to perform the contraction between momentum $q=k_1-k_3$ and the index corresponding to the axial vertex. This operation must produce surface terms corresponding to AV amplitudes, similar to other cases. Furthermore, even though this type of contribution is not visible at first glance, finite functions proportional to the squared mass should arise. That is a requirement to identify the amplitude PVV (3.247).

Once our expectations are clear, let us look closer at the expression derived directly from the contraction:

$$\begin{aligned} q^{\mu}T^{AVV}_{\mu\nu\alpha} &= 2iq^{\mu}p^{\beta}\left(\varepsilon_{\mu\nu\beta\rho}\Delta^{\rho}_{\alpha}-\varepsilon_{\mu\alpha\beta\rho}\Delta^{\rho}_{\nu}\right)-2iq^{\mu}\left(k_1+k_3\right)_{\rho}\varepsilon_{\mu\nu\alpha\beta}\Delta^{\beta\rho} \\ &+8iq^{\mu}p^{\beta}\left(\varepsilon_{\mu\nu\beta\rho}J^{\rho}_{3\alpha}-\varepsilon_{\mu\alpha\beta\rho}J^{\rho}_{3\nu}\right)-4i\left(p^2-p\cdot q\right)q^{\mu}\varepsilon_{\mu\nu\alpha\beta}J^{\beta}_{3} \\ &-2iq^{\mu}p^{\beta}\varepsilon_{\mu\nu\alpha\beta}\left\{q^2J_3+i\left(4\pi\right)^{-2}\left[\xi^{(0)}_{0}\left(p\right)+\xi^{(0)}_{0}\left(p-q\right)\right]\right\}. \end{aligned} \tag{3.376}$$

Unlike previous cases, there are no reductions of J-tensors since contractions involve the Levi-Civita symbol instead of external momenta. Besides, factorizing surface terms to recognize the required amplitudes is not possible. That occurs because we chose a trace expression prioritizing the μ-index back in the integrand analysis, and now this feature brought an inadequate index configuration that prevents identifications. Therefore, our strategy is to exchange positions of indices to find known ingredients.

Let us explore the 2nd-order J-tensor to illustrate this point. Following the reasoning observed when discussing Dirac traces (3.59), we construct a tensor with antisymmetry in five indices (ρ fixed) through the following Schouten identity

$$\varepsilon_{\mu\nu\beta\rho}J^{\rho}_{3\alpha}-\varepsilon_{\mu\alpha\beta\rho}J^{\rho}_{3\nu}=-\varepsilon_{\rho\alpha\mu\nu}J^{\rho}_{3\beta}-\varepsilon_{\nu\beta\rho\alpha}J^{\rho}_{3\mu}-\varepsilon_{\alpha\mu\nu\beta}J^{\rho}_{3\rho}. \tag{3.377}$$

By replacing this result on the relation among GF, the first two terms on the right-hand side generate momenta contractions. Hence, we must follow the procedure established for vector contractions and reduce finite contributions. These operations vanish most finite contributions, so the AVV contraction assumes the form

$$\begin{aligned} q^{\mu}T^{AVV}_{\mu\nu\alpha} &= 2iq^{\mu}p^{\beta}\left(\varepsilon_{\mu\nu\beta\rho}\Delta^{\rho}_{\alpha}-\varepsilon_{\mu\alpha\beta\rho}\Delta^{\rho}_{\nu}\right)-2iq^{\mu}\left(k_1+k_3\right)_{\rho}\varepsilon_{\mu\nu\alpha\beta}\Delta^{\beta\rho} \\ &-8iq^{\mu}p^{\beta}\varepsilon_{\mu\nu\alpha\beta}\left[J^{\rho}_{3\rho}+i\left(4\pi\right)^{-2}\xi^{(0)}_{0}\left(p-q\right)\right]. \end{aligned} \tag{3.378}$$

The index permutation above also brought an additional term depending on object $J^{\rho}_{3\rho}$. From definition (3.240), we take the J-tensor trace and identify reductions of finite structure functions (3.156) and (3.159):

$$J^{\rho}_{3\rho}=\frac{i}{\left(4\pi\right)^2}\left\{\left[p^2\xi^{(-1)}_{20}+\left(p\cdot q\right)\xi^{(-1)}_{11}\right]+\left[\left(p\cdot q\right)\xi^{(-1)}_{11}+q^2\xi^{(-1)}_{02}\right]-2\xi^{(0)}_{00}\right\}. \tag{3.379}$$

Although this structure resembles those of momenta contractions, we stress the presence of the finite function $\xi^{(0)}_{00}$. By replacing other reductions and expressing this contribution in terms of elements belonging to the $\xi^{(-1)}_{nm}$-family (3.160), we obtain the following trace:

$$J^{\rho}_{3\rho}=m^2J_3+\frac{i}{\left(4\pi\right)^2}\left[\frac{1}{2}-\xi^{(0)}_{0}\left(p-q\right)\right]. \tag{3.380}$$

Both the term proportional to the squared mass and the numerical factor remain when replacing this result within the AVV contraction. A comparison with Equation (3.247) shows that the first corresponds to the PVV amplitude[11]. With this identification, we finish explorations about finite contributions for now:

$$\begin{aligned} q^{\mu}T^{AVV}_{\mu\nu\alpha} &= 2iq^{\mu}p^{\beta}\left(\varepsilon_{\mu\nu\beta\rho}\Delta^{\rho}_{\alpha} - \varepsilon_{\mu\alpha\beta\rho}\Delta^{\rho}_{\nu}\right) - 2iq^{\mu}\left(k_1+k_3\right)_{\rho}\varepsilon_{\mu\nu\alpha\beta}\Delta^{\beta\rho} \\ &\quad -2mT^{PVV}_{\mu\nu} + \frac{1}{4\pi^2}\varepsilon_{\mu\nu\alpha\beta}q^{\mu}p^{\beta}. \end{aligned} \tag{3.381}$$

Alternatively, we could achieve this expression by making explicit the content of J-tensors from the beginning. Such a perspective would make calculations for this relation exceptionally simple. Even so, we chose to preserve the elements given by the systematization and follow a longer path. This reasoning established a routine, which will be fundamental to perform box contractions.

Extending this discussion to divergent contributions is direct if we note that the first structure of the equation above exhibits the same index configuration observed for 2nd-order J-tensors. Therefore, if the Schouten identity (3.377) applies to surface terms, index permutations produce the organization required to recognize the remaining amplitudes

$$\begin{aligned} q^{\mu}T^{AVV}_{\mu\nu\alpha} &= T^{AV}_{\nu\alpha}\left(k_2,k_3\right) - T^{AV}_{\alpha\nu}\left(k_1,k_2\right) - 2mT^{PVV}_{\nu\alpha} \\ &\quad -2iq^{\mu}p^{\beta}\varepsilon_{\mu\nu\alpha\beta}\left[\Delta^{\rho}_{\rho} + \frac{i}{8\pi^2}\right]; \end{aligned} \tag{3.382}$$

see Equation (3.229). Once again, we have an additional object Δ^{ρ}_{ρ} for this relation. We highlight that the only requirement to obtain this result is the validity of the integral linearity.

Our objective was to perform the axial vertex contraction for the AVV amplitude to verify the corresponding relation among GF; however, we found an additional contribution in the second row of the equation above. Differently from vector relations, this one is not automatic since it depends on a condition over the value attributed to surface terms. Its satisfaction occurs if the quantity in square brackets is null, which would imply the ensuing values for the surface term and its trace:

$$\Delta_{\rho\sigma} = -\frac{i}{32\pi^2}g_{\rho\sigma}, \quad \Delta^{\rho}_{\rho} = -\frac{i}{8\pi^2}. \tag{3.383}$$

We aim to extend these calculations to box contractions in the following subsection. For both cases, Dirac traces admit different expressions because they led to products involving the Levi-Civita symbol and metric tensors. We expect that the reasoning devel-

[11] In general, subamplitudes within AV^n might produce contributions belonging to PV^n-type amplitudes. That does not transpire here due to specific trace choices.

oped for the triangle also applies in the box context, so the chosen traces link to additional terms. Afterward, we discuss the source of this mathematical structure and investigate its implications.

3.5.2 Relations Among Green Functions - $AVVV$

This subsection aims to verify relations among GF derived for $AVVV$ contractions (3.305). As the corresponding expectations are cast in Equations (3.22)-(3.25), we simply transcribe them here:

$$(k_1 - k_4)^{\mu} T^{AVVV}_{\mu\nu\alpha\beta} \rightarrow T^{AVV}_{\nu\alpha\beta}(k_2, k_3, k_4) - T^{AVV}_{\beta\nu\alpha}(k_1, k_2, k_3) - 2mT^{PVVV}_{\nu\alpha\beta}, \quad (3.384)$$

$$(k_1 - k_2)^{\nu} T^{AVVV}_{\mu\nu\alpha\beta} \rightarrow T^{AVV}_{\mu\alpha\beta}(k_2, k_3, k_4) - T^{AVV}_{\mu\alpha\beta}(k_1, k_3, k_4), \quad (3.385)$$

$$(k_2 - k_3)^{\alpha} T^{AVVV}_{\mu\nu\alpha\beta} \rightarrow T^{AVV}_{\mu\nu\beta}(k_1, k_3, k_4) - T^{AVV}_{\mu\nu\beta}(k_1, k_2, k_4), \quad (3.386)$$

$$(k_4 - k_3)^{\beta} T^{AVVV}_{\mu\nu\alpha\beta} \rightarrow T^{AVV}_{\mu\nu\alpha}(k_1, k_2, k_3) - T^{AVV}_{\mu\nu\alpha}(k_1, k_2, k_4). \quad (3.387)$$

Although they have different levels of complexity, triangle and box calculations contain analogous ingredients. Notably, the systematization through J-tensors establishes a clear link between both cases. That strongly shapes the procedure adopted this time, so we introduce all properties of these tensors beforehand.

Momenta contractions occur subsequently, starting with those involving vector vertices. Observing the forms adopted for traces throughout this investigation, we expect them to exhibit reductions from the outset. That makes this context simpler even if numerous algebraic operations are necessary. The axial contraction requires index permutations additionally; thus, we approach this case carefully in the final subsubsection.

Properties of Finite Tensors

One remarkable ingredient of the systematization brought by IReg concerns structure functions used to describe the finite part of amplitudes. Those functions typical of four-point integrals were introduced in Subsection (3.3.4), where they receive integral representations characterized by three Feynman parameters. Furthermore, we derived reductions of these functions, in which case combinations constrained by the same sum of parameter powers $a + b + c$ lead to structures with decreased powers.

In Subsection (3.4.5), we computed four-point Feynman integrals and projected their finite content through the mentioned functions. Nonetheless, they did not appear randomly but grouped into symmetric objects following a constraint regarding parameter powers, the so-called J-tensors. Reductions appear inside momenta contractions and traces of them. Therefore, after performing these operations, we cast properties that concern this investigation below. Since the 4th-order tensor does not contribute to the studied amplitudes, we omit the corresponding information. We recall the notations for finite functions and tensors through the associations: $\xi_{ab} = \xi_{ab}(p,q)$, $\xi'_{ab} = \xi_{ab}(p,r)$, $\xi''_{ab} = \xi_{ab}(q,r)$, $\xi'''_{ab} = \xi_{ab}(q-p, r-p)$, and $\xi_{abc} = \xi_{abc}(p,q,r)$.

- First-order tensor - reducing $a+b+c=1$

$$2p^{\mu} J_{4\mu} = p^2 J_4 + J_3''' - J_3'' \quad (3.388)$$
$$2q^{\mu} J_{4\mu} = q^2 J_4 + J_3''' - J_3' \quad (3.389)$$
$$2r^{\mu} J_{4\mu} = r^2 J_4 + J_3''' - J_3 \quad (3.390)$$

- Second-order tensor - reducing $a+b+c=2$

$$2p^{\mu} J_{4\mu\nu} = p^2 J_{4\nu} + J_{3\nu}''' + p_{\nu} J_3''' - J_{3\nu}'' \quad (3.391)$$
$$2q^{\mu} J_{4\mu\nu} = q^2 J_{4\nu} + J_{3\nu}''' + p_{\nu} J_3''' - J_{3\nu}' \quad (3.392)$$
$$2r^{\mu} J_{4\mu\nu} = r^2 J_{4\nu} + J_{3\nu}''' + p_{\nu} J_3''' - J_{3\nu} \quad (3.393)$$
$$J_{4\mu\mu} = m^2 J_4 + J_3''' \quad (3.394)$$

- Third-order tensor - reducing $a+b+c=3$

$$2p^{\mu} J_{4\mu\nu\alpha} = p^2 J_{4\nu\alpha} + J_{3\nu\alpha}''' + p_{\nu} J_{3\alpha}''' + p_{\alpha} J_{3\nu}''' + p_{\nu\alpha} J_3''' - J_{3\nu\alpha}'' \quad (3.395)$$
$$2q^{\mu} J_{4\mu\nu\alpha} = q^2 J_{4\nu\alpha} + J_{3\nu\alpha}''' + p_{\nu} J_{3\alpha}''' + p_{\alpha} J_{3\nu}''' + p_{\nu\alpha} J_3''' - J_{3\nu\alpha}' \quad (3.396)$$
$$2r^{\mu} J_{4\mu\nu\alpha} = r^2 J_{4\nu\alpha} + J_{3\nu\alpha}''' + p_{\nu} J_{3\alpha}''' + p_{\alpha} J_{3\nu}''' + p_{\nu\alpha} J_3''' - J_{3\nu\alpha} \quad (3.397)$$
$$J_{4\mu\mu\nu} = m^2 J_{4\nu} + J_{3\nu}''' + p_{\nu} J_3''' \quad (3.398)$$

Although we already employed reductions of three-point functions, introducing different momenta configurations is necessary. For such purpose, recall the discussion developed when exploring 2nd-order standard tensors in the box context (3.4.7). These properties are cast in the sequence.

- Denominator D_{123} - $\xi_{ab} = \xi_{ab}(p,q)$

$$2p^{\mu} J_{3\mu} = p^2 J_3 - i(4\pi)^{-2} \left[\xi_0^{(0)}(p-q) - \xi_0^{(0)}(q) \right] \quad (3.399)$$
$$2q^{\mu} J_{3\mu} = q^2 J_3 - i(4\pi)^{-2} \left[\xi_0^{(0)}(p-q) - \xi_0^{(0)}(p) \right] \quad (3.400)$$
$$2p^{\mu} J_{3\mu\nu} = p^2 J_{3\nu} - i(4\pi)^{-2} \frac{1}{2} \left[(p+q)_{\nu} \xi_0^{(0)}(p-q) - q_{\nu} \xi_0^{(0)}(q) \right] \quad (3.401)$$
$$2q^{\mu} J_{3\mu\nu} = q^2 J_{3\nu} - i(4\pi)^{-2} \frac{1}{2} \left[(p+q)_{\nu} \xi_0^{(0)}(p-q) - p_{\nu} \xi_0^{(0)}(p) \right] \quad (3.402)$$
$$J_{3\rho}^{\rho} = m^2 J_3 + i(4\pi)^{-2} \left[\frac{1}{2} - \xi_0^{(0)}(p-q) \right] \quad (3.403)$$

- Denominator D_{124} - $\xi'_{ab} = \xi_{ab}(p,r)$

$$2p^{\mu}J'_{3\mu} = p^2 J'_3 - i(4\pi)^{-2}\left[\xi_0^{(0)}(p-r) - \xi_0^{(0)}(r)\right] \quad (3.404)$$

$$2r^{\mu}J'_{3\mu} = r^2 J'_3 - i(4\pi)^{-2}\left[\xi_0^{(0)}(p-r) - \xi_0^{(0)}(p)\right] \quad (3.405)$$

$$2p^{\mu}J'_{3\mu\nu} = p^2 J'_{3\nu} - i(4\pi)^{-2}\frac{1}{2}\left[(p+r)_{\nu}\xi_0^{(0)}(p-r) - r_{\nu}\xi_0^{(0)}(r)\right] \quad (3.406)$$

$$2r^{\mu}J'_{3\mu\nu} = r^2 J'_{3\nu} - i(4\pi)^{-2}\frac{1}{2}\left[(p+r)_{\nu}\xi_0^{(0)}(p-r) - p_{\nu}\xi_0^{(0)}(p)\right] \quad (3.407)$$

$$J'^{\rho}_{3\rho} = m^2 J'_3 + i(4\pi)^{-2}\left[\frac{1}{2} - \xi_0^{(0)}(p-r)\right] \quad (3.408)$$

- Denominator D_{134} - $\xi''_{ab} = \xi_{ab}(q,r)$

$$2q^{\mu}J''_{3\mu} = q^2 J''_3 - i(4\pi)^{-2}\left[\xi_0^{(0)}(q-r) - \xi_0^{(0)}(r)\right] \quad (3.409)$$

$$2r^{\mu}J''_{3\mu} = r^2 J''_3 - i(4\pi)^{-2}\left[\xi_0^{(0)}(q-r) - \xi_0^{(0)}(q)\right] \quad (3.410)$$

$$2q^{\mu}J''_{3\mu\nu} = q^2 J''_{3\nu} - i(4\pi)^{-2}\frac{1}{2}\left[(q+r)_{\nu}\xi_0^{(0)}(q-r) - r_{\nu}\xi_0^{(0)}(r)\right] \quad (3.411)$$

$$2r^{\mu}J''_{3\mu\nu} = r^2 J''_{3\nu} - i(4\pi)^{-2}\frac{1}{2}\left[(q+r)_{\nu}\xi_0^{(0)}(q-r) - q_{\nu}\xi_0^{(0)}(q)\right] \quad (3.412)$$

$$J''^{\rho}_{3\rho} = m^2 J''_3 + i(4\pi)^{-2}\left[\frac{1}{2} - \xi_0^{(0)}(q-r)\right] \quad (3.413)$$

- Denominator D_{234} - $\xi'''_{ab} = \xi_{ab}(q-p,r-p)$

$$2(q-p)^{\mu}J'''_{3\mu} = (q-p)^2 J'''_3 - i(4\pi)^{-2}\left[\xi_0^{(0)}(q-r) - \xi_0^{(0)}(r-p)\right] \quad (3.414)$$

$$2(r-p)^{\mu}J'''_{3\mu} = (r-p)^2 J'''_3 - i(4\pi)^{-2}\left[\xi_0^{(0)}(q-r) - \xi_0^{(0)}(q-p)\right] \quad (3.415)$$

$$\begin{aligned}2(q-p)^{\mu}J'''_{3\mu\nu} &= (q-p)^2 J'''_{3\nu} - \frac{1}{2}i(4\pi)^{-2}\times \\ &\times\left[(q+r-2p)_{\nu}\xi_0^{(0)}(q-r) - (r-p)_{\nu}\xi_0^{(0)}(r-p)\right]\end{aligned} \quad (3.416)$$

$$\begin{aligned}2(r-p)^{\mu}J'''_{3\mu\nu} &= (r-p)^2 J'''_{3\nu} - \frac{1}{2}i(4\pi)^{-2}\times \\ &\times\left[(q+r-2p)_{\nu}\xi_0^{(0)}(q-r) - (q-p)_{\nu}\xi_0^{(0)}(q-p)\right]\end{aligned} \quad (3.417)$$

$$J'''^{\rho}_{3\rho} = m^2 J'''_3 + i(4\pi)^{-2}\left[\frac{1}{2} - \xi_0^{(0)}(q-r)\right] \quad (3.418)$$

Vector Contractions

Proceeding to the explicit computation of relations among GF of the $AVVV$ function (3.305), let us consider vector vertices first. For them, a contraction with the corresponding momentum results in a difference between AVV triangles. Hence, using the expression attributed to this amplitude (3.258) gives hints for future calculations.

The most immediate implications concern terms whose index arrangements do not find correspondence inside the triangle. For instance, the $AVPP$ function fits this category for still being proportional to the metric tensor $g_{\alpha\beta}$ after contracting the index ν. When exploring other relations, this notion extends to similar amplitudes. Using reductions of 2nd and 1st-order J-tensors, we prove that these products indeed vanish

$$g_{\alpha\beta}p^{\nu}T_{\mu\nu}^{AVPP} = 0, \tag{3.419}$$
$$g_{\nu\beta}\left(q-p\right)^{\alpha}T_{\mu\alpha}^{APVP} = 0, \tag{3.420}$$
$$g_{\nu\alpha}\left(q-r\right)^{\beta}T_{\mu\beta}^{APPV} = 0. \tag{3.421}$$

Subsequently, look at those structures proportional to the Levi-Civita symbol having μ as the only free index. Comparing tensor (3.322) with the adequate sectors from $APVP$ and $APPV$ functions (3.352) shows that these contributions cancel out identically for the first contraction. Analogous structures arise for other contractions and cancel out in the same way. Therefore, we cast these identities in the sequence

$$\varepsilon_{\mu XYZ}\left[4p^{Z}T_{XY\alpha\beta}^{(12)}+p_{\beta}F_{4\alpha XYZ}^{(+,+)}-p_{\alpha}F_{4\beta XYZ}^{(+,-)}\right] = 0, \tag{3.422}$$
$$\varepsilon_{\mu XYZ}\left[4\left(q-p\right)^{Z}T_{XY\nu\beta}^{(13)}-\left(q-p\right)_{\beta}F_{4\nu XYZ}^{(-,+)}-\left(q-p\right)_{\nu}F_{4\beta XYZ}^{(+,-)}\right] = 0, \tag{3.423}$$
$$\varepsilon_{\mu XYZ}\left[4\left(q-r\right)^{Z}T_{XY\nu\alpha}^{(14)}-\left(q-r\right)_{\alpha}F_{4\nu XYZ}^{(-,+)}+\left(q-r\right)_{\nu}F_{4\alpha XYZ}^{(+,+)}\right] = 0. \tag{3.424}$$

Contractions assume the forms below when disregarding null objects:

$$\begin{aligned} p^{\nu}T_{\mu\nu\alpha\beta}^{AVVV} = {} & \varepsilon_{\mu\alpha XY}\left[4p^{\nu}T_{XY\nu\beta}^{(13)}+p_{X}T_{Y\beta}^{VPPV}+p_{\beta}F_{4XY}\right] \\ & +\varepsilon_{\mu\beta XY}\left[4p^{\nu}T_{XY\nu\alpha}^{(14)}+p_{X}T_{Y\alpha}^{VPVP}-p_{\alpha}F_{4XY}\right] \\ & +\varepsilon_{\mu\alpha\beta X}\left[2p_{X}T^{PPPP}-p^{\nu}T_{X\nu}^{VVPP}\right], \end{aligned} \tag{3.425}$$

$$\begin{aligned} \left(q-p\right)^{\alpha}T_{\mu\nu\alpha\beta}^{AVVV} = {} & \varepsilon_{\mu\nu XY}\left[4\left(q-p\right)^{\alpha}T_{XY\alpha\beta}^{(12)}-\left(q-p\right)^{X}T_{Y\beta}^{VPPV}-\left(q-p\right)_{\beta}F_{4XY}\right] \\ & +\varepsilon_{\mu\beta XY}\left[4\left(q-p\right)^{\alpha}T_{XY\nu\alpha}^{(14)}+\left(q-p\right)^{X}T_{Y\nu}^{VVPP}-\left(q-p\right)_{\nu}F_{4XY}\right] \\ & -\varepsilon_{\mu\nu\beta X}\left[2\left(q-p\right)^{X}T^{PPPP}+\left(q-p\right)^{\alpha}T_{X\alpha}^{VPVP}\right], \end{aligned} \tag{3.426}$$

$$\begin{aligned} \left(q-r\right)^{\beta}T_{\mu\nu\alpha\beta}^{AVVV} = {} & \varepsilon_{\mu\nu XY}\left[4\left(q-r\right)^{\beta}T_{XY\alpha\beta}^{(12)}-\left(q-r\right)^{X}T_{Y\alpha}^{VPVP}-\left(q-r\right)_{\alpha}F_{4XY}\right] \\ & +\varepsilon_{\mu\alpha XY}\left[4\left(q-r\right)^{\beta}T_{XY\nu\beta}^{(13)}-\left(q-r\right)^{X}T_{Y\nu}^{VVPP}+\left(q-r\right)_{\nu}F_{4XY}\right] \\ & +\varepsilon_{\mu\nu\alpha X}\left[2\left(q-r\right)^{X}T^{PPPP}-\left(q-r\right)^{\beta}T_{X\beta}^{VPPV}\right]. \end{aligned} \tag{3.427}$$

Our task becomes reducing all four-point finite functions, expecting that only structures associated with the triangle remain. Although the number of terms might bring complications, exploring each component separately is possible since different tensor arrangements do not mix. Nevertheless, be aware in this process that tensor subamplitudes carry contributions proportional to the scalar one.

Once these operations are clear from the previous subsection, let us just stress some details. The hierarchy associated with reductions must be strictly followed; therefore, we start with the highest-order structure function from four-point integrals $a+b+c=3$ and gradually decrease parameter powers. With this stage complete, it is necessary to process three-point structures using identity (3.371). That is possibly the most intricate part of these calculations, so using the AVV as a guide becomes essential; consult Equation (3.258). Meanwhile, reductions subtract each other for contractions dealing with differences between external momenta. That is a source of cancellations, decreasing our efforts when studying this sector. To exemplify, we present the first contraction in its final organization

$$\begin{aligned}
\left[p^{\nu}T^{AVVV}_{\mu\nu\alpha\beta}\right]_{\text{fin}} &= 8\varepsilon_{\mu\alpha XY}\Big\{(q-p)_X J'''_{3Y\beta} - q_X J''_{3Y\beta} - (q-p)_X (r-p)_\beta J'''_{3Y} \\
&\quad + q_X r_\beta J''_{3Y}\} + 8\varepsilon_{\mu\beta XY}\{(r-q)_X (J'''_{3Y\alpha} - J''_{3Y\alpha}) - q_X r_Y J''_{3\alpha} \\
&\quad + (q-p)_X (r-p)_Y J'''_{3\alpha}\} - 2\varepsilon_{\mu\alpha\beta X}\left\{\left(q^2 r_X - r^2 q_X\right) J''_3\right. \\
&\quad - 2q\cdot(q-r) J''_{3X} + 2(q-p)\cdot(q-r) J'''_{3X} \\
&\quad - \left[(q-p)^2 (r-p)_X - (r-p)^2 (q-p)_X\right] J'''_3 \\
&\quad \left. - i(4\pi)^{-2}\left[(q-p)^X \xi^{(0)}_0 (q-p) - q^X \xi^{(0)}_0 (q)\right]\right\}. \qquad (3.428)
\end{aligned}$$

We still have to analyze divergent structures to complete this analysis. As stated before, even though Feynman integrals depend on different standard objects, only one type of surface term appears within the $AVVV$ box. Our work summarizes into surveying substructures of this amplitude to find the corresponding contributions and organize them through algebraic operations. We exemplify this procedure for the first contraction:

$$\begin{aligned}
\left[p^{\nu}T^{AVVV}_{\mu\nu\alpha\beta}\right]_{\text{div}} &= 2p^{\nu}\left(\varepsilon_{\mu\alpha\beta X}\Delta^X_\nu + \varepsilon_{\mu\nu\alpha X}\Delta^X_\beta\right) \qquad (3.429) \\
&= -2\varepsilon_{\mu\beta XY}\left[(q-r)-(q-r)\right]^X \Delta^Y_\alpha - 2\varepsilon_{\mu\nu\alpha X}\left[(q-p)-q\right]^\nu \Delta^X_\beta \\
&\quad - 2\varepsilon_{\mu\alpha\beta X}\left[(k_2+k_4)-(k_1+k_4)\right]^\nu \Delta^X_\nu. \qquad (3.430)
\end{aligned}$$

At this point, identifying divergent and finite parts as those belonging to the triangle is straightforward (3.258). That extends to all cases; hence, all vector relations among GF apply regardless of the prescription adopted to evaluate surface terms:

$$p^{\nu}T^{AVVV}_{\mu\nu\alpha\beta} = T^{AVV}_{\mu\alpha\beta}(k_2,k_3,k_4) - T^{AVV}_{\mu\alpha\beta}(k_1,k_3,k_4), \qquad (3.431)$$

$$
\begin{aligned}
(q-p)^{\alpha}T^{AVVV}_{\mu\nu\alpha\beta} &= T^{AVV}_{\mu\nu\beta}\left(k_1,k_3,k_4\right)-T^{AVV}_{\mu\nu\beta}\left(k_1,k_2,k_4\right), &(3.432)\\
(q-r)^{\beta}T^{AVVV}_{\mu\nu\alpha\beta} &= T^{AVV}_{\mu\nu\alpha}\left(k_1,k_2,k_3\right)-T^{AVV}_{\mu\nu\alpha}\left(k_1,k_2,k_4\right). &(3.433)
\end{aligned}
$$

Axial Contraction

The remaining box relation arises from the contraction between the momentum $r = k_1 - k_4$ and the index corresponding to the axial vertex. Firstly, following the route established for vector cases, observe that structures associated with odd subamplitudes stand out from others. That is transparent when comparing terms where the metric has exclusively free indices; consult the final expressions for $AVVV$ (3.305) and $PVVV$ (3.282). Hence, our initial task is to verify the following expectation

$$
r^{\mu}\left[g_{\alpha\beta}T^{AVPP}_{\mu\nu}+g_{\nu\beta}T^{APVP}_{\mu\alpha}+g_{\nu\alpha}T^{APPV}_{\mu\beta}\right]=8im^2\left(g_{\kappa\nu}g_{\alpha\beta}-g_{\kappa\alpha}g_{\nu\beta}+g_{\kappa\beta}g_{\nu\alpha}\right)F_{4\kappa} \quad (3.434)
$$

We resort to the information established in Subsubsection (3.4.7) to accomplish this result. The first sector of the explored amplitudes features a three-index contraction involving the Levi-Civita symbol; thus, introducing another external momentum vanishes most contributions. Only the 2nd-order J-tensor (3.279) remains because it has terms on the metric tensor:

$$
\begin{aligned}
& -i\varepsilon_{\mu XYZ}r^{\mu}\left[g_{\alpha\beta}F^{(-,+)}_{4\nu XYZ}-g_{\nu\beta}F^{(+,+)}_{4\alpha XYZ}+g_{\nu\alpha}F^{(+,-)}_{4\beta XYZ}\right]\\
= \; & 8i\varepsilon_{\mu XYZ}r^{\mu}p^{X}q^{Y}\left(g_{\alpha\beta}J_{4\nu Z}-g_{\nu\beta}J_{4\alpha Z}+g_{\nu\alpha}J_{4\beta Z}\right)\\
= \; & 4ir^{\mu}p^{X}q^{Y}\left(g_{\alpha\beta}\varepsilon_{\mu\nu XY}-g_{\nu\beta}\varepsilon_{\mu\alpha XY}+g_{\nu\alpha}\varepsilon_{\mu\beta XY}\right)i\left(4\pi\right)^{-2}\xi^{(-1)}_{000}. \qquad (3.435)
\end{aligned}
$$

A finite function as $\xi^{(-1)}_{000}$ is typical of higher-order Feynman integrals; therefore, not compatible with intended identifications. Reduction (3.205) handles this situation while bringing the squared mass contribution necessary to find $F_{4\mu}$; we transcribe this property here

$$
\xi^{(-1)}_{000}=2m^2\xi^{(-2)}_{000}-\left[p^2\xi^{(-2)}_{100}+q^2\xi^{(-2)}_{010}+r^2\xi^{(-2)}_{001}\right]+\left[\xi^{(-1)}_{00}\right]^{\prime\prime\prime}. \qquad (3.436)
$$

Notwithstanding that the situation is similar to the other sector, it leads to a more complex expression due to the two-index contraction:

$$
\begin{aligned}
& -ir^{\mu}\left(g_{\alpha\beta}\varepsilon_{\mu\nu XY}-g_{\nu\beta}\varepsilon_{\mu\alpha XY}+g_{\nu\alpha}\varepsilon_{\mu\beta XY}\right)F_{4XY}\\
= \; & ir^{\mu}\left(g_{\alpha\beta}\varepsilon_{\mu\nu XY}-g_{\nu\beta}\varepsilon_{\mu\alpha XY}+g_{\nu\alpha}\varepsilon_{\mu\beta XY}\right)\times\\
& \times\left\{4\left[\left(q^2-q\cdot r\right)p_X-\left(p^2-p\cdot r\right)q_X\right]J_{4Y}\right.\\
& \left.+2p_Xq_Y\left(r^2J_4-J_3^{\prime\prime\prime}-J_3\right)\right\}. \qquad (3.437)
\end{aligned}
$$

Even so, both parts fit perfectly since functions constrained by $a+b+c=1$ compound the vector reduction $r^{\mu}J_{4\mu}$. Such an object cancels out all spare terms, completing the proof of relation (3.434). That corresponds to the first row from $PVVV$ amplitude (3.304).

As in triangle calculations (3.376), the remaining steps require index permutations through the symmetry properties of tensors. A crucial feature of these operations is that they generate additional contributions embodied in traces, which generate the expected contributions proportional to the squared mass.

To illustrate this procedure, we analyze finite functions whose parameter powers follow the condition $a+b+c=3$. They compound 3rd-order J_4-tensors found inside tensor combinations belonging to the box amplitude:

$$\begin{aligned}\left[r^{\mu}T^{AVVV}_{\mu\nu\alpha\beta}\right]_{a+b+c=3} &= 16r^{\mu}p^{X}\left(\varepsilon_{\mu\nu XY}J_{4\alpha Y\beta}-\varepsilon_{\mu\alpha XY}J_{4\nu Y\beta}\right)\\ &+16r^{\mu}q^{X}\left(\varepsilon_{\mu\alpha XY}J_{4\beta Y\nu}-\varepsilon_{\mu\beta XY}J_{4\alpha Y\nu}\right). \end{aligned} \tag{3.438}$$

Our reasoning consists of building an object exhibiting antisymmetry in five indices, a Schouten identity. Thus, considering only the first J_4-index as changeable, let us rearrange indices accordingly to the expression

$$\begin{aligned}\left[r^{\mu}T^{AVVV}_{\mu\nu\alpha\beta}\right]_{a+b+c=3} &= -16r^{\mu}p^{X}\left(\varepsilon_{\alpha\mu\nu X}J_{4YY\beta}+\varepsilon_{Y\alpha\mu\nu}J_{4XY\beta}+\varepsilon_{\nu XY\alpha}J_{4\mu Y\beta}\right)\\ &-16r^{\mu}q^{X}\left(\varepsilon_{\beta\mu\alpha X}J_{4YY\nu}+\varepsilon_{Y\beta\mu\alpha}J_{4XY\nu}+\varepsilon_{\alpha XY\beta}J_{4\mu Y\nu}\right). \end{aligned} \tag{3.439}$$

As all pieces are known, see Equations (3.395)-(3.398), the adequate replacements yield

$$\begin{aligned}&\left[r^{\mu}T^{AVVV}_{\mu\nu\alpha\beta}\right]_{a+b+c=3}\\ =\;&-16\varepsilon_{\alpha\mu\nu X}r^{\mu}p^{X}\left(m^{2}J_{4\beta}+J'''_{3\beta}+p_{\beta}J'''_{3}\right)\\ &-8\varepsilon_{Y\alpha\mu\nu}r^{\mu}\left(p^{2}J_{4Y\beta}+J'''_{3Y\beta}+p_{Y}J'''_{3\beta}+p_{\beta}J'''_{3Y}+p_{Y\beta}J'''_{3}-J''_{3Y\beta}\right)\\ &-8\varepsilon_{\nu XY\alpha}p^{X}\left(r^{2}J_{4Y\beta}+J'''_{3Y\beta}+p_{\beta}J'''_{3Y}-J_{3Y\beta}\right)\\ &-16\varepsilon_{\beta\mu\alpha X}r^{\mu}q^{X}\left(m^{2}J_{4\nu}+J'''_{3\nu}+p_{\nu}J'''_{3}\right)\\ &-8\varepsilon_{Y\beta\mu\alpha}r^{\mu}\left(q^{2}J_{4Y\nu}+J'''_{3Y\nu}+p_{Y}J'''_{3\nu}+p_{\nu}J'''_{3Y}+p_{Y\nu}J'''_{3}-J'_{3Y\nu}\right)\\ &-8\varepsilon_{\alpha XY\beta}q^{X}\left(r^{2}J_{4Y\nu}+J'''_{3Y\nu}+p_{Y}J'''_{3\nu}+p_{\nu}J'''_{3Y}+p_{Y\nu}J'''_{3}-J_{3Y\nu}\right). \end{aligned} \tag{3.440}$$

The next step is to track all finite contributions under the restriction $a+b+c=2$. After rearrangements and other algebraic operations, we obtain momenta contractions and traces of the 2nd-order J_4-tensor (3.391)-(3.394). These traces contain terms proportional to the squared mass that complete the content of four-point finite functions within $PVVV$ (there are some missing pieces on J_3). Except for this sector, other structure functions under this category disappear in the sequence through reductions of J_4-vectors (3.388)-(3.390). Although the process described in this paragraph is notably extensive, all steps are transparent and easily checked.

We must still explore those objects associated with three-point finite functions to perform the remaining identifications, including the AVV part (3.258). In addition to being quite extensive, this part also brings complications due to the different momenta configurations associated with the line notation. This discussion appears in detail when exploring 2nd-order standard tensors in the box context (3.4.7), while required tensor properties are at the outset of this subsection (3.5.2).

After fulfilling all reductions, we write for the finite sector

$$\begin{aligned} \left[r^{\mu}T^{AVVV}_{\mu\nu\alpha\beta}\right]_{\text{fin}} &= \left[T^{AVV}_{\nu\alpha\beta}\left(k_2,k_3,k_4\right) - T^{AVV}_{\beta\nu\alpha}\left(k_1,k_2,k_3\right)\right]_{\text{fin}} - 2mT^{PVVV}_{\nu\alpha\beta} \\ &\quad -2\varepsilon_{\nu\alpha\beta X}\left(k_1-k_2+k_3-k_4\right)^{X}\frac{i}{8\pi^2}. \end{aligned} \tag{3.441}$$

Among all components, let us emphasize the role played by traces $J^{\rho}_{3\rho}$ and $J'''^{\rho}_{3\rho}$ from Equations (3.403) and (3.418). First, their terms on the squared mass led to the missing pieces that completed the finite amplitude $PVVV$. Second, numerical factors are additional terms if one considers the original expectation for this relation. They correspond to the second line of the equation above and will receive more attention soon enough.

Lastly, we pursue divergent objects that remain in even subamplitudes after the axial vertex contraction:

$$\left[r^{\mu}T^{AVVV}_{\mu\nu\alpha\beta}\right]_{\text{div}} = 2\varepsilon_{\mu\alpha\beta X}r^{\mu}\Delta_{X\nu} + 2\varepsilon_{\mu\nu\alpha X}r^{\mu}\Delta_{X\beta}. \tag{3.442}$$

Although that differs significantly from the organization expected for the triangle (3.258), performing algebraic manipulations and exchanging index positions solve this situation. We add Schouten identities involving routings k_2 and k_3 since they are absent in this equation. That leads to the following structure

$$\begin{aligned} \left[r^{\mu}T^{AVVV}_{\mu\nu\alpha\beta}\right]_{\text{div}} &= -2\varepsilon_{\mu\nu XY}\left(k_4-k_3\right)^{X}\Delta_{\alpha Y} + 2\varepsilon_{\nu\alpha XY}\left(k_2-k_3\right)^{X}\Delta_{\beta Y} \\ &\quad -2\varepsilon_{\nu\alpha\beta X}\left(k_2+k_4\right)^{Y}\Delta_{XY} + 2\varepsilon_{\beta\alpha XY}\left(k_3-k_2\right)^{X}\Delta_{\nu Y} \\ &\quad -2\varepsilon_{\beta\nu XY}\left(k_1-k_2\right)^{X}\Delta_{\alpha Y} + 2\varepsilon_{\beta\nu\alpha X}\left(k_1+k_3\right)^{Y}\Delta_{XY} \\ &\quad -2\varepsilon_{\nu\alpha\beta X}\left(k_1-k_2+k_3-k_4\right)^{X}\Delta_{YY}, \end{aligned} \tag{3.443}$$

ultimately allowing the final identifications for the total amplitude

$$\begin{aligned} r^{\mu}T^{AVVV}_{\mu\nu\alpha\beta} &= T^{AVV}_{\nu\alpha\beta}\left(k_2,k_3,k_4\right) - T^{AVV}_{\beta\nu\alpha}\left(k_1,k_2,k_3\right) - 2mT^{PVVV}_{\nu\alpha\beta} \\ &\quad -2\varepsilon_{\nu\alpha\beta\sigma}\left(p-q+r\right)^{\sigma}\left[\Delta^{\rho}_{\rho} + \frac{i}{8\pi^2}\right]. \end{aligned} \tag{3.444}$$

We put additional terms together in the second line while writing their coefficients in terms of external momenta. Satisfying the axial relation among GF is not automatic since it requires the cancellations of these terms as an extra condition; i.e., it depends on the prescription adopted to evaluate the surface terms. Furthermore, note that the same condition was acknowledged in the triangle analysis (3.383).

3.5.3 Further Explorations on Relations Among GF

Previously, we analyzed relations among GF emerging from contractions involving amplitudes that are odd tensors. Relations obtained for vector vertices were automatic, which means their achievement does not depend on a prescription to evaluate divergent objects. In contrast, we found that axial relations apply under a condition for the surface term and its trace (3.383). That works as a requirement for maintaining the linearity of integration in this context.

Our first objective here is to understand the mechanisms that led to this outcome. In Subsection (3.2.4), we discussed roles played by vertices and Dirac traces. By endowing the μ index with a special role (3.57)-(3.58), we shaped the tensor sector and fixed the AVV integrand as (3.62). Posteriorly, when evaluating the axial relation among GF (also in μ), index permutations brought additional contributions to Equation (3.382). We also computed traces found inside the box amplitude by following the same strategy, and the corresponding axial contraction produced a similar situation (3.444).

Mathematical structures suggest a connection involving traces and the acknowledged results. Let us propose other trace arrangements and inquire about their implications over the triangle amplitude to clarify this subject. From this point on, we explore three AVV versions distinguished through numerical subindices

$$t_{1\mu\nu\alpha}^{AVV} \rightarrow \mathrm{tr}\left(\gamma_{\mu 5 A\nu B\alpha C}\right), \qquad t_{2\mu\nu\alpha}^{AVV} \rightarrow \mathrm{tr}\left(\gamma_{\mu A\nu 5 B\alpha C}\right), \qquad t_{3\mu\nu\alpha}^{AVV} \rightarrow \mathrm{tr}\left(\gamma_{\mu A\nu B\alpha 5 C}\right).$$

These associations specify the position to replace the chiral matrix definition, thus, prioritizing one free index among the options: μ, ν, and α.

Take the first version as a guide since it corresponds to the former integrand (3.56). Recognizing a Schouten identity with the prioritized index fixed is possible for these versions, as it occurred in Equation (3.59). Even if one ignores this property, integrating the amplitudes vanishes these sectors. Subsequently, our task is to organize integrands through standard tensors and vector subamplitudes, namely, VPP, SAP, and SPA. We already verified some properties of antisymmetric objects (3.65)-(3.66); therefore, using them leads to compact integrated expressions

$$T_{1\mu\nu\alpha}^{AVV} = 4i\varepsilon_{\mu\alpha XY} T_{3\nu;XY}^{(-)}\left(k_1;k_2,k_3\right) + 4i\varepsilon_{\mu\nu XY} T_{3\alpha;XY}^{(-)}\left(k_3;k_1,k_2\right) - i\varepsilon_{\mu\nu\alpha\beta} T_{\beta}^{VPP}, \quad (3.445)$$
$$T_{2\mu\nu\alpha}^{AVV} = 4i\varepsilon_{\nu\mu XY} T_{3\alpha;XY}^{(-)}\left(k_2;k_3,k_1\right) + 4i\varepsilon_{\nu\alpha XY} T_{3\mu;XY}^{(-)}\left(k_1;k_2,k_3\right) + i\varepsilon_{\mu\nu\alpha\beta} T_{\beta}^{SAP}, \quad (3.446)$$
$$T_{3\mu\nu\alpha}^{AVV} = 4i\varepsilon_{\alpha\nu XY} T_{3\mu;XY}^{(-)}\left(k_3;k_1,k_2\right) + 4i\varepsilon_{\alpha\mu XY} T_{3\nu;XY}^{(-)}\left(k_2;k_1,k_3\right) - i\varepsilon_{\mu\nu\alpha\beta} T_{\beta}^{SPA}. \quad (3.447)$$

These equations show how traces link to additional terms emerging in relations among GF. When prioritizing one vertex Γ_n, the corresponding free index $\mu_n = \{\mu, \nu, \alpha\}$ exclusively appears inside the Levi-Civita symbol for the tensor sector. Hence, contracting this same index does not immediately lead to reductions. Under these circumstances, we exchange index positions, and additional terms emerge through traces of rank-2 objects: J-tensor and surface term. Whereas other contractions are automatic, the nth relation among GF of the nth AVV version is not; these specific cases come as follows:

$$\begin{aligned} q^{\mu} T_{1\mu\nu\alpha}^{AVV} &= T_{\nu\alpha}^{AV}(k_2, k_3) - T_{\alpha\nu}^{AV}(k_1, k_2) - 2m T_{\nu\alpha}^{PVV} \\ &\quad -2iq^{\mu} p^{\beta} \varepsilon_{\mu\nu\alpha\beta} \left[\Delta_{\rho}^{\rho} + \frac{i}{8\pi^2} \right], \qquad (3.448) \\ p^{\nu} T_{2\mu\nu\alpha}^{AVV} &= T_{\mu\alpha}^{AV}(k_2, k_3) - T_{\mu\alpha}^{AV}(k_1, k_3) + 2iq^{\nu} p^{\beta} \varepsilon_{\mu\nu\alpha\beta} \left[\Delta_{\rho}^{\rho} + \frac{i}{8\pi^2} \right], \qquad (3.449) \\ (q-p)^{\alpha} T_{3\mu\nu\alpha}^{AVV} &= T_{\mu\nu}^{AV}(k_1, k_3) - T_{\mu\nu}^{AV}(k_1, k_2) + 2iq^{\alpha} p^{\beta} \varepsilon_{\mu\nu\alpha\beta} \left[\Delta_{\rho}^{\rho} + \frac{i}{8\pi^2} \right]. \qquad (3.450) \end{aligned}$$

Integrated subamplitudes were necessary to inspect relations for new triangle versions. If it interests the reader, follow the steps developed for the VPP (3.64) to express them as combinations of Feynman integrals. Posteriorly, the final forms emerge by replacing the necessary ingredients; consult Equation (3.257). Here, let us straightforwardly introduce these quantities:

$$\begin{aligned} T_{\beta}^{SAP} &= -2(k_1 + k_2)^{\rho} \Delta_{\beta\rho} - 2(p - 2q)_{\beta} I_{\log} \\ &\quad -4\left(q^2 - p \cdot q\right) J_{3\beta} - 2\left[p^2 q_{\beta} - q^2 p_{\beta} + 4m^2 (q-p)^2\right] J_3 \\ &\quad -2i(4\pi)^{-2} \left[(p-q)_{\beta} J_2(q-p) - q_{\beta} J_2(q)\right], \qquad (3.451) \end{aligned}$$

$$\begin{aligned} T_{\beta}^{SPA} &= 2(k_2 + k_3)^{\rho} \Delta_{\beta\rho} + 2(p+q) I_{\log} \\ &\quad +4(p \cdot q) J_{3\beta} - 2\left(p^2 q_{\beta} + q^2 p_{\beta} + 4m^2 p^2\right) J_3 \\ &\quad +2i(4\pi)^{-2} \left[p_{\beta} J_2(p) + q_{\beta} J_2(q)\right]. \qquad (3.452) \end{aligned}$$

This panorama concerns trace choices, having no strict relation with the vertex content. That becomes even clearer by extending this argumentation to all similar amplitudes (AVV, VAV, VVA, and AAA) since they all share the same tensor structure:

$$t_{\mu\nu\alpha}^{\Gamma\Gamma\Gamma} \to \mathrm{tr}\left(\gamma_{\mu}\gamma_5\gamma_A\gamma_{\nu}\gamma_B\gamma_{\alpha}\gamma_C\right) \frac{K_1^A K_2^B K_3^C}{D_{123}}. \qquad (3.453)$$

Regardless of its nature as an axial or a vector vertex, additional contributions arise for a contraction if the contracted index links to the vertex prioritized when taking the trace.

For instance, prioritizing the μ-index in the trace (3.57)-(3.58) makes the first relation among GF non-automatic for all four triangle amplitudes. Although this situation is unavoidable, we still can choose the position of additional terms by setting a specific trace expression.

Different integrands connect through algebraic operations, so one could expect them to lead to identical results. Nevertheless, that was not automatic after integration due to the divergent character of calculations. After observing this feature in momenta contractions, it is reasonable to compare different amplitude versions directly. With the aid of index permutations and other algebraic operations, we evaluate differences between versions

$$T^{AVV}_{i\mu\nu\alpha} - T^{AVV}_{j\mu\nu\alpha} = i\varepsilon_{\mu\nu\alpha\beta}P^{\beta}\left[\Delta^{\rho}_{\rho} + \frac{i}{8\pi^2}\right], \tag{3.454}$$

where $i \neq j$ refers to Equations (3.445)-(3.447) and P represents a linear combination of the external momenta p and q. The term between square brackets equals the additional terms acknowledged in contractions. Hence, opting for a prescription where the surface term follows condition (3.383) implies that all AVV versions collapse into one unique object while satisfying all relations among GF.

We still want to comment on the analysis regarding the box amplitude. Dirac traces also admitted different expressions in this case because they led to products involving the Levi-Civita symbol and the metric tensors. By endowing the μ index with a prioritized role, the organization at the integrand level puts this index exclusively in the Levi-Civita symbol while other terms cancel out identically. Renaming indices within these traces directly extends this notion to versions prioritizing other indices. That applies to any amplitude under this category as they share the tensor sector: $AVVV$, $AAAV$, and their permutations.

In general, for an amplitude version that prioritizes the index $\mu_n = \{\mu, \nu, \alpha, \beta\}$ in the traces, the nth relation among GF requires index permutations to identify momenta contractions and traces of 2nd-order tensors. Hence, using analogous traces on the right-hand side of these relations produces the additional term leading to condition (3.383). Explorations considering different trace versions on the left (box contraction) and on the right (triangles) might bring further information, so this study remains a future perspective. For this reason, we will not discuss the symmetry aspects of box correlators.

The following subsection links the current discussion with WIs, so we can inquire how the presence of surface terms reflects on the simultaneous analysis of both types of constraints.

3.5.4 Symmetries and Linearity

In Subsection (3.1.1), we derived algebraic identities among integrands of perturbative amplitudes. That suggests expectations through relations among Green Functions (GF) that should apply as a direct consequence of the linearity of integration. Hence, any violation of these relations would imply linearity breaking. We tested them in Subsection (3.5.1) for momenta contractions over the AVV triangle, verifying part of the cases without problems. Nonetheless, one relation among GF is not automatic for containing an additional contribution depending on a surface term.

We proved in Subsection (3.5.3) that choosing a trace expression sets the position of this additional contribution. By prioritizing one index when taking the trace, its contraction automatically produces the mentioned contributions. Although there are other trace possibilities, reference [48] shows that any other amplitude version combines those investigated here. Consequently, it would carry potentially violating terms coming from all combined parts. This overall situation has no relation with the vertex nature as being axial or vector.

Let us return to the original prospects regarding triangle WIs (3.26)-(3.28) to continue this inspection. They are consequences of the current algebra (3.2)-(3.3) and comprise symmetry implications over the complete amplitude. Hence, their verifications require symmetrizing final states and summing up direct and crossed diagrams. We already obtained the direct one (see Figure 3.1); thus, the crossed one arises by changing the role of indices $\mu \leftrightarrow \nu$ and external momenta $p \leftrightarrow q$.

With that clear, consider in a preliminary argument that the satisfaction of all relations among GF is automatic; i.e., they are valid without the need for conditions over divergent objects. Under this hypothesis, canceling differences between AV amplitudes would be our sole concern regarding WIs. Equations below follow the vertex order for AVV contractions to cast these structures:

$$
\begin{aligned}
& T^{AV}_{\nu\alpha}(k_2,k_3) - T^{AV}_{\alpha\nu}(k_1,k_2) \\
= \; & 2i\varepsilon_{\mu\nu\alpha\beta}\left[(p-q)^{\mu}(k_2+k_3)^{\rho} - p^{\mu}(k_1+k_2)^{\rho}\right]\Delta^{\beta}_{\rho}, \qquad (3.455)\\
& T^{AV}_{\mu\alpha}(k_2,k_3) - T^{AV}_{\mu\alpha}(k_1,k_3) \\
= \; & 2i\varepsilon_{\mu\nu\alpha\beta}\left[(q-p)^{\nu}(k_2+k_3)^{\rho} - q^{\nu}(k_1+k_3)^{\rho}\right]\Delta^{\beta}_{\rho}, \qquad (3.456)\\
& T^{AV}_{\mu\nu}(k_1,k_3) - T^{AV}_{\mu\nu}(k_1,k_2) \\
= \; & 2i\varepsilon_{\mu\nu\alpha\beta}\left[p^{\alpha}(k_1+k_2)^{\rho} - q^{\alpha}(k_1+k_3)^{\rho}\right]\Delta^{\beta}_{\rho}. \qquad (3.457)
\end{aligned}
$$

By eliminating surface terms $\Delta^{\beta}_{\rho} = 0$, one disappears with the AV amplitudes and guarantees the satisfaction of all WIs.

There are some details to address about the equations above. Even though similar structures arise for the crossed channel, combining these sectors is not feasible. Energy-momentum conservation attributes a physical meaning to differences of routings as external momenta, albeit not to routings themselves. That means these quantities are different for each channel (let us say k_i and k_i'), and there are no other connections involving them.

Under these circumstances, the discussion about symmetry implications applies channel by channel. Thus, we recall the referred WIs to cast *Expectations* over the triangle amplitude below. Momenta contractions associated with axial vertices should lead to a similar amplitude having a pseudoscalar vertex $AVV \to PVV$, while vector contractions should vanish $AVV \to 0$. Results different from these expressions represent symmetry violations at the quantum level and carry anomalous contributions.

- *Expectations* - Ward identities (WIs) anticipated from current algebra.

$$q^{\mu}T_{\mu\nu\alpha}^{AVV} \to -2mT_{\nu\alpha}^{PVV} \tag{3.458}$$

$$p^{\nu}T_{\mu\nu\alpha}^{AVV} \to 0 \tag{3.459}$$

$$(q-p)^{\alpha}\, T_{\mu\nu\alpha}^{AVV} \to 0 \tag{3.460}$$

It remains for us to evaluate the connection involving relations among GF and WIs explicitly. Since no prescription was adopted to evaluate the surface term up to this point, this analysis falls over the properties of this object. We stress two lines of reasoning while doing so.

First, maintaining the linearity of integration occurs through a prescription where the surface term assumes the finite non-zero value (3.383). That occurs if one uses linearity to verify directly that the surface term (3.133) has a finite trace

$$\Delta_{\rho}^{\rho} = 4\lambda^2 \int \frac{d^4k}{(2\pi)^4} \frac{1}{D_{\lambda}^3} = -\frac{i}{8\pi^2}, \tag{3.461}$$

computed with the aid of integral (3.220). This condition vanishes additional contributions acknowledged before; hence, amplitude versions obtained through different trace expressions coincide (3.454) and satisfy all relations among GF. Nevertheless, that violates all symmetry implications from WIs since the surface term itself is finite and non-zero. After computing the differences involving AVs in Equations (3.455)-(3.457), we cast these results in *Condition I* below. Comparing with the *Expectations*, observe that all contractions exhibit an anomalous contribution.

- *Condition I* - Linearity of integration leads to the finite non-zero value for the surface term $\Delta_{\rho\sigma} = -\frac{i}{32\pi^2} g_{\rho\sigma}$ and $\Delta_\rho^\rho = -\frac{i}{8\pi^2}$.

$$q^\mu T_{\mu\nu\alpha}^{AVV} = -2m T_{\nu\alpha}^{PVV} + \frac{1}{2\pi^2} \varepsilon_{\nu\alpha\rho\sigma} k_2^\rho q^\sigma \quad (3.462)$$

$$p^\nu T_{\mu\nu\alpha}^{AVV} = -\frac{1}{2\pi^2} \varepsilon_{\mu\alpha\rho\sigma} k_3^\rho p^\sigma \quad (3.463)$$

$$(q-p)^\alpha T_{\mu\nu\alpha}^{AVV} = \frac{1}{2\pi^2} \varepsilon_{\mu\nu\rho\sigma} k_1^\rho (q-p)^\sigma \quad (3.464)$$

On the other hand, it is possible to satisfy part of the WIs by adopting a prescription that eliminates surface terms. As mentioned before, that occurs in the case of Dimensional Regularization [25, 26, 27]. Non-automatic relations among GF are lost since this value does not cancel out additional contributions in contractions, characterizing a linearity violation. Meanwhile, canceling the AV amplitude saves part of the symmetry relations; *Condition II* below.

The possibility of changing the position of additional contributions by adopting other trace versions has significant consequences within this context. By eliminating surface terms, the first amplitude version preserves vector implications while bringing an anomalous term to the axial WI. This result is compatible with the usual perspective adopted in the literature since it is necessary to explain the phenomenon of the neutral pion decay into a pair of photons [6]. Alternatively, vector identities exhibit violations when it comes to the other two amplitude versions.

- *Condition II* - Preserving part of the Ward identities (WIs) leads to the null values $\Delta_{\rho\sigma} = 0$ and $\Delta_\rho^\rho = 0$. This time, we only cast the violated implications for each amplitude version.

$$q^\mu T_{1\mu\nu\alpha}^{AVV} = -2m T_{\nu\alpha}^{PVV} - \frac{1}{4\pi^2} \varepsilon_{\nu\alpha\rho\sigma} p^\rho q^\sigma \quad (3.465)$$

$$p^\nu T_{2\mu\nu\alpha}^{AVV} = -\frac{1}{4\pi^2} \varepsilon_{\mu\alpha\rho\sigma} p^\rho q^\sigma \quad (3.466)$$

$$(q-p)^\alpha T_{3\mu\nu\alpha}^{AVV} = \frac{1}{4\pi^2} \varepsilon_{\mu\nu\rho\sigma} p^\rho q^\sigma \quad (3.467)$$

3.6 Final Remarks and Conclusions

Throughout the third chapter, we investigated aspects of fermionic amplitudes that are odd tensors. The AVV triangle was our primary target since its anomalous character is a recurrent subject in the literature. We carefully examined its content and relations with other amplitudes, thus understanding new aspects of anomalies while emphasizing mathematical structures relevant to their discussion. We also extended this analysis to the $AVVV$ box because it contains similar tensor structures.

Firstly, let us remark on the crucial role of traces having one chiral matrix inside their argument in this context. They yield combinations of monomials built through products between the Levi-Civita symbol and metric tensors, in which case tensor properties allow different expressions. Although they are identical at the integrand level, the connection among corresponding versions for an integrated amplitude is not direct due to the divergent character of calculations. This feature has motivated authors to explore recipes for taking Dirac traces and study their implications [34, 35, 36].

To express this type of (odd) trace, one must suppress the dependence on the chiral matrix and compute the ensuing (even) trace. Such an operation requires employing one identity belonging to the set

$$\gamma_5\gamma_{[\mu_1\cdots\mu_r]} = \frac{i^{1+r(r+1)}}{(4-r)!}\varepsilon_{\mu_1\cdots\mu_4}\gamma^{[\mu_{r+1}\cdots\mu_4]}, \tag{3.468}$$

where the notation $\gamma_{[\mu_1\cdots\mu_r]}$ indicates antisymmetrized products of Dirac matrices. Reference [48] presents a broad discussion of this subject, approaching all versions of the four-dimensional triangle and inquiring about analogous cases in other space-time dimensions. Ultimately, the authors show that all amplitude expressions coming from these identities are combinations of more fundamental ones[12], those obtained through the chiral matrix definition (identity with $r=0$).

These ideas justify us targeting only these specific versions throughout this work. In truth, we replaced the definition in all six positions available to evaluate the trace containing six Dirac matrices plus the chiral one. Comparing neighboring positions made evident the presence of algebraic identities, which associate with null integrals when computing the triangle. Despite this being almost a trivial example, it outlines a strategy to pursue simplifications in more complex calculations. We used this tool when computing the box amplitude, achieving a clear view of its content and properties.

Replacing the chiral matrix definition in a particular position implies prioritizing one vertex in the trace. By doing so, all contributions having the corresponding index within

[12]That implies other versions carry anomalous terms in multiple vertices. For instance, one form identified through the combination $\frac{1}{2}\left(T^{AVV}_{1\mu\nu\alpha}+T^{AVV}_{2\mu\nu\alpha}\right)$ exhibits violations for contractions with both first and second vertices.

metric tensors cancel out. Hence, this index appears exclusively inside the Levi-Civita symbol, which is transparent by the provided organization. Observe how the trace choice shapes tensor contributions in the triangle versions from Equations (3.445)-(3.447). Although we did not present other versions here, that also occurs for the box amplitude. Posteriorly to the integration, index permutations are necessary when performing momenta contractions with the prioritized index. That is the mechanism inducing the presence of potentially violating terms in relations among Green functions. This reasoning allows the reverse way, choosing which index to prioritize aiming to position the additional contributions.

We stress the generality of these concepts by commenting on triangle amplitudes with similar tensor structures but different vertex configurations, namely, AVV, VAV, VVA, and AAA. Since they share the higher-order trace from Equation (3.56), opting for a trace expression shapes the tensor sector of these amplitudes equally, and our conclusions apply to all of them. When prioritizing the nth free index in the trace, one induces potentially violating terms in the nth momenta contraction. That does not depend on the character of the corresponding vertex as being axial or vector. The same situation occurs for box amplitudes, i.e., $AVVV$, $VAAA$, and their permutations. Again, further explorations are necessary to test the generality of the last statement.

Now, let us detail some aspects regarding integrated amplitudes. At the beginning of this chapter, we mentioned that integrals exhibiting power counting equal to or higher than linear are not translationally invariant. That means performing shifts on the integration variable requires adequate compensations to maintain the connection with the original expression. This feature implies the presence of surface terms in perturbative calculations, wholly expressed through the object $\Delta_{\mu\nu}$ in this investigation.

Take the AV bubble (3.229) as a preliminary study case. We observed a priori that it should be a null object since it was impossible to build an antisymmetric tensor exclusively using the external momentum. However, two-point amplitudes exhibit quadratic power counting in the physical dimension. Consequently, this amplitude admits the presence of a surface term proportional to an ambiguous combination of arbitrary labels $k_1 + k_2$. This type of contribution also arises for the AVV triangle (3.258), located inside the vector subamplitude (3.257).

Albeit with non-ambiguous coefficients, the $AVVV$ box exhibits the same surface term seen in the first two cases. Look into the complete amplitude (3.305) and its pertinent sectors (3.346) to find these objects. Their presence is characteristic of tensors with logarithmic power counting, as observed in Feynman integrals (3.237) and (3.276).

We also studied the implications of surface terms when exploring amplitude versions. Since they differ in the index arrangement set through trace choices, we had to permute indices to compare different possibilities. For the AVV triangle, this procedure emphasized the dependence on the surface term value, represented by the structure on the right-hand side of Equation (3.454). Canceling this contribution occurs if one assumes the *finite value* $\Delta_{\rho}^{\rho} = -i\left(8\pi^2\right)^{-1}$. We can interpret this constraint as a condition so all trace choices lead to one unique expression for the amplitude. Although we did not extend this argumentation, the involved tensors suggest that the box analysis is analogous.

Next, let us comment on the results achieved when performing momenta contractions. We identified the amplitudes from relations among Green functions directly in part of the cases. Nevertheless, as mentioned in the discussion about traces, potentially violating terms emerge in the nth momenta contraction of an amplitude that prioritizes the nth free index in the trace. Such additional contributions exhibit the same structure referred to in the previous paragraph. At least one relation among Green functions is not automatically satisfied but demands a condition over the surface term value to do so. Hence, the amplitude expression considering the *finite value* of the surface term satisfies all relations among Green functions. This outcome breaks all symmetry implications through Ward identities, which is transparent in the explicit values of these contractions (3.462)-(3.464). This part of the analysis also applies to the box amplitude.

On the other hand, adopting a prescription that sets surface terms as zero $\Delta_{\mu\nu} = 0$ preserves Ward identities for contractions that do not produce additional contributions. We acknowledge violations in the conditional relation among Green functions and the corresponding Ward identity. That is consistent with the impossibility of preserving chiral and gauge symmetry simultaneously. Furthermore, we clarify that it is possible to choose the position of the violation by adopting the trace expression accordingly. Equations (3.465)-(3.467) illustrate these possibilities for the triangle amplitude. Although we observed the same situation in the box amplitude, there are more possibilities to study before coming to a conclusion.

As a future perspective of this work, it is important to deepen the analysis of symmetry aspects. Reference [48] is a work in progress from T. J. Girardi, L. Ebani, and J. F. Thuorst and provides crucial information regarding low-energy implications of anomalous amplitudes. Explorations on the AVV triangle are particularly detailed, but the authors also extend this subject to analogous processes in other space-time dimensions.

Despite its similarities with the triangle, argumentations seem more intricate for the box amplitude. We observed that versions differ in their dependence on surface terms following the implications of trace choices. This feature reflects on potentially violating terms in contractions when prioritizing the first index in traces for all amplitudes within relations among Green functions. Nonetheless, other choices are possible and require further investigation.

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