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Aspects of Parity-Preserving Maxwell-Chern-Simons Theory: Vortices, Self-Duality, and Supersymmetry

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“ASPECTS OF PARITY-PRESERVING MAXWELL-CHERN-SIMONS THEORY: VORTICES, SELF-DUALITY, AND SUPERSYMMETRY”

WELLISSON BARBOSA DE LIMA

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"The meaning of life is just to be alive. It is so plain and so obvious and so simple. And yet, everybody rushes around in a great panic as if it were necessary to achieve something beyond themselves."

—Alan W. Watts

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I firmly believe that the appropriate response to the miraculous fact of existence should start with a feeling of gratitude. However, the directing of this feeling towards some particular people or things has more and more struck me as ungrateful, for lack of a better (softer) word, in the sense that we greatly overlook the unknown, uncontrollable, fortuitous, and potentially infinite causes that, not only happened but were, in fact, necessary for life to take its due course. Of course, this is an honest mistake. We do not know what we do not know. However, if humility is to be taken as a value, and I believe it should, I want to start by acknowledging and thanking everything there is. Everything, good or bad, that has happened, that is happening or will happen. If they weren't, then I wouldn't be. Let this be my expression of ultimate faith and trust in the universe. God, whatever that means, makes no mistakes.

Since I also don't want to make my friends and family unhappy, let me return to the protocol, and begin acknowledging some of those I clearly know have contributed to this work and my Ph.D. journey.

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Abstract

Part I of this thesis is devoted to a brief introduction of vortices in the paradigmatic scenario of the 3 - dimensional Abelian-Maxwell-Higgs model, followed by a short presentation of the Chern-Simons term and some of its relevant features.

In Part II starts our contribution, where we propose a parity invariant Maxwell-Chern-Simons $U(1) \times U(1)$ model coupled with scalar fields in $2+1$ dimensions, and show that it admits finite-energy topological vortices. We describe the main features of the model and find explicit numerical solutions for the equations of motion, considering different sets of parameters and analyzing some interesting particular regimes.

In Part III, we present the self-dual extension of the model. In this case, we show that the energy functional admits a Bogomol'nyi-type lower bound, whose saturation gives rise to first order self-duality equations. We perform a detailed analysis of this system, discussing its main features and exhibiting explicit numerical solutions corresponding to finite-energy topological vortices and non-topological solitons.

We remark that the structure of the theories follows naturally from the requirement of $\mathcal{P}$- and $\mathcal{T}$- invariance, a symmetry that is rarely envisaged in the context of Chern-Simons theories. In particular, the mixed Chern-Simons term plays an interesting role here, ensuring the main properties of the models and suggesting possible applications in condensed matter. Another distinctive aspect is that the topological vortices found here are characterized by two integer numbers.

Finally, in Part IV, we demonstrate that the self-dual model corresponds to the bosonic sector of an $\mathcal{N} = 2$ supersymmetric theory, we also indicate how the Bogomol'nyi bound and self-duality conditions arise naturally from supersymmetry.

Key Words: Field Theory, Gauge Theories, Chern-Simons Theories, Solitons in Field Theory, Vortices, Self-dual Vortices, Supersymmetry.

Resumo

A Parte I desta tese se dedica a uma breve introdução aos vórtices no cenário paradigmático do modelo Maxwell-Higgs abeliano em 3 dimensões, seguido de uma curta apresentaçao do termo de Chern-Simons e algumas de suas propriedades relevantes.

Na Parte II inicia-se nossa contribuição, onde propomos um modelo Maxwell-Chern-Simons $U(1) \times U(1)$ em $2 + 1$ dimensões, invariante sob paridade e reversão temporal, acoplado a campos escalares e mostramos que este admite soluções do tipo vórtices topológicos de energia finita. Descrevemos as principais propriedades do modelo e encontramos soluções numéricas explícitas para as equações de movimento, considerando diferentes conjuntos de parâmetros e, também, analisando alguns regimes particulares de interesse.

Na parte III, apresentamos a extensão auto-dual do modelo. Neste caso, nós mostramos que o funcional de energia admite um limite inferior do tipo Bogomol'nyi, cuja saturação leva a equações auto-duais de primeira ordem. Realizamos uma análise detalhada deste sistema, discutindo suas principais características e exibindo soluções numéricas explícitas que correspondem a configurações de energia finita do tipo vórtices topológicos e solitons não-topológicos.

Ressaltamos que a estrutura das teorias segue naturalmente do requerimento de invariância sob $\mathcal{P}$- e $\mathcal{T}$-, uma simetria que raramente é considerada no contexto de teorias de Chern-Simons. Em particular, o termo de Chern-Simons misto desempenha um papel interessante, garantindo as principais propriedades dos modelos e sugerindo possíveis aplicações na física da matéria condensada. Outro aspecto distintivo do modelo é que os vórtices topológicos que aqui aparecem são caracterizados por 2 números inteiros.

Finalmente, na Parte IV, nós demostramos que o modelo auto-dual corresponde ao setor bosônico de uma teoria supersimétrica $\mathcal{N} = 2$, também indicamos como o bound de Bogomol'nyi e as condições de auto-dualidade surgem naturalmente da supersimetria.

Palavras-Chave: Teoria de Campos, Teorias de Calibre, Teorias de Chern-Simons, Solitons em Teoria de Campos, Vórtices, Vórtices Auto-Duais, Supersimetria.

About this Ph.D. Thesis

This work is based on a single project that was divided into three parts. At first, it simply consisted of constructing a parity-preserving Maxwell-Chern-Simons electrodynamics with scalar matter and investigate the existence of topological vortex configurations. The complexity of a general approach led us to first consider the simplest scalar potential that would allow for the existence of vortices. The results led to the work (1).

The next step would be to consider a more general potential, instead we took an intermediary step in this regard, by considering a potential leading to self-dual vortices, which, on the other hand, led us to a class of theories that deserve investigation on their own merit. The main motivation, at the beginning, was the fact that self-dual configurations satisfy a set of first-order equations, much easier to hadle numerically. However, the richness and complexity underlying this type of theory proved to be much greater than simply to provide easier numerical computation. The results led to the published paper (2). This and the previous part of the project was done in strong collaboration with P. De Fabritiis.

Since the apple doesn't fall far from the tree, a natural next step turned out to be the provision of a supersymmetric origin for our self-dual model. More than mere attachment to symmetry, as we will see, there is also a well documented literature supporting this direction of investigation. More specifically, there is a deep connection between extended supersymmetry and self-duality. This work is still in progress, now also counting with the collaboration of J. P. S. Melo, and here we present some of the results obtained so far.

1. W.B. De Lima and P. De Fabritiis, *Vortices in a parity-invariant Maxwell-Chern-Simons model*, arXiv: 2205.10427 [hep-th]. Submitted for publication.

2. W.B. De Lima and P. De Fabritiis, *Self-dual Maxwell-Chern-Simons solitons in a parity-invariant scenario*, Phys. Lett. B 833, 137326 (2022).

Contents

Presentation and General Framework

Ever since the work of Nielsen and Olesen[10], vortex solutions became of much interest in the context of relativistic field theories. In that case, it consisted of a relativistic generalization of the Ginzburg-Landau theory for type II supercondutors, to which vortex solutions were first studied by Abrikosov [9]. The model proposed by [10] is the Abelian-Higgs model:

$$S_{AH} = \int d^3x \left[-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + |\left(\partial_\mu + ieA_\mu\right)\phi|^2 - \frac{\lambda}{4}\left(|\phi|^2 - v^2\right)^2 \right], \tag{1}$$

In general, the existence of non-trivial topological solutions in a field theory depend on the dimension of the spacetime and the gauge group. Vortices, in particular, are static topological configurations that might arise whenever the gauge (Lie) group has a non-trivial first homotopy group[1], that is, if you consider closed paths in the group space, not all of them can be continuously shrunk to a point. The simplest example of such a group is the abelian group $U(1)$.

A special feature of $U(1)$ gauge theories in 2+1 spacetime dimensions is that they also admit the Chern-Simons topological mass term [14, 15, 16, 17]:

$$S_{CS} = \int d^3x \left\{ \frac{\mu}{2} \epsilon^{\mu\nu\rho} A_\mu \partial_\nu A_\rho \right\}, \tag{2}$$

In fact, it can also be generalized to non-abelian gauge theories. Theories with such a term are known as Chern-Simons theories and are full of interesting and distinctive properties [21]. Among other things, it changes under gauge transformations by a total derivative, therefore keeping the action S_{CS} and the equations of motion gauge invariant; together with a Maxwell term, it gives the vector gauge boson a mass μ while preserving the gauge invariance (a special feature of this spacetime dimension); being a topological term, it doesn't contribute directly to the energy-momentum tensor as can be seen from the fact that no metric tensor appears in S_{CS} [2].

In [21] and refereces therein, one can see how Chern-Simons theories (abelian and non-abelian) with or without a Maxwell term can also support vortex solutions. The distinctive characteristic of vortices in Chern-Simons theories is that, firstly, they are charged, and secondly, their charge is proportional to the magnetic flux. Since, as we will see, the magnetic flux for vortex solutions is necessarily quantized, so are their charge.

Another property of (2) is that it changes sign under parity and time reversal transformations. Interestingly, but not unrelated fact about 2+1 dimensions is that, if you take a simple mass term for a Dirac spinor $m\overline{\psi}\psi$, it also changes sign under parity transfor-

[1] To be more accurate, actually we need a theory with spontaneous symmetry breaking such that the vaccum remains invariant under a subgroup H of the Lie group G, and the first homotopy group of the coset G/H must be non-trivial

[2] In Appendix B we show explictly how it cancels from the canonical energy-momentum tensor.

mation and time reversal. How these facts are related can be seen in [16] [21]. This does not mean though that parity conservation is lost in a Chern-Simons theory. The way to recover parity symmetry was indeed pointed out in [16] and later implemented by [60]. One simply doubles the degrees of freedom taking into account the relative sign change, having as result the parity preserving Chern-Simons action:

$$S_{ppCS} = \int d^3x \left\{ \frac{\mu}{2} \epsilon^{\mu\nu\rho} A_\mu^+ \partial_\nu A_\rho^+ - \frac{\mu}{2} \epsilon^{\mu\nu\rho} A_\mu^- \partial_\nu A_\rho^- \right\}, \tag{3}$$

Such that, under parity $A_\mu^\pm \to \mathcal{P}_\mu^\nu A_\nu^\mp$. What we will study in this work, which hasn't yet been done, are the vortices solutions in this parity preserving scenario of (Maxwell-) Chern-Simons theories. Before we get to that, we will first revise some essential aspects of vortices solutions in the simpler and more familiar scenario of the Abelian-Higgs model.

In the particular regime of the AH model (1) where $\lambda = 2e^2$ interesting things occur:

- $m_s = m_g$: The mass of the gauge boson ($\sqrt{2}ev$) becomes equal to the scalar mass ($\sqrt{\lambda}v$), which also implies that the vortices cease to interact. In contrast to $m_s > m_g (m_s < m_g)$ where the vortices were numerically shown to attract (repel).

- $E_{AH} \geq e\nu^2|\Phi|$ (Bogomol'nyi bound)[35]: The static energy functional becomes bounded by the flux.

- Self-duality equations, $E_{AH} = e\nu^2|\Phi| \Rightarrow$

$$(D_1 \pm iD_2)\,\phi = 0; \quad B = \pm e\left(|\phi|^2 - \nu^2\right); \tag{4}$$

  The saturation of the bound leads to a set of first-order coupled differential equations.

- The model admits an $\mathcal{N} = 2$ SUSY extension;

As we will see, all these properties will also hold for our parity-invariant Maxwell-Chern-Simons self-dual model. The main lesson of this work is to show that one does not need to abandon parity conservation to work with Chern-Simons theories. More than that, we can then explore the imposition of such a symmetry to completely determine our model, one more time demonstrating the power and elegance of being guided by symmetry principles in physics. *May beauty be our method...*

Speaking of symmetry, the relationship between self-duality and extended supersymmetry was first exhibited in a model by Witten and Olive in 1978 [46], and after many other instances Hlousek and Spector [127, 128, 129] were able to elaborate a general argument as to why this should always be the case, at least in 2+1 and 3+1 dimensions. This alone was sufficient to motivate the search for a supersymetric extension of our self-dual model, but as we will also see, condensed matter physics can already provide some interesting examples where supersymmetry emerges.

Chapter 1

Introduction

Vortices are ubiquitous in nature, appearing from the rotating water in a sink to the winds surrounding a tornado. Such configurations can also be found throughout the physics literature, as illustrated in Refs [1, 2, 3, 4, 5, 6, 7]. In field theory, vortices are defined as solitons and can appear whenever we have a continuous symmetry that is spontaneously broken and a vacuum manifold with a circular structure, as for example, in a (2+1)-dimensional abelian gauge theory in the Higgs phase [8].

In this sense, the first appearance of vortices in the literature was in the context of superconductivity, through the work of Abrikosov in 1957 [9]. In 1973, Nielsen and Olesen showed [10] that the Abelian-Higgs (AH) model in 2+1 dimensions (the relativistic generalization of the Ginzburg-Landau model [11]) admits finite-energy vortex solutions with a quantized magnetic flux. An exact vortex solution was found by de Vega and Schaposnik in 1976 [12], considering the particular relation between the couplings for which scalar and vector bosons have the same mass. The Abrikosov-Nielsen-Olesen (ANO) vortex described above is electrically neutral and, in fact, it was shown later by Julia and Zee in 1975 [13] that charged vortices with finite-energy cannot exist in the AH model.

A very interesting and subtle class of 2+1 topologically massive gauge theories was introduced in the early 80's [15, 16, 17], called nowadays Chern-Simons (CS) theories, after the pioneering work [14] (see also Refs. [18, 19, 20, 21]). The CS term is exclusive of odd-dimensions, typically $\mathcal{P}$- and $\mathcal{T}$- odd, and topological in nature. In 2+1 dimensions, it gives a gauge invariant mass to the gauge field, providing a mass gap that cures the infrared divergences of these theories, changing drastically their physical content and leading to a quantization of the ratio between the CS parameter and the gauge coupling. Over the years, CS theories have found applications all around physics, but the most famous breakthrough came with the work of Witten [22], about the relationship between CS theories and the Jones polynomial. For an introduction to CS physics, see Ref. [21]; for a review of vortices in this context, see Ref. [23].

It is well-known that a CS term has the property of flux attachment when coupled to matter fields, that is, it relates the electric charge with the magnetic flux. In 1986, it was shown that finite-energy charged vortices solutions exist in Abelian [24] and non-abelian [27, 25, 26] Higgs models in the presence of a CS term (see also Ref. [28]); the existence of quantum charged vortices has been shown in Ref. [29]. Interestingly enough, charged vortices can play an important role in condensed matter, for example, in the fractional quantum Hall effect [30], high-T_c superconductors [31], and superfluids [32].

In the pure CS limit, when the Maxwell kinetic term is absent, peculiar charged vortices were shown to exist [33], with magnetic field vanishing at the origin, instead of taking a finite value as usual. An interesting work studying vortices in a Maxwell-Chern-Simons-Higgs model, interpolating between AH model and pure CS-Higgs case was done in Ref. [34]. Self-dual vortices (and also non-topological solitons) satisfying first order equations, coming from the saturation of a Bogomol'nyi-type [35] lower bound for the energy, were found both in the pure CS limit [36, 37, 38] and in the Maxwell-CS model [39, 40, 41]. It is well-known that self-duality is closely related with $\mathcal{N} = 2$ supersymmetry [46, 47], and vortices were also studied in this scenario [50, 133, 53, 54]. For interesting reviews, see Refs. [55, 23]. This kind of soliton solutions can also be found in non-relativistic theories, as one can see for instance in Refs. [56, 57, 58, 59].

It is usually said that the presence of a CS term necessarily causes the violation of $\mathcal{P}$ and $\mathcal{T}$ symmetries. Although usually correct, this is not always true. In fact, it was already pointed out in [16, 17] and later shown by Hagen [60](see also Ref. [61]), that a gauge and parity-invariant CS theory can be constructed by essentially doubling the gauge degrees of freedom and adopting their respective CS terms with opposite signs. This discussion had as a background experiments suggesting parity-invariance in high-T_c superconductors[62, 63, 64], motivating the development of theoretical models for superconductivity agreeing with these results [65, 66, 67, 68]. A recent approach was proposed by Del Cima and Miranda [71] in the context of graphene physics (see also Ref. [69]). The authors considered a parity-preserving $U(1) \times U(1)$ massive quantum electrodynamics (QED) with two gauge fields having different behaviors under parity, and a CS term mixing them. Its massless version was studied in Ref. [73], and it was shown that it exhibits quantum parity conservation at all orders in perturbation theory [72]. Recently, it was shown in Ref. [74], that the massive version is ultraviolet finite, that is, exhibits vanishing β-functions associated to the gauge coupling constants and CS parameter, and also vanishing anomalous dimensions. Furthermore, it was shown that the model is parity and gauge anomaly free at all orders in perturbation theory. Interestingly, similar models with a mixed CS term find many applications in condensed matter [75, 76, 77, 78, 79, 80, 81, 82, 83, 84, 85, 86, 87].

Vortices in this context were already discussed in the literature before [88, 89, 90, 91, 92], but without Maxwell terms. In Ref. [88], the authors studied vortices in a $U(1) \times U(1)$ CS model coupled with scalar matter exhibiting fractional and mutual statistics. Following this work, the low energy dynamics of vortices was investigated in Ref. [89] (see also Ref. [92]) and hybrid anyons in Ref. [90]. Vortices in a CS theory coupled with fermions were studied in Ref. [91]. Finally, this subject is also investigated in the mathematical physics literature, as one can see for example in Refs. [93, 94, 95].

In the last few years, there have been several contributions to the literature of vortices, and here we briefly mention some of them. In Ref. [96], the authors reported a new topological vortex solution in a $U(1) \times U(1)$ Maxwell-Chern-Simons theory. Considering the situation in which one of the $U(1)$'s was spontaneously broken, they obtained a long-range force, protected at the quantum level by the Coleman-Hill theorem [97]. Another interesting development was achieved in Refs. [98, 99], where the authors used a systematic expansion in inverse powers of n to study giant vortices with large topological charge, observed experimentally in condensed matter systems [100, 101, 102]. In Ref. [103], the authors considered a $U(1) \times U(1)$, $\mathcal{N} = 2$ supersymmetric model in $2+1$ dimensions, investigating magnetic vortex formation and discussing applications of it. For some recent developments on vortex solutions within the gravitational context, see for instance Refs. [104, 105]. Other interesting recent works can be found in Refs. [106, 107, 108, 109, 110, 111].

In this work, we fill an important gap in the physics literature about vortices by considering them in a $\mathcal{P}$- and $\mathcal{T}$- preserving Maxwell-CS scenario. More specifically, we propose a $\mathcal{P}$- and $\mathcal{T}$- invariant Maxwell-CS $U(1)\times U(1)$ scalar QED in 2+1 dimensions, in analogy with the fermionic matter model studied in Ref. [71] and investigate the existence of topological vortices in the Higgs phase of this model. We also introduce an extension of this theory which allows for the existence of static self-dual soliton configurations. It is an endeavor that we deem interesting in its own sake, but not without any possibility of connection to recent investigations [112, 113, 114, 115, 116, 117].

Parts I is a warm-up to provide some more detailed background on what we are going to discuss, where Chapter 2 provides a brief overview of a vortex configuration using the simple Abelian-Higgs model, while in Chapter 3 we introduce the Chern-Simons term and some features of it. In Part II, Chapter 4 is devoted to the simplified model, where in Sec. 4.2, we introduce it and build the theoretical setup; in Sec. 4.3 we discuss general properties of the topological configurations considered; we present explicit vortex solutions in Sec. 4.4 and discuss its main features; the analysis of limiting cases is done in Sec. 4.5. Proceeding to Part III, in Chapter 6 we introduce and investigate the self-dual model; we present the self-duality equations in Sec. 6.3; the numerical topological and non-topological soliton solutions are shown and discussed in Sec. 6.4. We end our contribution in Part IV, where Ch. 8 provides some background of SUSY in 2+1 dimension, sections 8.3 and 8.5 provide two different derivations for the $\mathcal{N} = 2$ SUSY extension of the self-dual, while Ch. 9 briefly discuss the relationship between self-duality and extended SUSY, in general. Finally, in Cap. 10, we state our final remarks. We use natural units ($c = \hbar = 1$), the Minkowski metric[1] $\eta^{\mu\nu} = \text{diag}(+,-,-)$; and the Levi-Civita tensor $\epsilon^{012} \equiv -1$.

[1]Except in part IV, where we chose to follow [145] and use $\eta^{\mu\nu} = \text{diag}(-,+,+)$.

Part I

Warm-up

Chapter 2

The Abelian Maxwell-Higgs model in $(2+1)$ dimensions

2.1 Introduction

Let us consider the Abelian Higgs model in (2+1) dimensions. This theory describes a $U(1)$ gauge field with charged scalar matter and is an excellent laboratory to study the different phases of a gauge theory, topological solutions and dualities. In particular, there are peculiar solitons[1] called vortices, that typically appear in field theories with spontaneously broken symmetries where the vacuum manifold has a circular structure, and the Abelian Higgs model provides a simple and instructive example.

2.2 Presenting the model

2.2.1 Theoretical setup

The Abelian Higgs (AH) model can described by the lagrangian:

$$S = \int d^3x \left[-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + |D_\mu \phi|^2 - \frac{\lambda}{4} \left(|\phi|^2 - v^2 \right)^2 \right], \tag{2.1}$$

where we defined the covariant derivative as $D_\mu \phi = \partial_\mu \phi + ieA_\mu \phi$, being e the coupling constant. As usual, the field strength tensor is defined as $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. We remark that in $2+1$ dimensions with this parametrization, the gauge field and the coupling have dimension of $\sqrt{\text{mass}}$, that is, $[A_\mu] = [e] = M^{1/2}$. It will be sometimes useful to redefine the gauge field absorbing the coupling inside of it, $A_\mu \to \frac{1}{e} A_\mu$. In this case, notice that the mass dimension of the gauge field changes giving us $[A_\mu] = M$.

[1]Nonsingular solution of the classical equations of motion in Minkowski spacetime with finite energy.

The action above has a $U(1)$ gauge symmetry, with transformations given by

$$\phi \to \phi'(x) = e^{i\omega(x)}\phi(x), \quad A_\mu \to A'_\mu(x) = A_\mu(x) - \frac{1}{e}\partial_\mu\omega(x). \tag{2.2}$$

The gauge field transformation above can be easily obtained here imposing that the covariant derivative transforms in the same way as the scalar field, that is $(D_\mu\phi)' = e^{i\omega}D_\mu\phi$. The equations of motion associated with this system are given by:

$$D_\mu D^\mu \phi = -\frac{\lambda}{2}\left(|\phi|^2 - v^2\right)\phi, \quad \partial_\mu F^{\mu\nu} = ie\left(\phi^*(D_\nu\phi) - (D_\nu\phi)^*\phi\right) \tag{2.3}$$

The scalar potential of this model induces a non-trivial vacuum expectation value for the scalar field $\langle|\phi|^2\rangle = v^2$. Expanding around the non-trivial vacuum, we will see the Higgs mechanism taking place. In fact, as we choose a particular field configuration to be the vacuum, we observe that this object is not invariant under gauge transformations, even though the dynamics of the theory still is. This, by definition, means that the gauge symmetry is spontaneously broken, or to put it differently, that the theory enters into the Higgs phase, giving mass to the gauge field as well as to the scalar, having only massive particles in the spectrum and therefore being gapped.

2.2.2 Perturbative spectrum

Let us take a closer look at the spectrum of the system. From the canonical energy-momentum tensor, if we sum the total derivative $\partial_\rho\left(F^{\mu\rho}A^\nu\right)$ and use the equations of motion, we can obtain the following symmetric energy-momentum tensor:

$$T^{\mu\nu} = F^{\mu\rho}F_\rho^{\ \nu} + D^\mu\phi^* D^\nu\phi + D^\nu\phi^* D^\mu\phi - \eta^{\mu\nu}\mathcal{L}. \tag{2.4}$$

From this expression, we can immediately obtain the Hamiltonian of the system,

$$H = \int d^2x\, T^{00} = \int d^2x \left[\frac{1}{2}\left(F_{0i}\right)^2 + \frac{1}{4}\left(F_{ij}\right)^2 + |D_0\phi|^2 + |D_i\phi|^2 + \frac{\lambda}{4}\left(|\phi|^2 - v^2\right)^2\right] \tag{2.5}$$

The classical vacuum of the theory corresponds to the field configurations that minimize the energy. Looking at the above expression we can see that this can be achieved taking any field configuration such that $\langle|\phi|^2\rangle = v^2$ and such that A_μ is a pure gauge. In particular, one can choose for simplicity the vacuum configurations as (giving zero-energy):

$$\phi = v; \quad A_\mu = 0. \tag{2.6}$$

Expanding the scalar field around this non-trivial vacuum expectation value and using the exponential parametrization for it,

$$\phi(x) = \left(v + \frac{\rho(x)}{\sqrt{2}}\right) e^{i\theta(x)}, \tag{2.7}$$

we can perform a gauge transformation with $\omega(x) = -\theta(x)$ to gauge away the angular degree of freedom in the exponential and consider the so-called unitary gauge, where we keep only the physical degrees of freedom explicit, being a better setup to understand the spectrum of the theory.

In fact, with the above parametrization, we can rewrite the initial Lagrangian as:

$$\begin{aligned}\mathcal{L} = &-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}\partial_\mu\rho\partial^\mu\rho + e^2v^2A_\mu A^\mu - \frac{\lambda v^2}{2}\rho^2 \\ &+ \sqrt{2}e^2 v\rho A_\mu A^\mu - \frac{\sqrt{2}}{4}\lambda v\rho^3 + \frac{e^2}{2}\rho^2 A_\mu A^\mu - \frac{\lambda}{16}\rho^4.\end{aligned} \tag{2.8}$$

Through the Higgs mechanism, we se that the vector boson acquires a mass $m_V^2 = 2e^2v^2$ and the scalar boson acquires a mass $m_S^2 = \lambda v^2$.

The energy functional can be rewritten as

$$E = E_K + E_S, \tag{2.9}$$

where we defined the "kinetic" part of the energy as

$$E_K = \int d^2x \left[\frac{1}{2}\left(F_{0i}\right)^2 + |D_0\phi|^2\right], \tag{2.10}$$

and the "static" part of the energy as

$$E_S = \int d^2x \left[\frac{1}{4}\left(F_{ij}\right)^2 + |D_i\phi|^2 + \frac{\lambda}{4}\left(|\phi|^2 - v^2\right)^2\right] \tag{2.11}$$

Let us consider the gauge $A_0 = 0$. Notice that this condition does not fix completely the gauge, and we still have freedom to perform time-independent gauge transformations. With this choice, we have the following simplifications: $F_{0i} = \partial_0 A_i = \dot{A}_i$ and $D_0\phi = \partial_0\phi = \dot{\phi}$. We remember here the definitions $E^i = F^{i0}$ and $B = \epsilon^{ij}\partial_i A_j$.

Consider now the static limit, that is, consider $\partial_0 \equiv 0$. In this case, the kinetic energy will vanish and the energy functional will be given exclusively by the static one. The absolute minimum of this functional (corresponding to zero-energy solutions) will give us the classical vacuum of the theory. In this case, we have:

$$F_{ij} = 0; \quad D_i\phi = 0; \quad |\phi|^2 = v^2. \tag{2.12}$$

A particular solution to the above conditions can be chosen as (remember that $A_0 = 0$):

$$A_i = 0; \quad \phi = v. \tag{2.13}$$

We remark that we could have chosen any gauge transformation of the above configuration.

2.3 Topological configurations

2.3.1 Asymptotic conditions

Let us search for static soliton solutions in this model. That is, we are interested here in finite energy, static, regular solutions to the classical equations of motion. Looking at the static energy functional, we can immediately understand that in order to have a finite energy solution, we need to satisfy the vacuum conditions asymptotically. In fact, this is a necessary condition to guarantee the finiteness of the energy integral. Each term has to go to zero fast enough to compensate the integration measure ($d^2x \approx r dr d\theta$). That is, each term has to go to zero faster than $1/r^2$ to guarantee finiteness.

Therefore, imposing the finite-energy condition, we obtain the following asymptotic conditions:

$$F_{ij} \to 0; \quad D_i\phi \to 0; \quad |\phi| \to v. \tag{2.14}$$

From the scalar field condition, we immediately see that, the boundary condition fixes the radial part but still allows a non-trivial angular degree of freedom. In fact, we can write

$$\phi(r \to \infty, \theta) \longrightarrow \phi_\infty(\theta) = v e^{i\alpha_\infty(\theta)}, \tag{2.15}$$

where the function α_∞ depends only on θ, the angular variable parametrizing the direction in which we take the asymptotic limit. In fact, in two spatial dimensions, we can easily see that we can take the limit $r \to \infty$ in any direction we wish, parametrizing it by this angle θ. This amounts to the fact that, in the asymptotic limit, we have the equivalence $\partial\mathbb{R}^2 \equiv S^1_\infty$, that is the asymptotic boundary of the plane can be considered as an asymptotic "1-sphere" (circle) described in terms of an angle θ.

Now, to allow this angular dependence on the scalar field and accomplish the asymptotic condition $D_i\phi \to 0$, the gauge field must somehow compensate the scalar field asymptotic behavior. That is, $D_i\phi = \partial_i\phi + ieA_i\phi \to 0$ can be satisfied if we impose:

$$A_i \longrightarrow -\frac{1}{e}\partial_i\alpha_\infty \tag{2.16}$$

Notice that in this case the condition $F_{ij} \to 0$ is automatically satisfied.

Therefore, since in the asymptotic limit the scalar field takes a direction parametrized by an angle θ (that is, a point in S^1_∞) and attributes a phase to it through the function $e^{i\alpha_\infty(\theta)}$ (that is, an element of $U(1)$), we have the following map:

$$\phi_\infty : S^1_\infty \longrightarrow U(1) \simeq S^1_{\text{int}}. \tag{2.17}$$

As is well known, such maps can be grouped into equivalence classes according to whether they can be continuously deformed into one another or not. A map that describes this continuous deformation is called an homotopy, and the classes are called homotopy classes. With a suitable definition of product, one can show that the homotopy classes form a group, the homotopy group. This group provides a powerful way of characterizing topological spaces. In this particular example, the first homotopy group is isomorphic to the integers ($\Pi_1(S^1) = \mathbb{Z}$) and, therefore, the homotopy classes (consequently the maps therein) can be labeled by an integer number. The integer number labeling a given homotopy class is the so-called winding number W, and it says essentially the number of times we wrap around the circle S^1_{int} when we cover the circle S^1_∞ once. We can thus define:

$$W(\phi_\infty) = \frac{1}{2\pi}\int_0^{2\pi} d\theta\, \frac{d\alpha_\alpha}{d\theta} \tag{2.18}$$

It is possible to show that the $W(\phi_\infty)$ defined above is in fact an integer number.

The classical vacuum is the configuration $\phi = v = cte$, giving us an asymptotic ϕ_∞ that goes to the same point in any direction, being represented by $\alpha_\infty(\theta) = 0,\ \forall\theta$. On the other hand, we could consider the map that attributes at each asymptotic direction θ a phase equal to this angle. In this case we have $\alpha_\infty(\theta) = \theta$ and $\phi^{(1)}_\infty = ve^{i\theta}$. In fact, it is possible to show that we can write representative mappings for each homotopy class with winding n given by $\phi_\infty = ve^{in\theta}$. That is, any map with winding n can be continuously deformed in a representative map of the type written above for a given n, and mappings with different n cannot be deformed in each other by a continuous deformation.

Let us try to see this from a different perspective. Define the classical configuration space as the set of regular fields with finite static energy. As discussed above, such configurations can be classified by winding number, and a member of one class cannot be continuously deformed in a member of another class while keeping the energy finite (sometimes we say that there is an infinite energy barrier between them). Therefore, we can say that the configuration space consists of infinitely (but countable) many disconnected sets (the homotopy classes) labeled by the winding number. Since there is no continuous transformation taking a configuration of one homotopy class to another in this space, it is not possible to change the winding number by time evolution. It would not be suprising then to have somehow a conserved current associated with this quantity. Our tentative reasoning here for the existence of this conserved current has been topological rather then through Noether's theorem. We are searching for a topological conservation law.

In fact, defining $\hat{\phi} = \phi/|\phi|$, consider the topological current defined by:

$$J_T^\rho = \frac{i}{2\pi}\,\epsilon^{\mu\nu\rho}\,\partial_\mu\hat{\phi}^*\partial_\nu\hat{\phi}. \tag{2.19}$$

This current is identically conserved, without the need of using the equations of motion. The topological charge associated with this current can be written as:

$$\begin{aligned} Q_T &= \int d^2x\, J_T^0 = \frac{i}{2\pi}\int d^2x\,\epsilon^{0ij}\partial_i\hat{\phi}^*\partial_j\hat{\phi} \\ &= \frac{i}{2\pi}\int d^2x\,\epsilon^{0ij}\left[\partial_i\left(\hat{\phi}^*\partial_j\hat{\phi}\right) - \hat{\phi}^*\partial_i\partial_j\hat{\phi}\right] \\ &= \frac{i}{2\pi}\int_{S^1_\infty} dS\,\hat{r}_i\;\epsilon^{ij}\hat{\phi}^*\partial_j\hat{\phi} \end{aligned} \tag{2.20}$$

But, in the asymptotic limit we have $\hat{\phi} \to e^{i\alpha_\infty}$, with $\alpha_\infty = \alpha_\infty(\theta)$, then we have:

$$\begin{aligned} Q_T &= \frac{i}{2\pi}\int_{S^1_\infty} dS\,\hat{r}_i\;\epsilon^{ij}e^{-i\alpha_\infty}\left(i\partial_j\alpha_\infty\right)e^{i\alpha_\infty} \\ &= -\frac{1}{2\pi}\int rd\theta\,\hat{r}_i\,\epsilon^{ij}\,\hat{\theta}_j\frac{1}{r}\frac{d\alpha_\infty}{d\theta} = -\frac{1}{2\pi}\int d\theta\,\left[\hat{r}_i\,\epsilon^{ij}\epsilon_{jk}\hat{r}_k\right]\frac{d\alpha_\infty}{d\theta} \\ &= -\frac{1}{2\pi}\int d\theta\,\left[\hat{r}_i\,\left(-\delta^i_k\right)\hat{r}_k\right]\frac{d\alpha_\infty}{d\theta} = \frac{1}{2\pi}\int d\theta\,\frac{d\alpha_\infty}{d\theta} \\ &= W(\phi_\infty) \end{aligned} \tag{2.21}$$

Therefore, we can see that the topological charge is the winding number! Moreover, using the asymptotic behavior of the gauge field, $A_i \to -\frac{1}{e}\partial_i\alpha_\infty$, and Stokes theorem, we have:

$$\begin{aligned} Q_T &= -\frac{1}{2\pi}\int_{S^1_\infty} dS\,\hat{r}_i\;\epsilon^{ij}e^{-i\alpha_\infty}\left(\partial_j\alpha_\infty\right)e^{i\alpha_\infty} = -\frac{1}{2\pi}\int_{S^1_\infty} dS\,\hat{r}_i\;\epsilon^{ij}\left(-eA_j\right) \\ &= \frac{e}{2\pi}\int d^2x\,\epsilon^{ij}\partial_iA_j = \frac{e}{2\pi}\int d^2x\,B. \end{aligned} \tag{2.22}$$

Defining the dimensionless magnetic flux as $\Phi = e\int d^2x\,B$ and remembering that the topological charge is the winding number (and therefore an integer), we can conclude that the magnetic flux is quantized! The configurations with finite energy will be classified according to their quanta of magnetic flux.

$$\Phi = 2\pi\,Q_T = 2\pi n, \quad n \in \mathbb{Z}. \tag{2.23}$$

These are very general results, based on the topological properties obtained imposing the finite energy condition. Now, we need to take into account the dynamics of the system and search for solutions to the classical equations of motion. In the next section, we will investigate the original Nielsen-Olesen vortex solution in the Abelian Higgs model.

Finally, we want to call attention to an interesting point. In U(1) gauge theories in (2+1) dimensions, there is in general a global U(1) symmetry usually called $U(1)_{top}$. That is, we can define a topological current, given by

$$J^{\mu}_{top} = \frac{e}{4\pi}\epsilon^{\mu\nu\rho}F_{\nu\rho}, \tag{2.24}$$

such that in the absence of a "monopole operator", we have that this current is immediately conserved $\partial_\mu J^{\mu}_{top} = 0$ thanks to the Bianchi identity $\epsilon^{\mu\nu\rho}\partial_\mu F_{\nu\rho} = 0$. Considering the topological charge associated with it,

$$\begin{aligned} Q_{top} &= \int d^2x\, \tilde{J}^{0}_{top} = \frac{e}{4\pi}\int d^2x\, \epsilon^{ij}F_{ij} \\ &= \frac{e}{2\pi}\int d^2x\, \epsilon^{ij}\partial_i A_j = \frac{e}{2\pi}\int d^2x\, B = Q_T. \end{aligned} \tag{2.25}$$

We see agian that the conserved quantity associated with this topological current is the magnetic flux!

2.3.2 Vortex solutions

Let us consider the following radially symmetric *ansatz*:

$$\phi = vF(r)e^{in\theta}, \quad A_i = \frac{1}{er}\left[A(r) - n\right]\hat{\theta}_i. \tag{2.26}$$

Remember that we are adopting $A_0 = 0$ and considering the static limit $\partial_0 \equiv 0$.

From the equations of motion (2.3) we see that the $\nu = 0$ equation is trivial. As for the other equations, using the *ansatz*, we obtain:

$$A'' - \frac{A'}{r} = 2e^2v^2F^2A, \quad F'' + \frac{F'}{r} - \frac{A^2}{r^2}F = \frac{\lambda v^2}{2}(F^2 - 1)F \tag{2.27}$$

The prime denotes derivative w.r.t. the r variable. Defining the dimensionless parameter $\xi = \sqrt{\lambda}vr$, we can rewrite:

$$\ddot{A} - \frac{\dot{A}}{\xi} = \frac{2e^2}{\lambda}F^2A, \quad \ddot{F} + \frac{\dot{F}}{\xi} - \frac{A^2}{\xi^2}F = \frac{1}{2}(F^2 - 1)F \tag{2.28}$$

With the dot denoting derivative w.r.t ξ. The appropriate boundary conditions are: $F(0) = 0; A(0) = n; F(\infty) = 1$ and $A(\infty) = 0$. No analytical solution to these equations has been found[2], so one must resort to numerical solutions. Figure (2.1) exhibits one such a solution.

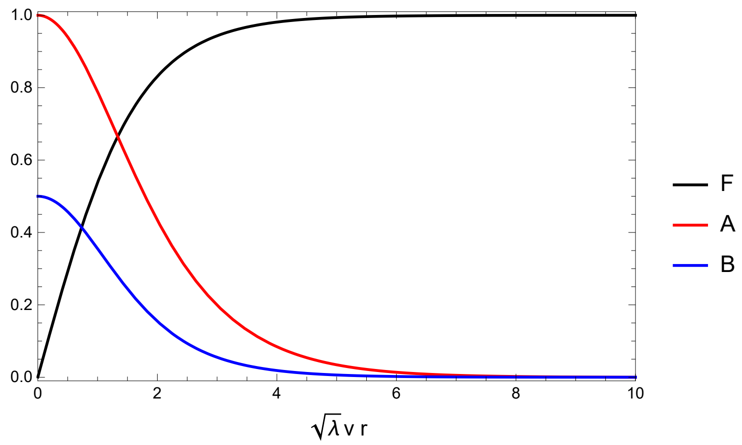

Figure 2.1: The scalar profile (solid black), the potential (solid red), and magnetic field ($B \equiv -\dot{A}/\xi$) (solid blue) for an AH vortex of winding n = 1 and taking $\lambda = 2e^2$.

One can prove that all the profiles approach their asymptotic values exponentially fast with a characteristic length given by the inverse mass $(\sqrt{\lambda}v)^{-1}$ for the scalar field and $(\sqrt{2}ev)^{-1}$ for the gauge field. Simple inspection of the solution show us how, in this case, the magnetic field is concentrated at the core of the vortex. As was already mentioned, the AH vortex is necessarily neutral [13] and it takes a Chern-Simons term for one to obtain charged vortices. Another consequence of this addition is the lowering of the magnetic field at the center of the vortex, so that in the pure CS limit it actually vanishes at the center[34], as we will see; in this scenario, for spherically symmetric vortices, the magnetic field is concentrated at a ring surrounding the core of the vortex. Now is an opportune moment for us to introduce the abelian Chern-Simons term.

[2]Some analytical results exist for the particular self-dual case $\lambda = 2e^2$. See [21] for a brief discussion on this.

Chapter 3

(Briefly) Introducing the Chern-Simons term

The requirement of gauge invariance in 2+1 dimensions allows one to construct the following quadratic action, besides the familiar Maxwell term:

$$S_{CS} = \int d^3x \mathcal{L}_{\mathcal{CS}} = \int d^3x \left\{ \frac{k}{2} \epsilon^{\mu\rho\nu} A_\mu \partial_\rho A_\nu - A_\mu J^\mu \right\} \tag{3.1}$$

Where $\epsilon^{\mu\rho\nu}$ is the completely anti-symmetric Levi-Civita symbol. Under gauge transformations $\delta A_\mu = \partial_\mu \Lambda$, we obtain:

$$\delta S_{CS} = \int d^3x \partial_\mu \left(\frac{k}{2} \epsilon^{\mu\rho\nu} \Lambda \partial_\rho A_\nu - \Lambda J^\mu \right) \tag{3.2}$$

Extremizing action (3.1) leads to the equation of motion:

$$\frac{k}{2} \epsilon^{\mu\rho\nu} F_{\rho\nu} = J^\mu \tag{3.3}$$

The first immediate consequence of the above equation is the non-propagation of degrees of freedom when $J^\mu = 0$, that is, a pure abelian Chern-Simons theory is non-propagating. The second, and distinctive, feature is evidenced by taking the $\mu = 0$ component of equation (3.3), considering $J^\mu = (\rho, \vec{J})$:

$$\rho = kB \Rightarrow Q = k\Phi \tag{3.4}$$

which is known as the flux attachment property. Every charged field configuration carries with it a magnetic flux and vice-versa. Several interesting phenomena can occur because of this [21], but the most direct for our purposes is to permit the existence of charged vortices. Moreover, due to the topological quantization of the flux of vortices, their charge is also quantized, *classically*!

3.1 Maxwell-Chern-Simons (MCS) electrodynamics

Another distinctive property of a CS term is to give mass to the photon without destroying gauge invariance, an exclusivity of 2+1 dimensions. As a consequence, contrary to 3+1 D that whereas a massless photon possesses 2 degrees of freedom (d.f.) and a massive one carries 3, both massless and massive carry only 1 d.f. in this lower dimensional case, unless a Higgs mechanism takes place, as we will see, in which case we get 2 d.f..

3.1.1 Massive photon without breaking gauge-invariance

Starting from the quadratic MCS action

$$S_{MCS} = \int d^3x \left\{ -\frac{1}{4e^2} F^{\mu\nu} F_{\mu\nu} + \frac{k}{2} \epsilon^{\mu\rho\nu} A_\mu \partial_\rho A_\nu \right\} \tag{3.5}$$

we derive the following equation of motion:

$$\partial_\nu F^{\nu\mu} + \frac{ke^2}{2} \epsilon^{\mu\rho\nu} F_{\rho\nu} = 0 \tag{3.6}$$

From (3.6) and defining the dual of $F_{\mu\nu}$, $\tilde{F}^\mu = \frac{1}{2}\epsilon^{\mu\nu\rho} F_{\nu\rho} \Rightarrow F_{\mu\nu} = \epsilon_{\mu\nu\rho} \tilde{F}^\rho$, it is not difficult to verify that:

$$\left[\Box + \left(ke^2 \right)^2 \right] \tilde{F}^\sigma = 0 \tag{3.7}$$

Where $\Box = \partial_\nu \partial^\nu$. We recognize (3.7) as the Proca equation with mass given by $m_{MCS} = ke^2$.

A less straightforward way of obtaining this result is from the propagator of the theory. We need only to write (3.5) as $\mathcal{L} = \Phi_i \mathcal{O}^{ij}(\partial) \Phi_j$, with $\mathcal{O}^{ij}(\partial)$ a differential operator. The propagator is simply $\left(\mathcal{O}^{-1}\right)^{ij}(x-y)$. The physical poles of this operator in momentum space provide the mass spectrum of the theory, that is, the squared mass of the excitations.

The development of the necessary steps are left to the appendix A.1 and here we only exhibit the final result:

$$\left(\mathcal{O}'^{-1} \right)_{\mu\nu} = e^2 \left(\frac{\eta_{\mu\nu} \Box - \partial_\mu \partial_\nu - ke^2 \epsilon_{\mu\nu\rho} \partial^\rho}{\Box \left(\Box + k^2 e^4 \right)} + \xi \frac{\partial_\mu \partial_\nu}{\Box^2} \right) \tag{3.8}$$

Now switching to momentum space through the definition

$$\left(\mathcal{O}'^{-1} \right)_{\mu\nu} \delta^3(x-y) = -\int \frac{d^3p}{(2\pi)^3} e^{-ip(x-y)} \Delta_{\mu\nu}(p) \tag{3.9}$$

we obtain

$$\Delta_{\mu\nu}(p) = e^2 \left[\frac{p^2 \eta_{\mu\nu} - p_\mu p_\nu - ike^2 \epsilon_{\mu\nu\rho} p^\rho}{p^2 \left(p^2 - k^2 e^4 \right)} + \xi \frac{p_\mu p_\nu}{\left(p^2 \right)^2} \right]. \tag{3.10}$$

The parameter ξ is due to gauge fixing, as explained at appendix A.1. The most direct way of reading the physical pole is from the coefficient that accompanies $\eta_{\mu\nu}$, from which we identify a pole at $p^2 = k^2 e^4 = m^2_{MCS}$, as expected.

The paradigmatic way of giving mass to a gauge field without breaking gauge invariance is through the Higgs mechanism, where a complex scalar field minimally coupled to the gauge boson acquires a non-null vacuum expectation value (v.e.v.), providing in this way a mass term to the vector boson. Nothing prevents us from using the same mechanism in the presence of both Maxwell and CS terms. Let's then consider:

$$\mathcal{L}_{\mathcal{MCSH}} = -\frac{1}{4e^2} F^{\mu\nu} F_{\mu\nu} + \frac{k}{2} \epsilon^{\mu\nu\rho} A_\mu \partial_\rho A_\nu + (D^\mu \phi)^* (D_\mu \phi) - V(|\phi|) \tag{3.11}$$

Where ϕ is the Higgs field and $D_\mu \phi = \partial_\mu \phi + i A_\mu \phi$ is its covariant derivative. $V(|\phi|)$ is a potential with spontaneous symmetry breaking, that is, it induces a non-trivial v.e.v. $\langle |\phi| \rangle = \upsilon$. From (3.11), we see that the term $(D^\mu \phi)^* (D_\mu \phi)$ contains $A^\mu A_\mu \phi^* \phi$ which, in its turn, produces $\upsilon^2 A^\mu A_\mu$ after the expansion of the fields around the configuration of minimum energy. We must investigate the effect of this term on the spectrum by, for example, studying the propagator of the theory

$$\mathcal{L}_{\mathcal{MCSH}}(A) = -\frac{1}{4e^2} F^{\mu\nu} F_{\mu\nu} + \frac{k}{2} \epsilon^{\mu\rho\nu} A_\mu \partial_\rho A_\nu + \upsilon^2 A^\mu A_\mu \tag{3.12}$$

Again, the details are left to Appendix A.2. Here it suffices to remember that we are interested in the coefficient with accompanies $\eta_{\mu\nu}$, which reads

$$a = e^2 \frac{\Box + 2\upsilon^2 e^2}{(\Box + 2\upsilon^2 e^2)^2 + k^2 e^4 \Box}$$

The poles can be read off from equation in momentum space ($\Box = -p^2$):

$$\left(-p^2 + 2\upsilon^2 e^2\right)^2 - k^2 e^4 p^2 = 0 \tag{3.13}$$

It gives us $p^2 = m^2_\pm$, where

$$m^2_\pm = 2\upsilon^2 e^2 + \frac{k^2 e^4}{2} \pm \frac{k e^2}{2} \sqrt{k^2 e^4 + 8\upsilon^2 e^2} = \frac{m^2_{MCS}}{4} \left(\sqrt{1 + \frac{4 m_H^2}{m^2_{MCS}}} \pm 1 \right)^2 \tag{3.14}$$

Being $m_{MCS} = k e^2$ the already familiar CS mass and $m_H^2 = 2\upsilon^2 e^2$ the Higgs mass scale. It is now evident the existence of 2 distinct massive d.f..

We close this subsection by pointing out that the Higgs mechanism can also take place in the pure CS limit of the theory. It can be achieved if we simply take the limit $e^2 \to \infty$ keeping k fixed.

Note that from the a coefficient (3.13) in momentums space we obtain:

$$
\begin{aligned}
a &= e^2 \frac{\Box + 2v^2e^2}{(\Box + 2v^2e^2)^2 + k^2e^4\Box} \Rightarrow e^2 \frac{p^2 - m_H^2}{(p^2 - m_-^2)(p^2 - m_+^2)} \\
e^2 \frac{p^2 - m_H^2}{(p^2 - m_-^2)(p^2 - m_+^2)} &= e^2 \left(\frac{m_-^2 - m_H^2}{m_-^2 - m_+^2} \right) \frac{1}{p^2 - m_-^2} + e^2 \left(\frac{m_H^2 - m_+^2}{m_-^2 - m_+^2} \right) \frac{1}{p^2 - m_+^2} \\
&= \frac{e^2}{2} \left(1 - \frac{1}{\sqrt{1 + \frac{8v^2}{k^2e^2}}} \right) \frac{1}{p^2 - m_-^2} + \frac{e^2}{2} \left(1 + \frac{1}{\sqrt{1 + \frac{8v^2}{k^2e^2}}} \right) \frac{1}{p^2 - m_+^2}
\end{aligned}
$$

Now taking the limit $e^2 \to \infty$:

$$
\begin{aligned}
\lim_{e^2 \to \infty} m_- &= \lim_{e^2 \to \infty} \frac{ke^2}{2} \left(\sqrt{1 + \frac{4(2e^2v^2)}{(ke^2)^2}} - 1 \right) \\
&= \lim_{e^2 \to \infty} \frac{ke^2}{2} \left(\sqrt{1 + \frac{8v^2}{k^2e^2}} - 1 \right) \\
&= \lim_{e^2 \to \infty} \frac{ke^2}{2} \left[1 + \frac{1}{2} \frac{8v^2}{k^2e^2} + \mathcal{O}\left(\frac{1}{e^4} \right) - 1 \right] \\
&= \frac{2v^2}{k}
\end{aligned}
$$

While $m_+ \to \infty$ in this limit, effectively decoupling from the theory. We are left with only 1 d.f. after the spontaneous symmetry breaking. This is no surprise, since in the pure CS limit, the gauge boson was non-propagating before the symmetry breaking. It became propagating after it ate up the would-be Goldstone boson.

We now turn the some aspects of fermions in 2+1 dimensions which will prove useful not only for our discussion of the CS term, but also to introduce the discrete Lorentz transformations of parity and time-reversal.

3.2 Fermions in 2+1 dimensions

We start from the Clifford algebra:

$$
\{\gamma^\mu, \gamma^\nu\} = \eta^{\mu\nu} \mathbb{1} \tag{3.15}
$$

where $\mu, \nu = 0, 1, 2$ and $\eta^{\mu\nu} = diag(+--)$. One possible representation (rep.) satisfying (3.15) is the Dirac rep.:

$$
\gamma^0 = \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \gamma^1 = i\sigma^1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad \gamma^2 = i\sigma^2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}
$$

One can check that they satisfy:

$$\gamma^\mu\gamma^\nu = \mathbb{1}\eta^{\mu\nu} + i\epsilon^{\mu\nu\rho}\eta_{\rho\sigma}\gamma^\sigma \quad , \qquad (\gamma^\mu)^\dagger = \gamma^0\gamma^\mu\gamma^0 \tag{3.16}$$

$$\mathrm{Tr}(\gamma^\mu\gamma^\nu\gamma^\rho) = 2i\epsilon^{\mu\nu\rho} \tag{3.17}$$

Where, $\epsilon^{012} = -1$. At least, two points should be highlighted. In 4 dimensions (3.17) is trivial, and its result in one lower dimension is what allows the appearance of a CS term from quantum corrections involving fermionic loops, unless some symmetry prevents it so. Secondly, chirality or handedness is not fundamental property, in the usual sense, in this dimension since $i\gamma^0\gamma^1\gamma^2 = \mathbb{1}$.

3.2.1 Parity ($\mathcal{P}$) and Time-Reversal ($\mathcal{T}$)

Starting from $\mathcal{P}$ transformations, we see that it must invert only one spatial component. If it inverts two, its determinant is 1, being therefore equivalent to a rotation. We shall take:

$$\begin{aligned} x^1 &\xrightarrow{\mathcal{P}} -x^1 \\ x^0, x^2 &\xrightarrow{\mathcal{P}} x^0, x^2. \end{aligned}$$

Or, simply $\mathcal{P}^\mu_\nu = diag(+-+)$. Let's see what this implies for the Dirac equation:

$$(i\gamma^\mu\partial_\mu - m)\Psi(x) = 0 \xrightarrow{\mathcal{P}} (i\gamma^\mu\partial^P_\mu - m)\Psi^P(x^P) = 0 \tag{3.18}$$

Where $\Psi^P(x^P)$ is the parity transformed Dirac spinor. Let's take $\Psi^P(x^P) = P\Psi(x)$ where P acts on spinor space. Using $\partial^P_\mu = \mathcal{P}^\nu_\mu\partial_\nu$, we obtain:

$$(iP^{-1}\gamma^\mu P\mathcal{P}^\nu_\mu\partial_\nu - m)\Psi(x) = 0 \tag{3.19}$$

If $P^{-1}\gamma^\mu P\mathcal{P}^\nu_\mu = \gamma^\nu$, we recover the original equation, but this implies:

$$\begin{aligned} P^{-1}\gamma^0 P = \gamma^0 &\Rightarrow \left[\gamma^0, P\right] = 0 \\ P^{-1}\gamma^1 P = -\gamma^1 &\Rightarrow \left\{\gamma^1, P\right\} = 0 \\ P^{-1}\gamma^2 P = \gamma^2 &\Rightarrow \left[\gamma^2, P\right] = 0 \end{aligned}$$

No 2x2 matrix satisfy these relations. One way out is to take $m = 0$

$$(iP^{-1}\gamma^\mu P\mathcal{P}^\nu_\mu\partial_\nu)\Psi(x) = 0 \tag{3.20}$$

so that we can impose $P^{-1}\gamma^\mu P\mathcal{P}^\nu_\mu = -\gamma^\nu$, implying

$$P^{-1}\gamma^0 P = -\gamma^0 \Rightarrow \left\{\gamma^0, P\right\} = 0$$
$$P^{-1}\gamma^1 P = \gamma^1 \Rightarrow \left[\gamma^1, P\right] = 0$$
$$P^{-1}\gamma^2 P = -\gamma^2 \Rightarrow \left\{\gamma^2, P\right\} = 0$$

Now, any $\mathcal{P} = \eta\gamma^1$ will do the job, provided η is a complex phase. We shall take $P = -i\gamma^1$. This leads to

$$\begin{aligned} \Psi &\xrightarrow{\mathcal{P}} -i\gamma^1\Psi \\ \overline{\Psi} = \Psi^\dagger\gamma^0 &\xrightarrow{\mathcal{P}} \overline{\Psi}i\gamma^1 \end{aligned} \tag{3.21}$$

and, therefore, a mass term like $\overline{\Psi}\Psi$ transforms under parity as a pseudo-scalar!

$$\overline{\Psi}\Psi \xrightarrow{\mathcal{P}} \overline{\Psi}i\gamma^1(-i\gamma^1)\Psi = -\overline{\Psi}\Psi$$

This suggests that if we want to construct a theory with fermions in 2+1 dimensions invariant under parity transformations, either the fermions are massless or both signs of the mass term appear, with the positive- (negative-) sign mass fermion transforming into negative (positive) one.

Considering now time-reversal, we start by remembering that it is a anti-linear and anti-unitary transformation, and as a consequence any complex number must be transformed into its complex conjugate. A quick way to see this is by combining special relativity and quantum mechanics. It is natural to expect that reversing time should reverse momentum, $\vec{p} \xrightarrow{\mathcal{T}} -\vec{p}$, but $E = \sqrt{\vec{p}^2 + m^2}$ remains then unaffected. Well, if the quantum mechanical relation $E \leftrightarrow i\frac{\partial}{\partial t}$ is to hold and, simultaneously, the energy should not change under $t \xrightarrow{\mathcal{T}} -t$ then $i \xrightarrow{\mathcal{T}} -i$, leaving the combination $i\frac{\partial}{\partial t}$ invariant. Applying all this to the Dirac equation and considering $\partial^T_\mu = \mathcal{T}^\nu_\mu \partial_\nu$, with $\mathcal{T}^\nu_\mu = diag(-++)$, we obtain

$$(i\gamma^\mu\partial_\mu - m)\Psi(x) = 0 \xrightarrow{\mathcal{T}} (i^T\gamma^\mu_T\partial^T_\mu - m)\Psi^T(x^T) = 0 \tag{3.22}$$

Time-reversal in spinor space will be represented by T, that is, $\Psi^T(x^T) = T\Psi(x)$

$$\begin{aligned} (i^T\gamma^\mu_T\partial^T_\mu - m)\Psi^T(x^T) &= 0 \\ \left[-iT^{-1}\left(\gamma^\mu\right)^* T\mathcal{T}^\nu_\mu\partial_\nu - m\right]\Psi(x) &= 0 \end{aligned} \tag{3.23}$$

Analogously to the case of parity transformation, imposing $-T^{-1}\left(\gamma^\mu\right)^* T\mathcal{T}^\nu_\mu = \gamma^\nu$ in order to reobtain the original equation leads to an impossibility. The way out is to set $m = 0$ and impose $-T^{-1}\left(\gamma^\mu\right)^* T\mathcal{T}^\nu_\mu = -\gamma^\nu$, instead. In this way, working with the Dirac rep. of the gamma matrices we obtain:

$$T^{-1}\gamma^0 T = -\gamma^0 \Rightarrow \left\{\gamma^0, T\right\} = 0$$
$$T^{-1}\gamma^1 T = -\gamma^1 \Rightarrow \left\{\gamma^1, T\right\} = 0$$
$$T^{-1}\gamma^2 T = \gamma^2 \Rightarrow \left[\gamma^2, T\right] = 0$$

that can be solved by taking $T = \gamma^2$, for instance. That is,

$$\begin{array}{rcl} \Psi & \xrightarrow{\mathcal{T}} & \gamma^2\Psi \\ \overline{\Psi} = \Psi^\dagger\gamma^0 & \xrightarrow{\mathcal{T}} & \overline{\Psi}\gamma^2 \end{array} \tag{3.24}$$

As a consequence, the mass term is also a pseudo-scalar under time-reversal transf.:

$$\overline{\Psi}\Psi \xrightarrow{\mathcal{T}} \overline{\Psi}\gamma^2\gamma^2\Psi = -\overline{\Psi}\Psi$$

An important fact is that the CS term is also a pseudo-scalar under $\mathcal{P}$ and $\mathcal{T}$

$$\epsilon^{\mu\nu\rho}A_\mu\partial_\nu A_\rho \xrightarrow{\mathcal{P},\mathcal{T}} -\epsilon^{\mu\nu\rho}A_\mu\partial_\nu A_\rho. \tag{3.25}$$

and that is not a mere coincidence. In fact, in section 3.4 we indicate how the quantum corrections naturally produce the CS term if one starts with a QED-like classical action. To check (3.25), one needs only to remember that $A_\mu^P = \mathcal{P}_\mu^\nu A_\nu$ and $A_\mu^T = -\mathcal{T}_\mu^\nu A_\nu$. The reason for this last minus sign can be seen, for example, from the fact the components of the electric field $E^i = -\partial_0 A^i - \partial_i A^0$ are scalars under time-reversal, therefore the A_i's (as opposed to A_0) should transform to compensate the transformation of ∂_0.

3.3 Poincaré algebra in 2+1 dimensions

The generators of the Poincaré group in 3 dimensions satisfy [144]:

$$\begin{array}{rcl} i\left[J^\mu, J^\nu\right] & = & \epsilon^{\mu\nu\rho}J_\rho; \\ i\left[J^\mu, P^\nu\right] & = & \epsilon^{\mu\nu\rho}P_\rho; \\ \left[P^\mu, P^\nu\right] & = & 0; \end{array} \tag{3.26}$$

Where $J^\mu = \frac{1}{2}\epsilon^{\mu\nu\rho}J_{\nu\rho}$ are the generators of the proper-orthochronous Lorentz subgroup. Under an infinitesimal transformation, the objects living in a finite dimensional representation of $J^{\mu\nu}$ will transform as: $\phi_a \to \phi'_a = \left[\delta_{ab} + \frac{i}{2}\omega_{\mu\nu}(J^{\mu\nu})_{ab}\right]\phi_b$, $a, b = 1, 2, ..., N$.

A field belonging to a finite dimensional rep. will transform, in general, as:

$$\phi'_a(x) = \left[e^{-\frac{i}{2}\omega_{\mu\nu}\Sigma^{\mu\nu}}\right]_{ab}\phi_b(\Lambda x) \tag{3.27}$$

Such that infinitesimally, to first order in $\omega_{\mu\nu} = -\omega_{\nu\mu}$, we get

$$\begin{aligned}\phi_a(x) \to \phi'_a(x) &= \left[\delta_{ab} - \frac{i}{2}\omega_{\mu\nu}(\Sigma^{\mu\nu})_{ab}\right]\phi_b(x^\mu + \omega^\mu_\nu x^\nu) \\ &= \left\{\delta_{ab} + \frac{i}{2}\omega_{\mu\nu}\left[\delta_{ab}i(x^\mu\partial^\nu - x^\nu\partial^\mu) - (\Sigma^{\mu\nu})_{ab}\right]\right\}\phi_b(x)\end{aligned}$$

And, therefore,

$$(J^{\mu\nu})_{ab} = \delta_{ab}i(x^\mu\partial^\nu - x^\nu\partial^\mu) - (\Sigma^{\mu\nu})_{ab} \tag{3.28}$$

The first contribution is the orbital component while the second is the spin component. With the Poincaré generators, we can construct the following Casimir invariants:

$$P^2 = P^\mu P_\mu, \qquad W = J^\mu P_\mu \tag{3.29}$$

We can easily check that W is invariant, in fact:

$$\begin{aligned}[J^\mu, W] &= [J^\mu, J^\nu P_\nu] = [J^\mu, J^\nu]P_\nu + J^\nu\eta_{\nu\rho}[J^\mu, P^\rho] \\ &= -i\epsilon^{\mu\nu\rho}J_\rho P_\nu - iJ_\rho\epsilon^{\mu\rho\nu}P_\nu = 0 \\ [P^\mu, W] &= [P^\mu, J^\nu P_\nu] = [P^\mu, J^\nu]P_\nu + J^\nu\eta_{\nu\rho}\cancelto{0}{[P^\mu, P^\rho]} \\ &= -i\epsilon^{\mu\nu\rho}P_\rho P_\nu = 0\end{aligned} \tag{3.30}$$

We can now use these invariant operators to define the one-particle states Φ, such that

$$P^2\Phi = m^2\Phi, \qquad W\Phi = -sm\Phi, \tag{3.31}$$

thus defining the mass m and spin s of particles. Note that, in the rest frame $p^\mu = (m, 0, 0)$, the invariant W acting on a one-particle state is simply $W = mJ^0 = mJ^{12}$, where J^{12} is the generator of rotations on the plane. Defining the one-particle state as J^{12} eigenstate with eigenvalue $-s$, we obtain (3.31). It should be stressed that: 1- W is a pseudo-scalar, then so is the eigenvalue s, therefore a particle with spin s is intrinsically different from a particle with spin $-s$, but they are related through $\mathcal{P}$ or $\mathcal{T}$ transformations; and 2- there is no reason, *a priori*, for s to be quantized like in the 3+1 dimensional case.

3.3.1 The spin $\frac{1}{2}$ rep.

The fundamental rep. of the Lorentz group is provided by the gamma matrices γ^μ through $i\Sigma^{\mu\nu} = \frac{i}{4}[\gamma^\mu, \gamma^\nu]$. In the dimension that we are working with $i\Sigma_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\lambda}\gamma^\lambda$, hence from (3.28) we can construct J^μ :

$$J^\mu = \frac{1}{2}\epsilon^{\mu\nu\rho}J_{\nu\rho} = i\epsilon^{\mu\nu\rho}x_\nu\partial_\rho - \frac{1}{2}\gamma^\mu \tag{3.32}$$

Now, to construct W we will also need P_μ which we will represent by $P_\mu = i\partial_\mu$

$$\begin{aligned} W = J^\mu P_\mu &= (i\epsilon^{\mu\nu\rho}x_\nu\partial_\rho - \frac{1}{2}\gamma^\mu)i\partial_\mu \\ &= -\cancelto{0}{\epsilon^{\mu\nu\rho}x_\nu\partial_\rho\partial_\mu} - \frac{i}{2}\gamma^\mu\partial_\mu = -\frac{i}{2}\gamma^\mu\partial_\mu \end{aligned} \tag{3.33}$$

Acting with W on the previously defined one-particle state, we shall obtain:

$$W\Psi = -ms\Psi \Rightarrow -\frac{i}{2}\gamma^\mu\partial_\mu\Psi = -ms\Psi \Rightarrow (i\gamma^\mu\partial_\mu - m2s)\,\Psi = 0 \tag{3.34}$$

Which is nothing other then Dirac equation if $s = \frac{1}{2}$. As we saw, the Dirac mass term changes sign under parity or time-reversal transformations, now it becomes clear that this change of sign comes from the change on the spin of the particle. Note that even the orbital angular momentum operator $l_z = i(x_1\partial_2 - x_2\partial_1)$ is odd under $\mathcal{P}$ ($x^1 \xrightarrow{\mathcal{P}} -x^1$) and $\mathcal{T}$ ($i \xrightarrow{\mathcal{T}} -i$), so it is reasonable that the same happens to spin ($s \xrightarrow{\mathcal{P},\mathcal{T}} -s$).

3.3.2 The spin 1 rep.

Let's now repeat the previous procedure considering the vector representation. We will ignore the orbital component since it does not contribute to the final answer. In the vector representation $V'^\mu = \Lambda^\mu{}_\nu V^\nu$ infinitesimally, we get:

$$\begin{aligned} V'^\mu &= (\delta^\mu_\nu + \omega^\mu_\nu)V^\nu \\ &= \left\{\delta^\mu_\nu + \frac{i}{2}\omega_{\alpha\beta}\left[-i(\eta^{\mu\alpha}\delta^\beta_\nu - \eta^{\mu\beta}\delta^\alpha_\nu)\right]\right\}V^\nu \end{aligned}$$

Implying then

$$(\Sigma^{\alpha\beta})_{\mu\nu} = -i(\delta^\alpha_\mu\delta^\beta_\nu - \delta^\beta_\mu\delta^\alpha_\nu) \Rightarrow (J^\mu)^{\alpha\beta}_{spin} = \frac{1}{2}\epsilon^{\mu\nu\rho}(\Sigma^{\alpha\beta})_{\nu\rho} = -i\epsilon^{\mu\alpha\beta} \tag{3.35}$$

Therefore $W^{\alpha\beta} = (J^\mu)^{\alpha\beta}P_\mu = (J^\mu)^{\alpha\beta}_{spin}P_\mu = \epsilon^{\mu\alpha\beta}\partial_\mu$. Applying it to the one-particle state:

$$W^{\alpha\beta}V_\beta = -smV^\alpha \Rightarrow \epsilon^{\mu\alpha\beta}\partial_\mu V_\beta = -smV^\alpha \tag{3.36}$$

Which is exactly the Maxwell-Chern-Simons equation (3.6), written in terms of the duall $\tilde{F}^\mu$, that is $\epsilon^{\mu\alpha\beta}\partial_\mu\tilde{F}_\beta = -ke^2\tilde{F}^\alpha$. Hence, one more time we have identified the mass $m_{MCS} = ke^2$ with the surplus of knowing that particle's spin is $s = 1$ (or $s = -1$, depending on the sign of k).

3.4 Chern-Simons term induced by quantum corrections: A sketch

In this section, we briefly go over some steps to show how a CS term can be generated by quantum corrections.

3.4.1 The quantum action in 10 seconds

The starting point is the object from which any quantum field theory extracts its information, the generating functional of the connected Green functions $W[J]$:

$$e^{\frac{i}{\hbar}W[J]} = \int D\Phi e^{\frac{i}{\hbar}\left\{S[\Phi]+\int d^D x J\Phi\right\}} \tag{3.37}$$

Here Φ collectively denotes all fields of the theory, $S[\Phi]$ is the classical action in D spacetime dimensions and $J(x)$ is an arbitrary external source. From it, one can define the quantum field

$$\Phi_q(x) = \frac{\delta W[J]}{\delta J(x)} = \Phi_q(x;J) \tag{3.38}$$

and assuming that we are able to invert the above relation to obtain $J(x) = J(x;\Phi_q(x))$, we can then proceed to define the quantum action $\Gamma[\Phi_q]$:

$$\Gamma[\Phi_q] = W[J] - \int d^D x J(x)\Phi_q \; ; \quad \frac{\delta\Gamma[\Phi_q]}{\delta\Phi_q(x)} = -J(x) \tag{3.39}$$

We are going to use an approximation to evaluate (3.37). The strategy is to expand the fields $\Phi = \Phi_c + \delta\Phi$ around a classical configuration Φ_c, that is, one that satisfies $\left[\frac{\delta S[\Phi]}{\delta\Phi(x)}\right]_{\Phi=\Phi_c} = -J(x)$, keeping only the quadratic terms of the fluctuations $\delta\Phi$. This is called a semi-classical approximation. We obtain:

$$e^{\frac{i}{\hbar}W[J]} \approx e^{\frac{i}{\hbar}\left(S[\Phi_c]+\int dxJ\Phi_c\right)} \int D\delta\Phi e^{\frac{i}{\hbar}\left\{\frac{1}{2}\int dxdy\delta\Phi(x)\left[\frac{\delta^2 S[\Phi]}{\delta\Phi(x)\delta\Phi(y)}\right]_{\Phi=\Phi_c}\delta\Phi(y)\right\}} \tag{3.40}$$

This approximation becomes exact in the limit $\hbar \to 0$. The main reason for this strategy being that a functional integral with a Gaussian-like integrand can be calculated. In fact, one can show that:

$$\int D\delta\Phi e^{\frac{i}{\hbar}\left\{\frac{1}{2}\int dxdy\delta\Phi_a(x)\left[\frac{\delta^2 S[\Phi]}{\delta\Phi_a(x)\delta\Phi_b(y)}\right]_{\Phi=\Phi_c}\delta\Phi_b(y)\right\}} = \mathcal{C}\left(\det\frac{\delta^2 S[\Phi]}{\delta\Phi_a(x)\delta\Phi_b(y)}\right)^{\sigma} \tag{3.41}$$

Where $\mathcal{C}$ has been introduced simply to absorb some irrelevant constant parameters, while $\sigma = \pm\frac{1}{2}, \pm 1$ is a parameter that depends on the nature of Φ_a: $-\frac{1}{2}$ $\left(+\frac{1}{2}\right)$ if it's a real bosonic (fermionic) field and $-1(+1)$ if it's a complex bosonic (fermionic) field. The final trick is to use the identity $\det A = e^{\mathrm{Tr}\,\mathrm{Ln}\,A}$, and rewrite

$$\left(\det \frac{\delta^2 S[\Phi]}{\delta\Phi_a(x)\delta\Phi_b(y)}\right)^\sigma = \left(e^{\mathrm{Tr}\,\mathrm{Ln}\,\frac{\delta^2 S[\Phi]}{\delta\Phi_a(x)\delta\Phi_b(y)}}\right)^\sigma = e^{\sigma\,\mathrm{Tr}\,\mathrm{Ln}\,\frac{\delta^2 S[\Phi]}{\delta\Phi_a(x)\delta\Phi_b(y)}} \tag{3.42}$$

With all this, we have thus obtained:

$$W[J] \approx S[\Phi_c] + \int d^D x J \Phi_c - i\hbar\sigma\,\mathrm{Tr}\,\mathrm{Ln}\,\frac{\delta^2 S[\Phi]}{\delta\Phi_a(x)\delta\Phi_b(y)} \tag{3.43}$$

Which, substituting back on (3.39), gives:

$$\Gamma[\Phi_q] \approx S[\Phi_q] - i\hbar\sigma\,\mathrm{Tr}\,\mathrm{Ln}\left[\frac{\delta^2 S[\Phi]}{\delta\Phi_a(x)\delta\Phi_b(y)}\right]_{\Phi=\Phi_q} + \mathcal{O}(\hbar^2) \tag{3.44}$$

3.4.2 CS from QED

From the result (3.44) we can now evaluate the leading quantum correction to the QED_3 action:

$$S[A,\Psi] = \int d^3x \left\{-\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \overline{\Psi}_\alpha \left(i\gamma^\mu D_\mu - m\right)_{\alpha\beta} \Psi_\beta\right\} \tag{3.45}$$

With $D_\mu = \partial_\mu + ieA_\mu$. Being a bit more explicit with the indices, $\left(i\gamma^\mu D_\mu - m\right)_{\alpha\beta}$ denotes:

$$\left(i\gamma^\mu D_\mu - m\right)_{\alpha\beta} = [i(\gamma^\mu)_{\alpha\beta}\partial_\mu - eA_\mu(\gamma^\mu)_{\alpha\beta} - m\delta_{\alpha\beta}] \tag{3.46}$$

Let us now focus on the corrections due to the fermion field:

$$\frac{\delta^2 S[A,\Psi]}{\delta\overline{\Psi}_\alpha(x)\delta\Psi_\beta(y)} = [i(\gamma^\mu)_{\alpha\beta}\partial_\mu - eA_\mu(x)(\gamma^\mu)_{\alpha\beta} - m\delta_{\alpha\beta}]\,\delta^3(x-y) \equiv O_{\alpha\beta}(x,y) \tag{3.47}$$

We need to evaluate

$$
\begin{aligned}
\operatorname{Tr}\operatorname{Ln}\left[(i\partial\!\!\!/ - m) - eA\!\!\!/\right] &= \operatorname{Tr}\operatorname{Ln}\left[(i\partial\!\!\!/ - m) - e(i\partial\!\!\!/ - m)(i\partial\!\!\!/ - m)^{-1}A\!\!\!/\right] \\
&= \operatorname{Tr}\operatorname{Ln}\left\{(i\partial\!\!\!/ - m)\left[\mathbb{1} - e(i\partial\!\!\!/ - m)^{-1}A\!\!\!/\right]\right\} \\
&= \operatorname{Tr}\operatorname{Ln}(i\partial\!\!\!/ - m) + \operatorname{Tr}\ln\left[\mathbb{1} - e(i\partial\!\!\!/ - m)^{-1}A\!\!\!/\right] \\
&= \operatorname{Tr}\operatorname{Ln}(i\partial\!\!\!/ - m) - \operatorname{Tr}\sum_{n=1}^{\infty}\frac{\left[e(i\partial\!\!\!/ - m)^{-1}A\!\!\!/\right]^n}{n}
\end{aligned}
$$

Slashed quantities have the usual definition $A\!\!\!/ = \gamma^\mu A_\mu$. We have also made use of the expansion $\ln(1-x) = -\sum\limits_{n=1}^{\infty}\frac{x^n}{n}$. The steps above are all formal operations that (sometimes) can be made proper sense, for example, in Fourier space. Actually, all we want to see is how the Chern-Simons term will arise. It will come from the term of order e^2. Let us first give some sense to the objects defined. Starting from the inverse of the Dirac operator:

$$
\left[(i\partial\!\!\!/ - m)^{-1}\right]_{\beta\sigma}(z,y) = \int \frac{d^3p}{(2\pi)^3} e^{-ip(z-y)} \frac{(p\!\!\!/ + m)_{\beta\sigma}}{p^2 - m^2} \tag{3.48}
$$

While

$$
[(i\partial\!\!\!/ - m)^{-1}A\!\!\!/]_{\alpha\beta}(x,y) = \int dz \frac{d^3p}{(2\pi)^3} e^{-ip(x-z)} \frac{(p\!\!\!/ + m)_{\alpha\beta}}{p^2 - m^2} A\!\!\!/_{\beta\sigma}(z)\delta^3(z-y) \tag{3.49}
$$

The indicated trace operation that we need to evaluate is both in physical space and spinor space, that is $\operatorname{Tr} O_{\alpha\beta}(x,y) \equiv \int d^3x O_{\alpha\alpha}(x,x)$. With all this considered, one can obtain from the calculation of the leading quantum correction (3.44) to order e^2, now taking $\sigma = -1$ and $\hbar = 1$:

$$
\begin{aligned}
&.i\frac{e^2}{2}\operatorname{Tr}[(i\partial\!\!\!/ - m)^{-1}A\!\!\!/(i\partial\!\!\!/ - m)^{-1}A\!\!\!/] = \\
&= i\frac{e^2}{2}\operatorname{Tr}\int dy \frac{d^3p}{(2\pi)^3}\frac{d^3p'}{(2\pi)^3} e^{-ip(x-y)-ip'y}\left(\frac{1}{p\!\!\!/ - m}\right)A\!\!\!/(p')\frac{d^3q}{(2\pi)^3}\frac{d^3q'}{(2\pi)^3}e^{-iq(y-z)-iq'z}\left(\frac{1}{q\!\!\!/ - m}\right)A\!\!\!/(q')
\end{aligned}
$$

Where we have denoted $\frac{p\!\!\!/+m}{p^2-m^2} \equiv \frac{1}{p\!\!\!/-m}$. After performing the trace in physical space, one obtains:

$$
\begin{aligned}
i\frac{e^2}{2}\operatorname{Tr}[(i\partial\!\!\!/ - m)^{-1}A\!\!\!/(i\partial\!\!\!/ - m)^{-1}A\!\!\!/] &= i\frac{e^2}{2}\operatorname{Tr}\int \frac{d^3p}{(2\pi)^3}\frac{d^3k}{(2\pi)^3}\frac{1}{k\!\!\!/ - m}A\!\!\!/(-p)\frac{1}{k\!\!\!/ + p\!\!\!/ - m}A\!\!\!/(p) \\
&= i\frac{e^2}{2}\int \frac{d^3p}{(2\pi)^3} A_\mu(-p)\Gamma^{\mu\nu}(p,m)A_\nu(p)
\end{aligned} \tag{3.50}
$$

With

$$\Gamma^{\mu\nu}(p,m) = \int \frac{d^3k}{(2\pi)^3} \operatorname{Tr}\left[\frac{\not{k}+m}{k^2-m^2}\gamma^\mu \frac{\not{k}+\not{p}+m}{(k+p)^2-m^2}\gamma^\nu\right] \tag{3.51}$$

Well, given its index structure and momentum dependence, the most general form for $\Gamma^{\mu\nu}(p,m)$ is:

$$\Gamma^{\mu\nu}(p,m) = f\eta^{\mu\nu} + gp^\mu p^\nu + hp_\rho \epsilon^{\mu\rho\nu} \tag{3.52}$$

We shall now show that h does not vanish. Indeed,

$$\operatorname{Tr}(\not{k}+m)\gamma^\mu(\not{k}+\not{p}+m)\gamma^\nu = \operatorname{Tr}(\not{k}\gamma^\mu\not{k}\gamma^\nu + \not{k}\gamma^\mu\not{p}\gamma^\nu + m\not{k}\gamma^\mu\gamma^\nu + m\gamma^\mu\not{k}\gamma^\nu + m\gamma^\mu\not{p}\gamma^\nu + m^2\gamma^\mu\gamma^\nu)$$

while

$$\operatorname{Tr}(m\not{k}\gamma^\mu\gamma^\nu + m\gamma^\mu\not{k}\gamma^\nu + m\gamma^\mu\not{p}\gamma^\nu) = -2imp_\rho\epsilon^{\mu\rho\nu}$$

Therefore, what we have shown is that

$$i\frac{e^2}{2}\int \frac{d^3p}{(2\pi)^3} A_\mu(-p)\Gamma^{\mu\nu}(p,m)A_\nu(p) \supset i\frac{e^2}{2}\operatorname{Tr}\int \frac{d^3p}{(2\pi)^3} A_\mu(-p)(-2im\epsilon^{\mu\rho\nu}p_\rho)\Gamma^{(2)}(m,p)A_\nu(p) \tag{3.53}$$

where

$$\Gamma^{(2)}(m,p) = \int \frac{d^3k}{(2\pi)^3}\frac{1}{k^2-m^2}\frac{1}{(k+p)^2-m^2}$$

(3.53) is essentially the Chern-Simons term in momentum space. Note the crucial dependence of the result on the fermion mass m. Actually, one can show that it depends only on the sign of the fermion mass. Therefore, in the presence of both positive and negative mass terms, one should not expect the arising of a Chern-Simons term from quantum corrections[1]. Moreover, care should be taken in the case of massless fermions, because the incorrect regularization of the infrared divergences might induce the CS term.

[1] Except if a parity-invariant Chern-Simons term exists, which it does in the presence of 2 gauge fields with appropriate transformations, as we will see.

Part II

Abelian vortices in a parity-invariant Maxwell-Chern-Simons-Higgs model

Chapter 4

The parity-invariant (Maxwell-) Chern-Simons model

4.1 Introduction

As pointed out in Chapter 1, one way to implement a parity preserving Chern-Simons theory is to suitably double the degrees of freedom, and the first of such implementation was given by Hagen [60], which we now present:

$$S = \int d^3x \left\{ \frac{\mu}{2} \epsilon^{\mu\nu\rho} A^+_\mu \partial_\nu A^+_\rho - \frac{\mu}{2} \epsilon^{\mu\nu\rho} A^-_\mu \partial_\nu A^-_\rho \right\}, \tag{4.1}$$

Where, in order to preserve parity symmetry, one must impose $A^\pm_\mu(x) \stackrel{P}{\rightarrow} A^{P\pm}_\mu(x^P) = \mathcal{P}_\mu{}^\nu A^\mp_\nu(x)$. Another way to construct a parity preserving Chern-Simons theories, which can be found in [70, 71, 72, 73, 74] for example, is to have a mixed Chern-Simons term:

$$S = \int d^3x \left\{ \mu \epsilon^{\mu\nu\rho} A_\mu \partial_\nu a_\rho \right\}, \tag{4.2}$$

Such that under parity transformation: $A_\mu(x) \stackrel{P}{\rightarrow} A^P_\mu(x^P) = \mathcal{P}_\mu{}^\nu A_\nu(x)$ and $a_\mu(x) \stackrel{P}{\rightarrow} a^P_\mu(x^P) = -\mathcal{P}_\mu{}^\nu a_\nu(x)$

As was said, to couple these fields with fermions while maintaining parity symmetry, one must also have both $s = \pm\frac{1}{2}$ fermions, each of which appears in the lagragian with mass terms $\mp m\overline{\psi}_\pm\psi_\pm$ and also transform into each other under parity transformation. Actually, as far as the Chern-Simons terms are concerned (4.1) and (4.2) are equivalent and can be mapped into each other through the transformations:

$$A^+_\mu = \frac{A_\mu + a_\mu}{\sqrt{2}}; \qquad A^-_\mu = \frac{A_\mu - a_\mu}{\sqrt{2}} \tag{4.3}$$

We will be working with the A_μ, a_μ variables. In [71, 72, 73] a $U_A(1) \times U_a(1)$ gauge theory for both vector fields coupled to fermions is considered as follows:

$$\begin{aligned} S = \int d^3x \Big\{ &-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{4}f^{\mu\nu}f_{\mu\nu} + \mu\epsilon^{\mu\nu\rho}A_\mu\partial_\nu a_\rho \\ &+ i\overline{\psi}_+ \not{D}\psi_+ + i\overline{\psi}_- \not{D}\psi_- - m(\overline{\psi}_+\psi_+ - \overline{\psi}_-\psi_-)\Big\} \end{aligned} \tag{4.4}$$

With conventions and definitions:

$$\begin{aligned} F_{\mu\nu} &= \partial_\mu A_\nu - \partial_\nu A_\mu, \quad f_{\mu\nu} = \partial_\mu a_\nu - \partial_\nu a_\mu \\ \not{D}\psi_\pm &= (\not{\partial} + ie\not{A} \pm ig\not{a})\psi_\pm, \quad \not{X} = \gamma^\mu X_\mu \\ \gamma^\mu &= (\sigma_3, -i\sigma_1, i\sigma_2), \quad \overline{\psi}_\pm = \psi^\dagger_\pm \gamma^0 \end{aligned} \tag{4.5}$$

The mass dimensions of the fields and parameters are : $[A] = [a] = [e] = [g] = \frac{1}{2}$; $[\psi_\pm] = [m] = [\mu] = 1$. Taking, without loss of generality, $m > 0$ the fermions ψ_+ and ψ_- correspond to massive representations of the 2+1 Poincaré group of spins $+\frac{1}{2}$ and $-\frac{1}{2}$, respectively. Analogously, considering $\mu > 0$,(4.1) taken together with the corresponding Maxwell terms for $A^\pm_\mu$ defines them as the spin ± 1 representations of the Poincaré group. Since spin is a pseudoscalar in 2+1 dimensions, no rotation or boost can change the spin of a particle, which makes it more analogous to the helicity of massless particles in 3+1 dimensions. The action (4.4) is invariant under the following transformations:

- $U_A(1) \times U_a(1)$:

$$\begin{cases} \psi_\pm \;\rightarrow\; \psi'_\pm = e^{i(\rho(x) \pm \xi(x))}\psi_\pm . \\ \overline{\psi}_\pm \;\rightarrow\; \overline{\psi}'_\pm = e^{-i(\rho(x) \pm \xi(x))}\overline{\psi}_\pm . \\ A_\mu \;\rightarrow\; A'_\mu = A_\mu - \frac{1}{e}\partial_\mu \rho(x) \; , \\ a_\mu \;\rightarrow\; a'_\mu = a_\mu - \frac{1}{g}\partial_\mu \xi(x) \; . \end{cases} \tag{4.6}$$

- Parity:

$$\begin{cases} x_\mu & \xrightarrow{P} \; x^P_\mu = \mathcal{P}_\mu{}^\nu x_\nu \qquad \text{, where } \mathcal{P}_\mu{}^\nu = diag(+-+) \\ A_\mu & \xrightarrow{P} \; A^P_\mu = \mathcal{P}_\mu{}^\nu A_\nu \\ a_\mu & \xrightarrow{P} \; a^P_\mu = -\mathcal{P}_\mu{}^\nu a_\nu \end{cases} \tag{4.7}$$

$$\begin{cases} \psi_\pm & \xrightarrow{P} \; \psi^P_\pm = -i\gamma^1\psi_\mp \\ \overline{\psi}_\pm & \xrightarrow{P} \; \overline{\psi}^P_\pm = i\overline{\psi}_\mp\gamma^1 \end{cases} , \tag{4.8}$$

4.2 Presenting the model

Let us pose the following question: What would be the equivalent system with scalar fields instead of fermionic matter? That is, can we construct a parity-invariant $U(1)\times U(1)$ gauge theory with a Chern-Simons term and charged scalar matter in analogy with the model described before? In the following, we will construct such a model and analyze its main properties.

4.2.1 Theoretical setup

Let us now propose our model, inspired by the model proposed in (4.4), aiming to construct a parity-preserving $U(1)\times U(1)$ gauge theory in (2+1) dimensions with charged scalar matter and a Chern-Simons term. The lagrangian is:

$$\begin{aligned} \mathcal{L} = &-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{4}f_{\mu\nu}f^{\mu\nu} + \mu\epsilon^{\mu\nu\rho}A_\mu\partial_\nu a_\rho \\ &+ |\left(\partial_\mu + ieA_\mu + iga_\mu\right)\phi_+|^2 + |\left(\partial_\mu + ieA_\mu - iga_\mu\right)\phi_-|^2 - V(|\phi_+|,|\phi_-|), \end{aligned} \tag{4.9}$$

where we naturally defined the field strength tensors as usual $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ and $f_{\mu\nu} = \partial_\mu a_\nu - \partial_\nu a_\mu$. Also defining the covariant derivatives acting on the scalar fields as,

$$D_\mu\phi_\pm = \partial_\mu\phi_\pm + ieA_\mu\phi_\pm \pm iga_\mu\phi_\pm. \tag{4.10}$$

From the above expression we immediately understand that the fields ϕ_+, ϕ_- have equal charge under the first $U(1)_A$ and opposite charges with respect to the second $U(1)_a$. Here as usual, e denotes the coupling associated with the gauge group $U(1)_A$ and g the coupling associated with the $U(1)_a$, where the subscript immediately tell us which gauge field is associated with each gauge group. We are taking charges ± 1 here for convenience.

We highlight the importance of Chern-Simons term $\mu\epsilon^{\mu\nu\rho}A_\mu\partial_\nu a_\rho$ to preserve parity and induce a topological gauge invariant mass term. The canonical mass dimensions of the fields and couplings here are given by: $[A] = [a] = [\phi_\pm] = \frac{1}{2}$; $[e^2] = [g^2] = [\mu] = 1$.

We can rewrite the above Lagrangian in another fashion if we define the following:

$$\Phi = \begin{pmatrix} \phi_+ \\ \phi_- \end{pmatrix} \tag{4.11}$$

Therefore the Lagrangian can be rewritten as

$$\begin{aligned} \mathcal{L} = &-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{4}f_{\mu\nu}f^{\mu\nu} + \mu\epsilon^{\mu\nu\rho}A_\mu\partial_\nu a_\rho \\ &+ |\left(\partial_\mu + ieA_\mu + iga_\mu\tau_3\right)\Phi|^2 - V(|\Phi|). \end{aligned} \tag{4.12}$$

The matrix τ_i being the i^{th} standard Pauli matrix. The Lagrangian (4.9) is by construction invariant under $U_A(1) \times U_a(1)$ gauge transformations. This implies that the potential $V(\phi_+, \phi_-)$ itself is gauge invariant and this requirement restricts its possible form. Furthermore, if we are constructing a parity-invariant model, we need to find what are the transformations of all its objects such that we can achieve this goal. We already discussed what are the parity transformations in the pure gauge sector that do the job, and we need to ask ourselves what are the parity transformations that we need to impose on the scalar sector to accomplish this goal.

We will find in the following that the scalar fields must swap their role realizing a $\mathbf{Z}_2$ transformation in the space of fields and extending the parity-transformation concept to realize this symmetry. From a practical point of view, the swapping of the scalar fields ϕ_+ and ϕ_- under parity is forced upon us by the requirement of parity invariance[1], and this behavior is reasonable if we remember the parity transformations of the matter fields in the fermionic version studied before, relating ψ_+ and ψ_-.

In fact, choosing an adequate potential $V(\phi_+, \phi_-)$, the Lagrangian (4.9) is invariant (by construction) under $U_A(1) \times U_a(1)$ gauge transformations and under parity transformations.

The explicit form of these transformations can be read below:

- $U_A(1) \times U_a(1)$:

$$\begin{cases} \phi_\pm \;\rightarrow\; \phi'_\pm = e^{i(\rho(x) \pm \xi(x))}\phi_\pm. \quad , \text{ or } \Phi \rightarrow \Phi' = e^{i\rho(x)}e^{i\xi(x)\tau_3}\Phi \\ \phi^*_\pm \;\rightarrow\; \phi'^*_\pm = e^{-i(\rho(x) \pm \xi(x))}\phi^*_\pm. \\ A_\mu \;\rightarrow\; A'_\mu = A_\mu - \frac{1}{e}\partial_\mu\rho(x)\ , \\ a_\mu \;\rightarrow\; a'_\mu = a_\mu - \frac{1}{g}\partial_\mu\xi(x) \qquad . \end{cases} \tag{4.13}$$

[1]Modulo an overall arbitrary complex phase.

- Parity:

$$\begin{cases} x_\mu & \xrightarrow{P} & x^P_\mu = \mathcal{P}_\mu{}^\nu x_\nu & \text{, where } \mathcal{P}_\mu{}^\nu = diag(+-+) \\ A_\mu & \xrightarrow{P} & A^P_\mu = \mathcal{P}_\mu{}^\nu A_\nu & \\ a_\mu & \xrightarrow{P} & a^P_\mu = -\mathcal{P}_\mu{}^\nu a_\nu & \\ \phi_\pm & \xrightarrow{P} & \phi^P_\pm = \zeta\phi_\mp & \text{, or } \Phi \xrightarrow{P} \Phi^P = \zeta\tau_1\Phi \end{cases} \tag{4.14}$$

The complex scalar fields ϕ_+ and ϕ_- transform under the gauge group as usual and in the quantum theory they will create particles of definite $U_A(1) \times U_a(1)$ charges. On the other hand, even relying on the analogy with the fermionic matter case and with the understanding that these are the transformations needed to ensure parity invariance in this model, the swapping of ϕ_+ and ϕ_- might seem somewhat artificial, since it does not follow solely from spacetime "rules", but it includes an extension of the parity concept in the space of fields. We remark that this is a consequence of our particular choice of variables, being possible to construct from them fields that transform under parity transformations as scalars or pseudo-scalars, a more familiar behavior if we are dealing with fields without spin.

In fact, we can propose the following definitions:

$$\sigma = \frac{\phi_+ + \phi_-}{\sqrt{2}}, \quad \pi = i\,\frac{\phi_+ - \phi_-}{\sqrt{2}} \tag{4.15}$$

It is immediate to see, using the parity transformations defined before, that σ transforms as a scalar and π transforms as a pseudo-scalar [2], that is,

$$\sigma \xrightarrow{P} \sigma, \quad \pi \xrightarrow{P} -\pi. \tag{4.16}$$

It is important to notice here that both σ and π are complex scalars, since ϕ_+ and ϕ_- are totally independent complex scalars fields, that are not related by complex conjugation.

Rewriting (4.9) in terms of these fields, we obtain:

$$\begin{aligned} \mathcal{L} = &-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{4}f_{\mu\nu}f^{\mu\nu} + \mu\epsilon^{\mu\nu\rho}A_\mu\partial_\nu a_\rho \\ &+ |\left(\partial_\mu + ieA_\mu + iga_\mu\tau_2\right)\Lambda|^2 - V(|\Lambda|), \end{aligned} \tag{4.17}$$

where we agglutinate both fields in the form

$$\Lambda = \begin{pmatrix} \sigma \\ \pi \end{pmatrix} \tag{4.18}$$

Writing in this different fashion, the gauge and parity transformations will be:

[2] Assuming we fix $\zeta = 1$

- $U_A(1) \times U_a(1)$:

$$\left\{ \Lambda \;\rightarrow\; \Lambda' = e^{i\rho(x)} e^{i\xi(x)\tau_2} \Lambda. \right. \tag{4.19}$$

- Parity:

$$\left\{ \Lambda \;\xrightarrow{P}\; \Lambda^P = \tau_3 \, \Lambda \right. \tag{4.20}$$

Particularly illuminating is the transformation (4.19) restricted to $\rho(x) = 0$, which reads:

$$\begin{pmatrix} \sigma' \\ \pi' \end{pmatrix} = e^{i\xi(x)\tau_2} \begin{pmatrix} \sigma \\ \pi \end{pmatrix} = \begin{pmatrix} \cos\xi(x) & \sin\xi(x) \\ -\sin\xi(x) & \cos\xi(x) \end{pmatrix} \begin{pmatrix} \sigma \\ \pi \end{pmatrix} \tag{4.21}$$

Infinitesimally, this means:

$$\begin{aligned} \delta\sigma &= \xi(x)\pi \\ \delta\pi &= -\xi(x)\sigma \end{aligned} \tag{4.22}$$

The meaning of the $U_a(1)$ now becomes clear as a $SO(2)$ rotation in the internal space consisting of a scalar a and pseudoscalar. Consistency with parity transformations in (4.22) imediately implies that $\xi(x)$ is pseudoscalar and it becomes natural that the gauging of such a symmetry introduces a pseudovector[3], that is exactly what we need in order to contruct a parity invariant mixed Chern-Simons term.

The model (4.9) leads to the equations of motion:

$$\begin{aligned} \partial_\mu F^{\mu\nu} + \mu\epsilon^{\nu\alpha\beta}\partial_\alpha a_\beta &= e\left(J^\nu_+ + J^\nu_-\right), \\ \partial_\mu f^{\mu\nu} + \mu\epsilon^{\nu\alpha\beta}\partial_\alpha A_\beta &= g\left(J^\nu_+ - J^\nu_-\right), \\ D_\mu D^\mu \phi_\pm &= -\frac{dV}{d\phi^*_\pm}, \end{aligned} \tag{4.23}$$

where we defined the currents as $J^\nu_\pm = i\left[\phi^*_\pm D^\nu \phi_\pm - D^\nu \phi^*_\pm \phi_\pm\right]$.

[3]Gauge fields are actually 1-forms, that is, they live on the space dual to that of vectors, but since we are working in flat Minkowski spacetime, we may use both names interchangeably having in mind the the metric allows us to go uniquely from one to the other.

Now, we can perform the previously mentioned field redefinition, this time in the gauge sector. We can propose the following objects:

$$A_\mu^+ = \frac{A_\mu + a_\mu}{\sqrt{2}}, \quad A_\mu^- = \frac{A_\mu - a_\mu}{\sqrt{2}}. \tag{4.24}$$

Doing so, we can rewrite the Lagrangian (4.9) using these variables as

$$\begin{aligned}\mathcal{L} = &-\frac{1}{4}F^+_{\mu\nu}F^{+\,\mu\nu} - \frac{1}{4}F^-_{\mu\nu}F^{-\,\mu\nu} + \frac{\mu}{2}\epsilon^{\mu\nu\rho}A^+_\mu\partial_\nu A^+_\rho - \frac{\mu}{2}\epsilon^{\mu\nu\rho}A^-_\mu\partial_\nu A^-_\rho \\ &+ |\left(\partial_\mu + iq_1A^+_\mu + iq_2A^-_\mu\right)\phi_+|^2 + |\left(\partial_\mu + iq_2A^+_\mu + iq_1A^-_\mu\right)\phi_-|^2 - V\left(|\phi_+|, |\phi_-|\right).\end{aligned} \tag{4.25}$$

Looking at the above Lagrangian we note the appearance of an object similar to a covariant derivative acting on the scalar fields,

$$\begin{aligned}\bar{D}_\mu\phi_+ &= \partial_\mu\phi_+ + iq_1A^+_\mu\phi_+ + iq_2A^-_\mu\phi_+ \\ \bar{D}_\mu\phi_- &= \partial_\mu\phi_- + iq_2A^+_\mu\phi_- + iq_1A^-_\mu\phi_-,\end{aligned} \tag{4.26}$$

built with the redefined vector fields A^+_μ and A^-_μ, that are not the fundamental gauge fields associated with the gauge group $U(1)_A \times U(1)_a$, but can be easily obtained from them by a unitary transformation $U = U^{-1}$ given by:

$$\begin{pmatrix} A^+_\mu \\ A^-_\mu \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} A_\mu \\ a_\mu \end{pmatrix}. \tag{4.27}$$

It is interesting to notice that written in this fashion, the scalar fields would have swapped effective charges, if we had been working with $U(1)_{A^+} \times U(1)_{A^-}$, that is, ϕ_+ would have effective charges (q_1, q_2) while ϕ_- would have (q_2, q_1), where we defined

$$q_1 = \frac{e+g}{\sqrt{2}}, \quad q_2 = \frac{e-g}{\sqrt{2}}. \tag{4.28}$$

Using the fields A^+_μ and A^-_μ we somehow diagonalize the gauge sector. The Chern-Simons terms now are separated, with opposite signs, a characteristic of parity-preserving Chern-Simons models, as already reported before. More than convenience, such a field redefinition allows us to understand an important aspect of the model. The field equations here are given by:

$$\begin{aligned}\partial_\mu F^{+\mu\nu} + \mu\epsilon^{\nu\alpha\beta}\partial_\alpha A^+_\beta &= q_1J^\nu_+ + q_2J^\nu_- \\ \partial_\mu F^{-\mu\nu} - \mu\epsilon^{\nu\alpha\beta}\partial_\alpha A^-_\beta &= q_2J^\nu_+ + q_1J^\nu_- \\ \bar{D}_\mu\bar{D}^\mu\phi_\pm &= -\frac{dV}{d\phi^*_\pm},\end{aligned} \tag{4.29}$$

with $J^\nu_\pm = i\left[\phi^*_\pm \bar{D}^\nu \phi_\pm - \phi_\pm \bar{D}^\nu \phi^*_\pm\right]$. Therefore, looking at the above equations, it is possible to conclude that A^+_μ and A^-_μ are the vector representations of the three-dimensional Poincaré group with spin equal to +1 and -1, respectively [119, 21]. Therefore, a parity-preserving Maxwell-Chern-Simons theory for a massive vector field is nothing but the process of considering both spin "polarizations". However, we shall stick to the field variables A_μ and a_μ that we started with.

The energy-momentum tensor can be computed (Appendix B) with the help of the equations of motion (4.23), resulting in:

$$\begin{aligned}
T^{\mu\nu} &= \eta^{\mu\nu}\frac{1}{4}F_{\alpha\beta}F^{\alpha\beta} - F^{\mu\beta}F^{\nu}{}_{\beta} + \eta^{\mu\nu}\frac{1}{4}f_{\alpha\beta}f^{\alpha\beta} - f^{\mu\beta}f^{\nu}{}_{\beta} \\
&+ D^\mu\phi^*_+ D^\nu\phi_+ + D^\mu\phi_+ D^\nu\phi^*_+ - \eta^{\mu\nu}|D_\alpha\phi_+|^2 \\
&+ D^\mu\phi^*_- D^\nu\phi_- + D^\mu\phi_- D^\nu\phi^*_- - \eta^{\mu\nu}|D_\alpha\phi_-|^2 \\
&+ \eta^{\mu\nu}V
\end{aligned} \tag{4.30}$$

In passing, we note the absence of a CS contribution to it. That is one of the reasons as to why it is often called a topological term. This could have been anticipated from the fact that no metric tensor $g_{\mu\nu} = \eta_{\mu\nu}$ is present in a CS term and that $T^{\mu\nu} \sim \frac{\delta S}{\delta g_{\mu\nu}}$, however an explicit cancellation without resorting to General Relativity can be seen at the Ap. B.

From (4.30), considering the 00^{th} component one can obtain the Hamiltonian, given by,

$$\begin{aligned}
T^{00} = \Bigg[&\frac{1}{2}F^2_{0i} + \frac{1}{4}F^2_{ij} + \frac{1}{2}f^2_{0i} + \frac{1}{4}f^2_{ij} + V \\
&+|D_0\phi_+|^2 + |D_0\phi_-|^2 + |D_i\phi_+|^2 + |D_i\phi_-|^2\Bigg].
\end{aligned} \tag{4.31}$$

Defining the electromagnetic fields associated with A_μ and a_μ,

$$\begin{aligned}
&E^i = F^{i0},\ \ B = \epsilon^{ij}\partial_i A_j, \\
&e^i = f^{i0}, \quad b = \epsilon^{ij}\partial_i a_j,
\end{aligned} \tag{4.32}$$

we can finally write the energy functional:

$$\begin{aligned}
H = \int d^2x \Bigg[&\frac{1}{2}\left(\vec{E}^2 + B^2\right) + \frac{1}{2}\left(\vec{e}^2 + b^2\right) + V \\
&+|D_0\phi_+|^2 + |D_0\phi_-|^2 + |D_i\phi_+|^2 + |D_i\phi_-|^2\Bigg]
\end{aligned} \tag{4.33}$$

For the discussion of the perturbative spectrum of the theory and its topological configurations with are going to consider a generic interaction potential $V(\phi_+, \phi_-)$ whose only pre-requisites we are going to impose, for now, are:

- Gauge (8.112) and parity (4.14) invariance;
- The potential induces a non-trivial vacuum expectation value (VEV) for ϕ_+ and ϕ_-;

The most general *renormalizable* potential compatible with these requirements is:

$$
\begin{aligned}
V = m^2 \left(|\phi_+|^2 + |\phi_-|^2\right) + \frac{M_1}{2} \left(|\phi_+|^4 + |\phi_-|^4\right) + M_2 |\phi_+|^2 |\phi_-|^2 \\
+ \frac{\lambda_1}{3} \left(|\phi_+|^6 + |\phi_-|^6\right) + \lambda_2 \left(|\phi_+|^2 |\phi_-|^4 + |\phi_-|^2 |\phi_+|^4\right)
\end{aligned}
\tag{4.34}
$$

Because of its generality, the parameters must be chosen carefully to ensure that the potential leads to stable vacua. It should be clear that, depending on the parameters, different vacua structures might appear, which could in principle lead to the spontaneous breaking of one, both, or none of the $U(1)$ symmetries. As a first step, let us choose the simplest scalar potential that leads to a spontaneously broken but parity-symmetric vacuum. Thus, we will consider, with $\lambda > 0$:

$$
V\left(\phi_+, \phi_-\right) = \frac{\lambda}{4} \left(|\phi_+|^2 - v^2\right)^2 + \frac{\lambda}{4} \left(|\phi_-|^2 - v^2\right)^2 . \tag{4.35}
$$

This is the simplest extension of the Abelian-Higgs potential for the case under study. Taking $v \neq 0$, it will clearly induce a non-trivial vacuum expectation value (VEV) for the scalar fields, putting the theory into the Higgs phase, where we have $\langle |\phi_\pm| \rangle = v$. This potential is not stable under quantum corrections, but this will not be an issue, since we are focusing on classical solutions. It should also be stressed that, the point here is to start with the minimum amount of ingredients necessary to obtain what we desire, which are vortex solutions in our case. The consideration of more general "physical" potentials, although desirable, should not drastically change the behavior of the topological vortex solutions. Therefore, by considering Eq. (4.35), we already have enough to extract the essential physical properties of the topological configurations we seek to investigate.

The equations of motion following from the Lagrangian are given by

$$
\begin{aligned}
\partial_\mu F^{\mu\nu} + \mu \epsilon^{\nu\alpha\beta} \partial_\alpha a_\beta &= e\left(J_+^\nu + J_-^\nu\right), \\
\partial_\mu f^{\mu\nu} + \mu \epsilon^{\nu\alpha\beta} \partial_\alpha A_\beta &= g\left(J_+^\nu - J_-^\nu\right), \\
D_\mu D^\mu \phi_\pm &= -\frac{dV}{d\phi_\pm^*},
\end{aligned}
\tag{4.36}
$$

where the currents are $J_\pm^\nu = i\left[\phi_\pm^* D^\nu \phi_\pm - \phi_\pm D^\nu \phi_\pm^*\right]$.

Let us take a look at the Gauss laws that this model presents. From the gauge fields equations of motion, and using $\rho_\pm = J^0_\pm$:

$$\vec{\nabla}\cdot\vec{E} + \mu b = e\left(\rho_+ + \rho_-\right),$$
$$\vec{\nabla}\cdot\vec{e} + \mu B = g\left(\rho_+ - \rho_-\right). \tag{4.37}$$

Defining the electric charge $Q = e\int d^2x\ (\rho_+ + \rho_-)$ and the g-electric charge $G = g\int d^2x\ (\rho_+ - \rho_-)$, and defining also the magnetic flux as $\Phi \equiv \int d^2x\, B$ and the g-magnetic flux as $\chi \equiv \int d^2x\, b$, we obtain upon integration:

$$Q = \mu\chi, \quad G = \mu\Phi. \tag{4.38}$$

That is, the electric charge associated with one gauge field is proportional to the magnetic flux associated with the other. It is well-known that there is a flux attachment caused by the CS term, but in our case this charge-flux relation happens between two different gauge fields. This mutual statistics behavior [61] is a distinctive feature of this class of models [88], but here we implement the flux attachment in a parity-invariant way. The mutual statistics behavior is nothing but the fact that, for instance, when a Q-charged particle revolves around a G-charged particle (equivalent to a double permutation), it picks up a phase $\propto \exp\left(i\oint\vec{A}.d\vec{x}\right) = \exp i\Phi = \exp\left(i\frac{G}{\mu}\right)$.

The vacuum configuration of the system is given by the absolute minimum of the energy functional (4.33), that can be achieved, for instance, considering $\phi_\pm = v$ and $A_\mu = a_\mu = 0$. In the unitary gauge we can write $\phi_\pm(x) = v + h_\pm(x)/\sqrt{2}$. The quadratic part of the Lagrangian here is given by

$$\begin{aligned}\mathcal{L}^{\text{quad}} &= -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{4}f_{\mu\nu}f^{\mu\nu} + \frac{1}{2}(\partial_\mu h_+)^2 + \frac{1}{2}(\partial_\mu h_-)^2 \\ &+ 2v^2\left(e^2 A_\mu A_\mu + g^2 a_\mu a_\mu\right) - \frac{\lambda v^2}{2}(h_+^2 + h_-^2) \\ &+ \frac{\mu}{2}\epsilon^{\mu\nu\rho}A_\mu\partial_\nu a_\rho + \frac{\mu}{2}\epsilon^{\mu\nu\rho}a_\mu\partial_\nu A_\rho.\end{aligned} \tag{4.39}$$

From the above expression we can immediately see that we have two degenerate massive scalars with $m_S = \sqrt{\lambda v^2}$. For the gauge quadratic part we can write

$$\mathcal{L}^{\text{quad}}_{\text{gauge}} = \frac{1}{2}\begin{pmatrix} A_\mu & a_\mu \end{pmatrix} O^{\mu\nu}\begin{pmatrix} A_\nu \\ a_\nu \end{pmatrix}, \tag{4.40}$$

where we defined the gauge dynamical operator

$$O^{\mu\nu} = \begin{pmatrix} \Box\Theta^{\mu\nu} + 4e^2v^2\eta^{\mu\nu} & \mu\epsilon^{\mu\rho\nu}\partial_\rho \\ \mu\epsilon^{\mu\rho\nu}\partial_\rho & \Box\Theta^{\mu\nu} + 4g^2v^2\eta^{\mu\nu} \end{pmatrix}. \tag{4.41}$$

After some manipulations (Appendix C), from the inverse of Eq. (4.41), one can find the dispersion relations $p_\pm^2 = m_\pm^2$, where:

$$m_\pm^2 = \frac{1}{2}[\mu^2 + 4v^2(e^2+g^2)]$$
$$\pm \frac{1}{2}\sqrt{[\mu^2 + 4v^2(e^2+g^2)]^2 - (8v^2eg)^2}. \quad (4.42)$$

It should be stressed that the above relation is necessarily real and non-negative, which ensures the absence of taquions and ghosts at *tree-level* in the model. We can see that the gauge fields will acquire mass contributions coming from the Higgs mechanism and also from the CS term. In particular, in the absence of a CS term ($\mu = 0$), we would have two massive vector bosons with $M_e = 2ev$ and $M_g = 2gv$. In the case without spontaneous symmetry breaking ($v = 0$), the Higgs mechanism does not take place and we find only a topological mass given by μ. In the absence of a Maxwell term, we obtain two copies of the dispersion relation $p^2 = 16e^2g^2v^4/\mu^2$, and we have degenerate gauge boson masses.

4.3 Topological Configurations

In order to have finite energy, each non-negative term in Eq. (4.33), from now on restricted to the static regime ($\partial_0 \equiv 0$), must asymptote to zero as $|\vec{x}| = r \to \infty$. These asymptotic conditions can be seen as boundary conditions for the fields at $S^1_\infty \equiv \partial\mathbb{R}^2$. In particular, the scalar fields must asymptote to the vacuum manifold, *i.e.*, with a fixed norm on the space of fields, but with phase freedom. In fact, since we have ϕ_+ and ϕ_-, there are two phase degrees of freedom in the asymptotic limit. This give us a map $\Phi_\infty : S^1_\infty \to S^1 \times S^1 \equiv U(1) \times U(1)$. Any such map can be classified by two integers determined by the fundamental homotopy group $\pi_1(S^1 \times S^1) \equiv \mathbb{Z} \times \mathbb{Z}$. Therefore we conclude that the finite-energy condition implies an homotopy classification leading to a labeling of the configurations by two integers.

In the asymptotic limit, we can take $\phi_\pm \to ve^{i\omega_\pm(\theta)}$ where θ parametrizes the circle S^1_∞, together with $A_i \to -\partial_i(\omega_+ + \omega_-)/2e$ and $a_i \to -\partial_i(\omega_+ - \omega_-)/2g$, to ensure that the covariant derivatives vanish at spatial infinity. To satisfy the remaining asymptotic conditions, we can take $A_0, a_0 \to 0$ as well as $\partial_i A_0, \partial_i a_0 \to 0$.

Let us define a (n,m)-vortex as a finite-energy static configuration obeying the boundary conditions stated above with the particular structure:

$$\phi_\pm \to ve^{i(m\pm n)\theta},$$
$$A_i \to -\frac{m}{e}\partial_i\theta,$$
$$a_i \to -\frac{n}{g}\partial_i\theta. \quad (4.43)$$

Where, in principle, we demand only that $m \pm n \in \mathbb{Z}$, allowing m and n to take simultaneously half-integer values. In the light of the natural doubling of degrees of freedom necessary to ensure parity invariance, the possibility of half-integer numbers should not be worrisome.

From the equations of motion, we already know that there is a relation between charges and magnetic fluxes. But, by definition, $\Phi = \int d^2x\, \epsilon^{ij}\partial_i A_j = \int_{S^1_\infty} dS\, \hat{r}_i\, \epsilon^{ij} A_j$. Upon using the asymptotic behavior of the gauge field and the relations $\hat{\theta}_i = \epsilon_{ij}\hat{r}_j = \epsilon_{ij}x^j/r$ and $\epsilon_{ij}\epsilon_{jk} = -\delta_{ik}$, we have, $\Phi = \int d\theta\, r\, \hat{r}_i\, \epsilon^{ij}\left(-\frac{m}{er}\epsilon_{jk}\hat{r}_k\right) = \frac{2\pi}{e}m$. Analogously for χ. Thus:

$$\Phi = \frac{2\pi}{e}m, \quad \chi = \frac{2\pi}{g}n. \tag{4.44}$$

Therefore, we can conclude that besides the magnetic flux associated with one gauge field being proportional to the electric charge of the other, they are all topologically quantized, and can be written as

$$Q = \frac{2\pi}{g}\,\mu\, n, \quad G = \frac{2\pi}{e}\,\mu\, m. \tag{4.45}$$

We propose the following (n,m)-vortex *ansatz*:

$$\begin{aligned}
\phi_\pm &= v\, F_\pm(r)\, e^{i(m\pm n)\theta}, \\
A_i &= \frac{1}{er}\,[A(r) - m]\,\hat{\theta}_i, \\
a_i &= \frac{1}{gr}\,[a(r) - n]\,\hat{\theta}_i, \\
A_0 &= \frac{1}{er}\,\alpha(r), \\
a_0 &= \frac{1}{gr}\,\beta(r).
\end{aligned} \tag{4.46}$$

To satisfy the asymptotic conditions, the functions above must satisfy the following boundary conditions:

$$F_\pm(\infty) = 1,\ A(\infty) = a(\infty) = 0. \tag{4.47}$$

We impose $F_\pm(0) = 0$, $A(0) = m$, $a(0) = n$, and also $\alpha(0) = \beta(0) = 0$ to avoid a singularity at the origin, except when $m = \pm n$, because in this case one of the scalar profiles can take a non-zero value at the origin. Under a parity transformation in the vortex configuration, we have $(m,n) \to (-m,n)$, $r \to r, \theta \to -\theta - \pi$ and $F_\pm \to \zeta_{m,n} F_\mp, A \to -A, a \to a, \alpha \to \alpha, \beta \to -\beta$. Where $\zeta_{m,n}$ is phase factor depending on the numbers m,n. We are not concerning ourselves with time-reversal invariance, since we will be interested only in static configurations.

The energy density functional, considering this *ansatz*, can be written as

$$\begin{aligned}\epsilon =& \frac{1}{2e^2r^2}\left[\dot{A}^2+\left(\dot{\alpha}-\frac{\alpha}{r}\right)^2\right]+\frac{1}{2g^2r^2}\left[\dot{a}^2+\left(\dot{\beta}-\frac{\beta}{r}\right)^2\right]\\ &+\frac{\lambda v^4}{4}\left[(F_+^2-1)^2+(F_-^2-1)^2\right]\\ &+\frac{v^2}{r^2}\left[F_+^2\left(\alpha+\beta\right)^2+F_-^2\left(\alpha-\beta\right)^2\right]\\ &+v^2\left[\dot{F}_+^2+\frac{F_+^2}{r^2}\left(A+a\right)^2+\dot{F}_-^2+\frac{F_-^2}{r^2}\left(A-a\right)^2\right].\end{aligned} \tag{4.48}$$

One can also compute the angular momentum of these finite-energy static vortex-like configurations, given by

$$J=-\int d^2x\,\epsilon^{ij}r_iT_{0j}. \tag{4.49}$$

Such that, using the *ansatz*, boundary conditions and equations of motion, in the static limit we can obtain (Appendix D) for the angular momentum of our (m,n)-vortices:

$$J=\frac{2\pi\mu}{eg}nm=\frac{QG}{2\pi\mu}. \tag{4.50}$$

We conclude that the angular momentum of these configurations is quantized, proportional to the product of charges, and fractional, exhibiting an anyonic nature.

Inserting this *ansatz* in the equations of motion, we obtain differential equations that must be solved in order to find an explicit solution. From the equations of motion, we obtain:

$$\ddot{\alpha}-\frac{\dot{\alpha}}{r}+\frac{\alpha}{r^2}+\mu\frac{e}{g}\dot{a}=\frac{M_e^2}{2}\left[\alpha\,\Delta F_+^2+\beta\,\Delta F_-^2\right], \tag{4.51}$$

$$\ddot{\beta}-\frac{\dot{\beta}}{r}+\frac{\beta}{r^2}+\mu\frac{g}{e}\dot{A}=\frac{M_g^2}{2}\left[\beta\,\Delta F_+^2+\alpha\,\Delta F_-^2\right]. \tag{4.52}$$

and,

$$\ddot{A}-\frac{\dot{A}}{r}+\mu\frac{e}{g}\left(\dot{\beta}-\frac{\beta}{r}\right)=\frac{M_e^2}{2}\left[A\,\Delta F_+^2+a\,\Delta F_-^2\right], \tag{4.53}$$

$$\ddot{a}-\frac{\dot{a}}{r}+\mu\frac{g}{e}\left(\dot{\alpha}-\frac{\alpha}{r}\right)=\frac{M_g^2}{2}\left[a\,\Delta F_+^2+A\,\Delta F_-^2\right], \tag{4.54}$$

where we defined $\Delta F_\pm^2=F_+^2\pm F_-^2$. The first two equations correspond to the $\nu=0$ components, and the last two to the $\nu=i$ components. From the scalar sector:

$$\ddot{F}_\pm+\frac{\dot{F}_\pm}{r}+\frac{F_\pm}{r^2}\left[(\alpha\pm\beta)^2-(A\pm a)^2\right]=\frac{m_S^2}{2}\left(F_\pm^2-1\right)F_\pm. \tag{4.55}$$

These are the differential equations that we need to solve considering the boundary conditions given in Eq. (4.47) and the initial conditions stated in sequence. We were not able to find an analytical solution for these equations, and therefore, in the next section we will present a few numerical solutions considering some particular cases that represent different possible scenarios.

In the above differential equations, one can note the appearance of a few mass scales, given by $m_S = \sqrt{\lambda v^2}$, $M_e = 2ev$, $M_g = 2gv$, and finally, μ. We can introduce the dimensionless coefficients $K_1 = \mu/m_S$, $K_2 = M_e/M_g = e/g$, and $K_3 = M_e/m_S$, writing the equations above using the dimensionless distance $x = m_S\, r$ (the derivatives from now on are with respect to x), in such a way that the differential equations can be written:

$$
\begin{aligned}
&\ddot{F}_{+}+\frac{\dot{F}_{+}}{x}+\frac{F_{+}}{x^2}\left[(\alpha+\beta)^2-(A+a)^2\right]=\frac{1}{2}\left(F_{+}^2-1\right)F_{+},\\
&\ddot{F}_{-}+\frac{\dot{F}_{-}}{x}+\frac{F_{-}}{x^2}\left[(\alpha-\beta)^2-(A-a)^2\right]=\frac{1}{2}\left(F_{-}^2-1\right)F_{-},\\
&\ddot{A}-\frac{\dot{A}}{x}+K_1K_2\left(\dot{\beta}-\frac{\beta}{x}\right)=\frac{K_3^2}{2}\left[A\,\Delta F_{+}^2+a\,\Delta F_{-}^2\right],\\
&\ddot{a}-\frac{\dot{a}}{x}+\frac{K_1}{K_2}\left(\dot{\alpha}-\frac{\alpha}{x}\right)=\frac{K_3^2}{2K_2^2}\left[a\,\Delta F_{+}^2+A\,\Delta F_{-}^2\right],\\
&\ddot{\alpha}-\frac{\dot{\alpha}}{x}+\frac{\alpha}{x^2}+K_1K_2\dot{a}=\frac{K_3^2}{2}\left[\alpha\,\Delta F_{+}^2+\beta\,\Delta F_{-}^2\right],\\
&\ddot{\beta}-\frac{\dot{\beta}}{x}+\frac{\beta}{x^2}+\frac{K_1}{K_2}\dot{A}=\frac{K_3^2}{2K_2^2}\left[\beta\,\Delta F_{+}^2+\alpha\,\Delta F_{-}^2\right].
\end{aligned}
\tag{4.56}
$$

Before diving headfirst in the numerical solutions for these differential equations, we can briefly analyze the asymptotic behavior of the vortex configurations. In fact, considering the asymptotic behaviors for the profiles $F_\pm \to 1$ and $A, a, \alpha, \beta \to 0$, we can write $F_\pm = 1 - \tilde{F}_\pm$, $A = 0 + \tilde{A}$, $a = 0 + \tilde{a}$, $\alpha = 0 + \tilde{\alpha}$ and $\beta = 0 + \tilde{\beta}$, where all the quantities with tilde are very small for large x. In this regime, we will consider only first order terms in the quantities with tilde, neglecting higher orders.

In this approximation, the first two equations in Eq. (4.56) become $\ddot{\tilde{F}} + \dot{\tilde{F}}/x - \tilde{F} = 0$, where we already used the expansion described above and neglected higher order terms. Notice that this is a modified Bessel equation, therefore we can write for the asymptotic behavior of the scalar profiles, $F(r) \approx 1 - C\mathcal{K}_0(m_S\, r)$, and conclude that the scalar fields will approach their asymptotic value exponentially with a characteristic decay length given by the scalar mass. In the same way, we can consider the third and last equations in Eq. (4.56). Using the same approximation discussed above, we obtain the following equations: $\ddot{\tilde{A}} - \frac{\dot{\tilde{A}}}{x} + K_1K_2\left(\dot{\tilde{\beta}} - \frac{\tilde{\beta}}{x}\right) = K_3^2\tilde{A}$ and $\ddot{\tilde{\beta}} - \frac{\dot{\tilde{\beta}}}{x} + \frac{\tilde{\beta}}{x^2} + \frac{K_1}{K_2}\dot{\tilde{A}} = \frac{K_3^2}{K_2^2}\tilde{\beta}$. These differential equations lead to the following asymptotic behavior in terms of the modified Bessel functions of the second kind:

$$
\begin{aligned}
A(r) &\approx C_\pm \, r \, \mathcal{K}_1(m_\pm r), \\
\beta(r) &\approx D_\pm \, \mathcal{K}_0(m_\pm \, r).
\end{aligned}
\tag{4.57}
$$

Therefore, the gauge profiles approach their asymptotic value exponentially, with a decay length given by the gauge field masses $m_\pm$, given in Eq. (4.42). The question of whether both m_+ and m_- are equally valid is a subtle one (see Refs. [33, 120, 121]), and should be investigated elsewhere. The same analysis can be done with the remaining equations and naturally gives us similar results.

4.4 Explicit numerical vortex solutions

In this section we will exhibit explicit numerical solutions for the differential equations presented in the last section. The general strategy adopted here is as follows. We propose to expand the profile functions $F_+, F_-, A, a, \alpha, \beta$ in powers of x around the origin, for example, $A(x) = \sum_k A_k x^k$. Plugging these expansions in the above differential equations and using the initial conditions, we can obtain constraints in the expansion coefficients. With these expansions near the origin at hand, we can proceed to search the numerical solutions that will also satisfy the boundary conditions at infinity using a shooting method. It is important to note that, since we have $A(0) = m$, $a(0) = n$, we need first of all to specify which (m, n)-vortex we are trying to find.

In general lines, for the equations and initial conditions considered here, there are six coefficients to be adjusted; the others vanish or can be found in terms of these six and of the mass quotients K_i. Roughly speaking, near the origin we obtained the following structure of expansions:

$$
\begin{aligned}
F_+(x) &= f_+ \, x^{|n+m|} + ..., \\
F_-(x) &= f_- \, x^{|n-m|} + ..., \\
A(x) &= m + A_2 x^2 + A_+ x^{2|n+m|+2} + A_- x^{2|n-m|+2} + ..., \\
a(x) &= n + a_2 x^2 + a_+ x^{2|n+m|+2} + a_- x^{2|n-m|+2} + ..., \\
\alpha(x) &= \alpha_1 x + \alpha_+ x^{2|n+m|+1} + \alpha_- x^{2|n-m|+1} + ..., \\
\beta(x) &= \beta_1 x + \beta_+ x^{2|n+m|+1} + \beta_- x^{2|n-m|+1} + ...,
\end{aligned}
\tag{4.58}
$$

where $f_+, f_-, A_2, a_2, \alpha_1, \beta_1$ are free parameters that are determined for each set of (m, n, K_1, K_2, K_3), in order to satisfy the asymptotic conditions at infinity.

In the following, we consider some examples representing distinctive classes of vortices.

For each case, we show explicit numerical solutions and analyze some aspects of them, stating the relevant parameters for the solution. In Sec. 4.4.1, we will analyze the situation where one of the integers is zero, using the case ($n = 1, m = 0$) as an example; In Sec. 4.4.2, we investigate the situation where m and n are equal and non-zero, adopting the case ($n = m = 1$) as illustration, and briefly commenting on ($m = n = 1/2$); In Sec. 4.4.3, we study the case where n and m are non-zero and different, using the case ($n = 1, m = 2$) as an example, and commenting on the case ($n = 3/2, m = 1/2$); Finally, in Sec. 4.4.4, we analyze solutions obtained with different coefficients K_i.

4.4.1 n=1, m=0

Let us focus first on the solutions with $n = 1$ and $m = 0$, since this is the simplest possible scenario. In this case, we obtain $\Phi = 0$, implying $G = 0$ and $J = 0$, but $\chi = 2\pi/g$, giving $Q = 2\pi\mu/g$. Thus, we would be dealing with configurations without magnetic flux, g-electric charge and angular momentum, but with non-trivial g-magnetic flux and electric charge.

Following the procedure described in the beginning of this section, we found a numerical solution for the full set of differential equations that has the property of giving equal profiles $F_+ = F_-$ and identically zero solutions for $A = \beta = 0$. This means that, for this simple ($n = 1, m = 0$) case, we found *a posteriori* that only half of the differential equations are non-trivial, and therefore in the numerical analysis we only considered these ones to simplify the analysis. The non-trivial profiles for the vortex solution are exhibited in Fig. 4.1.

Given this explicit solution, we can immediately plot the g-magnetic and electric fields related with this vortex solution, as one can see in Fig. 4.1. Notice that the g-magnetic field is finite, non-vanishing, and acquires its maximum value at the origin. The electric field is zero at the origin, maximum at a finite distance and vanishes asymptotically. This is exactly the situation reported in Ref. [24], where the authors considered an AH model in the presence of a CS term, and obtained a charged vortex solution. This is not a coincidence, because, although physically different, mathematically speaking we are in a similar situation, since we have exactly the same differential equations to be solved. But it should be stressed that, besides the parity-invariance of the model and different field content (for instance, we have two gauge fields instead of only one), our vortex solution has zero angular momentum, instead of a non-zero and fractional value as reported in Ref. [24]. The charge and g-current densities display a similar behavior, vanishing at the origin, attaining their maximum value at a finite distance and decaying asymptotically to zero. We remark that an equivalent situation occurs when we consider the case $n = 0, m = 1$.

We were not able to find numerical solutions for $m = 0$ or $n = 0$ with $F_+ \neq F_-$ and

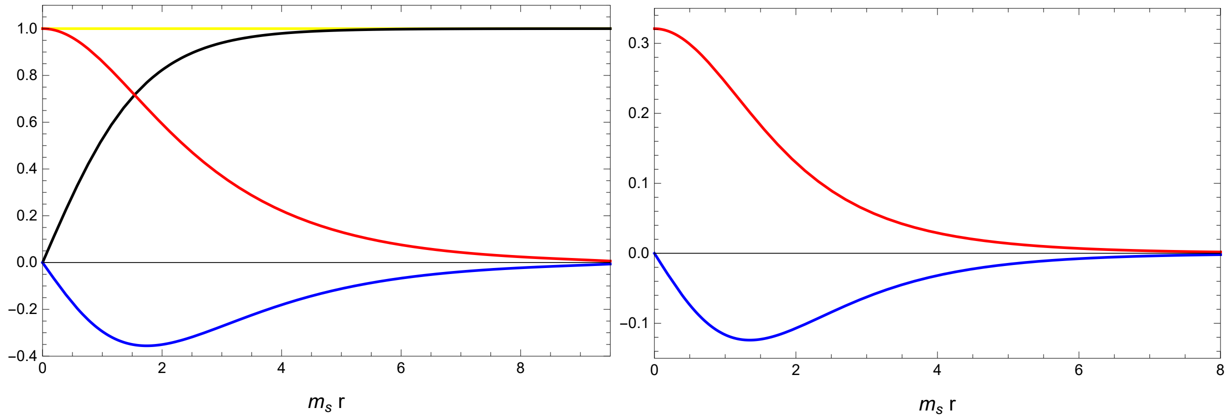

Figure 4.1: Topological vortex solution for $n = 1, m = 0$ and its physical fields in units as functions of $x = m_s r$. *Left figure*:Vortex solution for $n = 1, m = 0$. The scalar profile F is shown in black, and the gauge profiles a and α in red and blue, respectively, as functions of $x = m_S r$. The other profiles are identically zero. The relevant parameters here are: $F_1 = 0.58939309$, $a_2 = -0.16046967$, $\alpha_1 = -0.36281397$. *Right figure*: The g-magnetic (in red) and electric (in blue) fields as functions of $x = m_S r$ for the $n = 1, m = 0$ solution, in units of g/m_S^2 and e/m_S^2, respectively.

$A \neq 0, \beta \neq 0$. It seems that, at least in this simple scenario with vanishing m or n, there is a natural trivialization of a sector. One might wonder if this trivialization is somehow a consequence of taking the K_i parameters all equal to 1, since they represent quotients between mass scales appearing in our physical system, but it does not seems to be so. In fact, in Sec. 4.4.4, we will consider a few numerical solutions for different values of K_i, and in all cases we obtained similar scalar and gauge profiles, exhibiting the trivialization property reported above.

4.4.2 n=m=1

Now, let us search for solutions with $n = m = 1$. In this case, looking to Eq. (4.45) we immediately see that $Q = \frac{2\pi\mu}{g}$ and $G = \frac{2\pi\mu}{e}$. This vortex has a non-trivial angular momentum given by $J = \frac{2\pi\mu}{eg}$, differently from the previous solution. We report this vortex in Fig. 4.2.

Notice that we obtained *a posteriori* a simplified solution where $A = a$, $\alpha = \beta$, and $F_- = 1$. For the scalar profiles, it is important to remember that the exponential part of the scalar fields $\phi_\pm$ involves $m \pm n$. Therefore, the fact that F_- gives us a constant and F_+ displays a typical 2-vortex behavior ($\sim r^2$ near the origin) is an indication that the true winding numbers are given by $m + n$ and $m - n$, instead of m and n separately.

One can wonder again whether the trivial behavior of the gauge profiles is due to the choice of coefficients. Unlike the previous case, the answer is affirmative, at least with respect to the variation of K_2 governing the relationship between different gauge couplings. In fact, starting from the degenerate case and varying K_2, the solutions for

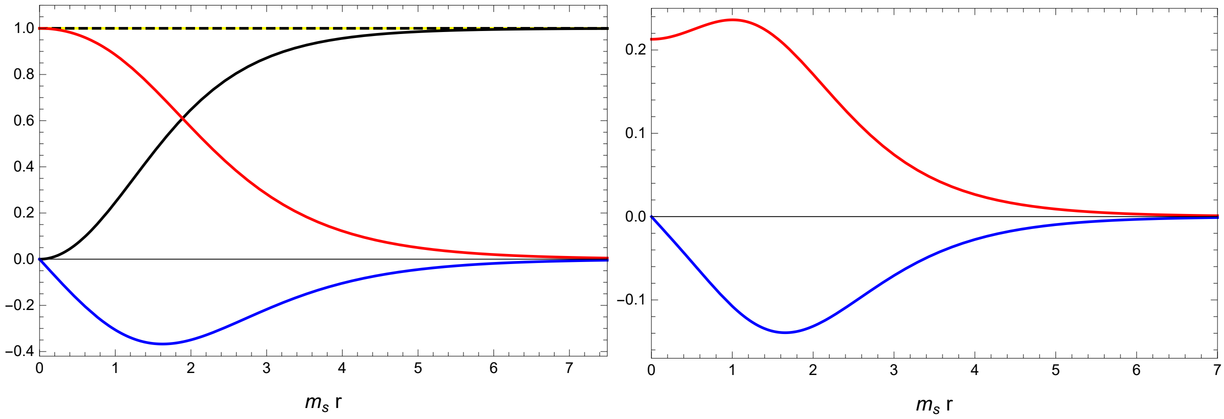

Figure 4.2: Topological vortex solution for $n = m = 1$ and its physical fields in units as functions of $x = m_s r$. *Left figure*: Vortex solution for $m = n = 1$. The scalar profile F_+ is shown in solid black, F_- in dashed black, and the gauge profiles a and α in red and blue respectively, as functions of $x = m_S\, r$. Notice that here we have $A = a$ and $\alpha = \beta$. The relevant parameters here are $F_{+2} = 0.28684863$, $F_{-0} = 1$, $A_2 = a_2 = -0.10644717$, $\alpha_1 = \beta_1 = -0.36047370$. *Right figure*: The magnetic (in red) and electric (in blue) fields as functions of $x = m_S\, r$ for the $n = m = 1$ solution, in units of e/m_S^2. Notice that here we have $B = b$ and $E_r = e_r$.

profiles A and a as well as α and β are not degenerate anymore; however, the scalar profiles do not present any appreciable qualitative change. Varying K_1 and K_3, we will find a behavior similar to the ones described in the last case, as depicted in Sec. 4.4.4.

Given the solution, we can plot its electric and magnetic fields in Fig. 4.2. The case $n = m = 1/2$ does not present any appreciable qualitative change in comparison with the solution presented here, except by the scalar profile near the origin, that displays a typical 1-vortex behavior, and by its lowest value of energy and angular momentum ($J = \pi\mu/2eg$). The energy hierarchy of our solutions will be shortly discussed in the next subsection.

4.4.3 n=1, m=2

Finally, we will consider the case $n = 1$ and $m = 2$. Here, we readily obtain $Q = \frac{2\pi\mu}{g}$ and $G = \frac{4\pi\mu}{e}$. Notice that we also have a non-vanishing angular momentum given by $J = \frac{4\pi\mu}{eg}$. In this case, we expect to see a totally novel result, since there are no simplifications in consequence of the choice of m and n.

The numerical solution obtained in this case is given in Fig. 4.3. As one can see, this time there is no degeneracy in the profiles, being all of them non-trivial. In the scalar profiles, notice that F_- displays a behavior near the origin characteristic of a 1-vortex, and F_+ of a 3-vortex.

The magnetic and electric fields (as well as the g-magnetic and g-electric) are shown in Fig. 4.3. For the first time, we observe an oscillating behavior in the electric and g-magnetic fields, and in particular, we see that there is a finite distance where they vanish. Since it is not clear which of the gauge fields (or which combination of them) describes observable electromagnetic phenomena, one should be careful before drawing any conclusion.

The case $n = 3/2, m = 1/2$ does not present any appreciable qualitative change in comparison with the solution presented here, except by the scalar profiles near the origin, since F_+ and F_- display a behavior typical of 2-vortex and 1-vortex solutions, respectively.

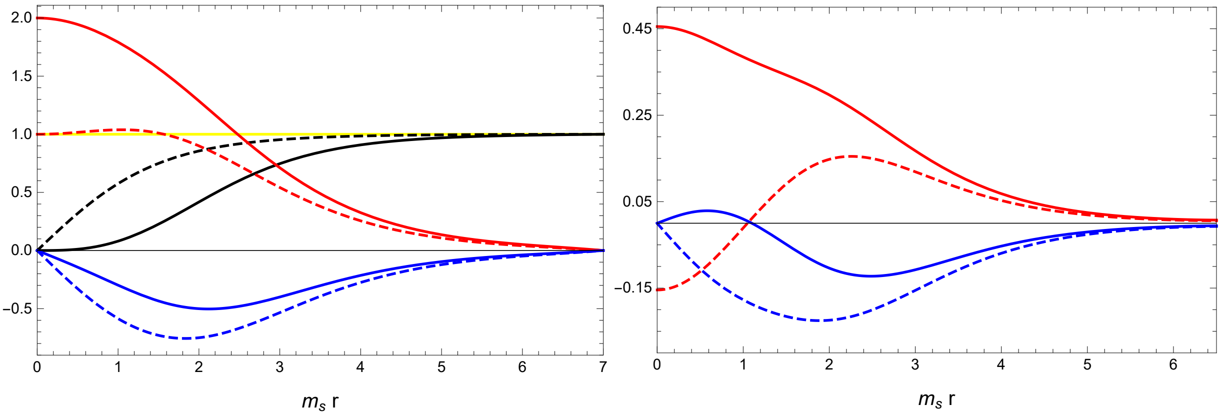

Figure 4.3: Topological vortex solution for $n = 1, m = 2$ and its physical fields in units as functions of $x = m_s r$. *Left figure*: Vortex solution for $n = 1, m = 2$. The scalar profile F_+ is shown in solid black, F_- in dashed black; the gauge profile A is shown in solid red, a in dashed red; the profile α is shown in solid blue, β in dashed blue; all of them are given as functions of $x = m_S r$. The relevant parameters here are $F_{+3} = 0.07723697$, $F_{-1} = 0.66377069$, $a_2 = 0.07718614$, $A_2 = -0.22754617$, $\alpha_1 = -0.27824800$, $\beta_1 = -0.68551826$. *Right figure*: The magnetic (solid red) and electric (solid blue) fields in units of e/m_S^2; the g-magnetic (dashed red) and g-electric (dashed blue) fields in units of g/m_S^2. All of them as functions of $x = m_S r$ for the $n = 1, m = 2$ solution.

At this point, equipped with all these vortex solutions, we can discuss their energy densities and highlight the mass hierarchy between them. Let us first call attention to the fact that we have been successful in finding finite-energy configurations, as one can immediately see in Fig. 4.4. From these energy densities, defining $M_{(m,n)}$ as the mass associated with the (m,n)-vortex, we obtained the following mass hierarchy in units of v^2 : $M_{(1/2,1/2)} \approx 1.31 < M_{(0,1)} \approx 2.27 < M_{(1,1)} \approx 2.92 < M_{(1/2,3/2)} \approx 3.87 < M_{(2,1)} \approx 5.70$. Interestingly enough, one can observe that $M_{(1/2,1/2)} + M_{(-1/2,1/2)} = 2M_{(1/2,1/2)} > M_{(0,1)}$. Remember that in the $(\pm 1/2, 1/2)$-vortex, $F_\pm$ is 1-vortex scalar profile, while $F_\mp$ lies in the vacuum, whereas in the $(0, 1)$-vortex both of them are typical 1-vortex scalar profiles. This suggests that there might be an attraction between these vortices. However, to truly understand the interactions between these vortices and conclusively assert this, a more thorough analysis should be done elsewhere, along the lines presented in Ref. [122], for example.

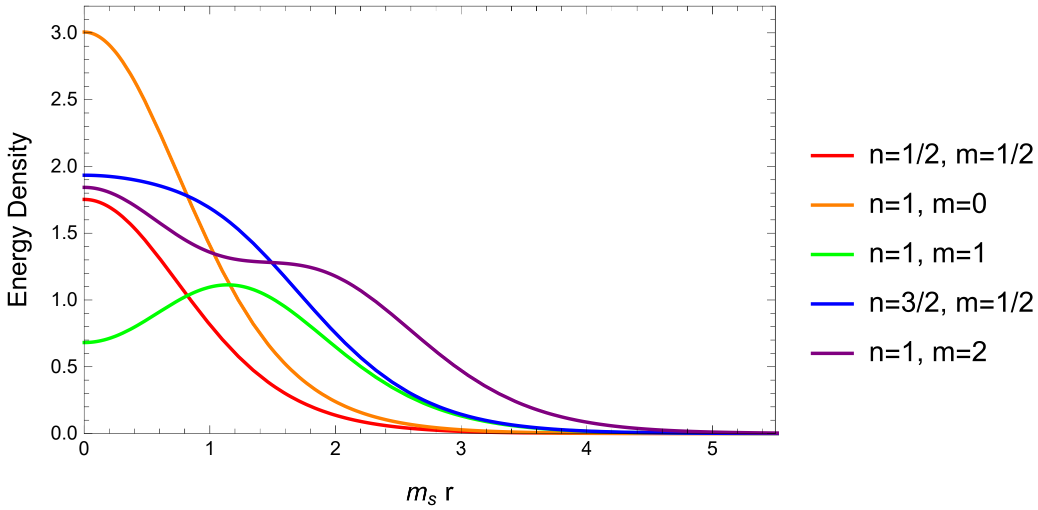

Figure 4.4: The energy density for the (m,n)-vortex solutions in units of $1/v^2\, m_S^2$. In red, (1/2,1/2); in orange, (1,0); in green, (1,1); in blue, (3/2,1/2); in purple, (1,2).

4.4.4 Vortex solutions for different K_i's

In this section, we investigate the existence of vortex solutions and their main properties upon varying the coefficients K_i. In the following, we will use as a reference the case $K_1 = K_2 = K_3 = 1$, already studied in the last sections, and change each K_i by a factor of two keeping the others fixed, to find different vortex solutions and compare their main features.

Focusing first in the case $n = 1, m = 0$, the variation of K_i led to qualitatively similar scalar and gauge profiles, and the trivialization property already highlighted before. As one can see from Fig. 4.5, the electric field qualitative behavior is the same for all the values considered: zero at the origin, attaining a finite non-zero maximum value at some distance and decaying to zero at large distances. Notice that by varying K_1, there are only small changes in the profile. By lowering K_2, we can observe a more pronounced decay and an improvement in its maximum value. On the other hand, by increasing K_3 we observe a sensible increase at the absolute value of the maximum electric field value, accompanied by a more pronounced decay and a small shift in the position where this maximum occur. For the g-magnetic field, the qualitative behavior is also the same as we vary K_i: attains a finite non-zero maximum value at the origin and decays monotonically as we increase the distance going to zero in the asymptotic limit. By increasing K_1, we see that the maximum value of the g-magnetic field diminishes, and this is compatible with the behavior observed in Ref. [34]. Lowering K_2 or increasing K_3, we observe a strong change in the maximum value of the g-magnetic field as well as a more pronounced decay as we go far from the origin. Lowering K_1, increasing K_2, or lowering K_3, as before, has the opposite effect, cf. Fig. 4.6.

Proceeding to the $n = m = 1$ solution, as already highlighted in the main text, the degeneracy that we have found is due to the equality of the couplings when $K_2 = 1$.

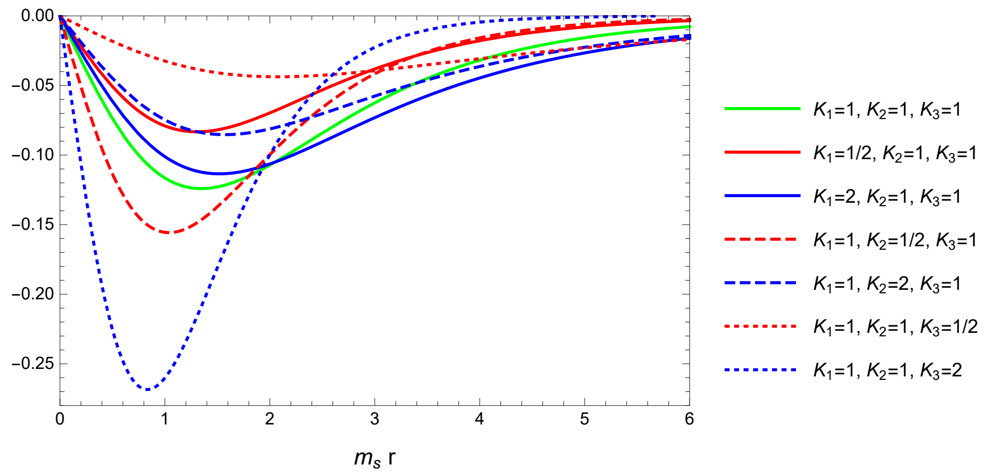

Figure 4.5: The electric fields associated with $n = 1, m = 0$ solution in units of e/m_S^2 for different values of (K_1, K_2, K_3). In solid green, $(1, 1, 1)$; in solid red, $(1/2, 1, 1)$; in solid blue, $(2, 1, 1)$; in dashed red, $(1, 1/2, 1)$; in dashed blue, $(1, 2, 1)$; in dotted red, $(1, 1, 1/2)$; in dotted blue, $(1, 1, 2)$.

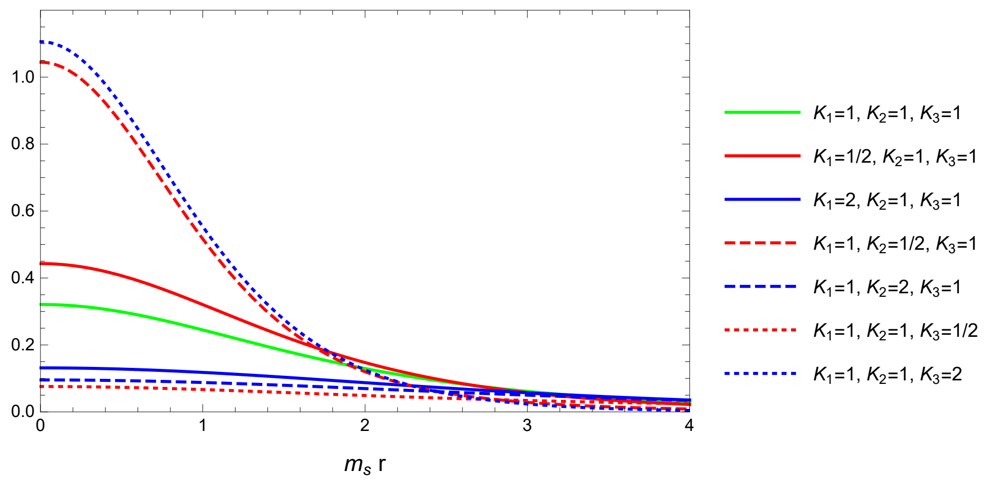

Figure 4.6: The g-magnetic fields associated with $n = 1, m = 0$ solution in units of g/m_S^2 for different values of (K_1, K_2, K_3). In solid green, $(1, 1, 1)$; in solid red, $(1/2, 1, 1)$; in solid blue, $(2, 1, 1)$; in dashed red, $(1, 1/2, 1)$; in dashed blue, $(1, 2, 1)$; in dotted red, $(1, 1, 1/2)$; in dotted blue, $(1, 1, 2)$.

When we depart from this simpler case, we find vortex solutions with $A \neq a$ and $\alpha \neq \beta$, naturally leading to different magnetic and g-magnetic (as well as electric and g-electric) fields, as one can see in Fig. 4.7. Upon varying K_1 and K_3, we observed the same behavior as described in the previous case.

Finally, we remark that in the case $n = 1, m = 2$ the variation of the coefficients K_i did not lead to any substantial difference from the cases already discussed here.

For completeness, it would be interesting to analyze what happens in some limiting cases of this model, for instance, when the CS terms or the Maxwell terms are absent. This analysis is done in the next section.

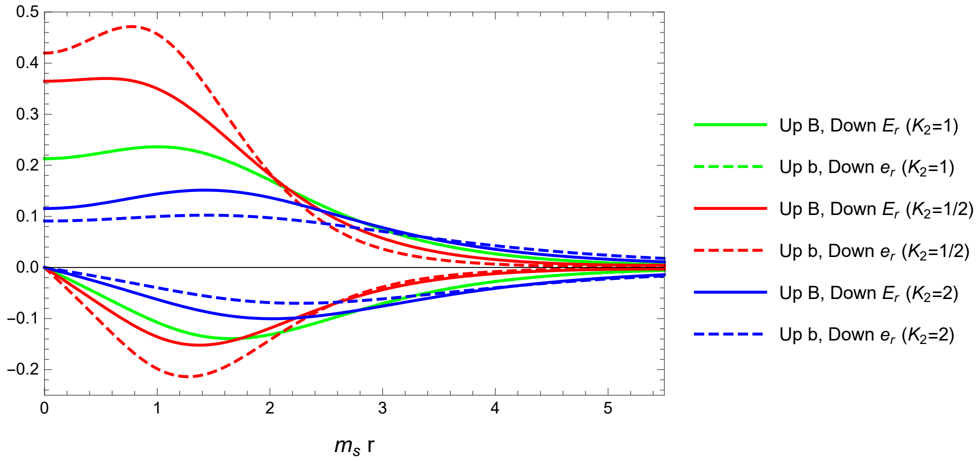

Figure 4.7: The magnetic (B), g-magnetic (b), electric (E_r) and g-electric (e_r) fields associated with the $m = n = 1$ solutions, for different values of K_2. The solid lines refer to B and E_r; the dashed lines refer to b and e_r. B and b are shown in the upper part; E_r and e_r are shown in the lower part. In green, $K_2 = 1$; in red, $K_2 = 1/2$; in blue, $K_2 = 2$.

4.5 Vortices in limiting cases

In this section, we study two particular limits of our model. First, we will briefly address the simpler case in which we do not have a CS term, that is, $\mu = 0$. From a practical point of view, this can be achieved by setting $K_1 = 0$, and the conclusions in this part will come straightforwardly. Notice that this scenario bears resemblance to the usual ANO vortex, since this is nothing but a scalar QED with two gauge fields and two scalars with different charges.

Second, we will analyze our model in the absence of Maxwell terms, with the gauge kinetic part given solely by the CS term. This allows us to solve the Gauss laws and write the time components of the gauge fields as functions of other quantities. This scenario, where the CS term dominates and the Maxwell terms can be neglected, could be seen as the low-energy regime of our model.

We remark that the results obtained in this section could be inferred by looking at the behavior of magnetic and electric fields when we changed the coefficient K_1 while keeping the others coefficients fixed, since this increases (or decreases) the importance of CS parameter with respect to the other scales of the system. Although it can give us a hint of what would happen in the limits considered here, it is important to remark that the passage from the model considered to the pure CS limit is a subtle one, as one can see for instance in Ref. [34], which justifies a separate investigation of the latter.

Now, we briefly state the results for $K_1 = \mu/m_S = 0$. We will consider the case $m = 0$ and $n = 1$ with $K_2 = K_3 = 1$ for definiteness, but we would have similar results in the other examples. The vortex solution *per se* does not exhibit any appreciable change in the profiles F and a as one can see in Fig. 4.8. But now we have $\alpha = 0$, and this fact is the most striking difference that appears in this regime. Since we do not have the CS

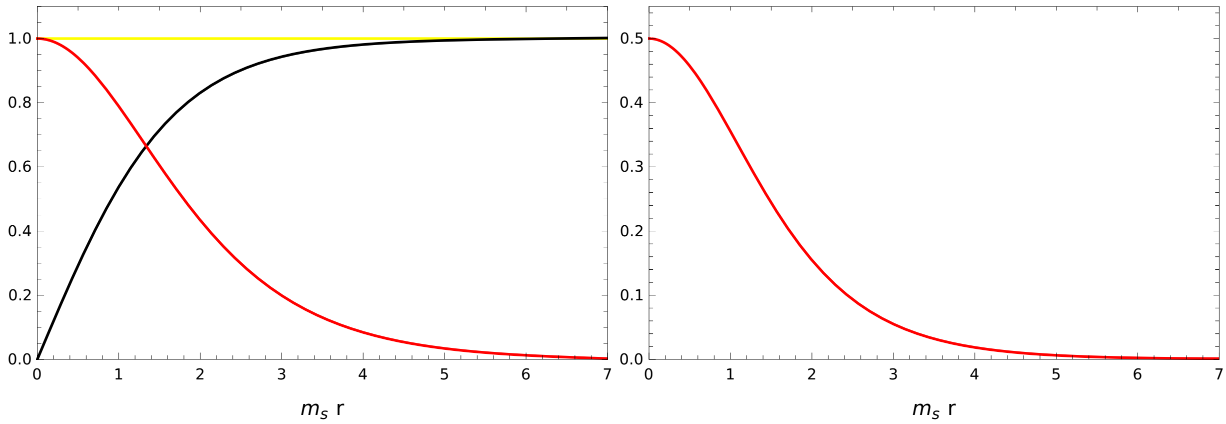

Figure 4.8: Vortex solution for $n = 1, m = 0$ in the pure Maxwell limit. and its physical fields in units as functions of $x = m_s r$. *Left figure*: The scalar profile F is shown in black and the gauge profile a in red, respectively, as functions of $x = m_S\, r$. The other profiles are identically zero. *Right figure*: The g-magnetic field in the pure Maxwell limit, in units of g/m_S^2, as a function of $x = m_S\, r$. The magnetic field as well as the electric and g-electric fields are zero here.

Gauss law constraint anymore, the electric field vanishes and we conclude that the vortex is neutral, as expected. The g-magnetic field in this regime is stronger in magnitude, but exhibit the usual profile, attaining a maximum at the origin and decaying as we increase x, as one can see in Fig. 4.8. This is in accordance with the already known results (see for example Ref. [34]).

Proceeding to the more interesting scenario in which we can neglect the Maxwell terms, the Gauss laws constraints become much simpler,

$$\begin{aligned} \mu b &= e\,(\rho_+ + \rho_-)\,, \\ \mu B &= g\,(\rho_+ - \rho_-)\,. \end{aligned} \tag{4.59}$$

Without Maxwell terms, we are able to obtain A_0 and a_0 directly from the other fields. In fact, we can find:

$$\begin{aligned} eA_0 &= \Lambda\big[eB\big(|\phi_+|^2 - |\phi_-|^2\big) - gb\big(|\phi_+|^2 + |\phi_-|^2\big)\big] \\ ga_0 &= \Lambda\big[gb\big(|\phi_+|^2 - |\phi_-|^2\big) - eB\big(|\phi_+|^2 + |\phi_-|^2\big)\big]\,, \end{aligned} \tag{4.60}$$

where we defined $\Lambda \equiv \mu/8eg|\phi_+|^2|\phi_-|^2$ for convenience. Plugging the *ansatz*, and writing in dimensionless variables using $x = m_S\, r$ and the coefficients K_i as before, we obtain the

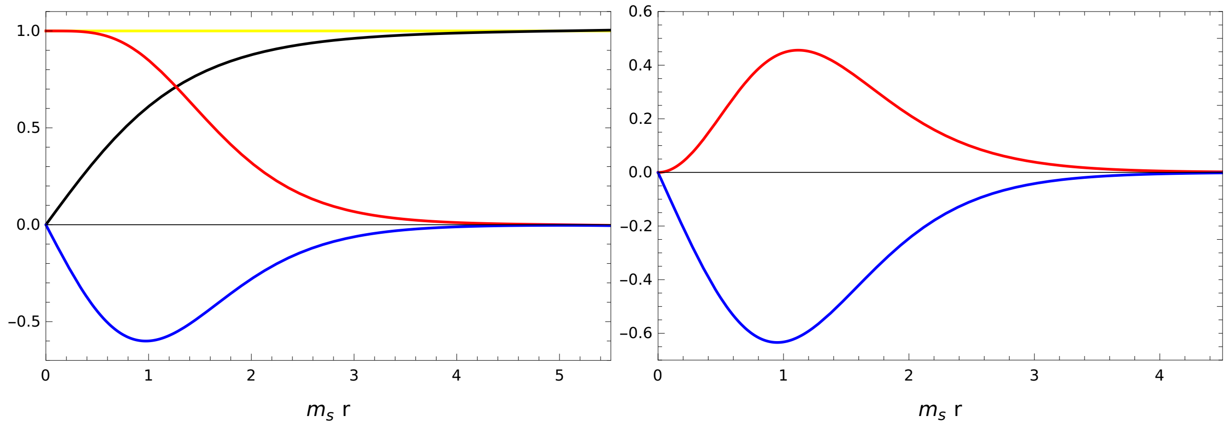

Figure 4.9: Vortex solution for $n = 1, m = 0$ in the pure CS limit and its physical fields in units as functions of $x = m_s r$. *Left figure*:Vortex solution for $n = 1, m = 0$ in the pure CS limit. The scalar profile F is shown in black; the gauge profiles a and α are shown in red and blue, respectively, as functions of $x = m_S r$. The other profiles are identically zero. *Right figure*: The g-magnetic (in red) and electric (in blue) fields as functions of $x = m_S r$ for the $n = 1, m = 0$ solution in the pure CS limit, in units of g/m_S^2 and e/m_S^2, respectively.

following expressions for α and β:

$$\begin{aligned}\alpha &= \frac{K_1 K_2}{2K_3^2}\frac{1}{F_+^2 F_-^2}\left[\dot{a}\left(F_+^2 + F_-^2\right) - \dot{A}\left(F_+^2 - F_-^2\right)\right],\\ \beta &= \frac{K_1 K_2}{2K_3^2}\frac{1}{F_+^2 F_-^2}\left[\dot{A}\left(F_+^2 + F_-^2\right) - \dot{a}\left(F_+^2 - F_-^2\right)\right].\end{aligned} \tag{4.61}$$

Now, we need only to plug these analytic expressions for α and β in the differential equations (4.56), ignoring the contributions coming from the Maxwell terms, and solve them for given m and n. Notice that we need only to care about the first four equations, since the last two are already satisfied when we write α and β as above.

Although this is a legitimate path to be followed, we simply solved the full set of differential equations in the absence of Maxwell contributions, without using explicitly the CS constraint, stated here only for completeness. In the following, we will exhibit the solution profiles and also the electric and magnetic (as well as g-electric and g-magnetic) fields associated with them. For all of them, we considered $K_1 = K_2 = K_3 = 1$ for simplicity.

The solution for the equations of motion in the pure CS regime for the case $n = 1, m = 0$ is given in Fig. 4.9; the electric and g-magnetic fields are shown in Fig. 4.9. Notice that they are zero at the origin, attains their maximum value at a finite distance and decays asymptotically, exactly as reported in Ref. [37], for example.

The $m = n = 1$ case gives very similar results, see Fig. 4.10. Remember that we are considering here the particular case in which $K_2 = 1$ and therefore we have degenerate

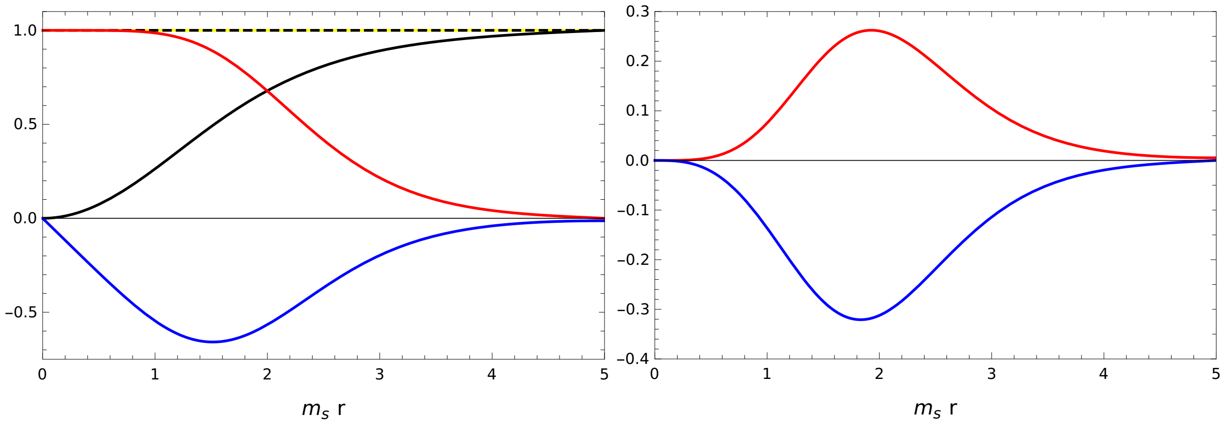

Figure 4.10: Vortex solution for $n = 1, m = 1$ in the pure CS limit and its physical fields in units as functions of $x = m_s r$. *Left figure*:The scalar profile F_+ is shown in solid black and F_- in dashed black; the gauge profiles A and α are shown in red and blue, respectively, as functions of $x = m_S\, r$. Notice that here we have $A = a$ and $\alpha = \beta$. *Right figure*: The magnetic (in red) and electric (in blue) fields as functions of $x = m_S\, r$ for the $n = m = 1$ solution in the pure CS limit, in units of e/m_S^2. Notice that here we have $B = b$ and $E_r = e_r$.

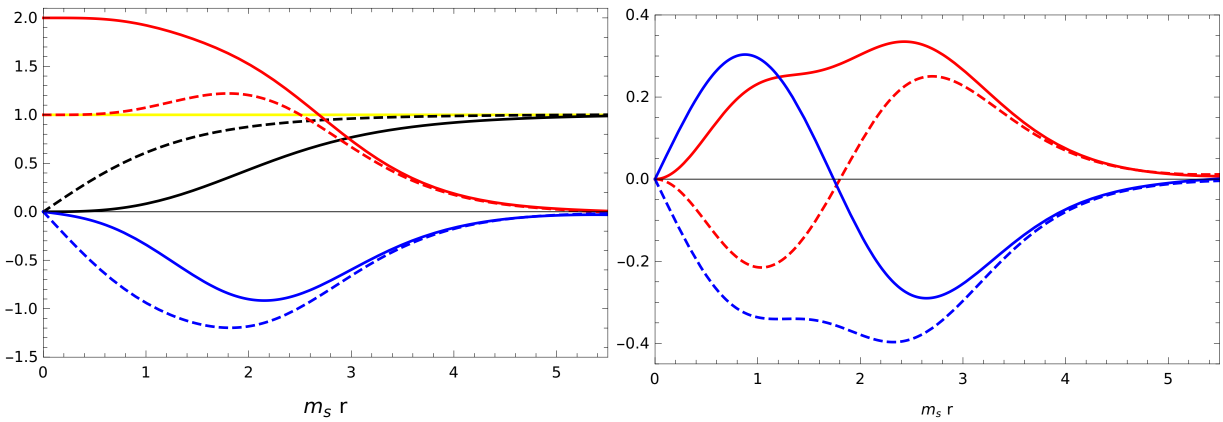

Figure 4.11: Vortex solution for $n = 1, m = 2$ in the pure CS limit and its physical fields in units as functions of $x = m_s r$. *Left figure*: The scalar profile F_+ is shown in solid black, F_- in dashed black; the gauge profile A is shown in solid red, a in dashed red; the profile α is shown in solid blue, β in dashed blue; all of them are given as functions of $x = m_S\, r$. *Right figure*: The magnetic (solid red) and electric (solid blue) fields in units of e/m_S^2; the g-magnetic (dashed red) and g-electric (dashed blue) fields in units of g/m_S^2. All of them as functions of $x = m_S\, r$ for the $n = 1, m = 2$ solution in the pure CS limit.

solutions, as we already discussed before.

The case $n = 1, m = 2$ presents a more complicated behavior, but it is reminiscent of the solution presented in the main text, as expected. In fact, the solutions are shown in Fig. 4.11 and the electric and magnetic (as well as g-electric and g-magnetic) fields are shown in Fig. 4.11. In particular, we still have non-trivial solutions for all profiles and an oscillating behavior for the fields.

Part III

Self-Dual Maxwell-Chern-Simons vortices in a parity-invariant scenario

Chapter 5

Introduction

The vortex configurations that we considered in Part II consisted of static solutions of the second order equations of motion, and their boundary conditions were derived from the requirement of having finite energy. There exists, however, an alternative approach. In a particular regime of a theory, called the self-dual Bogomol'nyi point [35], one can obtain an inequality between the energy of an arbitrary configuration and its magnetic flux. The idea then is to find those solutions whose energy is directly proportional to their magnetic flux, that is, those who exactly saturate Bogomol'nyi bound. In this way, the finiteness of energy follows immediately from finiteness of flux. In order for this energy-flux relation to hold, these solutions need to satisfy a set of first order differential equations known as self-duality equations, and the solutions are generically called self-dual solitons.

As already pointed out in the introduction, self-dual vortices (and also non-topological solitons) were found in pure Maxwell case [12], in the pure CS limit [36, 37, 38] and in the Maxwell-CS model [39, 40, 41]. What all these theories have in common is a specific relationship among the coupling constants (the self-dual point), which generally leads to mass degeneracies of their excitations. This is related to supersymmetry, but we shall have more to say about this later in the work. More about self-dual theories in this context can be found, for example, in [55].

While in the pure Maxwell case one needs a particular quartic potential to obtain the self-dual solutions, and in the pure CS case on needs a particular sixth order potential, in the presence of both Maxwell and CS an additional neutral scalar field is required to reach the self-dual point [39]. Due to the symmetries of our model, we will need two of them as we shall see.

Chapter 6

The self-dual model

6.1 Presenting the model

Let us then consider the following $\mathcal{P}$- and $\mathcal{T}$- invariant lagrangian:

$$\begin{aligned}\mathcal{L} = &-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{4}f_{\mu\nu}f^{\mu\nu} + \mu\epsilon^{\mu\nu\rho}A_\mu\partial_\nu a_\rho \\ &+ |D_\mu\phi_+|^2 + |D_\mu\phi_-|^2 + \frac{1}{2}\left(\partial_\mu N\right)^2 + \frac{1}{2}\left(\partial_\mu M\right)^2 \\ &- V\left(|\phi_+|, |\phi_-|, N, M\right),\end{aligned} \tag{6.1}$$

where the definitions are the same as before.

The full set of symmetry transformations of (6.1) is:

- $U(1)_A \times U(1)_a$:

$$\left\{\begin{array}{ll}\phi'_\pm &= e^{i\rho(x)}\phi_\pm, \\ \\ A'_\mu &= A_\mu - \frac{1}{e}\partial_\mu\rho(x)\ , \\ \\ a'_\mu &= a_\mu\ ;\end{array}\right. \qquad \left\{\begin{array}{ll}\phi'_\pm &= e^{\pm i\xi(x)}\phi_\pm, \\ \\ A'_\mu &= A_\mu\ , \\ \\ a'_\mu &= a_\mu - \frac{1}{g}\partial_\mu\xi(x)\ .\end{array}\right. \tag{6.2}$$

- Parity ($\mathcal{P}$) and Time-reversal ($\mathcal{T}$):

$$\left\{\begin{array}{ll}A^P_\mu &= \mathcal{P}_\mu^{\ \nu}A_\nu, \\ a^P_\mu &= -\mathcal{P}_\mu^{\ \nu}a_\nu, \\ \phi^P_\pm &= \zeta\,\phi_\mp,\end{array}\right. \qquad \left\{\begin{array}{ll}A^T_\mu &= -\mathcal{T}_\mu^{\ \nu}A_\nu, \\ a^T_\mu &= \mathcal{T}_\mu^{\ \nu}a_\nu, \\ \phi^T_\pm &= \eta\,\phi_\mp, .\end{array}\right.$$

where $\mathcal{P}_\mu^{\ \nu} = diag(+-+)$, $\mathcal{T}_\mu^{\ \nu} = diag(-++)$ and ζ, η are arbitrary complex phases.

The parity and time-reversal transformations of the neutral fields introduced are:

$$N^{P,T} = N, \qquad M^{P,T} = -M. \tag{6.3}$$

It should be mentioned that the action (4.9) is also invariant under time-reversal, but this symmetry played no particular role in what we have discussed so far. The same will be true in this case, mainly because we are interested in static configurations. However, the breaking of time-reversal symmetry is an active topic of discussion in some condensed matter systems and, as we will see, this is a phenomenon that is also present in our model.

To the purpose of investigating the existence of self-dual solitons, let us propose the following potential:

$$\begin{aligned} V = &(eN + gM)^2|\phi_+|^2 + (eN - gM)^2|\phi_-|^2 \\ &+ \frac{1}{2}\left[e\left(|\phi_+|^2 - |\phi_-|^2\right) - \mu M\right]^2 \\ &+ \frac{1}{2}\left[g\left(|\phi_+|^2 + |\phi_-|^2 - 2v^2\right) - \mu N\right]^2 . \end{aligned} \tag{6.4}$$

This potential is consistent with all the symmetries of the model, and despite not being the most general possibility, it arises naturally from the requirement of a Bogomol'nyi bound for the energy (Appendix E). Setting $|\phi_+| = |\phi_-|$, $M = 0$, $e = g$, and appropriately rescaling the remaining parameters, it exactly reproduces the potential proposed in [39], which is known to contain both pure Maxwell and pure CS self-dual vortices as limiting cases. Instead, if $N = M = 0$ and $e = g$, we can recover a particular instance of the potential used in Part II.

The equations of motion for this model are given by

$$\begin{aligned} \partial_\mu F^{\mu\nu} + \mu\epsilon^{\nu\alpha\beta}\partial_\alpha a_\beta &= e\left(J_+^\nu + J_-^\nu\right), \\ \partial_\mu f^{\mu\nu} + \mu\epsilon^{\nu\alpha\beta}\partial_\alpha A_\beta &= g\left(J_+^\nu - J_-^\nu\right), \\ D_\mu D^\mu \phi_\pm &= -\frac{dV}{d\phi_\pm^*}, \end{aligned} \tag{6.5}$$

supplemented by the equations for N and M

$$\begin{aligned} (\Box + \mu^2)N = &-2e\left[(eN + gM)|\phi_+|^2 + (eN - gM)|\phi_-|^2\right] \\ &+ \mu\left[g(|\phi_+|^2 + |\phi_-|^2 - 2v^2)\right] \\ (\Box + \mu^2)M = &-2g\left[(eN + gM)|\phi_+|^2 - (eN - gM)|\phi_-|^2\right] \\ &+ \mu\left[e(|\phi_+|^2 - |\phi_-|^2)\right] . \end{aligned} \tag{6.6}$$

The currents above are $J_\pm^\nu = i\left[\phi_\pm^* D^\nu\phi_\pm - \phi_\pm D^\nu\phi_\pm^*\right]$. As before, $E^i = F^{i0}$, $e^i = f^{i0}$, $B = \epsilon^{ij}\partial_i A_j$, and $b = \epsilon^{ij}\partial_i a_j$.

The theory (6.1) has the following energy functional:

$$
\begin{aligned}
H = \int d^2x \Big[& \frac{1}{2}\left(\vec{E}^2 + B^2\right) + \frac{1}{2}\left(\vec{e}^2 + b^2\right) + V(\phi_+, \phi_-, M, N) \\
& + |D_0\phi_+|^2 + |D_0\phi_-|^2 + |D_i\phi_+|^2 + |D_i\phi_-|^2 \\
& + \frac{1}{2}(\partial_0 M)^2 + \frac{1}{2}(\partial_0 N)^2 + \frac{1}{2}(\partial_i M)^2 + \frac{1}{2}(\partial_i N)^2 \Big]
\end{aligned}
\tag{6.7}
$$

With the definition (6.4), after some integrations by parts and making use of the equations of motion (Appendix E), (6.7) can be put in a very suggestive form:

$$
\begin{aligned}
H = \int d^2x \Big[& \frac{1}{2}\left(\vec{E} \pm \vec{\nabla} N\right)^2 + \frac{1}{2}\left(\vec{e} \pm \vec{\nabla} M\right)^2 + |D_\pm \phi_+|^2 + |D_\mp \phi_-|^2 + \frac{1}{2}(\partial_0 M)^2 + \frac{1}{2}(\partial_0 N)^2 \\
& + \frac{1}{2}\left\{B \pm \left[e\left(|\phi_+|^2 - |\phi_-|^2\right) - \mu M\right]\right\}^2 \\
& + \frac{1}{2}\left\{b \pm \left[g\left(|\phi_+|^2 + |\phi_-|^2 - 2v^2\right) - \mu N\right]\right\}^2 \\
& + |D_0\phi_+ \mp i(eN + gM)\phi_+|^2 + |D_0\phi_- \mp i(eN - gM)\phi_-|^2 \\
& \pm 2gbv^2 \Big]
\end{aligned}
\tag{6.8}
$$

What (6.8) clearly demonstrates is that the energy of the system satisfies the bound:

$$
H \geq 2v^2|g\chi|; \qquad \chi \equiv \int d^2xb \tag{6.9}
$$

That is, the theory naturally leads to a Bogomol'nyi-type bound to the energy functional, in fact, it was constructed to be so. Our main interest is to investigate the field configurations that saturate these bounds by satisfying the self-dual equations implied by (6.8). As we said at the introduction of this part, vortex solutions have been studied in various scenarios similar to the one we are considering, however none maintaining parity symmetry in the presence of a CS term. Vortices in a pure CS parity-preserving theory [91] have been considered, but in the presence of only fermionic matter. So what we are investigating here is the possibility of self-dual vortex solutions in a parity and time-reversal symmetric Maxwell-Chern-Simons theory in the presence of scalar matter, which is the most typical scenario for finding topological solutions, but which hadn't yet been considered by the literature.

First, we will investigate the spectrum around the possible vacua of the theory.

6.2 Perturbative Spectrum

Let us consider the vacuum configurations of the system, that is, the absolute minima of the energy functional. Looking at the Hamiltonian (6.7), we can see that the energy minimum can be achieved, for instance, with the following field configurations

$$
\begin{aligned}
&\phi_+, \phi_-, M, N = constants, \\
&A_\mu = a_\mu = 0,
\end{aligned}
\tag{6.10}
$$

provided that the constant fields ϕ_+, ϕ_-, M, N also minimize the potential (6.4), which in this case means $V = 0$. Inspection of (6.4) indicates the $V = 0$ if, and only if:

$$(eN + gM)^2|\phi_+|^2 = 0 \tag{6.11}$$

$$(eN - gM)^2|\phi_-|^2 = 0 \tag{6.12}$$

$$e\left(|\phi_+|^2 - |\phi_-|^2\right) - \mu M = 0 \tag{6.13}$$

$$g\left(|\phi_+|^2 + |\phi_-|^2 - 2v^2\right) - \mu N = 0 \tag{6.14}$$

Out of which only four possibilities arise:

1) (0,0) - Vacuum: $|\phi_+|^2 = |\phi_-|^2 = 0 \Rightarrow M = 0; N = -\frac{2gv^2}{\mu}$.

2) (1,1) - Vacuum: $|\phi_+|^2, |\phi_-|^2 \neq 0 \Rightarrow |\phi_+|^2 = |\phi_-|^2 = v^2; M = N = 0$.

3) (1,0) - Vacuum:$|\phi_+|^2 \neq 0; |\phi_-|^2 = 0 \Rightarrow |\phi_+|^2 = v^2; M = \frac{ev^2}{\mu}; N = -\frac{gv^2}{\mu}$.

4) (0,1) - Vacuum: $|\phi_+|^2 = 0; |\phi_-|^2 \neq 0 \Rightarrow |\phi_-|^2 = v^2; M = -\frac{ev^2}{\mu}; N = -\frac{gv^2}{\mu}$.

Expanding around these configurations and considering the quadratic part in the fluctuations, we can read the perturbative spectrum of the theory. It should be noted that the first two vacua will preserve the $\mathcal{P}$ and $\mathcal{T}$, but in principle the last two cases can give us the spontaneous breaking of them. Let us investigate in the following the spectrum around these vacua, considering each case separately.

(0,0)-Vacuum: $\left(\langle|\phi_+|^2\rangle = \langle|\phi_-|^2\rangle = \langle M\rangle = 0, \quad \langle N\rangle = -2gv^2/\mu\right)$

This is the unbroken vacuum. In this case, the charged scalar fields do not have non-trivial VEVs, and therefore the Higgs mechanism does not takes place. We are in the unbroken phase, with both U(1) gauge symmetries intact. This last fact can be made explicit, for example, by noting that under an infinitesimal gauge transformation the scalar fields transform like:

$$
\begin{aligned}
\delta\phi_+ &= i(\rho + \xi)\phi_+ \equiv i\omega_+\phi_+ \\
\delta\phi_- &= i(\rho - \xi)\phi_- \equiv i\omega_-\phi_-
\end{aligned}
$$

Where $\omega_\pm$ denote the general phase change that the fields $\phi_\pm$ undergo. We can ask what infinitesimal gauge transformation annihilates[1] the vacuum, that is:

$$\delta\phi_+^0 = i\omega_+\phi_+^0 = 0$$
$$\delta\phi_-^0 = i\omega_-\phi_-^0 = 0$$

Now, since in this case the vacuum can be parametrized by $\phi_+^0 = \phi_-^0 = 0$, one can see that for any infinitesimal $\omega_\pm$ the above relation is trivially satisfied. This shows how both U(1)'s remain unbroken.

The only field acquiring non-trivial VEV is the neutral scalar field N that we can write as $N = -2gv^2/\mu + \tilde{N}$. All the other fields can be thought as fluctuations around zero. The Lagrangian in this case becomes:

$$\begin{aligned}
\mathcal{L} = &-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{4}f_{\mu\nu}f^{\mu\nu} + \frac{\mu}{2}\epsilon^{\mu\rho\nu}A_\mu\partial_\rho a_\nu + \frac{\mu}{2}\epsilon^{\mu\rho\nu}a_\mu\partial_\rho A_\nu \\
&+ |\partial_\mu\phi_+|^2 - i\left(eA_\mu + ga_\mu\right)\left(\phi_+^*\partial_\mu\phi_+ - \phi_+\partial_\mu\phi_+^*\right) + \left(eA_\mu + ga_\mu\right)^2|\phi_+|^2 \\
&+ |\partial_\mu\phi_-|^2 - i\left(eA_\mu - ga_\mu\right)\left(\phi_-^*\partial_\mu\phi_- - \phi_-\partial_\mu\phi_-^*\right) + \left(eA_\mu - ga_\mu\right)^2|\phi_-|^2 \\
&+ \frac{1}{2}\left(\partial_\mu M\right)^2 + \frac{1}{2}\left(\partial_\mu N\right)^2 - V\left(\phi_+, \phi_-, M, N\right),
\end{aligned} \tag{6.15}$$

where the potential can be written here as

$$\begin{aligned}
V\left(\phi_+, \phi_-, M, N\right) = &\left(\frac{-2egv^2}{\mu} + e\tilde{N} + gM\right)^2|\phi_+|^2 + \left(\frac{-2egv^2}{\mu} + e\tilde{N} - gM\right)^2|\phi_-|^2 \\
&+ \frac{1}{2}\left[e\left(|\phi_+|^2 - |\phi_-|^2\right) - \mu M\right]^2 + \frac{1}{2}\left[g\left(|\phi_+|^2 + |\phi_-|^2\right) - \mu\tilde{N}\right]^2
\end{aligned} \tag{6.16}$$

Considering only the quadratic part of the above Lagrangian, we can write:

$$\begin{aligned}
\mathcal{L}^{\text{quad}} = &-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{4}f_{\mu\nu}f^{\mu\nu} + \frac{\mu}{2}\epsilon^{\mu\rho\nu}A_\mu\partial_\rho a_\nu + \frac{\mu}{2}\epsilon^{\mu\rho\nu}a_\mu\partial_\rho A_\nu \\
&+ |\partial_\mu\phi_+|^2 + |\partial_\mu\phi_-|^2 + \frac{1}{2}\left(\partial_\mu M\right)^2 + \frac{1}{2}\left(\partial_\mu N\right)^2 \\
&- \left(\frac{2egv^2}{\mu}\right)^2\left(|\phi_+|^2 + |\phi_-|^2\right) - \frac{\mu^2}{2}\left(M^2 + \tilde{N}^2\right)
\end{aligned} \tag{6.17}$$

The scalar sector exhibits two massive complex scalar fields with degenerate masses $m_{\phi_+} = m_{\phi_-} = 2egv^2/\mu$ (coming from the interaction with N that acquired a non-trivial VEV), a real scalar and a real pseudoscalar fields with degenerate masses $m_N = m_M = \mu$. In the (0,0)-Vacuum, the gauge symmetry is unbroken. It is important therefore to introduce gauge-fixing terms to have well-defined gauge-field propagators that will allow us to use the usual perturbative reasoning. For this reason, we supplement the above

[1]Borrowing a bit of quantum jargon.

Lagrangian with $\mathcal{L}_{gf} = -\frac{1}{2\alpha}(\partial^\mu A_\mu)^2 - \frac{1}{2\beta}(\partial^\mu a_\mu)^2$.

The gauge sector of the quadratic part (including gauge-fixing terms) can be written:

$$\mathcal{L}^{\text{quad}}_{\text{gauge}} = \frac{1}{2}\begin{pmatrix} A_\mu & a_\mu \end{pmatrix} O^{\mu\nu} \begin{pmatrix} A_\nu \\ a_\nu \end{pmatrix}, \tag{6.18}$$

where we defined the gauge dynamical operator as

$$O^{\mu\nu} = \begin{pmatrix} A^{\mu\nu} & B^{\mu\nu} \\ C^{\mu\nu} & D^{\mu\nu} \end{pmatrix}, \tag{6.19}$$

with

$$\begin{aligned} A^{\mu\nu} &= \Box\Theta^{\mu\nu} + \frac{\Box}{\alpha}\Omega^{\mu\nu} \\ B^{\mu\nu} &= C^{\mu\nu} = \mu S^{\mu\nu} \\ D^{\mu\nu} &= \Box\Theta^{\mu\nu} + \frac{\Box}{\beta}\Omega^{\mu\nu} \end{aligned} \tag{6.20}$$

The operators here are defined as usual

$$\Omega^{\mu\nu} = \frac{\partial^\mu\partial^\nu}{\Box}; \quad \Theta^{\mu\nu} = \eta^{\mu\nu} - \Omega^{\mu\nu}; \quad S^{\mu\nu} = \epsilon^{\mu\rho\nu}\partial_\rho \tag{6.21}$$

We want to analyze the poles of the gauge field propagators. For this purpose it is sufficient to study the diagonal part of the inverse gauge dynamical operator $O^{\mu\nu}$. Schematically we have,

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}^{-1} = \begin{pmatrix} (A - BD^{-1}C)^{-1} & * \\ * & (D - CA^{-1}B)^{-1} \end{pmatrix} \tag{6.22}$$

The step-by-step computation is presented in the appendix C. Here we report the results:

$$\begin{aligned} (A - BD^{-1}C)^{-1} &= \frac{1}{\Box + \mu^2}\Theta_{\mu\nu} + \frac{\alpha}{\Box}\Omega_{\mu\nu}, \\ (D - CA^{-1}B)^{-1} &= \frac{1}{\Box + \mu^2}\Theta_{\mu\nu} + \frac{\beta}{\Box}\Omega_{\mu\nu}. \end{aligned} \tag{6.23}$$

For instance, we can take the Landau gauge $\alpha, \beta \to 0$ to understand the spectrum since this is a physical information and therefore it is independent of a gauge choice. What we see is that we have two massive gauge bosons, with degenerate masses equal to μ. The only contribution to the gauge field masses is topological, coming from the Chern-Simons term, since there is no Higgs mechanism taking place. Despite the gauge fields being massive, considering the origin of this mass, there is still gauge symmetry. We remark that the gauge fields have degenerate masses with the two spinless excitations $\tilde{N}$

and M. It is important to stress here that the counting of the gauge degrees of freedom that we are preseting is more of a heurist argument, rather than a rigorous proof, for at least two reasons: 1) A more precise approach would be to couple the full propagator, off-diagonal parts included, with conserved currents, and evaluate the imaginary part of the residue of this amplitude at each of its poles. If this procedure results in zero, that means that no degrees of freedom are propagated. If it results in a positive number, then degrees of freedom are propagated and their number is given by the number of independent parameters present in the final result. If it results in a negative number, it means that the theory contains ghosts and unitarity is violated already at tree level. The reason we haven't done this here yet is, firstly because the result for the Maxwell-Chern-Simons propagators (6.23) is already known[71], and second because in this work we will concentrate on classical solutions of the theory, although the correct approach will be indispensable the moment we concern ourselves with the quantum theory. 2) The diagonal elements of the propagator can count as independent degrees of freedom only as long as no degrees of freedom are propagated by the off-diagonal elements (mixed propagators); when the latter happens, the correct approach would reveal the correct number of degrees of freedom propagated by the number of independent parameters in the result. For the vaccum that we are considering here, indeed no degrees of freedom are propagated by the mixed propagators, as is also already known[71].

(1,1) - Vacuum: $\left(\langle|\phi_+|^2\rangle = \langle|\phi_-|^2\rangle = v^2,\, \langle M\rangle = \langle N\rangle = 0\right)$

Now, with the vacuum configuration parametrized by $\phi_+^0 = -\phi_-^0 = v$ [2] it is clear that the only way to satisfy

$$\begin{aligned}\delta\phi_+^0 &= i\omega_+ v = 0\\ \delta\phi_-^0 &= -i\omega_- v = 0\end{aligned}$$

is by setting $\omega_+ = \omega_- = 0$. This is the totally broken vacuum. In this case both charged scalar fields have non-trivial VEVs and the Higgs mechanism takes place, the two U(1) gauge symmetries are spontaneously broken and the gauge fields will acquire another contribution to their masses, besides the topological one.

We can parametrize the charged scalar fields as

$$\phi_\pm(x) = \pm\left(v + \frac{\rho_\pm(x)}{\sqrt{2}}\right) e^{i\theta_\pm(x)} \tag{6.24}$$

Using the gauge transformation (8.112) with $\rho(x) \pm \xi(x) = -\theta_\pm(x)$, we can go to the unitary gauge where we gauge away the would-be Goldstone bosons and write the covariant

[2]The reason for this parametrization will be clear when we consider supersymmetry.

derivative on the charged scalar fields:

$$
\begin{aligned}
D_\mu \phi_+ &= \frac{1}{\sqrt{2}}\partial_\mu \rho_+ + i\left(eA_\mu + ga_\mu\right)\left(v + \frac{\rho_+}{\sqrt{2}}\right) \\
D_\mu \phi_- &= \frac{1}{\sqrt{2}}\partial_\mu \rho_- - i\left(eA_\mu - ga_\mu\right)\left(v + \frac{\rho_-}{\sqrt{2}}\right).
\end{aligned} \tag{6.25}
$$

The Lagrangian in this case can be written as

$$
\begin{aligned}
\mathcal{L} = &-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{4}f_{\mu\nu}f^{\mu\nu} + \frac{\mu}{2}\epsilon^{\mu\rho\nu}A_\mu\partial_\rho a_\nu + \frac{\mu}{2}\epsilon^{\mu\rho\nu}a_\mu\partial_\rho A_\nu \\
&+ \frac{1}{2}(\partial_\mu\rho_+)^2 + (eA_\mu + ga_\mu)^2\left(v + \frac{\rho_+}{\sqrt{2}}\right)^2 \\
&+ \frac{1}{2}(\partial_\mu\rho_-)^2 + (eA_\mu - ga_\mu)^2\left(v + \frac{\rho_-}{\sqrt{2}}\right)^2 \\
&+ \frac{1}{2}\left(\partial_\mu M\right)^2 + \frac{1}{2}\left(\partial_\mu N\right)^2 - V\left(\phi_+, \phi_-, M, N\right),
\end{aligned} \tag{6.26}
$$

where the potental here can be written as

$$
\begin{aligned}
V\left(\phi_+, \phi_-, M, N\right) = &\left(eN + gM\right)^2\left(v + \frac{\rho_+}{\sqrt{2}}\right)^2 + \left(eN - gM\right)^2\left(v + \frac{\rho_-}{\sqrt{2}}\right)^2 \\
&+ \frac{1}{2}\left[e\left[\left(v + \frac{\rho_+}{\sqrt{2}}\right)^2 - \left(v + \frac{\rho_-}{\sqrt{2}}\right)^2\right] - \mu M\right]^2 \\
&+ \frac{1}{2}\left[g\left[\left(v + \frac{\rho_+}{\sqrt{2}}\right)^2 + \left(v + \frac{\rho_-}{\sqrt{2}}\right)^2 - 2v^2\right] - \mu N\right]^2
\end{aligned} \tag{6.27}
$$

Considering only the quadratic part on the fluctuations in the unitary gauge, we can write

$$
\begin{aligned}
\mathcal{L}^{\text{quad}} = &-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{4}f_{\mu\nu}f^{\mu\nu} + \mu\epsilon^{\mu\nu\rho}A_\mu\partial_\nu a_\rho + 2v^2\left(e^2A_\mu A^\mu + g^2 a_\mu a^\mu\right) \\
&+ \frac{1}{2}(\partial_\mu\rho_+)^2 + \frac{1}{2}(\partial_\mu\rho_-)^2 + \frac{1}{2}(\partial_\mu M)^2 + \frac{1}{2}(\partial_\mu N)^2 \\
&- \frac{1}{2}\left[2v^2(e^2+g^2)(\rho_+^2+\rho_-^2) + (\mu^2+4v^2g^2)M^2 + (\mu^2+4v^2e^2)N^2\right. \\
&\left.+4v^2(g^2-e^2)\rho_+\rho_- - 2\sqrt{2}ve\mu(\rho_+-\rho_-)M - 2\sqrt{2}vg\mu(\rho_++\rho_-)N\right]
\end{aligned} \tag{6.28}
$$

The quadratic scalar sector can be rewritten in a compact form

$$
\mathcal{L}^{\text{quad}}_{\text{scalar}} = \frac{1}{2}\left(\partial^\mu\varphi\right)^T\partial_\mu\varphi - \frac{1}{2}\varphi^T\mathcal{M}^2\varphi, \quad \text{where} \quad \varphi = \begin{pmatrix} \rho_+ \\ \rho_- \\ M \\ N \end{pmatrix}
$$

And we have defined the squared mass matrix $\mathcal{M}^2$:

$$\mathcal{M}^2 = \begin{pmatrix} 2v^2(e^2+g^2) & 2v^2(g^2-e^2) & -\sqrt{2}ve\mu & -\sqrt{2}vg\mu \\ 2v^2(g^2-e^2) & 2v^2(e^2+g^2) & \sqrt{2}ve\mu & -\sqrt{2}vg\mu \\ -\sqrt{2}ve\mu & \sqrt{2}ve\mu & \mu^2+4v^2g^2 & 0 \\ -\sqrt{2}vg\mu & -\sqrt{2}vg\mu & 0 & \mu^2+4v^2e^2 \end{pmatrix}$$

The mass spectrum of the scalar sector is then given by the eigenvalues of $\mathcal{M}^2$ which are:

$$m_{\pm}^2 = \frac{1}{2}\left(\mu^2 + M_e^2 + M_g^2 \pm \sqrt{(\mu^2 + M_e^2 + M_g^2)^2 - 4M_e^2 M_g^2}\right) \tag{6.29}$$

Each one with multiplicity 2. The mass parameters are $M_e^2 = 4v^2e^2$ and $M_g^2 = 4v^2g^2$.

The gauge quadratic part can be written as before in eq. (6.18), but now we have:

$$\begin{aligned} A^{\mu\nu} &= \left(\Box + M_e^2\right)\Theta^{\mu\nu} + M_e^2\,\Omega^{\mu\nu}, \\ B^{\mu\nu} &= C^{\mu\nu} = \mu S^{\mu\nu}, \\ D^{\mu\nu} &= \left(\Box + M_g^2\right)\Theta^{\mu\nu} + M_g^2\,\Omega^{\mu\nu} \end{aligned} \tag{6.30}$$

Thus, we have in the gauge quadratic sector (in the unitary gauge $\alpha, \beta \to \infty$):

$$\begin{aligned} \left(A - BD^{-1}C\right)^{-1}_{\mu\nu} &= \frac{\Box + M_g^2}{\left(\Box + M_e^2\right)\left(\Box + M_g^2\right) + \mu^2\Box}\Theta_{\mu\nu} + \frac{1}{M_e^2}\Omega_{\mu\nu} \\ \left(D - CA^{-1}B\right)^{-1}_{\mu\nu} &= \frac{\Box + M_e^2}{\left(\Box + M_e^2\right)\left(\Box + M_g^2\right) + \mu^2\Box}\Theta_{\mu\nu} + \frac{1}{M_g^2}\Omega_{\mu\nu} \end{aligned} \tag{6.31}$$

The poles of the propagators, in momentum space ($\Box \to -p^2$), are exactly at $p^2 = m_{\pm}^2$ of (6.29). In summary, around the (1,1)-Vacuum we have 4 massive gauge and 4 massive scalar degrees of freedom with mass squared $m_{\pm}^2$ distributed equally.

It is important to notice that here we chose the unitary gauge for simplicity. There are contributions to the gauge fields masses coming from the Chern-Simons terms as well as the Higgs mechanism. The gauge symmetry is totally spontaneously broken and all the excitations are massive.

(1,0)-Vacuum: $\left(\langle|\phi_+|^2\rangle = v^2,\ \langle|\phi_-|^2\rangle = 0,\ \langle M\rangle = ev^2/\mu,\ \langle N\rangle = -gv^2/\mu\right)$

In this vacuum parity and time-reversal are broken and gauge symmetry seems to be only partially broken. The first way to confirm this and following the steps suggested before is to start by parametrizing the vacuum as $\phi_+ = v$ and $\phi_- = 0$ and ask how can we make

$$\delta\phi_+^0 = i\omega_+ v = 0$$
$$\delta\phi_-^0 = i\omega_- 0 = 0$$

It becomes clear that we must have $\omega_+ = \rho + \xi = 0$ while $\omega_- = \rho - \xi = 2\rho$ remains arbitrary. Thus, we can see that the U(1) symmetry that survives is, in its infinitesimal form:

$$\begin{aligned}
\delta\phi_+ &= 0\\
\delta\phi_- &= i2\rho\phi_-\\
\delta A_\mu &= -\frac{1}{e}\partial_\mu\rho\\
\delta a_\mu &= \frac{1}{g}\partial_\mu\rho
\end{aligned} \tag{6.32}$$

The ϕ_+ scalar field acquires a non-trivial VEV and we will expand around it using the exponential parametrization but again we use the gauge freedom to eliminate its phase, effectively gauging away the would-be Goldstone boson. Since $\langle|\phi_-|^2\rangle = 0$, we don't need to perfom any special parametrization for it .

So, in this scenario, we parametrize as follows:

$$\phi_+ = v + \frac{\rho_+}{\sqrt{2}}; \quad M = \frac{ev^2}{\mu} + \tilde{M}; \quad N = -\frac{gv^2}{\mu} + \tilde{N}$$

Such that the quadratic scalar sector will read:

$$\begin{aligned}
\mathcal{L}^{\text{quad}}_{\text{scalar}} =& |\partial_\mu\phi_-|^2 - \left(\frac{2egv^2}{\mu}\right)^2 |\phi_-|^2\\
&+ \frac{1}{2}(\partial_\mu\rho_+)^2 + \frac{1}{2}(\partial_\mu\tilde{M})^2 + \frac{1}{2}(\partial_\mu\tilde{N})^2\\
&- \frac{1}{2}\Big[2v^2(e^2+g^2)\rho_+^2 + (\mu^2 + 2v^2g^2)\tilde{M}^2 + (\mu^2 + 2v^2e^2)\tilde{N}^2\\
&-2\sqrt{2}ve\mu\rho_+\tilde{M} - 2\sqrt{2}vg\mu\rho_+\tilde{N} + 4v^2eg\tilde{M}\tilde{N}\Big]
\end{aligned} \tag{6.33}$$

Or, rewritting conveniently:

$$\begin{aligned}
\mathcal{L}^{\text{quad}}_{\text{scalar}} =& \; |\partial_\mu\phi_-|^2 - \left(\frac{2egv^2}{\mu}\right)^2 |\phi_-|^2\\
&+ \frac{1}{2}\left(\partial^\mu\tilde{\varphi}\right)^T \partial_\mu\tilde{\varphi} - \frac{1}{2}\tilde{\varphi}^T\tilde{\mathcal{M}}^2\tilde{\varphi}, \qquad \tilde{\varphi} = \begin{pmatrix}\rho_+\\ M\\ N\end{pmatrix}
\end{aligned}$$

Being:

$$\tilde{\mathcal{M}}^2 = \begin{pmatrix} 2v^2(e^2+g^2) & -\sqrt{2}ve\mu & -\sqrt{2}vg\mu \\ -\sqrt{2}ve\mu & \mu^2+2v^2g^2 & 2v^2eg \\ -\sqrt{2}vg\mu & 2v^2eg & \mu^2+2v^2e^2 \end{pmatrix}$$

Again, the mass spectrum can be read off from the eigenvalues of the mass matrix. The characteristic polynomial reads:

$$\lambda^3 - 2(\mu^2 + \tilde{M_e}^2 + \tilde{M_g}^2)\lambda^2 + (\mu^2 + \tilde{M_e}^2 + \tilde{M_g}^2)^2\lambda - 4\mu^2\tilde{M_e}^2\tilde{M_g}^2 = 0 \tag{6.34}$$

Where $\tilde{M_e}^2 = 2v^2e^2 = \frac{1}{2}M_e^2$ and $\tilde{M_g}^2 = 2v^2g^2 = \frac{1}{2}M_g^2$. Since the characteristic polynomial that determines them is of degree 3, the analysis is a little less straightforward (Appendix C) and the scalar mass spectrum is:

$$m_k^2 = \frac{2}{3}\left(\mu^2 + \tilde{M_e}^2 + \tilde{M_g}^2\right)\left(1 + \cos\left\{\frac{1}{3}\arccos\left[2\left(\frac{\sqrt[3]{\mu^2\tilde{M_e}^2\tilde{M_g}^2}}{\frac{\mu^2+\tilde{M_e}^2+\tilde{M_g}^2}{3}}\right)^3 - 1\right] - \frac{2\pi k}{3}\right\}\right),$$
$$\text{where } k = 0,1,2; \tag{6.35}$$

The spectrum (6.35) takes a very informative form if we consider the particular scenario where $e^2 = g^2 \Rightarrow \tilde{M_e}^2 = \tilde{M_g}^2$, which reads:

$$m^2 = \begin{cases} \mu^2 \\ m_\pm^2 = \frac{1}{2}\left(\mu^2 + 2M_e^2 \pm \sqrt{(\mu^2+2M_e^2)^2 - 4(M_e^2)^2}\right) \end{cases} \tag{6.36}$$

This doesn't come entirely as a suprise if one observes that, for instance, if $e = g$ then (6.33) can but written as:

$$\begin{aligned} \mathcal{L}_{\text{scalar}}^{\text{quad}} =& |\partial_\mu\phi_-|^2 - \left(\frac{2egv^2}{\mu}\right)^2|\phi_-|^2 + \frac{1}{2}(\partial_\mu Q)^2 - \frac{\mu^2}{2}Q^2 \\ &+ \frac{1}{2}(\partial_\mu\rho_+)^2 + \frac{1}{2}(\partial_\mu P)^2 \\ &- \frac{1}{2}\left[4v^2e^2\rho_+^2 + (\mu^2+4v^2e^2)P^2 - 4ve\mu\rho_+P\right] \end{aligned} \tag{6.37}$$

After the redefinitions $P = \frac{\tilde{N}+\tilde{M}}{\sqrt{2}}$ and $Q = \frac{\tilde{N}-\tilde{M}}{\sqrt{2}}$. That is, it's now obvious that the μ^2 mass comes from the scalar field Q whose quadratic part is diagonal after the symmetry breaking and the other two arise as eigenvalues of the squared mass matrix of the fields ρ_+ and P. Note that Q and P, unlinke $\tilde{N}$ and $\tilde{M}$ are not parity eigenstates but transform into each other, this will become more apparent when we consider the $(0,1)-$ vaccum. Finally, we observe the μ^2 is precisely the squared mass of the gauge bosons in the unbroken scenario. The meaning of this last fact will become clear after we analize the gauge sector of this vacuum, which we now turn to.

The expansion of ϕ_+ around its non-trivial VEV results in the following quadratic sector for the gauge fields:

$$\begin{aligned}\mathcal{L}_{\text{gauge}}^{\text{quad}} &= -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{4}f_{\mu\nu}f^{\mu\nu} + \frac{\mu}{2}\epsilon^{\mu\rho\nu}A_\mu\partial_\rho a_\nu + \frac{\mu}{2}\epsilon^{\mu\rho\nu}a_\mu\partial_\rho A_\nu \\ &+ \left(eA_\mu + ga_\mu\right)^2 v^2\end{aligned} \tag{6.38}$$

We, again, write it as:

$$\mathcal{L}_{\text{gauge}}^{\text{quad}} = \frac{1}{2}\begin{pmatrix} A_\mu & a_\mu \end{pmatrix}\begin{pmatrix} A^{\mu\nu} & B^{\mu\nu} \\ C^{\mu\nu} & D^{\mu\nu}\end{pmatrix}\begin{pmatrix} A_\nu \\ a_\nu \end{pmatrix}, \tag{6.39}$$

Now with

$$\begin{aligned} A^{\mu\nu} &= \left(\Box + \tilde{M_e}^2\right)\Theta^{\mu\nu} + \tilde{M_e}^2\,\Omega^{\mu\nu}, \\ B^{\mu\nu} &= C^{\mu\nu} = m^2\eta^{\mu\nu} + \mu S^{\mu\nu}, \\ D^{\mu\nu} &= \left(\Box + \tilde{M_g}^2\right)\Theta^{\mu\nu} + \tilde{M}_g^2\,\Omega^{\mu\nu}\end{aligned} \tag{6.40}$$

With the definitons as before, that is $\tilde{M_e}^2 = 2v^2e^2 = \frac{1}{2}M_e^2$, $\tilde{M_g}^2 = 2v^2g^2 = \frac{1}{2}M_g^2$ and also $m^2 = 2veg$. Note that, because $\tilde{M_e}^2\tilde{M_g}^2 = (m^2)^2$ we have:

$$\begin{pmatrix} \left(\Box + \tilde{M_e}^2\right)\Theta^{\mu\nu} + \tilde{M_e}^2\,\Omega^{\mu\nu} & m^2\eta^{\mu\nu} + \mu S^{\mu\nu} \\ m^2\eta^{\mu\nu} + \mu S^{\mu\nu} & \left(\Box + \tilde{M_g}^2\right)\Theta^{\mu\nu} + \tilde{M}_g^2\,\Omega^{\mu\nu}\end{pmatrix}\begin{pmatrix}\tilde{M_g}^2\partial_\nu f(x) \\ -m^2\partial_\nu f(x)\end{pmatrix} = 0 \tag{6.41}$$

Where $f(x)$ is any fuction whose derivatives are well defined and we have made use of the facts: $\Theta^{\mu\nu}\partial_\nu = S^{\mu\nu}\partial_\nu = 0$ and $\Omega^{\mu\nu}\partial_\nu = \eta^{\mu\nu}\partial_\nu = \partial^\mu$. What (6.41) tells us is that that operator has a null eigenvalue, which in turn means that it is not invertible, because its determinant must be zero. This is good news, since it consistently demonstrates in another way that there still remains a gauge symmetry. We must then supplement the gauge quadratic sector with gauge fixing terms so that we can invert the dynamical operator and find the propagator. We have to find the inverse of:

$$O^{\mu\nu} = \begin{pmatrix} A^{\mu\nu} & B^{\mu\nu} \\ C^{\mu\nu} & D^{\mu\nu} \end{pmatrix} \tag{6.42}$$

with

$$\begin{aligned} A^{\mu\nu} &= (\Box + \tilde{M_e}^2)\Theta^{\mu\nu} + \left(\tilde{M_e}^2 + \frac{\Box}{\alpha}\right)\Omega^{\mu\nu} \\ B^{\mu\nu} &= C^{\mu\nu} = m^2\eta^{\mu\nu} + \mu S^{\mu\nu} \\ D^{\mu\nu} &= (\Box + \tilde{M_g}^2)\Theta^{\mu\nu} + \left(\tilde{M_g}^2 + \frac{\Box}{\beta}\right)\Omega^{\mu\nu} \end{aligned} \tag{6.43}$$

The calculation is peformed in detail in Appendix C and here we present only the final answer for the diagonal elements of the inverse:

$$\begin{aligned} (A - BD^{-1}C)^{-1}_{\mu\nu} &= \frac{\left(\Box + \mu^2 + \tilde{M_e}^2 + \tilde{M_g}^2\right)\left(\Box + \tilde{M_g}^2\right)}{\Box\left(\Box + \mu^2 + \tilde{M_e}^2 + \tilde{M_g}^2\right)^2 + 4\mu^2\tilde{M_e}^2\tilde{M_g}^2}\Theta_{\mu\nu} \\ &+ \frac{2\mu m^2\left(\Box + \tilde{M_g}^2\right)}{\Box\left[\Box\left(\Box + \mu^2 + \tilde{M_e}^2 + \tilde{M_g}^2\right)^2 + 4\mu^2\tilde{M_e}^2\tilde{M_g}^2\right]}S_{\mu\nu} \\ &+ \frac{\alpha\left(\Box + \beta\tilde{M_g}^2\right)}{\Box + \alpha\tilde{M_e}^2 + \beta\tilde{M_g}^2}\Omega_{\mu\nu} \\ (D - CA^{-1}B)^{-1}_{\mu\nu} &= \frac{\left(\Box + \mu^2 + \tilde{M_e}^2 + \tilde{M_g}^2\right)\left(\Box + \tilde{M_e}^2\right)}{\Box\left(\Box + \mu^2 + \tilde{M_e}^2 + \tilde{M_g}^2\right)^2 + 4\mu^2\tilde{M_e}^2\tilde{M_g}^2}\Theta_{\mu\nu} \\ &+ \frac{2\mu m^2\left(\Box + \tilde{M_e}^2\right)}{\Box\left[\Box\left(\Box + \mu^2 + \tilde{M_e}^2 + \tilde{M_g}^2\right)^2 + 4\mu^2\tilde{M_e}^2\tilde{M_g}^2\right]}S_{\mu\nu} \\ &+ \frac{\beta\left(\Box + \alpha\tilde{M_e}^2\right)}{\Box + \alpha\tilde{M_e}^2 + \beta\tilde{M_g}^2}\Omega_{\mu\nu} \end{aligned} \tag{6.44}$$

It should be stressed that our use of the gauge fixing terms with parameters α and β is not so arbitrary, since they should be compatible with the partially fixed gauge that we already chose by making ϕ_+ real[3], however we will not worry ourselves with the details

[3] We could circumvent this problem by working in the t'Hooft gauge, but the analysis of the spectrum would have to be a little more cautious since gauge dependend poles would show up both in the scalar and gauge sector, and only a quantum analysis is able to show that they actually cancel each other out.

here, since it won't have any effect on the physical spectrum. The reader can consider them here simply as tools that allow us to invert the dynamical operator. The physical poles of the propagator, in momentum space, are the solutions for p^2 of the equation:

$$
\begin{aligned}
-p^2\left(-p^2+\mu^2+\tilde{M_e}^2+\tilde{M_g}^2\right)^2+4\mu^2\tilde{M_e}^2\tilde{M_g}^2&=0 \\
(p^2)^3-2(\mu^2+\tilde{M_e}^2+\tilde{M_g}^2)(p^2)^2+(\mu^2+\tilde{M_e}^2+\tilde{M_g}^2)^2p^2-4\mu^2\tilde{M_e}^2\tilde{M_g}^2&=0
\end{aligned}
\tag{6.45}
$$

Which is exactly (6.34) with $\lambda = p^2$, whose solutions are given by (6.35).

Once again, we can see more clearly what happens in the particular case when $e = g$. From (6.38) we can already see the simplification, but let's rewrite it as:

$$
\begin{aligned}
\mathcal{L}^{\text{quad}}_{\text{gauge}} = &-\frac{1}{4}F^{+}_{\mu\nu}F^{+\mu\nu}+\frac{\mu}{2}\epsilon^{\mu\rho\nu}A^{+}_{\mu}\partial_{\rho}A^{+}_{\nu}+\frac{1}{2}M_e^2A^{+}_{\mu}A^{+\mu} \\
&-\frac{1}{4}F^{-}_{\mu\nu}F^{-\mu\nu}-\frac{\mu}{2}\epsilon^{\mu\rho\nu}A^{-}_{\mu}\partial_{\rho}A^{-}_{\nu}
\end{aligned}
\tag{6.46}
$$

Where we have re-used the field redefinitions already presented before:

$$
A^{+}_{\mu}=\frac{A_\mu+a_\mu}{\sqrt{2}},\quad A^{-}_{\mu}=\frac{A_\mu-a_\mu}{\sqrt{2}}. \tag{6.47}
$$

We can see that, in this particular instance of $e = g$, A^{+}_{μ} is the one who receives a Proca mass term due to the spontanous breaking of the symmetry, which with the technology developed in Appendix C one can see that will lead to the same poles $m^2_{\pm}$ in (6.36). The responsible for the remaining gauge symmetry is the, now obvious, A^{-}_{μ} field because its mass comes only from the gauge invariant Chern-Simons term. In general, from the unbroken U(1) transformation (6.32), we can write:

$$
\delta\left(\frac{eA_\mu-ga_\mu}{\sqrt{e^2+g^2}}\right)=-\frac{1}{\sqrt{e^2+g^2}}\partial_\mu\omega_- \tag{6.48}
$$

By noticing that the U(1) remaining is essentially the phase freedom for ϕ_- ($e^{i\omega_-}\phi_-$) with $\omega_- = 2\rho$, the point of the above equation is simply to show that it is the particular combination appearing on its l.h.s. which transforms accordingly to what is to be expected from an abelian gauge field, *i.e.* , with a derivative of the phase argument. The scaling factor is just to get the dimensions straight and to reproduce what we obtained when setting $e = g$, for instance, it is not difficult to convince oneself that in this particular case, the charge of the field is actually $\sqrt{2}e$.

Summarizing, in the $(1, 0)$-Vacuum we have 2 scalar degrees of freedom coming from

ϕ_- with mass $m_{\phi_-} = 2egv^2/\mu$ just as in the unbroken case, another 3 from the other scalars with masses given by (6.35), and since the symmetry was only partially broken, instead of the gauge fields aquiring 2 more degrees of freedom like in the $(1,1)$-Vacuum, they get only one more giving a total of 3 with masses also given by (6.35).

(0,1)-Vacuum: $\left(\langle|\phi_+|^2\rangle = 0,\, \langle|\phi_-|^2\rangle = v^2,\, \langle M\rangle = -ev^2/\mu,\, \langle N\rangle = -gv^2/\mu\right)$

Here, we remark that the situation goes exactly as in the (1,0)-Vacuum as far the mass spectrum is concerned, that is, we also have: 2 scalar degrees of freedom with masses $2egv^2/\mu$, 3 with masses given by (6.35) and 3 vector degrees of freedom with masses also given by (6.35). The only difference is the role of each field in generating each mass but they coincide, not surprisingly, with the "parity trasformed" fields of the (1,0)-Vacuum as we now will see.

The appropriate field expansions around this vacuum is:

$$\phi_- = -\left(v + \frac{\rho_-}{\sqrt{2}}\right); \quad M = \frac{-ev^2}{\mu} + \tilde{M}; \quad N = -\frac{gv^2}{\mu} + \tilde{N}$$

Again, we removed the phase of the ϕ_- field making of the gauge freedom that we have. Substituing these into the lagragian, we get for the quadratic scalar sector:

$$\begin{aligned}\mathcal{L}^{\text{quad}}_{\text{scalar}} =& |\partial_\mu\phi_+|^2 - \left(\frac{2egv^2}{\mu}\right)^2 |\phi_+|^2 \\ &+ \frac{1}{2}(\partial_\mu\rho_-)^2 + \frac{1}{2}(\partial_\mu\tilde{M})^2 + \frac{1}{2}(\partial_\mu\tilde{N})^2 \\ &- \frac{1}{2}\Big[2v^2(e^2+g^2)\rho_-^2 + (\mu^2+2v^2g^2)\tilde{M}^2 + (\mu^2+2v^2e^2)\tilde{N}^2 \\ &+2\sqrt{2}ve\mu\rho_-\tilde{M} - 2\sqrt{2}vg\mu\rho_-\tilde{N} - 4v^2eg\tilde{M}\tilde{N}\Big]\end{aligned} \tag{6.49}$$

Or, again, rewritting conveniently:

$$\begin{aligned}\mathcal{L}^{\text{quad}}_{\text{scalar}} =& \; |\partial_\mu\phi_+|^2 - \left(\frac{2egv^2}{\mu}\right)^2 |\phi_+|^2 \\ &+ \frac{1}{2}\left(\partial^\mu\tilde{\varphi}\right)^T \partial_\mu\tilde{\varphi} - \frac{1}{2}\tilde{\varphi}^T\tilde{\mathcal{M}}^2\tilde{\varphi}, \quad \tilde{\varphi} = \begin{pmatrix}\rho_- \\ \tilde{M} \\ \tilde{N}\end{pmatrix}\end{aligned}$$

Being:

$$\tilde{\mathcal{M}}^2 = \begin{pmatrix} 2v^2(e^2+g^2) & \sqrt{2}ve\mu & -\sqrt{2}vg\mu \\ \sqrt{2}ve\mu & \mu^2+2v^2g^2 & -2v^2eg \\ -\sqrt{2}vg\mu & -2v^2eg & \mu^2+2v^2e^2 \end{pmatrix}$$

Which is nothing but the same squared mass matrix of the (1,0)-Vacuum with the replacement $e \to -e$, but note that in that case, the eigenvalues determining the mass spectrum depended only of e^2, therefore they remain the same here and are given by (6.35). Consistenly, the special case $e = g$ here takes the form:

$$\begin{aligned} \mathcal{L}^{\text{quad}}_{\text{scalar}} =& |\partial_\mu \phi_+|^2 - \left(\frac{2egv^2}{\mu}\right)^2 |\phi_+|^2 + \frac{1}{2}(\partial_\mu P)^2 - \frac{\mu^2}{2}P^2 \\ &+ \frac{1}{2}(\partial_\mu \rho_+)^2 + \frac{1}{2}(\partial_\mu Q)^2 \\ &- \frac{1}{2}\left[4v^2e^2\rho_-^2 + (\mu^2+4v^2e^2)Q^2 - 4ve\mu\rho_- Q\right] \end{aligned} \tag{6.50}$$

As we already suggested, the roles of P and Q have been reversed, but the spectrum in this special case remains as in (6.36). With respect to the spectrum of the gauge sector, the analysis is exactly the same as before but with $m^2 = 2veg \to -m^2 = -2veg$, but note that the mass spectrum has no business with m^2, since it depends only on $(m^2)^2 = 4v^2e^2g^2 = \tilde{M}_e^{\,2}\tilde{M}_g^{\,2}$. So the masses of the gauge sector also remain as given in (6.35). With regards to the special case $e = g$, the only difference is that the roles of the fields A_μ^+ and A_μ^- is swapped, reasonably so, but with no alteration with respect to the mass spectrum.

Regarding the partial breaking of the symmetry, the same reasoning follows. The vaccum configuration being parametrized by $\phi_+^0 = 0$ and $\phi_-^0 = -v$ implies that in order to satisfy

$$\begin{aligned} \delta\phi_+^0 &= i\omega_+ 0 = 0 \\ \delta\phi_-^0 &= -i\omega_- v = 0 \end{aligned}$$

we must have $\omega_- = \rho - \xi = 0$ and $\omega_+ = \rho + \xi = 2\rho$ remains arbitrary. The unbroken U(1) symmetry is in this case, infinitesimally:

$$
\begin{aligned}
\delta\phi_+ &= i2\rho\phi_+ \\
\delta\phi_- &= 0 \\
\delta A_\mu &= -\frac{1}{e}\partial_\mu\rho \\
\delta a_\mu &= -\frac{1}{g}\partial_\mu\rho
\end{aligned} \tag{6.51}
$$

The gauge field associated with this U(1) being

$$
\delta\left(\frac{eA_\mu + ga_\mu}{\sqrt{e^2+g^2}}\right) = -\frac{1}{\sqrt{e^2+g^2}}\partial_\mu\omega_+ \tag{6.52}
$$

which reduces to A_μ^+ when $e = g$.

One might have observed that around any of the vacua there exists several mass degeneracies. This is typically the case in supersymmetric theories. In fact, because the mass degeneracies are among particles whose spin differ by one integer instead of only half integer, it would be reasonable to suspect the presence of a SUSY $\mathcal{N} = 2$ behind the scenes here. Indeed, the model considered here there are topological solutions (vortices) that obey a set of Bogomol'nyi-type equations, the self-dual vortices; and they have a topologically conserved charge. Whenever that happens, it is possible to construct an $\mathcal{N} = 2$ supersymmetric version of the model with central charge equal to the topological charge and whose bosonic sector matches perfectly the self-dual theory [21, 55, 46, 127, 128, 129]. Later in this work we will show exactly how this is done. But if that is really the case, and at this point it is only a suspicion, we can conjecture that the scalar potential proposed here, although not being the most general allowed by renormalizability, doesn't get any quantum corrections, because supersymmetry prevents it from happening. This is a rather strong statement, but hopefully we will also be able to prove this in the future.

6.3 Topological configurations

Let us consider the 0^{th} component of the equations of motion (4.23) which gives us the Gauss laws:

$$\vec{\nabla} \cdot \vec{E} + \mu b = e(\rho_+ + \rho_-)$$
$$\vec{\nabla} \cdot \vec{e} + \mu B = g(\rho_+ - \rho_-). \tag{6.53}$$

Where $\rho_\pm = J^0_\pm$. Integrating in space, we can use Gauss theorem and discard the first term since we have vanishing electric fields in the asymptotic limit. We obtain,

$$\mu \int d^2x\, b = Q, \quad \mu \int d^2x\, B = G \tag{6.54}$$

where we naturally defined the electric and g-electric charges as $Q = e \int d^2x\ (\rho_+ + \rho_-)$, and $G = g \int d^2x\ (\rho_+ - \rho_-)$. We remember here that the canonical mass dimension of the objects here is given by $[A] = [a] = [e] = [g] = 1/2$ and $[\mu] = 1$. Notice that on the l.h.s of this equation we have a magnetic flux and a g-magnetic flux, respectively defined by $\int d^2x\, B = \Phi$ and $\int d^2x\, b = \chi$, and we conclude that the $\nu = 0$ equations give us,

$$\mu\, \chi = Q, \quad \mu\, \Phi = G. \tag{6.55}$$

Therefore from the equations of motion, we immediately see that the g-magnetic flux is proportional to the electric charge and the magnetic flux is proportional to the g-electric charge. This is the parity symmetric version of the distinctive feature of flux attachment of Chern-Simons theories. It plays an interesting role when we consider vortex configurations, so let's finally investigate them.

6.3.1 Asymptotic Conditions

What we are looking for are the static ($\partial_0 \equiv 0$) classical solutions that lead to a finite energy:

$$\begin{aligned} H = \int d^2x \Big[&\frac{1}{2}\left(\vec{E}^2 + B^2\right) + \frac{1}{2}\left(\vec{e}^2 + b^2\right) \\ &+ |D_0\phi_+|^2 + |D_0\phi_-|^2 + |D_i\phi_+|^2 + |D_i\phi_-|^2 \\ &+ \frac{1}{2}(\partial_i M)^2 + \frac{1}{2}(\partial_i N)^2 + V \Big] = \int d^2x\, \mathcal{H} = \int r\, dr\, d\theta\ \mathcal{H} \end{aligned} \tag{6.56}$$

To achieve finite energy, each term in $\mathcal{H}$ must have an asymptotic behavior sufficient to compensate the divergence of rdr in the measure an make the integral convergent. Therefore, each term must go to zero faster than $O(1/r^2)$ in order to guarantee the finiteness of the energy. It is clear that if the fields aproach one the 4 vacua that we studied on the last section at spatial infinity, then the contribution of the potential to the total energy will be finite. Nonetheless, for the sake of clarity we will restrict ourselves to the (1,1)-Vaccum, which is the most natural choice if one wishes to study topological vortices. However, it should be stressed that configurations whose asymptotic behavior tends to the other vacua might also lead to finite energy, for instance, non-topolgical vortices for the (0,0) -Vaccum and domain walls connecting the (1,0) and (0,1) vacua. These other configurations will be investigated in the future. Henceforth, we will be considering the configurations with asymptotic behavior as $|\vec{x}| = r \to \infty$ given by,

$$
\begin{aligned}
&|\phi_\pm|^2 \to v^2, \\
&|D_i\phi_\pm|^2, |D_0\phi_\pm|^2 \to 0, \\
&\vec{E}, \vec{e}, B, b, M, N \to 0, \\
&(\partial_i M)^2, (\partial_i N)^2 \to 0.
\end{aligned}
\tag{6.57}
$$

These conditions can be seen as boundary conditions for the fields in the boundary of space, that we can see as an asymptotic sphere $S^1_\infty = \partial\mathbb{R}^2$. From the first condition, even if the modulus of the scalar field is fixed at infinity, we still have angular freedom given by the angle θ that parametrizes the sphere at infinity (it can be seen as the direction in which you are going to the asymptotic limit $r \to \infty$), that is,

$$
\phi_\pm \to \phi_\pm^\infty(\theta) = v\, e^{i\omega_\pm(\theta)} \tag{6.58}
$$

Therefore, the asymptotic limit of the covariant derivatives is given by,

$$
D_i\phi_\pm \to iv(\partial_i\omega_\pm + eA_i \pm g a_i)e^{i\omega_\pm} \tag{6.59}
$$

Since $|D_i\phi_\pm|^2 \to 0$, we can obtain the asymptotic behavior of the gauge fields summing and subtracting these expressions, and given by the following pure gauge configurations:

$$
\begin{aligned}
A_i &\to -\frac{1}{2e}\partial_i(\omega_+ + \omega_-) \\
a_i &\to -\frac{1}{2g}\partial_i(\omega_+ - \omega_-).
\end{aligned}
\tag{6.60}
$$

With this asymptotic behavior, the condition in the magnetic fields will be immediately satisfied, and to accomplish the remaining conditions, it is sufficient to impose $A_0, a_0 \to 0$.

Let us discuss briefly what we have obtained here. Considering the static limit, the finite energy condition imposes an asymptotic behavior for the fields (6.57). In particular, the scalar fields have to obey the condition (6.58). For each direction in which we take the asymptotic limit, that is, for each angle θ parametrizing the sphere at infinity S^1_∞, the asymptotic fields will give essentially two $U(1)$ elements given by $e^{i\omega_+}$ and $e^{i\omega_-}$. Therefore, the asymptotic behavior of the scalar fields determines a function from the circle at infinity $S^1_\infty = \partial\mathbb{R}^2$ to the gauge group $U(1) \times U(1)$, that is,

$$\left(\phi_+^\infty(\theta), \phi_-^\infty(\theta)\right) : S^1_\infty \to U(1) \times U(1) \equiv S^1 \times S^1, \tag{6.61}$$

since topologically speaking $U(1)$ and S^1 are equivalents. These maps can be classified by homotopy classes, and in particular, the maps from the circle S^1 to the torus $S^1 \times S^1$ can be classified using two integers, since we have $\pi_1\left(S^1 \times S^1\right) = \mathbb{Z} \times \mathbb{Z}$. Mappings of different homotopy classes cannot be deformed in each other by a continuous transformation, and therefore give rise to inequivalent configurations. This is the topological origin of the stability of vortex solutions. Summarizing, we conclude that the finite energy condition imposes an asymptotic behavior for the fields, these asymptotic fields fall in different homotopy classes of configurations that cannot be deformed continuously in each other, and they can be classified here using 2 integer numbers, since we have $\phi_\pm^\infty : S^1_\infty \to S^1 \times S^1$ and we know that $\pi_1\left(S^1 \times S^1\right) = \mathbb{Z} \times \mathbb{Z}$.

6.3.2 Topological Vortices

Let us define a (n, m)- topological vortex as a finite energy static configuration obeying (6.57) with the following structure:

$$\begin{aligned}
&\phi_\pm \to v e^{i(m\pm n)\theta} \\
&A_i \to -\frac{m}{e}\partial_i\theta = -\frac{m}{er}\hat{\theta}_i \\
&a_i \to -\frac{n}{g}\partial_i\theta = -\frac{n}{gr}\hat{\theta}_i
\end{aligned} \tag{6.62}$$

where θ parametrizes the sphere at infinity as before, and we have $\vec{\nabla} = \hat{r}\frac{\partial}{\partial r} + \hat{\theta}\frac{1}{r}\frac{\partial}{\partial\theta}$.

Considering now the magnetic flux, we have,

$$\Phi = \int d^2x\, B = \int d^2x\, \epsilon^{ij}\partial_i A_j = \int_{S^1_\infty} dS\, \hat{r}_i\, \epsilon^{ij}\, A_j. \tag{6.63}$$

Now, taking into account the asymptotic behavior of the gauge fields since this integral is in the sphere at infinity and using the relation between spherical coordinates $\hat{\theta}_i = \epsilon_{ij}\hat{r}_j$, and the relation $\epsilon^{ij}\epsilon_{jk} = -\delta^i_k$ we have,

$$\Phi = \int_{S^1_\infty} r d\theta\, \hat{r}_i\, \epsilon^{ij}\left(-\frac{m}{er}\epsilon_{jk}\hat{r}_k\right) = \frac{2\pi}{e}m. \tag{6.64}$$

Performing the same steps, we find,

$$\chi = \frac{2\pi}{g} n. \tag{6.65}$$

Therefore, as expected from vortex solutions, their flux is quantized. And because of (6.55), their charges is also quantized:

$$Q = \frac{2\pi\mu}{g} n \quad G = \frac{2\pi\mu}{e} m. \tag{6.66}$$

6.3.3 Self-dual vortices

So far, we haven't used the fact that, with the help of the Gauss laws, the energy functional can be expressed as:

$$\begin{aligned} H = \int d^2x \Big[& \frac{1}{2}\left(\vec{E} \pm \vec{\nabla} N\right)^2 + \frac{1}{2}\left(\vec{e} \pm \vec{\nabla} M\right)^2 + |D_\pm \phi_+|^2 + |D_\mp \phi_-|^2 \\ & + \frac{1}{2}\left\{B \pm \left[e\left(|\phi_+|^2 - |\phi_-|^2\right) - \mu M\right]\right\}^2 + \frac{1}{2}(\partial_0 M)^2 \\ & + \frac{1}{2}\left\{b \pm \left[g\left(|\phi_+|^2 + |\phi_-|^2 - 2v^2\right) - \mu N\right]\right\}^2 + \frac{1}{2}(\partial_0 N)^2 \\ & + |D_0\phi_+ \mp i(eN + gM)\phi_+|^2 + |D_0\phi_- \mp i(eN - gM)\phi_-|^2 \\ & \pm 2gbv^2 \Big] \end{aligned} \tag{6.67}$$

Which in turn implies the inequality:

$$H \geq 2v^2 |g\chi|; \qquad \chi \equiv \int d^2x b \tag{6.68}$$

From (6.67) and (6.68) we have a straightforward way of making the energy finite by saturating the inequality (6.68) imposing the self-dual conditions:

$$D_\pm \phi_+ = 0 \tag{6.69}$$

$$D_\mp \phi_- = 0 \tag{6.70}$$

$$D_0\phi_+ \mp i(eN + gM)\phi_+ = 0 \tag{6.71}$$

$$D_0\phi_- \mp i(eN - gM)\phi_- = 0 \tag{6.72}$$

$$\partial_0 M = \partial_0 N = 0 \tag{6.73}$$

$$\vec{E} \pm \vec{\nabla} N = 0 \tag{6.74}$$

$$\vec{e} \pm \vec{\nabla} M = 0 \tag{6.75}$$

$$B \pm \left[e\left(|\phi_+|^2 - |\phi_-|^2\right) - \mu M\right] = 0 \tag{6.76}$$

$$b \pm \left[g\left(|\phi_+|^2 + |\phi_-|^2 - 2v^2\right) - \mu N\right] = 0 \tag{6.77}$$

A configuration satisfying these conditions has energy $H = 2v^2|g\chi|$. The upper (lower) sign corresponds to positive (negative) χ flux. Let us work out the implications of such equations for static ($\partial_0 \equiv 0$) configurations. Equations (6.73) are now trivially satisfied. Equations (6.71),(6.72),(6.74) and(6.75) are satified by taking:

$$\begin{aligned} A_0 &= \pm N, \\ a_0 &= \pm M \end{aligned} \tag{6.78}$$

Equations (6.69) and (6.70) can be rewritten as:

$$\begin{aligned} D_i\phi_+ &= \pm i\epsilon_{ij} D_j \phi_+ \\ D_i\phi_- &= \mp i\epsilon_{ij} D_j \phi_- \end{aligned} \tag{6.79}$$

Taking $\phi_+ = |\phi_+|e^{i\omega_+}$ and $\phi_- = |\phi_-|e^{i\omega_-}$, their combined effect results in:

$$\begin{aligned} eA_i &= \pm\frac{1}{2}\epsilon_{ij}\partial_j \ln\frac{|\phi_+|}{v} \mp \frac{1}{2}\epsilon_{ij}\partial_j \ln\frac{|\phi_-|}{v} - \frac{1}{2}\partial_i(\omega_+ + \omega_-) \\ ga_i &= \pm\frac{1}{2}\epsilon_{ij}\partial_j \ln\frac{|\phi_+|}{v} \pm \frac{1}{2}\epsilon_{ij}\partial_j \ln\frac{|\phi_-|}{v} - \frac{1}{2}\partial_i(\omega_+ - \omega_-) \end{aligned} \tag{6.80}$$

Where we have inserted, without any physical effect, the expectation value v inside the logarithms in order to make their argument dimensionless. Equations (6.80) then allow us define the spatial components of the vector potentials everywhere away from the zeroes of ϕ_+ and ϕ_-. Now acting on (6.80) with $\epsilon^{ki}\partial_i$ and remember that the magnetic and g-magnetic fields are defined by $B = \epsilon^{ki}\partial_k A_i$ and $b = \epsilon^{ki}\partial_k a_i$, we get:

$$\begin{aligned} eB &= \mp\frac{1}{2}\nabla^2 \ln\frac{|\phi_+|}{v} \pm \frac{1}{2}\nabla^2 \ln\frac{|\phi_-|}{v} \\ gb &= \mp\frac{1}{2}\nabla^2 \ln\frac{|\phi_+|}{v} \mp \frac{1}{2}\nabla^2 \ln\frac{|\phi_-|}{v} \end{aligned}$$

Substitution of these into (6.76) and (6.77) then results in:

$$\begin{aligned} \mp\frac{1}{2}\nabla^2 \ln\frac{|\phi_+|}{v} \pm \frac{1}{2}\nabla^2 \ln\frac{|\phi_-|}{v} &= \mp e\left[e\left(|\phi_+|^2 - |\phi_-|^2\right) - \mu M\right] \\ \mp\frac{1}{2}\nabla^2 \ln\frac{|\phi_+|}{v} \mp \frac{1}{2}\nabla^2 \ln\frac{|\phi_-|}{v} &= \mp g\left[g\left(|\phi_+|^2 + |\phi_-|^2 - 2v^2\right) - \mu N\right] \end{aligned}$$

After a bit of rearrangement:

$$
\begin{aligned}
\nabla^2 \ln \frac{|\phi_+|}{v} &= \left[(e^2+g^2)|\phi_+|^2 - (e^2-g^2)|\phi_-|^2 - \mu\left(eM + gN + \frac{2v^2 g}{\mu}\right)\right] \\
\nabla^2 \ln \frac{|\phi_-|}{v} &= \left[(e^2+g^2)|\phi_-|^2 - (e^2-g^2)|\phi_+|^2 - \mu\left(-eM + gN + \frac{2v^2 g}{\mu}\right)\right]
\end{aligned}
\tag{6.81}
$$

These need to be satisfied for the self-dual topological solutions together with the Gauss laws, which we have made use of in order to put the energy functional in the form (6.7). Remembering:

$$
\begin{aligned}
\vec{\nabla} \cdot \vec{E} + \mu b &= e(\rho_+ + \rho_-) \\
\vec{\nabla} \cdot \vec{e} + \mu B &= g(\rho_+ - \rho_-)
\end{aligned}
$$

Where $\rho_\pm = J^0_\pm = i\left[\phi^*_\pm D^0 \phi_\pm - D^0 \phi^*_\pm \phi_\pm\right]$. Therefore, from (6.71),(6.72),(6.74),(6.75),(6.76) and (6.77), we get:

$$
\begin{aligned}
\nabla^2 N - \mu^2 N &= -\mu\left[g\left(|\phi_+|^2 + |\phi_-|^2 - 2v^2\right)\right] + 2e\left[(eN+gM)|\phi_+|^2 + (eN-gM)|\phi_-|^2\right] \\
\nabla^2 M - \mu^2 M &= -\mu\left[e\left(|\phi_+|^2 - |\phi_-|^2\right)\right] + 2g\left[(eN+gM)|\phi_+|^2 - (eN-gM)|\phi_-|^2\right]
\end{aligned}
$$

Which is nothing other than exactly the static limit of the equations of motion (6.6) for M and N. For future use, it is helpful to write explicity the charge densities:

$$
\begin{aligned}
e(\rho_+ + \rho_-) &= \mp 2e\left[(eN+gM)|\phi_+|^2 + (eN-gM)|\phi_-|^2\right] \\
g(\rho_+ - \rho_-) &= \mp 2g\left[(eN+gM)|\phi_+|^2 - (eN-gM)|\phi_-|^2\right]
\end{aligned}
\tag{6.82}
$$

Now that the dust has settled, the set of equations we actually need to solve turns out to be these four:

$$
\begin{aligned}
\nabla^2 \ln \frac{|\phi_+|}{v} &= \left[(e^2+g^2)|\phi_+|^2 - (e^2-g^2)|\phi_-|^2 - \mu\left(eM + gN + \frac{2v^2 g}{\mu}\right)\right] \\
\nabla^2 \ln \frac{|\phi_-|}{v} &= \left[(e^2+g^2)|\phi_-|^2 - (e^2-g^2)|\phi_+|^2 - \mu\left(-eM + gN + \frac{2v^2 g}{\mu}\right)\right] \\
\nabla^2 N - \mu^2 N &= -\mu\left[g\left(|\phi_+|^2 + |\phi_-|^2 - 2v^2\right)\right] + 2e\left[(eN+gM)|\phi_+|^2 + (eN-gM)|\phi_-|^2\right] \\
\nabla^2 M - \mu^2 M &= -\mu\left[e\left(|\phi_+|^2 - |\phi_-|^2\right)\right] + 2g\left[(eN+gM)|\phi_+|^2 - (eN-gM)|\phi_-|^2\right]
\end{aligned}
\tag{6.83}
$$

6.3.4 Vortex Ansatz

Let us consider here the following radially symmetric *ansatz* for the scalar fields:

$$\begin{aligned} \phi_{\pm}(r,\theta) &= v\, F_{\pm}(r)\, e^{i(m\pm n)\theta}, \\ N(r,\theta) &= v\, \hat{N}\,(r)\,, \\ M(r,\theta) &= v\, \hat{M}\,(r)\,, \end{aligned} \tag{6.84}$$

where $m \pm n \in \mathbb{Z}$, and the profiles $F_{\pm}$, $\hat{N}$, and $\hat{M}$ are dimensionless. Plugging the *ansatz* above in Eqs. (6.80), we can obtain the gauge structure (here $\hat{\theta}_i = \epsilon_{ij} x^j / r$):

$$\begin{aligned} A_i(r,\theta) &= \frac{1}{er}\,[A(r) - m]\,\hat{\theta}_i, \\ a_i(r,\theta) &= \frac{1}{gr}\,[a(r) - n]\,\hat{\theta}_i, \end{aligned} \tag{6.85}$$

where we defined the gauge profiles as:

$$\begin{aligned} A(r) &= \pm\frac{1}{2}\left(\frac{rF'_{+}}{F_{+}} - \frac{rF'_{-}}{F_{-}}\right) \\ a(r) &= \pm\frac{1}{2}\left(\frac{rF'_{+}}{F_{+}} + \frac{rF'_{-}}{F_{-}}\right), \end{aligned} \tag{6.86}$$

or, equivalently:

$$\begin{aligned} F'_{+} &= \pm\,\frac{F_{+}\,(A+a)}{r}, \\ F'_{-} &= \mp\,\frac{F_{-}\,(A-a)}{r}. \end{aligned} \tag{6.87}$$

It should be stressed that, although the gauge field structure above (6.85) is the same of the last chapter, here it does not appear as an independent *ansatz* for the gauge fields, but it has its structure totally determined by the scalar fields *ansatz*, and as a consequence of the self-dual equations obtained by saturating the Bogomol'nyi bound.

Let us first discuss the profiles behavior at the origin. Looking at the gauge structure (6.85), in order to avoid a singularity at the origin, we must have $A(0) = m$ and $a(0) = n$. Using Eq.(6.87) we see that they need to satisfy

$$\begin{aligned} (n+m)\,F_{+}(0) &= 0, \\ (n-m)\,F_{-}(0) &= 0. \end{aligned} \tag{6.88}$$

These considerations imply the following behavior:

$$\begin{cases} F_+(r) \approx r^{\pm(n+m)} \\ F_-(r) \approx r^{\pm(n-m)} \end{cases} \quad \text{as } r \to 0. \tag{6.89}$$

Therefore, to ensure that the fields have a regular behavior at the origin, we must have $\pm n > |m|$. Notice that if we take $n = 0$, we cannot ensure a regular behavior at the origin for both fields simultaneously, unless we also set $m = 0$, in which case $F_+(0)$ and $F_-(0)$ remain undetermined. Finally, if we consider $n = -m \neq 0$, then $F_+(0)$ is undetermined while $F_-(0) = 0$; if we consider $n = m \neq 0$, then $F_-(0)$ is undetermined while $F_+(0) = 0$. It should be noted that the behavior of $\hat{N}$ and $\hat{M}$ near the origin will follow from their equations of motion, once the behavior of F_+ and F_- for small r are determined.

Now, we proceed to the discussion of the asymptotic conditions. The energy contribution coming from the potential implies that for any finite-energy configurations, we must have $F_+(\infty)$ and $F_-(\infty)$ equal to 0 or 1. Furthermore, the covariant derivatives contribution to the energy functional include the following terms:

$$E \supset 2\pi v^2 \int dr \left[\frac{F_+^2 \left(A+a\right)^2}{r} + \frac{F_-^2 \left(A-a\right)^2}{r} \right] \tag{6.90}$$

Therefore, from the finite-energy condition, we find the following asymptotic conditions:

$$\begin{aligned} &[A(\infty) + a(\infty)]\, F_+(\infty) = 0, \\ &[A(\infty) - a(\infty)]\, F_-(\infty) = 0. \end{aligned} \tag{6.91}$$

First of all, let us consider the case in which the scalar profiles asymptote to the $(1,1)$-vacuum, that is, when $F_+(\infty) = F_-(\infty) = 1$. In this case, we are dealing with topological vortices, and we must have $A(\infty) = a(\infty) = 0$. These configurations have quantized magnetic fluxes ($\Phi = \frac{2\pi}{e} m$ and $\chi = \frac{2\pi}{g} n$), charges ($Q = \frac{2\pi}{g}\mu n$ and $G = \frac{2\pi}{e}\mu m$) and energy ($E = 2gv^2|\chi| = 4\pi v^2|n|$).

Furthermore, we consider the case in which we asymptote to the $(0,0)$-vacuum, that is, when we have $F_+(\infty) = F_-(\infty) = 0$. In this case, we can generically assume:

$$\begin{cases} F_+(r) \approx \frac{1}{r^{\pm(\alpha+\beta)}} \\ F_-(r) \approx \frac{1}{r^{\pm(\alpha-\beta)}} \end{cases} \quad \text{as } r \to \infty \text{ and with } \pm\alpha > |\beta|; \tag{6.92}$$

Which in its turn implies $a(\infty) = -\alpha$ and $A(\infty) = -\beta$. In this case, we get for the fluxes:

(n,m)	$(\frac{1}{2},\frac{1}{2})$		$(1,0)$		$(\frac{3}{2},\frac{1}{2})$		$(0,0)$
	NT	T	NT	T	NT	T	NT
$E(1/4\pi v^2)$	5.38	0.50	5.76	1.00	8.53	1.50	3.25
$J(eg/2\pi\mu)$	-4.78	0.25	0.81	0.00	0.75	0.75	0.75
$\Phi(e/2\pi)$	1.53	0.50	-0.17	0.00	0.50	0.50	-0.23
$\chi(g/2\pi)$	5.38	0.50	5.76	1.00	8.53	1.50	3.25

Table 6.1: Physical properties of topological vortices (T) and non-topological solitons (NT) for different values of n and m.

$$\chi = \int d^2x b = \frac{2\pi}{g}(n+\alpha);$$
$$\Phi = \int d^2x B = \frac{2\pi}{e}(m+\beta); \tag{6.93}$$

That is, they no longer need to be quantized. These are known as non-topological vortices. The case $\alpha = 0$ makes sense only if $\beta = 0$, and we fall back to the situation with $F_+(\infty) = F_-(\infty) = 1$. There is also the hybric case where, for example, $F_-(\infty) = 1$ while $F_+(\infty) = 0$, which we can consider by taking $\beta = \alpha$, which is also a non-topological vortex; the reversed situation being simply $\beta = -\alpha$. These will not be detailed here, because we are only concerned with the parity-preserving scenario.

This concludes all the acceptable asymptotic behaviors compatible with the requirement of the energy being finite for self-dual topological solutions. Any of them, being self-dual, saturate the energy bound, meaning that their energy is equal to:

$$E = 4\pi v^2 |n+\alpha| \tag{6.94}$$

The angular momentum of these configurations is given by $J = \int d^2x\, \epsilon^{ij} r_i T_{0j}$. In Appendix D, we found the following expression for the angular momentum of the finite-energy, static, rotationally symmetric vortices: $J = \frac{2\pi\mu}{eg}\left[A(0)a(0) - A(\infty)a(\infty)\right]$. Nothing's changed here. We have $A(0) = m, a(0) = n$ and also $A(\infty) \equiv -\beta, a(\infty) \equiv -\alpha$. Thus, we can rewrite this expression in the following way:

$$J = \frac{2\pi\mu}{eg}\left(n\,m - \alpha\,\beta\right) = \frac{QG}{2\pi\mu} - \frac{Q}{e}\beta - \frac{G}{g}\alpha. \tag{6.95}$$

This is in agreement with the result found in Ref.[88].

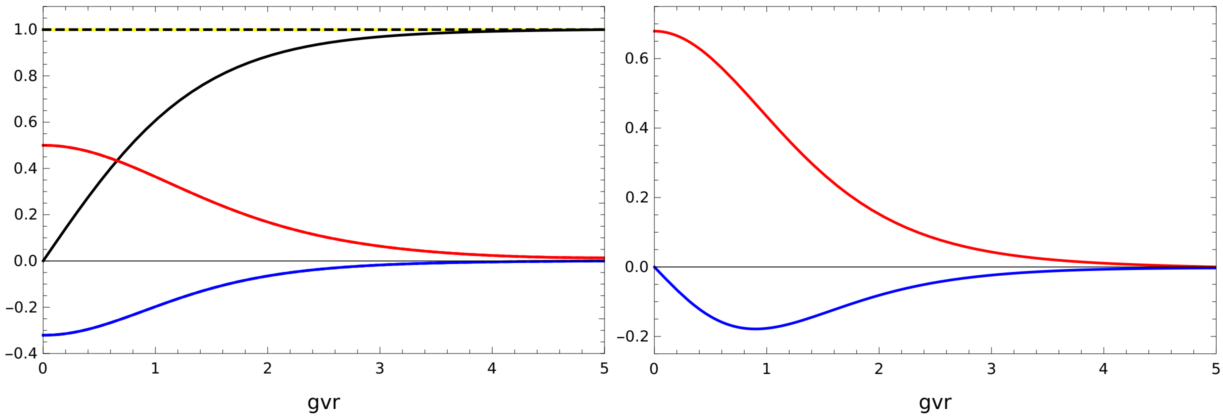

Figure 6.1: Topological vortex solution for $n = m = 1/2$ and its physical fields in units of gv^2 as functions of $x = gv\,r$. *Left figure*: F_+ and F_- are shown in solid and dashed black, $N = M$ in blue, and $A = a$ in red, respectively. *Right figure*: In red, the magnetic field; in blue, the electric field. Notice that in this case we have $B = b$ and $E_r = e_r$.

6.4 Explicit solutions and discussion

In this section, we exhibit explicit numerical solutions for the self-duality equations. First of all, we rewrite the differential equations using dimensionless quantities given by $x = gv\,r$, $\gamma = \mu/gv$ and $\kappa = e/g$. After the dust has settled, the differential equations are:

$$\begin{aligned}
\nabla_x^2 \ln F_+^2 &= (1+\kappa^2)F_+^2 + (1-\kappa^2)F_-^2 - \gamma\kappa\hat{M} - \gamma\kappa\hat{N} - 2, \\
\nabla_x^2 \ln F_-^2 &= (1+\kappa^2)F_-^2 + (1-\kappa^2)F_+^2 + \gamma\kappa\hat{M} - \gamma\kappa\hat{N} - 2, \\
\nabla_x^2 \hat{N} &= -\gamma\left(F_+^2 + F_-^2 - 2\right) + 2\kappa^2\hat{N}\left(F_+^2 + F_-^2\right) \\
&\quad + 2\kappa\hat{M}\left(F_+^2 - F_-^2\right) + \gamma^2\hat{N}, \\
\nabla_x^2 \hat{M} &= -\gamma\kappa\left(F_+^2 - F_-^2\right) + 2\hat{M}\left(F_+^2 + F_-^2\right) \\
&\quad + 2\kappa\hat{N}\left(F_+^2 - F_-^2\right) + \gamma^2\hat{M},
\end{aligned} \tag{6.96}$$

The general strategy adopted here is the following: we expand the profile functions $F_+, F_-, \hat{N}, \hat{M}$ in powers of x around the origin, using the generic notation $A(x) = \sum_k A_k x^k$. Applying these expansions in the differential equations and using the initial conditions, we can find constraints in the expansion coefficients. With these expressions at hand, we can search for numerical solutions that also satisfy the asymptotic boundary conditions using a shooting method. In general lines, for the differential equations and initial conditions considered here, there are only 4 free parameters to be numerically determined by demanding the appropriate boundary conditions at infinity.

In the following, we consider some examples with the lowest possible values for n and m that represent each possible class of solutions. The topological vortices (asymptoting

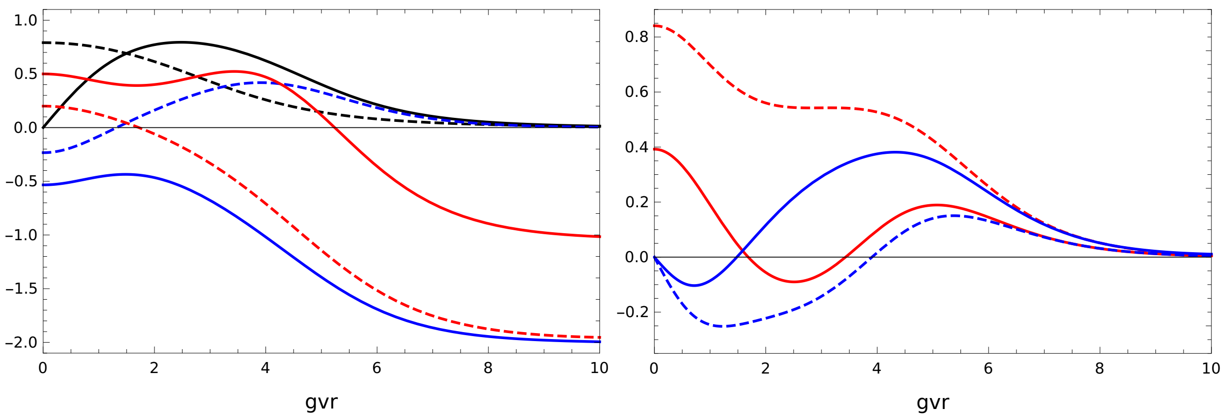

Figure 6.2: Non-topological soliton for $n = 1/2, m = 1/2$ and its physical fields in units of gv^2, as functions of $x = gv\,r$. *Left figure*: F_+ and F_- are shown in solid and dashed black, N and M in solid and dashed blue, A and $0.4a$ in solid and dashed red, respectively (a was rescaled to facilitate the visualization). Here we have $\beta \simeq 1.03$ and $\alpha \simeq 4.88$. *Right figure*: The magnetic (solid red), g-magnetic (dashed red), electric (solid blue) and g-electric (dashed blue) fields.

to the $(1,1)$-vacuum) and non-topological solitons (asymptoting to the $(0,0)$-vacuum), with its physical fields (*i.e.*, electric, magnetic, g-electric and g-magnetic), for the cases $(n,m) = (\frac{1}{2},\frac{1}{2}),(1,0),(\frac{3}{2},\frac{1}{2})$ and $(0,0)$ are shown in Figs. 6.1, 6.2, 6.3, 6.4, 6.5, 6.6, 6.7, respectively. Their relevant physical properties are shown in Table 6.1. The charges are not shown there, but can immediately be found remembering that $Q = \mu\chi$ and $G = \mu\Phi$. Here we adopt $\gamma = \kappa = 1$ for simplicity, but in the end of this section we comment about the relevant changes in the solutions when we vary these coefficients.

The topological vortices have quantized physical properties while non-topological solitons do not, and the later have energy bigger than the former. The angular momentum for topological vortices is quantized, proportional to the product of the charges and fractional, exhibiting an anyonic nature. For $n = m = 0$, the only solution asymptoting to the $(1,1)$-vacuum is the trivial one.

The multiplicity of zeroes of the scalar field is related to the winding number of the vortex. Therefore, the power-law behavior of F_+ and F_- in Eq. (6.89) clearly indicates that the true winding numbers are given by $n+m$ and $n-m$, instead of m and n separately, as is clearly illustrated in the explicit solutions that we found.

The most distinctive signature of a symmetry in a system is the presence of a degeneracy in the spectrum, hence it is reasonable to expect that the parity invariance of our model should reproduce this effect. To this end, we state how the vortex solutions change under parity transformations: $(n,m) \to (n,-m)$, $r \to r, \theta \to -\theta - \pi, F_\pm \to F_\mp, M \to -M, N \to N, \beta \to -\beta, \alpha \to \alpha$, being all the others directly inferred from the self-duality equations.

Considering the self-dual topological vortices, that is, that satisfy $E \propto |n|$, it is imme-

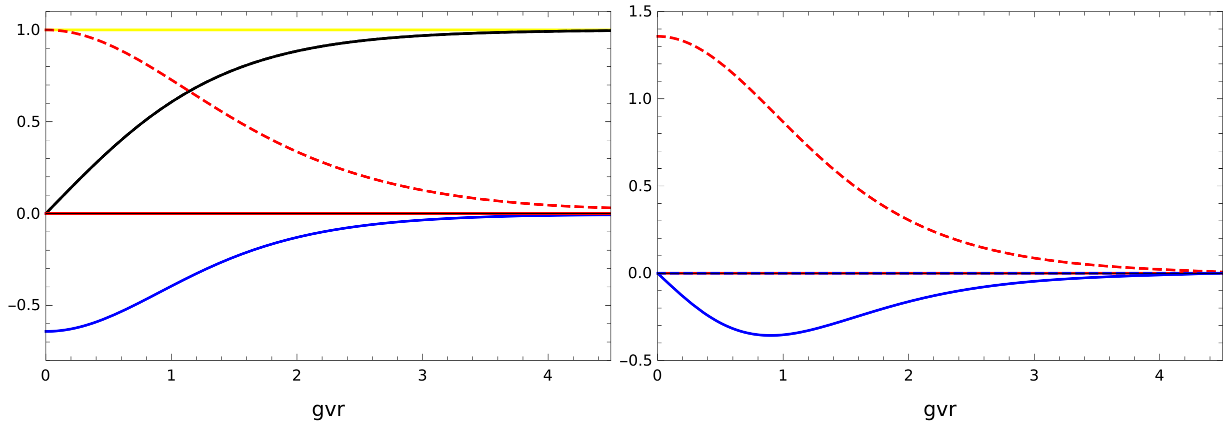

Figure 6.3: Topological vortex for $n = 1, m = 0$ and its physical fields in units of gv^2, as functions of $x = gv\,r$. *Left figure*: $F_+ = F_-$ are shown in black, N and M in solid and dashed blue, A and a in solid and dashed red, respectively. Here we have $A = M = 0$. *Right figure*: In red, the g-magnetic field; in blue, the electric field. Here have $B = e_r = 0$.

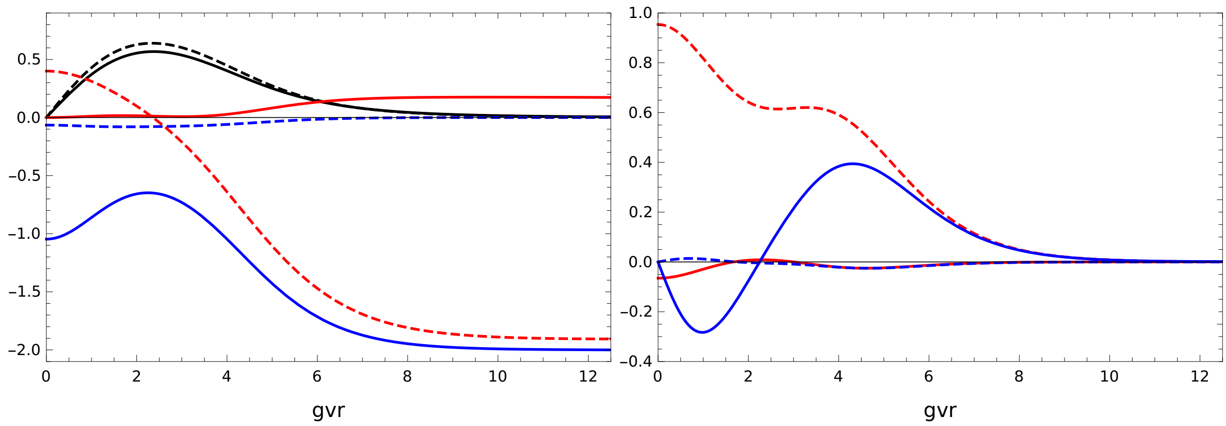

Figure 6.4: Non-topological soliton for $n = 1, m = 0$ and its physical fields in units of gv^2, as functions of $x = gv\,r$. *Left figure*: F_+ and F_- are shown in solid and dashed black, N and M in solid and dashed blue, A and $0.4a$ in solid and dashed red, respectively (a was rescaled to facilitate the visualization). Here we have $\beta \simeq -0.17$ and $\alpha \simeq 4.76$. *Right figure*: The magnetic (solid red), g-magnetic (dashed red), electric (solid blue), and g-electric (dashed blue) fields.

diate to conclude that a given solution and its parity-transformed version have the same energy. But the complete independence of the energy from m suggests a much greater degeneracy. In fact, from the condition of regularity of the solutions as $r \to 0$, we observed that $\pm n \geq |m|$, which in turn implies that, for $n > 0$ ($n < 0$) there are $2n + 1$ ($2|n| + 1$) solutions of the same energy. Since the energy does not depend on the sign of n, we obtain a $2(2|n| + 1)$-fold degeneracy. It is reasonable to speculate whether this comes from a larger symmetry group. In the light of previous comments, a good candidate would be supersymmetry or, given the structure of the degeneracy, an internal $SU(2)$. This investigation should be pursued elsewhere. The above discussion does not apply to

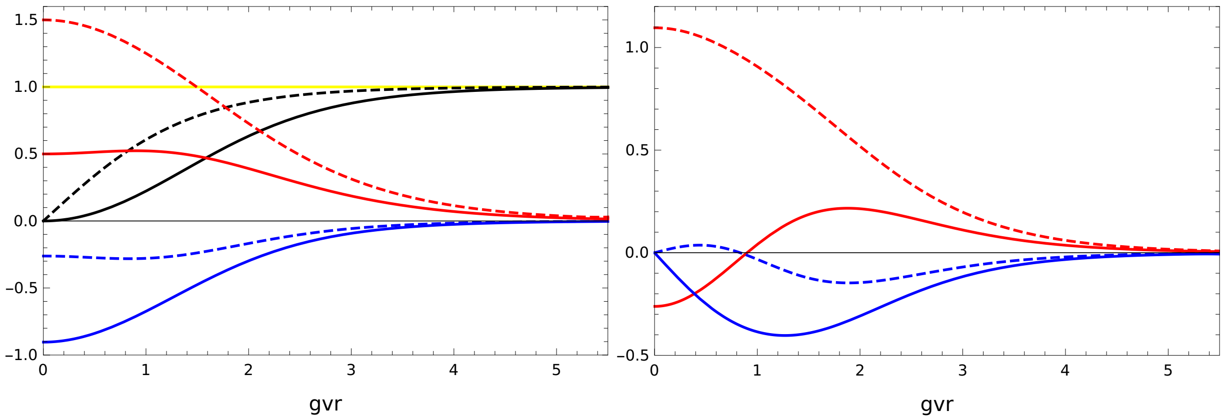

Figure 6.5: Topological vortex for $n = 3/2, m = 1/2$ and its physical fields in units of gv^2, as functions of $x = gv\,r$. *Left figure*: F_+ and F_- are shown in solid and dashed black, N and M in solid and dashed blue, A and a in solid and dashed red, respectively. *Right figure*: The magnetic (solid red), g-magnetic (dashed red), electric (solid blue), and g-electric (dashed blue) fields.

the non-topological solitons.

In Part II, we studied the energies of different vortices, obtaining the following result: $M_{(1/2,1/2)} + M_{(1/2,-1/2)} = 2M_{(1/2,1/2)} > M_{(1,0)}$, where $M_{(n,m)}$ is the mass associated with the (n,m)- topological vortex. The left-hand side of the inequality represents the static energy of well-separated F_+ and F_- vortices of winding 1, while the right-hand side is their energy when superimposed at the origin. Therefore, the inequality suggested a possible attraction between these vortices. Now, in the self-dual model studied here, on the other hand, $2M_{(1/2,1/2)} = M_{(1,0)}$, indicating that these vortices do not interact with each other, allowing, for example, the existence of static multi-vortex configurations, as it is usually the case for self-dual models.

In this section we considered $\gamma = \kappa = 1$ for simplicity, but the existence of solitons here is not conditioned to this assumption, and we were able to find solutions for different values of these coefficients. Interestingly enough, keeping κ fixed and increasing γ, we see that the magnetic field at the origin decreases; decreasing γ, the magnetic field increases, cf. Fig. 6.8. Since $\gamma \propto \mu$, this suggests that it would reach a maximum value in the pure Maxwell limit and go to zero in the pure CS limit, as it happens in the usual Maxwell-CS case [34]. It is well-known that in the absence of a CS term, the vortices are electrically neutral, therefore having zero electric field. In fact, we observed that in decreasing γ, the maximum value of the electric field diminished, in accordance with what is expected. Furthermore, keeping γ fixed and considering $\kappa \neq 1$, we can see that for $n = m = 1/2$, the electric and magnetic fields will not be degenerate anymore, cf. Fig. 6.8. In the other examples considered, taking $\kappa \neq 1$ does not lead to significant qualitative changes.

Finally, we also found solitons asymptoting to the parity-breaking $(1,0)$- and $(0,1)$-vacua. Given the rich vacuum structure of this theory, in principle, one could also find

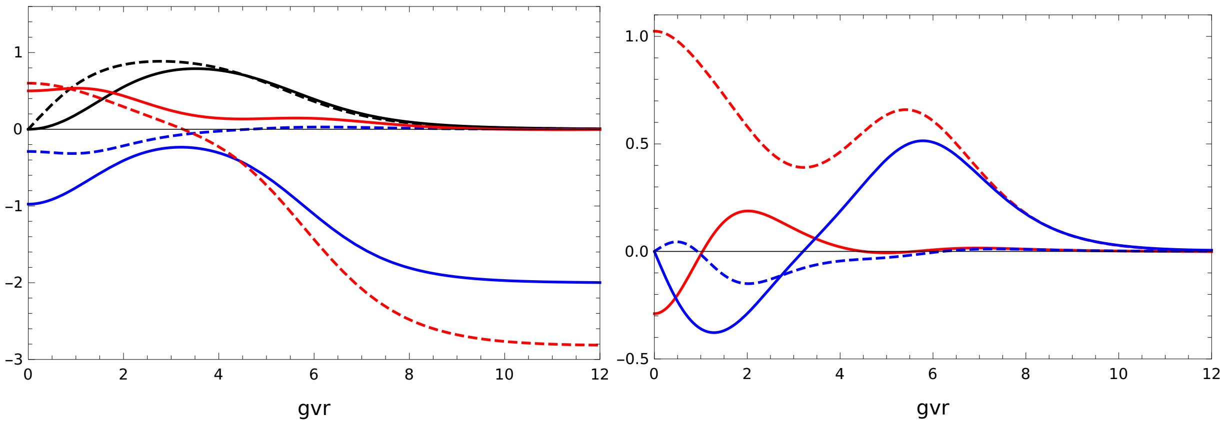

Figure 6.6: Non-topological soliton for $n = 3/2, m = 1/2$ and its physical fields in units of gv^2, as functions of $x = gv\,r$. *Left figure*: F_+ and F_- are shown in solid and dashed black; N and M in solid and dashed blue; A and $0.4a$ in solid and dashed red, respectively (a was rescaled to facilitate the visualization). Here we have $\beta \simeq 0.00$ and $\alpha \simeq 7.03$. *Right figure*: The magnetic (solid red), g-magnetic (dashed red), electric (solid blue) and g-electric (dashed blue) fields

domain walls connecting any pair of degenerate vacua. These last were not discussed here for reasons of scope, but Fig. 6.9 exhibits one solution asymptoting to the $(1, 0)$-vacuum. As one might have noticed, some of our solutions display an intriguing oscillating behavior for the electric and magnetic fields. This is due to our choice to work with the field variables A_μ and a_μ. In fact, in terms of the previously defined A^+_μ and A^-_μ, the electric and magnetic fields display a much more familiar behavior, as one can see for example in Fig. 6.10. This "improvement" consistently happened to every solution, at least when $\kappa = e/g = 1$, which is reasonable in the light of the interpretation we have already given to A^+_μ and A^-_μ.

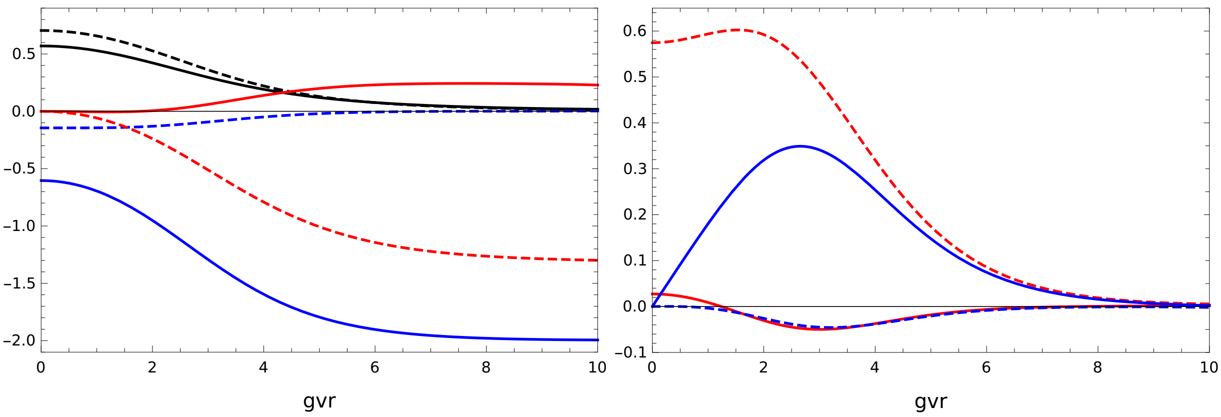

Figure 6.7: Non-topological soliton for $n = m = 0$ and its physical fields in units of gv^2, as functions of $x = gv\,r$. *Left figure*: F_+ and F_- are shown in solid and dashed black; N and M in solid and dashed blue; A and $0.4a$ in solid and dashed red, respectively (a was rescaled to facilitate the visualization). Here we have $\beta \simeq -0.23$ e $\alpha \simeq 3.25$. *Right figure*: The magnetic (solid red), g-magnetic (dashed red), electric (solid blue) and g-electric (dashed blue) fields.

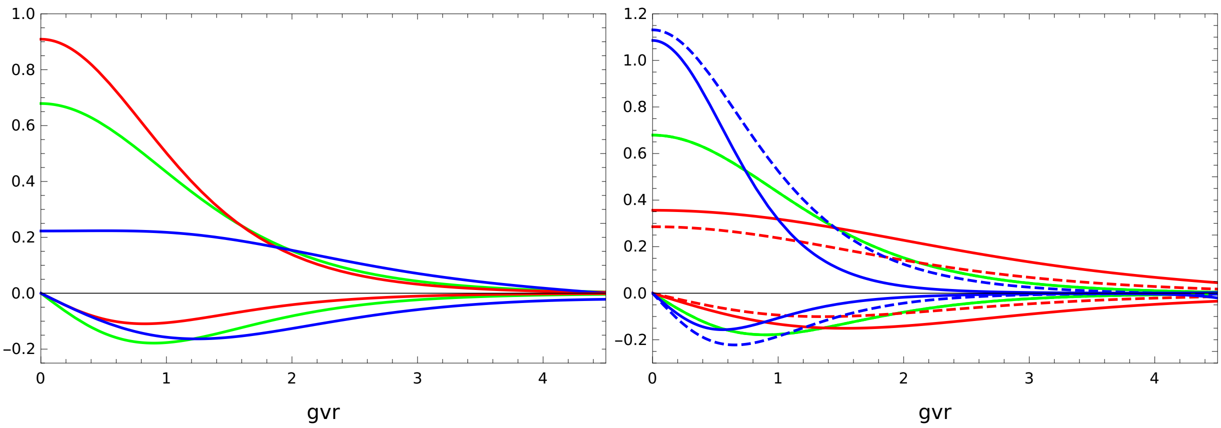

Figure 6.8: Physical fields associated with the $n = m = 1/2$ topological vortex. *Left figure*: The electric (lower half-plane) and magnetic (upper half-plane) fields for $\kappa = 1$ and $\gamma = 1$ (green), 0.5 (red), 2 (blue). *Right figure*: The magnetic, g-magnetic, electric and g-electric fields, for $\gamma = 1$ and $\kappa = 1$ (green), 0.5 (red), 2 (blue). The solid lines refer to B and E_r; the dashed lines to b and e_r. B and b are shown in the upper half-plane; E_r and e_r in the lower half-plane.

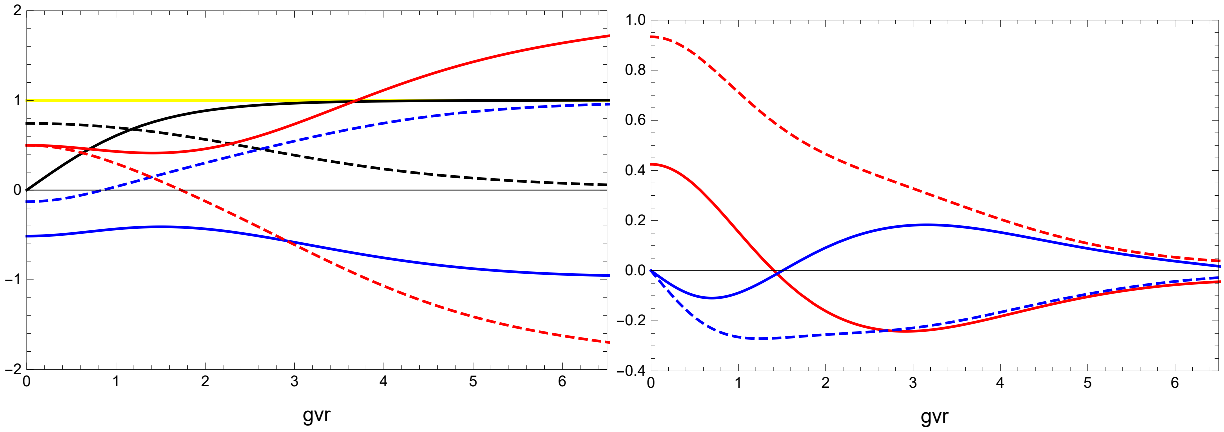

Figure 6.9: Non-topological soliton asymptoting to (1,0)-vacuum for $n = m = 1/2$ and its physical fields in units of gv^2, as functions of $x = gv\,r$. *Left figure*: F_+ and F_- are shown in solid and dashed black; N and M in solid and dashed blue; A and a in solid and dashed red, respectively . Here we have $\beta \simeq -1.79$ e $\alpha \simeq 1.76$. *Right figure*: The magnetic (solid red), g-magnetic (dashed red), electric (solid blue) and g-electric (dashed blue) fields.

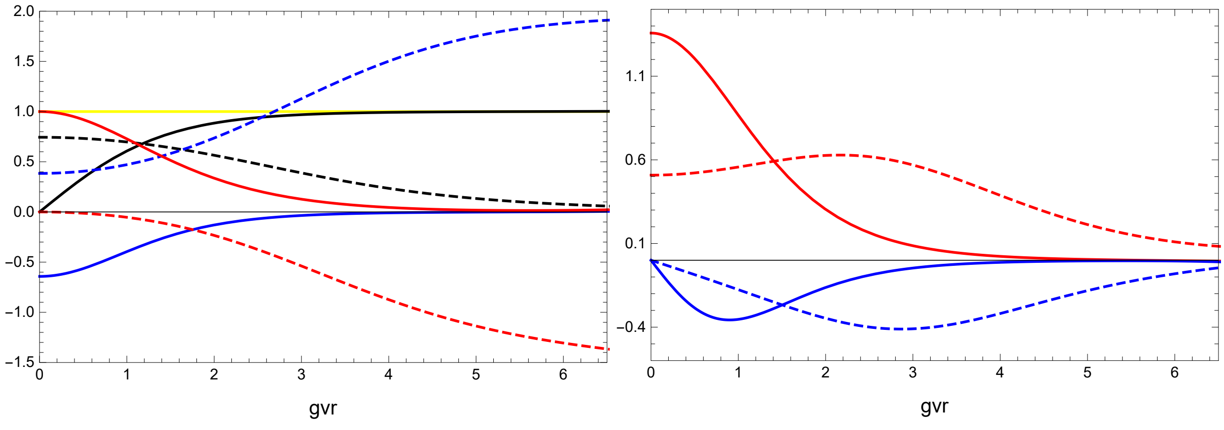

Figure 6.10: Non-topological soliton asymptoting to (1,0)-vacuum for $n = m = 1/2$ and its physical fields in units of gv^2, as functions of $x = gv\,r$. *Left figure*: F_+ and F_- are shown in solid and dashed black; $N + M$ and $M - N$ in solid and dashed blue; $A + a$ and $0.4(a - A)$ (rescaled to facilitate the visualization) in solid and dashed red, respectively . *Right figure*: $B + b$ (solid red), $b - B$ (dashed red), $E + e$ (solid blue) and $e - E$ (dashed blue) fields.

Part IV

$\mathcal{N} = 2$ supersymmetric Maxwell-Chern-Simons model with parity conservation

Chapter 7

Introduction

So far, we have only flirted with the idea of a supersymmetric origin for our self-dual model. Some clues to this were the fact that the only parameters appearing in the potential (6.4) were the gauge couplings e, g, and the expectation value v; the strict positivity of the potential; and, more evidently, the several mass degeneracies both in the spectrum around the vacua as well as for the masses of the topological vortices. This part of the thesis is dedicated to proving that, indeed, our self-dual model corresponds to the bosonic sector of an $\mathcal{N} = 2$ supersymmetric model. As we will briefly discuss, although an interesting and beautiful result, this is hardly a surprise.

In 1977, Di Vecchia and Ferrara [45], followed by Witten and Olive in 1978 [46], considered a number of 1+1 and 3+1 dimensional theories exploring the relationship between supersymmetry and self-dual first order equations. In particular, [46] evidenced the extended nature of this supersymmetry as well as the presence of a central charge equal to the topological charge in the models considered. As a consequence, a Bogomol'nyi-like bound for the static energy (mass) was found. The classical configurations saturating such bound were precisely the self-dual solitons.

This topic would return in the early 90's, now in 2+1 dimensions. In 1990, the self-dual Higgs-Chern-Simons [36, 37, 38] model with its particular sixth order potential and the Maxwell-Chern-Simons [39] self-dual model were first derived from the requirement of a Bogolmol'nyi bound. Soon after, supersymmetry was brought into the scene demonstrating that both pure CS [47, 48, 49] and Maxwell-CS [50] self-dual models were part of a larger extended supersymmetric theory. As highlighted in [47], the imposition of $\mathcal{N} = 2$ extended supersymmetry is sufficient to completely determine the potential. The elegant derivation in [49] using $\mathcal{N} = 2$ superspace makes it even more evident. Another approach was used in [47], where a second set of supersymmetry transformations was imposed on a manifest $\mathcal{N} = 1$ SUSY invariant theory, one more time completely determining the potential. We shall make use of both these approaches.

The success of all this derivations was then proved to be no coincidence. From 1992 to 1993, Hlousek and Spector demonstrated why $\mathcal{N} = 1$ supersymmetry plus a topologically conserved current implies $\mathcal{N} = 2$ supersymmetry with central charge equal to topological charge, together with a derivation of the Bogomol'nyi bound and self-dual equations from the extended SUSY algebra. [127, 128, 129].

The successful derivation of self-dual models from supersymmetry, now appropriately justified, kept happening. In 1994, it was the familiar self-dual Maxwell-Higgs model that was shown to be part of a bigger supersymmetric puzzle [130]. Starting from an $\mathcal{N} = 1$ invariant theory, it was shown that a particular relation between the coupling constants should hold in order to extend it to $\mathcal{N} = 2$, the same relation needed for the existence of a Bogomol'nyi bound.

One last interesting example, and evidently not exhausting the list, is the Maxwell CS theory with magnetic moment interaction [42, 43, 44], which was shown to admit self-dual topological configurations when the couplings were appropriately chosen. Then, it was given a supersymmetric framework in two different ways. First, in 1996 [131], imposing the appropriate relations for the couplings in an $\mathcal{N} = 1$ Susy invariant theory, thus allowing it to be extended to $N = 2$. Secondly, in 1999 via dimensional reduction[132, 133] from $\mathcal{N} = 1, D = 3 + 1$ to $\mathcal{N} = 2, D = 2 + 1$.

It becomes clear that seeking an supersymmetric extension to our model is nothing but natural. Moreover, the importance of such investigation lies beyond the simple (and necessary) exploration of the known relationship between supersymmetry and self-duality. Indeed, condensed-matter physics can already provide several instances where supersymmetry, for example, it is known to have applications in graphene physics [134, 135, 136, 137] and to dynamically emerge in condensed matter systems [138, 139, 140, 141]. Within this context, one can easily investigate, for example, the propagation of fermionic degrees of freedom around a vortex background, something that can be readily obtained from the supersymmetric transformation of the vortex solution. Investigations along these lines have already been considered in the context of cosmic strings [142, 143], for instance.

In the next chapters, we follow closely the definitions and conventions of [145], including the Minkowski metric $\eta_{\mu\nu} = diag(- + +)$.

Chapter 8

Supersymmetry in 2+1 dimensions

8.1 Why supersymmetry?

Let's consider the Poincaré algebra:

$$[P^\mu, P^\nu] = 0; \tag{8.1}$$

$$[M^{\mu\nu}, P^\rho] = i\left(\eta^{\mu\rho}P^\nu - \eta^{\nu\rho}P^\mu\right) \tag{8.2}$$

$$[M^{\mu\nu}, M^{\rho\sigma}] = i\left(\eta^{\mu\rho}M^{\nu\sigma} + \eta^{\nu\sigma}M^{\mu\rho} - \eta^{\mu\sigma}M^{\nu\rho} - \eta^{\nu\rho}M^{\mu\sigma}\right) \tag{8.3}$$

Where P^μ are the generators of translations, $M^{\mu\nu}$ of boosts and rotations.

Now, any theoretical particle physicist who takes symmetry as valuable principle would at some point consider if this is the most general algebra possible for spacetime symmetries within the context of quantum field theory. The Coleman-Mandula theorem was proved exactly to answer such inquiry, and the response can be stated in the following way [146]: *"The most general **Lie-algebra** of symmetries of the S-matrix contains the energy-momentum operator P_μ, the Lorentz rotation generators $M_{\mu\nu}$, and a finite number of Lorentz scalar operators B_l, i. e.*

$$[P_\mu, B_l] = [M_{\mu\nu}, B_l] = 0,$$

where the B_l's constitute a Lie-algebra

$$[B_l, B_m] = iC_{lm}{}^k B_k$$

and $C_{lm}{}^k$ are the structure constants of a Lie-algebra of a compact internal symmetry group (e.g. SU(2))." Provided that some other reasonable physical assumptions are made. Supersymmetry arises formally as natural extension of this theorem in the form of the Haag-Lopusanski-Sohnius theorem by relaxing one of the conditions: to allow the Lie-algebra to become a Graded (or Super)-Lie-algebra, that is, by including anticommutators.

Therefore, making use of the gamma matrices $\{\gamma^\mu, \gamma^\nu\} = -2\eta^{\mu\nu}$ and the spinor representation of the Lorentz generators $(M^{\mu\nu})^\alpha{}_\beta = \frac{i}{4}[\gamma^\mu, \gamma^\nu]^\alpha{}_\beta \equiv i\,(\Sigma^{\mu\nu})^\alpha{}_\beta$, we include in the Poincaré algebra:

$$[P_\mu, Q_\alpha] = 0; \tag{8.4}$$
$$[M^{\mu\nu}, Q^\alpha] = i\,(\Sigma^{\mu\nu})^\alpha{}_\beta\, Q^\beta \tag{8.5}$$
$$\{Q_\alpha, Q_\beta\} = 2\,(C\gamma^\mu)_{\alpha\beta}\, P_\mu \tag{8.6}$$

Where Q^α is a Majorana spinor, generator of supersymmetry transformations, as we will see in more detail soon.

In fact, we are also allowed to consider $\mathcal{N}$-extended supersymmetry:

$$\{Q^I_\alpha, Q^J_\beta\} = 2\delta^{IJ}\,(C\gamma^\mu)_{\alpha\beta}\, P_\mu \tag{8.7}$$

With $I, J = 1, 2, ..., \mathcal{N}$. That is, to include $\mathcal{N}$ different generators.

The set of all these commutators and anti-commutators constitutes the algebra of the so-called Super-Poincaré group. As a matter of fact, this is actually not the most general set of possible symmetries of Minkowski spacetime. Indeed, the Lorentz group is born out of the invariance of $ds'^2 = ds^2 = \eta_{\mu\nu}dx^\mu dx^\nu$, and if we also relax this condition by demanding only the invariance of the light-cone $ds^2 = 0$, we would get the conformal group, which has the Lorentz group as a sub-group. Had we started with the conformal group to begin with, we would have arrived at the (Super-Conformal)x(Internal Symmetry) group as the most general possible group for a reasonable field theory in Minkowski spacetime. The Super-Poincaré shall suffice for our purposes.

8.2 $\mathcal{N} = 1$ Supersymmetry in 2+1 dimensions

The symmetries of Minkowski spacetime in 2+1 dimensions, equipped with the metric $\eta_{\mu\nu} = \mathrm{diag}\,(-,+,+)$, include Lorentz transformations $x \to \Lambda x$ such that $\Lambda^t\eta\Lambda = \eta$. Among them, there are discrete transformations (parity and time reversal). The set of transformations that are connected with the identity form a subgroup, having $\det\Lambda = +1$ and $\Lambda^0{}_0 > 0$. This is the so-called proper-orthocronous Lorentz sub-group, $SO(1,2)$. Such group has a double cover called $\mathrm{Spin}(1,2)$, (that is, $SO(1,2) \simeq \mathrm{Spin}(1,2)/\mathbb{Z}_2$) and this group happens to be isomorphic to $SL(2,\mathbb{R})$, the set of 2×2 real matrices with unit determinant. Therefore, if one wants to understand the Lorentz group representations in 2+1 dimensions, one should study the $SL(2,\mathbb{R})$ representations. The fundamental representation of this group acts on real two-component spinors[1] $\psi^\alpha = (\psi^+, \psi^-)$, and

[1]Strictly speaking, the group $SO(1,2)$ does not have any spinor representations, but its double cover, $\mathrm{Spin}(1,2)$, has.

these will be the fundamental objects used to construct Supersymmetry representations in what follows.

8.2.1 Spinor representation of the Lorentz group

The $SL(2,\mathbb{R})$ is a connected non-compact simple real Lie group of dimension 3 whose Lie algebra (denoted as $sl\,(2,\mathbb{R})$) is the algebra of all real, traceless 2×2 matrices. The Lorentz spinors ψ^α transform under the action of an element $A \in SL(2,\mathbb{R})$, that can be written as the exponential of the algebra as $A = e^{-\frac{1}{2}\omega_{\mu\nu}\Sigma^{\mu\nu}}$, where $\omega_{\mu\nu} = -\omega_{\nu\mu}$ are 3 coefficients characterizing the specific transformation, and $\Sigma^{\mu\nu}$ provide a basis for the $sl\,(2,\mathbb{R})$ algebra in the spinorial representation, that is:

$$\psi^\alpha \rightarrow \psi'^\alpha = \left[e^{-\frac{1}{2}\omega_{\mu\nu}\Sigma^{\mu\nu}}\right]^\alpha_{\ \beta}\psi^\beta. \tag{8.8}$$

Let us introduce the Dirac matrices in the Majorana representation

$$\gamma^0 = \sigma_y, \quad \gamma^1 = i\sigma_x, \quad \gamma^2 = i\sigma_z, \tag{8.9}$$

and satisfying

$$\gamma^\mu\gamma^\nu = -\eta^{\mu\nu}\mathbb{1}_{2\times 2} + i\epsilon^{\mu\nu\rho}\gamma_\rho \tag{8.10}$$

where σ_i are the usual Pauli matrices, and remembering that we have fixed $\epsilon^{012} = -1$. Note that all the Dirac matrices here are imaginary and obey the Clifford algebra:

$$\{\gamma^\mu, \gamma^\nu\} = -2\eta^{\mu\nu}\mathbb{1}_{2\times 2}. \tag{8.11}$$

Now, we can construct a basis for the $sl\,(2,\mathbb{R})$, with 3 linearly independent traceless 2×2 real matrices, using the Dirac matrices as:

$$\Sigma^{\mu\nu} = \frac{1}{4}\left[\gamma^\mu, \gamma^\nu\right] = \frac{i}{2}\epsilon^{\mu\nu\rho}\gamma_\rho. \tag{8.12}$$

Explicitly:

$$\begin{aligned}
\Sigma^{01} &= \frac{1}{4}\left[\gamma^0, \gamma^1\right] = \frac{1}{2}\sigma_z = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \\
\Sigma^{12} &= \frac{1}{4}\left[\gamma^1, \gamma^2\right] = \frac{i}{2}\sigma_y = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \\
\Sigma^{20} &= \frac{1}{4}\left[\gamma^2, \gamma^0\right] = \frac{1}{2}\sigma_x = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.
\end{aligned} \tag{8.13}$$

$SO(1,2)$ and its double cover $Spin(1,2)$ share the same algebra, denoted $so(1,2)$. Since there is an isomorphism between $Spin(1,2)$ and $SL(2,\mathbb{R})$, one can conclude that the Lie algebras $so(1,2)$ and $sl(2,\mathbb{R})$ are isomorphic. The Lie algebra $so(1,2)$ is given by 3 antisymmetric generators $M_{\mu\nu}$ such that:

$$[M^{\mu\nu}, M^{\rho\sigma}] = i\left(\eta^{\mu\rho}M^{\nu\sigma} + \eta^{\nu\sigma}M^{\mu\rho} - \eta^{\mu\sigma}M^{\nu\rho} - \eta^{\nu\rho}M^{\mu\sigma}\right), \tag{8.14}$$

which comes to be satisfied by $M^{\mu\nu} = i\Sigma^{\mu\nu}$, see Appendix F.

Following [145], we will take as the invariant tensor of $SL(2,\mathbb{R})$, the hermitian and antisymmetric matrix[2] C, responsible for the raising and lowering of spinorial indices, defined as:

$$C_{\alpha\beta} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = -C_{\beta\alpha} = -C^{\alpha\beta}. \tag{8.15}$$

The conventions for raising and lowering spinorial indices are:

$$\psi^\alpha = C^{\alpha\beta}\psi_\beta, \quad \psi_\alpha = \psi^\beta C_{\beta\alpha}, \quad \psi^2 = \frac{1}{2}\psi^\alpha\psi_\alpha \tag{8.16}$$

And it should be stressed that *all our spinors are anticommuting (Grassmann)*. Sometimes, symmetrization and anti-symmetrization of spinorial indices will be implicitly denoted by $A_{(\alpha}B_{\beta)} = A_\alpha B_\beta + A_\beta B_\alpha$ and $A_{[\alpha}B_{\beta]} = A_\alpha B_\beta - A_\beta B_\alpha$

Some useful identities are:

$$C_{\alpha\beta}C^{\gamma\delta} = \delta_\alpha{}^\gamma\delta_\beta{}^\delta - \delta_\beta{}^\gamma\delta_\alpha{}^\delta, \quad A_{[\alpha}B_{\beta]} = -C_{\alpha\beta}A^\lambda B_\lambda \tag{8.17}$$

In particular, the first identity implies $C_{\alpha\beta}C^{\beta\gamma} = -\delta_\alpha{}^\gamma$, therefore $(C^{-1})^{\alpha\beta} = C^{\beta\alpha}$.

The matrix C is precisely the charge-conjugation matrix of Dirac's theory, which here we take to satisfy:

$$\gamma^{\mu t} = -C\gamma^\mu C^{-1} \tag{8.18}$$

It implies in the following definition for charge conjugate spinors:

$$\psi^c = -C^{-1}\overline{\psi}^t \Leftrightarrow \overline{\psi^c} = \psi^t C, \tag{8.19}$$

where $\overline{\psi} = \psi^\dagger\gamma^0$ is the Dirac conjugate, as usual. It is not difficult to verify the in our conventions $\psi^c = \psi^*$, that is, the charge conjugate spinor is simply the complex conjugate spinor, therefore, any Majorana spinor $\psi^c = \psi$ will have real components.

[2]Note that, for C as defined we have: $A \in SL(2,\mathbb{R}) \implies ACA^t = \det(A)\,C = C$, exactly as an invariant tensor should behave.

One of the many properties derived from (8.18) is that $C\gamma^\mu$ (as well as $\gamma^\mu C^{-1}$) is a symmetric matrix. Now, note that any symmetric bispinor $P_{\alpha\beta}$ has only 3 independent components, the same as a three-vector P_μ, and indeed, one can construct a map from three-vectors to symmetric bi-spinors via:

$$P_{\alpha\beta} = -(C\gamma^\mu)_{\alpha\beta}P_\mu \Leftrightarrow P^{\alpha\beta} = (\gamma^\mu C^{-1})^{\alpha\beta}P_\mu \Leftrightarrow P^\alpha_{\ \beta} = (\gamma^\mu)^\alpha_{\ \beta}P_\mu \Leftrightarrow P_\alpha^{\ \beta} = (\gamma^{\mu t})_\alpha^{\ \beta}P_\mu \tag{8.20}$$

In fact, one can verify that the Lorentz transformation of $P_{\alpha\beta}$ induces the correct Lorentz transformation $P'_\mu = \Lambda_\mu^{\ \nu}P_\nu$[3]. In terms of $P_{\alpha\beta}$, we can rewrite

$$\{Q_\alpha, Q_\beta\} = -2P_{\alpha\beta} \tag{8.21}$$

8.2.2 Superspace and Superfields

A group theoretical way of constructing the Euclidean plane is to take the Euclidean group, consisting of rotations ($\mathcal{R} \in SO(2)$) on the plane and translations ($\vec{a} = (a^1, a^2) \in \mathbb{R}^2$) such that the composition of two arbitrary group elements $(\mathcal{R}', \vec{a'})$ and $(\mathcal{R}, \vec{a})$ is given by $(\mathcal{R}', \vec{a'}).(\mathcal{R}, \vec{a}) = (\mathcal{R}'\mathcal{R}, \mathcal{R}'\vec{a} + \vec{a'})$, and establish the following equivalence relation: $(\mathcal{R}', \vec{a'}) \sim (\mathcal{R}, \vec{a})$ whenever there exists an $(\tilde{\mathcal{R}}, \vec{0})$ in the Euclidean group such that $(\mathcal{R}', \vec{a'}) = (\mathcal{R}, \vec{a}).(\tilde{\mathcal{R}}, \vec{0})$. The equivalence classes constructed in this way are in one-to-one correspondence to points in the Euclidean plane. Roughly speaking, this idea can be expressed as *Euclidean Plane= Euclidean Group/Rotations*. Similarly, Minkowski spacetime can be constructed as *Poincaré Group/ Lorentz transformations*. As a natural extension of this idea, the superspace is what one obtains formally as the elements of the Super-Poincaré group after identifying those related by Lorentz transformations, that is, *Superspace = Super-Poincaré Group/* $SO(1,2)$. We shall take a more intuitive approach by analogy with the fact that one can represent the Lorentz generators $M_{\mu\nu}$ using the coordinates and derivatives of Minkowski spacetime, more specifically $M_{\mu\nu} = -i(x_\mu\partial_\nu - x_\nu\partial_\mu)$. Our task is to find a similar representation for the two-component spinor charges Q^α, and for that purpose we now introduce the two-component anticommuting (Grassmann) coordinates and derivatives $\theta^\alpha, \partial_\alpha \equiv \frac{\partial}{\partial\theta^\alpha}$:

$$\{\theta^\alpha, \theta^\beta\} = \{\partial_\alpha, \partial_\beta\} = 0,\ \{\theta^\alpha, \partial_\beta\} = \delta_\beta^{\ \alpha} \tag{8.22}$$

The coordinates θ^α are taken to be Majorana spinors implying, in our case, that they have 2 independent real components ($\theta^{\alpha *} = \theta^\alpha$). On the other hand, because of our choice of C, $\theta_\alpha \equiv \theta^\beta C_{\beta\alpha}$ satisfies $\theta^*_\alpha = -\theta_\alpha$. Our conventions also imply the following properties:

$$\partial_\alpha\theta_\beta = C_{\alpha\beta}, \quad \partial^\alpha\theta^\beta = C^{\alpha\beta}, \quad \partial_\alpha\theta^2 = \theta_\alpha, \quad \partial^2\theta^2 = -1, \quad \theta_\alpha\theta_\beta = -C_{\alpha\beta}\theta^2. \tag{8.23}$$

[3]Spinor indices will be indicated by letters from the beginning of the greek alphabet $\alpha, \beta, \gamma, \delta$... while Lorentz indices will be from the middle on μ, ν, ξ, ρ...

The Superspace, in 2+1 dimensions, is the space parametrized by the coordinates $(x, \theta) = (x^{\alpha\beta}, \theta^{\alpha})$[4].

A Superfield is defined as a function $\Phi(x, \theta)$ depending on the superspace coordinates, behaving under an infinitesimal Susy transformation with spinorial parameter ϵ^{α} as:

$$\delta_{\epsilon}\Phi = -i\epsilon^{\alpha}Q_{\alpha}\Phi. \tag{8.24}$$

Let us investigate what could be the Susy generator representation as a differential operator acting on superfields. First of all, we know that they must close the Susy algebra, $\{Q_{\alpha}, Q_{\beta}\} = -2P_{\alpha\beta}$. Inspired by the momentum operator representation as a differential operator, $P_{\alpha\beta} = -i\partial_{\alpha\beta}$, we can intuit that the supercharge will include something like ∂_{α}. In fact, we can represent it as

$$Q_{\alpha} = i\left(\partial_{\alpha} - i\theta^{\beta}\partial_{\alpha\beta}\right) \tag{8.25}$$

What are the consequences of a Susy transformation on the superspace? Let us perform a (pure) Susy transformation with parameter ϵ^{α}, we obtain:

$$\Phi(x + \delta x, \theta + \delta\theta) = e^{-i\epsilon^{\alpha}Q_{\alpha}}\Phi(x, \theta). \tag{8.26}$$

On the one hand, we have:

$$\Phi(x + \delta x, \theta + \delta\theta) = \Phi(x, \theta) + \delta x^{\alpha\beta}\partial_{\alpha\beta}\Phi(x, \theta) + \delta\theta^{\alpha}\partial_{\alpha}\Phi(x, \theta). \tag{8.27}$$

On the other hand, since $Q_{\alpha} = i\left(\partial_{\alpha} - i\theta^{\beta}\partial_{\alpha\beta}\right)$, we have:

$$e^{-i\epsilon^{\alpha}Q_{\alpha}}\Phi(x, \theta) = \Phi(x, \theta) - i\epsilon^{\alpha}Q_{\alpha}\Phi(x, \theta) = \Phi(x, \theta) + \epsilon^{\alpha}\partial_{\alpha}\Phi(x, \theta) - i\epsilon^{\alpha}\theta^{\beta}\partial_{\alpha\beta}\Phi(x, \theta). \tag{8.28}$$

Remembering that $\partial_{\alpha\beta}$ is symmetric in its indices, and comparing both last expressions, we can immediately obtain what is the consequence in superspace after performing a Susy transformation:

$$\begin{aligned} x'^{\alpha\beta} &= x^{\alpha\beta} - \frac{i}{2}\left(\epsilon^{\alpha}\theta^{\beta} + \epsilon^{\beta}\theta^{\alpha}\right), \\ \theta'^{\alpha} &= \theta^{\alpha} + \epsilon^{\alpha}. \end{aligned} \tag{8.29}$$

Therefore, a Susy transformation can be seen as a translation in superspace. It is remarkable that it affects the spacetime even if there is no translation generator acting here, and this is an immediate consequence of the fact that the anti-commutator of two Susy

[4] Most often, we shall refer to the bosonic coordinates and derivatives as $x^{\alpha\beta}$ and $\partial_{\alpha\beta}$ instead of x^{μ} and ∂_{μ}, having in mind the natural map between bi-spinors and three-vectors.

generators is proportional to the momentum operator, *i.e.*, we have $\{Q_\alpha, Q_\beta\} = -2P_{\alpha\beta}$.

In fact, one can verify that the commutation of two Susy transformations gives a translation. Notice that, due to the Grassmanian nature of the objects:

$$\begin{aligned}\delta_\epsilon\delta_\eta\Phi &= i\epsilon^\alpha Q_\alpha\left(i\eta^\beta Q_\beta\Phi\right) = +\epsilon^\alpha\eta^\beta Q_\alpha Q_\beta\Phi,\\ \delta_\eta\delta_\epsilon\Phi &= i\eta^\beta Q_\beta\left(i\epsilon^\alpha Q_\alpha\Phi\right) = -\epsilon^\alpha\eta^\beta Q_\beta Q_\alpha\Phi,\end{aligned}$$

Then,

$$\left(\delta_\epsilon\delta_\eta - \delta_\eta\delta_\epsilon\right)\Phi = \epsilon^\alpha\eta^\beta\{Q_\alpha, Q_\beta\}\Phi = -2\epsilon^\alpha\eta^\beta P_{\alpha\beta}\Phi. \tag{8.30}$$

Therefore,

$$[\delta_\epsilon, \delta_\eta] = 2i\epsilon^\alpha\eta^\beta\partial_{\alpha\beta}. \tag{8.31}$$

Suppose we have a superfield Φ, that is, a function of superspace $\Phi = \Phi(x, \theta)$ that transforms as $\delta_\epsilon\Phi = -i\epsilon^\alpha Q_\alpha\Phi$ under a SUSY transformation with an infinitesimal spinorial parameter ϵ^α. Since we have $[Q_\alpha, P_{\beta\gamma}] = 0$, and we can represent the momentum operator as $P_{\alpha\beta} = -i\partial_{\alpha\beta}$, we can see that $\partial_{\alpha\beta}\Phi$ is also a superfield. However, the same is not true for $\partial_\alpha\Phi$, because of the anti-commutation relation $\{Q_\alpha, Q_\beta\} = -2P_{\alpha\beta}$. Thus, one can ask: How can I define a derivative that is covariant under SUSY transformations? If there is one, it should satisfy:

$$\delta_\epsilon\left(D_\alpha\Phi\right) = D_\alpha\left(\delta_\epsilon\Phi\right). \tag{8.32}$$

The answer turns out to be:

$$D_\alpha = \partial_\alpha + i\theta^\beta\partial_{\alpha\beta}. \tag{8.33}$$

Using the Susy algebra, one can show that $[D_\alpha, P_{\beta\gamma}] = \{D_\alpha, Q_\beta\} = 0$. The covariant derivative satisfies the following properties:

$$\begin{aligned}&D_\alpha D_\beta = i\partial_{\alpha\beta} + C_{\beta\alpha}D^2, \quad D^\beta D_\alpha D_\beta = 0, \quad \partial^{\alpha\gamma}\partial_{\beta\gamma} = \delta_\beta^\alpha\Box,\\ &D^2 D_\alpha = -D_\alpha D^2 = i\partial_{\alpha\beta}D^\beta, \quad \left(D^2\right)^2 = \Box.\end{aligned} \tag{8.34}$$

These will be useful when constructing SUSY invariant actions.

A generic superfield can be written in a Taylor series in the Grassmann coordinate as:

$$\Psi_{\alpha\beta\ldots}(x, \theta) = A_{\alpha\beta\ldots}(x) + \theta^\sigma B_{\sigma\alpha\beta\ldots}(x) - \theta^2 C_{\alpha\beta\ldots}(x). \tag{8.35}$$

The anti-commutativity of the θ^α's automatically forbids any power of θ greater than 2.

In the above expression, the functions of spacetime A, B, and C are called the *components* of the superfield Ψ. The SUSY transformation of Ψ can naturally be expressed by the transformation of its components, that will also tell us about the degrees of freedom that are carried by each Susy representation in each kind of superfield. When we propose Susy invariant actions, after the integration over the Grassman coordinates, what will remain is an usual action for the component fields.

The definition of integration over Grassmann coordinates is taken to be:

$$\int d^2\theta = \int \frac{1}{2} d\theta^\alpha d\theta_\alpha \equiv \partial^2 = \frac{1}{2}\partial^\alpha\partial_\alpha \tag{8.36}$$

The motivation for this definition is the usual one, that is, it has the same 2 properties of a definite bosonic integral from $-\infty$ to ∞, namely 1) independence of the variable of integration after the integration is carried out, and 2) translation invariance $\theta \to \theta + \delta$. The superspace integration of a generic superfield $\Psi_{\alpha\beta\ldots}(x,\theta)$ gives us:

$$\int d^3x d^2\theta\ \Psi_{\alpha\beta\ldots}(x,\theta) = \int d^3x\ \partial^2\Psi_{\alpha\beta\ldots}(x,\theta) = \int d^3x\ C_{\alpha\beta\ldots}(x) \tag{8.37}$$

The simplest $\mathcal{N}=1$ Susy representation is given by the scalar superfield. Its Taylor expansion in θ is:

$$\Phi(x,\theta) = \phi(x) + \theta^\alpha\psi_\alpha(x) - \theta^2 F(x). \tag{8.38}$$

Imposing a Susy transformation of the form $\delta_\epsilon\Phi = -i\epsilon^\alpha Q_\alpha\Phi$ and remembering that $Q_\alpha = i(\partial_\alpha - i\theta^\beta\partial_{\beta\alpha})$ implies:

$$\begin{aligned}
\delta\phi &= \epsilon^\alpha\psi_\alpha,\\
\delta\psi_\alpha &= \epsilon^\beta\left(i\partial_{\alpha\beta}\phi + C_{\alpha\beta}F\right),\\
\delta F &= i\epsilon^\alpha\partial_\alpha{}^\beta\psi_\beta.
\end{aligned} \tag{8.39}$$

There is another extremely useful technique to perform computations in SUSY theories, the projection technique. In principle, one can always take any expression involving superfields and expand in θ, but this can become cumbersome pretty fast. A more efficient procedure is to use the following component projections for the components of $\Phi(x,\theta) = \phi(x) + \theta^\alpha\psi_\alpha(x) - \theta^2 F(x)$:

$$\begin{aligned}
\phi(x) &= \Phi(x,\theta)|_{\theta=0},\\
\psi_\alpha(x) &= D_\alpha\Phi(x,\theta)|_{\theta=0},\\
F(x) &= D^2\Phi(x,\theta)|_{\theta=0}.
\end{aligned} \tag{8.40}$$

From now on, the vertical bar will always implicitly mean evaluation at $\theta=0$.

We now remark that, since $iQ_\alpha + D_\alpha = 2i\theta^\beta\partial_{\beta\alpha}$, we can write

$$-iQ_\alpha| = D_\alpha| \tag{8.41}$$

and use this expression together with the various properties obeyed by D_α (8.34) to obtain the component fields transformations. As an example, let's check the SUSY variation of ψ_α:

$$\begin{aligned}\delta\psi_\alpha &= -i\epsilon^\beta Q_\beta D_\alpha\Phi| \\ &= \epsilon^\beta D_\beta D_\alpha\Phi| \\ &= \epsilon^\beta\left(i\partial_{\beta\alpha}\Phi + C_{\alpha\beta}D^2\Phi\right)| \\ &= \epsilon^\beta\left(i\partial_{\alpha\beta}\phi + C_{\alpha\beta}F\right)\end{aligned} \tag{8.42}$$

Finally, we remark that by means of the projection |, we are able to write:

$$\int d^3x\, d^2\theta\, \Phi(x,\theta) = \int d^3x\, \partial^2\Phi(x,\theta) = \int d^3xD^2\Phi(x,\theta)|, \tag{8.43}$$

and this will greatly facilitate our task of constructing SUSY invariant actions.

8.2.3 Susy invariant actions

One possible utility of the superspace formalism comes from the following fact: An action S constructed as the integral in full superspace of a scalar superfield is invariant under SUSY transformations. Indeed, take the scalar superfield $f\left(\Phi, D_\alpha\Phi, ...\right)$. Whatever it is its dependence on other superfields, by definition, it deserves the expansion $f(x,\theta) = \phi_f(x) + \theta^\alpha\psi_{f\alpha}(x) - \theta^2F_f(x)$, so that starting from the action

$$S = \int d^3x\, d^2\theta\; f\left(\Phi, D_\alpha\Phi, ...\right), \tag{8.44}$$

we have that

$$S = \int d^3x\, D^2 f\left(\Phi, D_\alpha\Phi, ...\right)| = \int d^3xF_f(x). \tag{8.45}$$

But, as we saw, the SUSY variation of an F component is a total divergence, $\partial_\alpha{}^\beta(\epsilon^\alpha\psi_{f\beta})$, therefore leaving the action invariant under such transformation.

Therefore, to write a SUSY invariant action all we need is to write the integral in full superspace of a scalar function of superfields and its covariant derivatives. Furthermore, we need to pay attention to the mass dimension of the terms in order to achieve a renormalizable action. Of course, the superfields must be chosen in such a way to describe the desired dynamics, and typically will involve fields and their supersymmetric partners.

In the following subsections, we investigate the superfields that are used to describe the different supermultiplets and the action that governs their dynamics. We will concentrate on: i) *scalar multiplet*: it consists of a real scalar and a two-component Majorana spinor, and it is described by a real scalar superfield (superhelicity 0). This is the representation where matter sits; ii) *vector multiplet* : it consists of a real vector and a real two-component Majorana spinor, and it is described by a real spinor superfield (superhelicity 1/2). This is the representation where gauge particles sit.

Scalar multiplet

The simplest SUSY representation is the scalar multiplet, described by the real scalar superfield $\Phi(x,\theta) = \phi(x) + \theta^\alpha\psi_\alpha(x) - \theta^2 F(x)$, consisting of a real scalar ϕ and a two-component Majorana spinor ψ (F is an auxiliary scalar field and does not propagate degrees of freedom).

By analogy with the ordinary field theory case, we propose the following kinetic term for the real scalar superfield:

$$S^{\Phi}_{kin} = -\frac{1}{2}\int d^3x\, d^2\theta\, \left(D_\alpha\Phi\right)^2 . \tag{8.46}$$

In the above expression, one should remember the convention for spinors $\psi^2 = \frac{1}{2}\psi^\alpha\psi_\alpha$. The mass dimension of these objects, to have a dimensionless action in our units, must be: $[\phi] = \frac{1}{2}$ and $[\psi] = 1$, since we know[5] that $[\theta] = -\frac{1}{2}$. Integrating D^α by parts in the expression above, we can rewrite it in an equivalent form,

$$S^{\Phi}_{kin} = \frac{1}{2}\int d^3x\, d^2\theta\, \Phi D^2\Phi. \tag{8.47}$$

To obtain the component expression, we can expand each superfield in θ, act with the derivatives and integrate over the Grassmann coordinates. Alternatively, we can use the more practical procedure described previously, which we now exhibit:

$$\begin{aligned} S^{\Phi}_{kin} &= \frac{1}{2}\int d^3x\, d^2\theta\, \Phi D^2\Phi = \frac{1}{2}\int d^3x D^2\left(\Phi D^2\Phi\right)| \\ &= \frac{1}{4}\int d^3x D^\alpha\left(D_\alpha\Phi D^2\Phi + \Phi D_\alpha D^2\Phi\right)| \\ &= \frac{1}{2}\int d^3x \left(D^2\Phi D^2\Phi + D^\alpha\Phi D_\alpha D^2\Phi + \Phi(D^2)^2\Phi\right)| \\ &= \frac{1}{2}\int d^3x \left(\Phi\Box\Phi + iD^\alpha\Phi\partial_\alpha{}^\beta D_\beta\Phi + D^2\Phi D^2\Phi\right)| \\ &= \frac{1}{2}\int d^3x \left[\phi\Box\phi + i\psi^\alpha\partial_\alpha{}^\beta\psi_\beta + F^2\right]. \end{aligned} \tag{8.48}$$

[5]Consistency demands that the mass dimension of ∂_α, appearing in Q_α for example, should equal the that of $\theta^\beta\partial_{\alpha\beta}$. Since $[\partial_{\alpha\beta}] = 1$, that implies $[\theta] = -1/2$.

Therefore, the action S^{Φ}_{kin} describes the dynamics of a free massless real scalar field ϕ and a free massless Majorana fermion ψ. Notice that the field F is in fact non-propagating, however it is what allows for the linear representation of supersymmetry transformations as in (8.39). It is possible to eliminate the auxiliar field F through its equation of motion, however supersymmetry can only be achieved after that if one also imposes the equations of motion on the other fields, thus realizing on-shell supersymmetry.

The action (8.48) gives only the kinetic term for the degrees of freedom present in the scalar multiplet, but the world cannot be described only by free massless fields. How could we include interactions and mass for these objects? We can achieve part of this by including a superpotential term in the Lagrangian, that is,

$$S^{\Phi}_{int} = \int d^3x\, d^2\theta\ f(\Phi). \tag{8.49}$$

Performing the same trick as before:

$$\begin{aligned}
S^{\Phi}_{int} = \int d^3x d^2\theta f(\Phi) &= \int d^3x D^2 f(\Phi)| \\
&= \frac{1}{2}\int d^3x D^{\alpha} D_{\alpha} f(\Phi)| \\
&= \frac{1}{2}\int d^3x D^{\alpha}\left[f'(\Phi) D_{\alpha}\Phi\right]| \\
&= \frac{1}{2}\int d^3x \left[f''(\Phi)(D^{\alpha}\Phi)(D_{\alpha}\Phi) + f'(\Phi) D^{\alpha} D_{\alpha}\Phi\right]| \\
&= \int d^3x \left[f''(\Phi)(D_{\alpha}\Phi)^2 + f'(\Phi) D^2\Phi\right]| \\
&= \int d^3x \left[f'(\phi)F + f''(\phi)\psi^2\right].
\end{aligned} \tag{8.50}$$

Where $f(\phi)$ denotes $f(\Phi)|$, and the same for $f'(\phi)$ and $f''(\phi)$. The derivatives are taken with respect to Φ. In a renormalizable model, $f(\Phi)$ can be at most quartic. Considering a particular case, $f(\Phi) = \frac{1}{2}m\Phi^2 + \frac{1}{6}\lambda\Phi^3$, one can obtain for the component action,

$$S^{\Phi}_{int} = \int d^3x \left[m\left(\psi^2 + \phi F\right) + \lambda\left(\phi\psi^2 + \frac{1}{2}\phi^2 F\right)\right]. \tag{8.51}$$

Upon using the equations of motion for the auxiliary field, $F + m\phi + \frac{1}{2}\lambda\phi^2 = 0$, we can find the complete action $S^{\Phi} = S^{\Phi}_{kin} + S^{\Phi}_{int}$, that is:

$$S^{\Phi} = \int d^3x \left[\frac{1}{2}\phi\Box\phi + \frac{i}{2}\psi^{\alpha}\partial_{\alpha}{}^{\beta}\psi_{\beta} - \frac{m^2}{2}\phi^2 + m\psi^2 - \frac{\lambda m}{2}\phi^3 + \lambda\phi\psi^2 - \frac{\lambda^2}{8}\phi^4\right]. \tag{8.52}$$

Looking at the above expression, even if SUSY is not explicit, it manifests itself through special relations between masses and couplings. Thus, even if we did not know the origin of the model, this degeneracy would suggest the existence of some symmetry behind it.

Of course one can generalize the model present above by introducing more superfields, a different superpotential, or a more complicated kinetic term. We will not consider these cases here by reasons of scope, but after introducing the gauge superfield, we will study the situation where these different superfields are minimally coupled.

Vector Multiplet

Another simple SUSY representation is given by the vector multiplet, described by the real spinor superfield $\Gamma_\alpha(x,\theta) = i\theta^\beta A_{\alpha\beta}(x) - 2\theta^2\lambda_\alpha(x)$, consisting of a massless vector field $A_{\alpha\beta} = -(C\gamma^\mu)_{\alpha\beta}A_\mu$ and a Majorana two-component spinor λ^α (here we are already writing Γ_α in the so-called Wess-Zumino gauge, as will be discussed below). It should be stressed that, in 2+1 dimensions, a gauge field propagates only 1 physical degree of freedom, and so does a Majorana spinor. We will only discuss Abelian gauge theories in this work, and with the purpose of introducing the spinor gauge superfield, we now consider the gauge symmetry group is $U(1)$.

To introduce the real spinor superfield, let us first introduce the complex scalar superfield as a doublet of real scalar superfields $\Phi(\Phi^*) = \Phi_1 + i\Phi_2(\Phi_1 - i\Phi_2)$ that transforms under a constant phase rotation as:

$$\begin{aligned} \Phi' &= e^{iK}\Phi, \\ \Phi^{*'} &= \Phi^* e^{-iK}. \end{aligned} \tag{8.53}$$

One can write the free kinetic term for the complex scalar superfield in total analogy with the real case, using $S = \frac{1}{2}\int d^3x\, d^2\theta\, D^\alpha\Phi^* D_\alpha\Phi = \int d^3x\, d^2\theta\, |D_\alpha\Phi|^2$. Notice that this kinetic action is naturally invariant under these constant phase transformations described above.

We now extend this idea to a *local invariance in superspace*, where K is promoted to a real scalar superfield $K(x,\theta) = \omega(x) + \theta^\alpha\sigma_\alpha(x) - \theta^2\tau(x)$. Analogously to the usual case, we need covariantize the spinor derivative D_α to include a superfield that will play the role of a gauge field, transforming in a specific way as to compensate the matter field transformation. That is, we extend the notion of derivative upon including a spinor gauge connection as:

$$D_\alpha \to \nabla_\alpha = D_\alpha - i\Gamma_\alpha. \tag{8.54}$$

To be gauge covariant in the usual sense, we need to have the following transformation property under a local phase transformation with a real scalar superfield K:

$$\Phi'(x,\theta) = e^{iK(x,\theta)}\Phi(x,\theta) \implies (\nabla_\alpha\Phi)' = e^{iK(x,\theta)}\left(\nabla_\alpha\Phi\right). \tag{8.55}$$

Thus,

$$e^{iK}D_\alpha\Phi + i(D_\alpha K)e^{iK}\Phi - i\Gamma'_\alpha e^{iK}\Phi = e^{iK}D_\alpha\Phi - e^{iK}i\Gamma_\alpha\Phi \tag{8.56}$$

From which we can obtain immediately, since we are considering only the abelian case:

$$\delta\Gamma_\alpha = \Gamma'_\alpha - \Gamma_\alpha = D_\alpha K. \tag{8.57}$$

The supergauge transformations are thus summarized as

$$\begin{aligned}\Phi'(x,\theta) &= e^{iK(x,\theta)}\Phi(x,\theta),\\ \Gamma'_\alpha(x,\theta) &= \Gamma_\alpha(x,\theta) + D_\alpha K(x,\theta).\end{aligned} \tag{8.58}$$

We can perform a rescaling $\Gamma_\alpha \to e\Gamma_\alpha$ to introduce the coupling constant. The action of ∇_α will be given as $D_\alpha \mp i\Gamma_\alpha$, with - or + as we are acting on Φ or Φ^*, respectively.

Therefore, to write the kinetic term invariant under local phase transformations, we can perform a minimal coupling by simply substituting $D_\alpha \to \nabla_\alpha$, and thus writing:

$$S^\Phi_{gauge} = -\frac{1}{2}\int d^3x\, d^2\theta\ \nabla^\alpha\Phi^*\nabla_\alpha\Phi = -\int d^3x\, d^2\theta\ |\nabla\Phi|^2. \tag{8.59}$$

The general form of the real spinor superfield $\Gamma_\alpha(x,\theta)$ can be written as

$$\Gamma_\alpha(x,\theta) = \chi_\alpha(x) + \theta^\beta\left[C_{\alpha\beta}B(x) + iA_{\alpha\beta}(x)\right] - \theta^2\left[2\lambda_\alpha(x) - i\partial_\alpha{}^\beta\chi_\beta(x)\right]. \tag{8.60}$$

From the above expression, we can find the components by projection as

$$\chi_\alpha = \Gamma_\alpha|, \quad B = \frac{1}{2}D^\alpha\Gamma_\alpha|, \quad A_{\alpha\beta} = -\frac{i}{2}D_{(\alpha}\Gamma_{\beta)}|, \quad \lambda_\alpha = \frac{1}{2}D^\beta D_\alpha\Gamma_\beta|. \tag{8.61}$$

First of all, we need to understand what is the precise supergauge transformation that transforms this general Γ_α in the one presented at the beginning of this subsection. The Wess-Zumino (WZ) gauge is the choice that explicitly exhibits the physical degrees of freedom described of this superfield. It is important to keep in mind that this supergauge choice is *not* invariant under SUSY transformations.

Performing a supergauge transformation with parameter $K = \omega + \theta\sigma - \theta^2\tau$, we have:

$$\begin{aligned}\Gamma'_\alpha &= \Gamma_\alpha + D_\alpha K\\ &= \chi_\alpha + \theta^\beta\left[C_{\alpha\beta}B + iA_{\alpha\beta}\right] - \theta^2\left[2\lambda_\alpha - i\partial_\alpha{}^\beta\chi_\beta\right]\\ &+ \sigma_\alpha + \theta^\beta\left[C_{\alpha\beta}\tau + i\partial_{\alpha\beta}\omega\right] - \theta^2\left[0 - i\partial_\alpha{}^\beta\sigma_\beta\right].\end{aligned} \tag{8.62}$$

That is, if we write $\delta\Gamma_\alpha = D_\alpha K$ for the second line, we find:

$$\delta\chi_\alpha = \sigma_\alpha, \quad \delta B = \tau, \quad \delta A_{\alpha\beta} = \partial_{\alpha\beta}\omega, \quad \delta\lambda_\alpha = 0. \tag{8.63}$$

From the above expressions, we see that performing a supergauge transformation, we can perform arbitrary shifts on χ and B, that is, we can choose σ_α and τ to set them as we wish. In particular, we can choose $\sigma_\alpha = -\chi_\alpha$ and $\tau = -B$ to set them to zero. The gauge in which we take $\chi = B = 0$ is the WZ gauge, where we have:

$$\Gamma_\alpha = \theta^\beta i A_{\alpha\beta} - \theta^2 2\lambda_\alpha \tag{8.64}$$

This gauge reveals the physical content of the vector multiplet, and we still have a residual gauge freedom that allows us to perform a gauge transformation $A'_{\alpha\beta} = A_{\alpha\beta} + \partial_{\alpha\beta}\omega$.

Now, we need to give dynamics to the gauge superfield. Usually, this is done by the Maxwell term, through the field strength, that can be defined using the gauge covariant derivative, and we need to figure out what is the analogous object here. Let us introduce the superfield strength W_α that will do the job:

$$W_\alpha = \frac{1}{2} D^\beta D_\alpha \Gamma_\beta. \tag{8.65}$$

A few comments are in order. First, note that W_α is gauge-invariant. Indeed, one can gauge transform $\Gamma_\alpha \to \Gamma_\alpha + D_\alpha K$ and remember that $D^\beta D_\alpha D_\beta = 0$. Using the last identity, we can also show that $D^\alpha W_\alpha = 0$.

By its very definition, we can obtain the component expansion of the superfield strength W_α, noting that we can use the WZ gauge for Γ_α to facilitate the computations since W_α is a gauge-invariant object. We obtain:

$$W_\alpha = \lambda_\alpha + \theta^\beta \hat{F}_{\beta\alpha} - i\theta^2 \partial_\alpha{}^\beta \lambda_\beta \tag{8.66}$$

Having defined $\hat{F}_{\alpha\beta} = \hat{F}_{\beta\alpha} \equiv -\frac{1}{2}\left(\partial_\alpha{}^\gamma A_{\gamma\beta} + \partial_\beta{}^\gamma A_{\gamma\alpha}\right) = -(C\Sigma^{\mu\nu})_{\alpha\beta} F_{\mu\nu} = -\frac{i}{2}\epsilon^{\mu\nu\rho}(C\gamma_\rho)_{\alpha\beta} F_{\mu\nu}$.

Therefore, the super-field-strength (also called sometimes as gaugino superfield) describes the degrees of freedom of one real vector and one real two-component spinor. We can define its components by projection as

$$\lambda_\alpha = W_\alpha|, \quad \hat{F}_{\alpha\beta} = D_\alpha W_\beta| = D_\beta W_\alpha|. \tag{8.67}$$

It is important to keep in mind the constraint given by the Bianchi identity, $D^\alpha W_\alpha = 0$, and its consequence, $D^2 W_\alpha = i\partial_\alpha{}^\beta W_\beta$.

In possession of this object, we can now build an action that will give rise to the supersymmetric dynamics of gauge fields. We propose the following action:

$$S^A_{kin} = \frac{1}{2e^2} \int d^3x\, d^2\theta\ W^2 \tag{8.68}$$

Using the projection technique we obtain,

$$S^A_{kin} = \frac{1}{2e^2} \int d^3x\, D^2 W^2| = \frac{1}{2e^2} \int d^3x \left[W^\alpha D^2 W_\alpha - \frac{1}{2} \left(D^\alpha W^\beta \right) \left(D_\alpha W_\beta \right) \right] | \tag{8.69}$$

The first term can be rewritten in components using the property $D^2 W_\alpha = i\partial_\alpha{}^\beta W_\beta$ and the projection $\lambda_\alpha = W_\alpha|$. The second term can be written remembering that $F_{\alpha\beta} = D_\alpha W_\beta|$. Therefore, we have for the kinetic gauge action a Maxwell term for the gauge field and the usual kinetic term for the gaugino:

$$S^A_{kin} = \frac{1}{e^2} \int d^3x \left[-\frac{1}{4} \hat{F}^{\alpha\beta} \hat{F}_{\alpha\beta} + \frac{i}{2} \lambda^\alpha \partial_\alpha{}^\beta \lambda_\beta \right] \tag{8.70}$$

Translating the Maxwell term to vector-index notation, we find

$$\hat{F}^{\alpha\beta} \hat{F}_{\alpha\beta} = \left[\left(\Sigma^{\mu\nu} C^{-1} \right)^{\alpha\beta} F_{\mu\nu} \right] \left[- \left(C \Sigma^{\rho\sigma} \right)_{\beta\alpha} F_{\rho\sigma} \right] = -\operatorname{Tr} \left(\Sigma^{\mu\nu} C^{-1} C \Sigma^{\rho\sigma} \right) F_{\mu\nu} F_{\rho\sigma} \tag{8.71}$$

But we know that $\operatorname{Tr} \left(\Sigma^{\mu\nu} \Sigma^{\rho\sigma} \right) = -\frac{1}{2} \left(\eta^{\mu\rho} \eta^{\nu\sigma} - \eta^{\mu\sigma} \eta^{\nu\rho} \right)$, thus

$$\hat{F}^{\alpha\beta} \hat{F}_{\alpha\beta} = \frac{1}{2} \left(\eta^{\mu\rho} \eta^{\nu\sigma} - \eta^{\mu\sigma} \eta^{\nu\rho} \right) F_{\mu\nu} F_{\rho\sigma} = F^{\mu\nu} F_{\mu\nu}, \tag{8.72}$$

giving us the usual Maxwell term, now with vector indices instead of spinorial ones.

There is room for an extra term in 2+1 dimensions, a gauge-invariant mass term that will give rise to the famous Chern-Simons term. In fact, with Γ^α and W_α in hands, we can write:

$$S^A_\mu = \frac{1}{e^2} \int d^3x\, d^2\theta \left[\frac{1}{2} \mu\, \Gamma^\alpha W_\alpha \right] \tag{8.73}$$

Performing a gauge transformation $\Gamma^\alpha W_\alpha \to \Gamma^\alpha W_\alpha + (D^\alpha K) W_\alpha$. Integrating D^α by parts, we can use $D^\alpha W_\alpha = 0$ and see that S^A_μ is gauge-invariant up to a total spinor derivative.

After using the projection technique and the WZ gauge, one obtains:

$$S^A_\mu = \frac{\mu}{e^2} \int d^3x \left[\lambda^\beta \lambda_\beta - \frac{i}{2} A^{\alpha\beta} \hat{F}_{\alpha\beta} \right] \tag{8.74}$$

The second term is the Chern-Simons, giving a gauge-invariant mass for the gauge field.

Let us write the Chern-Simons term in the usual form. We can rewrite,

$$A^{\alpha\beta}\hat{F}_{\alpha\beta} = \left[\left(\gamma^\mu C^{-1}\right)^{\alpha\beta} A_\mu\right]\left[-\left(C\Sigma^{\nu\rho}\right)_{\beta\alpha} F_{\nu\rho}\right] = -\,\mathrm{Tr}\left(\gamma^\mu C^{-1} C\Sigma^{\nu\rho}\right) A_\mu F_{\nu\rho} \tag{8.75}$$

But from $\mathrm{Tr}\left(\gamma^\mu \Sigma^{\nu\rho}\right) = -i\epsilon^{\mu\nu\rho}$, we have

$$-\frac{i}{2}A^{\alpha\beta}\hat{F}_{\alpha\beta} = \frac{1}{2}\epsilon^{\mu\nu\rho}A_\mu F_{\nu\rho} = \epsilon^{\mu\nu\rho}A_\mu \partial_\nu A_\rho \tag{8.76}$$

Therefore, the Chern-Simons term takes the expected form $\mathcal{L} \supset \mu\epsilon^{\mu\nu\rho}A_\mu\partial_\nu A_\rho$.

Matter + Gauge

In this subsection, we explore the minimal coupling in a supersymmetric scenario. The action for a complex scalar superfield Φ minimally coupled to the real spinor gauge superfield Γ_α, is given by

$$\begin{aligned} S^{\Phi}_{gauge} &= -\frac{1}{2}\int d^3x\, d^2\theta\ \left(\nabla^\alpha \Phi^*\right)\left(\nabla_\alpha \Phi\right) \\ &= -\frac{1}{2}\int d^3x\, D^2\left[\left(D^\alpha + i\Gamma^\alpha\right)\Phi^*\right]\left[\left(D_\alpha - i\Gamma_\alpha\right)\Phi\right] \end{aligned} \tag{8.77}$$

To obtain the component Lagrangian, we can simply expand all the terms above, or we can proceed by projection. However, there is a more efficient procedure that we can perform here, the so-called *covariant projection.* The components defined by the covariant projection are physically equivalent to the ordinary ones, only differing by a gauge dependent field redefinition. Nonetheless, they provide an equally valid description of the theory.

Defining the components by covariant projection means to assign:

$$\phi = \Phi|, \quad \psi_\alpha = \nabla_\alpha \Phi|, \quad F = \nabla^2\Phi|. \tag{8.78}$$

Also, when acting on a gauge invariant quantity, we are allowed to write

$$\int d^3x\, d^2\theta = \int d^3x\, D^2| = \int d^3x\, \nabla^2|. \tag{8.79}$$

As before, we can integrate by parts the covariant derivative to write the action in a different form:

$$S^{\Phi}_{gauge} = \int d^3x\, d^2\theta\ \left[\Phi^* \nabla^2 \Phi\right] \tag{8.80}$$

Using the Leibnitz rule for the gauge covariant derivative, we obtain:

$$
\begin{aligned}
\nabla^2\left[\Phi^*\nabla^2\Phi\right] &= \frac{1}{2}\nabla^\beta\left[\nabla_\beta\Phi^*\nabla^2\Phi+\Phi^*\nabla_\beta\nabla^2\Phi\right] \\
&= \frac{1}{2}\left[\nabla^\beta\nabla_\beta\Phi^*\nabla^2\Phi-\nabla_\beta\Phi^*\nabla^\beta\nabla^2\Phi+\nabla^\beta\Phi^*\nabla_\beta\nabla^2\Phi+\Phi^*\nabla^\beta\nabla_\beta\nabla^2\Phi\right] \\
&= \left[\nabla^2\Phi^*\nabla^2\Phi+\nabla^\beta\Phi^*\nabla_\beta\nabla^2\Phi+\Phi^*\nabla^2\nabla^2\Phi\right]. \qquad (8.81)
\end{aligned}
$$

There are important properties satisfied by the gauge covariant derivative, given by,

$$
\begin{aligned}
\nabla_\alpha\nabla^2 &= i\nabla_\alpha{}^\beta\nabla_\beta+iW_\alpha, \\
\nabla^2\nabla_\alpha &= -i\nabla_\alpha{}^\beta\nabla_\beta-2iW_\alpha, \\
\left(\nabla^2\right)^2 &= \square-iW^\alpha\nabla_\alpha. \qquad (8.82)
\end{aligned}
$$

where $\square=\nabla^{\alpha\beta}\nabla_{\alpha\beta}$ is the covariant d'Alembertian, also defining $\nabla_{\alpha\beta}=\partial_{\alpha\beta}-i\Gamma_{\alpha\beta}$, with $\Gamma_{\alpha\beta}=-\frac{i}{2}D_{(\alpha}\Gamma_{\beta)}$. Note, for example, that $\nabla_{\alpha\beta}|=-(C\gamma^\mu)_{\alpha\beta}(\partial_\mu-iA_\mu)$.

The first term is immediate, since $F=\nabla^2\Phi|$. From the second term, we have:

$$
\begin{aligned}
\nabla^\alpha\Phi^*\nabla_\alpha\nabla^2\Phi| &= \nabla^\alpha\Phi^*\left(i\nabla_\alpha{}^\beta\nabla_\beta\Phi+iW_\alpha\Phi\right)| \\
&= -i\nabla^\alpha\Phi^*\nabla_{\alpha\beta}\nabla^\beta\Phi+i(\nabla^\alpha\Phi^*)W_\alpha\Phi| \\
&= -i\nabla^\alpha\Phi^*\partial_{\alpha\beta}\nabla^\beta\Phi+\frac{i}{2}\nabla^\alpha\Phi^*(D_{(\alpha}\Gamma_{\beta)})\nabla^\beta\Phi+i(\nabla^\alpha\Phi^*)W_\alpha\Phi| \\
&= i(\psi^*)^\alpha\partial_\alpha{}^\beta\psi_\beta+(\psi^*)^\alpha A_\alpha{}^\beta\psi_\beta+i(\psi^*)^\alpha\lambda_\alpha\phi \qquad (8.83)
\end{aligned}
$$

Since by projection, we have $\psi^\alpha=\nabla^\alpha\Phi|$, $2iA_{\alpha\beta}=D_{(\alpha}\Gamma_{\beta)}|$, $\lambda^\alpha=W^\alpha|,\phi=\Phi|$. Finally, from the last term we can obtain:

$$
\begin{aligned}
\Phi^*\nabla^2\nabla^2\Phi| &= \Phi^*\square\Phi-i\Phi^*W^\alpha\nabla_\alpha\Phi| \\
&= \frac{1}{2}\Phi^*\left(\nabla^{\alpha\beta}\nabla_{\alpha\beta}\right)\Phi|-i\phi^*\lambda^\alpha\psi_\alpha \\
&= \frac{1}{2}\phi^*\left(\partial^{\alpha\beta}-iA^{\alpha\beta}\right)\left(\partial_{\alpha\beta}-iA_{\alpha\beta}\right)\phi-i\phi^*\lambda^\alpha\psi_\alpha \qquad (8.84)
\end{aligned}
$$

Therefore, putting all together, we can write the action in components as:

$$
S^\Phi_{gauge}=\int d^3x\left[\frac{1}{2}\phi^*D^{\alpha\beta}D_{\alpha\beta}\phi+i(\psi^*)^\alpha D_\alpha{}^\beta\psi_\beta+i(\psi^*)^\alpha\lambda_\alpha\phi-i\phi^*\lambda^\alpha\psi_\alpha+F^*F\right], \quad (8.85)
$$

where we defined the usual gauge covariant derivative as $D_{\alpha\beta}=\partial_{\alpha\beta}-iA_{\alpha\beta}=-(C\gamma^\mu)_{\alpha\beta}D_\mu$. Notice that this gives us the kinetic term for the scalar and fermion fields, minimally coupled with the gauge field, plus an Yukawa-like term . A more complete picture can be given by the sum of all the actions discussed here, paying attention to adopt a superpotential that is invariant under gauge transformations.

The scalar kinetic term can be rewritten using vector indices. Indeed, using $\mathrm{Tr}\,(\gamma^\mu\gamma^\nu) = -2\eta^{\mu\nu}$, we have

$$D^{\alpha\beta}D_{\alpha\beta} = \left(\gamma^\mu C^{-1}\right)^{\alpha\beta}(-C\gamma^\nu)_{\beta\alpha}D_\mu D_\nu = -\,\mathrm{Tr}\,(\gamma^\mu\gamma^\nu)\,D_\mu D_\nu = 2D^\mu D_\mu. \tag{8.86}$$

To restore the coupling constant, remember that we need to rescale $\Gamma_\alpha \to e\Gamma_\alpha$, what can be equivalently done by rescaling its components $A_{\alpha\beta} \to eA_{\alpha\beta}, \lambda_\alpha \to e\lambda_\alpha$. Therefore,

$$\begin{aligned} S^{\Phi}_{gauge} &= -\frac{1}{2}\int d^3x\, d^2\theta\, \left[\nabla^\alpha\Phi^*\nabla_\alpha\Phi\right] = \int d^3x\, d^2\theta\, \left[\Phi^*\nabla^2\Phi\right] \\ &= \int d^3x\, \left[\phi^* D^\mu D_\mu\phi + i(\psi^*)^\alpha D_\alpha^{\ \beta}\psi_\beta + ie(\psi^*)^\alpha\lambda_\alpha\phi - ie\phi^*\lambda^\alpha\psi_\alpha + F^*F\right]. \end{aligned} \tag{8.87}$$

8.3 The $\mathcal{N} = 2$ model

The supersymmetric technology developed so far is not new and is based on [145]. The point is that we now have all the necessary ingredients to answer the question we previously posed. Is the our self-dual model the bosonic sector of an $\mathcal{N} = 2$ supersymmetric theory? The answer is *yes* and we shall show how to construct the full theory.

As was pointed out earlier, by writing the action as an integral in full superspace of a scalar superfield, the theory is automatically invariant under $\mathcal{N} = 1$ supersymmetry. Therefore, we begin the construction from

$$S = \int d^3x d^2\theta\ \mathcal{L}(x,\theta), \tag{8.88}$$

such that the lagrangian above contains:

$$\begin{aligned} \mathcal{L}(x,\theta) = &\frac{1}{4}(W^A)^\alpha(W^A)_\alpha + \frac{1}{4}(W^a)^\alpha(W^a)_\alpha - \frac{1}{4}D^\alpha S D_\alpha S - \frac{1}{4}D^\alpha R D_\alpha R \\ &- \frac{1}{2}\nabla^\alpha\Phi_+^*\nabla_\alpha\Phi_+ - \frac{1}{2}\nabla^\alpha\Phi_-^*\nabla_\alpha\Phi_- + \frac{\mu}{2}(\Gamma^A)^\alpha W^a_\alpha + f(\Phi_+,\Phi_-,S,R). \end{aligned} \tag{8.89}$$

Where $\nabla^\pm_\alpha\Phi_\pm = \left(D_\alpha - ie\Gamma^A_\alpha \mp ig\Gamma^a_\alpha\right)\Phi_\pm$ and $f(\Phi_+,\Phi_-,S,R)$ is an arbitrary superpotential, for now. The superfields are defined as:

$$\Phi_\pm = \varphi_\pm + \theta^\alpha(\psi_\pm)_\alpha - \theta^2 F_\pm \tag{8.90}$$

$$\Gamma^A_\alpha = i\theta^\beta A_{\alpha\beta} - 2\theta^2\Lambda_\alpha \quad \text{(WZ)} \tag{8.91}$$

$$\Gamma^a_\alpha = i\theta^\beta a_{\alpha\beta} - 2\theta^2\lambda_\alpha \quad \text{(WZ)} \tag{8.92}$$

$$S = N + \theta^\alpha\xi_\alpha - \theta^2 G \tag{8.93}$$

$$R = M + \theta^\alpha\zeta_\alpha - \theta^2 H \tag{8.94}$$

Or, using projections:

$$\varphi_\pm = \Phi_\pm|, \quad (\psi_\pm)_\alpha = \nabla_\alpha \Phi_\pm|, \quad F_\pm = \nabla^2 \Phi_\pm| \tag{8.95}$$

$$N = S|, \quad \xi_\alpha = D_\alpha S|, \quad G = D^2 S| \tag{8.96}$$

$$M = R|, \quad \zeta_\alpha = D_\alpha R|, \quad H = D^2 R| \tag{8.97}$$

$$A_{\alpha\beta} = -\frac{i}{2} D_{(\alpha}\Gamma^A_{\beta)}|, \quad \Lambda_\alpha = \frac{1}{2} D^\beta D_\alpha \Gamma^A_\beta| \tag{8.98}$$

$$a_{\alpha\beta} = -\frac{i}{2} D_{(\alpha}\Gamma^a_{\beta)}|, \quad \lambda_\alpha = \frac{1}{2} D^\beta D_\alpha \Gamma^a_\beta| \tag{8.99}$$

The fermions $\Lambda^\alpha, \lambda^\alpha, \xi^\alpha$ and ζ^α are Majorana. As for the super field-strengths:

$$W^A_\alpha = \frac{1}{2} D^\beta D_\alpha \Gamma^A_\beta = \Lambda_\alpha + \theta^\beta \hat{F}_{\alpha\beta} - i\theta^2 \partial_\alpha^{\ \beta} \Lambda_\beta \tag{8.100}$$

$$W^a_\alpha = \frac{1}{2} D^\beta D_\alpha \Gamma^a_\beta = \lambda_\alpha + \theta^\beta \hat{f}_{\alpha\beta} - i\theta^2 \partial_\alpha^{\ \beta} \lambda_\beta \tag{8.101}$$

$$\Lambda_\alpha = W^A_\alpha|, \quad \hat{F}_{\alpha\beta} = D_\alpha W^A_\beta| = D_\beta W^A_\alpha| \tag{8.102}$$

$$\lambda_\alpha = W^a_\alpha|, \quad \hat{f}_{\alpha\beta} = D_\alpha W^a_\beta| = D_\beta W^a_\alpha| \tag{8.103}$$

And the analogues of (8.82) in this case are (considering $\epsilon = \pm$):

$$[(\nabla^\epsilon)^\alpha, (\nabla^\epsilon)_{\alpha\beta}] = -3eW^A_\beta - \epsilon 3gW^a_\beta, \tag{8.104}$$

$$(\nabla^\epsilon)^2(\nabla^\epsilon)_\beta + (\nabla^\epsilon)_\beta(\nabla^\epsilon)^2 = -ieW^A_\beta - \epsilon igW^a_\beta, \tag{8.105}$$

$$\frac{1}{2}(\nabla^\epsilon)_\beta(\nabla^\epsilon)_\lambda(\nabla^\epsilon)^\beta = ieW^A_\beta + \epsilon igW^a_\beta, \tag{8.106}$$

$$(\nabla^\epsilon)_\beta(\nabla^\epsilon)^2 = i(\nabla^\epsilon)_\beta^{\ \alpha}(\nabla^\epsilon)_\alpha + ieW^A_\beta + \epsilon igW^a_\beta \tag{8.107}$$

$$(\nabla^\epsilon)^2(\nabla^\epsilon)_\beta = -i(\nabla^\epsilon)_\beta^{\ \alpha}(\nabla^\epsilon)_\alpha - 2ieW^A_\beta - \epsilon 2igW^a_\beta. \tag{8.108}$$

Considering our discussion in the last subsection, the choice of terms in (8.89) should now be clear. They are necessary to reproduce the correct kinetic terms[6]

$$\begin{aligned}\mathcal{L} \supset &-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{4}f_{\mu\nu}f^{\mu\nu} + \mu\epsilon^{\mu\nu\rho}A_\mu\partial_\nu a_\rho - \frac{1}{2}(\partial_\mu N)^2 - \frac{1}{2}(\partial_\mu M)^2 \\ &- |(\partial_\mu - ieA_\mu - iga_\mu)\phi_+|^2 - |(\partial_\mu - ieA_\mu + iga_\mu)\phi_-|^2\end{aligned}$$

Before proceeding, let us now try to understand the action of parity on superspace. In 2+1 dimensions, the matrix P acting as parity in spinor space must satisfy $P^{-1}\gamma^\mu P\mathcal{P}^\nu_\mu = -\gamma^\nu$ and we have set $P = -i\gamma^1$. We shall now impose that the grassmann coordinate of superspace transforms as $\theta \xrightarrow{P} \theta^P = -i\gamma^1\theta$; making the indices explicit $(\theta^P)^\alpha = -i(\gamma^1)^\alpha_{\ \beta}\theta^\beta$. Let's work out some consequences of this. For example, that $(\theta^P)^2 = -\theta^2$.

[6]Up to the modification of the metric which now is $\eta_{\mu\nu} = \mathrm{diag}(-++)$ and the sign of the gauge couplings appearing inside the covariant derivative, however it is clear that (6.1) and (8.89) are physically equivalent, as far as the kinetic terms are concerned.

$$(\theta^P)^2 = \frac{1}{2}(\theta^P)^\alpha \theta^P_\alpha = \frac{1}{2}(\theta^P)^\alpha (\theta^P)^\beta C_{\beta\alpha} = \frac{1}{2}(\gamma^1)^\alpha{}_\delta (\gamma^1)^\beta{}_\epsilon C_{\alpha\beta}\theta^\delta\theta^\epsilon = \frac{1}{2}C_{\delta\epsilon}\theta^\delta\theta^\epsilon = -\theta^2$$

The reason why we care about this is that, as a consequence $\int d^2\theta = \partial^2 \xrightarrow{P} -\int d^2\theta$. That is to say, the fermionic integration measure of superspace is parity-odd. Therefore, in order to construct a parity-even action we need $\mathcal{L}(x,\theta) \xrightarrow{P} -\mathcal{L}(x,\theta)$. With this in mind and also to obtain the correct parity transformations of the familiar fields $\phi_\pm, M, N, A_\mu$ and a_μ we set:

$$\Phi_\pm \xrightarrow{P} -\Phi_\mp, \quad S \xrightarrow{P} S, \quad R \xrightarrow{P} -R. \tag{8.109}$$

Together with:

$$\Gamma^A \xrightarrow{P} i\gamma^1\Gamma^A, \quad \Gamma^a \xrightarrow{P} -i\gamma^1\Gamma^a \tag{8.110}$$

These sets of transformations imply:

$$\begin{cases} \varphi^P_\pm &= -\varphi_\mp, \quad \psi^P_\pm = -i\gamma^1\psi_\mp \\ A^P_\mu &= \mathcal{P}^\nu_\mu A_\nu, \quad a^P_\mu = -\mathcal{P}^\nu_\mu a_\nu \\ N^P &= N, \qquad M^P = -M \\ X^P &= -i\gamma^1 X, \quad Y^P = i\gamma^1 Y \\ X &= \{\Lambda, \zeta\}, \quad Y = \{\lambda, \xi\} \end{cases}$$

We also expect the Abelian group $U(1)_A \times U(1)_a$ to be a gauge symmetry of our theory, and here we implement it in a supersymmetric way via:

$$\begin{cases} \Phi'_\pm &= e^{iK}\Phi_\pm, \\ \delta\Gamma^A_\alpha &= \frac{1}{e}D_\alpha K. \end{cases} \quad \begin{cases} \Phi'_\pm &= e^{\pm iL}\Phi_\pm, \\ \delta\Gamma^a_\alpha &= \frac{1}{g}D_\alpha L. \end{cases} \tag{8.111}$$

Throughout this section we are using this symmetry to fix the Wess-Zumino gauge. After which, the only remaining gauge symmetry is, taking their combined effect:

$$\begin{cases} \phi'_\pm &= e^{i(\rho(x)\pm\xi(x))}\phi_\pm. \\ \psi'_\pm &= e^{i(\rho(x)\pm\xi(x))}\psi_\pm. \\ \delta A_\mu &= \frac{1}{e}\partial_\mu\rho(x)\ , \\ \delta a_\mu &= \frac{1}{g}\partial_\mu\xi(x)\ . \end{cases} \tag{8.112}$$

The super-action constructed from (8.89) is invariant under parity and gauge transformations (after fixing WZ gauge) and generates the desired kinetic terms; it was constructed to be so. The only remaining piece of the puzzle is the superpotential, $f(\Phi_+, \Phi_-, S, R)$, which we demand to satisfy the same symmetries. Not only that, but we also require $\mathcal{N}=2$ Supersymmetry, here implemented as the following set of transformation with a constant Majorana fermion parameter η:

$$\delta\Phi_\pm = \pm i\eta^\alpha\nabla_\alpha\Phi_\pm, \quad \delta\Gamma^A_\alpha = -2\eta_\alpha S, \quad \delta\Gamma^a_\alpha = -2\eta_\alpha R, \quad \delta S = \eta^\alpha W^A_\alpha, \quad \delta R = \eta^\alpha W^a_\alpha. \tag{8.113}$$

These transformations were obtained in analogy with [50, 51][7] and their are originated from a known method of obtaining $\mathcal{N}=2$ SUSY in 3 D from $\mathcal{N}=1$ SUSY in 4 D [52].

These transformations are sufficient to determine the potential:

$$f(\Phi_+,\Phi_-,S,R) = \mu SR - \Phi_+^*\Phi_+(eS+gR) + \Phi_-^*\Phi_-(eS-gR) + 2gv^2R. \tag{8.114}$$

Hence, the $\mathcal{N}=2$ supersymmetric parity-invariant Maxwell-Chern-Simons model is:

$$\begin{aligned} S = \int d^3x d^2\theta \Big\{ & \frac{1}{4}(W^A)^\alpha(W^A)_\alpha + \frac{1}{4}(W^a)^\alpha(W^a)_\alpha - \frac{1}{4}D^\alpha S D_\alpha S - \frac{1}{4}D^\alpha R D_\alpha R \\ & -\frac{1}{2}\nabla^\alpha\Phi_+^*\nabla_\alpha\Phi_+ - \frac{1}{2}\nabla^\alpha\Phi_-^*\nabla_\alpha\Phi_- + \frac{\mu}{2}(\Gamma^A)^\alpha W^a_\alpha \\ & +\mu SR - \Phi_+^*\Phi_+(eS+gR) + \Phi_-^*\Phi_-(eS-gR) + 2gv^2R \Big\} \end{aligned} \tag{8.115}$$

Now, making extensive use of the projection technique we can obtain it in terms of the component fields[8]

$$\begin{aligned} S = \int d^3x \Big\{ & -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} - \frac{1}{4}f^{\mu\nu}f_{\mu\nu} + \mu\varepsilon^{\mu\nu\rho}A_\mu\partial_\nu a_\rho + \frac{1}{2}N\Box N + \frac{1}{2}M\Box M \\ & + \phi_+^*D^\mu D_\mu\phi_+ + \phi_-^*D^\mu D_\mu\phi_- - \phi_+^*\phi_+(eN+gM)^2 - \phi_-^*\phi_-(eN-gM)^2 \\ & -\frac{1}{2}\left[e(|\phi_+|^2-|\phi_-|^2)-\mu M\right]^2 - \frac{1}{2}\left[g(|\phi_+|^2+|\phi_-|^2-2v^2)-\mu N\right]^2 \\ & +\frac{i}{2}\Lambda^\alpha\partial_\alpha{}^\beta\Lambda_\beta + \frac{i}{2}\lambda^\alpha\partial_\alpha{}^\beta\lambda_\beta + \mu\Lambda^\alpha\lambda_\alpha + \frac{i}{2}\xi^\alpha\partial_\alpha{}^\beta\xi_\beta + \frac{i}{2}\zeta^\alpha\partial_\alpha{}^\beta\zeta_\beta + \mu\xi^\alpha\zeta_\alpha \\ & + i(\psi_+^*)^\alpha\mathcal{D}_\alpha{}^\beta(\psi_+)_\beta + i(\psi_-^*)^\alpha\mathcal{D}_\alpha{}^\beta(\psi_-)_\beta \\ & + i(\psi_+^*)^\alpha\left(e\Lambda_\alpha+g\lambda_\alpha\right)\phi_+ - i\phi_+^*\left(e\Lambda^\alpha+g\lambda^\alpha\right)(\psi_+)_\alpha \\ & + i(\psi_-^*)^\alpha\left(e\Lambda_\alpha-g\lambda_\alpha\right)\phi_- - i\phi_-^*\left(e\Lambda^\alpha-g\lambda^\alpha\right)(\psi_-)_\alpha \\ & - \left[\phi_+(\psi_+^*)^\alpha+\phi_+^*(\psi_+)^\alpha\right](e\xi_\alpha+g\zeta_\alpha) + \left[\phi_-(\psi_-^*)^\alpha+\phi_-^*(\psi_-)^\alpha\right](e\xi_\alpha-g\zeta_\alpha) \\ & - (\psi_+^*)^\alpha(\psi_+)_\alpha(eN+gM) + (\psi_-^*)^\alpha(\psi_-)_\alpha(eN-gM) \Big\}. \end{aligned} \tag{8.116}$$

The first three lines correspond to the parity-preserving Maxwell-Chern-Simons self-

[7] In our case, ensuring that these transformations are consistent with parity transformations.

[8] After eliminating the auxiliary fields $F_\pm, G$, and H using their equations of motion.

dual model, while the remaining has been given to us by supersymmetry. We shall have more to say about the fermionic sector later.

Let us rewrite the above action using the more familiar Dirac notation. Remember that we are working in the Majorana representation of the gamma matrices, which implies, as a consequence, that $\psi^c = \psi^*$ for any spinor ψ. Also remembering our definition of charge-conjugate spinor:

$$\psi^c = -C^{-1}\overline{\psi}^t \Leftrightarrow (\psi^c)^\alpha = C^{\alpha\beta}\overline{\psi}_\beta = (\psi^*)^\alpha \tag{8.117}$$

We now take the indices of the Dirac conjugate to be raised and lowered as usual, that is to say, $\overline{\psi}^\alpha = C^{\alpha\beta}\overline{\psi}_\beta$ and $\overline{\psi}_\alpha = \overline{\psi}^\beta C_{\beta\alpha}$. Finally, we also need to consider $\psi^*_\alpha = (\psi_\alpha)^* = (\psi^\beta C_{\beta\alpha})^* = -(\psi^*)^\beta C_{\beta\alpha}$, which means to say that, because the matrix C is imaginary, the complex conjugate spinor gains a sign when lowering its index. All this leads to the following identifications:

$$\begin{aligned} (\psi^*)^\alpha &= \overline{\psi}^\alpha, \quad \psi^*_\alpha = -\overline{\psi}_\alpha, \qquad &&\text{for complex Dirac spinors} \\ \psi^\alpha &= \overline{\psi}^\alpha, \quad \psi_\alpha = \overline{\psi}_\alpha, \qquad &&\text{for Majorana spinors} \end{aligned} \tag{8.118}$$

Recalling the fact that, for Majorana spinors in our definitions $(\psi^*)^\alpha = \psi^\alpha$ while $\psi^*_\alpha = -\psi_\alpha$. One last step is to define the fermionic bilinears:

$$\begin{aligned} \overline{\psi}\chi &\equiv \overline{\psi}_\alpha \chi^\alpha \\ \overline{\psi}\gamma^\mu\chi &\equiv \overline{\psi}_\alpha (\gamma^\mu)^\alpha{}_\beta \chi^\beta \end{aligned}$$

All this allows us to write:

$$\begin{aligned} S = \int d^3x \Big\{ &-\frac{1}{4}F^{\mu\nu}F_{\mu\nu} - \frac{1}{4}f^{\mu\nu}f_{\mu\nu} + \mu\varepsilon^{\mu\nu\rho}A_\mu\partial_\nu a_\rho + \frac{1}{2}N\Box N + \frac{1}{2}M\Box M \\ &+ \phi_+^* D^\mu D_\mu \phi_+ + \phi_-^* D^\mu D_\mu \phi_- - \phi_+^*\phi_+(eN+gM)^2 - \phi_-^*\phi_-(eN-gM)^2 \\ &- \frac{1}{2}\left[e(|\phi_+|^2 - |\phi_-|^2) - \mu M\right]^2 - \frac{1}{2}\left[g(|\phi_+|^2 + |\phi_-|^2 - 2v^2) - \mu N\right]^2 \\ &+ \frac{i}{2}\overline{\Lambda}\partial\!\!\!/\Lambda + \frac{i}{2}\overline{\lambda}\partial\!\!\!/\lambda - \mu\overline{\Lambda}\lambda + \frac{i}{2}\overline{\xi}\partial\!\!\!/\xi + \frac{i}{2}\overline{\zeta}\partial\!\!\!/\zeta - \mu\overline{\xi}\zeta \\ &+ i\overline{\psi}_+ D\!\!\!\!/\,\psi_+ + i\overline{\psi}_- D\!\!\!\!/\,\psi_- \\ &- i\overline{\psi}_+\left(e\Lambda + g\lambda\right)\varphi_+ + i\varphi_+^*\left(e\overline{\Lambda} + g\overline{\lambda}\right)\psi_+ - i\overline{\psi}_-\left(e\Lambda - g\lambda\right)\varphi_- + i\varphi_-^*\left(e\overline{\Lambda} - g\overline{\lambda}\right)\psi_- \\ &+ \overline{\psi}_+\left(e\xi + g\zeta\right)\varphi_+ + \varphi_+^*\left(e\overline{\xi} + g\overline{\zeta}\right)\psi_+ - \overline{\psi}_-\left(e\xi - g\zeta\right)\varphi_- - \varphi_-^*\left(e\overline{\xi} - g\overline{\zeta}\right)\psi_- \\ &+ \overline{\psi}_+\psi_+(eN+gM) - \overline{\psi}_-\psi_-(eN-gM)\Big\}. \end{aligned} \tag{8.119}$$

Where $X\!\!\!/ = \gamma^\mu X_\mu$ as usual. Now defining

$$\Omega \equiv \frac{\Lambda + i\xi}{\sqrt{2}}, \quad \Delta \equiv \frac{\lambda + i\zeta}{\sqrt{2}} \quad \Rightarrow \quad \Omega^c \equiv \frac{\Lambda - i\xi}{\sqrt{2}}, \quad \Delta^c \equiv \frac{\lambda - i\zeta}{\sqrt{2}} \tag{8.120}$$

we can state our final result, after some partial integrations,

$$\begin{aligned} S = \int d^3x \Big\{ & -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} - \frac{1}{4}f^{\mu\nu}f_{\mu\nu} + \mu\varepsilon^{\mu\nu\rho}A_\mu\partial_\nu a_\rho - \frac{1}{2}(\partial_\mu N)^2 - \frac{1}{2}(\partial_\mu M)^2 \\ & - |(\partial_\mu - ieA_\mu - iga_\mu)\phi_+|^2 - |(\partial_\mu - ieA_\mu + iga_\mu)\phi_-|^2 \\ & - |\phi_+|^2(eN + gM)^2 - |\phi_-|^2(eN - gM)^2 - \frac{1}{2}\left[e(|\phi_+|^2 - |\phi_-|^2) - \mu M\right]^2 \\ & - \frac{1}{2}\left[g(|\phi_+|^2 + |\phi_-|^2 - 2v^2) - \mu N\right]^2 \\ & + i\overline{\Omega}\partial\!\!\!/\Omega + i\overline{\Delta}\partial\!\!\!/\Delta - \mu\left(\overline{\Omega}\Delta + \overline{\Delta}\Omega\right) \\ & + i\overline{\psi}_+\gamma^\mu(\partial_\mu - ieA_\mu - iga_\mu)\psi_+ + i\overline{\psi}_-(\partial_\mu - ieA_\mu + iga_\mu)\psi_- \\ & - i\sqrt{2}\overline{\psi}_+(e\Omega + g\Delta)\phi_+ + i\sqrt{2}\phi_+^*\left(e\overline{\Omega} + g\overline{\Delta}\right)\psi_+ \\ & - i\sqrt{2}\overline{\psi}_-(e\Omega^c - g\Delta^c)\phi_- + i\sqrt{2}\phi_-^*\left(e\overline{\Omega^c} - g\overline{\Delta^c}\right)\psi_- \\ & + \overline{\psi}_+\psi_+(eN + gM) - \overline{\psi}_-\psi_-(eN - gM)\Big\}. \end{aligned} \tag{8.121}$$

We have thus demonstrated, by construction, that the self-dual model corresponds to the bosonic sector of an $\mathcal{N} = 2$ supersymmetric model.[9]

In terms of the component fields, the original supersymmetry transformation $\delta X = -i\epsilon^\alpha Q_\alpha X$ (X being any superfield) and the second set supersymmetry of transformations (8.113) can be expressed as:

$$\left\{ \begin{array}{ll} \delta\varphi_\pm & = \bar{\epsilon}\psi_\pm \\ \delta\psi_\pm & = -i\gamma^\mu D_\mu\varphi_\pm\epsilon + \epsilon F_\pm \\ \delta N & = \bar{\epsilon}\xi \\ \delta\xi & = -i\gamma^\mu\partial_\mu N\epsilon + \epsilon G \\ \delta M & = \bar{\epsilon}\zeta \\ \delta\zeta & = -i\gamma^\mu\partial_\mu M\epsilon + \epsilon H \\ \delta A^\mu & = i\bar{\epsilon}\gamma^\mu\Lambda \\ \delta\Lambda & = -\frac{i}{2}\epsilon^{\mu\nu\rho}F_{\mu\nu}\gamma_\rho\epsilon \\ \delta a^\mu & = i\bar{\epsilon}\gamma^\mu\lambda \\ \delta\lambda & = -\frac{i}{2}\epsilon^{\mu\nu\rho}f_{\mu\nu}\gamma_\rho\epsilon \end{array} \right. \qquad \left\{ \begin{array}{ll} \delta\varphi_\pm & = \mp i\overline{\eta}\psi_\pm \\ \delta\psi_\pm & = \pm\gamma^\mu D_\mu\varphi_\pm\eta \mp i\eta F_\pm \\ \delta N & = -\overline{\eta}\Lambda \\ \delta\xi & = -\frac{i}{2}\epsilon^{\mu\nu\rho}F_{\mu\nu}\gamma_\rho\eta \\ \delta M & = -\overline{\eta}\lambda \\ \delta\zeta & = -\frac{i}{2}\epsilon^{\mu\nu\rho}f_{\mu\nu}\gamma_\rho\eta \\ \delta A^\mu & = i\overline{\eta}\gamma^\mu\xi \\ \delta\Lambda & = i\gamma^\mu\partial_\mu N\eta - \eta G \\ \delta a^\mu & = i\overline{\eta}\gamma^\mu\zeta \\ \delta\lambda & = i\gamma^\mu\partial_\mu M\eta - \eta H \end{array} \right.$$

[9]The correspondence can be made exact by taking $e, g, N, M \to -e, -g, -N, -M$, but, as it was said, they're physically the same theory.

Where $G = [e(|\phi_+|^2 - |\phi_+|^2) - \mu M], H = [g(|\phi_+|^2 + |\phi_+|^2 - 2v^2) - \mu N]$, and $F_\pm = \pm(eN \pm gM)\phi_\pm$.

We can use the two Majorana spinor parameters ϵ and η to define a single complex Dirac spinor parameter $\vartheta \equiv \epsilon + i\eta$, also implying $\overline{\vartheta} = \overline{\epsilon} - i\overline{\eta}$, $\vartheta^c = \epsilon - i\eta$ and $\overline{\vartheta^c} = \overline{\epsilon} + i\overline{\eta}$, in terms of which we can express the transformations as:

$$\begin{cases} \delta\varphi_+ &= \overline{\vartheta}\psi_+ \\ \delta\varphi_- &= \overline{\vartheta^c}\psi_- \\ \delta\psi_+ &= -i\not{D}\varphi_+\vartheta + \vartheta^c(eN+gM)\varphi_+ \\ \delta\psi_- &= -i\not{D}\varphi_-\vartheta^c - \vartheta(eN-gM)\varphi_- \\ \delta A^\mu &= \frac{i}{\sqrt{2}}\left(\overline{\vartheta}\gamma^\mu\Omega - \overline{\Omega}\gamma^\mu\vartheta\right) \\ \delta a^\mu &= \frac{i}{\sqrt{2}}\left(\overline{\vartheta}\gamma^\mu\Delta - \overline{\Delta}\gamma^\mu\vartheta\right) \\ \delta N &= -\frac{i}{\sqrt{2}}\left(\overline{\vartheta}\Omega - \overline{\Omega}\vartheta\right) \\ \delta M &= -\frac{i}{\sqrt{2}}\left(\overline{\vartheta}\Delta - \overline{\Delta}\vartheta\right) \\ \delta\Omega &= \frac{1}{\sqrt{2}}\left\{-\frac{i}{2}\epsilon^{\mu\nu\rho}F_{\mu\nu}\gamma_\rho + \not{\partial}N + i\left[e(|\varphi_+|^2 - |\varphi_-|^2) - \mu M)\right]\right\}\vartheta \\ \delta\Delta &= \frac{1}{\sqrt{2}}\left\{-\frac{i}{2}\epsilon^{\mu\nu\rho}f_{\mu\nu}\gamma_\rho + \not{\partial}M + i\left[g(|\varphi_+|^2 + |\varphi_-|^2 - 2v^2) - \mu N)\right]\right\}\vartheta \end{cases}$$

Finally, considering the newly defined Ω and Δ, the parity transformations are now summarized to be:

$$\begin{cases} \varphi^P_\pm &= -\varphi_\mp, \quad \psi^P_\pm = -i\gamma^1\psi_\mp \\ A^P_\mu &= \mathcal{P}^\nu_\mu A_\nu, \quad a^P_\mu = -\mathcal{P}^\nu_\mu a_\nu \\ N^P &= N, \qquad M^P = -M \\ \Omega^P &= -i\gamma^1 X, \quad \Delta^P = i\gamma^1 Y \end{cases}$$

And we just complement it with the parity transformation of the Dirac parameter of $\mathcal{N}=2$ SUSY transformations $\vartheta^P = -i\gamma^1\vartheta^c$.

As said previously, this method of derivation was inspired by references [50, 51]. Although successful, it suffers from the same minor drawback as the bosonic self-dual Maxwell-Chern-Simons model, being the *adhoc* introduction of fields. In the self-dual case, the introduction of N and M is necessary to obtain self-duality equations and a Bogomol'nyi bound, while in the supersymmetric scenario, the introduction of the superfields S and R is fundamental to achieve $\mathcal{N}=2$ SUSY through (8.113).

Now, inspired by the work [49], we are going to derive the same model from a more elegant approach, via manifest $\mathcal{N}=2$ supersymmetry in a $\mathcal{N}=2$ superspace formalism. The subject of superspace for extended supersymmetry is a very intricate and delicate one, however our task is greatly facilitated by the *almost* direct relationship of $\mathcal{N}=1$ SUSY in 4 dimensions and $\mathcal{N}=2$ SUSY in 3 dimensions, so that the superspace formalism of the former can be *almost* directly translated to the latter.

8.4 $\mathcal{N} = 2$ Supersymmetry in 2+1 dimensions

As stated in section 8.1, the Haag-Lopusanski-Sohnius theorem, here considered in 2+1 dimensions, allows for the following extension of the Poincaré algebra:

$$\{Q^I_\alpha, Q^J_\beta\} = -2P_{\alpha\beta}\delta^{IJ} + A^{IJ}\epsilon_{\alpha\beta}, \quad \left[Q^I_\alpha, P_{\beta\gamma}\right] = 0, \quad \left[Q^I_\alpha, A^{JK}\right] = 0. \tag{8.122}$$

with $I, J = 1, 2, ..., \mathcal{N}$.

In the above expression, we have $A^{IJ} = -A^{JI}$, and these objects are called *central charges*. They were not discussed so far, because they were not relevant for our previous derivation, but, in fact, the above form is now the most general extension of the Poincaré algebra, in the context of the Coleman-Mandula and Haag-Lopusanski-Sohnius theorems.

The set of transformations which mix the supercharges $Q'^I_\alpha = R^I{}_J Q^J_\alpha$ but leaves the Susy algebra invariant forms a group, the so-called R-symmetry group. In 2+1 dimensions, this is given by $SO(\mathcal{N})$ (in contrast, the 3+1 dimensional analog would be $SU(\mathcal{N})$). It is interesting to note at this point that for $\mathcal{N} = 2$ in 2+1 dimensions, the R-symmetry group is $SO(2) \simeq U(1)$, giving the same result as the case $\mathcal{N} = 1$ in 3+1 dimensions.

In the particular case in which $\mathcal{N} = 2$, since any antisymmetric object with two indices can be written using the invariant tensor $\epsilon^{IJ} = -\epsilon^{JI}$, we can write $A^{IJ} = iT\epsilon^{IJ}$. In the following, we will narrow our scope, and discuss the construction of $\mathcal{N} = 2$ extended superspace in detail.

8.4.1 $\mathcal{N} = 2$ Superspace

The main idea here is to construct an $\mathcal{N} = 2$ version of superspace, to accomplish in extended supersymmetry what we have already achieved in simple SUSY, that is, to have a formalism that allow us to construct in a simple way an $\mathcal{N} = 2$ supersymmetric invariant action, namely, by writing the integral in full superspace of a scalar function of superfields and their covariant derivatives, since their SUSY variation integrated in full superspace will be a total derivative that can be safely ignored inside the action.

The extended superspace idea basically consists in associating Grassmann variables θ^α_I with each Supersymmetry generator Q^I_α in such a way that a SUSY transformation generated by the supercharges Q^I_α can be seen as a translation in the superspace, this understood as the extended set of coordinates (x^μ, θ^α_I). In 2+1 dimensions, considering $\mathcal{N} = 1$, we have only one supercharge Q^α (a two-components Majorana spinor), and thus we need only to add a two-component real spinorial coordinate θ^α. In the case $\mathcal{N} = 2$, there are two supercharges Q^1_α and Q^2_α, and thus we need to add 2 two-component real spinors θ^α_1 e θ^α_2, obtaining 4 Grassmanian coordinates. This is very similar to what occurs in $\mathcal{N} = 1$ SUSY in 3+1 dimensions, where we grade the Poincaré algebra by introducing a pair of supercharges that are two-component complex spinors (Q_α and $\bar{Q}^{\dot{\alpha}}$), that transforms under Lorentz according with the $(1/2, 0)$ and $(0, 1/2)$ representations, respectively. Thus, we need introduce there 2 pair of Grassman coordinates that are two-component complex spinors (θ_α and $\bar{\theta}^{\dot{\alpha}}$) and transforms under Lorentz in a way analogous to their respective supercharge, in a total of 4 Grassman coordinates as before. However, notice that in the 3+1 dimensional case the dotted and undotted indices are different in nature, and in the 2+1 dimensional case, as the relevant group is $SL\left(2, \mathbb{R}\right)$ admitting only the fundamental real representation, these indices are of the same nature, and therefore can be contracted.

The SUSY algebra can be realized by representing its generators as differential operators acting on functions of superspace, the superfields $\Phi_{\alpha\beta\ldots}(x, \theta)$, as:

$$Q^I_\alpha = i\left(\partial^I_\alpha - i\theta^{\beta I}\partial_{\beta\alpha}\right), \quad P_{\alpha\beta} = -i\partial_{\alpha\beta}. \tag{8.123}$$

It is possible to define the SUSY covariant derivatives here by

$$D^I_\alpha = \partial^I_\alpha + i\theta^{\beta I}\partial_{\beta\alpha}. \tag{8.124}$$

These derivatives will commute (in the graded sense) with the SUSY generators (*i.e.*, $\left\{D^I_\alpha, Q^J_\beta\right\} = \left[D^I_\alpha, P_{\beta\gamma}\right] = 0$) and satisfy an algebra similar to the one satisfied by the supercharges (that is, they satisfy $\{D^I_\alpha, D^J_\beta\} = 2i\partial_{\alpha\beta}\delta^{IJ}$ and also $\left[D^I_\alpha, \partial_{\beta\gamma}\right] = 0$).

Let us rewrite the $\mathcal{N} = 2$ superspace coordinates in a such a way that we can take advantage of the similarity between this space and the 3+1 dimensional $\mathcal{N} = 1$ superspace, in order to facilitate future developments.

Define complex anti-commuting coordinates as

$$\theta^\alpha \equiv \theta^\alpha_1 - i\theta^\alpha_2, \quad \bar{\theta}^\alpha \equiv \theta^\alpha_1 + i\theta^\alpha_2, \tag{8.125}$$

Note here that, while $(\theta^*)^\alpha = \bar{\theta}^\alpha$, $\theta^*_\alpha = -\bar{\theta}_\alpha$ due to our definition of $C_{\alpha\beta}$ and also defining $\bar{\theta}_\alpha = \bar{\theta}^\beta C_{\beta\alpha}$. Barred objects are to be understood only in the sense just defined and not to be confused with Dirac conjugate $\overline{\psi} = \psi^\dagger\gamma^0$. Only at the very end, we shall provide the relationship between them. The fermionic derivatives are defined as

$$\partial_\alpha = \frac{1}{2}\left(\partial^{(1)}_\alpha + i\partial^{(2)}_\alpha\right), \quad \bar{\partial}_\alpha = \frac{1}{2}\left(\partial^{(1)}_\alpha - i\partial^{(2)}_\alpha\right), \tag{8.126}$$

in such a way that the following properties hold

$$\partial_\alpha\theta^\beta = \delta^\beta_\alpha, \quad \partial_\alpha\bar{\theta}^\beta = 0, \quad \bar{\partial}_\alpha\bar{\theta}^\beta = \delta^\beta_\alpha, \quad \bar{\partial}_\alpha\theta^\beta = 0. \tag{8.127}$$

Therefore, the different coordinates and derivatives behave as expected.

Let us define the supercharges in this framework as

$$Q_\alpha = \frac{1}{2}\left(Q_\alpha^{(1)} + iQ_\alpha^{(2)}\right), \quad \bar{Q}_\alpha = \frac{1}{2}\left(Q_\alpha^{(1)} - iQ_\alpha^{(2)}\right). \tag{8.128}$$

Adopting a similar definition for the covariant derivatives, we can immediately find

$$D_\alpha = \partial_\alpha + \frac{1}{2} i\bar{\theta}^\beta \partial_{\beta\alpha}, \quad \bar{D}_\alpha = \bar{\partial}_\alpha + \frac{1}{2} i\theta^\beta \partial_{\beta\alpha}. \tag{8.129}$$

The definitions made above are done in such a way that the SUSY algebra reads

$$\{Q_\alpha, \bar{Q}_\beta\} = -P_{\alpha\beta}, \quad \{Q_\alpha, Q_\beta\} = \{\bar{Q}_\alpha, \bar{Q}_\beta\} = 0. \tag{8.130}$$

and that the covariant derivatives satisfy

$$\{D_\alpha, \bar{D}_\beta\} = i\partial_{\alpha\beta}, \quad \{D_\alpha, D_\beta\} = \{\bar{D}_\alpha, \bar{D}_\beta\} = 0. \tag{8.131}$$

The structure built above bears a close resemblance with the $\mathcal{N} = 1$ SUSY algebra in 3+1 dimensions. Of course, here we don't have dotted and undotted indices, because the relevant group here is $SL(2, \mathbb{R})$ that does not admit an anti-fundamental representation inequivalent to the fundamental one, as the case in the 3+1 dimensional case where the relevant group is $SL(2, \mathbb{C})$, as it was already highlighted above. Interestingly, here we can write contractions as $\bar{\theta}^\alpha \theta_\alpha$, what is not allowed in 3+1 dimensions, because upon contracting dotted and undotted objects, we would not obtain an object invariant under $SL(2, \mathbb{C})$, therefore also not Lorentz invariant.

One can define the superspace integrals

$$\int d^4\theta = \int d^2\theta \, d^2\bar{\theta}, \quad \int d^2\theta = \frac{1}{2}\int d\theta^\alpha d\theta_\alpha, \quad \int d^2\bar{\theta} = \frac{1}{2}\int d\bar{\theta}^\alpha d\bar{\theta}_\alpha \tag{8.132}$$

These integrals act like derivatives, as is usual with Grassman integrals, and the usual projection trick can be applied, for instance: $\int d^2\theta ... = D^2 ...|_{\theta=\bar{\theta}=0}$ and $\int d^2\bar{\theta} ... = \bar{D}^2 ...|_{\theta=\bar{\theta}=0}$.

8.4.2 $\mathcal{N} = 2$ Superfields

The simplest superfields that we need to build in $\mathcal{N} = 2$ SUSY invariant theories in 2+1 dimensions (corresponding to $\mathcal{N} = 2$ SUSY *irreps*), are the complex chiral superfield $\Phi(x, \theta, \bar{\theta})$ and the real scalar superfield $V(x, \theta, \bar{\theta})$, representing the $\mathcal{N} = 2$ scalar and gauge supermultiplets, respectively.

Chiral Superfield

Let us define the chiral scalar superfield by the SUSY covariant condition

$$\bar{D}_\alpha \Phi(x, \theta, \bar{\theta}) = 0 \tag{8.133}$$

In order to solve this constraint, we introduce the chiral variables $x_{L,R}$ defined by

$$x_L^{\alpha\beta} = x^{\alpha\beta} + \frac{i}{4}\left(\theta^\alpha \bar{\theta}^\beta + \theta^\beta \bar{\theta}^\alpha\right), \quad x_R^{\alpha\beta} = x^{\alpha\beta} - \frac{i}{4}\left(\theta^\alpha \bar{\theta}^\beta + \theta^\beta \bar{\theta}^\alpha\right). \tag{8.134}$$

A superfield that depends only on the chiral superspace $\Phi = \Phi(x_L, \theta)$ will automatically solve the chiral constraint, thus obtaining a chiral scalar superfield. In fact, one can find

$$\bar{\partial}_\alpha \Phi(x_L, \theta) = \frac{\partial x_L^{\beta\gamma}}{\partial \bar{\theta}^\alpha} \frac{\partial}{\partial x_L^{\beta\gamma}} \Phi + \bar{\partial}_\alpha \Phi = -\frac{i}{2}\theta^\beta \partial^L_{\alpha\beta}, \tag{8.135}$$

and also

$$\partial_{\alpha\beta} = \frac{\partial x_L^{\gamma\delta}}{\partial x^{\alpha\beta}} \frac{\partial}{\partial x_L^{\beta\gamma}} = \partial^L_{\alpha\beta}, \tag{8.136}$$

thus one find at the end

$$\bar{D}_\alpha \Phi = \bar{\partial}_\alpha \Phi + \frac{i}{2}\theta^\beta \partial_{\alpha\beta}\Phi = -\frac{i}{2}\theta^\beta \partial^L_{\alpha\beta}\Phi + \frac{i}{2}\theta^\beta \partial^L_{\alpha\beta}\Phi = 0. \tag{8.137}$$

The superfield $\Phi = \Phi(x_L, \theta)$ automatically solves the chiral constraint $\bar{D}_\alpha \Phi = 0$, and analogously the superfield $\bar{\Phi} = \bar{\Phi}(x_R, \bar{\theta})$ satisfies the anti-chiral constraint $D_\alpha \bar{\Phi} = 0$. To avoid any risk of confusion, we remark that although the chiral superfield $\Phi(x_L, \theta)$ does not have explicit dependence on $\bar{\theta}$, it has an implicit one on it through x_L.

Performing a Taylor expansion on the Grassman variable θ, we obtain

$$\Phi(x_L, \theta) = \phi(x_L) + \theta^\alpha \psi_\alpha(x_L) - \theta^2 F(x_L). \tag{8.138}$$

Therefore, the chiral superfield carry information of 1 complex scalar ϕ, 1 two-component complex fermion ψ and 1 complex auxiliary scalar F, giving us a total of 2 bosonic and 2 fermionic degrees of freedom on-shell. This superfield represents the $\mathcal{N} = 2$ scalar supermultiplet, where matter will sit in our theories.

From the perspective of $\mathcal{N} = 1$ SUSY representations, the scalar multiplet (superspin 0) is formed by 1 real scalar and 1 two-component real fermion, giving us 1 bosonic and 1 fermionic degrees of freedom on-shell. Thus, one can see that the degrees of freedom of an $\mathcal{N} = 2$ scalar multiplet is equivalent to the d.o.f. of two $\mathcal{N} = 1$ real scalar multiplets.

The free action associated with the chiral scalar superfield can be found in analogy with the $\mathcal{N}=1$ case in 3+1 dimensions, and by dimensional analysis is given by

$$S_{\text{free}}^{\text{scalar}} = \int d^3x d^4\theta\, \bar{\Phi}\Phi \tag{8.139}$$

Let us expand the above expression in components. We will consider $x_L^{\alpha\beta} = x^{\alpha\beta} + \delta x^{\alpha\beta}$, where we defined $\delta x^{\alpha\beta} = \frac{i}{4}\left(\theta^\alpha\bar{\theta}^\beta + \theta^\beta\bar{\theta}^\alpha\right)$ and then expand $\Phi(x_L)$ around x, that is, $\Phi(x+\delta x) = \Phi(x) + \delta x^{\alpha\beta}\partial_{\alpha\beta}\Phi + \frac{1}{2}\delta x^{\alpha\beta}\delta x^{\gamma\delta}\partial_{\alpha\beta}\partial_{\gamma\delta}\Phi$, thus

$$\begin{aligned}\Phi &= \phi + \theta\psi - \theta^2 F + \frac{i}{4}\left(\theta^\alpha\bar{\theta}^\beta + \theta^\beta\bar{\theta}^\alpha\right)\partial_{\alpha\beta}\left(\phi + \theta\psi\right) - \frac{1}{32}\left(\theta^\alpha\bar{\theta}^\beta + \theta^\beta\bar{\theta}^\alpha\right)\left(\theta^\gamma\bar{\theta}^\delta + \theta^\delta\bar{\theta}^\gamma\right)\partial_{\alpha\beta}\partial_{\gamma\delta}\phi \\ &= \phi + \theta\psi - \theta^2 F + \frac{i}{2}\theta^\alpha\bar{\theta}^\beta\partial_{\alpha\beta}\phi - \frac{i}{2}\theta^2\bar{\theta}^\alpha\partial_\alpha{}^\beta\psi_\beta + \frac{1}{4}\theta^2\bar{\theta}^2\Box\phi. \end{aligned} \tag{8.140}$$

In the same way, we can write

$$\begin{aligned}\bar{\Phi} &= \bar{\phi} + \bar{\theta}\bar{\psi} - \bar{\theta}^2 \bar{F} - \frac{i}{4}\left(\theta^\alpha\bar{\theta}^\beta + \theta^\beta\bar{\theta}^\alpha\right)\partial_{\alpha\beta}\left(\bar{\phi} + \bar{\theta}\bar{\psi}\right) - \frac{1}{32}\left(\theta^\alpha\bar{\theta}^\beta + \theta^\beta\bar{\theta}^\alpha\right)\left(\theta^\gamma\bar{\theta}^\delta + \theta^\delta\bar{\theta}^\gamma\right)\partial_{\alpha\beta}\partial_{\gamma\delta}\bar{\phi} \\ &= \bar{\phi} + \bar{\theta}\bar{\psi} - \bar{\theta}^2 \bar{F} - \frac{i}{2}\theta^\alpha\bar{\theta}^\beta\partial_{\alpha\beta}\bar{\phi} - \frac{i}{2}\bar{\theta}^2\theta^\alpha\partial_\alpha{}^\beta\bar{\psi}_\beta + \frac{1}{4}\theta^2\bar{\theta}^2\Box\bar{\phi} \end{aligned} \tag{8.141}$$

In is instructive to observe how $\bar{\Phi} = \Phi^*$ by noting for example that $(\theta\psi)^* = (\theta^\alpha\psi_\alpha)^* = \psi_\alpha^*\theta^{*\alpha} = \theta^{*\alpha}(-\psi_\alpha^*) = \bar{\theta}\bar{\psi}$, consistently with our previous definitions.

In the action, we have an integral over the full superspace, therefore the only possible contributions will come from $\bar{\theta}^2\theta^2$. This includes,

$$\begin{aligned} S \supset \int d^3x d^4\theta\, \Big[&\frac{1}{4}\theta^\alpha\bar{\theta}^\beta\theta^\gamma\bar{\theta}^\delta\, \partial_{\alpha\beta}\phi\, \partial_{\gamma\delta}\bar{\phi} + \bar{\theta}^2\theta^2\bar{F}F \\ &- \frac{i}{2}\theta\psi\,\theta^\alpha\bar{\theta}^\beta\,\partial_{\alpha\beta}\,\bar{\theta}\bar{\psi} + \frac{i}{2}\bar{\theta}\bar{\psi}\,\theta^\alpha\bar{\theta}^\beta\,\partial_{\alpha\beta}\,\theta\psi\Big] \end{aligned} \tag{8.142}$$

The part with F give us immediately $\bar{F}F$ after integration. The part with fermions is as follows:

$$\begin{aligned} A &= -\frac{i}{2}\theta\psi\,\theta^\alpha\bar{\theta}^\beta\,\partial_{\alpha\beta}\,\bar{\theta}\bar{\psi} = +\frac{i}{2}\theta^\alpha\bar{\theta}^\beta\left(\theta^x\bar{\theta}^y\right)\psi_x\partial_{\alpha\beta}\bar{\psi}_y \\ &= -\frac{i}{2}\theta^\alpha\theta^x\bar{\theta}^\beta\bar{\theta}^y\psi_x\partial_{\alpha\beta}\bar{\psi}_y = -\frac{i}{2}C^{\alpha x}C^{\beta y}\bar{\theta}^2\theta^2\psi_x\partial_{\alpha\beta}\bar{\psi}_y \\ &= +\frac{i}{2}\bar{\theta}^2\theta^2\psi^\alpha\partial_\alpha{}^\beta\bar{\psi}_\beta \end{aligned} \tag{8.143}$$

Thus, the fermionic part will give us $\int d^4\theta\,(A + ...) = +\frac{i}{2}\psi^\alpha\partial_\alpha{}^\beta\bar{\psi}_\beta + \frac{i}{2}\bar{\psi}^\alpha\partial_\alpha{}^\beta\psi_\beta$. For the scalar part we have $\theta^\alpha\bar{\theta}^\beta\partial_{\alpha\beta}\phi\,\theta^\gamma\bar{\theta}^\delta\partial_{\gamma\delta}\bar{\phi} = -\bar{\theta}^2\theta^2\partial_{\alpha\beta}\phi\partial^{\beta\alpha}\bar{\phi}$, but we know that $\partial_{\alpha\beta}\phi\partial^{\beta\alpha}\bar{\phi} = \left(\gamma^\mu C^{-1}\right)^{\alpha\beta}\left(-C\gamma^\nu\right)_{\beta\alpha}\partial_\mu\phi\partial_\nu\bar{\phi} = -\operatorname{Tr}(\gamma^\mu\gamma^\nu)\partial_\mu\phi\partial_\nu\bar{\phi} = 2\partial_\mu\phi\partial^\mu\bar{\phi}$. Therefore, we have from this term a contribution $-\frac{1}{2}\partial_\mu\phi\partial^\mu\bar{\phi}$.

We also have from the scalar contribution:

$$B = \frac{1}{4}\bar{\theta}^2\theta^2 \left(\bar{\phi}\Box\phi + \phi\Box\bar{\phi}\right) = \bar{\theta}^2\theta^2 \left(-\frac{1}{2}\partial_\mu\bar{\phi}\partial^\mu\phi\right). \tag{8.144}$$

Therefore, putting all together, we find that the free action for the chiral superfield in components is given by

$$S^{\text{scalar}}_{\text{free}} = \int d^3x \left[-\partial^\mu\bar{\phi}\partial_\mu\phi + i\bar{\psi}_\alpha \left(\gamma^\mu\right)^\alpha{}_\beta \partial_\mu\psi^\beta + \bar{F}F\right]. \tag{8.145}$$

In order to introduce interactions, we can add an integral in the half-superspace of a function that only depends on Φ (in this case we say that it is *holomorphic*), together with its hermitian conjugate. This function $W(\Phi)$ is usually called a *superpotential.*

$$S_{\text{int}} = \int d^3x\, d^2\theta\; W(\Phi) + h.c. \tag{8.146}$$

Joining this two pieces, we obtain an interacting theory in 2+1 dimensions for the $\mathcal{N}=2$ scalar supermultiplet. In the following we will introduce the superfield representing the gauge supermultiplet, and then we will be able to study also the interaction between gauge and matter.

Gauge Superfield

The $\mathcal{N}=2$ gauge supermultiplet is formed by 1 real scalar, 1 two-component complex fermion and 1 gauge field. On-shell, this gives us 2 bosonic and 2 fermionic degrees of freedom.

Let us define the real scalar superfield $V(x,\theta,\bar{\theta})$ representing this $\mathcal{N}=2$ SUSY *irrep*, defined as a real superfield without any other constraint. In principle, this superfield would have 16 degrees of freedom, but after imposing gauge transformations, some will be eliminated. The unconstrained real scalar superfield can be written as[10]:

$$\begin{aligned} V = {} & C + \theta\chi + \bar{\theta}\bar{\chi} + i\theta^2 M - i\bar{\theta}^2\bar{M} + \theta\gamma^\mu\bar{\theta}A_\mu + \theta\bar{\theta}\sigma \\ & + \theta^2\bar{\theta}^\alpha\left(\bar{\lambda}_\alpha - \frac{i}{2}\partial_\alpha{}^\beta\chi_\beta\right) + \bar{\theta}^2\theta^\alpha\left(\lambda_\alpha - \frac{i}{2}\partial_\alpha{}^\beta\bar{\chi}_\beta\right) + \theta^2\bar{\theta}^2\left(D + \frac{1}{4}\Box C\right) \end{aligned} \tag{8.147}$$

The above choice of components was only done to simplify what comes next, but without loss of generality. Let us define the following Abelian supergauge transformation:

$$V' = V + i\left(\Lambda - \bar{\Lambda}\right), \tag{8.148}$$

[10]As in $\mathcal{N}=1$ superspace: $\theta^2 = \frac{1}{2}\theta^\alpha\theta_\alpha, \bar{\theta}^2 = \frac{1}{2}\bar{\theta}^\alpha\bar{\theta}_\alpha$. While $\theta\bar{\theta} = \theta^\alpha\bar{\theta}_\alpha$ and $\theta\gamma^\mu\bar{\theta} = \theta_\alpha(\gamma^\mu)^\alpha{}_\beta\bar{\theta}^\beta$.

where $\Lambda \approx (\phi, \psi, F)$ is a chiral scalar superfield, such that we have the real superfield

$$i\left(\Lambda-\bar{\Lambda}\right)=i\left(\phi-\bar{\phi}\right)+i\left(\theta\psi-\bar{\theta}\bar{\psi}\right)-i\left(\theta^{2}F-\bar{\theta}^{2}\bar{F}\right)-\frac{1}{4}\left(\theta^{\alpha}\bar{\theta}^{\beta}+\theta^{\beta}\bar{\theta}^{\alpha}\right)\partial_{\alpha\beta}\left(\phi+\bar{\phi}\right)$$
$$+\frac{1}{2}\theta^{2}\bar{\theta}^{\alpha}\partial_{\alpha}{}^{\beta}\psi_{\beta}-\frac{1}{2}\bar{\theta}^{2}\theta^{\alpha}\partial_{\alpha}{}^{\beta}\bar{\psi}_{\beta}+\frac{i}{4}\theta^{2}\bar{\theta}^{2}\square\left(\phi-\bar{\phi}\right) \tag{8.149}$$

Thus, performing the supergauge transformation described above, we find

$$\begin{aligned}
&C\to C+i\left(\phi-\bar{\phi}\right),\\
&\chi\to\chi+i\psi,\\
&M\to M-F,\\
&\lambda\to\lambda,\\
&D\to D,
\end{aligned} \tag{8.150}$$

Notice that we have λ and D invariants under such transformation, thanks to the convenient parametrization that we have adopted for the real superfield. Looking at the above transformations, it is easy to see that we can conveniently choose the chiral superfield components in such a way that we eliminate C, χ and M. This is a supergauge choice called Wess-Zumino (WZ) gauge. In fact, we must choose $Im(\phi) = C/2$, $\psi = i\chi$ and $F = M$. There is still freedom to choose $Re(\phi)$, and this freedom is associated with the $U(1)$ gauge symmetry. In the gauge sector, we have the following transformation:

$$\theta\gamma^{\mu}\bar{\theta}A_{\mu}\to\theta\gamma^{\mu}\bar{\theta}A_{\mu}-\frac{1}{4}\left(\theta^{\alpha}\bar{\theta}^{\beta}+\theta^{\beta}\bar{\theta}^{\alpha}\right)\partial_{\alpha\beta}\left(\phi+\bar{\phi}\right) \tag{8.151}$$

Notice that $\theta\gamma^{\mu}\bar{\theta}A_{\mu}=\theta_{\alpha}\left(\gamma^{\mu}\right)^{\alpha}{}_{\beta}\bar{\theta}^{\beta}A_{\mu}=\theta_{\alpha}A^{\alpha}{}_{\beta}\bar{\theta}^{\beta}=-\theta^{\alpha}A_{\alpha\beta}\bar{\theta}^{\beta}$. Thus, the above transformation can be written as: $-\theta^{\alpha}A_{\alpha\beta}\bar{\theta}^{\beta}\to-\theta^{\alpha}A_{\alpha\beta}\bar{\theta}^{\beta}-\frac{1}{2}\left(\theta^{\alpha}\bar{\theta}^{\beta}+\theta^{\beta}\bar{\theta}^{\alpha}\right)\partial_{\alpha\beta}Re(\phi)$, giving us precisely the transformation of an Abelian gauge field, that is,

$$A_{\alpha\beta}\to A_{\alpha\beta}+\partial_{\alpha\beta}\omega,\quad\text{where}\quad\omega=Re(\phi) \tag{8.152}$$

Adopting the WZ supergauge described above, we can write the real scalar superfield as:

$$V_{\rm WZ}\left(x,\theta,\bar{\theta}\right)=\theta^{\alpha}\bar{\theta}_{\alpha}\,\sigma(x)-\theta^{\alpha}\bar{\theta}^{\beta}A_{\alpha\beta}+\theta^{2}\bar{\theta}^{\alpha}\,\bar{\lambda}_{\alpha}(x)+\bar{\theta}^{2}\theta^{\alpha}\,\lambda_{\alpha}(x)+\theta^{2}\bar{\theta}^{2}\,D(x). \tag{8.153}$$

In the above expression, we have 2 bosonic and 2 fermionic degrees of freedom on-shell, since D is only an auxiliary field. Therefore, the WZ gauge allows us to see the physical degrees of freedom of the $\mathcal{N}=2$ gauge supermultiplet represented here by the $\mathcal{N}=2$ real scalar superfield $V\left(x,\theta,\bar{\theta}\right)$.

From the point of view of $\mathcal{N}=1$ representations, the gauge supermultiplet (superspin 1/2), formed by 1 gauge boson $A_{\alpha\beta}$, 1 two-component complex fermion λ_α and 1 real scalar σ, can be seen as an $\mathcal{N}=1$ gauge multiplet (containing 1 gauge boson and 1 two-component real fermion) together with an $\mathcal{N}=1$ scalar multiplet (containing 1 real scalar and 1 two-component real fermion). We can also understand the $\mathcal{N}=2$ gauge supermultiplet in 3 dimensions as the dimensional reduction of the $\mathcal{N}=1$ gauge multiplet in 4 dimensions, where the real scalar σ plays the role of the A_3 component of the gauge field.

In the WZ gauge, there is a huge simplification. In fact, we can find

$$\begin{aligned} V_{WZ}^2 &= \left(\theta\bar{\theta}\right)\left(\theta\bar{\theta}\right)\sigma^2 + \theta^\alpha\bar{\theta}^\beta\theta^\gamma\bar{\theta}^\delta A_{\alpha\beta}A_{\gamma\delta} - 2\left(\theta\bar{\theta}\right)\theta^\alpha\bar{\theta}^\beta A_{\alpha\beta}\sigma \\ &= -2\theta^2\bar{\theta}^2\sigma^2 - \theta^2\bar{\theta}^2 A_{\alpha\beta}A^{\alpha\beta} + 2\theta^2\bar{\theta}^2\sigma A^\beta{}_\beta \\ &= -2\theta^2\bar{\theta}^2\sigma^2 - 2\theta^2\bar{\theta}^2 A_\mu A^\mu, \end{aligned} \tag{8.154}$$

because we have $A^{\alpha\beta}A_{\alpha\beta} = \left(\gamma^\mu C^{-1}\right)^{\alpha\beta}\left(-C\gamma^\nu\right)_{\beta\alpha} A_\mu A_\nu = -\operatorname{Tr}\left(\gamma^\mu\gamma^\nu\right) A_\mu A_\nu = 2A_\mu A^\mu$, and also $A^\beta{}_\beta = \left(\gamma^\mu\right)^\beta{}_\beta A_\mu = tr(\gamma^\mu)A_\mu = 0$. Furthermore, we see that $V_{WZ}^n = 0$ for any $n > 2$. Therefore, in the WZ gauge, we can expand the exponential as $e^V = 1 + V_{WZ} + \frac{1}{2}V_{WZ}^2$, which will be useful when considering the gauge interaction with matter.

- **Maxwell**

Now, we need to understand what is the action describing the dynamics of this real scalar superfield $V(x,\theta,\bar{\theta})$. The Super-Maxwell action can be written in two different ways, using the scalar field strength $\Sigma = \bar{D}^\alpha\left(e^{-V}D_\alpha e^V\right)$ or the spinorial field strength $W_\alpha = \bar{D}^2\left(e^{-V}D_\alpha e^V\right)$, the latter being closer to the 4d $\mathcal{N}=1$ case, as follows

$$S_{\text{free}}^{\text{gauge}} = -\frac{1}{4e^2}\int d^3x d^4\theta\ \Sigma^2 = -\frac{1}{4e^2}\int d^3x d^2\theta\ W^\alpha W_\alpha. \tag{8.155}$$

In the Abelian case, since the real scalar objects do commute, we can simply write: $\Sigma = \bar{D}^\alpha D_\alpha V$ and $W_\alpha = \bar{D}^2 D_\alpha V$. To see that both options give the same result, we need only to verify that we have $\int d^2\bar{\theta}\,\Sigma^2 = \bar{D}^2\Sigma^2 = W^\alpha W_\alpha$, but this is not hard to check, if we are careful with the Leibniz rule and remember that $\psi^\alpha\psi^\beta = -C^{\alpha\beta}\psi^2$ and also that three or more $\bar{D}$ gives automatically zero. Let us focus a bit on the latter action. First of all, we see that $\bar{D}^\alpha W_\alpha = 0$, and thus the spinor field strength is a chiral superfield, since $\bar{D}^\alpha\bar{D}^2 \equiv 0$, following from $\{\bar{D},\bar{D}\} = 0$. Second, W_α is a gauge-invariant object. In fact, using $\bar{D}\Lambda = D\bar{\Lambda} = 0$,

$$\begin{aligned} W_\alpha \to W'_\alpha = \bar{D}^2 D_\alpha\left[V + i\left(\Lambda - \bar{\Lambda}\right)\right] &= W_\alpha + \frac{i}{2}\bar{D}^\beta\bar{D}_\beta D_\alpha\Lambda \\ &= W_\alpha - \frac{1}{2}\bar{D}^\beta\{\bar{D}_\beta, D_\alpha\}\Lambda = W_\alpha. \end{aligned} \tag{8.156}$$

Therefore, any action constructed using W_α will be gauge-invariant. What are the components of W_α and Σ? As we have already said, in the Abelian case, we can write

$W_\alpha = \bar{D}^2 \left(e^{-V} D_\alpha e^V\right) = \bar{D}^2 D_\alpha V$, and since this object is gauge-invariant, we can use the expression in the WZ gauge. Thus, we need to compute

$$D_\alpha V = \bar{\theta}_\alpha \sigma - \bar{\theta}^\beta A_{\alpha\beta} + \theta_\alpha \bar{\theta}^\beta \bar{\lambda}_\beta + \bar{\theta}^2 \lambda_\alpha + \theta_\alpha \bar{\theta}^2 D - \frac{i}{2}\bar{\theta}^2 \theta^\beta \partial_{\alpha\beta}\sigma + \frac{i}{2}\bar{\theta}^2\theta^\delta \partial_\alpha{}^\beta A_{\beta\delta} + \frac{i}{2}\bar{\theta}^2\theta^2\partial_\alpha{}^\beta \bar{\lambda}_\beta \tag{8.157}$$

$$\begin{aligned} \bar{D}_x D_\alpha V = & \, C_{x\alpha}\sigma - A_{\alpha x} - \theta_\alpha \bar{\lambda}_x + \bar{\theta}_x \lambda_\alpha - \theta_\alpha \bar{\theta}_x D - \frac{i}{2}\bar{\theta}_x \theta^\beta \partial_{\alpha\beta}\sigma + \frac{i}{2}\bar{\theta}_x \theta^\delta \partial_\alpha{}^\beta A_{\beta\delta} + \frac{i}{2}\bar{\theta}_x \theta^2 \partial_\alpha{}^\beta \bar{\lambda}_\beta \\ & + \frac{i}{2}\theta^y \bar{\theta}_\alpha \partial_{xy}\sigma - \frac{i}{2}\theta^y \bar{\theta}^\beta \partial_{xy} A_{\alpha\beta} + \frac{i}{2}\theta^2 \bar{\theta}^\beta \partial_{x\alpha}\bar{\lambda}_\beta + \frac{i}{2}\bar{\theta}^2 \theta^y \partial_{xy}\lambda_\alpha + \frac{i}{2}\theta^2\bar{\theta}^2 \partial_{x\alpha} D \\ & + \frac{1}{4}\theta^2\bar{\theta}^2 \partial_x{}^\beta \partial_{\alpha\beta}\sigma - \frac{1}{4}\theta^2\bar{\theta}^2 \partial_x{}^\delta \partial_\alpha{}^\beta A_{\beta\delta} \end{aligned} \tag{8.158}$$

From the above expression, we can compute $\Sigma = \bar{D}^\alpha D_\alpha V = C^{\alpha x}\left(\bar{D}_x D_\alpha V\right)$ and find

$$\begin{aligned} \Sigma = & -2\sigma + \theta\bar{\lambda} + \bar{\theta}\lambda + \theta\bar{\theta}D + \frac{i}{2}\theta^2\bar{\theta}^\alpha \partial_\alpha{}^\beta \bar{\lambda}_\beta + \frac{i}{2}\bar{\theta}^2\theta^\alpha \partial_\alpha{}^\beta \lambda_\beta \\ & - \frac{i}{2}\theta^\gamma \bar{\theta}^\alpha \left(\partial_\alpha{}^\beta A_{\beta\gamma} + \partial_\gamma{}^\beta A_{\beta\alpha}\right) + \frac{1}{2}\theta^2\bar{\theta}^2 \Box\sigma \end{aligned} \tag{8.159}$$

From the above expression, one can find

$$\int d^4\theta\, \Sigma^2 = -2\sigma\Box\sigma - 2D^2 - i\bar{\lambda}^\alpha \partial_\alpha{}^\beta \lambda_\beta - i\lambda^\alpha \partial_\alpha{}^\beta \bar{\lambda}_\beta + \hat{F}^{\alpha\gamma}\hat{F}_{\alpha\gamma}$$

Remembering the definition $\hat{F}_{\alpha\gamma} = -\frac{1}{2}\left(\partial_\alpha{}^\beta A_{\beta\gamma} + \partial_\gamma{}^\beta A_{\beta\alpha}\right)$. But this last part, as we already saw, can be written as $\hat{F}^{\alpha\gamma}\hat{F}_{\alpha\gamma} = F^{\mu\nu}F_{\mu\nu}$ with $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$.

Putting all together, we obtain

$$S^{\text{Maxwell}}_{\mathcal{N}=2} = -\frac{1}{4e^2}\int d^3x\, d^4\theta\, \Sigma^2 = \frac{1}{e^2}\int d^3x \left[-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}\sigma\Box\sigma + \frac{i}{2}\bar{\lambda}_\alpha \partial^\alpha{}_\beta \lambda^\beta + \frac{1}{2}D^2\right] \tag{8.160}$$

Therefore, we have obtained precisely what was expected for the free action with the degrees of freedom described by the $\mathcal{N} = 2$ gauge supermultiplet.

After some computation, one can also find $W_\alpha = \bar{D}^2 D_\alpha V$:

$$\begin{aligned} W_\alpha = & -\lambda_\alpha - \theta_\alpha D + i\theta^\beta \partial_{\alpha\beta}\sigma - \frac{i}{2}\theta^\gamma \partial_\alpha{}^\beta A_{\beta\gamma} - \frac{i}{2}\theta^\gamma \partial_\gamma{}^\beta A_{\beta\alpha} - i\theta^2 \partial_\alpha{}^\beta \bar{\lambda}_\beta - \frac{i}{2}\theta^x\bar{\theta}^y \partial_{xy}\lambda_\alpha \\ & + \frac{i}{2}\theta^2\bar{\theta}^\beta \partial_{\beta\alpha} D - \frac{1}{2}\theta^2\bar{\theta}_\alpha \Box\sigma + \frac{1}{4}\theta^2\bar{\theta}^\beta \Box A_{\alpha\beta} - \frac{1}{4}\theta^2\bar{\theta}^\gamma \partial_{\gamma\delta}\partial_{\alpha\beta}A^{\beta\delta} - \frac{1}{4}\theta^2\bar{\theta}^2 \Box\lambda_\alpha \end{aligned} \tag{8.161}$$

It would be instructive rewrite the super-Maxwell action now using the field strength W_α. In principle, one can write $W_\alpha = W_\alpha(x_L, \theta)$ using chiral coordinates since it is a chiral superfield. In fact, W_α can be written as:

$$W_\alpha(x_L,\theta) = -\lambda_\alpha - \theta_\alpha D + i\theta^\beta \partial_{\alpha\beta}\sigma + i\theta^\gamma \hat{F}_{\alpha\gamma} - i\theta^2 \partial_\alpha{}^\beta \bar{\lambda}_\beta \tag{8.162}$$

Where it is understood that the components are functions of x_L, and the full expression can naturally be obtained by expanding around x to extract the $\bar{\theta}$ dependence. Thus,

$$W^\alpha W_\alpha = \left(2i\lambda^\alpha \partial_\alpha{}^\beta \bar{\lambda}_\beta + 2D^2 - \partial_{\alpha\beta}\sigma\partial^{\alpha\beta}\sigma - \hat{F}^{\alpha\beta}\hat{F}_{\alpha\beta} - 2\partial^{\alpha\beta}\sigma\hat{F}_{\alpha\beta}\right)\theta^2 \tag{8.163}$$

Remember that $\int d^2\theta\, \theta^2 = -1$. The last term gives us

$$\partial^{\alpha\beta}\sigma\hat{F}_{\alpha\beta} = \left(\gamma^\mu C^{-1}\right)^{\alpha\beta}\partial_\mu\sigma\left(-C\Sigma^{\nu\rho}\right)_{\alpha\beta}F_{\nu\rho} = -\operatorname{Tr}\left(\gamma^\mu\Sigma^{\nu\rho}\right)\partial_\mu\sigma F_{\nu\rho}$$

. But since $\operatorname{Tr}\left(\gamma^\mu\Sigma^{\nu\rho}\right) = -i\epsilon^{\mu\nu\rho}$, we have $\partial^{\alpha\beta}\sigma f_{\alpha\beta} = i\epsilon^{\mu\nu\rho}\partial_\mu\sigma F_{\nu\rho} = \partial_\mu(i\epsilon^{\mu\nu\rho}\sigma F_{\nu\rho}) - i\sigma\epsilon^{\mu\nu\rho}\partial_\mu F_{\nu\rho}$, giving only a total derivative, because of Bianchi identity. Therefore,

$$-\frac{1}{4}\int d^2\theta\, W^\alpha W_\alpha = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{i}{2}\bar{\lambda}_\alpha\partial^\alpha{}_\beta\lambda^\beta + \frac{1}{2}D^2 - \frac{1}{2}\partial_\mu\sigma\partial^\mu\sigma - \partial_\mu(...) \tag{8.164}$$

That gives us exactly the same expression as before if we discard the boundary term.

- **Chern-Simons**

In 2+1 dimensions a new possibility arises. We can contract $\bar{D}^\alpha V$ with $D_\alpha V$ thanks to the $SL(2,\mathbb{R})$ structure, something that would not be possible in 3+1 dimensions because $SL(2,\mathbb{C})$ has 2 inequivalent representations, and the contraction of dotted and undotted indices would not give rise to an invariant. Therefore, we can propose the following term:

$$S^{\text{CS}}_{\mathcal{N}=2} = -\frac{\mu}{2e^2}\int d^3x d^4\theta\; V\Sigma = \frac{\mu}{2e^2}\int d^3x d^4\theta\; \bar{D}^\alpha V D_\alpha V \tag{8.165}$$

The action above is the $\mathcal{N}=2$ incarnation of the Chern-Simons term. Performing a gauge transformation,

$$\begin{aligned}\bar{D}^\alpha V D_\alpha V \to\ & \bar{D}^\alpha\left[V + i\left(\Lambda - \bar{\Lambda}\right)\right] D_\alpha\left[V + i\left(\Lambda - \bar{\Lambda}\right)\right] \\ &= \bar{D}^\alpha V D_\alpha V - i\bar{D}^\alpha\bar{\Lambda}D_\alpha V + i\bar{D}^\alpha V D_\alpha\Lambda + \bar{D}^\alpha\bar{\Lambda}D_\alpha\Lambda \\ &= \bar{D}^\alpha V D_\alpha V + D(...) + \bar{D}(...),\end{aligned} \tag{8.166}$$

where we used integration by parts in the SUSY covariant derivatives and used the property $\{\bar{D}^\alpha, D_\alpha\} = 0$, as well as the definition of chiral superfields $\bar{D}_\alpha\Lambda = D_\alpha\bar{\Lambda} = 0$. Therefore, the Chern-Simons action is gauge invariant up to total SUSY covariant derivatives.

The component form of the above action can be obtained by using the explicit form of V and Σ and performing the integration over $d^4\theta$. In fact, we have,

$$\begin{aligned} V\,\Sigma \supset &-4\sigma D\,\theta^2\bar\theta^2 + \bar\theta^2\theta^\alpha\lambda_\alpha\theta^\beta\bar\lambda_\beta + \theta^2\bar\theta^\alpha\bar\lambda_\alpha\bar\theta^\beta\lambda_\beta - \theta^\alpha\bar\theta^\beta A_{\alpha\beta}\theta^\gamma\bar\theta_\gamma D \\ &+ \frac{i}{2}\theta^\alpha\bar\theta^\beta A_{\alpha\beta}\theta^\gamma\bar\theta^\delta\left(\partial_\delta{}^\omega A_{\omega\gamma} + \partial_\gamma{}^\omega A_{\omega\delta}\right) \\ &= \theta^2\bar\theta^2\left[-4\sigma D - 2\bar\lambda^\alpha\lambda_\alpha + iA^{\gamma\delta}\hat F_{\delta\gamma}\right] \end{aligned} \tag{8.167}$$

And we already saw in (8.76) that the last term correctly reproduces the CS term. Therefore, putting all together, we find for the $\mathcal{N}=2$ Chern-Simons action:

$$S^{\rm CS}_{\mathcal{N}=2} = -\frac{\mu}{2e^2}\int d^3x\,d^4\theta\;V\,\Sigma = \frac{1}{e^2}\int d^3x\left[\mu\epsilon^{\mu\nu\rho}A_\mu\partial_\nu A_\rho + \mu\,\bar\lambda^\alpha\lambda_\alpha + \mu\,2\sigma D\right]. \tag{8.168}$$

Notice that besides the usual terms already present in the $\mathcal{N}=1$ version of the CS term and the complex nature of λ_α, we also have a term like σD appearing. The presence of this extra real scalar field σ is somehow an imprint of the underlying $\mathcal{N}=2$ SUSY.

We can briefly consider the mixed Chern-Simons term, because it will be useful later. In this case, we consider $\tilde V = \theta^\alpha\bar\theta_\alpha\,\omega - \theta^\alpha\bar\theta^\beta a_{\alpha\beta} + \theta^2\bar\theta^\alpha\,\bar\zeta_\alpha + \bar\theta^2\theta^\alpha\,\zeta_\alpha + \theta^2\bar\theta^2\,d$ as another gauge superfield, and try to compute the mixed object $\int d^4\theta\,\tilde V\Sigma$.

$$\begin{aligned} \tilde V\Sigma \supset\ & \theta^\alpha\bar\theta_\alpha\theta^\beta\bar\theta_\beta\,\omega D + i\theta^\alpha\bar\theta_\alpha\theta^\gamma\bar\theta^\beta\,\omega\hat F_{\beta\gamma} - \theta^\alpha\bar\theta^\beta\theta^\gamma\bar\theta_\gamma\,Da_{\alpha\beta} - i\theta^\alpha\bar\theta^\beta\theta^\gamma\bar\theta^\delta\,a_{\alpha\beta}\hat F_{\delta\gamma} \\ &+ \theta^2\bar\theta^\alpha\bar\zeta_\alpha\bar\theta^\beta\lambda_\beta + \bar\theta^2\theta^\alpha\zeta_\alpha\theta^\beta\bar\lambda_\beta - 2\theta^2\bar\theta^2\,\sigma d. \end{aligned} \tag{8.169}$$

Thus, after integration, we find

$$\int d^4\theta\,\tilde V\Sigma = +ia_{\alpha\beta}\hat F^{\beta\alpha} - \bar\zeta^\alpha\lambda_\alpha - \bar\lambda^\alpha\zeta_\alpha - 2\omega D - 2\sigma d \tag{8.170}$$

Therefore, we have for the mixed Chern-Simons term:

$$\begin{aligned} S^{mix-CS}_{\mathcal{N}=2} &= -\frac{\mu}{2eg}\int d^3x\,d^4\theta\,\tilde V\Sigma \\ &= \frac{1}{eg}\int d^3x\left[\mu\epsilon^{\mu\nu\rho}a_\mu\partial_\nu A_\rho + \frac{\mu}{2}\bar\zeta^\alpha\lambda_\alpha + \frac{\mu}{2}\bar\lambda^\alpha\zeta_\alpha + \mu\,\omega D + \mu\,\sigma d\right] \end{aligned} \tag{8.171}$$

Notice that in the limit of equal gauge superfields, we recover the CS term obtained before.

Finally, we remark that nobody can forbid us to consider a term analogous to the Fayet-Iliopoulos (FI) term in 3+1 SUSY, given simply by

$$S^{FI}_{\mathcal{N}=2} = \frac{1}{e}\int d^3x\,d^4\theta\,\xi V = \frac{1}{e}\int d^3x\,\xi D \tag{8.172}$$

Such a term is manifestly SUSY invariant since it is an integral in full superspace of a superfield, it is real, and can be shown to be invariant under supergauge transformations (this is the reason why we can adopt the WZ gauge to write it in components). In fact, it transforms like $V \to V + i\left(\Lambda - \bar{\Lambda}\right)$, and the chiral Λ and anti-chiral $\bar{\Lambda}$ superfields are killed by part of the Grassman integrals, $\int d^2\bar{\theta} \approx \bar{D}^2$ and $\int d^2\theta \approx D^2$, respectively. Moreover, this term can play a role in the discussion of vortices, since it might allow the scalar field to get a non-trivial expectation value, putting the system into the Higgs phase.

Matter + Gauge

The natural next step is to investigate the action for matter superfields minimally coupled with abelian gauge superfields. Inspired by the 3+1-dimensional case, this can be achieved through the following action

$$S = \int d^3x\, d^4\theta\, \bar{\Phi} e^V \Phi \tag{8.173}$$

Under a supergauge transformation, the matter superfields transform like $\Phi \to e^{-i\Lambda}\Phi$ and $\Phi \to \bar{\Phi} e^{i\bar{\Lambda}}$, thus we can immediately check the gauge invariance of this term:

$$\bar{\Phi} e^V \Phi \to \bar{\Phi} e^{i\bar{\Lambda}} e^{V + i\left(\Lambda - \bar{\Lambda}\right)} e^{-i\Lambda}\Phi = \bar{\Phi} e^V \Phi. \tag{8.174}$$

Being supergauge-invariant, we can choose the WZ gauge where $e^{V_{WZ}} = 1 + V_{WZ} + \frac{1}{2}V_{WZ}^2$:

$$\int d^3x\, d^4\theta \bar{\Phi} e^V \Phi = \int d^3x\, d^4\theta \left[\bar{\Phi}\Phi + \bar{\Phi} V \Phi + \frac{1}{2}\bar{\Phi} V^2 \Phi\right] \tag{8.175}$$

The first contribution is the usual kinetic term for the chiral superfield. Since we have $\frac{1}{2}V_{WZ}^2 = -\theta^2\bar{\theta}^2\sigma^2 - \theta^2\bar{\theta}^2 A_\mu A^\mu$, already exhausting the Grassman coordinates, the last term can only give us $-\bar{\phi}\phi A_\mu A^\mu - \bar{\phi}\phi\sigma^2$ after integration. For the $\bar{\Phi} V \Phi$ term, we have:

$$\begin{aligned} \bar{\Phi} V \Phi \supset \left(\theta^\alpha \bar{\theta}_\alpha \sigma - \theta^\alpha \bar{\theta}^\beta A_{\alpha\beta}\right) \left[\frac{i}{2}\theta^\alpha \bar{\theta}^\beta \left(\bar{\phi}\partial_{\alpha\beta}\phi - \phi\partial_{\alpha\beta}\bar{\phi}\right) + \theta^\alpha \psi_\alpha \bar{\theta}^\beta \bar{\psi}_\beta\right] \\ + \theta^2 \bar{\theta}^\alpha \bar{\lambda}_\alpha \bar{\theta}^\beta \bar{\psi}_\beta \phi + \bar{\theta}^2 \theta^\alpha \lambda_\alpha \theta^\beta \psi_\beta \bar{\phi} + \theta^2 \bar{\theta}^2 D \bar{\phi}\phi \end{aligned} \tag{8.176}$$

Thus, integrating in the Grassman coordinates we obtain

$$\int d^4\theta\, \bar{\Phi} V \Phi = \bar{\psi}^\alpha \psi_\alpha \sigma + \frac{i}{2} A_{\alpha\beta}\left(\bar{\phi}\partial^{\alpha\beta}\phi - \phi\partial^{\alpha\beta}\bar{\phi}\right) - \bar{\psi}_\alpha A^\alpha{}_\beta \psi^\beta - \bar{\lambda}^\alpha \bar{\psi}_\alpha \phi - \lambda^\alpha \psi_\alpha \bar{\phi} + D\bar{\phi}\phi \tag{8.177}$$

We can rewrite the second term using vector indices:

$$
\begin{aligned}
\frac{i}{2}A_{\alpha\beta}\bar{\phi}\partial^{\alpha\beta}\phi &= \frac{i}{2}\left(-C\gamma^{\mu}\right)_{\alpha\beta}\left(\gamma^{\nu}C^{-1}\right)^{\alpha\beta}A_{\mu}\bar{\phi}\partial_{\nu}\phi \\
&= -\frac{i}{2}\,\mathrm{Tr}\left(\gamma^{\mu}\gamma^{\nu}\right)A_{\mu}\bar{\phi}\partial_{\nu}\phi \\
&= iA_{\mu}\bar{\phi}\partial^{\mu}\phi
\end{aligned}
\tag{8.178}
$$

Thus, $\frac{i}{2}A_{\alpha\beta}\left(\bar{\phi}\partial^{\alpha\beta}\phi - \phi\partial^{\alpha\beta}\bar{\phi}\right) = iA_{\mu}\left(\bar{\phi}\partial^{\mu}\phi - \phi\partial^{\mu}\bar{\phi}\right)$. We also have already computed $\frac{1}{2}\bar{\Phi}V^2\Phi = -\bar{\phi}\phi A_{\mu}A^{\mu} - \bar{\phi}\phi\sigma^2$, and the $\bar{\Phi}\Phi$ contribution is

$$
\int d^4\theta\,\bar{\Phi}\Phi = -\partial^{\mu}\bar{\phi}\partial_{\mu}\phi + i\bar{\psi}_{\alpha}\left(\gamma^{\mu}\right)^{\alpha}{}_{\beta}\,\partial_{\mu}\psi^{\beta} + \bar{F}F \tag{8.179}
$$

Defining $\mathcal{D}^{\alpha}{}_{\beta} = \partial^{\alpha}{}_{\beta} + iA^{\alpha}{}_{\beta}$ for the gauge covariant derivative acting on ϕ, ψ (with the opposite sign for $\bar{\phi}, \bar{\psi}$), we have $-\mathcal{D}_{\mu}\bar{\phi}\mathcal{D}^{\mu}\phi = -\partial_{\mu}\bar{\phi}\partial^{\mu}\phi + iA_{\mu}\left(\bar{\phi}\partial^{\mu}\phi - \phi\partial^{\mu}\bar{\phi}\right) - A_{\mu}A^{\mu}\bar{\phi}\phi$. Putting all together, we obtain:

$$
\begin{aligned}
S^{\mathcal{N}=2}_{G+M} = \int d^3x\Big[&- \partial^{\mu}\bar{\phi}\partial_{\mu}\phi + i\bar{\psi}_{\alpha}\left(\gamma^{\mu}\right)^{\alpha}{}_{\beta}\,\partial_{\mu}\psi^{\beta} + \bar{F}F - \bar{\phi}\phi A_{\mu}A^{\mu} - \bar{\phi}\phi\sigma^2 \\
&+ \bar{\psi}^{\alpha}\psi_{\alpha}\sigma + iA_{\mu}\left(\bar{\phi}\partial^{\mu}\phi - \phi\partial^{\mu}\bar{\phi}\right) - \bar{\psi}_{\alpha}A^{\alpha}{}_{\beta}\psi^{\beta} - \bar{\lambda}^{\alpha}\bar{\psi}_{\alpha}\phi - \lambda^{\alpha}\psi_{\alpha}\bar{\phi} + D\bar{\phi}\phi\Big]
\end{aligned}
\tag{8.180}
$$

Therefore, the matter superfield minimally coupled with a gauge superfield in $\mathcal{N} = 2$ is:

$$
\begin{aligned}
S^{\mathcal{N}=2}_{G+M} = \int d^3x\,d^4\theta\,\bar{\Phi}e^{V}\Phi = \int d^3x\Big[&- \mathcal{D}_{\mu}\bar{\phi}\mathcal{D}^{\mu}\phi - \bar{\phi}\phi\sigma^2 + \bar{F}F + D\bar{\phi}\phi \\
&+ i\bar{\psi}_{\alpha}\mathcal{D}^{\alpha}{}_{\beta}\psi^{\beta} + \bar{\psi}^{\alpha}\psi_{\alpha}\sigma - \bar{\lambda}^{\alpha}\bar{\psi}_{\alpha}\phi - \lambda^{\alpha}\psi_{\alpha}\bar{\phi}\Big]
\end{aligned}
\tag{8.181}
$$

The most general action for one chiral superfield matter minimally coupled with an Abelian gauge superfield in $\mathcal{N} = 2$ superspace will be given by the sum of all the terms discussed above: the kinetic term with minimal coupling, the Maxwell term, the CS term, the FI term, and the superpotential for the chiral superfield. This can be written as

$$
S = \int d^3x\left\{\int d^4\theta\Big[- \frac{1}{4}\Sigma^2 + \bar{\Phi}e^{V}\Phi - \frac{\mu}{2}V\Sigma + \xi V\Big] + \int d^2\theta\,W\left(\Phi\right) + \int d^2\bar{\theta}\,\bar{W}\left(\bar{\Phi}\right)\right\} \tag{8.182}
$$

It is important to notice that when we have charged matter superfields, in order to have gauge-invariance for the action, the superpotential must be gauge-invariant. But the superpotential must also be holomorphic on the chiral superfield, thus, if we are considering a model with only one chiral superfield, a superpotential term is not allowed, since this would not be gauge-invariant. We must have at least two chiral superfields with opposite charges, in order to build an holomorphic superpotential that is gauge-invariant.

- **Adjusting gauge conventions**

Here we briefly address how to deal with different gauge conventions. The first thing to notice is that if we want to introduce the gauge coupling, we need only to perform the substitution $V \to eV$. The same is true with respect to the sign convention. Therefore, we propose to adopt the following redefinition: $V \to -eV$. In component language, this amounts to redefine all V components by the same quantity. That is: $A_{\alpha\beta} \to -eA_{\alpha\beta}$, $\lambda \to -e\lambda$, $\sigma \to -e\sigma$, and $D \to -eD$. For the gauge transformations, we propose:

$$V \to V + \frac{i}{e}\left(\Lambda - \bar{\Lambda}\right), \quad \Phi \to e^{i\Lambda}\Phi, \quad \bar{\Phi} \to \bar{\Phi}e^{-i\bar{\Lambda}} \tag{8.183}$$

In such a way that $\bar{\Phi}e^{-eV}\Phi \to \bar{\Phi}e^{-i\bar{\Lambda}}e^{-eV-i\left(\Lambda-\bar{\Lambda}\right)}e^{i\Lambda}\Phi = \bar{\Phi}e^{-eV}\Phi$ is still gauge-invariant.

Therefore, performing his redefinition, we obtain for the main actions in components:

$$S^{\text{Maxwell}}_{\mathcal{N}=2} = -\frac{1}{4}\int d^3x\, d^4\theta\, \Sigma^2 = \int d^3x \left[-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}\sigma\Box\sigma + \frac{i}{2}\bar{\lambda}_\alpha \partial^\alpha{}_\beta \lambda^\beta + \frac{1}{2}D^2\right] \tag{8.184}$$

$$S^{\text{CS}}_{\mathcal{N}=2} = -\frac{\mu}{2}\int d^3x\, d^4\theta\; V\,\Sigma = \int d^3x \left[\mu\epsilon^{\mu\nu\rho}A_\mu\partial_\nu A_\rho + \mu\,\bar{\lambda}^\alpha\lambda_\alpha + \mu\,2\sigma D\right]. \tag{8.185}$$

Notice that in the pure gauge sector, the only change was an overall factor in the action.

$$\begin{aligned} S^{\mathcal{N}=2}_{G+M} = \int d^3x\, d^4\theta\, \bar{\Phi}e^{-eV}\Phi = \int d^3x\Big[&-\mathcal{D}_\mu\bar{\phi}\mathcal{D}^\mu\phi + \bar{F}F - e^2\sigma^2\bar{\phi}\phi - eD\bar{\phi}\phi - e\sigma\bar{\psi}^\alpha\psi_\alpha \\ &+ i\bar{\psi}_\alpha\mathcal{D}^\alpha{}_\beta\psi^\beta + e\bar{\lambda}^\alpha\bar{\psi}_\alpha\phi + e\lambda^\alpha\psi_\alpha\bar{\phi}\Big]. \end{aligned} \tag{8.186}$$

The gauge covariant derivative now reads: $\mathcal{D}^\alpha{}_\beta = \partial^\alpha{}_\beta - ieA^\alpha{}_\beta$.

Parity

In the context of $\mathcal{N} = 2$ superspace, we now define the parity transformation of the Grassmann coordinates as

$$(\theta^P)^\alpha = -i(\gamma^1)^\alpha{}_\beta\bar{\theta}^\beta, \quad (\bar{\theta}^P)^\alpha = -i(\gamma^1)^\alpha{}_\beta\theta^\beta \Rightarrow (\partial^P)^\alpha = i(\gamma^1)^\alpha{}_\beta\bar{\partial}^\beta, \quad (\bar{\partial}^P)^\alpha = i(\gamma^1)^\alpha{}_\beta\partial^\beta$$

And, as a consequence, the parity transformed of a chiral superfield $D^\alpha\Phi = 0$ is necessarily

$$D^\alpha\Phi = 0 \xrightarrow{P} (D^P)^\alpha\Phi^P = i(\gamma^1)^\alpha{}_\beta\bar{D}^\beta\Phi^P = 0 \Rightarrow \bar{D}^\alpha\Phi^P = 0$$

an anti-chiral superfield. Moreover, the integral in full superspace measure $\int d^3x d^4\theta$ becomes parity-even.

Now with all the necessary tools in hand, we can turn to the second derivation of our $\mathcal{N} = 2$ SUSY extension.

8.5 The $\mathcal{N}=2$ model (second derivation)

Let us consider here Φ_+ and Φ_-, $\mathcal{N}=2$ chiral and anti-chiral superfields, respectively, to accommodate matter, and two $\mathcal{N}=2$ gauge superfields V_A and V_a associated with an Abelian $U(1)_A \times U(1)_a$ gauge symmetry. Explicitly, using chiral (anti-chiral) coordinates

$$\Phi_+(x_L,\theta) = \phi_+(x_L) + \theta\psi_+(x_L) - \theta^2 F_+(x_L), \tag{8.187}$$
$$\Phi_-\left(x_R,\bar\theta\right) = \phi_-(x_R) + \bar\theta\psi_-(x_R) - \bar\theta^2 F_-(x_R). \tag{8.188}$$

For the gauge superfields, adopting the WZ supergauge, we have:

$$V_A\left(x,\theta,\bar\theta\right) = \theta^\alpha\bar\theta_\alpha\, N(x) - \theta^\alpha\bar\theta^\beta A_{\alpha\beta}(x) + i\sqrt{2}\theta^2\bar\theta^\alpha\,\Omega_\alpha(x) - i\sqrt{2}\bar\theta^2\theta^\alpha\,\bar\Omega_\alpha(x) + \theta^2\bar\theta^2\, G(x), \tag{8.189}$$
$$V_a\left(x,\theta,\bar\theta\right) = \theta^\alpha\bar\theta_\alpha\, M(x) - \theta^\alpha\bar\theta^\beta a_{\alpha\beta}(x) + i\sqrt{2}\theta^2\bar\theta^\alpha\,\Delta_\alpha(x) - i\sqrt{2}\bar\theta^2\theta^\alpha\,\bar\Delta_\alpha(x) + \theta^2\bar\theta^2\, H(x). \tag{8.190}$$

In the above expressions, remember that V_A carries information about the $\mathcal{N}=1$ superfields Γ^A_α and S, while V_a carries information about Γ^a_α and R, where we defined:

$$\Phi_\pm = \phi_\pm + \theta^\alpha(\psi_\pm)_\alpha - \theta^2 F_\pm \tag{8.191}$$
$$\Gamma^A_\alpha = i\theta^\beta A_{\alpha\beta} - 2\theta^2\Lambda_\alpha \tag{8.192}$$
$$\Gamma^a_\alpha = i\theta^\beta a_{\alpha\beta} - 2\theta^2\lambda_\alpha \tag{8.193}$$
$$S = N + \theta^\alpha\xi_\alpha - \theta^2 G \tag{8.194}$$
$$R = M + \theta^\alpha\zeta_\alpha - \theta^2 H, \tag{8.195}$$

and where the complex two-component spinor Ω is constructed with the real two-component spinors Λ and ξ; analogously, the complex spinor Δ is composed by real spinors λ and ζ.

The $\mathcal{N}=2$ SUSY transformations performed in the $\mathcal{N}=1$ superfields are given by:

$$\begin{aligned}
\delta\Phi_+ &= +i\eta^\alpha\nabla_\alpha\Phi_+, \quad \delta\Gamma^A_\alpha = -2\eta_\alpha S, \quad \delta S = \eta^\alpha W^A_\alpha,\\
\delta\Phi_- &= -i\eta^\alpha\nabla_\alpha\Phi_-, \quad \delta\Gamma^a_\alpha = -2\eta_\alpha R, \quad \delta R = \eta^\alpha W^a_\alpha.
\end{aligned} \tag{8.196}$$

The Maxwell terms in the $\mathcal{N}=2$ formalism will be given by

$$S^{\mathrm{A}}_{\mathcal{N}=2} = -\frac{1}{4}\int d^3x\, d^4\theta\, \Sigma_A^2 = \int d^3x\left[-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}N\Box N + i\Omega_\alpha\partial^\alpha{}_\beta\bar\Omega^\beta + \frac{1}{2}G^2\right] \tag{8.197}$$
$$S^{\mathrm{a}}_{\mathcal{N}=2} = -\frac{1}{4}\int d^3x\, d^4\theta\, \Sigma_a^2 = \int d^3x\left[-\frac{1}{4}f_{\mu\nu}f^{\mu\nu} + \frac{1}{2}M\Box M + i\Delta_\alpha\partial^\alpha{}_\beta\bar\Delta^\beta + \frac{1}{2}H^2\right], \tag{8.198}$$

where the scalar field strengths are naturally given by $\Sigma_A = \bar D^\alpha D_\alpha V_A$ and $\Sigma_a = \bar D^\alpha D_\alpha V_a$.

The mixed Chern-Simons term is given by

$$\begin{aligned}
S^{CS}_{\mathcal{N}=2} &= -\frac{\mu}{2}\int d^3x\, d^4\theta\, V_a \Sigma_A \\
&= \int d^3x \left[\mu\epsilon^{\mu\nu\rho} a_\mu \partial_\nu A_\rho + \mu \Delta^\alpha \bar{\Omega}_\alpha + \mu \Omega^\alpha \bar{\Delta}_\alpha + \mu\, MG + \mu\, NH\right]
\end{aligned} \tag{8.199}$$

For the minimal coupling between gauge and matter, we will propose to include $\bar{\Phi}_+ e^{-eV_A - gV_a}\Phi_+$ and $\bar{\Phi}_- e^{+eV_A - gV_a}\Phi_-$. The reason is that under parity, we have

$$\Phi_\pm \xrightarrow{P} -\Phi_\mp, \quad V_A \xrightarrow{P} -V_A, \quad V_a \xrightarrow{P} V_a$$

. Thus we will propose the following parity-invariant terms, when taken in combination:

$$\begin{aligned}
S^{\Phi_+}_{\mathcal{N}=2} = \int d^4\theta\, \bar{\Phi}_+ e^{-eV_A - gV_a}\Phi_+ = &-\mathcal{D}_\mu \bar{\phi}_+ \mathcal{D}^\mu \phi_+ + i\bar{\psi}_{+\alpha} \mathcal{D}^\alpha{}_\beta \psi_+^\beta - \bar{\psi}_+^\alpha \psi_{+\alpha}\left(eN + gM\right) \\
&+ \bar{F}_+ F_+ - \bar{\phi}_+\phi_+ \left(eN + gM\right)^2 - \bar{\phi}_+\phi_+\left(eG + gH\right) \\
&+ i\sqrt{2}\phi_+ \bar{\psi}_+^\alpha \left(e\Omega_\alpha + g\Delta_\alpha\right) - i\sqrt{2}\bar{\phi}_+ \psi_+^\alpha \left(e\bar{\Omega}_\alpha + g\bar{\Delta}_\alpha\right)
\end{aligned} \tag{8.200}$$

$$\begin{aligned}
S^{\Phi_-}_{\mathcal{N}=2} = \int d^4\theta\, \bar{\Phi}_- e^{+eV_A - gV_a}\Phi_- = &-\mathcal{D}_\mu \bar{\phi}_- \mathcal{D}^\mu \phi_- + i\bar{\psi}_{-\alpha} \mathcal{D}^\alpha{}_\beta \psi_-^\beta + \bar{\psi}_-^\alpha \psi_{-\alpha}\left(eN - gM\right) \\
&+ \bar{F}_- F_- - \bar{\phi}_-\phi_- \left(eN - gM\right)^2 + \bar{\phi}_-\phi_-\left(eG - gH\right) \\
&- i\sqrt{2}\bar{\phi}_- \psi_-^\alpha \left(e\Omega_\alpha - g\Delta_\alpha\right) + i\sqrt{2}\phi_- \bar{\psi}_-^\alpha \left(e\bar{\Omega}_\alpha - g\bar{\Delta}_\alpha\right)
\end{aligned} \tag{8.201}$$

The covariant derivatives act as $\mathcal{D}^\alpha{}_\beta X_\pm = \left(\partial^\alpha{}_\beta - ieA^\alpha{}_\beta \mp iga^\alpha{}_\beta\right) X_\pm$, with $X = \{\phi, \psi\}$.

In this model, Φ_+ has charges $(+1, +1)$ and Φ_- has charges $(+1, -1)$ under the gauge group $U(1)_A \times U(1)_a$, implying that there is no combination of Φ_+ and Φ_- (without using their conjugates) that is completely neutral. Therefore, it is not possible to write down a holomorphic superpotential $W(\Phi_+, \Phi_-)$ that preserves the gauge symmetry of the model.

In principle, it is possible to write down a Fayet-Iliopoulos term for each of them. But to preserve parity, we will see that only one of them will be allowed (with V_a), since its behavior under parity transformations compensates the sign change that occurs in the Grassman measure (in this case, none). That is, $S^{FI}_{\mathcal{N}=2} = \int d^3x\, d^4\theta\, \xi V_a = \int d^3x\, \xi H$.

Collecting all the terms with the auxiliary superfields G and H, we have:

$$\mathcal{L}_{aux} = \frac{1}{2}G^2 + \frac{1}{2}H^2 + \mu MG + \mu NH - eG\left(|\phi_+|^2 - |\phi_-|^2\right) - gH\left(|\phi_+|^2 + |\phi_-|^2\right) + \xi H \tag{8.202}$$

Performing the variation with respect to each of them, we obtain:

$$G = e\left(|\phi_+|^2 - |\phi_-|^2\right) - \mu M, . \tag{8.203}$$

$$H = g\left(|\phi_+|^2 + |\phi_-|^2\right) - \mu N - \xi. \tag{8.204}$$

Substituting the above equations back in $\mathcal{L}_{aux}$, we obtain (considering $\xi = 2gv^2$):

$$\mathcal{L}_{aux} = -\frac{1}{2}\left[e\left(|\phi_+|^2 - |\phi_-|^2\right) - \mu M\right]^2 - \frac{1}{2}\left[g\left(|\phi_+|^2 + |\phi_-|^2 - 2v^2\right) - \mu N\right]^2 \tag{8.205}$$

Therefore, putting all together, we have for the complete $\mathcal{N} = 2$ action

$$S = \int d^3x\, d^4\theta \Big[-\frac{1}{4}\Sigma_A^2 - \frac{1}{4}\Sigma_a^2 - \frac{\mu}{2}V_a\Sigma_A + 2gv^2 V_a + \bar{\Phi}_+ e^{-eV_A - gV_a}\Phi_+ + \bar{\Phi}_- e^{+eV_A - gV_a}\Phi_- \Big] \tag{8.206}$$

In components, the above action reads:

$$\begin{aligned}
\mathcal{L} = & -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{4}f_{\mu\nu}f^{\mu\nu} + \mu\epsilon^{\mu\nu\rho}A_\mu\partial_\nu a_\rho + \frac{1}{2}N\Box N + \frac{1}{2}M\Box M - \mathcal{D}_\mu\bar{\phi}_+\mathcal{D}^\mu\phi_+ - \mathcal{D}_\mu\bar{\phi}_-\mathcal{D}^\mu\phi_- \\
& + i\bar{\psi}_{+\alpha}\mathcal{D}^\alpha{}_\beta\psi_+^\beta + i\bar{\psi}_{-\alpha}\mathcal{D}^\alpha{}_\beta\psi_-^\beta + i\Omega_\alpha\partial^\alpha{}_\beta\bar{\Omega}^\beta + i\Delta_\alpha\partial^\alpha{}_\beta\bar{\Delta}^\beta + \mu\Omega^\alpha\bar{\Delta}_\alpha + \mu\Delta^\alpha\bar{\Omega}_\alpha \\
& + i\sqrt{2}\bar{\psi}_+^\alpha\left(e\Omega_\alpha + g\Delta_\alpha\right)\phi_+ - i\sqrt{2}\psi_+^\alpha\left(e\bar{\Omega}_\alpha + g\bar{\Delta}_\alpha\right)\bar{\phi}_+ \\
& + i\sqrt{2}\bar{\psi}_-^\alpha\left(e\bar{\Omega}_\alpha - g\bar{\Delta}_\alpha\right)\phi_- - i\sqrt{2}\psi_-^\alpha\left(e\Omega_\alpha - g\Delta_\alpha\right)\bar{\phi}_- \\
& - \bar{\psi}_+^\alpha\psi_{+\alpha}\left(eN + gM\right) + \bar{\psi}_-^\alpha\psi_{-\alpha}\left(eN - gM\right) - |\phi_+|^2\left(eN + gM\right)^2 - |\phi_-|^2\left(eN - gM\right)^2 \\
& - \frac{1}{2}\left[e\left(|\phi_+|^2 - |\phi_-|^2\right) - \mu M\right]^2 - \frac{1}{2}\left[g\left(|\phi_+|^2 + |\phi_-|^2 - 2v^2\right) - \mu N\right]^2.
\end{aligned} \tag{8.207}$$

Up to this point any barred object denoted simply complex conjugation, in the sense of $(X^*)^\alpha = \bar{X}^\alpha$ and $X^*_\alpha = -\bar{X}_\alpha$. However, **coincidentally**, because of our conventions and definitions, this is exactly the same map relating complex conjugation and Dirac conjugation (8.118). Hence, from now on, a barred spinor denotes a Dirac conjugate, that is $\overline{\Psi} = \Psi^\dagger\gamma^0$.

And we will adopt the conventions:

$$\overline{\psi}\chi = \overline{\psi}_\alpha\chi^\alpha \tag{8.208}$$

$$\overline{\chi}\not{V}\psi = \overline{\chi}_\alpha V^\alpha{}_\beta\psi^\beta = \overline{\chi}_\alpha(\gamma^\mu)^\alpha{}_\beta V_\mu\psi^\beta \tag{8.209}$$

where we have taken into consideration the natural map between Lorentz vectors and bi-spinors $\not{V} \equiv \gamma^\mu V_\mu \Leftrightarrow V^\alpha{}_\beta = (\gamma^\mu)^\alpha{}_\beta V_\mu$.

With all this in hand, we can finally write our model in the two equivalent foms:

$$
\begin{aligned}
S =& \int d^3x\, d^4\theta \left(-\frac{1}{4}\Sigma_A^2 - \frac{1}{4}\Sigma_a^2 - \frac{\mu}{2} V_a \Sigma_A + 2gv^2 V_a + \bar{\Phi}_+ e^{-eV_A - gV_a} \Phi_+ + \bar{\Phi}_- e^{+eV_A - gV_a} \Phi_- \right) \\
=& \int d^3x \left\{ -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{4} f_{\mu\nu} f^{\mu\nu} + \mu \epsilon^{\mu\nu\rho} A_\mu \partial_\nu a_\rho \right. \\
& - \frac{1}{2} \left(\partial_\mu N\right)^2 - \frac{1}{2} \left(\partial_\mu M\right)^2 - |\mathcal{D}_\mu \phi_+|^2 - |\mathcal{D}_\mu \phi_-|^2 \\
& + i\overline{\psi}_+ \not{\mathcal{D}} \psi_+ + i\overline{\psi}_- \not{\mathcal{D}} \psi_- + i\overline{\Omega} \not{\partial} \Omega + i\overline{\Delta} \not{\partial} \Delta - \mu \overline{\Omega} \Delta - \mu \overline{\Delta} \Omega \\
& - i\sqrt{2} \overline{\psi}_+ \left(e\Omega + g\Delta\right) \phi_+ + i\sqrt{2} \phi_+^* \left(e\overline{\Omega} + g\overline{\Delta}\right) \psi_+ \\
& - i\sqrt{2} \overline{\psi}_- \left(e\Omega^c - g\Delta^c\right) \phi_- + i\sqrt{2} \phi_-^* \left(e\overline{\Omega^c} - g\overline{\Delta^c}\right) \psi_- \\
& + \overline{\psi}_+ \psi_+ \left(eN + gM\right) - \overline{\psi}_- \psi_- \left(eN - gM\right) - |\phi_+|^2 \left(eN + gM\right)^2 - |\phi_-|^2 \left(eN - gM\right)^2 \\
& \left. - \frac{1}{2} \left[e\left(|\phi_+|^2 - |\phi_-|^2 \right) - \mu M \right]^2 - \frac{1}{2} \left[g\left(|\phi_+|^2 + |\phi_-|^2 - 2v^2 \right) - \mu N \right]^2 \right\}
\end{aligned} \tag{8.210}
$$

In some cases, we took the liberty of exploring the equality $(X^c)^\alpha = (X^*)^\alpha = \overline{X}^\alpha$, which is true in our conventions and definitions.

We have thus completed the second derivation of our $\mathcal{N} = 2$ supersymmetric parity-preserving Maxwell-Chern-Simons model. This time without having to add any fields, instead, everything follows naturally from the $\mathcal{N} = 2$ superspace formalism. Hopefully, this derivation will also aid others in deriving $\mathcal{N} = 2$ SUSY invariant theories in a more direct way.

As a next step in this thesis, we will investigate the fermionic spectrum of the theory and confirm one of the hallmarks of unbroken supersymmetry, namely the degeracy of masses between fermions and bosons.

8.6 Fermionic spectrum

Considering then the model

$$
\begin{aligned}
S = \int d^3x \Big\{ & -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} - \frac{1}{4}f^{\mu\nu}f_{\mu\nu} + \mu\varepsilon^{\mu\nu\rho}A_\mu\partial_\nu a_\rho - \frac{1}{2}(\partial^\mu N)^2 - \frac{1}{2}(\partial^\mu M)^2 \\
& - |\left(\partial_\mu - ieA_\mu - iga_\mu\right)\phi_+|^2 - |\left(\partial_\mu - ieA_\mu + iga_\mu\right)\phi_-|^2 \\
& + i\overline{\psi}_+\left(\partial\!\!\!/ - ieA\!\!\!/ - iga\!\!\!/\right)\psi_+ + i\overline{\psi}_-\left(\partial\!\!\!/ - ieA\!\!\!/ + iga\!\!\!/\right)\psi_- \\
& + i\overline{\Omega}\partial\!\!\!/\Omega + i\overline{\Delta}\partial\!\!\!/\Delta - \mu\left(\overline{\Omega}\Delta + \overline{\Delta}\Omega\right) \\
& - i\sqrt{2}\overline{\psi}_+\left(e\Omega + g\Delta\right)\phi_+ + i\sqrt{2}\phi_+^*\left(e\overline{\Omega} + g\overline{\Delta}\right)\psi_+ \\
& - i\sqrt{2}\overline{\psi}_-\left(e\Omega^c - g\Delta^c\right)\phi_- + i\sqrt{2}\phi_-^*\left(e\overline{\Omega^c} - g\overline{\Delta^c}\right)\psi_- \\
& + \overline{\psi}_+\psi_+(eN + gM) - \overline{\psi}_-\psi_-(eN - gM) \\
& - |\phi_+|^2(eN + gM)^2 - |\phi_-|^2(eN - gM)^2 \\
& - \frac{1}{2}\left[e(|\phi_+|^2 - |\phi_-|^2) - \mu M\right]^2 - \frac{1}{2}\left[g(|\phi_+|^2 + |\phi_-|^2 - 2v^2) - \mu N\right]^2 \Big\}.
\end{aligned}
$$

the corresponding energy functional is

$$
\begin{aligned}
H = \int d^2x \Big\{ & \frac{1}{2}\left(\vec{E}^2 + B^2\right) + \frac{1}{2}\left(\vec{e}^2 + b^2\right) + \frac{1}{2}(\partial_0 M)^2 + \frac{1}{2}(\partial_0 N)^2 + \frac{1}{2}(\partial_i M)^2 + \frac{1}{2}(\partial_i N)^2 \\
& + |D_0\phi_+|^2 + |D_0\phi_-|^2 + |D_i\phi_+|^2 + |D_i\phi_-|^2 \\
& - i\psi_+^\dagger\vec{\alpha}.\left(\vec{\nabla} - ie\vec{A} - ig\vec{a}\right)\psi_+ - i\psi_-^\dagger\vec{\alpha}.\left(\vec{\nabla} - ie\vec{A} + ig\vec{a}\right)\psi_- \\
& - \psi_+^\dagger\left(eA_0 + ga_0\right)\psi_+ - \psi_-^\dagger\left(eA_0 - ga_0\right)\psi_- \\
& - i\Omega^\dagger\vec{\alpha}.\vec{\nabla}\Omega - i\Delta^\dagger\vec{\alpha}.\vec{\nabla}\Delta + \mu\left(\overline{\Omega}\Delta + \overline{\Delta}\Omega\right) \\
& + i\sqrt{2}\overline{\psi}_+\left(e\Omega + g\Delta\right)\phi_+ - i\sqrt{2}\phi_+^*\left(e\overline{\Omega} + g\overline{\Delta}\right)\psi_+ \\
& + i\sqrt{2}\overline{\psi}_-\left(e\Omega^c - g\Delta^c\right)\phi_- - i\sqrt{2}\phi_-^*\left(e\overline{\Omega^c} - g\overline{\Delta^c}\right)\psi_- \\
& - \overline{\psi}_+\psi_+(eN + gM) + \overline{\psi}_-\psi_-(eN - gM) \\
& + |\phi_+|^2(eN + gM)^2 + |\phi_-|^2(eN - gM)^2 \\
& + \frac{1}{2}\left[e(|\phi_+|^2 - |\phi_-|^2) - \mu M\right]^2 + \frac{1}{2}\left[g(|\phi_+|^2 + |\phi_-|^2 - 2v^2) - \mu N\right]^2 \Big\}.
\end{aligned}
\tag{8.211}
$$

Even though it is not immediate to verify the strict positivity of this functional, the underlying supersymmetry of the model guarantee us so, as we will see. The minimum of the scalar potential is achieved if, and only if:

$$
\begin{aligned}
&(eN+gM)^2|\phi_+|^2=0,\\
&(eN-gM)^2|\phi_-|^2=0,\\
&e\left(|\phi_+|^2-|\phi_-|^2\right)-\mu M=0,\\
&g\left(|\phi_+|^2+|\phi_-|^2-2v^2\right)-\mu N=0.
\end{aligned}
\tag{8.212}
$$

Out of which four possibilities arise:

$$
\begin{aligned}
&\bullet\,(0,0):|\phi_+|^2\!=\!|\phi_-|^2\!=\!0;M\!=\!0;N\!=\!-\frac{2gv^2}{\mu},\\
&\bullet\,(1,1):|\phi_+|^2\!=\!|\phi_-|^2\!=\!v^2;M\!=\!N\!=\!0,\\
&\bullet\,(0,1):|\phi_+|^2\!=\!0;|\phi_-|^2\!=\!v^2;M\!=\!-\frac{ev^2}{\mu};N\!=\!-\frac{gv^2}{\mu},\\
&\bullet\,(1,0):|\phi_+|^2\!=\!v^2;|\phi_-|^2\!=\!0;M\!=\!\frac{ev^2}{\mu};N\!=\!-\frac{gv^2}{\mu}.
\end{aligned}
\tag{8.213}
$$

Together with these conditions, the energy minimum is achieved by setting all the fermions to zero and the gauge fields to a pure gauge configuration $A_\mu = a_\mu = 0$, for example.

Let us verify the spectrum of the perturbations around the configurations of minimum energy, $\Phi = \Phi_{\text{vacuum}} + \tilde{\Phi}$, by retaining only the quadratic terms in the fields.

8.6.1 (0,0)-Vacuum

In this case, it suffices to take the expansion $N = -\frac{2gv^2}{\mu} + \tilde{N}$

$$
\begin{aligned}
\mathcal{L}^{\text{quad}} &= -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{4}f_{\mu\nu}f^{\mu\nu} + \mu\epsilon^{\mu\rho\nu}A_\mu\partial_\rho a_\nu \\
&\quad - \frac{1}{2}\left(\partial_\mu M\right)^2 - \frac{1}{2}\left(\partial_\mu \tilde{N}\right)^2 - \frac{\mu^2}{2}\left(M^2+\tilde{N}^2\right) \\
&\quad + i\overline{\Omega}\partial\!\!\!/\Omega + i\overline{\Delta}\partial\!\!\!/\Delta - \mu\left(\overline{\Omega}\Delta + \overline{\Delta}\Omega\right) \\
&\quad - |\partial_\mu\phi_+|^2 - |\partial_\mu\phi_-|^2 - \left(\frac{2egv^2}{\mu}\right)^2\left(|\phi_+|^2+|\phi_-|^2\right) \\
&\quad + i\overline{\psi}_+\partial\!\!\!/\psi_+ + i\overline{\psi}_-\partial\!\!\!/\psi_- - \frac{2egv^2}{\mu}\left(\overline{\psi}_+\psi_+ - \overline{\psi}_-\psi_-\right)
\end{aligned}
\tag{8.214}
$$

Immediately, one can see the existence of 4 bosonic degrees of freedom (d.o.f.) and 4 fermionic d.o.f. with mass $\frac{2egv^2}{\mu}$. From the analysis already made for the bosonic case, we are also aware of the existence of another 4 bosonic d.o.f. with mass μ, 2 from the scalars $M, \tilde{N}$ and 2 from the gauge fields. We can expect that the two-component spinors Ω and Δ will carry the corresponding fermionic d.o.f. with mass μ. Indeed, the structure is identical to the decomposition of a Dirac spinor of mass μ into two left and right Weyl spinors in 4 dimensions, therefore in the same way that in this case the left and right spinors carry mass μ, it also holds for Ω and Δ.

8.6.2 (1,1)-Vacuum

Here, using the unitary gauge, we can take the expansion:

$$\phi_{\pm}(x) = \pm\left(v + \frac{\rho_{\pm}(x)}{\sqrt{2}}\right) \tag{8.215}$$

In this situation, the quadratic terms are:

$$\begin{aligned}\mathcal{L}^{\text{quad}} &= -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{4}f_{\mu\nu}f^{\mu\nu} + \mu\epsilon^{\mu\nu\rho}A_\mu\partial_\nu a_\rho + 2v^2\left(e^2A_\mu A^\mu + g^2a_\mu a^\mu\right)\\
&-\frac{1}{2}(\partial_\mu\rho_+)^2 - \frac{1}{2}(\partial_\mu\rho_-)^2 - \frac{1}{2}(\partial_\mu M)^2 - \frac{1}{2}(\partial_\mu N)^2\\
&-\frac{1}{2}\Big[2v^2(e^2+g^2)(\rho_+^2+\rho_-^2) + (\mu^2+4v^2g^2)M^2 + (\mu^2+4v^2e^2)N^2\\
&+4v^2(g^2-e^2)\rho_+\rho_- - 2\sqrt{2}ve\mu(\rho_+-\rho_-)M - 2\sqrt{2}vg\mu(\rho_++\rho_-)N\Big]\\
&+i\overline{\psi}_+\partial\!\!\!/\psi_+ + i\overline{\psi^c}_-\partial\!\!\!/\psi^c_- + i\overline{\Omega}\partial\!\!\!/\Omega + i\overline{\Delta}\partial\!\!\!/\Delta - \mu\left(\overline{\Omega}\Delta + \overline{\Delta}\Omega\right)\\
&-i\sqrt{2}\overline{\psi}_+\left(e\Omega+g\Delta\right)v + i\sqrt{2}v\left(e\overline{\Omega}+g\overline{\Delta}\right)\psi_+ - i\sqrt{2}\overline{\psi^c}_-\left(e\Omega-g\Delta\right)v + i\sqrt{2}v\left(e\overline{\Omega}-g\overline{\Delta}\right)\psi^c_-\end{aligned}$$

Conveniently, we have made use of the following properties valid for any two spinors $\chi_1, \chi_2 : \overline{\chi_1}\partial\!\!\!/\chi_1 = \overline{\chi_1^c}\partial\!\!\!/\chi_1^c$ e $\overline{\chi_1}\chi_2 = \overline{\chi_2^c}\chi_1^c$.

Around this vaccum, we previously found 8 bosonic d.o.f. with masses

$$m_{\pm}^2 = \frac{1}{2}\left(\mu^2 + M_e^2 + M_g^2 \pm \sqrt{(\mu^2+M_e^2+M_g^2)^2 - 4M_e^2M_g^2}\right) \tag{8.216}$$

equally distributed among the gauge field (which now have 2 d.o.f each, after symmetry breaking) and the scalars. The mass scales are defined as $M_e^2 = 4v^2e^2$ e $M_g^2 = 4v^2g^2$.

Considering now only the fermionic sector, let us define the following 8 component spinor:

$$\Xi = \begin{pmatrix}\psi_+\\ \psi^c_-\\ \Omega\\ \Delta\end{pmatrix}, \quad \overline{\Xi} = \begin{pmatrix}\overline{\psi}_+ & \overline{\psi^c}_- & \overline{\Omega} & \overline{\Delta}\end{pmatrix} \tag{8.217}$$

In such a way that we can rewrite the quadratic part as:

$$\mathcal{L}^{\text{quad}} \supset \bar{\Xi} \begin{pmatrix} i\partial\!\!\!/ & 0 & -i\sqrt{2}ev\mathbb{1}_{2\times 2} & -i\sqrt{2}gv\mathbb{1}_{2\times 2} \\ 0 & i\partial\!\!\!/ & -i\sqrt{2}ev\mathbb{1}_{2\times 2} & i\sqrt{2}gv\mathbb{1}_{2\times 2} \\ i\sqrt{2}ev\mathbb{1}_{2\times 2} & i\sqrt{2}ev\mathbb{1}_{2\times 2} & i\partial\!\!\!/ & -\mu\mathbb{1}_{2\times 2} \\ i\sqrt{2}gv\mathbb{1}_{2\times 2} & -i\sqrt{2}gv\mathbb{1}_{2\times 2} & -\mu\mathbb{1}_{2\times 2} & i\partial\!\!\!/ \end{pmatrix} \Xi \tag{8.218}$$

The dispersion relation can be obtained from the poles of the propagator, defined as the inverse of the dynamical operator above. In its turn, the poles of the propagator can be directly obtained from the determinant of the dynamical operator. To that purpose, we shall make use of the property

$$\det\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \det(A - BD^{-1}C)\,\det(D) \tag{8.219}$$

We begin by noting that $\det D = \left(\Box - \mu^2\right)^2$ and

$$D^{-1} = \begin{pmatrix} i\partial\!\!\!/ & -\mu\mathbb{1}_{2\times 2} \\ -\mu\mathbb{1}_{2\times 2} & i\partial\!\!\!/ \end{pmatrix}^{-1} = \frac{1}{\Box - \mu^2}\begin{pmatrix} i\partial\!\!\!/ & \mu\mathbb{1}_{2\times 2} \\ \mu\mathbb{1}_{2\times 2} & i\partial\!\!\!/ \end{pmatrix}$$

From it, we obtain the operator:

$$A - BD^{-1}C = \frac{1}{\Box - \mu^2}\begin{pmatrix} \left[\Box - \mu^2 - 2v^2(e^2+g^2)\right] i\partial\!\!\!/ + 4v^2 eg\mu & -2v^2(e^2-g^2)i\partial\!\!\!/ \\ -2v^2(e^2-g^2)i\partial\!\!\!/ & \left[\Box - \mu^2 - 2v^2(e^2+g^2)\right] i\partial\!\!\!/ - 4v^2 eg\mu \end{pmatrix} \tag{8.220}$$

One more time making use of (8.219), but noting that in this case C and D commute, it allows us to simplify $\det(A - BD^{-1}C)\det D = \det(AD - BC)$ and to obtain:

$$\begin{aligned} \det(A - BD^{-1}C) &= \frac{1}{(\Box - \mu^2)^4}\left\{\left[\Box - \mu^2 - 2v^2(e^2+g^2)\right]^2\Box - 16v^4e^2g^2\mu^2 - 4v^4(e^2-g^2)^2\Box\right\}^2 \\ &= \frac{1}{(\Box - \mu^2)^4}\left[(\Box - \mu^2 - 4v^2e^2)(\Box - \mu^2 - 4v^2g^2)\Box - 16v^4e^2g^2\mu^2\right]^2 \end{aligned}$$

Finally, we get:

$$\det(A - BD^{-1}C)\det D = \left[\frac{(\Box - \mu^2 - 4v^2e^2)(\Box - \mu^2 - 4v^2g^2)\Box - 16v^4e^2g^2\mu^2}{\Box - \mu^2}\right]^2$$

Written in this way, it is easy to check that $\Box = \mu^2$ is a root of the numerator of the above expression, which means that we can factor out a $(\Box - \mu^2)$ to be canceled with the denominator. This is equivalent to performing the polinomial division indicated within the brackets, considering $\Box$ as the variable. The result of this division is:

$$\begin{aligned}\det(A - BD^{-1}C)\det D &= \left\{\Box^2 - \left[\mu^2 + 4v^2(e^2+g^2)\right]\Box + 16v^4e^2g^2\right\}^2 \\ &= \left\{\Box^2 - \left[\mu^2 + M_e^2 + M_g^2\right]\Box + M_e^2 M_g^2\right\}^2 \end{aligned} \tag{8.221}$$

The roots of the determinant above, representing the poles of the propagator, are precisely the $\Box = m_\pm^2$ in (8.216).

8.6.3 (1,0)-Vacuum

In this last case that we will consider, the expansions can be given by:

$$\phi_+ = v + \frac{\rho_+}{\sqrt{2}}; \quad M = \frac{ev^2}{\mu} + \tilde{M}; \quad N = -\frac{gv^2}{\mu} + \tilde{N}$$

Which gives us the following bilinear sector:

$$\begin{aligned}\mathcal{L}^{\text{quad}}_{\text{scalar}} = &- |\partial_\mu \phi_-|^2 - \left(\frac{2egv^2}{\mu}\right)^2 |\phi_-|^2 \\ &- \frac{1}{2}(\partial_\mu \rho_+)^2 - \frac{1}{2}(\partial_\mu \tilde{M})^2 - \frac{1}{2}(\partial_\mu \tilde{N})^2 \\ &- \frac{1}{2}\Big[2v^2(e^2+g^2)\rho_+^2 + (\mu^2 + 2v^2g^2)\tilde{M}^2 + (\mu^2 + 2v^2e^2)\tilde{N}^2 \\ &-2\sqrt{2}ve\mu\rho_+\tilde{M} - 2\sqrt{2}vg\mu\rho_+\tilde{N} + 4v^2eg\tilde{M}\tilde{N}\Big] \\ &+ i\overline{\psi}_-\partial\!\!\!/\psi_- + \frac{2egv^2}{\mu}\overline{\psi}_-\psi_- \\ &+ i\overline{\psi}_+\partial\!\!\!/\psi_+ + i\overline{\Omega}\partial\!\!\!/\Omega + i\overline{\Delta}\partial\!\!\!/\Delta - \mu\left(\overline{\Omega}\Delta + \overline{\Delta}\Omega\right) \\ &- i\sqrt{2}\overline{\psi}_+\left(e\Omega + g\Delta\right)v + i\sqrt{2}v\left(e\overline{\Omega} + g\overline{\Delta}\right)\psi_+ \end{aligned}$$

Briefly recalling the spectrum in the bosonic sector of this case, we have two d.o.f with mass $m^2 = \left(\frac{2egv^2}{\mu}\right)^2$ and 6 (3 scalars + 3 gauge fields) with masses:

$$m_k^2 = \frac{2}{3}\left(\mu^2 + \tilde{M_e}^2 + \tilde{M_g}^2\right)\left(1 + \cos\left\{\frac{1}{3}\arccos\left[2\left(\frac{\sqrt[3]{\mu^2\tilde{M_e}^2\tilde{M_g}^2}}{\frac{\mu^2+\tilde{M_e}^2+\tilde{M_g}^2}{3}}\right)^3 - 1\right] - \frac{2\pi k}{3}\right\}\right),$$

where $k = 0, 1, 2;$ (8.222)

Which is just complicated way of stating the roots of the characteristic equation:

$$\lambda^3 - 2(\mu^2 + \tilde{M_e}^2 + \tilde{M_g}^2)\lambda^2 + (\mu^2 + \tilde{M_e}^2 + \tilde{M_g}^2)^2\lambda - 4\mu^2\tilde{M_e}^2\tilde{M_g}^2 = 0 \tag{8.223}$$

Where $\tilde{M_e}^2 = 2v^2e^2 = \frac{1}{2}M_e^2$ and $\tilde{M_g}^2 = 2v^2g^2 = \frac{1}{2}M_g^2$. We shall verify that the same polynomial will also arise on the fermionic sector. The 2 fermionic d.o.f with squared mass $m^2 = \left(\frac{2egv^2}{\mu}\right)^2$ are already self evident. Analogously to the previous subsection, let us define now:

$$\Xi = \begin{pmatrix} \psi_+ \\ \Omega \\ \Delta \end{pmatrix}, \quad \overline{\Xi} = \begin{pmatrix} \overline{\psi}_+ & \overline{\Omega} & \overline{\Delta} \end{pmatrix} \tag{8.224}$$

And we obtain the bilinear sector:

$$\mathcal{L}^{\text{quad}} \supset \overline{\Xi} \begin{pmatrix} i\partial\!\!\!/ & -i\sqrt{2}ev\mathbb{1}_{2\times 2} & -i\sqrt{2}gv\mathbb{1}_{2\times 2} \\ i\sqrt{2}ev\mathbb{1}_{2\times 2} & i\partial\!\!\!/ & -\mu\mathbb{1}_{2\times 2} \\ i\sqrt{2}gv\mathbb{1}_{2\times 2} & -\mu\mathbb{1}_{2\times 2} & i\partial\!\!\!/ \end{pmatrix} \Xi \tag{8.225}$$

One more time we will be using the property (8.219), since it is not necessary that the matrices A, B, C and D be square matrices, it suffices that all necessary operations are well defined. To this end, using the same definition of D from the previous subsection, but now with $A = i\partial\!\!\!/$, B and C *mutatis mutandis*. In this way, we obtain

$$A - BD^{-1}C = \frac{1}{\Box - \mu^2}\left\{\left[\Box - \mu^2 - 2v^2(e^2+g^2)\right] i\partial\!\!\!/ - 4v^2eg\mu\right\}$$

Resorting again to our "utility belt", let's make use of the following property valid for matrices $M_{2\times 2}$: $\det M = \frac{1}{2}\left[(\text{Tr}M)^2 - \text{Tr}M^2\right]$, because with it we can use $\text{Tr}\,(\gamma^\mu) = 0$ and $\text{Tr}(\gamma^\mu\gamma^\nu) = -2\eta^{\mu\nu}$. A bit of algebra gives us the determinant:

$$\det\left(A - BD^{-1}C\right)\det D = \left\{-\left[\Box - \mu^2 - 2v^2(e^2+g^2)\right]^2\Box + 16v^4e^2g^2\mu^2\right\} \tag{8.226}$$

$$= \left[-\left(\Box - \mu^2 - \tilde{M_e}^2 - \tilde{M_g}^2\right)^2\Box + 4\tilde{M_e}^2\tilde{M_g}^2\mu^2\right] \tag{8.227}$$

$$= -\Box^3 + 2\left(\mu^2 + \tilde{M_e}^2 + \tilde{M_g}^2\right)\Box^2 - \left(\mu^2 + \tilde{M_e}^2 + \tilde{M_g}^2\right)^2\Box + 4\tilde{M_e}^2\tilde{M_g}^2\mu^2 \tag{8.228}$$

The condition $\det\left(A - BD^{-1}C\right)\det D = 0$ gives us exactly the characteristic equation (8.223) (with $\Box \equiv \lambda$) and, in this way, we have identified 6 more fermionic d.o.f with masses (8.222). The parity invariance of the model ensures that the (0,1)-Vaccum will not give nothing new with respect to the spectrum. This completes our verification of the fermionic spectrum.

Chapter 9

$\mathcal{N} = 2$ Supersymmetry and Self-Duality

So far, the only relationship we explicitly saw between $\mathcal{N} = 2$ supersymmetry and self-duality was from our derivations of the self-dual model from a SUSY invariant setting. In this final chapter, we shall verify how in fact these two are deeply related. Starting with how the extended supersymmetric algebra with central charge implies a Bogomol'nyi-like bound.

Let's consider again the $\mathcal{N} = 2$ SUSY algebra with central charge:

$$\{Q^I_\alpha, Q^J_\beta\} = -2P_{\alpha\beta}\delta^{IJ} + iT\epsilon^{IJ}C_{\alpha\beta}. \tag{9.1}$$

Considering (9.1), we define the following combinations

$$Q_\alpha = \frac{1}{2}\left(Q^{(1)}_\alpha + iQ^{(2)}_\alpha\right), \quad \bar{Q}_\alpha = \frac{1}{2}\left(Q^{(1)}_\alpha - iQ^{(2)}_\alpha\right). \tag{9.2}$$

and observe that

$$\left\{Q_\alpha, \bar{Q}_\beta\right\} = -P_{\alpha\beta} - \frac{T}{2}C_{\alpha\beta}, \quad \{Q_\alpha, Q_\beta\} = \left\{\bar{Q}_\alpha, \bar{Q}_\beta\right\} = 0 \tag{9.3}$$

From these, we now define

$$a_\pm = \frac{1}{2\sqrt{E}}\left(Q^1 \pm iQ^2\right), \quad a^\dagger_\pm = \frac{1}{2\sqrt{E}}\left(\bar{Q}^1 \mp i\bar{Q}^2\right) \tag{9.4}$$

So that

$$\begin{aligned}\left\{a_\pm, a^\dagger_\pm\right\} &= \frac{1}{4E}\left\{Q^1 \pm iQ^2, \bar{Q}^1 \mp i\bar{Q}^2\right\} \\ &= \frac{1}{4E}\left[-P^{11} - P^{22} \mp i\left(-P^{12} - \frac{T}{2}C^{12}\right) \pm i\left(-P^{21} - \frac{T}{2}C^{21}\right)\right]\end{aligned}$$

We need the object $P^{\alpha\beta} = P_\mu\left(\gamma^\mu C^{-1}\right)^{\alpha\beta}$. Since we are working with $\gamma^0 = \sigma_y, \gamma^1 =$

$i\sigma_x, \gamma^2 = i\sigma_z$, and $C = \sigma_y = -C^{\alpha\beta}$, from $P_\mu = (-E, P_x, P_y)$ we get:

$$P^{\alpha\beta} = \begin{pmatrix} -E - P_x & P_y \\ P_y & -E + P_x \end{pmatrix} \tag{9.5}$$

Therefore,

$$\left\{a_\pm, a_\pm^\dagger\right\} = \frac{1}{4E}\left[2E \mp T\right] \tag{9.6}$$

The l.h.s of (9.6), when we take an expection value, is necessarily semi-positive, which means then that:

$$E \geq \pm\frac{T}{2} \Leftrightarrow E \geq \frac{|T|}{2} \tag{9.7}$$

which states that the energy of any representation of the algebra (particle or soliton) is bounded by the central charge. Firstly, we note that the usual result of $E \geq 0$ in supersymmetric theories is recovered when $T = 0$. Sencondly, note the similarity of this result with the bound (6.9) . It becomes exactly a Bogomol'nyi bound the moment we identify the central charge T with the topological charge. As we are going to argue, this is always the case when we have $\mathcal{N} = 1$ SUSY and a topologically conserved charge.

When the saturation of the bound occurs, that is, for configurations where $T = \pm 2E$, then $\left\{a_\pm, a_\pm^\dagger\right\} = 0$ while $\left\{a_\mp, a_\mp^\dagger\right\} = 1$. This means that, in this case, the operators $a_\pm$, and $a_\pm^\dagger$ should then realized to be zero. The functional expression for $a_\pm$ and $a_\pm^\dagger$ in terms of the component fields leads exactly to the self-duality equations. That is to say

$$\left(Q^1 \pm iQ^2\right)|\psi\rangle = 0 \Rightarrow \text{Self-duality equations}$$

We are not going to verify this explicitly for our model in this work, since it's already long enough, but in ([46, 47, 129, 130],) one can check how this goes.

As a final act, we argue (following [127, 128, 129]) that *"a theory with $\mathcal{N} = 1$ supersymmetry and a topological conservation law automatically has $\mathcal{N} = 2$ supersymmetry with a central charge, with the topological charge of the theory appearing as the central charge."*

As we saw in Chapter 2, a topologically conserved current can be expressed as $J^\mu_{top} = \epsilon^{\mu\nu\rho}\partial_\nu A_\rho$, that is to say, it is conserved regardless of the equations of motion. Note that A_μ and $A'_\mu = A_\mu + \partial_\mu\omega$ lead to the same topologically conserved current, therefore we are going to use this freedom to fix the gauge to $\partial_\mu A^\mu = 0$. Following the program of [127], assuming $\mathcal{N} = 1$ SUSY with generator $Q^{(1)}_\alpha$, let's perform the transformation $-iS^\mu_\alpha = [Q^{(1)}_\alpha, A_\mu]$. The object S^μ_α obtained in this way is a conserved vector-spinor current. The possibilities of such structure are limited by the Haag-Lopusanski-Sohnius theorem. First we observe that $\{Q_\alpha^{(1)}, S_\beta^\mu\} = -iJ^\mu_{top}C_{\alpha\beta} + (\mathrm{ST})^\mu_{\alpha\beta}$[1], therefore, if the theory possesses configurations of non-trivial topological charge $T \equiv \int d^2x J^0_{top}$, then S^μ_α cannot be realized to be trivial. It also cannot be the current associated with the original supersymmetry, that is to say that, it's not true that $Q_\alpha^{(1)} = \int d^2x S^0_\alpha$, because if it were, then $\{Q_\alpha^{(1)}, S_\beta^\mu\}$ should generate the energy-momentum tensor [2], which it does not. The only remaining option is that S^μ_α is the conserved current associated with a second supersymmetry, $Q_\alpha^{(2)} = \int d^2x S^0_\alpha$. As a consequence, $\{Q_\alpha^{(1)}, Q_\beta^{(2)}\} = -iTC_{\alpha\beta}$, which in comparison with (9.1) confirms the identification of the central charge of the algebra with topological charge, $T = \int d^2x J^0_{top}$.

To give some concreteness to some of the points just mentioned, let's start by considering the following vector $\mathcal{N} = 1$ superfield

$$\Sigma^\mu = j^\mu + \theta^\alpha S^\mu_\alpha - \theta^2 J^\mu \tag{9.8}$$

The supersymmetry transformations of the component fields read:

$$\begin{aligned}
\delta j^\mu &= \epsilon^\alpha S^\mu_\alpha \\
\delta S^\mu_\alpha &= \epsilon^\beta \left(i\partial_{\alpha\beta} j^\mu + C_{\alpha\beta} J^\mu \right), \\
\delta J^\mu &= i\epsilon^\alpha \partial_\alpha{}^\beta S^\mu_\beta.
\end{aligned} \tag{9.9}$$

Alternatively, if we define the variation of any component ϕ_i as $\delta\phi_i = -i\left[\epsilon^\alpha Q_\alpha^{(1)}, \phi_i\right]$, the first two transformations above can be rewritten as:

$$\left[Q_\alpha^{(1)}, j^\mu\right] = iS^\mu_\alpha \ , \quad \left\{Q_\alpha^{(1)}, S^\mu_\beta\right\} = -iJ^\mu C_{\alpha\beta} - \partial_{\alpha\beta} j^\mu \tag{9.10}$$

The last step is to accommodate the potential A_μ within Σ^μ in such a way that $D^2\Sigma^\mu| = J^\mu = J^\mu_{top} = \epsilon^{\mu\nu\rho}\partial_\nu A_\rho$. The answer is provided by first observing that one can construct a supersymmetric topologically conserved current $\mathcal{J}_\alpha$, $D^\alpha \mathcal{J}_\alpha = 0$, from a gauge spinor superfield Γ^α via

$$\mathcal{J}_\alpha = \frac{1}{2} D^\beta D_\alpha \Gamma_\beta \Rightarrow -\frac{i}{2} D_{(\alpha}\mathcal{J}_{\beta)}| = -(C\gamma^\mu)_{\alpha\beta}(J_{top})_\mu$$

such that $\Gamma'_\alpha = \Gamma_\alpha + D_\alpha \Omega$ lead to the same current. And secondly, from Γ_α one constructs

$$\Sigma^\mu = -\frac{i}{2} D_\alpha (\gamma^\mu)^\alpha{}_\beta \Gamma^\beta \quad \text{with the gauge choice} \quad D^\alpha \Gamma_\alpha = 0 \tag{9.11}$$

The Σ^μ thus constructed satisfies $\partial_\mu \Sigma^\mu = 0$ and $D^2\Sigma^\mu| = J^\mu_{top}$, and all the previous reasoning follows from it.

[1] ST stands for a Schwinger term, which, for our purposes, it suffices to say that its integral in space for $\mu = 0$ vanishes

[2] In order to correctly reproduce $\{Q_\alpha^{(1)}, Q_\beta^{(1)}\} = -2P_{\alpha\beta}$

One might wonder what does the conserved current $j^\mu = \Sigma^\mu|$ corresponds to. Defining the charge $G = \int d^2x j^0$, the algebra (9.10) implies $[Q^{(1)}_\alpha, G] = iQ^{(2)}_\alpha$. Now, suppose that in (9.1) we define new charges $Q'^I = e^{i\omega G} Q^I e^{-i\omega G} = R^I{}_J(\omega) Q^J$, where $R \in SO(2)$. Then

$$\begin{aligned}\{Q'^I_\alpha, Q'^J_\beta\} &= R^I{}_K R^J{}_L \{Q^K_\alpha, Q^L_\beta\} \\ &= R^I{}_K R^J{}_L \left(-2P_{\alpha\beta}\delta^{KL} + iT\epsilon^{KL} C_{\alpha\beta}\right) \\ &= -2P_{\alpha\beta}\delta^{IJ} + iT\epsilon^{IJ} C_{\alpha\beta}\end{aligned} \tag{9.12}$$

This proves the existence of an R-symmetry, here acting as an $SO(2) \simeq U(1)$ group. By taking the parametrization $R(\omega) = e^{i\omega\sigma_2}$, we can express this symmetry infinitesimally as $[Q^1_\alpha, G] = iQ^2_\alpha$ and $[Q^2_\alpha, G] = -iQ^1_\alpha$. Now it becomes clear that j^μ is nothing other than the current associated with the $SO(2) \simeq U(1)$ R-symmetry.

Note that, the R-symmetry algebra together with (9.1), imply

$$\left[Q^1_\alpha, G\right] = iQ^2_\alpha, \quad \{Q^{(1)}_\alpha, Q^{(2)}_\beta\} = -iTC_{\alpha\beta},$$

which essentially state that the R-symmetry charge G, the second SUSY charge Q^2_α, and the central charge T belong to same supermultiplet of the first supersymmetry Q^1_α. This hint is what motivates the existence and construction of Σ^μ in the first place.

Chapter 10

Final Remarks and Prospectives

In this work, we considered a parity and time-reversal invariant Maxwell-Chern-Simons $U(1) \times U(1)$ model coupled with charged scalars in 2+1 dimensions, and investigated the existence of topological vortices in this scenario. We described the main features of the model and discussed general properties of topological configurations that could be present in it. Using an appropriate *ansatz* and the equations of motion, we obtained the relevant differential equations and solved them numerically. We explicitly analyzed three examples that are representatives of the possible solutions and showed explicit vortex configurations for each case, describing their main properties such as the electric and magnetic fields related with each particular solution. We therefore conclude that there are vortex solutions in this novel class of Maxwell-CS models. We also investigated a self-dual version of the model. We obtained a Bogomol'nyi bound for the energy, whose saturation led us to first-order self-duality equations. We exhibited explicit numerical solutions corresponding to topological vortices and non-topological solitons, and discussed their main properties. Next, we demonstrated, using $\mathcal{N} = 1$ and $\mathcal{N} = 2$ superspace formalism, that the self-dual model corresponds to the bosonic sector of an $\mathcal{N} = 2$ supersymmetric theory.

There are many directions to be explored, for example, it would be interesting to analyze the quantization of the CS parameter, as well as studying these models in compact manifolds; a torus, for example. A thorough investigation concerning the interaction between these vortices, answering the question whether they attract or repel, would also be enlightening, together with a deeper understanding of the phases of the theory (screening or confining).

The presence of two complex scalars raises the question of whether there might be a global symmetry in the model, which could possibilitate the existence of semi-local vortices [149], or even if one can construct a parity-invariant model with this feature.

It is of utmost importance to improve the model introduced in Chapter 4.2, allowing the proper investigation of its quantum aspects, to study more general potentials leading to different spontaneous breaking patterns. Another interesting aspect is to consider the existence of dualities in this context.

Furthermore, the product structure of the angular momentum and the presence of two gauge potentials lead us to speculate about a relation between these charged vortices and Dirac monopoles, in the spirit of Refs. [123, 124, 125].

Interestingly enough, models similar to the ones considered here can find many applications in condensed matter [75, 76, 77, 78, 79, 80, 81, 82, 83, 84, 85, 86, 87], and it would be exciting to find a physical system accurately described by our model, allowing it to be experimentally realized.

It is also important to investigate the physics of the parity and time-reversal breaking $(1,0)$- and $(0,1)$-vacua, and in particular, to study solitons asymptoting to them, as well as the existence of domain walls connecting the degenerate phases of this model. The role of parity and time-reversal in superconductors is a topic that has been attracting much interest recently [112, 113, 114], and hopefully our model could find some use in this subject.

Regarding the supersymmetric extension, a lot remains to be investigated. For example, to obtain the $\mathcal{N}=2$ SUSY generators as functional of the fields in order to verify that they satisfy the correct algebra, also allowing to precisely identify the identity of the central charge with the topological charge, which we here conjecture to be the parity-even flux $\chi = \int d^2x\ b$. A quantum investigation of the model would be informative as to the stability of the self-dual scalar potential, for instance. One can also investigate the (non(?)) renormalizability of the physical parameters of the theory, which in its turn illuminates on the quantum validity of the Bogomol'nyi bound $M = E_(static) = 2v^2|g\chi|$. The zero-energy configurations existing as perturbations around a vortex-background, known as zero-modes, are particularly interesting in the supersymmetric scenario [50], so a study of the bosonic and fermionic zero-modes of our self-dual model is in order. Their quantization provides information on the moduli-space of vortex solutions, in the case of bosonic zero-modes, and on their degeneracy, for the fermionic ones. We mentioned that $\mathcal{N}=2$ SUSY in 3 dimensions can be obtained through dimensional reduction from $\mathcal{N}=1$ SUSY in 4 d. Considering that a self-dual Maxwell-Chern-Simons model can be obtained from dimensional reduction of the Lorentz violating Carrol-Field-Jackiw electrodynamics [147] and that this theory has recently been given an interesting supersymmetric extension [148], the dimensional reduction of a parity-preserving version of such a theory should provide another derivation of our model. More interestingly, if the fermionic bilinears associated with the Lorentz-symmetry violation present in [148] somehow survive this reduction process, it raises the question of what effects it produces in the lower dimensional theory.

Once fermions are introduced into the scenery, one might also consider their role when studying the theory in manifolds with non-trivial topology. Specifically, how exotic spinors [150, 151] emerge in such a supersymmetric scenario. This is an interesting direction of investigation, even without the imposition of parity-symmetry, in a first moment.

The appearance of supersymmetry in graphene and other low-dimensional condensed matter systems [134, 136, 137, 138, 139, 140, 141], motivates the search for a concrete applicability of our model in some physical system. Parity and/or time-reversal symmetric superconductors, or also with room for the spontaneous breaking of these discreet symmetries, seem to be an ideal candidate for that purpose. Even maybe cosmological applicability, along lines already investigated by our group in the context of cosmic strings [142, 143], for instance.

Appendix A

MCS propagator

A.1 Without Higgs

Putting the lagrangian in the form $\mathcal{L} = \Phi_i \mathcal{O}^{ij}(\partial)\Phi_j$:

$$
\begin{aligned}
\mathcal{L}_{\mathcal{MCS}} &= -\frac{1}{4e^2}F^{\mu\nu}F_{\mu\nu} + \frac{k}{2}\epsilon^{\mu\rho\nu}A_\mu\partial_\rho A_\nu \\
&= -\frac{1}{4e^2}\left(\partial^\mu A^\nu - \partial^\nu A^\mu\right)\left(\partial_\mu A_\nu - \partial_\nu A_\mu\right) + \frac{k}{2}\epsilon^{\mu\rho\nu}A_\mu\partial_\rho A_\nu \\
&= -\frac{1}{2e^2}\left(\partial^\mu A^\nu \partial_\mu A_\nu - \partial^\mu A^\nu \partial_\nu A_\mu\right) + \frac{k}{2}\epsilon^{\mu\rho\nu}A_\mu\partial_\rho A_\nu \\
&= -\frac{1}{2e^2}\left(-A^\nu\partial^\mu\partial_\mu A_\nu + A^\nu\partial^\mu\partial_\nu A_\mu\right) + \frac{k}{2}\epsilon^{\mu\rho\nu}A_\mu\partial_\rho A_\nu - \cancelto{\text{“0”}}{\frac{1}{2e^2}\partial^\mu\left(A^\nu\partial_\mu A_\nu - A^\nu\partial_\nu A_\mu\right)} \\
&= \frac{1}{2e^2}\left(A^\nu\Box A_\nu - A^\nu\partial^\mu\partial_\nu A_\mu\right) + \frac{k}{2}\epsilon^{\mu\rho\nu}A_\mu\partial_\rho A_\nu \\
&= \frac{1}{2e^2}\left(A_\mu\eta^{\mu\nu}\Box A_\nu - A_\mu\partial^\mu\partial^\nu A_\nu\right) + \frac{k}{2}\epsilon^{\mu\rho\nu}A_\mu\partial_\rho A_\nu \\
&= \frac{1}{2}A_\mu\left[\frac{1}{e^2}\left(\eta^{\mu\nu}\Box - \partial^\mu\partial^\nu\right) + k\epsilon^{\mu\rho\nu}\partial_\rho\right]A_\nu = \frac{1}{2}A_\mu\left[\frac{\Box}{e^2}\left(\eta^{\mu\nu} - \frac{\partial^\mu\partial^\nu}{\Box}\right) + k\epsilon^{\mu\rho\nu}\partial_\rho\right]A_\nu \\
&= \frac{1}{2}A_\mu\left[\frac{\Box}{e^2}\left(\delta^\mu_\nu - \frac{\partial^\mu\partial_\nu}{\Box}\right) + k\eta_{\nu\sigma}\epsilon^{\mu\rho\sigma}\partial_\rho\right]A^\nu = \frac{1}{2}A_\mu\mathcal{O}^\mu_\nu A^\nu
\end{aligned}
$$

Where $\mathcal{O}^\mu_\nu = \frac{\Box}{e^2}\Theta^\mu_\nu + kS^\mu_\nu$, being $\Theta^\mu_\nu \equiv \left(\delta^\mu_\nu - \frac{\partial^\mu\partial_\nu}{\Box}\right)$ the transverse operator while $S^\mu_\nu \equiv \eta_{\nu\sigma}\epsilon^{\mu\rho\sigma}\partial_\rho$.

Before proceeding, we need to verify the "algebra" satisfied by these operators. The known algebra satisfied by the transverse and longitudinal $\Omega^\mu_\nu \equiv \frac{\partial^\mu\partial_\nu}{\Box}$ is :

$$\Theta^\mu_\rho\Theta^\rho_\nu = \Theta^\mu_\nu, \quad \Omega^\mu_\rho\Omega^\rho_\nu = \Omega^\mu_\nu, \quad \Theta^\mu_\rho\Omega^\rho_\nu = \Omega^\mu_\rho\Theta^\rho_\nu = 0$$

Now evaluating the other products:

$$
\begin{aligned}
&\rightarrowtail S^\mu_\rho \Omega^\rho_\nu = \eta_{\rho\sigma}\epsilon^{\mu\lambda\sigma}\partial_\lambda \frac{\partial^\rho \partial_\nu}{\Box} = \frac{\epsilon^{\mu\lambda\sigma}\partial_\lambda\partial_\sigma\partial_\nu}{\Box} = 0 = \Omega^\mu_\rho S^\rho_\nu \\
&\rightarrowtail S^\mu_\rho \Theta^\rho_\nu = S^\mu_\rho \left(\delta^\rho_\nu - \Omega^\rho_\nu\right) = S^\mu_\nu = \Theta^\mu_\rho S^\rho_\nu \\
&\rightarrowtail S^\mu_\rho S^\rho_\nu = \left(\eta_{\rho\sigma}\epsilon^{\mu\lambda\sigma}\partial_\lambda\right)\left(\eta_{\nu\beta}\epsilon^{\rho\alpha\beta}\partial_\alpha\right) = \eta_{\rho\sigma}\eta_{\nu\beta}\epsilon^{\mu\lambda\sigma}\epsilon^{\rho\alpha\beta}\partial_\lambda\partial_\alpha = \delta^\rho_\sigma\delta^\beta_\nu\epsilon^{\mu\lambda\sigma}\epsilon_{\rho\alpha\beta}\partial_\lambda\partial^\alpha \\
&\qquad = \epsilon^{\mu\lambda\sigma}\epsilon_{\sigma\alpha\nu}\partial_\lambda\partial^\alpha = \left(\delta^\mu_\alpha\delta^\lambda_\nu - \delta^\mu_\nu\delta^\lambda_\alpha\right)\partial_\lambda\partial^\alpha = \partial^\mu\partial^\nu - \delta^\mu_\nu\Box \\
&\qquad = -\Box\left(\delta^\mu_\nu - \frac{\partial^\mu\partial_\nu}{\Box}\right) = -\Box\Theta^\mu_\nu
\end{aligned}
$$

The results are summarized in Table 1.

	Θ	Ω	S
Θ	Θ	0	S
Ω	0	Ω	0
S	S	0	$-\Box\Theta$

Table A.1: Product of operators

We cannot immediately invert the operator $\mathcal{O}^\mu_\nu = \frac{\Box}{e^2}\Theta^\mu_\nu + kS^\mu_\nu$. The reason being that its determinant is zero. A quick way to see this is through the decomposition $A^\mu = \Theta^\mu_\nu A^\nu + \Omega^\mu_\nu A^\nu = A^\mu_T + A^\mu_L$. The longitudinal spurious gauge component[1] A^μ_L is an eigenvector of $\mathcal{O}^\mu_\nu$ with null eigenvalue. In fact, from the Tab. (A.1), it is clear that

$$
\mathcal{O}^\mu_\nu A^\nu_L = \left(\frac{\Box}{e^2}\Theta^\mu_\nu + kS^\mu_\nu\right)\Omega^\nu_\rho A^\rho = 0
$$

One way to circumvent this problem is by adding the following *gauge fixing* term to $\mathcal{L}_{\mathcal{MCS}}$:

$$
\mathcal{L}_{gf} = -\frac{1}{2e^2\xi}\left(\partial_\mu A^\mu\right)^2 \tag{A.1}
$$

Where ξ is an arbitrary gauge parameter and any observable quantity must be independent of it. Note that

$$
\begin{aligned}
\mathcal{L}_{gf} &= -\frac{1}{2e^2\xi}\left(\partial_\mu A^\mu\right)^2 = -\frac{1}{2e^2\xi}\partial_\mu A^\mu \partial_\nu A^\nu = \frac{1}{2e^2\xi}A^\mu\partial_\mu\partial_\nu A^\nu - \frac{1}{2e^2\xi}\cancelto{\text{“0”}}{\partial_\mu\left(A^\mu\partial_\nu A^\nu\right)} \\
&= \frac{1}{2e^2\xi}A_\mu\partial^\mu\partial_\nu A^\nu = \frac{1}{2e^2\xi}A_\mu\Box\frac{\partial^\mu\partial_\nu}{\Box}A^\nu = \frac{1}{2e^2\xi}A_\mu\Box\Omega^\mu_\nu A^\nu
\end{aligned} \tag{A.2}
$$

Therefore,

$$
\mathcal{L}_{\mathcal{MCS}} + \mathcal{L}_{gf} = \frac{1}{2}A_\mu\left(\frac{\Box}{e^2}\Theta^\mu_\nu + kS^\mu_\nu + \frac{\Box}{e^2\xi}\Omega^\mu_\nu\right)A^\nu = \frac{1}{2}A_\mu\mathcal{O}'^\mu{}_\nu A^\nu \tag{A.3}
$$

[1] Because A^μ_T is, in fact, gauge invariant.

Let's find the inverse of $\mathcal{O}'^{\mu}_{\ \nu}$, that is, the operator $\left(\mathcal{O}'^{-1}\right)^{\nu}_{\ \rho}$ such that:

$$\mathcal{O}'^{\mu}_{\ \nu}\left(\mathcal{O}'^{-1}\right)^{\nu}_{\ \rho} = \delta^{\mu}_{\rho} \tag{A.4}$$

Motivated by the algebra of Tab. (A.1), we can construct $\left(\mathcal{O}'^{-1}\right)$ out of Θ, Ω and S. We propose:

$$\left(\mathcal{O}'^{-1}\right)^{\nu}_{\ \rho} = a\Theta^{\nu}_{\rho} + b\Omega^{\nu}_{\rho} + cS^{\nu}_{\rho} \tag{A.5}$$

Our goal is to determine a, b, and c such that (A.4) holds. So:

$$\begin{aligned}
\mathcal{O}'^{\mu}_{\ \nu}\left(\mathcal{O}'^{-1}\right)^{\nu}_{\ \rho} &= \left(\frac{\Box}{e^2}\Theta^{\mu}_{\nu} + kS^{\mu}_{\nu} + \frac{\Box}{e^2\xi}\Omega^{\mu}_{\nu}\right)\left(a\Theta^{\nu}_{\rho} + b\Omega^{\nu}_{\rho} + cS^{\nu}_{\rho}\right) \\
&= \frac{\Box}{e^2}a\Theta^{\mu}_{\rho} + \frac{\Box}{e^2}cS^{\mu}_{\rho} + kaS^{\mu}_{\rho} - kc\Box\Theta^{\mu}_{\rho} + \frac{\Box}{e^2\xi}b\Omega^{\mu}_{\rho} \\
&= \left(\frac{\Box}{e^2}a - kc\Box\right)\Theta^{\mu}_{\rho} + \frac{\Box}{e^2\xi}b\Omega^{\mu}_{\rho} + \left(\frac{\Box}{e^2}c + ka\right)S^{\mu}_{\rho}
\end{aligned}$$

Now using that $\Theta^{\mu}_{\rho} = \delta^{\nu}_{\rho} - \Omega^{\mu}_{\rho}$,

$$\mathcal{O}'^{\mu}_{\ \nu}\left(\mathcal{O}'^{-1}\right)^{\nu}_{\ \rho} = \left(\frac{\Box}{e^2}a - kc\Box\right)\delta^{\mu}_{\rho} + \left(\frac{\Box}{e^2\xi}b - \frac{\Box}{e^2}a + kc\Box\right)\Omega^{\mu}_{\rho} + \left(\frac{\Box}{e^2}c + ka\right)S^{\mu}_{\rho}$$

Comparing with (A.4), we must have:

$$\begin{cases} \frac{\Box}{e^2}a - kc\Box = 1 \\ \frac{\Box}{e^2\xi}b - \frac{\Box}{e^2}a + kc\Box = 0 \\ \frac{\Box}{e^2}c + ka = 0 \end{cases} \tag{A.6}$$

$$\frac{\Box}{e^2}c + ka = 0 \Rightarrow c = -\frac{ke^2}{\Box}a$$

$$\frac{\Box}{e^2}a - kc\Box = 1 \Rightarrow \frac{\Box}{e^2}a + k^2e^2a = 1 \Rightarrow a\left(\frac{\Box + k^2e^4}{e^2}\right) = 1 \Rightarrow a = \frac{e^2}{\Box + k^2e^4}$$

$$c = -\frac{ke^2}{\Box}\frac{e^2}{\Box + k^2e^4}$$

$$\frac{\Box}{e^2\xi}b - \frac{\Box}{e^2}a + kc\Box = 0 \Rightarrow \frac{\Box}{e^2\xi}b - \frac{\Box}{\Box + k^2e^4} - k^2e^2\frac{e^2}{\Box + k^2e^4} = 0 \Rightarrow \frac{\Box}{e^2\xi}b - 1 = 0 \Rightarrow b = \xi\frac{e^2}{\Box}$$

Therefore,

$$\begin{aligned}
\left(\mathcal{O}'^{-1}\right)^{\nu}_{\ \rho} &= \frac{e^2}{\Box + k^2e^4}\Theta^{\nu}_{\rho} + \xi\frac{e^2}{\Box}\Omega^{\nu}_{\rho} - \frac{ke^2}{\Box}\frac{e^2}{\Box + k^2e^4}S^{\nu}_{\rho} \\
&= \frac{e^2}{\Box + k^2e^4}\delta^{\nu}_{\rho} + \left(\xi\frac{e^2}{\Box} - \frac{e^2}{\Box + k^2e^4}\right)\Omega^{\nu}_{\rho} - \frac{ke^2}{\Box}\frac{e^2}{\Box + k^2e^4}S^{\nu}_{\rho}
\end{aligned}$$

$$\begin{aligned}
\left(\mathcal{O}'^{-1}\right)_{\mu\nu} &= \frac{e^2}{\Box + k^2e^4}\eta_{\mu\nu} + \left(\xi\frac{e^2}{\Box} - \frac{e^2}{\Box + k^2e^4}\right)\Omega_{\mu\nu} - \frac{ke^2}{\Box}\frac{e^2}{\Box + k^2e^4}S_{\mu\nu} \\
&= \frac{e^2}{\Box + k^2e^4}\eta_{\mu\nu} + \left(\xi\frac{e^2}{\Box} - \frac{e^2}{\Box + k^2e^4}\right)\frac{\partial_\mu\partial_\nu}{\Box} - \frac{ke^2}{\Box}\frac{e^2}{\Box + k^2e^4}\epsilon_{\mu\nu\rho}\partial^{\rho} \\
&= e^2\left(\frac{\eta_{\mu\nu}\Box - \partial_\mu\partial_\nu - ke^2\epsilon_{\mu\nu\rho}\partial^{\rho}}{\Box\left(\Box + k^2e^4\right)} + \xi\frac{\partial_\mu\partial_\nu}{\Box^2}\right) \qquad \text{(A.7)}
\end{aligned}$$

To give a mathematically more meaningful sense to (A.7), let's define the propagator in momentum space:

$$\left(\mathcal{O}'^{-1}\right)_{\mu\nu}\delta^3(x-y) = -\int\frac{d^3p}{(2\pi)^3}e^{-ip(x-y)}\Delta_{\mu\nu}(p) \qquad \text{(A.8)}$$

$$\begin{aligned}
\left(\mathcal{O}'^{-1}\right)_{\mu\nu}\delta^3(x-y) &= e^2\left(\frac{\eta_{\mu\nu}\Box - \partial_\mu\partial_\nu - ke^2\epsilon_{\mu\nu\rho}\partial^{\rho}}{\Box\left(\Box + k^2e^4\right)} + \xi\frac{\partial_\mu\partial_\nu}{\Box^2}\right)\int\frac{d^3p}{(2\pi)^3}e^{-ip(x-y)} \\
&= \int\frac{d^3p}{(2\pi)^3}e^2\left(\frac{\eta_{\mu\nu}\left(-p^2\right) + p_\mu p_\nu + ike^2\epsilon_{\mu\nu\rho}p^{\rho}}{-p^2\left(-p^2 + k^2e^4\right)} - \xi\frac{p_\mu p_\nu}{\left(-p^2\right)^2}\right)e^{-ip(x-y)} \\
&= -\int\frac{d^3p}{(2\pi)^3}e^{-ip(x-y)}e^2\left(\frac{p^2\eta_{\mu\nu} - p_\mu p_\nu - ike^2\epsilon_{\mu\nu\rho}p^{\rho}}{p^2\left(p^2 - k^2e^4\right)} + \xi\frac{p_\mu p_\nu}{\left(p^2\right)^2}\right)
\end{aligned}$$

Where we used $\partial_\mu e^{-ip(x-y)} = -ip_\mu e^{-ip(x-y)}$ and the integral representation $\delta^3(x-y) = \int\frac{d^3p}{(2\pi)^3}e^{-ip(x-y)}$.

We have finally obtained:

$$\Delta_{\mu\nu}(p) = e^2\left(\frac{p^2\eta_{\mu\nu} - p_\mu p_\nu - ike^2\epsilon_{\mu\nu\rho}p^{\rho}}{p^2\left(p^2 - k^2e^4\right)} + \xi\frac{p_\mu p_\nu}{\left(p^2\right)^2}\right) \qquad \text{(A.9)}$$

A.2 With Higgs

$$\begin{aligned}
\mathcal{L}_{\mathcal{MCSH}}(A) &= -\frac{1}{4e^2}F^{\mu\nu}F_{\mu\nu} + \frac{k}{2}\epsilon^{\mu\rho\nu}A_\mu\partial_\rho A_\nu + \upsilon^2 A^\mu A_\mu \\
&= \frac{1}{2}A_\mu\left[\frac{\Box}{e^2}\left(\delta^\mu_\nu - \frac{\partial^\mu\partial_\nu}{\Box}\right) + k\eta_{\nu\sigma}\epsilon^{\mu\rho\sigma}\partial_\rho\right]A^\nu + \upsilon^2 A^\mu A_\mu \\
&= \frac{1}{2}A_\mu\left[\frac{(\Box + 2\upsilon^2 e^2)}{e^2}\delta^\mu_\nu - \frac{\Box}{e^2}\Omega^\mu_\nu + kS^\mu_\nu\right]A^\nu \\
&= \frac{1}{2}A_\mu\left[\alpha\delta^\mu_\nu + \beta\Omega^\mu_\nu + kS^\mu_\nu\right]A^\nu = \frac{1}{2}A_\mu \mathcal{O}'^\mu{}_\nu A^\nu
\end{aligned}$$

We must find $\left(\mathcal{O}'^{-1}\right)^\nu_\rho = a\delta^\nu_\rho + b\Omega^\nu_\rho + cS^\nu_\rho$ such that (A.4) holds. We are now using δ^ν_ρ instead of Θ^ν_ρ by means of the identity $\Theta + \Omega = \mathbb{1}$. Carrying on,

$$\begin{aligned}
\mathcal{O}'^\mu{}_\nu\left(\mathcal{O}'^{-1}\right)^\nu_\rho &= \left(\alpha\delta^\mu_\nu + \beta\Omega^\mu_\nu + kS^\mu_\nu\right)\left(a\delta^\nu_\rho + b\Omega^\nu_\rho + cS^\nu_\rho\right) \\
&= \alpha a\delta^\mu_\rho + \alpha b\Omega^\mu_\rho + \alpha cS^\mu_\rho + \beta a\Omega^\mu_\rho + \beta b\Omega^\mu_\rho + kaS^\mu_\rho - kc\Box\Theta^\mu_\rho \\
&= (\alpha a - kc\Box)\,\delta^\mu_\rho + (\alpha b + \beta a + \beta b + kc\Box)\,\Omega^\mu_\rho + (\alpha c + ka)\,S^\mu_\rho
\end{aligned}$$

Comparing with (A.4), we must take

$$\begin{cases} \alpha a - kc\Box = 1 \\ \alpha b + \beta a + \beta b + kc\Box = 0 \\ \alpha c + ka = 0 \end{cases} \tag{A.10}$$

$$\alpha c + ka = 0 \Rightarrow c = -\frac{ka}{\alpha}$$

$$\alpha a - kc\Box = 1 \Rightarrow \alpha a + \frac{k^2 a}{\alpha}\Box = 1 \Rightarrow a\left(\frac{\alpha^2 + k^2\Box}{\alpha}\right) = 1 \Rightarrow a = \frac{\alpha}{\alpha^2 + k^2\Box}$$

Remembering now that

$$\alpha = \frac{\Box + 2\upsilon^2 e^2}{e^2}$$

then,

$$\begin{aligned}
a &= \frac{\Box + 2\upsilon^2 e^2}{e^2}\frac{1}{\left(\frac{\Box + 2\upsilon^2 e^2}{e^2}\right)^2 + k^2\Box} \\
&= e^2\frac{\Box + 2\upsilon^2 e^2}{\left(\Box + 2\upsilon^2 e^2\right)^2 + k^2 e^4\Box}
\end{aligned}$$

Since our only interest is in the denominator of the coefficient, let us focus on it

$$
\begin{aligned}
\left(\Box + 2\upsilon^2 e^2\right)^2 + k^2 e^4 \Box &= \Box^2 + 4\upsilon^2 e^2 \Box + 4\upsilon^4 e^4 + k^2 e^4 \Box \\
&= \Box^2 + \left(4\upsilon^2 e^2 + k^2 e^4\right) \Box + 4\upsilon^4 e^4 \\
&= \left(\Box + 2\upsilon^2 e^2 + \frac{k^2 e^4}{2}\right)^2 - 2\upsilon^2 k^2 e^6 - \frac{k^4 e^8}{4} \\
&= \left(\Box + 2\upsilon^2 e^2 + \frac{k^2 e^4}{2}\right)^2 - \frac{k^2 e^4}{4}\left(k^2 e^4 + 8\upsilon^2 e^2\right)
\end{aligned}
$$

$$
= \left(\Box + 2\upsilon^2 e^2 + \frac{k^2 e^4}{2} - \frac{ke^2}{2}\sqrt{k^2 e^4 + 8\upsilon^2 e^2}\right)\left(\Box + 2\upsilon^2 e^2 + \frac{k^2 e^4}{2} + \frac{ke^2}{2}\sqrt{k^2 e^4 + 8\upsilon^2 e^2}\right)
$$

In momentum space ,$\Box \Rightarrow -p^2$:

$$
\begin{aligned}
\Rightarrow \quad & \left[p^2 - \left(2\upsilon^2 e^2 + \frac{k^2 e^4}{2} - \frac{ke^2}{2}\sqrt{k^2 e^4 + 8\upsilon^2 e^2}\right)\right]\left[p^2 - \left(2\upsilon^2 e^2 + \frac{k^2 e^4}{2} + \frac{ke^2}{2}\sqrt{k^2 e^4 + 8\upsilon^2 e^2}\right)\right] \\
\Rightarrow \quad & \left(p^2 - m_-^2\right)\left(p^2 - m_+^2\right)
\end{aligned}
$$

Where,

$$
\begin{aligned}
m_\pm^2 &= 2\upsilon^2 e^2 + \frac{k^2 e^4}{2} \pm \frac{ke^2}{2}\sqrt{k^2 e^4 + 8\upsilon^2 e^2} \\
&= m_H^2 + \frac{m_{MCS}^2}{2} \pm \frac{m_{MCS}}{2}\sqrt{m_{MCS}^2 + 4m_H^2} \\
&= m_H^2 + \frac{m_{MCS}^2}{2} \pm \frac{m_{MCS}^2}{2}\sqrt{1 + \frac{4m_H^2}{m_{MCS}^2}} \\
&= \frac{m_{MCS}^2}{4}\left(\frac{4m_H^2}{m_{MCS}^2} + 2 \pm 2\sqrt{1 + \frac{4m_H^2}{m_{MCS}^2}}\right) \\
&= \frac{m_{MCS}^2}{4}\left[\left(1 + \frac{4m_H^2}{m_{MCS}^2}\right) \pm 2\sqrt{1 + \frac{4m_H^2}{m_{MCS}^2}} + 1\right] \\
&= \frac{m_{MCS}^2}{4}\left(\sqrt{1 + \frac{4m_H^2}{m_{MCS}^2}} \pm 1\right)^2 \qquad \text{(A.11)}
\end{aligned}
$$

$$
\Rightarrow m_\pm = \frac{m_{MCS}}{2}\left(\sqrt{1 + \frac{4m_H^2}{m_{MCS}^2}} \pm 1\right) \qquad \text{(A.12)}
$$

Appendix B

The energy-momentum tensor

From Noether's theorem, the symmetry under space-time translations imply the conservation law $\partial_\mu T^{\mu\nu} = 0$, where:

$$T^{\mu\nu} = \frac{\partial \mathcal{L}}{\partial\left(\partial_\mu \Phi_i\right)} \partial^\nu \Phi_i - \eta^{\mu\nu} \mathcal{L} \tag{B.1}$$

With Φ_i collectively denotes all the fields of the lagrangian $\mathcal{L}$. Now, considering

$$\begin{aligned}\mathcal{L} = & -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{4} f_{\mu\nu} f^{\mu\nu} + \frac{\mu}{2} \epsilon^{\mu\nu\rho} A_\mu \partial_\nu a_\rho + \frac{\mu}{2} \epsilon^{\mu\nu\rho} a_\mu \partial_\nu A_\rho \\ & + \left|D_\mu \phi_+\right|^2 + \left|D_\mu \phi_-\right|^2 - V,\end{aligned} \tag{B.2}$$

we have:

$$\begin{aligned}\frac{\partial \mathcal{L}}{\partial\left(\partial_\mu A_\beta\right)} &= -F^{\mu\beta} + \frac{\mu}{2} \epsilon^{\alpha\mu\beta} a_\alpha, \\ \frac{\partial \mathcal{L}}{\partial\left(\partial_\mu a_\beta\right)} &= -f^{\mu\beta} + \frac{\mu}{2} \epsilon^{\alpha\mu\beta} A_\alpha, \\ \frac{\partial \mathcal{L}}{\partial\left(\partial_\mu \phi^{\pm}\right)} &= D^\mu \phi_\pm^*, \\ \frac{\partial \mathcal{L}}{\partial\left(\partial_\mu \phi^{*\pm}\right)} &= D^\mu \phi_\pm.\end{aligned}$$

From these we get:

$$
\begin{aligned}
T^{\mu\nu} &= -F^{\mu\beta}\partial^{\nu}A_{\beta} - f^{\mu\beta}\partial^{\nu}a_{\beta} \\
+\frac{\mu}{2}\epsilon^{\alpha\mu\beta}a_{\alpha}\partial^{\nu}A_{\beta} + \frac{\mu}{2}\epsilon^{\alpha\mu\beta}A_{\alpha}\partial^{\nu}a_{\beta} \\
+D^{\mu}\phi_{+}^{*}\partial^{\nu}\phi_{+} + D^{\mu}\phi_{+}\partial^{\nu}\phi_{+}^{*} \\
+D^{\mu}\phi_{-}^{*}\partial^{\nu}\phi_{-} + D^{\mu}\phi_{-}\partial^{\nu}\phi_{-}^{*} - \eta^{\mu\nu}\mathcal{L}
\end{aligned}
$$

Now we substitute $\partial^{\nu}A_{\beta} = F^{\nu}{}_{\beta} + \partial_{\beta}A^{\nu}$ and $\partial^{\nu}a_{\beta} = f^{\nu}{}_{\beta} + \partial_{\beta}a^{\nu}$ in the first line and then integrate by parts.

$$
\begin{aligned}
T^{\mu\nu} &= -F^{\mu\beta}F^{\nu}{}_{\beta} - f^{\mu\beta}f^{\nu}{}_{\beta} \\
-F^{\mu\beta}\partial_{\beta}A^{\nu} - f^{\mu\beta}\partial_{\beta}a^{\nu} \\
+\frac{\mu}{2}\epsilon^{\alpha\mu\beta}a_{\alpha}\partial^{\nu}A_{\beta} + \frac{\mu}{2}\epsilon^{\alpha\mu\beta}A_{\alpha}\partial^{\nu}a_{\beta} \\
+D^{\mu}\phi_{+}^{*}\partial^{\nu}\phi_{+} + D^{\mu}\phi_{+}\partial^{\nu}\phi_{+}^{*} \\
+D^{\mu}\phi_{-}^{*}\partial^{\nu}\phi_{-} + D^{\mu}\phi_{-}\partial^{\nu}\phi_{-}^{*} - \eta^{\mu\nu}\mathcal{L} \\
&= -F^{\mu\beta}F^{\nu}{}_{\beta} - f^{\mu\beta}f^{\nu}{}_{\beta} \\
-\left(\partial_{\beta}F^{\beta\mu}\right)A^{\nu} - \left(\partial_{\beta}f^{\beta\mu}\right)a^{\nu} \\
+\frac{\mu}{2}\epsilon^{\alpha\mu\beta}a_{\alpha}\partial^{\nu}A_{\beta} + \frac{\mu}{2}\epsilon^{\alpha\mu\beta}A_{\alpha}\partial^{\nu}a_{\beta} \\
+D^{\mu}\phi_{+}^{*}\partial^{\nu}\phi_{+} + D^{\mu}\phi_{+}\partial^{\nu}\phi_{+}^{*} \\
+D^{\mu}\phi_{-}^{*}\partial^{\nu}\phi_{-} + D^{\mu}\phi_{-}\partial^{\nu}\phi_{-}^{*} - \eta^{\mu\nu}\mathcal{L}
\end{aligned}
\tag{B.3}
$$

Making use of the equations of motion (4.23) and the definition of the currents $J^{\pm}$, we get:

$$
\begin{aligned}
T^{\mu\nu} &= -F^{\mu\beta}F^{\nu}{}_{\beta} - f^{\mu\beta}f^{\nu}{}_{\beta} \\
+\left(\mu\epsilon^{\mu\alpha\beta}\partial_{\alpha}a_{\beta}\right)A^{\nu} + \left(\mu\epsilon^{\mu\alpha\beta}\partial_{\alpha}A_{\beta}\right)a^{\nu} \\
+\frac{\mu}{2}\epsilon^{\alpha\mu\beta}a_{\alpha}\partial^{\nu}A_{\beta} + \frac{\mu}{2}\epsilon^{\alpha\mu\beta}A_{\alpha}\partial^{\nu}a_{\beta} \\
+D^{\mu}\phi_{+}^{*}D^{\nu}\phi_{+} + D^{\mu}\phi_{+}D^{\nu}\phi_{+}^{*} \\
+D^{\mu}\phi_{-}^{*}D^{\nu}\phi_{-} + D^{\mu}\phi_{-}D^{\nu}\phi_{-}^{*} - \eta^{\mu\nu}\mathcal{L}
\end{aligned}
$$

Substituting again $\partial^{\nu}A_{\beta} = F^{\nu}{}_{\beta} + \partial_{\beta}A^{\nu}$ and $\partial^{\nu}a_{\beta} = f^{\nu}{}_{\beta} + \partial_{\beta}a^{\nu}$, but now on the third line:

$$
\begin{aligned}
T^{\mu\nu} = &-F^{\mu\beta}F^{\nu}{}_{\beta} - f^{\mu\beta}f^{\nu}{}_{\beta} \\
&+ \left(\mu\epsilon^{\mu\alpha\beta}\partial_\alpha a_\beta\right) A^\nu + \left(\mu\epsilon^{\mu\alpha\beta}\partial_\alpha A_\beta\right) a^\nu \\
&+\frac{\mu}{2}\epsilon^{\alpha\mu\beta}a_\alpha F^{\nu}{}_{\beta} + \frac{\mu}{2}\epsilon^{\alpha\mu\beta}A_\alpha f^{\nu}{}_{\beta} \\
&+\frac{\mu}{2}\epsilon^{\alpha\mu\beta}a_\alpha \partial_\beta A^\nu + \frac{\mu}{2}\epsilon^{\alpha\mu\beta}A_\alpha \partial_\beta a^\nu \\
&+D^\mu\phi_+^* D^\nu\phi_+ + D^\mu\phi_+ D^\nu\phi_+^* \\
&+D^\mu\phi_-^* D^\nu\phi_- + D^\mu\phi_- D^\nu\phi_-^* - \eta^{\mu\nu}\mathcal{L}
\end{aligned}
$$

$$
\begin{aligned}
T^{\mu\nu} = &-F^{\mu\beta}F^{\nu}{}_{\beta} - f^{\mu\beta}f^{\nu}{}_{\beta} \\
&\mu\epsilon^{\mu\alpha\beta}\left(\partial_\alpha a_\beta\right) A^\nu + \mu\epsilon^{\mu\alpha\beta}\left(\partial_\alpha A_\beta\right) a^\nu \\
&+\frac{\mu}{2}\epsilon^{\alpha\mu\beta}a_\alpha F^{\nu}{}_{\beta} + \frac{\mu}{2}\epsilon^{\alpha\mu\beta}A_\alpha f^{\nu}{}_{\beta} \\
&-\frac{\mu}{2}\epsilon^{\mu\alpha\beta}\left(\partial_\alpha a_\beta\right) A^\nu - \frac{\mu}{2}\epsilon^{\mu\alpha\beta}\left(\partial_\alpha A_\beta\right) a^\nu \\
&+D^\mu\phi_+^* D^\nu\phi_+ + D^\mu\phi_+ D^\nu\phi_+^* \\
&+D^\mu\phi_-^* D^\nu\phi_- + D^\mu\phi_- D^\nu\phi_-^* - \eta^{\mu\nu}\mathcal{L}
\end{aligned}
$$

In the last passage, after integrating by parts we also performed a relabelling of the indexes for convenience.

$$
\begin{aligned}
T^{\mu\nu} = &-F^{\mu\beta}F^{\nu}{}_{\beta} - f^{\mu\beta}f^{\nu}{}_{\beta} \\
&+\frac{\mu}{4}\epsilon^{\mu\alpha\beta}\left(f_{\alpha\beta}A^\nu - 2A_\alpha f^{\nu}{}_{\beta}\right) \\
&+\frac{\mu}{4}\epsilon^{\mu\alpha\beta}\left(F_{\alpha\beta}a^\nu - 2a_\alpha F^{\nu}{}_{\beta}\right) \\
&+D^\mu\phi_+^* D^\nu\phi_+ + D^\mu\phi_+ D^\nu\phi_+^* \\
&+D^\mu\phi_-^* D^\nu\phi_- + D^\mu\phi_- D^\nu\phi_-^* - \eta^{\mu\nu}\mathcal{L}
\end{aligned}
$$

If we now consider explicitly the Chern-Simons terms that are inside $\mathcal{L}$, we will end up with the following structure for either A^μ and a^μ:

$$
\frac{\mu}{4}\left[\epsilon^{\mu\alpha\beta}\left(f_{\alpha\beta}A^\nu - 2A_\alpha f^{\nu}{}_{\beta}\right) - \eta^{\mu\nu}\left(\epsilon^{\alpha\beta\rho}A_\alpha f_{\beta\rho}\right)\right] \tag{B.4}
$$

$$\frac{\mu}{4}\left[\epsilon^{\mu\alpha\beta}\left(F_{\alpha\beta}a^{\nu}-2a_{\alpha}F^{\nu}{}_{\beta}\right)-\eta^{\mu\nu}\left(\epsilon^{\alpha\beta\rho}a_{\alpha}F_{\beta\rho}\right)\right] \tag{B.5}$$

We will now prove that these expressions vanishes identically. For that purpose, it is convenient to work with the dual fields $\tilde{X}^{\mu}\equiv\frac{1}{2}\epsilon^{\mu\alpha\beta}X_{\alpha\beta}$, $X^{\mu\nu}=\epsilon^{\mu\nu\rho}\tilde{X}_{\rho}$, with $X=F,f$. So (B.4) can be rewritten as:

$$\begin{aligned}
&=\left(\tilde{f}^{\mu}A^{\nu}-A_{\alpha}\epsilon^{\mu\alpha\beta}\epsilon^{\nu}{}_{\beta\rho}\tilde{f}^{\rho}\right)-\eta^{\mu\nu}A_{\alpha}\tilde{f}^{\alpha}\\
&=\tilde{f}^{\mu}A^{\nu}-\eta^{\mu\nu}A_{\alpha}\tilde{f}^{\alpha}+A_{\alpha}\left(\eta^{\mu\nu}\delta^{\alpha}_{\rho}-\delta^{\mu}_{\rho}\eta^{\alpha\nu}\right)\tilde{f}^{\rho}\\
&=0
\end{aligned}$$

The same holds for (B.5).

This proves the well known fact that Chern-Simons terms do not contribute explictly to the energy-momentum tensor and we are left with:

$$\begin{aligned}
T^{\mu\nu}=&\left(\eta^{\mu\nu}\frac{1}{4}F_{\alpha\beta}F^{\alpha\beta}-F^{\mu\beta}F^{\nu}{}_{\beta}\right)\\
&+\left(\eta^{\mu\nu}\frac{1}{4}f_{\alpha\beta}f^{\alpha\beta}-f^{\mu\beta}f^{\nu}{}_{\beta}\right)\\
&+D^{\mu}\phi_{+}^{*}D^{\nu}\phi_{+}+D^{\mu}\phi_{+}D^{\nu}\phi_{+}^{*}-\eta^{\mu\nu}|D_{\alpha}\phi_{+}|^{2}\\
&+D^{\mu}\phi_{-}^{*}D^{\nu}\phi_{-}+D^{\mu}\phi_{-}D^{\nu}\phi_{-}^{*}-\eta^{\mu\nu}|D_{\alpha}\phi_{-}|^{2}\\
&+\eta^{\mu\nu}V
\end{aligned} \tag{B.6}$$

The energy-momentum tensor after the addition of the two neutral scalar fields M and N reads simply:

$$\begin{aligned}
T^{\mu\nu}=&\left(\eta^{\mu\nu}\frac{1}{4}F_{\alpha\beta}F^{\alpha\beta}-F^{\mu\beta}F^{\nu}{}_{\beta}\right)\\
&+\left(\eta^{\mu\nu}\frac{1}{4}f_{\alpha\beta}f^{\alpha\beta}-f^{\mu\beta}f^{\nu}{}_{\beta}\right)\\
&+D^{\mu}\phi_{+}^{*}D^{\nu}\phi_{+}+D^{\mu}\phi_{+}D^{\nu}\phi_{+}^{*}-\eta^{\mu\nu}|D_{\alpha}\phi_{+}|^{2}\\
&+D^{\mu}\phi_{-}^{*}D^{\nu}\phi_{-}+D^{\mu}\phi_{-}D^{\nu}\phi_{-}^{*}-\eta^{\mu\nu}|D_{\alpha}\phi_{-}|^{2}\\
&+\partial^{\mu}M\partial^{\nu}M-\eta^{\mu\nu}\frac{1}{2}\left(\partial_{\alpha}M\right)^{2}\\
&+\partial^{\mu}N\partial^{\nu}N-\eta^{\mu\nu}\frac{1}{2}\left(\partial_{\alpha}N\right)^{2}\\
&+\eta^{\mu\nu}V
\end{aligned} \tag{B.7}$$

Appendix C

Broken phase propagator and Scalar Mass Spectrum of (1,0) and (0,1) Vacua

Given

$$O^{\mu\nu} = \begin{pmatrix} A^{\mu\nu} & B^{\mu\nu} \\ C^{\mu\nu} & D^{\mu\nu} \end{pmatrix} \tag{C.1}$$

with

$$\begin{aligned} A^{\mu\nu} &= (\Box + M_e^2)\Theta^{\mu\nu} + \left(M_e^2 + \frac{\Box}{\alpha}\right)\Omega^{\mu\nu} \\ B^{\mu\nu} &= C^{\mu\nu} = m^2\eta^{\mu\nu} + \mu S^{\mu\nu} \\ D^{\mu\nu} &= (\Box + M_g^2)\Theta^{\mu\nu} + \left(M_g^2 + \frac{\Box}{\beta}\right)\Omega^{\mu\nu} \end{aligned} \tag{C.2}$$

$$\Omega^{\mu\nu} = \frac{\partial^\mu\partial^\nu}{\Box}; \quad \Theta^{\mu\nu} = \eta^{\mu\nu} - \Omega^{\mu\nu}; \quad S^{\mu\nu} = \epsilon^{\mu\rho\nu}\partial_\rho \tag{C.3}$$

we want to determine:

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}^{-1} = \begin{pmatrix} (A - BD^{-1}C)^{-1} & * \\ * & (D - CA^{-1}B)^{-1} \end{pmatrix} \tag{C.4}$$

Since A and D have a similar structure, we start by determining the inverse of a generic operator

$$o^{\mu\nu} = A\Theta^{\mu\nu} + B\Omega^{\mu\nu} + CS^{\mu\nu} \tag{C.5}$$

The task of finding o^{-1} is possible because the operators $\Theta^{\mu\nu}$, $\Omega^{\mu\nu}$ and $S^{\mu\nu}$ close an "algebra". It is straighforward to determine the algebra, which is the following:

	Θ	Ω	S
Θ	Θ	0	S
Ω	0	Ω	0
S	S	0	$-\Box\Theta$

Table C.1: Algebra of operators $\Theta^{\mu\nu}$, $\Omega^{\mu\nu}$ and $S^{\mu\nu}$

Proposing as an ansaz for the inverse $o^{-1} = a\Theta^{\mu\nu} + b\Omega^{\mu\nu} + cS^{\mu\nu}$, and imposing that the product $o^{\mu\nu}o^{-1}_{\nu\rho} = \delta^{\mu}{}_{\rho}$ we find:

$$o^{-1}_{\mu\nu} = \frac{A}{A^2 + \Box C^2}\Theta_{\mu\nu} + \frac{1}{B}\Omega_{\mu\nu} - \frac{C}{A^2 + \Box C^2}S_{\mu\nu} \tag{C.6}$$

Applying this to the operators $A_{\mu\nu}$ and $D_{\mu\nu}$ we get:

$$A^{-1}_{\mu\nu} = \frac{1}{\Box + M_e^2}\Theta_{\mu\nu} + \frac{\alpha}{\Box + \alpha M_e^2}\Omega_{\mu\nu} \tag{C.7}$$

$$D^{-1}_{\mu\nu} = \frac{1}{\Box + M_g^2}\Theta_{\mu\nu} + \frac{\beta}{\Box + \beta M_g^2}\Omega_{\mu\nu} \tag{C.8}$$

From these and the table C.1, we can determine $(D - CA^{-1}B)_{\mu\nu}$ and $(A - BD^{-1}C)_{\mu\nu}$:

$$\begin{aligned}(A - BD^{-1}C)_{\mu\nu} = {}& \frac{\left[(\Box + \alpha M_e^2)\left(\Box + \beta M_g^2\right) - \alpha\beta(m^2)^2\right]}{\alpha\left(\Box + \beta M_g^2\right)}\Omega_{\mu\nu} \\ & \frac{\left[(\Box + M_e^2)\left(\Box + M_g^2\right) + \mu^2\Box - (m^2)^2\right]}{\Box + M_g^2}\Theta_{\mu\nu} - \frac{2\mu m^2}{\Box + M_g^2}S_{\mu\nu}\end{aligned} \tag{C.9}$$

$$\begin{aligned}(D - CA^{-1}B)_{\mu\nu} = {}& \frac{\left[(\Box + \alpha M_e^2)\left(\Box + \beta M_g^2\right) - \alpha\beta(m^2)^2\right]}{\beta\left(\Box + \alpha M_e^2\right)}\Omega_{\mu\nu} \\ & \frac{\left[(\Box + M_e^2)\left(\Box + M_g^2\right) + \mu^2\Box - (m^2)^2\right]}{\Box + M_e^2}\Theta_{\mu\nu} - \frac{2\mu m^2}{\Box + M_e^2}S_{\mu\nu}\end{aligned} \tag{C.10}$$

Making use again of (C.5) and (C.6), we find the propagators:

$$
\begin{aligned}
(A - BD^{-1}C)^{-1}_{\mu\nu} = & \frac{\left[\left(\Box + M_e^2\right)\left(\Box + M_g^2\right) + \mu^2\Box - (m^2)^2\right]}{\left[\left(\Box + M_e^2\right)\left(\Box + M_g^2\right) + \mu^2\Box - (m^2)^2\right]^2 + 4(m^2)^2\mu^2\Box}\left(\Box + M_g^2\right)\Theta_{\mu\nu} \\
& + \frac{2\mu m^2}{\left[\left(\Box + M_e^2\right)\left(\Box + M_g^2\right) + \mu^2\Box - (m^2)^2\right]^2 + 4(m^2)^2\mu^2\Box}\left(\Box + M_g^2\right)S_{\mu\nu} \\
& + \frac{\alpha\left(\Box + \beta M_g^2\right)}{\left[\left(\Box + \alpha M_e^2\right)\left(\Box + \beta M_g^2\right) - \alpha\beta(m^2)^2\right]}\Omega_{\mu\nu}
\end{aligned} \tag{C.11}
$$

$$
\begin{aligned}
(D - CA^{-1}B)^{-1}_{\mu\nu} = & \frac{\left[\left(\Box + M_e^2\right)\left(\Box + M_g^2\right) + \mu^2\Box - (m^2)^2\right]}{\left[\left(\Box + M_e^2\right)\left(\Box + M_g^2\right) + \mu^2\Box - (m^2)^2\right]^2 + 4(m^2)^2\mu^2\Box}\left(\Box + M_e^2\right)\Theta_{\mu\nu} \\
& + \frac{2\mu m^2}{\left[\left(\Box + M_e^2\right)\left(\Box + M_g^2\right) + \mu^2\Box - (m^2)^2\right]^2 + 4(m^2)^2\mu^2\Box}\left(\Box + M_e^2\right)S_{\mu\nu} \\
& + \frac{\beta\left(\Box + \alpha M_e^2\right)}{\left[\left(\Box + \alpha M_e^2\right)\left(\Box + \beta M_g^2\right) - \alpha\beta(m^2)^2\right]}\Omega_{\mu\nu}
\end{aligned} \tag{C.12}
$$

As we saw, the characteristic polynomial determining the scalar mass spectrum around the (1,0) vaccum is given by:

$$
\lambda^3 - 2(\mu^2 + \tilde{M_e}^2 + \tilde{M_g}^2)\lambda^2 + (\mu^2 + \tilde{M_e}^2 + \tilde{M_g}^2)^2\lambda - 4\mu^2\tilde{M_e}^2\tilde{M_g}^2 = 0 \tag{C.13}
$$

Where $\tilde{M_e}^2 = 2v^2e^2 = \frac{1}{2}M_e^2$ and $\tilde{M_g}^2 = 2v^2g^2 = \frac{1}{2}M_g^2$. It is in the standard form:

$$
\begin{aligned}
& a\lambda^3 + b\lambda^2 + c\lambda + d = 0; \\
& a = 1; \quad b = -2(\mu^2 + \tilde{M_e}^2 + \tilde{M_g}^2); \\
& c = \left(\mu^2 + \tilde{M_e}^2 + \tilde{M_g}^2\right)^2 \text{ and } d = -4\mu^2\tilde{M_e}^2\tilde{M_g}^2.
\end{aligned} \tag{C.14}
$$

Note that $c = \frac{b^2}{4}$. First, we want to make sure that all roots are real. It is helpful to perform the redefinition $\lambda = \tilde{\lambda} - \frac{b}{3a}$ such that (C.13) takes the form of a depressed cubic:

$$\begin{aligned} a\lambda^3 + b\lambda^2 + c\lambda + d = 0 &\Rightarrow \tilde{\lambda}^3 + p\tilde{\lambda} + q = 0 \\ p = \frac{3ac - b^2}{3a^2} \quad &\text{, in our case} \quad p = -\frac{b^2}{12}; \\ q = \frac{2b^3 - 9abc + 27a^2 d}{27a^3} &\rightarrow q = \frac{-(1/4)b^3 + 27d}{27}; \end{aligned} \tag{C.15}$$

To ensure that all roots are real, the discriminant (Δ) must be greater or equal to zero, and it is given by:

$$\Delta = -(4p^3 + 27q^2)$$

Therefore, we must have:

$$\begin{aligned} 4p^3 + 27q^2 &\leq 0 \\ 4\left(-\frac{b^2}{12}\right)^3 + 27\left[\frac{-(1/4)b^3 + 27d}{27}\right]^2 &\leq 0 \\ \left(\frac{b}{3}\right)^3 &\leq 2d \\ \left(\frac{-2(\mu^2 + \tilde{M_e}^2 + \tilde{M_g}^2)}{3}\right)^3 &\leq 2\left(-4\mu^2 \tilde{M_e}^2 \tilde{M_g}^2\right) \\ \sqrt[3]{\mu^2 \tilde{M_e}^2 \tilde{M_g}^2} &\leq \frac{\mu^2 + \tilde{M_e}^2 + \tilde{M_g}^2}{3} \end{aligned} \tag{C.16}$$

That (C.16) holds follow from the fact that the geometric mean (l.h.s.) is always less or equal to the arithmetic mean (r.h.s.).

The next physical requirement is that the roots must all be non-negative to prevent us from tachyons in the theory.

The three roots of the depressed cubic can be expressed, without the need of complex coeficients[1], by the François Viète formula:

$$\tilde{\lambda}_k = 2\sqrt{-\frac{p}{3}}\cos\left[\frac{1}{3}\arccos\left(\frac{3q}{2p}\sqrt{-\frac{3}{p}}\right) - \frac{2\pi k}{3}\right], \quad \text{where } k = 0, 1, 2; \tag{C.17}$$

Since we are looking for $\lambda_k = \tilde{\lambda}_k - \frac{b}{3a}$, we need that:

[1]Even though the roots may be all real, they cannot be all be expressed *algebraically* with only real numbers

$$\tilde{\lambda}_k \geq \frac{b}{3a} \xrightarrow{a=1} \tilde{\lambda}_k \geq \frac{b}{3} \tag{C.18}$$

But from (C.17), we have:

$$\begin{gathered}
-2\sqrt{-\frac{p}{3}} \leq \tilde{\lambda}_k \leq 2\sqrt{-\frac{p}{3}} \\
-2\sqrt{\frac{b^2}{36}} \leq \tilde{\lambda}_k \leq 2\sqrt{\frac{b^2}{36}} \\
-\frac{|b|}{3} \leq \tilde{\lambda}_k \leq \frac{|b|}{3} \\
b < 0 \Rightarrow \frac{b}{3} \leq \tilde{\lambda}_k \leq -\frac{b}{3}
\end{gathered}$$

Therefore, all the roots are safely real, non-negative and given by:

$$\lambda_k = \frac{2}{3}\left(\mu^2 + \tilde{M_e}^2 + \tilde{M_g}^2\right)\left(1 + \cos\left\{\frac{1}{3}\arccos\left[2\left(\frac{\sqrt[3]{\mu^2 \tilde{M_e}^2 \tilde{M_g}^2}}{\frac{\mu^2 + \tilde{M_e}^2 + \tilde{M_g}^2}{3}}\right)^3 - 1\right] - \frac{2\pi k}{3}\right\}\right), \quad \text{where } k = 0, 1, 2; \tag{C.19}$$

Appendix D

Calculation of the vortex's angular momentum

In 2+1 dimensions the total angular momentum (here a pseudo-scalar) is given by:

$$J = -\int d^2x \epsilon^{ij} x_i T_{0j} \tag{D.1}$$

Since we are interested only in static configurations, that is, for which $\partial_0 = 0$, the density of linear momentum of the fields, T_{0j}, will be:

$$\begin{aligned}
T_{0j} &= -F_0{}^l F_{jl} - f_0{}^l f_{jl} \\
&\quad + D_0\phi_+^* D_j\phi_+ + D_0\phi_+ D_j\phi_+^* \\
&\quad + D_0\phi_-^* D_j\phi_- + D_0\phi_- D_j\phi_-^* \\
&= \left(F^{l0}F_{jl} + f^{l0}f_{jl}\right) \\
&\quad + 2\mathfrak{Re}\left(D_0\phi_+^* D_j\phi_+ + D_0\phi_-^* D_j\phi_-\right) \\
&= \epsilon_{jl}\left(E^l B + e^l b\right) \\
&\quad + 2\mathfrak{Re}\left(D_0\phi_+^* D_j\phi_+ + D_0\phi_-^* D_j\phi_-\right)
\end{aligned}$$

The contribution to the total angular momentum coming solely from the gauge fiels is thus:

$$\begin{aligned}
-J_g &= \int d^2x \epsilon^{ij} x_i \epsilon_{jl}\left(E^l B + e^l b\right) \\
&= \int d^2x x^l \left(E^l B + e^l b\right) = \int d^2x \left[\left(\vec{r}.\vec{E}\right) B + \left(\vec{r}.\vec{e}\right) b\right]
\end{aligned}$$

Now focusing our attention on the contribution coming from the scalars, we start by observing that:

$$
\begin{aligned}
D_j\phi_\pm &= \left[\partial_j + i\left(eA_j \pm ga_j\right)\right]|\phi_\pm|e^{i\omega_\pm} \\
&= \left[\partial_j|\phi_\pm| + i\left(eA_j \pm ga_j + \partial_j\omega_\pm\right)|\phi_\pm|\right]e^{i\omega_\pm}
\end{aligned}
$$

Where we have adpted the polar form of the complex scalars. Taking into consideration the static limit me have:

$$
D_0\phi^*_\pm D_j\phi_\pm = -i(eA_0 \pm ga_0)|\phi_\pm|e^{-i\omega_\pm}\left[\partial_j|\phi_\pm| + i\left(eA_j \pm ga_j + \partial_j\omega_\pm\right)|\phi_\pm|\right]e^{i\omega_\pm}
$$

Such that

$$
\begin{aligned}
\mathfrak{Re}\left(D_0\phi^*_\pm D_j\phi_\pm\right) &= \left(eA_0 \pm ga_0\right)\left(eA_j \pm ga_j + \partial_j\omega_\pm\right)|\phi_\pm|^2 \\
&= \left(eA_0 \pm ga_0\right)\left(e\overline{A}_j \pm g\overline{a}_j\right)|\phi_\pm|^2
\end{aligned}
$$

In the equation above, the barred gauge fields were defined as to include the contribution coming from the phases of the scalars, that is, $e\overline{A}_j \pm g\overline{a}_j = eA_j \pm ga_j + \partial_j\omega_\pm$. It is just a gauge transformation. In terms of the *ansatz* they would be simply $\overline{A}_i = (1/er)A(r)\hat{\theta}_i$ and $\overline{a}_i = (1/gr)a(r)\hat{\theta}_i$. Now we write:

$$
\begin{aligned}
&2\mathfrak{Re}\left(D_0\phi^*_+ D_j\phi_+ + D_0\phi^*_- D_j\phi_-\right) = \\
&= 2(eA_0 + ga_0)\left(e\overline{A}_j + g\overline{a}_j\right)|\phi_+|^2 + 2(eA_0 - ga_0)\left(e\overline{A}_j - g\overline{a}_j\right)|\phi_-|^2 \\
&= \overline{A}_j 2e\left[(eA_0 + ga_0)|\phi_+|^2 + (eA_0 - ga_0)|\phi_-|^2\right] + \\
&+ \overline{a}_j 2g\left[(eA_0 + ga_0)|\phi_+|^2 - (eA_0 - ga_0)|\phi_-|^2\right]
\end{aligned}
$$

But the densities $\rho_\pm = i\left(\phi^*_\pm D_0\phi_\pm - \phi_\pm D_0\phi^*_\pm\right) = -2\mathfrak{Im}\left(\phi^*_\pm D_0\phi_\pm\right)$ in the static limit are exaclty $\rho_\pm = -2(eA_0 \pm ga_0)|\phi_\pm|^2$. This allows us to write:

$$
2\mathfrak{Re}\left(D_0\phi^*_+ D_j\phi_+ + D_0\phi^*_- D_j\phi_-\right) = -\overline{A}_j e(\rho_+ + \rho_-) - \overline{a}_j g(\rho_+ - \rho_-)
$$

From the Gauss laws, we have:

$$\vec{\nabla} \cdot \vec{E} + \mu b = e(\rho_+ + \rho_-)$$
$$\vec{\nabla} \cdot \vec{e} + \mu B = g(\rho_+ - \rho_-).$$

Therefore,

$$2\mathfrak{Re}\left(D_0\phi_+^* D_j\phi_+ + D_0\phi_-^* D_j\phi_-\right) = -\overline{A}_j\left(\vec{\nabla} \cdot \vec{E} + \mu b\right) - \overline{a}_j\left(\vec{\nabla} \cdot \vec{e} + \mu B\right)$$

We are now ready to evaluate its contribution to angular momentum:

$$\begin{aligned}
-J_s &= \int d^2x \epsilon^{ij} x_i 2\mathfrak{Re}\left(D_0\phi_+^* D_j\phi_+ + D_0\phi_-^* D_j\phi_-\right) \\
&= -\int d^2x \epsilon^{ij} x_i \left[\overline{A}_j\left(\vec{\nabla} \cdot \vec{E} + \mu b\right) + \overline{a}_j\left(\vec{\nabla} \cdot \vec{e} + \mu B\right)\right] \\
&= \int d^2x \left(\epsilon^{ji}\frac{x^i}{r}\right) r \left[\overline{A}^j\left(\vec{\nabla} \cdot \vec{E} + \mu b\right) + \overline{a}^j\left(\vec{\nabla} \cdot \vec{e} + \mu B\right)\right] \\
&= \int d^2x r\hat{\theta}_j \left[\overline{A}^j\left(\vec{\nabla} \cdot \vec{E} + \mu b\right) + \overline{a}^j\left(\vec{\nabla} \cdot \vec{e} + \mu B\right)\right] \\
&= \int d^2x r \left[\overline{A}_\theta\left(\vec{\nabla} \cdot \vec{E} + \mu b\right) + \overline{a}_\theta\left(\vec{\nabla} \cdot \vec{e} + \mu B\right)\right]
\end{aligned}$$

In polar coordinates, we have $\vec{E} = E_r\hat{r} + E_\theta\hat{\theta}$, such that:

$$\vec{\nabla} \cdot \vec{E} = \frac{1}{r}\frac{\partial\left(rE_r\right)}{\partial r} + \frac{1}{r}\frac{\partial E_\theta}{\partial \theta}$$

And for $b = \epsilon^{ij}\partial_i\overline{a}_j$, in polar coordinates, we have:

$$b = \frac{1}{r}\left[\frac{\partial\left(r\overline{a}_\theta\right)}{\partial r} - \frac{\partial \overline{a}_r}{\partial \theta}\right]$$

Analagously for $\vec{e}$ and B. Now we *assume* radial symmetry, that is, $\partial_\theta = 0$ for any non-vanishing component. This allows us to write:

$$\begin{aligned}
-J_s &= \int d^2x \epsilon^{ij} x_i 2\mathfrak{Re}\left(D_0\phi_+^* D_j\phi_+ + D_0\phi_-^* D_j\phi_-\right) \\
&= \int d^2x r\left[\overline{A}_\theta\left(\vec{\nabla}\cdot\vec{E}+\mu b\right)+\overline{a}_\theta\left(\vec{\nabla}\cdot\vec{e}+\mu B\right)\right] \\
&= \int d^2x\left\{r\overline{A}_\theta\left[\frac{1}{r}\frac{d\left(rE_r\right)}{dr}\right]+r\overline{a}_\theta\left[\frac{1}{r}\frac{d\left(re_r\right)}{dr}\right]\right\} \\
&+\mu\int d^2x\left\{r\overline{A}_\theta\frac{1}{r}\left[\frac{d\left(r\overline{a}_\theta\right)}{dr}\right]+r\overline{a}_\theta\frac{1}{r}\left[\frac{d\left(r\overline{A}_\theta\right)}{dr}\right]\right\}
\end{aligned}$$

$$\begin{aligned}
-J_s &= \int d\theta\int dr\left\{r\overline{A}_\theta\left[\frac{d\left(rE_r\right)}{dr}\right]+r\overline{a}_\theta\left[\frac{d\left(re_r\right)}{dr}\right]\right\} \\
&+\mu\int d\theta\int dr\left\{r\overline{A}_\theta\left[\frac{d\left(r\overline{a}_\theta\right)}{dr}\right]+\left[\frac{d\left(r\overline{A}_\theta\right)}{dr}\right]r\overline{a}_\theta\right\} \\
&= -\int d\theta\int rdr\left\{\left[\frac{1}{r}\frac{d\left(r\overline{A}_\theta\right)}{dr}\right]rE_r+\left[\frac{1}{r}\frac{d\left(r\overline{a}_\theta\right)}{dr}\right]re_r\right\} \\
&+\mu\int d\theta\int dr\frac{d}{dr}\left[\left(r\overline{A}_\theta\right)\left(r\overline{a}_\theta\right)\right] \\
&= -\int d^2x\left[\left(\vec{r}.\vec{E}\right)B+\left(\vec{r}.\vec{e}\right)b\right] \\
&+2\pi\mu\left[r^2\overline{A}_\theta\overline{a}_\theta\right]_0^\infty
\end{aligned}$$

We obtain thus the total angular momentum:

$$J = J_g + J_s = -2\pi\mu\left[r^2\overline{A}_\theta\overline{a}_\theta\right]_0^\infty = \frac{2\pi\mu}{eg}\left[A(0)a(0)-A(\infty)a(\infty)\right] \tag{D.2}$$

For topological vortices, $A(0)=m$ and $a(0)=n$ while $A(\infty)=a(\infty)=0$, therefore we finally arrive at the desired result for the total angular momentum of the vortex:

$$J = \frac{2\pi\mu}{eg}nm = \frac{Q}{e}m = \frac{G}{g}n = \frac{QG}{2\pi\mu} \tag{D.3}$$

For non-topological solitons, as we will see, $A(\infty)=-\beta$ and $a(\infty)=-\alpha$, two real numbers.

Appendix E

Constructing the self-dual potential

In order to construct a model that naturally leads to Bogomol'nyi-type bounds for the energy functional, we follow the strategy of [?] and introduce two neutral fields N (scalar) and M (pseudoscalar), in the form:

$$\begin{aligned}\mathcal{L} = &- \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{4}f_{\mu\nu}f^{\mu\nu} + \mu\epsilon^{\mu\nu\rho}A_\mu\partial_\nu a_\rho \\ &+ |\left(\partial_\mu + ieA_\mu + iga_\mu\right)\phi_+|^2 + |\left(\partial_\mu + ieA_\mu - iga_\mu\right)\phi_-|^2 \\ &+ \frac{1}{2}(\partial_\mu M)^2 + \frac{1}{2}(\partial_\mu N)^2 - V(|\phi_+|, |\phi_-|, M, N),\end{aligned} \tag{E.1}$$

The Lagrangian (E.1) leads to the energy functional:

$$\begin{aligned}H = \int d^2x \Big[&\frac{1}{2}\left(\vec{E}^2 + B^2\right) + \frac{1}{2}\left(\vec{e}^2 + b^2\right) + V(\phi_+, \phi_-, M, N) \\ &+|D_0\phi_+|^2 + |D_0\phi_-|^2 + |D_i\phi_+|^2 + |D_i\phi_-|^2 \\ &+ \frac{1}{2}(\partial_0 M)^2 + \frac{1}{2}(\partial_0 N)^2 + \frac{1}{2}(\partial_i M)^2 + \frac{1}{2}(\partial_i N)^2\Big]\end{aligned} \tag{E.2}$$

Let's take the definition of the currents $J^\nu_\pm$ that appear in the equations of motion:

$$\begin{aligned}\partial_\mu F^{\mu\nu} + \mu\epsilon^{\nu\alpha\beta}\partial_\alpha a_\beta &= e\left(J^\nu_+ + J^\nu_-\right), \\ \partial_\mu f^{\mu\nu} + \mu\epsilon^{\nu\alpha\beta}\partial_\alpha A_\beta &= g\left(J^\nu_+ - J^\nu_-\right),\end{aligned} \tag{E.3}$$

With $J^\nu_\pm = i\left[\phi^*_\pm D^\nu\phi_\pm - D^\nu\phi^*_\pm\phi_\pm\right]$. Then

$$\partial_i(J_\pm)_j = i\left[\partial_i\phi^*_\pm D_j\phi_\pm + \phi^*_\pm\partial_i\left(D_j\phi_\pm\right) - \partial_i\left(D_j\phi^*_\pm\right)\phi_\pm - D_j\phi^*_\pm\partial_i\phi_\pm\right] \tag{E.4}$$

From the definition of the covariant derivatives, we have: $\partial_i\phi_\pm = D_i\phi_\pm - ieA_i\phi_\pm \mp iga_i\phi_\pm$, so substituing back in (E.4):

$$\begin{aligned}\partial_i(J_\pm)_j &= i\left[D_i\phi_\pm^* D_j\phi_\pm + \phi_\pm^*\partial_i\left(D_j\phi_\pm\right) - \partial_i\left(D_j\phi_\pm^*\right)\phi_\pm - D_j\phi_\pm^* D_i\phi_\pm\right]\\ &\quad + i\left[\left(+ieA_i\phi_\pm^* \pm iga_i\phi_\pm^*\right)D_j\phi_\pm - D_j\phi_\pm\left(-ieA_i\phi_\pm \mp iga_i\phi_\pm\right)\right]\\ &= i\left[D_i\phi_\pm^* D_j\phi_\pm + \phi_\pm^* D_i\left(D_j\phi_\pm\right) - D_i\left(D_j\phi_\pm^*\right)\phi_\pm - D_j\phi_\pm^* D_i\phi_\pm\right]\end{aligned} \tag{E.5}$$

Contracting with the spatial anti-symmetric levi-civita symbol $\epsilon^{0ij} \equiv \epsilon^{ij} \to \epsilon^{12} = -\epsilon^{21} = -1$:

$$\epsilon^{ij}\partial_i(J_\pm)_j = 2i\epsilon^{ij}\left(D_i\phi_\pm^* D_j\phi_\pm\right) + i\epsilon^{ij}\left[\phi_\pm^* D_i\left(D_j\phi_\pm\right) - D_i\left(D_j\phi_\pm^*\right)\phi_\pm\right] \tag{E.6}$$

But note that:

$$\begin{aligned}\epsilon^{ij}D_i(D_j\phi_\pm) &= \epsilon^{ij}(\partial_i + ieA_i \pm iga_i)(\partial_j + ieA_j \pm iga_j)\phi_\pm\\ &= \left(ie\epsilon^{ij}\partial_i A_j \pm ig\epsilon^{ij}\partial_i a_j\right)\phi_\pm\\ &= i(eB \pm gb)\phi_\pm\end{aligned} \tag{E.7}$$

Which means then:

$$\epsilon^{ij}\partial_i(J_\pm)_j = 2i\epsilon^{ij}\left(D_i\phi_\pm^* D_j\phi_\pm\right) - 2(eB \pm gb)|\phi_\pm|^2 \tag{E.8}$$

Now, we observe that the following terms $|D_i\phi_\pm|^2$ which appear in the energy functional can be rewritten as:

$$\begin{aligned}|D_i\phi|^2 &= D_1\phi^* D_1\phi + D_2\phi^* D_2\phi\\ &= \left[(D_1 \mp iD_2)\phi^*\right]\left[(D_1 \pm iD_2)\phi\right] \mp iD_1\phi^* D_2\phi \pm iD_2\phi^* D_1\phi\\ &= |(D_1 \pm iD_2)\phi|^2 \pm i\epsilon^{ij}D_i\phi^* D_j\phi\end{aligned} \tag{E.9}$$

In the result above either sign would work for both $\phi_\pm$, that's why we did not specify which field we were using during the derivation and also to prevent confusion. With the last two results, we can write:

$$|D_i\phi_+|^2 = |(D_1 \pm iD_2)\phi_+|^2 \pm \frac{1}{2}\epsilon^{ij}\partial_i(J_+)_j \pm (eB + gb)|\phi_+|^2$$
$$|D_i\phi_-|^2 = |(D_1 \pm iD_2)\phi_-|^2 \pm \frac{1}{2}\epsilon^{ij}\partial_i(J_-)_j \pm (eB - gb)|\phi_+|^2 \qquad \text{(E.10)}$$

We want to substitute $|D_i\phi_+|^2 + |D_i\phi_-|^2$ by the r.h.s. of (E.10), but there is an ambiguity as to which sign should we use. The ambiguity is partially eliminated by the requirement of parity symmetry, if we observe that:

$$\begin{aligned}(D_1 + iD_2)\phi_+ &= [(\partial_1 + ieA_1 + iga_1) + i(\partial_2 + ieA_2 + iga_2)]\,\phi_+ \\ &\xrightarrow{P} [(-\partial_1 - ieA_1 + iga_1) + i(\partial_2 + ieA_2 - iga_2)]\,\phi_- = \\ &= -\,[(\partial_1 + ieA_1 - iga_1) - i(\partial_2 + ieA_2 - iga_2)]\,\phi_- \\ &= -(D_1 - iD_2)\phi_- \end{aligned} \qquad \text{(E.11)}$$

This suggests that, in order to preserve the parity symmetry of (E.2), we must take the combinations with reversed sign, that is:

$$\begin{aligned} H = \int d^2x \Big[&\frac{1}{2}\left(\vec{E}^2 + B^2\right) + \frac{1}{2}\left(\vec{e}^2 + b^2\right) + V(\phi_+, \phi_-, M, N) \\ &+ |D_0\phi_+|^2 + |D_0\phi_-|^2 + |(D_1 \pm iD_2)\phi_+|^2 + |(D_1 \mp iD_2)\phi_-|^2 \\ &+ \frac{1}{2}(\partial_0 M)^2 + \frac{1}{2}(\partial_0 N)^2 + \frac{1}{2}(\partial_i M)^2 + \frac{1}{2}(\partial_i N)^2 \\ &\pm (eB + gb)|\phi_+|^2 \mp (eB - gb)|\phi_-|^2 \Big] \end{aligned} \qquad \text{(E.12)}$$

Where we have dropped a surface term. We can now make use of the neutral fields that we introduced earlier to write:

$$\begin{aligned} \frac{1}{2}\vec{E}^2 + \frac{1}{2}(\vec{\nabla}N)^2 &= \frac{1}{2}\left(\vec{E} \pm \vec{\nabla}N\right)^2 \mp \vec{E}.\vec{\nabla}N \\ &= \frac{1}{2}\left(\vec{E} \pm \vec{\nabla}N\right)^2 \pm N\vec{\nabla}.\vec{E} \mp \underbrace{\cancel{\vec{\nabla}.(N\vec{E})}}_{0} \\ &= \frac{1}{2}\left(\vec{E} \pm \vec{\nabla}N\right)^2 \pm N\left(-\mu b + eJ^0_+ + eJ^0_-\right) \end{aligned} \qquad \text{(E.13)}$$

Where we have made use of the Gauss law ($\nu = 0$ in E.3). Analogously,

$$\frac{1}{2}\vec{e}^2 + \frac{1}{2}(\vec{\nabla}M)^2 = \frac{1}{2}\left(\vec{e} \pm \vec{\nabla}M\right)^2 \pm M\left(-\mu B + gJ^0_+ - gJ^0_-\right) \qquad \text{(E.14)}$$

Because the l.h.s of the equation above is even under parity, so is the r.h.s.. That's precisely why we took M to be a pseudoscalar.

Plugging these results in the energy functional, we get:

$$
\begin{aligned}
H = \int d^2x \Big[& \frac{1}{2}\left(\vec{E} \pm \vec{\nabla} N\right)^2 + \frac{1}{2}B^2 + \frac{1}{2}\left(\vec{e} \pm \vec{\nabla} M\right)^2 + \frac{1}{2}b^2 + V(\phi_+, \phi_-, M, N) \\
& + |D_0\phi_+|^2 + |D_0\phi_-|^2 + |(D_1 \pm iD_2)\phi_+|^2 + |(D_1 \mp iD_2)\phi_-|^2 \\
& + \frac{1}{2}(\partial_0 M)^2 + \frac{1}{2}(\partial_0 N)^2 \\
& \pm (eB + gb)|\phi_+|^2 \mp (eB - gb)|\phi_-|^2 \\
& \pm N\left(-\mu b + eJ^0_+ + eJ^0_-\right) \pm M\left(-\mu B + gJ^0_+ - gJ^0_-\right)\Big] \qquad \text{(E.15)}
\end{aligned}
$$

Note the presence of the following terms:

$$
\begin{aligned}
|D_0\phi_+|^2 \pm eNJ^0_+ \pm gMJ^0_+ &= |D_0\phi_+|^2 \pm (eN + gM)i\left[\phi^*_+ D^0\phi_+ - D^0\phi^*_+\phi_+\right] \\
&= \left[D_0\phi_+ \mp i(eN + gM)\phi_+\right]\left[D_0\phi^*_+ \pm i(eN + gM)\phi^*_+\right] \\
&\quad - (eN + gM)^2|\phi_+|^2 \\
&= |D_0\phi_+ \mp i(eN + gM)\phi_+|^2 - (eN + gM)^2|\phi_+|^2
\end{aligned}
$$

And similarly:

$$
|D_0\phi_-|^2 \pm eNJ^0_- \pm (-g)MJ^0_- = |D_0\phi_- \mp i(eN - gM)\phi_-|^2 - (eN - gM)^2|\phi_-|^2
$$

Therefore:

$$
\begin{aligned}
H = \int d^2x \Big[& \frac{1}{2}\left(\vec{E} \pm \vec{\nabla} N\right)^2 + \frac{1}{2}B^2 + \frac{1}{2}\left(\vec{e} \pm \vec{\nabla} M\right)^2 + \frac{1}{2}b^2 + V(\phi_+, \phi_-, M, N) \\
& + |D_0\phi_+ \mp i(eN + gM)\phi_+|^2 + |D_0\phi_- \mp i(eN - gM)\phi_-|^2 \\
& + |(D_1 \pm iD_2)\phi_+|^2 + |(D_1 \mp iD_2)\phi_-|^2 \\
& + \frac{1}{2}(\partial_0 M)^2 + \frac{1}{2}(\partial_0 N)^2 - (eN + gM)^2|\phi_+|^2 - (eN - gM)^2|\phi_-|^2 \\
& \pm (eB + gb)|\phi_+|^2 \mp (eB - gb)|\phi_-|^2 \\
& \pm N(-\mu b) \pm M(-\mu B)\Big] \qquad \text{(E.16)}
\end{aligned}
$$

One last trick and we're ready to go. We are going to induce a non-trivial VEV for the fields $\phi_\pm$.

$$
\begin{aligned}
H = \int d^2x \Big[& \frac{1}{2}\left(\vec{E} \pm \vec{\nabla} N\right)^2 + \frac{1}{2}B^2 + \frac{1}{2}\left(\vec{e} \pm \vec{\nabla} M\right)^2 + \frac{1}{2}b^2 + V(\phi_+, \phi_-, M, N) \\
& + |D_0\phi_+ \mp i(eN + gM)\phi_+|^2 + |D_0\phi_- \mp i(eN - gM)\phi_-|^2 \\
& + |(D_1 \pm iD_2)\phi_+|^2 + |(D_1 \mp iD_2)\phi_-|^2 \\
& + \frac{1}{2}(\partial_0 M)^2 + \frac{1}{2}(\partial_0 N)^2 - (eN + gM)^2|\phi_+|^2 - (eN - gM)^2|\phi_-|^2 \\
& \pm (eB + gb)\left(|\phi_+|^2 - v_+^2\right) \mp (eB - gb)\left(|\phi_-|^2 - v_-^2\right) \\
& \pm N\,(-\mu b) \pm M\,(-\mu B) \pm v_+^2(eB + gb) \mp v_-^2(eB - gb)\Big]
\end{aligned} \tag{E.17}
$$

At last, we observe that:

$$
\begin{aligned}
& \frac{1}{2}B^2 \pm eB\left(|\phi_+|^2 - v_+^2\right) \mp eB\left(|\phi_-|^2 - v_-^2\right) \pm M\,(-\mu B) = \\
& = \frac{1}{2}\left\{B \pm \left[e\left(|\phi_+|^2 - v_+^2\right) - e\left(|\phi_-|^2 - v_-^2\right) - \mu M\right]\right\}^2 \\
& - \frac{1}{2}\left[e\left(|\phi_+|^2 - v_+^2\right) - e\left(|\phi_-|^2 - v_-^2\right) - \mu M\right]^2
\end{aligned} \tag{E.18}
$$

And the same for the b:

$$
\begin{aligned}
& \frac{1}{2}b^2 \pm gb\left(|\phi_+|^2 - v_+^2\right) \pm gb\left(|\phi_-|^2 - v_-^2\right) \pm N\,(-\mu b) = \\
& = \frac{1}{2}\left\{b \pm \left[g\left(|\phi_+|^2 - v_+^2\right) + g\left(|\phi_-|^2 - v_-^2\right) - \mu N\right]\right\}^2 \\
& - \frac{1}{2}\left[g\left(|\phi_+|^2 - v_+^2\right) + g\left(|\phi_-|^2 - v_-^2\right) - \mu N\right]^2
\end{aligned} \tag{E.19}
$$

So that we can finally write the energy functional in the very suggestive form:

$$
\begin{aligned}
H = \int d^2x \Bigg[& \frac{1}{2}\left(\vec{E} \pm \vec{\nabla} N\right)^2 + \frac{1}{2}\left(\vec{e} \pm \vec{\nabla} M\right)^2 + |D_\pm \phi_+|^2 + |D_\mp \phi_-|^2 + \frac{1}{2}(\partial_0 M)^2 + \frac{1}{2}(\partial_0 N)^2 \\
& + \frac{1}{2}\left\{B \pm \left[e\left(|\phi_+|^2 - v_+^2\right) - e\left(|\phi_-|^2 - v_-^2\right) - \mu M\right]\right\}^2 \\
& + \frac{1}{2}\left\{b \pm \left[g\left(|\phi_+|^2 - v_+^2\right) + g\left(|\phi_-|^2 - v_-^2\right) - \mu N\right]\right\}^2 \\
& + |D_0 \phi_+ \mp i(eN + gM)\phi_+|^2 + |D_0 \phi_- \mp i(eN - gM)\phi_-|^2 \\
& \pm eB(v_+^2 - v_-^2) \pm gb(v_+^2 + v_-^2) \\
& + V(\phi_+, \phi_-, M, N) - (eN + gM)^2 |\phi_+|^2 - (eN - gM)^2 |\phi_-|^2 \\
& - \frac{1}{2}\left[e\left(|\phi_+|^2 - v_+^2\right) - e\left(|\phi_-|^2 - v_-^2\right) - \mu M\right]^2 \\
& - \frac{1}{2}\left[g\left(|\phi_+|^2 - v_+^2\right) + g\left(|\phi_-|^2 - v_-^2\right) - \mu N\right]^2 \Bigg]
\end{aligned} \tag{E.20}
$$

The final touch is that if we want a parity symmetric energy functional, we must take $v_+^2 = v_-^2 = v^2$, and we arrive at the desired result:

$$
\begin{aligned}
H = \int d^2x \Bigg[& \frac{1}{2}\left(\vec{E} \pm \vec{\nabla} N\right)^2 + \frac{1}{2}\left(\vec{e} \pm \vec{\nabla} M\right)^2 + |D_\pm \phi_+|^2 + |D_\mp \phi_-|^2 + \frac{1}{2}(\partial_0 M)^2 + \frac{1}{2}(\partial_0 N)^2 \\
& + \frac{1}{2}\left\{B \pm \left[e\left(|\phi_+|^2 - |\phi_-|^2\right) - \mu M\right]\right\}^2 \\
& + \frac{1}{2}\left\{b \pm \left[g\left(|\phi_+|^2 + |\phi_-|^2 - 2v^2\right) - \mu N\right]\right\}^2 \\
& + |D_0 \phi_+ \mp i(eN + gM)\phi_+|^2 + |D_0 \phi_- \mp i(eN - gM)\phi_-|^2 \\
& \pm 2gbv^2 \\
& + V(\phi_+, \phi_-, M, N) - (eN + gM)^2 |\phi_+|^2 - (eN - gM)^2 |\phi_-|^2 \\
& - \frac{1}{2}\left[e\left(|\phi_+|^2 - |\phi_-|^2\right) - \mu M\right]^2 \\
& - \frac{1}{2}\left[g\left(|\phi_+|^2 + |\phi_-|^2 - 2v^2\right) - \mu N\right]^2 \Bigg]
\end{aligned} \tag{E.21}
$$

To cancel the last terms and obtain a Bogomol'nyi-like bound for the energy functional H, one defines the potential:

$$
\begin{aligned}
V(\phi_+, \phi_-, M, N) = {} & (eN + gM)^2 |\phi_+|^2 + (eN - gM)^2 |\phi_-|^2 \\
& + \frac{1}{2}\left[e\left(|\phi_+|^2 - |\phi_-|^2\right) - \mu M\right]^2 \\
& + \frac{1}{2}\left[g\left(|\phi_+|^2 + |\phi_-|^2 - 2v^2\right) - \mu N\right]^2
\end{aligned} \tag{E.22}
$$

Appendix F

Fundamental rep. of the Lorentz algebra in 2+1 dimensions

The Lie algebra $so(1,2)$ is given by 3 antisymmetric generators $M_{\mu\nu}$ such that[1]:

$$[M^{\mu\nu}, M^{\rho\sigma}] = i\left(\eta^{\mu\rho}M^{\nu\sigma} + \eta^{\nu\sigma}M^{\mu\rho} - \eta^{\mu\sigma}M^{\nu\rho} - \eta^{\nu\rho}M^{\mu\sigma}\right). \tag{F.1}$$

We show below that this algebra is satisfied by $M^{\mu\nu} = i\Sigma^{\mu\nu}$. In fact, one can write $\Sigma^{\mu\nu} = \frac{1}{4}\left[\gamma^\mu, \gamma^\nu\right]$ and use $[\gamma^\mu, \gamma^\nu] = 2i\epsilon^{\mu\nu\rho}\gamma_\rho$, finding $\Sigma^{\mu\nu} = \frac{i}{2}\epsilon^{\mu\nu\rho}\gamma_\rho$, thus

$$\begin{aligned}
[\Sigma^{\mu\nu}, \Sigma^{\rho\sigma}] = & -\frac{1}{4}\varepsilon^{\mu\nu x}\varepsilon^{\rho\sigma y}\left[\gamma_x, \gamma_y\right] \\
= & -\frac{i}{2}\epsilon^{\mu\nu x}\epsilon^{\rho\sigma y}\epsilon^{abc}\eta_{xa}\eta_{yb}\gamma_c.
\end{aligned} \tag{F.2}$$

But we see that $\frac{i}{2}\epsilon^{abc}\eta_{xa}\eta_{yb}\gamma_c = \Sigma_{xy}$ and we know that

$$-\epsilon^{\mu\nu x}\epsilon^{\rho\sigma y} = \eta^{\mu\rho}\left(\eta^{\nu\sigma}\eta^{xy} - \eta^{\nu y}\eta^{x\sigma}\right) - \eta^{\mu\sigma}\left(\eta^{\nu\rho}\eta^{xy} - \eta^{\nu y}\eta^{x\rho}\right) + \eta^{\mu y}\left(\eta^{\nu\rho}\eta^{x\sigma} - \eta^{\nu\sigma}\eta^{x\rho}\right). \tag{F.3}$$

Since Σ_{xy} is antisymmetric in x, y it gives zero when contracted with η_{xy}. Thus we have

$$[\Sigma^{\mu\nu}, \Sigma^{\rho\sigma}] = \eta^{\mu\rho}\Sigma^{\nu\sigma} + \eta^{\nu\sigma}\Sigma^{\mu\rho} - \eta^{\mu\sigma}\Sigma^{\nu\rho} - \eta^{\nu\rho}\Sigma^{\mu\sigma} \tag{F.4}$$

Thus,

$$\begin{aligned}
[i\Sigma^{\mu\nu}, i\Sigma^{\rho\sigma}] &= -\left[\Sigma^{\mu\nu}, \Sigma^{\rho\sigma}\right] = -\left(\eta^{\mu\rho}\Sigma^{\nu\sigma} + \eta^{\nu\sigma}\Sigma^{\mu\rho} - \eta^{\mu\sigma}\Sigma^{\nu\rho} - \eta^{\nu\rho}\Sigma^{\mu\sigma}\right) \\
&= i\left[\eta^{\mu\rho}i\Sigma^{\nu\sigma} + \eta^{\nu\sigma}i\Sigma^{\mu\rho} - \eta^{\mu\sigma}i\Sigma^{\nu\rho} - \eta^{\nu\rho}i\Sigma^{\mu\sigma}\right]
\end{aligned} \tag{F.5}$$

[1] here we are considering the metric $\eta_{\mu\nu} = diag\,(-,+,+)$

Therefore, we see that $M^{\mu\nu} = i\Sigma^{\mu\nu} = -\frac{1}{2}\epsilon^{\mu\nu\rho}\gamma_\rho$ closes the algebra, providing the spinorial representation of the Lorentz generators.

Let us propose that the supercharge transforms under Lorentz according with

$$\begin{aligned}[M^{\mu\nu}, Q^\alpha] &= i\left(\Sigma^{\mu\nu}\right)^\alpha{}_\beta Q^\beta \\ &= -\frac{1}{2}\epsilon^{\mu\nu\rho}\left(\gamma_\rho\right)^\alpha{}_\beta Q^\beta \end{aligned} \tag{F.6}$$

Defining the rotation generator as $J \equiv M^{12}$ and the boost generators as $K^i \equiv M^{0i}$, we can see explicitly the effect of such Lorentz transformations by performing the following

$$\Phi' = e^{\frac{i}{2}\omega_{\mu\nu}M^{\mu\nu}}\Phi. \tag{F.7}$$

When Φ are spinors, we have in particular $\Psi' = e^{\frac{i}{2}\omega_{\mu\nu}(i\Sigma^{\mu\nu})}\Psi$. For the angular momentum and the boosts generators in this representation, we have

$$\begin{aligned} J \equiv i\Sigma^{12} &= -\frac{1}{2}\epsilon^{120}\gamma_0 = -\frac{1}{2}\sigma_y, \\ K^x \equiv i\Sigma^{01} &= -\frac{1}{2}\epsilon^{012}\gamma_2 = \frac{i}{2}\sigma_z, \\ K^y \equiv i\Sigma^{02} &= -\frac{1}{2}\epsilon^{021}\gamma_1 = -\frac{i}{2}\sigma_x. \end{aligned} \tag{F.8}$$

Notice that the angular momentum above is Hermitian and the boosts generators are anti-hermitian, as expected. In particular, we see that we have only one rotation generator, thus two rotations in the plane always commute. For the boosts, we have

$$\left[K^i, K^j\right] = \left[M^{0i}, M^{0j}\right] = i\left(\eta^{00}M^{ij}\right) = -iM^{ij}. \tag{F.9}$$

We can rewrite the above expression using $M^{ij} = -\epsilon^{ij}J$, since we have $\epsilon^{12} = \epsilon^{012} = -1$ and also $J = M^{12}$. In this case, we have $[K^i, K^j] = i\epsilon^{ij}J$. Analogously, we have

$$\left[J, K^i\right] = \left[M^{12}, M^{0i}\right] = i\left(\eta^{2i}M^{10} - \eta^{1i}M^{20}\right) = i\left(\eta^{1i}K^2 - \eta^{2i}K^1\right). \tag{F.10}$$

By the same reasoning used above, we can write $[J, K^i] = -i\epsilon^{ij}K^j$. Therefore we have:

$$[J, J] = 0, \quad \left[J, K^i\right] = -i\epsilon^{ij}K^j, \quad \left[K^i, K^j\right] = i\epsilon^{ij}J. \tag{F.11}$$

To implement a rotation in the plane, we can use the unitary operator (adopting the matrix notation $\varepsilon = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and defining the rotation parameter $\theta \equiv \omega_{12}$)

$$R = e^{i\omega_{12}\left(i\Sigma^{12}\right)} = e^{i\theta J} = e^{-i\theta\sigma_y/2} = e^{-\theta\epsilon/2} \approx \mathbb{1} - \frac{\theta}{2}\varepsilon - \frac{1}{2}\left(\frac{\theta}{2}\right)^2 \mathbb{1} + \frac{1}{6}\left(\frac{\theta}{2}\right)^3 \varepsilon + ...$$
$$\approx \cos\left(\frac{\theta}{2}\right)\mathbb{1} - \sin\left(\frac{\theta}{2}\right)\varepsilon, \tag{F.12}$$

that seems to realize a rotation of $\frac{\theta}{2}$ on the plane.

The commutation between the angular momentum and the Susy generator will be

$$[J, Q^\alpha] \equiv \left[M^{12}, Q^\alpha\right] = -\frac{1}{2}\left(\sigma_y\right)^\alpha{}_\beta Q^\beta = -\frac{i}{2}\epsilon_{\alpha\beta}Q^\beta. \tag{F.13}$$

In the above expresison we have $-i\epsilon_{\alpha\beta}Q^\beta = Q^\beta C_{\beta\alpha} = Q_\alpha$, thus we obtain

$$[J, Q^\alpha] = \frac{1}{2}Q_\alpha. \tag{F.14}$$

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