Centro Brasileiro de Pesquisas Físicas-CBPF
COSMO

The Odd 2D Bubbles, 4D Triangles, and Einstein and Weyl Anomalies in 2D Gravitational Fermionic amplitudes: The Role of Breaking Integration Linearity for Anomalies

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Doctoral Thesis

Rio de Janeiro
2023

MINISTÉRIO DA CIÊNCIA,TECNOLOGIA E INOVAÇÃO

GOVERNO FEDERAL

UNIÃO E RECONSTRUÇÃO

“THE ODD 2D BUBBLES, 4D TRIANGLES, AND EINSTEIN AND WEYL ANOMALIES IN 2D GRAVITATIONAL FERMIONIC AMPLITUDES: THE ROLE OF BREAKING INTEGRATION LINEARITY FOR ANOMALIES”

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Tese de Doutorado em Física apresentada no Centro Brasileiro de Pesquisas Físicas do Ministério da Ciência Tecnologia e Inovação. Fazendo parte da banca examinadora os seguintes professores:

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Rio de Janeiro, 19 de abril de 2023.

"Do *mainstream* não se exige nada, mas da crítica razoável se quer que se mostre até a existência dos átomos com a qual se escreve."
(José Fernando Thuorst)

Acknowledgments

There are many people that I am grateful for being part of this journey, especially to...

- My mother for teaching me to be strong and resilient. For helping you go through this phase of uncertainty and lack of scholarship without questioning my choice to continue. My father, who even though he is no longer here, taught me from an early age that study and dedication would get me where I wanted to go. To my brothers Patrícia, Rodrigo, and Ana and my lifelong friend André, for always being with me, smiling in the victories, and taking me through the falls (which were not few). To my nephews Alice and Léo for existing and complementing my life in a unique way. To my cousin Alexandre for being my family and my haven in Rio.

- My two workmates: José, for being my tireless companion on this journey and never letting me forget who I am and where I wanted to go. To Thalis for bringing me wisdom, friendship, and dedication. From room 1012 that we shared at INPE-UFSM, early mornings at CCNE... UFLA, USP... to the corridors of Diracstan, more than a decade passed... a PhD made us stronger. If one had given up halfway through... maybe this thesis would never exist.

- Professor Orimar, for our many discussions (and those we didn't have), which left me with an immense desire to explore the hidden paths of quantum field theories. And also for having introduced me and strengthened my relationship with Professor Helayel, who is an example for everyone who knows him as a teacher and as a human being. To Professor Tião, for his great contribution to the discussions that fostered this thesis.

- Miguel, for the best years I had in Rio and for being so generous to me. To Guilherme, Erich, Pablo, Jade, Ivana, and many other friends I made at CBPF... between "muretas" and discussions, the days were much better for having you around. To Luiza for being one of my favorite people in the world and Flores for our "reunion" for life.

- My brother-in-law Rodrigo and friends from Estância Velha, Carol, Lari, Fe, and Chuca, for bringing me back to reality when I needed it.

- The APG José Leite Lopes. To Bete, Ricardo (in memoriam), and Cláudia for always being willing to help me with the CBPF bureaucracies.
- All the professors, especially those at UFSM, UFLA and CBPF who helped me to become the physicist I am today.
- CNPQ, for the financial support from this trip called doctorate; without that, it would never be possible.
- Dilma and Lula, who allowed me to study and get to a quality public university. And finally, to the Brazilian people who once again dreamed of better days (which will come!).

Abstract

We investigated Relations Among Green Functions defined in the context of an alternative strategy for coping with the divergences, also called the Implicit Regularization Method (IREG). This procedure does not use specific rules for the context being investigated: the mathematical content (divergent and finite) will remain intact until the calculations end. The divergent part will be organized through standardized objects free of physical quantities. In contrast, the finite part is projected in a class of well-behaved functions that carry all the amplitudes' physical content. That relations arise in fermionic amplitudes in even space-time dimensions, where anomalous tensors connect to finite amplitudes as in the bubbles and triangles in two and four dimensions. Those tensors depend on surface terms, whose non-zero values arise from finite amplitudes as requirements of consistency with the linearity of integration and uniqueness. Maintaining these terms implies breaking momentum-space homogeneity and, in a later step, the Ward identities. Meanwhile, eliminating them allows more than one mathematical expression for the same amplitude. That is a consequence of choices related to the involved Dirac traces. Independently of divergences, it is impossible to satisfy all symmetry implications by simultaneously requiring vanishing surface terms and linearity. Then we approach the 1-loop level fermionic correction for the propagation of the graviton in a space-time $D = 1 + 1$ through the action of a Weyl fermion in curved space-time. In this context, gravitational anomalies arise, and the amplitudes investigated have the highest degree of divergence quadratic. That imposes a substantial algebraic effort; however, the conclusions are in agreement with the non-gravitational amplitudes. At the end of the calculations, we show how it is possible to fix the value of the divergent part through the relations imposed for amplitudes.

Keywords: Anomalies, Gravitational Anomalies, Divergences, Implicit Regularization.

Resumo

Investigamos Relações entre Funções de Green definidas no contexto de uma estratégia alternativa para lidar com as divergências, também conhecida como Método de Regularização Implícita (IREG). Este procedimento não utiliza regras específicas para o contexto que está sendo investigado: o conteúdo matemático (divergente e finito) permanecerá intacto até o final dos cálculos. A parte divergente será organizada através de objetos padronizados livres de grandezas físicas. Em contraste, a parte finita é projetada em uma classe de funções bem comportadas que carregam todo o conteúdo físico das amplitudes. Essas relações surgem em amplitudes fermiônicas em dimensões espaço-temporais pares, onde tensores anômalos se conectam a amplitudes finitas como nas bolhas e triângulos em duas e quatro dimensões. Esses tensores dependem de termos de superfície, cujos valores diferentes de zero surgem de amplitudes finitas como requisitos de consistência com a linearidade de integração e unicidade. Manter esses termos implica quebrar a homogeneidade do espaço-momento e, em uma etapa posterior, as Identidades de Ward. Entretanto, eliminá-los permite mais de uma expressão matemática para a mesma amplitude. Isso é consequência de escolhas relacionadas aos traços de Dirac envolvidos. Independentemente das divergências, é impossível satisfazer todas as implicações de simetria exigindo simultaneamente termos de superfície nulos e linearidade. Em seguida, abordamos a correção fermiônica ao nível 1-loop para a propagação do gráviton em um espaço-tempo $D = 1 + 1$ através da ação de um férmion de Weyl em um espaço-tempo curvo. Nesse contexto, surgem as anomalias gravitacionais, sendo que as amplitudes investigadas apresentam o maior grau de divergência quadrática. Isso impõe um esforço algébrico substancial; no entanto, as conclusões estão de acordo com as amplitudes sem acoplamento derivativo. Ao final dos cálculos, mostramos como é possível fixar o valor da parte divergente através das relações impostas para as amplitudes.

Palavras-chave: Anomalias, Anomalias Gravitacionais, Divergências, Regularização Implícita.

Acronyms and Abbreviations

QFT - Quantum Field Theory
2D - Two Dimensions
4D - Four Dimensions
RAGFs - Relations Among Green Functions
WIs - Ward Identities
IREG - Implicit Regularization
DR - Dimensional Regularization
LHS - Left-Hand Side
RHS - Right-Hand Side
S - Scalar
P - Pseudoscalar
A- Axial
V - Vector

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[1] Throughout this section, we factored out three-point rank-two finite amplitudes from the discussion.

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Chapter 1

Introduction

Since their inception, anomalies have played an important role in Quantum Field Theories (QFTs). The authors [1, 2, 3, 4] first met the subject in the forties and fifties. Then, it was rediscovered in two dimensions ($2D$) by Johnson [5]; through the non-conservation of the axial current in the two-point functions. And in four dimensions ($4D$) in the context of the *ABJ anomaly of the triangle's graph* [6, 7, 8]. In this case, it manifests when two vector currents couple to an axial current via a fermionic propagator loop. The anomalous term (i.e., not expected from the canonical equations) in the divergence of the axial current that violates the PCAC (partial conservation of the axial current) would be responsible for the decay rate of some mesons, including the electromagnetic decay of the neutral pion, $\pi^0 \to \gamma\gamma$, observed experimentally. Later, many studies considered perturbative and non-perturbative approaches to investigate these phenomena. Among them, the Fujikawa interpretation of the path-integral measure [9], heat kernel [10], and cohomological methods [11].

It is well-known that anomalies prevent the quantum counterparts of Noether currents from satisfying their classical conservation laws, which break Ward Identities (WI). Meanwhile, these constraints are necessary to ensure the perturbative renormalizability of gauge models. That also applies to theories with spontaneous symmetry breaking as the Standard Model [12, 13]. The anomaly cancellation mechanism corroborates with the number of quark generations that ultimately implies the prediction of the top quark, for example, see the book [11], and the maintenance of the renormalizability of the standard model ensures internal consistency of the theory.

Similarly, there are anomalies present when fermionic fields couple to gravitational fields. Delbourgo and Salam in [14] and Kimura in [15] established that in the physical dimension, $D = 1 + 3$, two gravitons contribute to the axial anomaly from a triangle diagram. Two energy-momentum tensors couple to an axial current via a fermionic propagator loop. This anomaly would indicate [16] the impossibility of obtaining a gauge theory in a gravitational context unless there is an anomaly cancellation mechanism.

Alvarez-Gaumé and Witten also show in [16] that the violation of the diffeomorphism invariance (Einstein anomalies) at $D = 4k+2$ occurs in "purely gravitational" anomalies, without gauge coupling, in curved spacetime for Weyl fermions with spin 1/2 or 3/2 coupled to the gravitational field via energy-momentum tensor. When there is a violation of the conformal symmetry as we have the Weyl anomaly (or trace anomaly). Capper and Duff in [17, 18] studied such anomalies in the graviton propagation by interaction with photons and Weyl fermions at the 1-loop level, and more recently, the contribution of the Pontryagin density to the Weyl anomalies has been revisited by Bonara et al., [19], [20], and [21]. Furthermore, for gravitation, we have Lorentz anomalies: They signify an antisymmetric part in the energy-momentum tensor, in even dimensions, in particular $2D$, they can be traded by the Einstein anomalies [11] using the local Bardeen-Zummino polynomial [22]. The same polynomial transforms the consistency into the covariant form for anomalies.

Among the places where anomalies manifest, we have the perturbative scenario for correlators of axial and vector currents that are divergent odd tensors. Some of them AV^n amplitudes in $d = 2n$ dimensions, which cannot satisfy all WIs, (see [23]). These are $(n+1)$th-rank tensors of odd-parity and functions of n momenta variables. Consequently, they have a set of low-energy theorems obtained through momenta contractions. In one loop, they contain Dirac traces having two more gamma matrices than the number of dimensions. These traces are linear combinations of monomials in Levi-Civita tensor and metric, displaying equivalent expressions that differ regarding index arrangement, signs, and the number of monomials. In addition, the power counting of the integrals indicates the presence of surface terms, making these structures depend on the graph's momenta routing (outside the amplitude AV in $d = 2$). Since perturbative solutions admit arbitrary choices for routings and Dirac traces, the final results show many possibilities.

This last proposition is inseparable from the fact that divergences are the rule to get model predictions of QFT in perturbation theory. Regularization methods are adopted to obtain information about the amplitudes' kinematic dependence and symmetry consequences. Some examples of these techniques are Cut-off, Pauli-Villars, Analytic Regularization, Dimensional Regularization (DR) [24, 25], High Covariant Regularization [26, 27], Differential Renormalization ([28]). However, these regularization methods can compromise the theory's predictive power by modifying amplitudes and making the divergent structures finite. Beyond its limits of applicability in theories involving the chiral matrix, manipulations not guaranteed to the original expressions take effect as shifts in the integration variable[1]. Furthermore, new methods to deal with multi-loop calculations aiming for algorithmic implementation of precision numerical predictions [29], [30]. The prescription also may prescribe rules, not inherent to Feynman's ones, for which properties of the algebras are valid or not [31, 32, 33, 34, 35, 36].

[1] Take the DR as an example; it eliminates surface terms as a condition to achieve symmetry preservation.

On the other hand, tensor Feynman integrals exhibiting diverging power counting have surface terms. For the linearly diverging ones, a shift in the integration variable requires compensation through non-zero surface terms [37], [38], and [11]. They cannot be free-shifted and need arbitrary labels for internal momenta. Energy-momentum conservation sets differences in the routings as functions of the physical momenta; however, internal momenta are arbitrary (by themselves and their sums) and may assume non-covariant expressions [39]. Since non-zero surface terms imply the breaking of translational symmetry in the momentum space and this operation is needed to prove WIs, other symmetries violations also occur. By exploring tensor properties, we investigate symmetry maintenance and its relation with the mathematical content of the diagrams. That materializes into a discussion about the linearity of integration and choices for perturbative solutions related to their uniqueness[2].

For one of our purposes, we use a general model coupling spin-1/2 fermions (through their bilinear and without derivatives, eventually with fermions of distinct masses) with boson fields of even and odd parity (spins 0 and 1). The n-vertex polygon graphs of spin-1/2 internal propagators are one part of the analysis, specifically the $2D$-AV and VA bubbles, $4D$-AVV, VAV, VVA, and AAA. In the e-print [40], the extension to the $6D$-$AVVV$ box is also explored with the same conclusions. In two dimensions, the AV-VA amplitudes worked with arbitrary masses; the author has the publication [41].

The amplitudes are obtained within a procedure to handle divergent and finite integrals introduced in the Ph.D. thesis of O.A. Battistel [42]. Several investigations applied this strategy in $2D$, $4D$, $6D$, and $5D$. This method has no limit of applicability; without specific rules to the context being investigated. We can use it for theories in even and odd dimensions simultaneously, in addition to careful investigation into chiral theories [43, 44] [45, 46] [47] [48] [49]. Other investigations use the name Implicit Regularization (IREG), having a similar approach [50, 51, 52, 53].

This procedure uses a general identity to isolate divergences that do not interfere with Feynman's rules. Since we do not evaluate divergent integrals explicitly, amplitudes are not modified at any stage of calculations. Also, we use arbitrary routings for the momenta of internal lines. In this strategy, we devise a notational scheme to systematize finite integrals and their divergent parts based on previous works on the subject [54], [55], and [56]. Three relevant ingredients to our discussion are irreducible divergent objects, tensor surface terms, and finite functions. The only assumption is linearity applies to the Feynman integrals, which manifests through Relations Among Green Functions (RAGFs). This aspect is one of the main points of this investigation.

In this way, having studied, in the last instance, chiral anomalies in two and four dimensions, we proceed to see how the conclusions extend for the two-dimensional grav-

[2] To uniqueness, which needs a particular definition to work its consequences, we provide it along the thesis.

itational anomalies [16, 57, 58, 59, 60]. To that end, we explore couplings with currents involving derivatives in the fermion field. The physical scenario is described by a model from a Weyl fermion coupled to a background gravitational field using the same model as the references [61, 62]. In an expansion around the Minkowski metric, the matter field induces corrections through the two-point function of its (linearized) stress tensor. Taking advantage of the strategy, we write all the expressions similar to the case without derivative coupling, which point to many similarities for the elements in the root of symmetry violations.

By carrying intact the divergent content, until the end of all computations, our stance on the perturbative amplitudes enables a detailed view of the elements that yield different results. It also clarifies the connection among the surface terms in amplitudes with ambiguities of routings, traces, and symmetry violations. Any interpretation of divergences that sets surface terms as zero for even amplitudes makes their results symmetric concerning the symmetries related to momenta contractions but not metric contractions. Nevertheless, these prescriptions break integration linearity for odd amplitudes since equal integrands give rise to different integrals. Hence, an uncountable number of tensors follows from the same expression.

On the other hand, by adopting the value of the surface term that preserves linearity, all manipulation on the traces provides one and only one tensor of the routing variables. Therefore the physical interpretation requires arbitrary parameters to fix the symmetries. The freedom allows us to improve the known and desired content of the results (for non-derivative couplings). However, the consequence is that even amplitudes will more often violate their WIs if they ask universality to play a role.

We organized the work as follows. In Chapter (2), we have the general model, definitions, and a preliminary discussion. Chapter (3) discusses the strategy to handle the amplitudes, where we define irreducible objects, tensor surface terms, and finite parts. The compilation of the effects of traces and surface terms in $2D$ appears in the Chapters (4; 5) through complete and independent computation of all the quantities related to RAGFs. The consequences of the results preserving linearity or saving translational symmetry are presented and interpreted in light of low-energy theorems. Chapter (6) deals with all odd triangles in $4D$, their RAGFs, and the concept of uniqueness. The Sections (6.2) and (6.3) deal with general properties of low-energy theorems and offer a proposition that connects linearity, low-energy behavior of finite amplitudes and surface terms. Chapter (7 and 8) extend the propositions to a gravitational scenario. In the last Chapter (9), we discuss some points implied by the investigation for other scenarios.

Chapter 2

Notation, Definitions, Model and Preliminaries

Feynman rules, vertices, and propagators employed in this investigation come from a model where fermionic currents couple to bosonic fields of even and odd parity $\{\Phi(x), V_\mu(x), \Pi(x), A_\mu(x)\}$ through the general interacting action

$$\mathcal{S}_I = \int \mathrm{d}^{2n}x \left[e_S S(x)\Phi(x) + e_\Pi P(x)\Pi(x) + e_V J^\mu(x) V_\mu(x) + e_A J^\mu_*(x) A_\mu(x)\right]. \quad (2.1)$$

The currents $\{S, P, J_\mu, J_{*\mu}\}$ are bilinears in the fermionic fields $J_{i;ab}(x) = \left(\bar{\psi}_a \Gamma_i \psi_b\right)(x)$. They deliver the vertices proportional[1] to

$$\Gamma_i \in (S, P, V, A) = (1, \gamma_*, \gamma_\mu, \gamma_*\gamma_\mu), \quad (2.2)$$

where γ_μ are the generators of the Clifford algebra of Dirac matrices satisfying $\{\gamma^{\mu_1}, \gamma^{\mu_2}\} = 2g^{\mu_1\mu_2}$. The chiral matrix, which is the algebra's highest-weight element, satisfies $\{\gamma_*, \gamma^{\mu_k}\} = 0$ and assumes the explicit form

$$\gamma_* = i^{n-1}\gamma_0\gamma_1\cdots\gamma_{2n-1} = \frac{i^{n-1}}{(2n)!}\varepsilon_{\nu_1\cdots\nu_{2n}}\gamma^{\nu_1\cdots\nu_{2n}}. \quad (2.3)$$

We often adopt a merging notation to products of matrices $\gamma^{\nu_1\cdots\nu_{2n}} = \gamma^{\nu_1}\gamma^{\nu_2}\cdots\gamma^{\nu_{2n}}$, adapting to Lorentz indexes $\mu_1\mu_2\cdots\mu_s = \mu_{12\cdots s}$ when convenient. The behavior under the permutation of the indexes is determined by the objects: $g_{\mu_1\mu_2} = g_{\mu_{12}} = g_{\mu_{21}}$ or $\varepsilon_{\mu_1\mu_2\cdots\mu_{2n}} = \varepsilon_{\mu_{12\cdots 2n}} = -\varepsilon_{\mu_{21}\cdots\mu_{2n}}$. For the $2n$-dimensional, follow the normalization $\varepsilon^{0123\cdots 2n-1} = 1$.

The algebra elements are the antisymmetrized products of gamma matrices

$$\gamma_{[\mu_1\cdots\mu_r]} = \frac{1}{r!}\sum_{\pi\in S_r} \mathrm{sign}(\pi)\,\gamma_{\mu_{\pi(1)}\cdots\mu_{\pi(r)}}. \quad (2.4)$$

They satisfy general identities as seen in the appendix of the reference [63]:

$$\gamma_*\gamma_{[\mu_1\cdots\mu_r]} = \frac{i^{n-1+r(r+1)}}{(2n-r)!}\varepsilon_{\mu_1\cdots\mu_r}{}^{\nu_{r+1}\cdots\nu_{2n}}\gamma_{[\nu_{r+1}\cdots\nu_{2n}]}. \quad (2.5)$$

[1]The proportionality comes from the coupling constants $\{e_S, e_\Pi, e, e_A\}$, taken as the unit for our purposes.

These identities are needed when taking traces with the chiral matrix. For products of tensors, we adopted the antisymmetrization notation

$$A_{[\alpha_1\cdots\alpha_r}B_{\alpha_{r+1}\cdots\alpha_s]} = \frac{1}{s!}\sum_{\pi\in S_s}\text{sign}(\pi)A_{\alpha_{\pi(1)}\cdots\alpha_{\pi(r)}}B_{\alpha_{\pi(r+1)}\cdots\alpha_{\pi(s)}}, \tag{2.6}$$

where the normalization factor does not interfere with the used identities.

The spinorial Feynman propagators come from the standard kinetic term of Dirac fermions

$$S_F\left(K_i\right) = \frac{1}{(\not{K}_i - m_i + i0^+)} = \frac{(\not{K}_i + m_i)}{D_i}, \tag{2.7}$$

where $D_i = K_i^2 - m_i^2$ with $K_i = k + k_i$ and m_i corresponding the mass of the i-particle. The momentum k is the unrestricted loop momentum while k_i are routings that keep track of the flux of external momenta through the graph, see [39][2]. They cannot be written as a function of the kinematical data in divergent integrals. In our approach, they codify conditions of the satisfaction of symmetries or lack thereof. Nonetheless, their differences relate to external momenta through the definition

$$p_{ij} = k_i - k_j, \tag{2.8}$$

using momenta conservation in the vertices of the diagram in figure (2.1).

$T^{\Gamma_1\Gamma_2\cdots\Gamma_{n_1}} =$

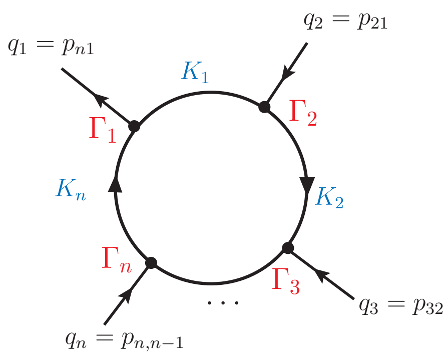

Figure 2.1: General diagram for the one-loop amplitudes of this work.

The integrand of these amplitudes follows from Feynman rules

$$t^{\Gamma_1\Gamma_2\cdots\Gamma_{n_1}}\left(k_1,\cdots,k_{n_1}\right) = \text{tr}[\Gamma_1 S_F(K_1)\Gamma_2 S_F(K_2)\cdots\Gamma_{n_1}S_F(K_{n_1})]. \tag{2.9}$$

That is a well-defined function of the external momenta and sums undetermined by momentum conservation

$$P_{ij} = k_i + k_j. \tag{2.10}$$

[2]Consult section (4.1) for a comment on the arbitrariness of these routings.

Often we adopt the simplification $S(i) \equiv S_F(K_i)$, where the numerical index i represents all parameters of the corresponding line. The total amplitude comes from integration in the loop momenta

$$T^{\Gamma_1\Gamma_2\cdots\Gamma_s}(1,\cdots,s) = \int \frac{\mathrm{d}^{2n}k}{(2\pi)^{2n}} t^{\Gamma_1\Gamma_2\cdots\Gamma_s}(1,\cdots,s). \tag{2.11}$$

When replacing the specific vertex operators Γ_i from (2.2), the notation accompanies the Lorentz indexes in order with the operators. In addition, we set aside the minus signs for closed loops.

2.1 Relation Among Green Functions (RAGF)

As a part of the investigation, we establish identities among Green functions that display Lorentz indices of vector and axial currents. These are commonly called Relations Among Green Functions (RAGFs) and have been used in investigations in the IREG scenario [43][45][49]. They can be considered conditions on the linearity of integration even before WIs are asked to play some role in perturbation amplitudes.

Let us take the amplitude AV^{r-1} to introduce these relations since they are part of our analysis,

$$t^{AV\cdots V}_{\mu_1\mu_2\cdots\mu_r} = \mathrm{tr}[\gamma_*\gamma_{\mu_1}S(1)\gamma_{\mu_2}S(2)\cdots\gamma_{\mu_r}S(r)]. \tag{2.12}$$

When contracted with $p_{21}^{\mu_2}$ in the vector vertex γ_{μ_2}, we remove one propagator using $K_i = k + k_i$ and $S^{-1}(i) = \not{K}_i - m$ through the standard manipulation

$$\not{p}_{ab} = \not{K}_a - \not{K}_b = S^{-1}(a) - S^{-1}(b) + (m_a - m_b) \tag{2.13}$$

This result leads to the vector RAGF, a difference between two amplitudes built out of the same rules

$$p_{21}^{\mu_2} t^{AV\cdots V}_{\mu_1\mu_2\cdots\mu_r} = [t^{AV\cdots V}_{\mu_1\hat{\mu}_2\cdots\mu_r}(1,\hat{2},\cdots,r) - t^{AV\cdots V}_{\mu_1\hat{\mu}_2\cdots\mu_r}(\hat{1},2,\cdots,r)] + (m_2 - m_1)\, t^{ASV\cdots V}_{\mu_1\hat{\mu}_2\cdots\mu_r}. \tag{2.14}$$

The "hats" mean the omission of the propagator corresponding to that routing and the vertices corresponding to the Lorentz indexes. In other words, the RHS contains lower-point functions that are in general more singular under integration (but not always). Now, observe the contraction of the axial vertex with $p_{r1}^{\mu_1}$

$$\begin{aligned} p_{r1}^{\mu_1} t^{AV\cdots V}_{\mu_{12}\cdots\mu_r} &= \mathrm{tr}[S(r)\gamma_* S^{-1}(r)S(1)\gamma_{\mu_2}S(2)\cdots\gamma_{\mu_{r-1}}S(r-1)\gamma_{\mu_r}] \\ &\quad -\mathrm{tr}[\gamma_*\gamma_{\mu_2}S(2)\cdots\gamma_{\mu_r}S(r)]. \end{aligned} \tag{2.15}$$

Using the commutation of the chiral and Dirac matrices that implies in the identity

$$S(r)\gamma_* S^{-1}(r) = (-\gamma_* - 2mS(r)\gamma_*), \tag{2.16}$$

leading to the axial RAGF

$$p_{r1}^{\mu_1} t_{\mu_{12}\cdots\mu_r}^{AV\cdots V} = [t_{\mu_r\hat{\mu}_1\mu_2\cdots\mu_{r-1}}^{AV\cdots V}(1,2,\cdots,\hat{r}) - t_{\hat{\mu}_1\mu_2\cdots\mu_r}^{AV\cdots V}(\hat{1},2,\cdots,r)] - (m_r + m_1)\, t_{\mu_2\cdots\mu_r}^{PV\cdots V}. \tag{2.17}$$

After integration, the relations achieved above become

$$p_{r1}^{\mu_1} T_{\mu_{12}\cdots\mu_r}^{AV\cdots V} = [T_{\mu_r\hat{\mu}_1\cdots\mu_{r-1}}^{AV\cdots V}(1,2,\cdots,\hat{r}) - T_{\hat{\mu}_1\cdots\mu_r}^{AV\cdots V}(\hat{1},2,\cdots,r)] - (m_r + m_1)\, T_{\mu_2\cdots\mu_r}^{PV\cdots V} \tag{2.18}$$

$$p_{21}^{\mu_2} T_{\mu_{12}\cdots\mu_r}^{AV\cdots V} = [T_{\mu_1\hat{\mu}_2\cdots\mu_r}^{AV\cdots V}(1,\hat{2},\cdots,r) - T_{\mu_1\hat{\mu}_2\cdots\mu_r}^{AV\cdots V}(\hat{1},2,\cdots,r)] + (m_2 - m_1)\, T_{\mu_1\hat{\mu}_2\cdots\mu_r}^{ASV\cdots V}. \tag{2.19}$$

These equations embody assumptions of linearity of integration in perturbative computations; however, this characteristic is not guaranteed for divergent amplitudes. We expose this scenario through complete calculations of amplitudes and their relations. Although these equations are a structural property of the operations, they are not a priori linked to the particularities of the model and its symmetries. However, after summing up all contributions from the crossed diagrams (if applicable), the properties for the total sum of lower-point Green functions coming from the momenta contraction should make the expression correspond to the WIs.

The WIs are equations satisfied by Green functions as a consequence of continuous symmetries of the action. They are valid in perturbative approximations built on Feynman rules unless they are inevitably anomalous. They arise from the joint application of the algebra of quantized currents and equations of motion to these currents: $\partial_\mu J^\mu = 0$ and $\partial_\mu J_*^\mu = -2miP$. Their expressions in the position space for axial and vector WIs are

$$\partial_{\mu_1}^{x_1} \left\langle J_*^{\mu_1}(x_1) J_{\mu_2}(x_2) \cdots J_{\mu_r}(x_r) \right\rangle = -2mi \left\langle P(x_1) J_{\mu_2}(x_2) \cdots J_{\mu_r}(x_r) \right\rangle, \tag{2.20}$$

$$\partial_{\mu_2}^{x_2} \left\langle J_{*\mu_1}(x_1) J^{\mu_2}(x_2) \cdots J_{\mu_r}(x_r) \right\rangle = 0, \tag{2.21}$$

where $\langle \cdots \rangle = \langle 0 | T[\cdots] | 0 \rangle$ is an abbreviation for the time ordering of the currents. In our notation for perturbative amplitudes, we would have analogous equations

$$q_1^{\mu_1} T_{\mu_{12}\cdots\mu_r}^{A\to V\cdots V} = -2m T_{\mu_2\cdots\mu_r}^{P\to V\cdots V}; \quad q_2^{\mu_2} T_{\mu_{12}\cdots\mu_r}^{A\to V\cdots V} = 0; \cdots \quad q_r^{\mu_r} T_{\mu_{12}\cdots\mu_r}^{A\to V\cdots V} = 0. \tag{2.22}$$

The arrow means the mentioned sum of contributions. The connection involving RAGFs and WIs is straightforward, so that violations of RAGFs imply violations of WIs. This way, maintaining all WIs depends on satisfying all RAGFs while having translational invariance in the momentum space. We show how this requirement is impossible for a class of amplitudes as those introduced in the sequence. These objects share similar tensor structures, contain diverging surface terms, and produce the same consequences regards anomalies in their specific dimensions. All of them are divergent odd tensors: they have logarithmic power counting in 2D and linear power counting in 4D.

- The $2D$ Bubbles: $T^{AV}_{\mu_{12}}$; $T^{VA}_{\mu_{12}}$;
- The $4D$ Triangles: $T^{AVV}_{\mu_{123}}$; $T^{VAV}_{\mu_{123}}$; $T^{VVA}_{\mu_{123}}$; $T^{AAA}_{\mu_{123}}$;

In the second part starting in the Chapter (7), we explore the consequences in a gravitational scenario, we will also consider the perturbative amplitudes with derivative coupling in 2D (defined in the same chapter). They have linear and quadratic power counting and appear in associated with the study of Einstein and Weyl anomalies.

- The Gravitational Amplitudes Even: $T^{VV}_{\mu_{12};\alpha_1}$; $T^{VV}_{\mu_{12};\alpha_{12}}$; $T^{AA}_{\mu_{12};\alpha_1}$; $T^{AA}_{\mu_{12};\alpha_{12}}$;
- The Gravitational Amplitudes Odd: $T^{AV}_{\mu_{12};\alpha_1}$; $T^{AV}_{\mu_{12};\alpha_{12}}$;

To compute these amplitudes, we have to take the Dirac traces. After that, any amplitude is expressed as linear combinations of bare Feynman integrals following the definition[3],[4]

$$\bar{J}^{(2n)\mu_1\mu_2\cdots\mu_{n_1}}_{n_2}(1,2,\cdots,n_2) = \int \frac{\mathrm{d}^{2n}k}{(2\pi)^{2n}} \frac{K_i^{\mu_1}\cdots K_i^{\mu_{n_1}}}{D_1 D_2\cdots D_{n_2}}. \tag{2.23}$$

These integrals have power counting $\omega = 2n + n_1 - 2n_2$, where n_1 is the tensor rank and n_2 is the number of denominators. A set of five types of integrals arise within each amplitude, which is the subject of subsection (3.2). But first, we develop a procedure to deal with divergent quantities in the sequence.

[3] We simplify the dependence of the functions on their arguments $f(k_1, k_2, \cdots) = f(1, 2, \cdots)$, omitting them if it is clear.

[4] Changing from a reference routing k_j to another k_i is a matter of recognizing the definition of p_{ij} in (2.8) and writing $K_i = K_j + p_{ij}$.

Chapter 3

Procedure to Handle the Divergences and the Finite Integrals

Before presenting the strategy to solve the divergent amplitudes, let us digress into the divergent-integrals issue in QFT. It is well-known that the products of propagators that are not regular distribution are ill-defined in general. A good example is the equation

$$\int \frac{\mathrm{d}^4 k}{(2\pi)^4} \mathrm{tr}[S_F(k)\, S_F(k-p)] = \int \mathrm{d}^4 x \mathrm{tr}[\hat{S}_F(x)\, \hat{S}_F(-x)] \mathrm{e}^{ip\cdot x}. \tag{3.1}$$

The LHS displays a divergent convolution of two Feynman propagators in momentum space. The RHS is the Fourier transform of a product of propagators in position space. So both sides do not define distributions because when the point-wise product of distributions does not exist, the convolution product of their Fourier transform does not also.

These short-distance UV singularities manifest in divergences of loop momentum integrals. Their origins trace back to multiplications of distributions by discontinuous step function in the chronological ordering of operators in the interaction picture. That leads, through the Wick theorem, to the Feynman rules; see [64, 65], originally in Epstein and Glaser [66]. Although the undefined Feynman diagrams can be circumvented by carefully studying the splitting of distributions with causal support in the setting of causal perturbation theory [67, 68, 69] (where no divergent integral appears at all), we work with Feynman rules in the context of regularizations.

We use the systematic procedure known as Implicit Regularization (IREG) to handle the divergences. Its development dates back to the late 1990s in the Ph.D. thesis of O.A. Battistel [42], having its first investigations in the references [70, 71]. Its objective is to keep the connection at all times with the expression of the "bare" Feynman rules while removing physical parameters (i.e., routings and masses) from divergent integrals and putting them in strictly finite integrals. The divergent ones do not suffer any modification besides an organization through surface terms and irreducible scalar integrals.

This objective is realized by noticing that all Feynman integrals depend on the propagators-like structures $D_i = [(k+k_i)^2 - m^2]$ defined in equation (2.7). Thus, by introducing a parameter λ^2, it is possible to construct an identity to separate quantities depending on physical parameters

$$\frac{1}{D_i} = \frac{1}{D_\lambda + A_i} = \frac{1}{D_\lambda}\frac{1}{[1-(-A_i/D_\lambda)]}, \tag{3.2}$$

where $D_\lambda = (k^2 - \lambda^2)$ and $A_i = 2k \cdot k_i + (k_i^2 + \lambda^2 - m^2)$. Now, we use the sum of the geometric progression of order N and ratio $(-A_i/D_\lambda)$ to write

$$\frac{1}{[1-(-A_i/D_\lambda)]} = \sum_{r=0}^{N}(-A_i/D_\lambda)^r + (-A_i/D_\lambda)^{N+1}\frac{1}{[1-(-A_i/D_\lambda)]}. \tag{3.3}$$

Immediately it is possible to determine the asymptotic behavior at infinity of the powers $(-A_i/D_\lambda)^r$ as $\|k\|^{-r}$. Observe that those terms in the summation sign depend on the routings only in the numerator through a polynomial.

With the help of equations (3.3) and (3.2), we get

$$\frac{1}{D_i} = \sum_{r=0}^{N}(-1)^r\frac{A_i^r}{D_\lambda^{r+1}} + (-1)^{N+1}\frac{A_i^{N+1}}{D_\lambda^{N+1}D_i}. \tag{3.4}$$

As this identity is valid for arbitrary N, choosing N as equal to or greater than the power counting is possible. The integration of the last term is finite under these circumstances, exhibiting dependence on the external momenta $p_{ij} = k_i - k_j$ when treating a product of propagators. The parameters λ^2 generate a connection between divergent and finite parts of integrals. That implies specific behavior to the divergent scalar integrals that is straightforwardly satisfied. We adopt the mass of the propagator $\lambda^2 = m^2$ as the scale[1].

To modularize the analysis, we organize divergences without modifications in the first subsection. After that, we introduce the finite functions necessary to express the amplitudes. Lastly, we introduce integrals pertinent to this work, discussing some examples.

3.1 Divergent Terms

After applying the identity (3.4), we express the Feynman integrals through surface terms, irreducible divergent objects, and finite functions. Divergent terms follow the structure of the summation part of the identity and appear as a set of pure integration-momentum integrals

$$\int \frac{\mathrm{d}^{2n}k}{(2\pi)^{2n}}\frac{1}{D_\lambda^a}, \quad \int \frac{\mathrm{d}^{2n}k}{(2\pi)^{2n}}\frac{k_{\mu_1}k_{\mu_2}}{D_\lambda^{a+1}}, \cdots \quad \int \frac{\mathrm{d}^{2n}k}{(2\pi)^{2n}}\frac{k_{\mu_1}k_{\mu_2}\cdots k_{\mu_{2b-1}}k_{\mu_{2b}}}{D_\lambda^{a+b}}, \tag{3.5}$$

with $n \geq a$. Since they have the same power counting, combining them into surface terms is always possible

$$-\frac{\partial}{\partial k^{\mu_1}}\frac{k_{\mu_2}\cdots k_{\mu_{2n}}}{D_\lambda^a} = 2a\frac{k_{\mu_1}k_{\mu_2}\cdots k_{\mu_{2n}}}{D_\lambda^{a+1}} - g_{\mu_1\mu_2}\frac{k^{\mu_3}\cdots k^{\mu_{2n}}}{D_\lambda^a} - \text{permutations}. \tag{3.6}$$

[1]The identity is independent of the parameter λ^2, which is clear when taking the derivative with this parameter.

Observing the equation above, note that a surface term combines into lower-order surface terms. That produces a chain of associations, leading to scalar integrals that encode the divergent content of the original expression. They preserve the possibility or not of shifting the integration variable, which means we are trading the freedom of the operation of translation in the momentum space for the arbitrary choice of the routings in these perturbative corrections. These surface terms are always present for linear and higher divergent or logarithmic-divergent tensor integrals. Although their coefficients depend on ambiguous momenta (2.10) in the first case, only external momenta (2.8) appear in the second.

We define combinations that arise for this investigation for the abelian chiral anomalies as follows

$$\Delta^{(2n)}_{(n+1);\mu_1\mu_2}(\lambda^2) = \int \frac{\mathrm{d}^{2n}k}{(2\pi)^{2n}} \left(\frac{2nk_{\mu_1}k_{\mu_2}}{D^{n+1}_\lambda} - g_{\mu_1\mu_2}\frac{1}{D^n_\lambda} \right) = -\int \frac{\mathrm{d}^{2n}k}{(2\pi)^{2n}} \frac{\partial}{\partial k^{\mu_1}} \frac{k_{\mu_2}}{D^n_\lambda}, \tag{3.7}$$

where the superscript $n = 1, 2$ indicates respectively two and four dimensions. The corresponding irreducible scalar comes from the definition

$$I^{(2n)}_{\log}\left(\lambda^2\right) = \int \frac{\mathrm{d}^{2n}k}{(2\pi)^{2n}} \frac{1}{D^n_\lambda}. \tag{3.8}$$

The separation highlights diverging structures and organizes them without performing any analytic operation. Moreover, it makes evident that the divergent content is a local polynomial in the ambiguous and physical momenta obtained without expansions or limits.

For the gravitational case, the integrals show superior power counting; the iterative use of this systematization from the first tensor term allows to recombine of all the tensor integrals in terms of surface plus scalar integrals, whose coefficients are symmetrical combinations of the metric tensor,

$$\Delta^{(2)}_{2\mu_{12}} = \int \frac{\mathrm{d}^2k}{(2\pi)^2} \left[\frac{2k_{\mu_{12}}}{D^2_\lambda} - g_{\mu_{12}} \frac{1}{D_\lambda} \right] = -\int \frac{\mathrm{d}^2k}{(2\pi)^2} \frac{\partial}{\partial k^{\mu_1}} \frac{k_{\mu_2}}{D_\lambda}. \tag{3.9}$$

The 4th-rank surface term

$$\Box^{(2)}_{3\mu_{1234}} = \int \frac{\mathrm{d}^2k}{(2\pi)^2} \left[\frac{8k_{\mu_{1234}}}{D^3_\lambda} - \frac{g_{(\mu_{12}}k_{\mu_{12})}}{D^2_\lambda} \right] = -\frac{1}{2}\sum_{i=1}^{4} \int \frac{\mathrm{d}^2k}{(2\pi)^2} \frac{\partial}{\partial k^{\mu_i}} \frac{k_{\mu_1 \cdots \hat{\mu}_i \cdots \mu_4}}{D^2_\lambda} \tag{3.10}$$

and the longest one, the 6th-rank surface term

$$\Sigma^{(2)}_{4\mu_{123456}} = \int \frac{\mathrm{d}^2k}{(2\pi)^2} \left[\frac{48k_{\mu_{123456}}}{D^4_\lambda} - \frac{8}{3}\frac{g_{(\mu_{12}}k_{\mu_{1234})}}{D^3_\lambda} \right] = -\frac{4}{3}\sum_{i=1}^{6} \int \frac{\mathrm{d}^2k}{(2\pi)^2} \frac{\partial}{\partial k^{\mu_i}} \frac{k_{\mu_1 \cdots \hat{\mu}_i \cdots \mu_6}}{D^3_\lambda}. \tag{3.11}$$

For the symmetrization of indices, we use

$$A_{(\alpha_1 \cdots \alpha_r} B_{\alpha_{r+1} \cdots \alpha_s)} = \sum_{\pi \in S^{non}_s} A_{\alpha_{\pi(1)} \cdots \alpha_{\pi(r)}} B_{\alpha_{\pi(r+1)} \cdots \alpha_{\pi(s)}}. \tag{3.12}$$

In our notation, $S_s^{non} \subset S_s$ is a subgroup of the permutation group of s elements that does not count terms that are already symmetric. It means the total sum has all terms that make the tensor completely antisymmetric without repetition of terms with a coefficient equal to the unit. We are using the convention of condensing the indices $k_{\mu_1} \cdots k_{\mu_n} = k_{\mu_{1\cdots n}}$ and the same for vector k. These surface terms, therefore, have the character of being explicitly completely symmetric, a handy property in computations. Beyond the logarithmic objects defined above also appear quadratically divergent integrals organized in the objects:

$$\Delta^{(2)}_{1\mu_{12}} = \int \frac{\mathrm{d}^2 k}{(2\pi)^2} \left[\frac{2k_{\mu_{12}}}{D_\lambda} - g_{\mu_1\mu_2} \log \frac{(k^2 - m^2)}{k^2} \right] \tag{3.13}$$

$$\Box^{(2)}_{2\mu_{1234}} = \int \frac{\mathrm{d}^2 k}{(2\pi)^2} \left[\frac{4k_{\mu_{1234}}}{D^2_\lambda} - \frac{g_{(\mu_{12}} k_{\mu_{34})}}{D_\lambda} \right]. \tag{3.14}$$

And the quadratic scalar

$$I^{(2)}_{\mathrm{quad}} = \int \frac{\mathrm{d}^2 k}{(2\pi)^2} \log \frac{(k^2 - m^2)}{k^2}. \tag{3.15}$$

Important note: the complete symmetrization of the indices that appear as the product of the metrics can cause the expressions for the surface terms to have dozens of terms. For the sake of clarity, let us define the combinations,

$$W_{4\mu_{123456}} = \Sigma^{(2)}_{4\mu_{123456}} + \frac{1}{3} g_{(\mu_{12}} \Box^{(2)}_{3\mu_{3456})} + \frac{1}{3} g_{(\mu_{12}} g_{\mu_{34}} \Delta^{(2)}_{2\mu_{56})} \tag{3.16}$$

$$W_{3\mu_{1234}} = \Box^{(2)}_{3\mu_{1234}} + \frac{1}{2} g_{(\mu_{12}} \Delta^{(2)}_{2\mu_{34})} \tag{3.17}$$

$$W_{2\mu_{1234}} = \Box^{(2)}_{2\mu_{1234}} + \frac{1}{2} g_{(\mu_{12}} \Delta^{(2)}_{1\mu_{34})}. \tag{3.18}$$

The first row has sixty-one terms, while the second and third rows have seven terms. They allow us to write the integrals often present in the separation of divergent terms as

$$\begin{aligned}
\int \frac{\mathrm{d}^2 k}{(2\pi)^2} \frac{48 k_{\mu_{123456}}}{D^4_\lambda} &= W_{4\mu_{123456}} + g_{(\mu_{12}} g_{\mu_{34}} g_{\mu_{56})} I^{(2)}_{\log} \\
\int \frac{\mathrm{d}^2 k}{(2\pi)^2} \frac{8 k_{\mu_{1234}}}{D^3_\lambda} &= W_{3\mu_{1234}} + g_{(\mu_{12}} g_{\mu_{34})} I^{(2)}_{\log} \\
\int \frac{\mathrm{d}^2 k}{(2\pi)^2} \frac{4 k_{\mu_{1234}}}{D^2_\lambda} &= W_{2\mu_{1234}} + g_{(\mu_{12}} g_{\mu_{34})} I^{(2)}_{\mathrm{quad}} \\
\int \frac{\mathrm{d}^2 k}{(2\pi)^2} \frac{2 k_{\mu_{12}}}{D^2_\lambda} &= \Delta^{(2)}_{2\mu_{12}} + g_{\mu_{12}} I^{(2)}_{\log} \\
\int \frac{\mathrm{d}^2 k}{(2\pi)^2} \frac{2 k_{\mu_{12}}}{D_\lambda} &= \Delta^{(2)}_{1\mu_{12}} + g_{\mu_{12}} I^{(2)}_{\mathrm{quad}}.
\end{aligned} \tag{3.19}$$

For the trace of $W_{4\mu_{123456}}$ and $W_{3\mu_{1234}}$, we begin with

$$W^{\rho}_{4\rho\mu_{1234}} = \Sigma^{\rho}_{4\rho\mu_{1234}} + \frac{10}{3} \Box_{3\mu_{1234}} + \frac{1}{3} g_{(\mu_{12}} \Box^{\rho}_{3\mu_{34})\rho} + \frac{8}{3} g_{(\mu_{12}} \Delta_{2\mu_{34})} + \frac{1}{3} g_{(\mu_{12}} g_{\mu_{34})} \Delta^{\rho}_{2\rho} \tag{3.20}$$

$$W^{\rho}_{3\rho\mu_{12}} = \Box^{\rho}_{3\rho\mu_{12}} + 3\Delta_{2\mu_{12}} + \frac{1}{2} g_{\mu_{12}} \Delta^{\rho}_{2\rho}. \tag{3.21}$$

They arise from a simple combinatorial analysis: For $g_{(\mu_{12}} \Box_{3\mu_{3456})}$ there are fifteen terms where only in one of the indices μ_{56} appears in the metric and six terms where both indices appear in $\Box_{3\mu_{3456}}$, the remaining ones have the indices μ_5 or μ_6 in the metric and the other in the surface term. In the first and last set of permutations, we get a factor of ten for $\Box_{3\mu_{1234}}$, and the other six generate a complete symmetric combination of the trace and metric, namely

$$g^{\mu_{56}} g_{(\mu_{12}} \Box_{3\mu_{3456})} = 10 \Box_{3\mu_{1234}} + g_{(\mu_{12}} \Box^{\rho}_{3\mu_{34})\rho}. \tag{3.22}$$

As for the term $g_{(\mu_{12}} g_{\mu_{34}} \Delta_{2\mu_{56})}$, they are forty-five terms, in eighteen of them the μ_{56} indices are in the metric and twenty-four the metric and the surface term share them. These terms generate a factor of eight multiplied by the symmetric combinations of $g_{(\mu_{12}} \Delta_{2\mu_{34})}$, the remaining three yield the total result

$$g^{\mu_{56}} g_{(\mu_{12}} g_{\mu_{34}} \Delta_{2\mu_{56})} = 8 g_{(\mu_{12}} \Delta_{2\mu_{34})} + g_{(\mu_{12}} g_{\mu_{34})} \Delta^{\rho}_{2\rho}, \tag{3.23}$$

where $\Delta^{\rho}_{2\rho}$ is the trace of the divergent object.

As a last observation, two essential combinations appear in the verification process of RAGF, resulting from traces with the metric. It is possible to immediately express the features of W-tensors defined above in the following ways

$$2W^{\rho}_{3\rho\mu_{12}} - 8\Delta^{(2)}_{2\mu_{12}} = [2(\Box^{(2)\rho}_{3\rho\mu_{12}} - \Delta^{(2)}_{2\mu_{12}}) - g_{\mu_{12}} \Delta^{(2)\rho}_{2\rho}] + 2 g_{\mu_{12}} \Delta^{(2)\rho}_{2\rho} \tag{3.24}$$

$$\begin{aligned} 3W^{\rho}_{4\rho\mu_{1234}} - 18 W_{3\mu_{1234}} \;=\; & [3\Sigma^{(2)\rho}_{4\rho\mu_{1234}} - 8\Box^{(2)}_{3\mu_{1234}} - g_{(\mu_{12}} g_{\mu_{34})} \Delta^{(2)\rho}_{2\rho}] \\ & + g_{(\mu_{12}} [\Box^{(2)\rho}_{3\rho\mu_{34})} - \Delta^{(2)}_{2\mu_{34})} - \frac{1}{2} g_{\mu_{34})} \Delta^{(2)\rho}_{2\rho}] + 3 g_{(\mu_{12}} g_{\mu_{34})} \Delta^{(2)\rho}_{2\rho}. \end{aligned} \tag{3.25}$$

Its determination follows from the combinatorial analysis of the terms symmetrized in their definitions. The term $g_{(\mu_2\alpha_2} g_{\nu_{12})} \Delta^{\rho}_{2\rho}$ inside the parentheses is equal to $2 g_{(\mu_2\alpha_2} g_{\nu_{12})} \Delta^{\rho}_{2\rho}$ due to metric degeneracy. The term

$$g_{(\mu_{12}} [\Box^{(2)\rho}_{3\rho\mu_{34})} - \Delta^{(2)}_{2\mu_{34})} - \frac{1}{2} g_{\mu_{34})} \Delta^{(2)\rho}_{2\rho}] \tag{3.26}$$

represents the six permutations for it to be completely symmetric. When one splits it into three terms, the last one is symmetric with just three terms of the type $g_{\mu_{12}} g_{\mu_{34}}$. Hence we get a factor of one instead of a half, which is identical to the combination we have begun. This arrangement makes the expression similar to the one shown for the trace of W_3.

These relations were exposed here because the expansion on the basic surface terms becomes excessively long and unnecessary. The surface terms in the leading integrals (highest rank-tensor) do not need expansion. The RAGF conditions of satisfaction only require these terms to be ranked by their indices and the number of contractions, as we will see in the Chapter on gravitational two-point functions.

3.2 Finite Functions

3.2.1 Two Dimensions

After separating the finite part, we solve the integrals through techniques of perturbative calculations and project their results into a family of functions. Two-point basic functions assume the form

$$Z_{n_1}^{(-1)} = \int_0^1 \mathrm{d}x \frac{x^{n_1}}{Q}; \tag{3.27}$$

$$Z_{n_1}^{(0)} = \int_0^1 \mathrm{d}x x^{n_1} \log \frac{Q}{-\lambda^2}, \tag{3.28}$$

with $n_i \in \mathbb{N}$, and the Q is a polynomial given by

$$Q\left(q^2, m_2, m_1\right) = q^2 x\left(1-x\right) + \left(m_1^2 - m_2^2\right) x - m_1^2. \tag{3.29}$$

An important point that will be explored is $q^2 = 0$ for equal masses $m_1 = m_2$, where

$$Z_{n_1}^{(-1)}\left(0\right) = -\frac{1}{m^2\left(n_1+1\right)}; \quad Z_{n_1}^{(0)}\left(0\right) = 0. \tag{3.30}$$

And the combination between $Z_1^{(-1)}$ and $Z_0^{(-1)}$ given by

$$\left[\left(m_1^2 - m_2^2\right) Z_1^{(-1)} - m_1^2 Z_0^{(-1)}\right]_{q^2=0} = \int_0^1 \mathrm{d}x \frac{\left(m_1^2 - m_2^2\right) x - m_1^2}{Q\left(0, m_2, m_1\right)} = 1; \tag{3.31}$$

It has a nice limit that will appear in investigating the AV of different masses.

Reductions: $Z_{n_1}^{(k)}$ in both parameters and the ones required for this work are

$$Z_0^{(0)} = \log \frac{m_2^2}{\lambda^2} + 2q^2 Z_2^{(-1)} - \left(q^2 + m_1^2 - m_2^2\right) Z_1^{(-1)} \tag{3.32}$$

$$2q^2 Z_1^{(-1)} = \left(q^2 + m_1^2 - m_2^2\right) Z_0^{(-1)} + \log \frac{m_1^2}{m_2^2} \tag{3.33}$$

$$q^2 Z_{n_1+2}^{(-1)} = \left(q^2 + m_1^2 - m_2^2\right) Z_{n_1+1}^{(-1)} - m_1^2 Z_{n_1}^{(-1)} - \frac{1}{\left(n_1+1\right)}, \tag{3.34}$$

with $n_1 \geq 0$. In the gravitational setting (where only equal masses integrals will are explored), we have the function

$$Z_0^{(1)} = \int_0^1 \mathrm{d}x Q \log \frac{Q}{-\lambda^2}. \tag{3.35}$$

Adopting $m_1 = m_2 = \lambda$, the reductions needed for that scenario are

$$Z_0^{(1)} - m^2 = 2q^2 Z_2^{(0)} - q^2 Z_1^{(0)} \tag{3.36}$$

$$2Z_1^{(0)} = Z_0^{(0)} \tag{3.37}$$

$$(n_1+3)q^2 Z_{n_1+2}^{(0)} = (n_1+2)q^2 Z_{n_1+1}^{(0)} - (n_1+1)m^2 Z_{n_1}^{(0)} - \frac{(n_1+1)}{(n_1+2)(n_1+3)} q^2, \tag{3.38}$$

with $n_1 \geq 0$.

3.2.2 Four Dimensions

For the three-point amplitudes[2], we have the polynomial

$$Q\left(p,q,m_2,m_3,m_1\right) = p^2x_1\left(1-x_1\right)+q^2x_2\left(1-x_2\right)-2\left(p\cdot q\right)x_1x_2 + \left(m_1^2-m_2^2\right)x_1+\left(m_1^2-m_3^2\right)x_2-m_1^2. \tag{3.39}$$

And the corresponding basic functions,

$$Z^{(-1)}_{n_1n_2} = \int_0^1 \mathrm{d}x_1\int_0^{1-x_1}\mathrm{d}x_2\frac{x_1^{n_1}x_2^{n_2}}{Q} \tag{3.40}$$

$$Z^{(0)}_{n_1n_2} = \int_0^1 \mathrm{d}x_1\int_0^{1-x_1}\mathrm{d}x_2 x_1^{n_1}x_2^{n_2}\log\frac{Q}{-\lambda^2}. \tag{3.41}$$

At the point where all bilinears are zero, and for equal masses $m_1=m_2=m_3$, they satisfy

$$Z^{(-1)}_{n_1n_2}\left(0\right) = -\frac{n_1!n_2!}{m^2\left[\left(n_1+n_2+2\right)!\right]};\quad Z^{(0)}_{n_1n_2}\left(0\right)=0. \tag{3.42}$$

Writing the parameters in terms of derivatives of the polynomials and using partial integration follows relations among these functions. More precisely, they are reductions of involved parameter powers n_1+n_2 for equation (3.40) (see Appendices **??**). They were approached in the papers [54][55][56]. This resource is necessary for the operations performed throughout this investigation.

Let us start by making the derivative of the Q polynomial for equal masses concerning the parameter x_i and multiplying by $1/Q$; we construct the result

$$x_1^{n_1}x_2^{n_2}\frac{\partial}{\partial x_1}\log Q = -2\left[p^2x_1^{n_1+1}x_2^{n_2}+\left(p\cdot q\right)x_1^{n_1}x_2^{n_2+1}\right]\frac{1}{Q}+p^2\frac{x_1^{n_1}x_2^{n_2}}{Q} \tag{3.43}$$

$$x_1^{n_1}x_2^{n_2}\frac{\partial}{\partial x_2}\log Q = -2\left[q^2x_1^{n_1}x_2^{n_2+1}+\left(p\cdot q\right)x_1^{n_1+1}x_2^{n_2}\right]\frac{1}{Q}+q^2\frac{x_1^{n_1}x_2^{n_2}}{Q}. \tag{3.44}$$

When integrating $\int_0^{1-x_1}\mathrm{d}x_2$, in some cases, we need to commute the integral and a derivative. The upper limit of the integral is not a constant; in that situation, we applied it to the Leibnitz formula

$$\int_{a(x)}^{b(x)}\mathrm{d}z\frac{\partial}{\partial x}F\left(x,z\right) = \frac{\partial}{\partial x}\int_{a(x)}^{b(x)}\mathrm{d}zF\left(x,z\right)-\left[F\left(x,b\left(x\right)\right)\frac{\partial b\left(x\right)}{\partial x}-F\left(x,a\left(x\right)\right)\frac{\partial a\left(x\right)}{\partial x}\right]. \tag{3.45}$$

For our purposes $b'\left(x\right)=-1$ and $a'\left(x\right)=0$, hence

$$\int_0^{b(x)}\mathrm{d}z\frac{\partial}{\partial x}F\left(x,z\right) = \frac{\partial}{\partial x}\int_0^{b(x)}\mathrm{d}zF\left(x,z\right)+F\left(x,b\left(x\right)\right). \tag{3.46}$$

The limits of integration will bring a binomial expansion as well

$$\left(1-x_1\right)^{n_2} = \sum_{s=0}^{n_2}\left(-1\right)^s\binom{n_2}{s}x_1^s. \tag{3.47}$$

[2]These polynomials can be written in terms of Symanzik polynomials constructed using the spanning trees and two-forests of the graph.

Through the application of these elements, it is derived the formulae

$$\begin{aligned}2[p^2 Z^{(-1)}_{n_1+1;n_2} + (p\cdot q)\, Z^{(-1)}_{n_1;n_2+1}] &= p^2 Z^{(-1)}_{n_1;n_2} + (1-\delta_{n_1 0})\, n_1 Z^{(0)}_{n_1-1,n_2} \\ &+\delta_{n_1 0} Z^{(0)}_{n_2}(q) - \sum_{s=0}^{n_2} (-1)^s \binom{n_2}{s} Z^{(0)}_{n_1+s}(q-p)\end{aligned} \tag{3.48}$$

$$\begin{aligned}2[q^2 Z^{(-1)}_{n_1;n_2+1} + (p\cdot q)\, Z^{(-1)}_{n_1+1;n_2}] &= q^2 Z^{(-1)}_{n_1;n_2} + (1-\delta_{n_2 0})\, n_2 Z^{(0)}_{n_1;n_2-1} \\ &+\delta_{n_2 0} Z^{(0)}_{n_1}(p) - \sum_{s=0}^{n_2} (-1)^s \binom{n_2}{s} Z^{(0)}_{n_1+s}(q-p)\,.\end{aligned} \tag{3.49}$$

They represent a reduction in n_i from a situation of $n_1+n_2+1 \to n_1+n_2$ appearing in the RAGFs and WI verifications. It is also necessary to use another reduction

$$2Z^{(0)}_{00} = \left[p^2 Z^{(-1)}_{10} + q^2 Z^{(-1)}_{01}\right] - 2m^2 Z^{(-1)}_{00} - 1 + 2Z^{(0)}_{1}(q-p)\,. \tag{3.50}$$

That comes from the previous ones and the use of

$$\frac{1}{2} = -p^2 Z^{(-1)}_{20} - q^2 Z^{(-1)}_{02} - p^2 Z^{(-1)}_{10} - q^2 Z^{(-1)}_{01} - 2\,(p\cdot q)\, Z^{(-1)}_{11} - m^2 Z^{(-1)}_{00} \tag{3.51}$$

from integrating the identity $\frac{Q}{Q}=1$. This set of mathematical results is enough to develop any computation concerning the finite parts in this thesis.

3.3 Basis of Feynman Integrals

At the end of Section (2), we introduced a set of $(n+1)$-point amplitudes in $2n$ dimensions. In the same context, equation (2.23) presented a general definition for integrals that appear after taking Dirac traces. We describe in a nutshell those that arise within the amplitudes. At two dimensions, the needed integrals are defined by

$$\left[\bar{J}^{(2)}_1(k_i)\,;\bar{J}^{(2)\mu_1}_1(k_i)\,;\bar{J}^{(2)\mu_{12}}_1(k_i)\,;\bar{J}^{(2)\mu_{123}}_1(k_i)\right] = \int\frac{d^2k}{(2\pi)^2}\frac{(1;\ K_i^{\mu_1};\ K_{iii}^{\mu_{12}};\ K_{iii}^{\mu_{123}})}{D_i} \tag{3.52}$$

$$\left[\bar{J}^{(2)}_2;\bar{J}^{(2)\mu_1}_2;\bar{J}^{(2)\mu_{12}}_2;\bar{J}^{(2)\mu_{123}}_2,\bar{J}^{(2)\mu_{1234}}_2\right] = \int\frac{d^2k}{(2\pi)^2}\frac{(1;\ K_1^{\mu_1};\ K_{111}^{\mu_{12}};\ K_{111}^{\mu_{123}};\ K_{1111}^{\mu_{1234}})}{D_{12}}. \tag{3.53}$$

And at four dimensions, we define the functions with two and three propagators

$$\left[\bar{J}^{(4)}_2;\bar{J}^{(4)\mu_1}_2\right] = \int\frac{d^4k}{(2\pi)^4}\frac{(1;\ K_i^{\mu_1})}{D_{ij}}, \tag{3.54}$$

$$\left[\bar{J}^{(4)}_3;\bar{J}^{(4)\mu_1}_3;\bar{J}^{(4)\mu_{12}}_3\right] = \int\frac{d^4k}{(2\pi)^4}\frac{(1;\ K_1^{\mu_1};\ K_1^{\mu_1}K_1^{\mu_2})}{D_{123}}. \tag{3.55}$$

We use the conventions $D_{12\cdots i} = D_1 D_2\cdots D_i$, and $K_i = k+k_i$ with $K^{\nu_{a_1}\cdots\nu_{a_n}}_{a_1\cdots a_n} = K^{\nu_{a_1}}_{a_1}\cdots K^{\nu_{a_n}}_{a_n}$, where $a_i\in\{1,\cdots,n\}$. For the case of integrals with fewer propagators of each dimension, it is necessary to specify the momenta.

3.3.1 Two Dimensions

The power counting of n-point integrals associated with the chiral anomaly from odd amplitudes in two dimensions are

$$\begin{cases} \omega(J_2^{(2)}) = -2 \\ \omega(J_2^{(2)\mu_1}) = -1 \\ \omega(J_2^{(2)\mu_{12}}) = 0 \end{cases} ; \quad \begin{cases} \omega(J_1^{(2)}) = 0 \\ \omega(J_1^{(2)\mu_1}) = 1 \end{cases} ; \tag{3.56}$$

The power counting for integrals associated with derivative coupling for n-point integrals

$$\begin{cases} \omega(J_2^{(2)\mu_{123}}) = 1 \\ \omega(J_2^{(2)\mu_{1234}}) = 2 \end{cases} ; \quad \begin{cases} \omega(J_1^{(2)\mu_{12}}) = 2 \\ \omega(J_1^{(2)\mu_{123}}) = 3 \end{cases} ; \tag{3.57}$$

Some integrals contain finite and divergent parts, so we adopt the overbar to indicate such a feature. For instance, in $2n$ dimensions, the integral $\bar{J}_n^{(2n)}$ contains a diverging object and finite contributions labeled as $J_n^{(2n)}$. The presence of the overbar distinguishes the complete integral from its finite content. That also means they coincide for strictly finite integrals, namely $\bar{J}_{n+1}^{(2n)\mu_1} = J_{n+1}^{(2n)\mu_1}$ and $\bar{J}_{n+1}^{(2n)} = J_{n+1}^{(2n)}$.

The one-point integrals in (3.56), are obtained using the identity (3.4) with $N = 1$

$$\frac{1}{D_i} = \frac{1}{D_\lambda} - \frac{A_i}{D_\lambda^2} + \frac{A_i^2}{D_\lambda^2 D_i}. \tag{3.58}$$

When Integrating the finite parts and identifying the divergent objects as (3.8) and (3.9)

$$\bar{J}_1^{(2)}(k_i) = I_{\log}^{(2)}\left(\lambda^2\right) - \frac{i}{4\pi}\log\frac{m_i^2}{\lambda^2} \tag{3.59}$$

$$\bar{J}_{1\mu_1}^{(2)}(k_i) = -k_i^{\nu_1}\Delta_{2\nu_1\mu_1}^{(2)}\left(\lambda^2\right). \tag{3.60}$$

The two integrals show logarithmic divergence. The last one corresponds to a pure surface term. The argument of $I_{\log}\left(\lambda^2\right)$ object may be transformed by

$$\frac{1}{\left(k^2-\lambda^2\right)} = \frac{1}{\left(k^2-m_i^2\right)} - \frac{\left(m_i^2-\lambda^2\right)}{\left(k^2-\lambda^2\right)\left(k^2-m_i^2\right)}. \tag{3.61}$$

This identification implies a scale relation between the divergent and finite part

$$I_{\log}^{(2)}\left(\lambda^2\right) = I_{\log}^{(2)}\left(m_i^2\right) + \frac{i}{(4\pi)}\log\frac{m_i^2}{\lambda^2}. \tag{3.62}$$

The scalar one be written as $\bar{J}_1^{(2)}(k_i) = I_{\log}^{(2)}(m_i^2)$. For more details, see Appendix (C).

For the two-point integrals with the power counting given by (3.56), we have

$$J_2^{(2)} = \frac{i}{4\pi}[Z_0^{(-1)}(q, m_2, m_1)] \tag{3.63}$$

$$J_2^{(2)\mu_1} = \frac{i}{4\pi}[-q^{\mu_1}Z_1^{(-1)}] \tag{3.64}$$

$$J_2^{(2)\mu_1\mu_2} = \frac{i}{4\pi}\left[-\frac{1}{2}g^{\mu_1\mu_2}Z_0^{(0)} + q^{\mu_1}q^{\mu_2}Z_2^{(-1)}\right] \tag{3.65}$$

$$\bar{J}_2^{(2)\mu_1\mu_2} = J_2^{(2)\mu_1\mu_2} + \frac{1}{2}\left[\Delta_2^{(2)\mu_1\mu_2} + g^{\mu_1\mu_2}I_{\log}^{(2)}\right]. \tag{3.66}$$

Arguments were omitted since they are the same for all integrals. The two-point divergent integral is obtained by applying the identity (3.4) with $N = 0$; its complete calculation is performed in the Appendix (B.2).

Reductions 2D: For Chapters (4) and (5), we will need the reductions listed above

$$2q_{\mu_1}J_2^{(2)\mu_{12}} = -\left(q^2+m_1^2-m_2^2\right)J_2^{(2)\mu_2} - \frac{i}{4\pi}q^{\mu_2}\log\left(m_2^2/\lambda^2\right) \tag{3.67}$$

$$g_{\mu_{12}}J_2^{(2)\mu_{12}} = \left(\frac{i}{4\pi}+m_1^2J_2^{(2)}\right) - \frac{i}{4\pi}\log\left(m_2^2/\lambda^2\right) \tag{3.68}$$

$$2q_{\mu_1}J_2^{(2)\mu_1} = -\left(q^2+m_1^2-m_2^2\right)J_2^{(2)} + \frac{i}{4\pi}\log\left(m_2^2/m_1^2\right) \tag{3.69}$$

$$q^2(2J_{2\mu_2}^{(2)}+q_{\mu_2}J_2^{(2)}) = -q_{\mu_2}\left(m_1^2-m_2^2\right)J_2^{(2)} - \frac{i}{4\pi}q_{\mu_2}\log\left(m_1^2/m_2^2\right). \tag{3.70}$$

In Chapter (7), in addition to the functions introduced above, it is necessary to the single mass of 3rd-rank integral, obtained by applying the identity with $N = 1$:

$$\bar{J}_{2\mu_{123}}^{(2)} = J_{2\mu_{123}}^{(2)} - \frac{1}{4}P^{\nu_1}W_{3\mu_{123}\nu_1} + \frac{1}{4}\left(P-q\right)_{(\mu_1}\Delta_{2\mu_{23})} - \frac{1}{4}q_{(\mu_1}g_{\mu_{23})}I_{\log} \tag{3.71}$$

$$J_{2\mu_{123}}^{(2)} = -\frac{i}{4\pi}\left[-\frac{1}{2}q_{(\mu_1}g_{\mu_{23})}Z_1^{(0)} + q_{\mu_{123}}Z_3^{(-1)}\right]. \tag{3.72}$$

And 4th-rank integral, using $N = 2$ in (3.4):

$$\begin{aligned}\bar{J}_{2\mu_{12}\alpha_{12}}^{(2)} = {} & J_{2\mu_{12}\alpha_{12}}^{(2)} + \frac{1}{4}W_{2\mu_{12}\alpha_{12}} + \frac{1}{4}g_{(\mu_{12}\alpha_{12})}I_{\text{quad}} \\ & -\frac{1}{24}\left[q^2g_{(\mu_{12}\alpha_{12})} - 4q_{(\mu_{12}}g_{\alpha_{12})}\right]I_{\log} \\ & +\frac{1}{48}\left(3P^{\nu_{12}}+q^{\nu_{12}}\right)W_{4\mu_{12}\alpha_{12}\nu_{12}} \\ & -\frac{1}{16}\left(P^2+q^2\right)W_{3\mu_{12}\alpha_{12}} - \frac{1}{8}P^{\nu_1}\left(P-q\right)_{(\mu_1}W_{3\mu_2\alpha_{12})\nu_1} \\ & +\frac{1}{8}\left[\left(P-q\right)_{\mu_1}\left(P-q\right)_{(\alpha_1}\Delta_{2\alpha_2)\mu_2} + \left(P-q\right)_{\mu_2}\left(P-q\right)_{(\alpha_1}\Delta_{2\alpha_2)\mu_1}\right] \\ & +\frac{1}{8}\left(P-q\right)_{\alpha_1}\left(P-q\right)_{\alpha_2}\Delta_{2\mu_{12}} + \frac{1}{8}\left(P-q\right)_{\mu_1}\left(P-q\right)_{\mu_2}\Delta_{2\alpha_{12}}\end{aligned} \tag{3.73}$$

$$J_{2\mu_{12}\alpha_{12}}^{(2)} = \frac{i}{4\pi}\left\{\frac{1}{4}g_{(\mu_{12}}g_{\alpha_{12})}\left[Z_0^{(1)}-m^2\right] - \frac{1}{2}g_{(\mu_{12}}q_{\alpha_{12})}Z_2^{(0)} + q_{\mu_{12}}q_{\alpha_{12}}Z_4^{(-1)}\right\}. \tag{3.74}$$

We use index condensation notation for momentum, $q_{\mu_1}...q_{\mu_n} = q_{\mu_1...\mu_n}$, as well as for metric $g_{\mu_1\mu_2} = g_{\mu_{12}}$. Remembering that $q_{(\mu_{12}}g_{\alpha_{12})}$ is the symmetric combination.

Using the reduction of the last section, we derive the identities

$$2q^{\mu_1}J_{2\mu_{123}}^{(2)} = -q^2J_{2\mu_2\mu_3}^{(2)} \tag{3.75}$$

$$2q^{\mu_1}J_{2\mu_{1234}}^{(2)} = -q^2J_{2\mu_{234}}^{(2)}. \tag{3.76}$$

And the contraction with the metric tensor given by

$$2g_{\mu_{12}}J_2^{(2)\mu_{123}} = 2m^2J_2^{(2)\mu_3} - \frac{i}{4\pi}q^{\mu_3} \tag{3.77}$$

$$2g_{\mu_{12}}J_2^{(2)\mu_{1234}} = 2m^2J_2^{(2)\mu_{34}} + \frac{i}{4\pi}\frac{1}{6}\left[3q^{\mu_{34}} - \theta^{\mu_{34}}\left(q\right)\right]. \tag{3.78}$$

3.3.2 Four Dimensions

As to the four-dimensional integral, we have the following power counting

$$\begin{cases} \omega(J_3^{(4)}) = -2 \\ \omega(J_3^{(4)\mu_1}) = -1 \\ \omega(J_3^{(4)\mu_{12}}) = 0 \end{cases} ; \quad \begin{cases} \omega(J_2^{(4)}) = 0 \\ \omega(J_2^{(4)\mu_1}) = 1 \end{cases} ; \tag{3.79}$$

The scalar and vector three-point functions are finite: $\bar{J}_3^{(4)} = J_3^{(4)}$and $\bar{J}_3^{(4)\mu_1} = J_3^{(4)\mu_1}$. We compute the case with the highest power counting to illustrate some features of our treatment. The four-dimensional vector two-point integral,

$$\bar{J}_2^{(4)\mu_1} = \int \frac{\mathrm{d}^4 k}{(2\pi)^4} \frac{K_i^{\mu_1}}{D_{ij}} \tag{3.80}$$

has linear power counting, which requires using the identity (3.4) with $N = 1$, as (3.58). Its replacement allows rewriting the integrand

$$\begin{aligned} \frac{K_i^{\mu_1}}{D_{ij}} &= \frac{K_i^{\mu_1}}{D_\lambda^2} - \frac{(A_i + A_j)\, K_i^{\mu_1}}{D_\lambda^3} \\ &+ \left[\frac{A_i A_j}{D_\lambda^4} + \frac{A_i^2}{D_\lambda^3 D_i} + \frac{A_j^2}{D_\lambda^3 D_j} - \frac{A_i A_j^2}{D_\lambda^4 D_j} - \frac{A_j A_i^2}{D_\lambda^4 D_i} + \frac{A_i^2 A_j^2}{D_\lambda^4 D_{ij}} \right] K_i^{\mu_1}. \end{aligned} \tag{3.81}$$

After applying the integration sign, we gather the purely divergent integrals and integrate the remaining finite integrals.

This result exhibits all elements presented before. We organize the local divergences through surface terms and irreducible scalars,

$$\bar{J}_2^{(4)\mu_1} = J_2^{(4)\mu_1}(p_{ji}) - \frac{1}{2} [P_{ji}^{\nu_1} \Delta_{3\nu_1}^{(4)\mu_1} + p_{ji}^{\mu_1} I_{\log}^{(4)}], \tag{3.82}$$

while integrating the finite part without restrictions,

$$J_2^{(4)\mu_1}(p_{ji}) = \frac{i}{(4\pi)^2} p_{ji}^{\mu_1} Z_1^{(0)}(p_{ij}^2, m^2), \tag{3.83}$$

where $p_{ij} = k_i - k_j$ and $P_{ij} = k_i + k_j$ (2.8-2.10). For completeness, the scalar integral,

$$\bar{J}_2^{(4)} = I_{\log}^{(4)} + J_2^{(4)}(p_{ji}). \tag{3.84}$$

Following our organization, its finite part is given by

$$J_2^{(4)}(p_{ij}) = -\frac{i}{(4\pi)^2} Z_0^{(0)}(p_{ij}^2, m^2).$$

Three-Point: We need scalar, vector, and tensor integrals.

$$J_3^{(4)} = i\,(4\pi)^{-2}\,[Z_{00}^{(-1)}(p,q)] \tag{3.85}$$

$$J_{3\mu_1}^{(4)} = i\,(4\pi)^{-2}\,[-p_{\mu_1} Z_{10}^{(-1)} - q_{\mu_1} Z_{01}^{(-1)}] \tag{3.86}$$

$$J_{3\mu_1\mu_2}^{(4)} = i\,(4\pi)^{-2} \left[p_{\mu_{12}} Z_{20}^{(-1)} + q_{\mu_{12}} Z_{02}^{(-1)} + p_{(\mu_1} q_{\mu_2)} Z_{11}^{(-1)} - \frac{1}{2} g_{\mu_{12}} Z_{00}^{(0)} \right] \tag{3.87}$$

$$\bar{J}^{(4)}_{3\mu_1\mu_2} = J^{(4)}_{3\mu_1\mu_2} + \frac{1}{4}(\Delta^{(4)}_{3\mu_{12}} + g_{\mu_{12}} I^{(4)}_{\log}). \tag{3.88}$$

Of these, only the tensor integral is divergent, where we used $N = 0$, in (3.4). It is worth mentioning that the arguments p and q are only general variables that tag the entries of the functions; they must be carefully substituted for the ones that appear in a particular part of the investigation. In four dimensions, we will adopt $p = p_{21}$ and $q = p_{31}$.

Reductions 4D: The three points obey the reductions of the previous section as the two-point functions. Therefore it is possible to show that the tensors J satisfy

$$2p^{\mu_1} J^{(4)}_{3\mu_1} = -p^2 J^{(4)}_3 + [J^{(4)}_2(q) - J^{(4)}_2(q-p)] \tag{3.89}$$

$$2q^{\mu_1} J^{(4)}_{3\mu_1} = -q^2 J^{(4)}_3 + [J^{(4)}_2(p) - J^{(4)}_2(q-p)]. \tag{3.90}$$

And for the tensor integrals

$$2p^{\mu_1} J^{(4)}_{3\mu_1\mu_2} = -p^2 J^{(4)}_{3\mu_2} + [J^{(4)}_{2\mu_2}(q) + J^{(4)}_{2\mu_2}(q-p) + q_{\mu_2} J^{(4)}_2(q-p)] \tag{3.91}$$

$$2q^{\mu_1} J^{(4)}_{3\mu_1\mu_2} = -q^2 J^{(4)}_{3\mu_2} + [J^{(4)}_{2\mu_2}(p) + J^{(4)}_{2\mu_2}(q-p) + q_{\mu_2} J^{(4)}_2(q-p)]. \tag{3.92}$$

In addition to the trace contraction

$$g^{\mu_1\mu_2} J^{(4)}_{3\mu_1\mu_2} = m^2 J^{(4)}_3 + \frac{i}{2(4\pi)^2} + J^{(4)}_2(q-p)\,. \tag{3.93}$$

In sections where a specific dimension is handled, we drop the super-index in $J^{(d)}$ integrals.

We will also need the reductions of the Z-functions for the case of different masses

$$Z^{(1)}_0 = \left(-m_2^2\right)\left[\log\frac{m_2^2}{\lambda^2} - 1\right] + 2p^2 Z^{(0)}_2 - \left(p^2 + m_1^2 - m_2^2\right) Z^{(0)}_1 \tag{3.94}$$

$$2q^2 Z^{(0)}_1 = \left(q^2 + m_1^2 - m_2^2\right) Z^{(0)}_0 + m_2^2 \log\frac{m_2^2}{\lambda^2} - m_1^2 \log\frac{m_1^2}{\lambda^2} + \left(m_1^2 - m_2^2\right) \tag{3.95}$$

$$q^2 Z^{(0)}_{n+2} = \frac{(n+2)}{(n+3)}\left(q^2 + m_1^2 - m_2^2\right) Z^{(0)}_{n+1} - \frac{(n+1)}{(n+3)} m_1^2 Z^{(0)}_n \tag{3.96}$$

$$+\frac{1}{(n+3)} m_2^2 \log\frac{m_2^2}{\lambda^2} - \frac{1}{(n+3)(n+2)}\left[\frac{(n+1)}{(n+3)} q^2 + \left(m_2^2 - m_1^2\right)\right]. \tag{3.97}$$

All the results of this Session also will be used to determine under what conditions the Einstein and Weyl anomalies manifest themselves in the gravitational amplitudes. However, in the following two Chapters, we will verify the explicit form of the odd two-dimensional and four-dimensional abelian chiral amplitudes. After doing this, we will extend the results to the two-dimensional gravitational case.

Chapter 4

Two-Dimensional AV-VA Functions

In this section, we compute amplitudes of two Lorentz indices to establish the connection between linearity, symmetries, and low-energy implications, which materialize through Relations Among Green Functions (RAGFs) and Ward Identities (WIs). It is also defined what we mean by uniqueness, exploring examples that evoke this concept. Since the involved amplitudes exhibit logarithmic power counting, they depend only on the difference between routings and not on the arbitrary sums; then, we adopt $q = p_{21} = k_2 - k_1$.

Our first step, therefore, is to clarify the mentioned connection. After introducing the model 2, we showed how to establish identities among the amplitudes integrands (2.14)-(2.17). The integration should produce RAGFs for the vector and axial vertexes

$$q^{\mu_2} T^{AV}_{\mu_{12}} = T^{A}_{\mu_1}(1) - T^{A}_{\mu_1}(2) \tag{4.1}$$
$$q^{\mu_1} T^{VA}_{\mu_{12}} = T^{A}_{\mu_2}(1) - T^{A}_{\mu_2}(2). \tag{4.2}$$

$$q^{\mu_1} T^{AV}_{\mu_{12}} = T^{A}_{\mu_2}(1) - T^{A}_{\mu_2}(2) - 2mT^{PV}_{\mu_2} \tag{4.3}$$
$$q^{\mu_2} T^{VA}_{\mu_{12}} = T^{A}_{\mu_1}(1) - T^{A}_{\mu_1}(2) + 2mT^{VP}_{\mu_1}. \tag{4.4}$$

These contractions are direct implications of the integral linearity, and conditions to their validity are the subject of the first subsection. Meanwhile, WIs require vanishing the axial one-point functions above. That occurs because the formal current-conservation equations require it (2.21 and 2.20).

Moreover, if these symmetry constraints are valid, the general structure of these amplitudes as odd tensors implies kinematic properties to the scalar invariants F_i as,

$$T^{AV}_{\mu_{12}} = \varepsilon_{\mu_1\mu_2} F_1 + \varepsilon_{\mu_1\nu} q^{\nu} q_{\mu_2} F_2 + \varepsilon_{\mu_2\nu} q^{\nu} q_{\mu_1} F_3. \tag{4.5}$$

Contracting with the external momenta in the respective indexes yields

$$q^{\mu_2} T^{AV}_{\mu_{12}} = \varepsilon_{\mu_1\nu} q^{\nu} (q^2 F_2 + F_1), \tag{4.6}$$
$$q^{\mu_1} T^{AV}_{\mu_{12}} = \varepsilon_{\mu_2\nu} q^{\nu} (q^2 F_3 - F_1). \tag{4.7}$$

The vector conservation in the first equation implies $F_1 = -q^2 F_2$, whose replacement in the second equation produces

$$q^{\mu_1} T^{AV}_{\mu_{12}} = \varepsilon_{\mu_2 \nu} q^\nu q^2 (F_3 + F_2). \tag{4.8}$$

Hence, if invariants do not have poles in $q^2 = 0$, we have a low-energy implication for axial contraction. If axial WI is satisfied, this implication falls on the PV amplitude

$$q^{\mu_1} T^{AV}_{\mu_{12}}\Big|_{q^2=0} = 0 = -2m T^{PV}_{\mu_2}\Big|_{q^2=0} =: \varepsilon_{\mu_2 \nu} q^\nu \Omega^{PV}(q^2 = 0), \tag{4.9}$$

being Ω^{PV} is the form factor associated with PV. The deduction of this last behavior requires the validity of both WIs, so it has the same status as a symmetry property. The reciprocal form of this statement appears by exchanging the order of the arguments. If the axial WI is selected first, it implies $F_1 = q^2 F_3 - \Omega^{PV}$ in (4.7). Its replacement in the vector contraction (4.6) gives the low-energy implication for the contraction with the index of the vector current

$$q^{\mu_2} T^{AV}_{\mu_{12}}\Big|_{q^2=0} = -\varepsilon_{\mu_1 \nu} q^\nu \Omega^{PV}(q^2 = 0). \tag{4.10}$$

With this scenario in hand, our objective is their analysis in the light of explicit integration (2.11). From definition (2.9), the general integrand of two-point amplitudes is

$$\begin{aligned} t^{\Gamma_1 \Gamma_2} &= K^{\nu_{12}}_{12} \mathrm{tr}[\Gamma_1 \gamma_{\nu_1} \Gamma_2 \gamma_{\nu_2}] \frac{1}{D_{12}} + m^2 \mathrm{tr}[\Gamma_1 \Gamma_2] \frac{1}{D_{12}}, \\ &\quad + m K^{\nu_1}_1 \mathrm{tr}[\Gamma_1 \gamma_{\nu_1} \Gamma_2] \frac{1}{D_{12}} + m K^{\nu_1}_2 \mathrm{tr}[\Gamma_1 \Gamma_2 \gamma_{\nu_1}] \frac{1}{D_{12}}. \end{aligned} \tag{4.11}$$

Specific versions emerge after choosing the vertices and keeping the non-zero traces:

$$t^{AV}_{\mu_{12}} = K^{\nu_{12}}_{12} \mathrm{tr}(\gamma_* \gamma_{\mu_1 \nu_1 \mu_2 \nu_2}) \frac{1}{D_{12}} + m^2 \mathrm{tr}(\gamma_* \gamma_{\mu_1 \mu_2}) \frac{1}{D_{12}}, \tag{4.12}$$

$$t^{VA}_{\mu_{12}} = K^{\nu_{12}}_{12} \mathrm{tr}(\gamma_* \gamma_{\mu_1 \nu_1 \mu_2 \nu_2}) \frac{1}{D_{12}} - m^2 \mathrm{tr}(\gamma_* \gamma_{\mu_1 \mu_2}) \frac{1}{D_{12}}. \tag{4.13}$$

As the trace of four gamma matrices is a linear combination of the metric and the Levi-Civita tensor, various expressions emerge through substitutions involving the following versions of the identity (2.5):

$$2\gamma_* = \varepsilon_{\nu_{12}} \gamma^{\nu_{12}}; \tag{4.14}$$

$$\gamma_* \gamma_{\mu_1} = -\varepsilon_{\mu_1 \nu_1} \gamma^{\nu_1}; \tag{4.15}$$

$$\gamma_* \gamma_{[\mu_1 \mu_2]} = -\varepsilon_{\mu_1 \mu_2}. \tag{4.16}$$

They lead to expressions that are not automatically equal after integration. To unfold this rationale, let us apply the chiral matrix definition in form $2\gamma_* = \varepsilon^{ef} \gamma_{ef}$ to write

$$\mathrm{tr}(\gamma_* \gamma_{abcd}) = \frac{1}{2} \varepsilon^{ef} \mathrm{tr}(\gamma_{efabcd}) \tag{4.17}$$

$$= 2[-g_{ab}\varepsilon_{cd} + g_{ac}\varepsilon_{bd} - g_{ad}\varepsilon_{bc} - g_{bc}\varepsilon_{ad} + g_{bd}\varepsilon_{ac} - g_{cd}\varepsilon_{ab}]. \tag{4.18}$$

We explore two equivalent sorting of indices $(a, b, c, d) = (\mu_1, \nu_1, \mu_2, \nu_2)$ and $(a, b, c, d) = (\mu_2, \nu_2, \mu_1, \nu_1)$, corresponding to the substitution of the chiral matrix definition around the first and second vertices. The traces differ by signs of terms but are equivalent. To study them, we perform the contractions with $K_{12}^{\nu_{12}} = K_1^{\nu_1} K_2^{\nu_2}$ and write the equations

$$\begin{aligned} K_{12}^{\nu_{12}} \mathrm{tr}(\gamma_* \gamma_{\mu_1} \gamma_{\nu_1} \gamma_{\mu_2} \gamma_{\nu_2}) &= -2\varepsilon_{\mu_1 \nu_1} \left(K_{1\mu_2} K_2^{\nu_1} + K_{2\mu_2} K_1^{\nu_1} \right) - 2\varepsilon_{\mu_2 \nu_1} \left(K_{1\mu_1} K_2^{\nu_1} - K_{2\mu_1} K_1^{\nu_1} \right) \\ &\quad +2\varepsilon_{\mu_1 \mu_2} \left(K_1 \cdot K_2 \right) + 2 g_{\mu_1 \mu_2} \varepsilon_{\nu_1 \nu_2} K_{12}^{\nu_{12}}, \end{aligned} \tag{4.19}$$

$$\begin{aligned} K_{12}^{\nu_{12}} \mathrm{tr}(\gamma_* \gamma_{\mu_2} \gamma_{\nu_2} \gamma_{\mu_1} \gamma_{\nu_1}) &= +2\varepsilon_{\mu_1 \nu_1} \left(K_{1\mu_2} K_2^{\nu_1} - K_{2\mu_2} K_1^{\nu_1} \right) - 2\varepsilon_{\mu_2 \nu_1} \left(K_{1\mu_1} K_2^{\nu_1} + K_{2\mu_1} K_1^{\nu_1} \right) \\ &\quad -2\varepsilon_{\mu_1 \mu_2} \left(K_1 \cdot K_2 \right) - 2 g_{\mu_1 \mu_2} \varepsilon_{\nu_1 \nu_2} K_{12}^{\nu_{12}}. \end{aligned} \tag{4.20}$$

The general form (4.11) shows that combining the bilinears with mass terms associated with $\mathrm{tr}(\gamma_* \gamma_{\mu_{12}}) = -2\varepsilon_{\mu_{12}}$ leads to scalar two-point amplitudes identified as

$$t^{PP} = q^2 \frac{1}{D_{12}} - \frac{1}{D_1} - \frac{1}{D_2}, \tag{4.21}$$

$$t^{SS} = (4m^2 - q^2) \frac{1}{D_{12}} + \frac{1}{D_1} + \frac{1}{D_2}. \tag{4.22}$$

The following reduction was used for these integrands

$$S_{ij} = K_i \cdot K_j - m^2 = \frac{1}{2}(D_i + D_j - p_{ij}^2). \tag{4.23}$$

It is possible to express all other contributions in terms of the same object, a standard tensor present similarly in all explored dimensions

$$t_{\mu_2}^{(s_1)\nu_1} = \left(K_{1\mu_2} K_2^{\nu_1} + s_1 K_{2\mu_2} K_1^{\nu_1} \right) \frac{1}{D_{12}}. \tag{4.24}$$

where the $s_1 = \pm$. The tensors that arise from the expression above are given by

$$t_{\mu_{12}}^{(+)} = 2 \frac{K_{1\mu_1} K_{1\mu_2}}{D_{12}} + q_{(\mu_1} K_{1\mu_2)} \frac{1}{D_{12}} \tag{4.25}$$

$$t_{\mu_{12}}^{(-)} = q_{[\mu_2} K_{1\mu_1]} \frac{1}{D_{12}}. \tag{4.26}$$

Nevertheless, anticipating a connection with higher dimensions, we opt to write the last term as a pseudo-scalar function

$$t^{SP} = -t^{PS} = \varepsilon_{\nu_1 \nu_2} t^{(-)\nu_1 \nu_2} = 2 \frac{\varepsilon_{\nu_1 \nu_2} K_1^{\nu_1} K_2^{\nu_2}}{D_{12}}$$

using $\varepsilon_{\nu_1 \nu_2} K_1^{\nu_1} K_2^{\nu_2} = \varepsilon_{\nu_1 \nu_2} p_{21}^{\nu_2} K_1^{\nu_1}$ and then the definition of the vector integral for equal masses, proportional to $p_{21}^{\nu_1} Z_1$, results in

$$T^{SP} = 2\varepsilon_{\nu_1 \nu_2} q^{\nu_2} J_2^{\nu_1} = 0.$$

Therefore, given both versions for the four-matrix trace, we have the corresponding versions for the AV amplitude

$$(t_{\mu_{12}}^{AV})_1 = -2\varepsilon_{\mu_1 \nu_1} t_{\mu_2}^{(+)\nu_1} - \varepsilon_{\mu_1 \mu_2} t^{PP} - 2\varepsilon_{\mu_2 \nu_1} t_{\mu_1}^{(-)\nu_1} + 2 g_{\mu_1 \mu_2} t^{SP}, \tag{4.27}$$

$$(t_{\mu_{12}}^{AV})_2 = -2\varepsilon_{\mu_2 \nu_1} t_{\mu_1}^{(+)\nu_1} - \varepsilon_{\mu_1 \mu_2} t^{SS} + 2\varepsilon_{\mu_1 \nu_1} t_{\mu_2}^{(-)\nu_1} - 2 g_{\mu_1 \mu_2} t^{SP}. \tag{4.28}$$

The same happens to the VA amplitude

$$(t^{VA}_{\mu_{12}})_1 = -2\varepsilon_{\mu_1\nu_1}t^{(+)\nu_1}_{\mu_2} + \varepsilon_{\mu_1\mu_2}t^{SS} - 2\varepsilon_{\mu_2\nu_1}t^{(-)\nu_1}_{\mu_1} + g_{\mu_1\mu_2}t^{SP} \tag{4.29}$$
$$(t^{VA}_{\mu_{12}})_2 = -2\varepsilon_{\mu_2\nu_1}t^{(+)\nu_1}_{\mu_1} + \varepsilon_{\mu_1\mu_2}t^{PP} + 2\varepsilon_{\mu_1\nu_1}t^{(-)\nu_1}_{\mu_2} - g_{\mu_1\mu_2}t^{SP}. \tag{4.30}$$

As mentioned at the beginning of the section, integrated amplitudes depend exclusively on the external momentum q. That precludes the construction of some 2nd-order tensors, which cancels out terms like $t^{(-)}$ and SP. Further examination of the general form (4.11) allows the identification of even amplitudes

$$t^{VV}_{\mu_1\mu_2} = (2t^{(+)}_{\mu_1\mu_2} + g_{\mu_1\mu_2}t^{PP}) \tag{4.31}$$
$$t^{AA}_{\mu_1\mu_2} = (2t^{(+)}_{\mu_1\mu_2} - g_{\mu_1\mu_2}t^{SS}). \tag{4.32}$$

Hence, the integration provides the relations among odd and even amplitudes

$$(T^{AV}_{\mu_{12}})_1 = -\varepsilon_{\mu_1}{}^{\nu_1}(T^{VV}_{\nu_1\mu_2}); \qquad (T^{AV}_{\mu_{12}})_2 = -\varepsilon_{\mu_2}{}^{\nu_1}(T^{AA}_{\mu_1\nu_1}); \tag{4.33}$$
$$(T^{VA}_{\mu_{12}})_1 = -\varepsilon_{\mu_1}{}^{\nu_1}(T^{AA}_{\nu_1\mu_2}); \qquad (T^{VA}_{\mu_{12}})_2 = -\varepsilon_{\mu_2}{}^{\nu_1}(T^{VV}_{\mu_1\nu_1}). \tag{4.34}$$

Although we did not detail, following the same steps produced both VA versions. These associations are directly achieved at the integrand level using (4.15), the identity $\gamma_*\gamma_{\mu_i} = -\varepsilon_{\mu_i}{}^{\nu_1}\gamma_{\nu_1}$ in the adequate position. We need a clear distinction among versions since their comparison is not automatic for integrated amplitudes due to their diverging character.

We also use the last identity $\gamma_*\gamma_{[\mu_1\mu_2]} = -\varepsilon_{\mu_1\mu_2}$ to introduce the third version for the discussed amplitudes. Replacing the form $\gamma_*\gamma_{\mu_1}\gamma_{\nu_1} = -\varepsilon_{\mu_1\nu_1} + g_{\mu_1\nu_1}\gamma_*$ in the traces produces the results

$$(t^{AV}_{\mu_{12}})_3 = -\frac{1}{2}[\varepsilon_{\mu_1}{}^{\nu_1}(t^{VV}_{\nu_1\mu_2}) + \varepsilon_{\mu_2}{}^{\nu_1}(t^{AA}_{\mu_1\nu_1})] - \varepsilon_{\mu_2\nu_1}t^{(-)\nu_1}_{\mu_1} + \varepsilon_{\mu_1\nu_1}t^{(-)\nu_1}_{\mu_2}, \tag{4.35}$$
$$(t^{VA}_{\mu_{12}})_3 = -\frac{1}{2}[\varepsilon_{\mu_1}{}^{\nu_1}(t^{AA}_{\nu_1\mu_2}) + \varepsilon_{\mu_2}{}^{\nu_1}(t^{VV}_{\mu_1\nu_1})] - \varepsilon_{\mu_2\nu_1}t^{(-)\nu_1}_{\mu_1} + \varepsilon_{\mu_1\nu_1}t^{(-)\nu_1}_{\mu_2}. \tag{4.36}$$

Since $t^{(-)\nu}_{\mu}$ tensors vanish after integration, different versions with each other as follows

$$(T^{AV}_{\mu_{12}})_3 = \frac{1}{2}[(T^{AV}_{\mu_{12}})_1 + (T^{AV}_{\mu_{12}})_2]; \quad (T^{VA}_{\mu_{12}})_3 = \frac{1}{2}[(T^{VA}_{\mu_{12}})_1 + (T^{VA}_{\mu_{12}})_2]. \tag{4.37}$$

This particular aspect receives further attention in the section (6). The investigation developed by the article [72] uses this version in equation (85). It illustrates how any possible expression follows from versions one and two.

Before proceeding, we need integrated expressions. Their obtainment occurs by replacing the results of appendix (C) in the integrated versions of structures (4.21),(4.22), and (4.24). The scalar two-point functions assume the forms

$$T^{PP} = q^2J_2 - 2I_{\log}, \tag{4.38}$$
$$T^{SS} = \left(4m^2 - p^2\right)J_2 + 2I_{\log}. \tag{4.39}$$

And the symmetric sign tensor is

$$T^{(+)}_{\mu_{12}} = 2(\bar{J}_{2\mu_{12}} + q_{\mu_1} J_{2\mu_2}) \tag{4.40}$$

$$= 2\theta_{\mu_{12}}(q)\left(m^2 J_2 + \frac{i}{4\pi}\right) - \frac{1}{2} g_{\mu_{12}} q^2 J_2 + (\Delta_{2\mu_{12}} + g_{\mu_{12}} I_{\log}), \tag{4.41}$$

where $\theta_{\alpha\lambda}(q) = (g_{\alpha\lambda} q^2 - q_\alpha q_\lambda)/q^2$ is the transversal projector. We put these pieces together to compound 2nd-order even tensors

$$T^{VV}_{\mu_1\mu_2} = 2\Delta_{2\mu_1\mu_2} + 4\theta_{\mu_1\mu_2}\left(m^2 J_2 + \frac{i}{4\pi}\right), \tag{4.42}$$

$$T^{AA}_{\mu_1\mu_2} = 2\Delta_{2\mu_1\mu_2} + 4\theta_{\mu_1\mu_2}\left(m^2 J_2 + \frac{i}{4\pi}\right) - g_{\mu_1\mu_2}\left(4m^2 J_2\right), \tag{4.43}$$

which lead to the versions for the AV amplitude

$$(T^{AV}_{\mu_{12}})_1 = -2\varepsilon_{\mu_1}{}^{\nu}\Delta_{2\mu_2\nu} - 4\varepsilon_{\mu_1\nu}\theta^{\nu}_{\mu_2}\left(m^2 J_2 + \frac{i}{4\pi}\right), \tag{4.44}$$

$$(T^{AV}_{\mu_{12}})_2 = -2\varepsilon_{\mu_2}{}^{\nu}\Delta_{2\mu_1\nu} - 4\varepsilon_{\mu_2\nu}\theta^{\nu}_{\mu_1}\left(m^2 J_2 + \frac{i}{4\pi}\right) - \varepsilon_{\mu_1\mu_2}\left(4m^2 J_2\right). \tag{4.45}$$

Two-point functions within axial RAGFs are finite and related through the expressions

$$T^{PV}_{\mu} = -T^{VP}_{\mu} = \varepsilon_{\mu\nu} q^{\nu}\left[-2m J_2(q)\right], \tag{4.46}$$

$$T^{PA}_{\mu} = -T^{AP}_{\mu} = -\varepsilon_{\mu\nu}(T^{PV})^{\nu}. \tag{4.47}$$

Whereas one-point functions are pure surface terms proportional to the routing k_i ,

$$T^{A}_{\mu}(i) = -\varepsilon_{\mu}{}^{\nu_1} T^{V}_{\nu_1}(i) = 2\varepsilon_{\mu}{}^{\nu_1} k_i^{\nu_2}\Delta_{2\nu_1\nu_2}. \tag{4.48}$$

Even though the integrands are equivalent, the same does not apply to integrated functions. In the case of even amplitudes (VV and AA), expressions depend on the prescription adopted for evaluating divergences. That also occurs for odd amplitudes (AV and VA), but they rely on the version for the trace. Using the chiral matrix definition around the first or the second vertexes brings implications for the index arrangement in finite and divergent parts. This perspective produced identities originally, but now the connection is not automatic. That becomes clear when we subtract the AV expressions

$$(T^{AV}_{\mu_{12}})_1 - (T^{AV}_{\mu_{12}})_2 = -2(\varepsilon_{\mu_1\nu}\Delta^{\nu}_{2\mu_2} - \varepsilon_{\mu_2\nu}\Delta^{\nu}_{2\mu_1}) \tag{4.49}$$

$$-4(\varepsilon_{\mu_1\nu_1}\theta^{\nu}_{\mu_2} - \varepsilon_{\mu_2\nu_1}\theta^{\nu}_{\mu_1})\left(m^2 J_2 + \frac{i}{4\pi}\right) + 4\varepsilon_{\mu_1\mu_2} m^2 J_2.$$

We use Schouten identities in 2D to rearrange indexes in the finite part and in surface terms. Through the antisymmetry of the Levi-Civita tensor, we have explicitly

$$\varepsilon_{\mu_1\nu}\Delta^{\nu}_{2\mu_2} + \varepsilon_{\mu_2\mu_1}\Delta^{\nu}_{2\nu} + \varepsilon_{\nu\mu_2}\Delta^{\nu}_{2\mu_1} = 0 = \varepsilon_{[\mu_1\nu}\Delta^{\nu}_{2\mu_2]}, \tag{4.50}$$

$$\varepsilon_{\mu_1\nu}\theta^{\nu}_{\mu_2} + \varepsilon_{\mu_2\mu_1}\theta^{\nu}_{\nu} + \varepsilon_{\nu\mu_2}\theta^{\nu}_{\mu_1} = 0 = \varepsilon_{[\mu_1\nu}\theta^{\nu}_{\mu_2]}. \tag{4.51}$$

So, the difference reduces to

$$(T^{AV}_{\mu_{12}})_1 - (T^{AV}_{\mu_{12}})_2 = -\varepsilon_{\mu_1\mu_2}\left(2\Delta^{\alpha}_{2\alpha} + \frac{i}{\pi}\right). \tag{4.52}$$

The integration linearity requires this difference to vanish identically, constraining the value of $\Delta^{\alpha}_{2\alpha}$. That represents a link between linearity and the uniqueness of perturbative solutions. Now, we analyze the role the surface terms play regarding the RAGFs.

4.1 Verification and Consequences of the RAGFs

We perform contractions with momentum for the integrated amplitudes to analyze the RAGFs, starting with even functions because they relate to the odd ones. These operations produce the difference between vector one-point functions (2.14), and that occurs identically. After contracting the integrated VV, finite parts cancel out due to $q^{\mu_2}\theta^{\nu}_{\mu_2} = 0$, and only a surface term remains. The comparison with the V function (4.48) leads directly to the expected relation

$$\begin{aligned} q^{\mu_1}T^{VV}_{\mu_{12}} &= 2q^{\nu_1}\Delta_{2\mu_2\nu_1} = [T^{V}_{\mu_2}(1) - T^{V}_{\mu_2}(2)] && (4.53)\\ q^{\mu_1}T^{AA}_{\mu_{12}} + 2mT^{PA}_{\mu_2} &= 2q^{\nu_1}\Delta_{2\mu_2\nu_1} = [T^{V}_{\mu_2}(1) - T^{V}_{\mu_2}(2)]. && (4.54)\end{aligned}$$

The same occurs with the AA. In this case, finite function PA and surface term appear.

Now, we turn our attention to relations for odd amplitudes (4.1)-(4.4). Taking first version of AV (4.44), the contraction with vector vertex yields

$$q^{\mu_2}(T^{AV}_{\mu_{12}})_1 = -2\varepsilon_{\mu_1\nu_1}q^{\nu_2}\Delta^{\nu_1}_{2\nu_2} = [T^{A}_{\mu_1}(1) - T^{A}_{\mu_1}(2)]. \tag{4.55}$$

Again, identifying the axial amplitude (4.48) is straightforward and does not require conditions. That differs from the axial contraction, which needs the rearranging of indexes,

$$q^{\mu_1}(T^{AV}_{\mu_{12}})_1 = -2q^{\mu_1}\varepsilon_{\mu_1\nu}\Delta^{\nu}_{2\mu_2} - 4q^{\mu_1}\varepsilon_{\mu_1\nu}\theta^{\nu}_{\mu_2}\left(m^2J_2 + \frac{i}{4\pi}\right). \tag{4.56}$$

After employing (4.50)-(4.51), reminding that $\theta^{\nu}_{\nu} = 1$, we have

$$q^{\mu_1}(T^{AV}_{\mu_{12}})_1 = [T^{A}_{\mu_2}(k_1) - T^{A}_{\mu_2}(k_2)] - 2mT^{PV}_{\mu_2} + \varepsilon_{\mu_2\nu_1}q^{\nu_1}\left(2\Delta^{\alpha}_{2\alpha} + \frac{i}{\pi}\right), \tag{4.57}$$

where PV has the form (4.46). The last term prevents automatic satisfaction of this relation, conditioning the value assumed by the surface term. This situation also occurs for the second version (4.45); however, the additional term is on the vector contraction

$$\begin{aligned} q^{\mu_2}(T^{AV}_{\mu_{12}})_2 &= [T^{A}_{\mu_1}(k_1) - T^{A}_{\mu_1}(k_2)] + \varepsilon_{\mu_1\nu}q^{\nu}\left(2\Delta^{\alpha}_{2\alpha} + \frac{i}{\pi}\right) && (4.58)\\ q^{\mu_1}(T^{AV}_{\mu_{12}})_2 &= [T^{A}_{\mu_2}(k_1) - T^{A}_{\mu_2}(k_2)] - 2mT^{PV}_{\mu_2}. && (4.59)\end{aligned}$$

This pattern repeats for the VA amplitude: additional terms arise in the same contractions

$$q^{\mu_1}(T^{VA}_{\mu_{12}})_1 = \varepsilon_{\mu_2\nu_1}q^{\nu_1}\left(2\Delta^{\alpha}_{2\alpha}+\frac{i}{\pi}\right)+T^{A}_{\mu_2}(1)-T^{A}_{\mu_2}(2) \tag{4.60}$$

$$q^{\mu_2}(T^{VA}_{\mu_{12}})_2 = \varepsilon_{\mu_1\nu_1}q^{\nu_1}\left(2\Delta^{\alpha}_{2\alpha}+\frac{i}{\pi}\right)+T^{A}_{\mu_1}(1)-T^{A}_{\mu_1}(2)+2mT^{VP}_{\mu_1}. \tag{4.61}$$

RAGFs, deduced as identities for integrands, represent integration linearity within this context. Even amplitudes automatically satisfy the relations since they do not depend on the surface term value. On the other hand, odd amplitudes require the condition[1]

$$\Delta^{\alpha}_{2\alpha} = -i\,(2\pi)^{-1}. \tag{4.62}$$

This term emerges for the contraction with the vertex that defines the amplitude version (the position of use of the chiral matrix definition). Besides, choosing this finite value for surface terms ensures that the AV's are equal (4.52), clarifying the relation between linearity and uniqueness. Any formula to the Dirac traces leads to one unique answer that respects the linearity of integration. Nevertheless, this condition sets non-zero values for one-point functions (4.48), affecting symmetry implications through WIs. That occurs for all relations in this subsection since amplitudes depend on the surface term. This subject receives attention in the sequence.

4.2 Ward Identities

In the model, we discussed the divergence of axial and vector currents (2.20)-(2.21), indicating implications through WIs for perturbative amplitudes. The adopted strategy translates these implications as restrictions over RAGFs, which link linearity and symmetries. This subsection analyses such connection with particular attention to the anomalous amplitudes, known for the impossibility of satisfying all WIs simultaneously.

Adopting a prescription that eliminates surface terms reduces all RAGFs for even amplitudes to the corresponding WIs. For odd amplitudes, this condition satisfies those WIs corresponding to automatic RAGFs while violating the others. Observe the first version of AV to clarify this statement. Identifying the relations was automatic to the vector RAGF; however, the axial RAGF gets an additional term. Hence, the zero value for the surface term satisfies the vector WI while violating the axial WI. We see the opposite for the second version, which breaks vector WI. Both identities are disregarded for the third version since it is a composition of the first two. See all the results in the Table 4.1. The same arguments are applied to the VA. Under this perspective, selecting an amplitude version would choose the vertex for symmetry violation. Furthermore, this value for surface terms breaks the integration linearity (in anomalous case).

[1]Since the third version is a combination, see (4.37), all vertices have potentially violated terms.

Table 4.1: Violations for vanishing surface term in each version.

$q^{\mu_1}(T^{AV}_{\mu_{12}})_1 = -2mT^{PV}_{\mu_2} + (i/\pi)\,\varepsilon_{\mu_2\nu_1}q^{\nu_1}$	$q^{\mu_2}(T^{AV}_{\mu_{12}})_1 = 0$
$q^{\mu_1}(T^{AV}_{\mu_{12}})_2 = -2mT^{PV}_{\mu_2}$	$q^{\mu_2}(T^{AV}_{\mu_{12}})_2 = (i/\pi)\,\varepsilon_{\mu_1\nu_1}q^{\nu_1}$
$q^{\mu_1}(T^{AV}_{\mu_{12}})_3 = -2mT^{PV}_{\mu_2} + (i/2\pi)\,\varepsilon_{\mu_2\nu_1}q^{\nu_1}$	$q^{\mu_2}(T^{AV}_{\mu_{12}})_3 = (i/2\pi)\,\varepsilon_{\mu_1\nu_1}q^{\nu_1}$
$q^{\mu_1}T^{VV}_{\mu_{12}} = 0$	$q^{\mu_2}T^{VV}_{\mu_{12}} = 0$
$q^{\mu_1}T^{AA}_{\mu_{12}} = -2mT^{PA}_{\mu_2}$	$q^{\mu_2}T^{AA}_{\mu_{12}} = 2mT^{AP}_{\mu_2}$

In contrast, by choosing the value that preserves linearity (4.62), different amplitude versions collapse into one unique form[2] (4.52). However, that violates all WIs for odd and even amplitudes since they depend on the value of the surface term; see Table 4.2.

Table 4.2: Violations for unique amplitudes

$q^{\mu_1}T^{AV}_{\mu_{12}} = -2mT^{PV}_{\mu_2} + (i/2\pi)\,\varepsilon_{\mu_2\nu}q^{\nu}$	$q^{\mu_2}T^{AV}_{\mu_{12}} = (i/2\pi)\,\varepsilon_{\mu_1\nu}q^{\nu}$
$q^{\mu_1}T^{VV}_{\mu_{12}} = -(i/2\pi)\,q_{\mu_2}$	$q^{\mu_2}T^{VV}_{\mu_{12}} = -(i/2\pi)\,q_{\mu_2}$
$q^{\mu_1}T^{AA}_{\mu_{12}} = -2mT^{PA}_{\mu_2} - (i/2\pi)\,q_{\mu_2}$	$q^{\mu_2}T^{AA}_{\mu_{12}} = 2mT^{AP}_{\mu_2} - (i/2\pi)\,q_{\mu_2}$

Low-energy properties of finite functions are fundamental to this analysis. Under the hypothesis that both WIs for the AV amplitude apply, we established the kinematical behavior in zero of Ω^{PV} as being zero (4.9). Nevertheless, employing the PV expression (4.46) and the limit (3.30), we have

$$\Omega^{PV}(0) = 4m^2 J_2\big|_0 = \frac{i}{\pi}m^2 Z_0^{(-1)}(0) = -\frac{i}{\pi}. \tag{4.63}$$

That means the hypothesis is false. Hence, when satisfying the vector WI, the axial WI violation is the value corresponding to the negative of $\Omega^{PV}(0)$. The other expectation (4.10) leads to the reciprocal: satisfying the axial WI implies violating the vector WI.

The scenario can be understood by noting a general 2nd-order odd tensor

$$F_{\mu_1\mu_2} = \varepsilon_{\mu_1\mu_2}F_1 + \varepsilon_{\mu_1\nu}q^{\nu}q_{\mu_2}F_2 + \varepsilon_{\mu_2\nu}q^{\nu}q_{\mu_1}F_3, \tag{4.64}$$

exhibits a feature when contracted with the momentum: we get two equations that are strict consequences of its tensor properties

$$q^{\mu_1}F_{\mu_1\mu_2} = \varepsilon_{\mu_2\nu}q^{\nu}V_1\left(q^2\right) = \varepsilon_{\mu_2\nu}q^{\nu}\left(q^2F_3 - F_1\right) \tag{4.65}$$

$$q^{\mu_2}F_{\mu_1\mu_2} = \varepsilon_{\mu_1\nu}q^{\nu}V_2\left(q^2\right) = \varepsilon_{\mu_1\nu}q^{\nu}\left(q^2F_2 + F_1\right). \tag{4.66}$$

If form factors are free of kinematic singularities observed in the explicit forms of the amplitudes, we have the implication at zero

$$V_1(0) + V_2(0) = 0. \tag{4.67}$$

[2] The version $(AV)_3$ happens to be independent of value of the surface term. Parametrizing $\Delta_{2\mu\nu} = ag_{\mu\nu}$ in its equation, we get an expression independent of coefficient a and equal to the unique form.

If one of the terms vanishes, the other must do so. Otherwise, if one of the $V_i\left(q^2\right)$ relates to a finite function (PV or VP), an additional constant must appear as compensation within the last equation. Nevertheless, these statements are inconsistent with the satisfaction of both WIs, which only occurs if linearity of integration holds with null surface terms. Thus, the low-energy behavior of these finite functions is the source of anomalous terms in amplitudes (AV-VA) and not their perturbative ambiguity.

But ambiguities relate to the low-energy implications. Under the condition of linearity and considering surface terms in the general tensor, this limit implies the constraint $2\Delta_{2\alpha}^{\alpha} = \Omega^{PV}(0)$. Such an aspect will be fully explored in the section considering odd triangles in the physical dimension. Conclusions similar to those drawn here anticipate the presence of anomalies and linearity breaking in this new circumstances. However, now we will explore the same two-dimensional scenario but consider a model where different species of massive fermions interact and what generalities we can obtain from this context.

Chapter 5

The AV of Two Distinct Masses

To show that the behavior of amplitudes is independent of masses, let us explore the universe where different species of massive fermions interact. At the end of this Chapter, we answer the question: Can amplitudes be obtained as consistent with their expected symmetry properties? The generalization of this work is published in the paper [41].

The n-point fermionic functions with different masses follow (2.9), where the mass indexes follow the momentum; In this scenario, the argument of the propagator i accounts for the routing and the mass running in the internal lines, viz., $S\left(i\right) \equiv S\left(K_i, m_i\right) = \left(\not{K}_i - m_i\right)^{-1}$. The expansion in terms of traces is given by

$$\begin{aligned} t^{\Gamma_1\Gamma_2} &= K_{12}^{\nu_{12}}\mathrm{tr}[\Gamma_1\gamma_{\nu_1}\Gamma_2\gamma_{\nu_2}]\frac{1}{D_{12}} + m_1 m_2\mathrm{tr}[\Gamma_1\Gamma_2]\frac{1}{D_{12}} \\ &\quad + m_2 K_1^{\nu_1}\mathrm{tr}[\Gamma_1\gamma_{\nu_1}\Gamma_2]\frac{1}{D_{12}} + m_1 K_2^{\nu_1}\mathrm{tr}[\Gamma_1\Gamma_2\gamma_{\nu_1}]\frac{1}{D_{12}}. \end{aligned} \tag{5.1}$$

The first relevant point concerns versions one and two as independent equations for odd amplitudes, just as for equal masses. The expressions established in (4.33) also apply,

$$(T^{AV}_{\mu_1\mu_2})_1 = -\varepsilon_{\mu_1}{}^{\nu_1} T^{VV}_{\nu_1\mu_2} \qquad (T^{AV}_{\mu_1\mu_2})_2 = -\varepsilon_{\mu_2}{}^{\nu_1} T^{AA}_{\mu_1\nu_1}. \tag{5.2}$$

That happens to two masses since the T^{SP} function and tensor $T^{(-)}_{\mu_2\mu_2}$ are identically zero. They are proportional to the vector integral $J_2^{\nu_1} = -i\left(4\pi\right)^{-1} q^{\nu_1} Z_1\left(q, m_1, m_2\right)$. Explicitly,

$$T^{(-)}_{\mu_{12}} = q_{[\mu_2} J_{2\mu_1]}\left(q, m_1, m_2\right) = 0 \tag{5.3}$$

$$T^{SP} = 2\varepsilon_{\nu_1\nu_2} q^{\nu_2} J_2^{\nu_1}\left(q, m_1, m_2\right) = 0. \tag{5.4}$$

Effectively amounts to the validity for different masses regarding the general expression obtainable through γ_* definition, as (4.27) and (4.28).

Expressions to 2nd-order tensors are written through scalar sub-amplitudes $\Gamma_1\Gamma_2 = SS$ and $\Gamma_1\Gamma_2 = PP$. To obtain these structures, we use the identity for the distinct fermions,

$$2K_2 \cdot K_1 = D_1 + D_2 + \left(m_1^2 + m_2^2 - q^2\right). \tag{5.5}$$

Employing (3.59) to one-point integrals[1], we have

$$T^{PP} = [+q^2 - (m_1 - m_2)^2]J_2 - \left[2I_{\log}(\lambda^2)\right] + \frac{i}{4\pi}\left[\log\left(m_1^2/\lambda^2\right) + \log\left(m_2^2/\lambda^2\right)\right] \quad (5.6)$$

$$T^{SS} = [-q^2 + (m_1 + m_2)^2]J_2 + \left[2I_{\log}(\lambda^2)\right] - \frac{i}{4\pi}\left[\log\left(m_1^2/\lambda^2\right) + \log\left(m_2^2/\lambda^2\right)\right] \quad (5.7)$$

From the equations above, a relation that connects the sub-amplitudes is

$$T^{PP} + T^{SS} = 4m_1 m_2 J_2.$$

While the tensorial part is compiled in the sign tensor (4.25),

$$T^{(+)}_{\mu_{12}} = 2\bar{J}_{2\mu_1\mu_2} + q_{(\mu_1}J_{2\mu_2)} = 2\bar{J}_{2\mu_1\mu_2} + 2q_{\mu_1}J_{2\mu_2}, \quad (5.8)$$

Evoking (3.66), we get the functional structure to equal masses,

$$2T^{(+)}_{\mu_{12}} = 4(J_{2\mu_1\mu_2} + q_{\mu_1}J_{2\mu_2}) + 2\Delta_{2\mu_{12}}(\lambda^2) + 2g_{\mu_{12}}I_{\log}(\lambda^2). \quad (5.9)$$

However, differences emerge in reducing the basic functions of two masses.

With these tools in hand, it is straightforward to express 2nd-order tensor amplitudes: The first one is the Double-Vector (VV), given by

$$\begin{aligned} T^{VV}_{\mu_1\mu_2} &= 2T^{(+)}_{\mu_1\mu_2} + g_{\mu_{12}}T^{PP} \quad (5.10)\\ &= 2\left[\Delta_{2\mu_{12}}\left(\lambda^2\right)\right] + 4(J_{2\mu_1\mu_2} + q_{\mu_2}J_{2\mu_1}) + g_{\mu_{12}}[q^2 - (m_1 - m_2)^2]J_2 \\ &\quad + \frac{i}{4\pi}g_{\mu_{12}}\left[\log\left(m_1^2/\lambda^2\right) + \log\left(m_2^2/\lambda^2\right)\right]. \end{aligned}$$

To show the elegance of the method, we also can write the amplitude in terms of $Z_n^{(-1)}$,

$$\begin{aligned} T^{VV}_{\mu_1\mu_2} &= 2[\Delta_{2\mu_{12}}\left(\lambda^2\right)] + \frac{i}{\pi}\theta_{\mu_{12}}[1 + m_1^2 Z_0^{(-1)} - \left(m_1^2 - m_2^2\right)Z_1^{(-1)}] \\ &\quad + \frac{i}{2\pi}g_{\mu_{12}}\left(m_1 - m_2\right)[\left(m_1 + m_2\right)Z_1^{(-1)} - m_1 Z_0^{(-1)}]. \end{aligned}$$

It used reductions for $Z_k^{(n)}$ that are complementary to using J-integrals. They occur when we perform contractions to investigate symmetry relations. The expression for the Double-Axial Green Function (AA) is

$$\begin{aligned} T^{AA}_{\mu_1\mu_2} &= T^{VV}_{\mu_1\mu_2} - g_{\mu_1\mu_2}\left(T^{SS} + T^{PP}\right) \quad (5.11)\\ &= +2\Delta_{2\mu_{12}} + 4(J_{2\mu_1\mu_2} + q_{\mu_2}J_{2\mu_1}) + g_{\mu_{12}}[q^2 - (m_1 + m_2)^2]J_2 \\ &\quad + \frac{i}{4\pi}g_{\mu_{12}}\left[\log\left(m_1^2/\lambda^2\right) + \log\left(m_2^2/\lambda^2\right)\right]. \end{aligned}$$

From even amplitudes can be to express the odd ones: the first version and the second version for distinct masses are

$$\begin{aligned} (T^{AV}_{\mu_1\mu_2})_1 &= -2\varepsilon_{\mu_1\nu_1}\Delta^{\nu_1}_{2\mu_2} - 4\varepsilon_{\mu_1\nu_1}(J^{\nu_1}_{2\mu_2} + q_{\mu_2}J_2^{\nu_1}) - \varepsilon_{\mu_1\mu_2}[q^2 - (m_1 - m_2)^2]J_2 \\ &\quad - \frac{i}{4\pi}\varepsilon_{\mu_1\mu_2}\left[\log\left(m_1^2/\lambda^2\right) + \log\left(m_2^2/\lambda^2\right)\right] \quad (5.12) \end{aligned}$$

[1] See D_1 and D_2 in the expression (5.5); when we substitute this identity, these terms always cancel one of the propagators, reducing the function from two to one-point.

$$(T^{AV}_{\mu_1\mu_2})_2 = -2\varepsilon_{\mu_2\nu_1}\Delta^{\nu_1}_{2\mu_1} - 4\varepsilon_{\mu_2\nu_1}(J^{\nu_1}_{2\mu_1} + q_{\mu_1}J^{\nu_1}_2) + \varepsilon_{\mu_1\mu_2}[q^2 - (m_1+m_2)^2]J_2 + \frac{i}{4\pi}\varepsilon_{\mu_1\mu_2}\left[\log\left(m_1^2/\lambda^2\right) + \log\left(m_2^2/\lambda^2\right)\right]. \quad (5.13)$$

One-index two-point amplitudes coming from RAGFs for odd amplitudes: Performing the traces and writing $K_2 = K_1 + q$ to get the integrand for $\Gamma_1\Gamma_2 = AS$ and $\Gamma_1\Gamma_2 = PV$. Thus, by our defintions, we get the finite amplitudes

$$T^{PV}_{\mu_2} = 2\varepsilon_{\mu_2\nu}\left[(m_2 - m_1)\, J^{\nu}_2 - m_1 q^{\nu} J_2\right] = -T^{VP}_{\mu_2} \quad (5.14)$$

$$T^{AS}_{\mu_1} = -2\varepsilon_{\mu_1\nu}\left[(m_1 + m_2)\, J^{\nu}_2 + m_1 q^{\nu} J_2\right] = T^{SA}_{\mu_1}. \quad (5.15)$$

The same procedure applies to the two amplitudes coming from RAGFs for even ones

$$T^{SV}_{\mu_2} = 2\left[(m_1 + m_2)\, J_{2\mu_2} + m_1 q_{\mu_2} J_2\right] = T^{VS}_{\mu_2} \quad (5.16)$$

$$T^{PA}_{\mu_2} = -2\left[(m_2 - m_1)\, J_{2\mu_2} - m_1 q_{\mu_2} J_2\right] = -T^{AP}_{\mu_2}. \quad (5.17)$$

A last point is the ubiquitous presence of the one-point differences; to them, we adopt one more notation to simplify the expressions. They are the same as the equal mass case because they are proportional to $\bar{J}_{1\mu}(k_1)$ that remain a pure surface-term

$$T^{A}_{(-)\mu_i} = T^{A}_{\mu_i}(k_1) - T^{A}_{\mu_i}(k_2) = -2\varepsilon_{\mu_i\nu_1}q^{\nu_2}\Delta^{\nu_1}_{2\nu_2} \quad (5.18)$$

$$T^{V}_{(-)\mu_i} = T^{V}_{\mu_i}(k_1) - T^{V}_{\mu_i}(k_2) = 2q^{\nu_1}\Delta_{2\nu\mu_i}. \quad (5.19)$$

Where we first time define the difference between axial one-point functions as $T^{A}_{(-)\mu_i} = T^{A}_{\mu_i}(k_1) - T^{A}_{\mu_i}(k_2)$. The other one-point function that appears is the scalar one

$$T^{S}(k_i) = 2m_i\bar{J}_1(k_i) = 2m_i I_{\log}\left(m_i^2\right) = 2m_i\left[I_{\log}\left(\lambda^2\right) - (i/4\pi)\log\left(m_i^2/\lambda^2\right)\right]. \quad (5.20)$$

Following this, we will study RAGFs to odd and even amplitudes and the effects over these relations due to two species of massive fermions in the currents; since the divergent of the vector current is connected to the scalar density, it is not strictly conserved now. Later, an expansion of the discussion of the low-energy theorem to the AV amplitude and its relation to WI and integration linearity is exposed.

5.1 Relations Among Green Functions

RAFGs will be used as fundamental mathematical tools to provide essential insights into the behavior of the amplitudes in question and how their properties relate.

Odd amplitudes: To explore the mechanism, take the definition

$$t^{AV}_{\mu_{12}} = \mathrm{tr}[\gamma_*\gamma_{\mu_1}S(1)\,\gamma_{\mu_2}S(2)] \quad (5.21)$$

and contract with q^{μ_2}. Next, is it possible to apply the identity

$$\not{q} = (\not{K}_2 - m_2) - (\not{K}_1 - m_1) + (m_2 - m_1). \quad (5.22)$$

We yield a relation between one- and two-point amplitudes

$$q^{\mu_2} t^{AV}_{\mu_{12}} = \mathrm{tr}[\gamma_* \gamma_{\mu_1} S(1)] - \mathrm{tr}[\gamma_* \gamma_{\mu_1} S(2)] + (m_2 - m_1)\, \mathrm{tr}[\gamma_* \gamma_{\mu_1} S(1)\, S(2)] \quad (5.23)$$
$$= t^{A}_{(-)\mu_1} + (m_2 - m_1)\, t^{AS}_{\mu_1}. \quad (5.24)$$

The procedure to obtain the vector contraction is similar, namely

$$q^{\mu_1} t^{AV}_{\mu_{12}} = t^{A}_{(-)\mu_2} - (m_1 + m_2)\, t^{PV}_{\mu_2}. \quad (5.25)$$

With further exploration, let us introduce the second contractions for amplitudes,

$$q^{\mu_2} q^{\mu_1} t^{AV}_{\mu_{12}} = q^{\mu_2} t^{A}_{(-)\mu_2} - (m_1 + m_2)\, t^{PP} \quad (5.26)$$
$$q^{\mu_1} q^{\mu_2} t^{AV}_{\mu_{12}} = q^{\mu_1} t^{A}_{(-)\mu_1} + (m_2 - m_1)\, t^{SS}. \quad (5.27)$$

In parallel to the equal mass scenario, we have RAGFs for even tensors. Regarding these RAGFs, we have two-point functions that are not present for equal masses since they are proportional to the mass difference,

$$q^{\mu_1} t^{VV}_{\mu_1\mu_2} = t^{V}_{(-)\mu_2} + (m_2 - m_1)\, t^{SV}_{\mu_2} \quad (5.28)$$
$$q^{\mu_2} t^{VV}_{\mu_1\mu_2} = t^{V}_{(-)\mu_1} + (m_2 - m_1)\, t^{VS}_{\mu_1}. \quad (5.29)$$

We have an additional term proportional to the contraction with SV for two contractions

$$q^{\mu_2} q^{\mu_1} t^{VV}_{\mu_1\mu_2} = q^{\mu_2} t^{V}_{(-)\mu_2} + (m_2 - m_1)\, q^{\mu_2} t^{SV}_{\mu_2}. \quad (5.30)$$

For the double-axial one, the simple and double contraction with the momentum obeys

$$q^{\mu_1} t^{AA}_{\mu_1\mu_2} = t^{V}_{(-)\mu_2} - (m_1 + m_2)\, t^{PA}_{\mu_2} \quad (5.31)$$
$$q^{\mu_2} t^{AA}_{\mu_1\mu_2} = t^{V}_{(-)\mu_1} + (m_2 + m_1)\, t^{AP}_{\mu_1}. \quad (5.32)$$

$$q^{\mu_2} q^{\mu_1} t^{AA}_{\mu_1\mu_2} = q^{\mu_2} t^{V}_{(-)\mu_2} - (m_1 + m_2)\, q^{\mu_2} t^{PA}_{\mu_2}. \quad (5.33)$$

RAGF Verification: The axial amplitudes exhibit a nontrivial behavior, as is expected, since equal masses are a particular case. Here, the vector and the axial currents are not conserved and are proportional to a difference and the sum of the masses,

$$\partial_\mu J^\mu = i\,(m_a - m_b)\, \bar{\psi}_a \psi_b \quad (5.34)$$
$$\partial_\mu J^\mu_* = -i\,(m_a + m_b)\, \bar{\psi}_a \psi_b. \quad (5.35)$$

So, in these amplitudes, we will focus our attention now.

Version one: Contracting the expression (5.12), terms proportional to the vector integral vanishes by the symmetry of indices $\varepsilon_{\nu_1\nu_2} q^{\nu_2} J^{\nu_1}_2 = 0$, so we have

$$q^{\mu_1} (T^{AV}_{\mu_1\mu_2})_1 = 2q^{\nu_2} \varepsilon_{\nu_1\nu_2} \Delta^{\nu_1}_{2\mu_2} + 4\varepsilon_{\nu_1\nu_2} q^{\nu_2} J^{\nu_1}_{2\mu_2} + \varepsilon_{\mu_2\nu} q^{\nu} [q^2 - (m_1 - m_2)^2] J_2 \quad (5.36)$$
$$+ (i/4\pi)\, \varepsilon_{\mu_2\nu} q^{\nu} [\log(m_1^2/\lambda^2) + \log(m_2^2/\lambda^2)].$$

We need to exchange the indices in $J_{2\mu_2}^{\nu_1}$ (in the first line) employing Schouten identity as

$$\varepsilon_{\nu_1\nu_2}q^{\nu_2}J_{2\mu_2}^{\nu_1}+\varepsilon_{\mu_2\nu_1}q^{\nu_2}J_{2\nu_2}^{\nu_1}+\varepsilon_{\nu_2\mu_2}q^{\nu_2}J_{2\nu_1}^{\nu_1}=0. \tag{5.37}$$

Two types of contractions arise from equations (3.67)-(3.68) introduced in Section (3.3),

$$\begin{aligned} 2q^{\nu_2}J_{2\nu_2}^{\nu_1} &= -\left(q^2+m_1^2-m_2^2\right)J_2^{\nu_1}-(i/4\pi)\,q^{\nu_1}\log\left(m_2^2/\lambda^2\right) \\ g_{\nu_{12}}J_2^{\nu_{12}} &= i/4\pi+m_1^2J_2-(i/4\pi)\log\left(m_2^2/\lambda^2\right). \end{aligned} \tag{5.38}$$

Using the results above, we lead to the expression:

$$\begin{aligned} q^{\mu_1}(T_{\mu_1\mu_2}^{AV})_1 &= 2q^{\nu_2}\varepsilon_{\nu_1\nu_2}\Delta_{2\mu_2}^{\nu_1}+\frac{i}{\pi}\varepsilon_{\mu_2\nu_1}q^{\nu_1} \\ &+\varepsilon_{\mu_2\nu_1}q^2(2J_2^{\nu_1}+q^{\nu_1}J_2)+(i/4\pi)\,\varepsilon_{\mu_2\nu}q^{\nu}\log\left(m_1^2/m_2^2\right) \\ &+\varepsilon_{\mu_2\nu_1}\{2(m_1^2-m_2^2)J_2^{\nu_1}+q^{\nu_1}[4m_1^2-(m_1-m_2)^2]J_2\}. \end{aligned} \tag{5.39}$$

The identity $\varepsilon_{[\nu_1\nu_2}\Delta_{2\mu_2]}^{\nu_1}=0$ allows adjusting indices and recognizing one-point functions together with relation for finite vectors and scalar two-point integrals of two masses

$$q^2\left(2J_2^{\nu}+q^{\nu}J_2\right)=-q^{\nu}\left(m_1^2-m_2^2\right)J_2-(i/4\pi)\,q^{\nu}\log\left(m_1^2/m_2^2\right). \tag{5.40}$$

Doing it some more algebraic operations, we produce the result for this contraction,

$$\begin{aligned} q^{\mu_1}(T_{\mu_1\mu_2}^{AV})_1 &= -2\varepsilon_{\mu_1\nu_1}q^{\nu_2}\Delta_{2\nu_2}^{\nu_1}+\varepsilon_{\mu_1\nu_2}q^{\nu_2}(2\Delta_{2\nu_1}^{\nu_1}+i/\pi) \\ &+2\,(m_1+m_2)\,\varepsilon_{\mu_2\nu_1}[(m_1-m_2)J_2^{\nu_1}+q^{\nu_1}m_1J_2]. \end{aligned} \tag{5.41}$$

Recalling the PV functions of two masses and one-point differences means

$$q^{\mu_1}(T_{\mu_1\mu_2}^{AV})_1=T_{(-)\mu_2}^{A}-(m_1+m_2)T_{\mu_2}^{PV}+\varepsilon_{\mu_2\nu_2}q^{\nu_2}(2\Delta_{2\nu_1}^{\nu_1}+i/\pi). \tag{5.42}$$

The contraction with the second vertex in the same version starts with

$$\begin{aligned} q^{\mu_2}(T_{\mu_1\mu_2}^{AV})_1 &= -2\varepsilon_{\mu_1\nu_1}q^{\nu_2}\Delta_{2\nu_2}^{\nu_1}-2\varepsilon_{\mu_1\nu_1}(2q^{\nu_2}J_{2\nu_2}^{\nu_1}+2q^2J_2^{\nu_1}) \\ &-\varepsilon_{\mu_1\nu_1}q^{\nu}[q^2-(m_1-m_2)^2]J_2 \\ &-(i/4\pi)\,\varepsilon_{\mu_1\mu_2}q^{\mu_2}\left[\log\left(m_1^2/\lambda^2\right)+\log\left(m_2^2/\lambda^2\right)\right]; \end{aligned} \tag{5.43}$$

here, the reductions occur directly, see $q^{\nu_2}J_{2\nu_2}^{\nu_1}$. Using (3.67), we get

$$q^{\mu_2}(T_{\mu_1\mu_2}^{AV})_1=-2\varepsilon_{\mu_1\nu_1}q^{\nu_2}\Delta_{2\nu_2}^{\nu_1}-2\varepsilon_{\mu_1\nu_1}\left(m_2-m_1\right)\left[\left(m_1+m_2\right)J_2^{\nu_1}+m_1q^{\nu_1}J_2\right], \tag{5.44}$$

where all the elements of the RAGF can be identified in the final result,

$$q^{\mu_2}(T_{\mu_1\mu_2}^{AV})_1=T_{(-)\mu_1}^{A}+\left(m_2-m_1\right)T_{\mu_1}^{AS}. \tag{5.45}$$

Note that RAGF is automatically satisfied and does not have an additional term as (5.42).

Version two: To the second one apply the same considerations: Starting with q^{μ_1},

$$q^{\mu_1}(T^{AV}_{\mu_1\mu_2})_2 = -2\varepsilon_{\mu_2\nu_1}q^{\mu_1}\Delta^{\nu_1}_{2\mu_1} - (i/4\pi)\,\varepsilon_{\mu_2\nu}q^{\nu}\left[\log\left(m_1^2/\lambda^2\right)+\log\left(m_2^2/\lambda^2\right)\right] -4\varepsilon_{\mu_2\nu_1}(q^{\mu_1}J^{\nu_1}_{2\mu_1}+q^2J^{\nu_1}_{2}) - \varepsilon_{\mu_2\nu}q^{\nu}[q^2-(m_1+m_2)^2]J_2. \quad (5.46)$$

Reducing the integrals in a direct way as $q^{\mu_1}J^{\nu_1}_{2\mu_1}$ and recognizing the terms follows

$$q^{\mu_1}(T^{AV}_{\mu_1\mu_2})_1 = T^{A}_{(-)\mu_2} - (m_1+m_2)\,T^{PV}_{\mu_2}. \quad (5.47)$$

The relation in the second vertex (vectorial) appears to have the same behavior as the equation (5.36). The terms can not be identified directly; see the equation below

$$q^{\mu_2}(T^{AV}_{\mu_1\mu_2})_2 = -2q^{\nu_2}\varepsilon_{\nu_1\nu_2}\Delta^{\nu_1}_{2\mu_1} + 4\varepsilon_{\nu_1\nu_2}(J^{\nu_1}_{2\mu_1}+q_{\mu_1}J^{\nu_1}_{2}) + \varepsilon_{\mu_1\nu}q^{\nu}[q^2-(m_1+m_2)^2]J_2 + (i/4\pi)\,\varepsilon_{\mu_1\nu}q^{\nu}\left[\log\left(m_1^2/\lambda^2\right)+\log\left(m_2^2/\lambda^2\right)\right] \quad (5.48)$$

Again, we have to switch the indices of place what will amount to the apperance of a conditioning factor in its RAGFs, namely,

$$q^{\mu_2}(T^{AV}_{\mu_1\mu_2})_2 = T^{A}_{(-)\mu_1} + (m_2-m_1)\,T^{AS}_{\mu_1} + \varepsilon_{\mu_1\nu_1}q^{\nu_1}(2\Delta^{\nu_2}_{2\nu_2}+i/\pi). \quad (5.49)$$

Equivalence: To be complete, we must evaluate the difference between the versions (5.12) and (5.13). Taking their full expression and subtracting one from another

$$(T^{AV}_{\mu_1\mu_2})_1 - (T^{AV}_{\mu_1\mu_2})_2 = 2[\varepsilon_{\mu_2\nu_1}\Delta^{\nu_1}_{2\mu_1} - \varepsilon_{\mu_1\nu_1}\Delta^{\nu_1}_{2\mu_2}] - 2\varepsilon_{\mu_1\mu_2}[q^2-\left(m_1^2+m_2^2\right)]J_2 +4[\varepsilon_{\mu_2\nu_1}(J^{\nu_1}_{2\mu_1}+q_{\mu_1}J^{\nu_1}_{2}) - \varepsilon_{\mu_1\nu_1}(J^{\nu_1}_{2\mu_2}+q_{\mu_2}J^{\nu_1}_{2})] -(i/2\pi)\varepsilon_{\mu_1\mu_2}[\log(m_1^2/\lambda^2)+\log(m_2^2/\lambda^2)]; \quad (5.50)$$

thereby employing the Schouten identity in the second line above, we have

$$4\varepsilon_{\mu_2\nu_1}(J^{\nu_1}_{2\mu_1}+q_{\mu_1}J^{\nu_1}_{2}) - 4\varepsilon_{\mu_1\nu_1}(J^{\nu_1}_{2\mu_2}+q_{\mu_2}J^{\nu_1}_{2}) = 4\varepsilon_{\mu_2\mu_1}(J^{\nu_1}_{2\nu_1}+q_{\nu_1}J^{\nu_1}_{2}).$$

With the help of reductions, it is relatively easy to show exactly

$$(T^{AV}_{\mu_1\mu_2})_1 - (T^{AV}_{\mu_1\mu_2})_2 = 2[\varepsilon_{\mu_2\nu_1}\Delta^{\nu_1}_{2\mu_1} - \varepsilon_{\mu_1\nu_1}\Delta^{\nu_1}_{2\mu_2}] + (i/\pi)\,\varepsilon_{\mu_2\mu_1}. \quad (5.51)$$

Apllying $\varepsilon_{[\mu_2\nu_1}\Delta^{\nu_1}_{2\mu_1]}=0$, this result naturally also may be expressed as

$$(T^{AV}_{\mu_1\mu_2})_1 - (T^{AV}_{\mu_1\mu_2})_2 = \varepsilon_{\mu_2\mu_1}(2\Delta^{\nu_1}_{2\nu_1}+i/\pi). \quad (5.52)$$

Another way to systematize the RAGFs that will be used in Chapter (7) is to notice that every time the index is contracted with the one remaining in the even amplitude, the relation is satisfied. Therefore we can use the above relation to exchange the versions when contracting with the index in the vertex used to define the version

$$q^{\mu_1}(T^{AV}_{\mu_1\mu_2})_1 = q^{\mu_1}(T^{AV}_{\mu_1\mu_2})_2 + \varepsilon_{\mu_2\mu_1}q^{\mu_1}\left(2\Delta^{\nu}_{2\nu}+i/\pi\right) = T^{A}_{(-)\mu_2} - 2\,(m_1+m_2)\,T^{PV}_{\mu_2} + \varepsilon_{\mu_2\nu}q^{\nu}\left(2\Delta^{\alpha}_{2\alpha}+i/\pi\right) \quad (5.53)$$

$$\begin{aligned} q^{\mu_2}(T^{AV}_{\mu_1\mu_2})_2 &= q^{\mu_2}(T^{AV}_{\mu_1\mu_2})_1 - \varepsilon_{\mu_2\mu_1}q^{\mu_2}(2\Delta^{\alpha}_{2\alpha} + i/\pi) \\ &= T^{A}_{(-)\mu_2} - 2(m_1 - m_2)T^{AS}_{\mu_2} + \varepsilon_{\mu_1\nu}q^{\nu}(2\Delta^{\alpha}_{2\alpha} + i/\pi). \end{aligned} \tag{5.54}$$

These features are notable in two dimensions. In four dimensions, we also establish relations among versions (three of them). However, in that scenario, the odd amplitudes do not collapse in a direct connection to even ones. We have to check the RAGFs explicitly.

Even Amplitudes: The relations to the even amplitudes are easy to check,

$$\begin{aligned} q^{\mu_1}T^{VV}_{\mu_1\mu_2} &= 2q^{\mu_1}\Delta_{2\mu_{12}} + (i/4\pi)\, q_{\mu_2}\left[\log\left(m_1^2/\lambda^2\right) + \log\left(m_2^2/\lambda^2\right)\right] \\ &\quad +4\left(q^{\nu_1}J_{2\mu_1\nu_1} + q_{\mu_2}q^{\nu_1}J_{2\nu_1}\right) + q_{\mu_2}[q^2 - (m_1 - m_2)^2]J_2. \end{aligned} \tag{5.55}$$

Using the same operations in J_2-integrals as applied to the odd amplitudes follows

$$\begin{aligned} q^{\mu_1}T^{VV}_{\mu_1\mu_2} &= T^{V}_{(-)\mu_2} - 2\,(m_1 - m_2)\left[(m_1 + m_2)\,J_{2\mu_2} + m_1 q_{\mu_2}J_2\right] \\ &= T^{V}_{(-)\mu_2} + (m_2 - m_1)\,T^{SV}_{\mu_2} \end{aligned} \tag{5.56}$$

$$q^{\mu_2}T^{VV}_{\mu_1\mu_2} = T^{V}_{(-)\mu_1} + (m_2 - m_1)\,T^{VS}_{\mu_1}. \tag{5.57}$$

For the AA-amplitude (5.11), the two relations follows by

$$\begin{aligned} q^{\mu_1}T^{AA}_{\mu_1\mu_2} &= T^{V}_{(-)\mu_2} - (m_1 + m_2)\,T^{PA}_{\mu_2} \\ q^{\mu_2}T^{AA}_{\mu_1\mu_2} &= T^{V}_{(-)\mu_2} + (m_2 + m_1)\,T^{AP}_{\mu_1}. \end{aligned} \tag{5.58}$$

See PA in (5.17); we could have expressed only in term of one since they differ by a sign.

The double-contraction for the even amplitudes (5.30) and (5.33) is associated with finite one-rank amplitudes. By themselves their relations are

$$q^{\mu_1}t^{VS}_{\mu_1} = +(m_2 - m_1)\,t^{SS} + [t^S(1) - t^S(2)] \tag{5.59}$$

$$q^{\mu_1}t^{AP}_{\mu_1} = -(m_2 + m_1)\,t^{PP} - [t^S(1) + t^S(2)]. \tag{5.60}$$

The LHS is finite, but the RHS shows a log-divergent object $I_{\log}$. Nonetheless, in our strategy, it is an exact and straightforward algebraic step to verify them. Using as an example the following equation

$$q^{\mu_1}T^{VS}_{\mu_1} = 2\,(m_1 + m_2)\,q^{\mu_1}J_{2\mu_1} + 2m_1q^2J_2. \tag{5.61}$$

Applying Eq. (3.69) in order to reduce the two-masses vector integral, we have

$$q^{\mu_1}T^{VS}_{\mu_1} = -(m_2 - m_1)\,[q^2 - (m_1 + m_2)^2]J_2 + (i/4\pi)\,(m_1 + m_2)\log\left(m_2^2/m_1^2\right). \tag{5.62}$$

The last term can be manipulated by the scale relation (3.62), viz.,

$$(i/4\pi)\log\left(m_2^2/m_1^2\right) = I_{\log}\left(m_1^2\right) - I_{\log}\left(m_2^2\right), \tag{5.63}$$

which through an organization of the terms produces the following expression

$$q^{\mu_1}T^{VS}_{\mu_1} = (m_2-m_1)\{[-q^2+(m_1+m_2)^2]J_2+\left[I_{\log}(m_1^2)+I_{\log}(m_2^2)\right]\} \\ +2m_1 I_{\log}(m_1^2)-2m_2 I_{\log}\left(m_2^2\right). \quad (5.64)$$

We can rewrite the first term as the SS amplitude (5.7) and organize the result

$$q^{\mu_1}T^{VS}_{\mu_1} = (m_2-m_1)T^{SS} \\ +2m_1[I_{\log}(\lambda^2)-(i/4\pi)\log\left(m_1^2/\lambda^2\right)] \\ -2m_2[I_{\log}(\lambda^2)-(i/4\pi)\log\left(m_2^2/\lambda^2\right)]. \quad (5.65)$$

The scalar one-point function is given in (5.20). Hence we verify that the two last lines correspond to the difference between them, representing the satisfaction of its RAGF,

$$q^{\mu_1}T^{VS}_{\mu_1} = (m_2-m_1)T^{SS}+T^S(1)-T^S(2). \quad (5.66)$$

Note that in these case, the difference between scalar one-point functions does not cancel and depends on the individual masses.

The $q^\mu T^{AP}_\mu$ works under the same manipulations used in VS, starting with

$$q^{\mu_1}T^{AP}_{\mu_1} = 2(m_2-m_1)q^{\mu_1}J_{2\mu_1}-2m_1 q^2 J_2. \quad (5.67)$$

Through of the relation estabilish in (3.69), the equation above results in

$$q^{\mu_1}T^{AP}_{\mu_1} = (m_1+m_2)[(m_1-m_2)^2-q^2]J_2+(i/4\pi)(m_2-m_1)\log\left(m_2^2/m_1^2\right). \quad (5.68)$$

Rewriten the first term by (5.6) and organize the result

$$q^{\mu_1}T^{AP}_{\mu_1} = -(m_1+m_2)T^{PP} \quad (5.69)$$
$$-2m_1[I_{\log}(\lambda^2)-(i/4\pi)\log\left(m_1^2/\lambda^2\right)] \quad (5.70)$$
$$-2m_2[I_{\log}(\lambda^2)-(i/4\pi)\log\left(m_2^2/\lambda^2\right)].$$

The two last lines now appear as the sum of scalar one-point functions, namely

$$q^{\mu_1}T^{AP}_{\mu_1} = -(m_1+m_2)T^{PP}-[T^S(1)+T^S(2)]. \quad (5.71)$$

For equal masses, the term to one-point functions is proportional to the masses' sum.

As explored in the chapter for equal masses, it is possible to obtain properties for the amplitudes by combining their general tensor structures with their symmetry relations or Ward's identities. These results are not restricted to perturbative solutions and should remain valid even for exact solutions. The VS function is constructed from a vector with an external vector, $T^{VS}_\mu = q_\mu F_1(q^2)$, where $F_1(q^2)$ is an invariant function. This form allows us to state a low-energy limit for this amplitude contracting the equation, viz, $q^\mu T^{VS}_\mu = q^2 F_1(q^2)$. Then, $q^\mu T^{VS}_\mu\big|_{q^2=0} = 0$, since $F_1(q^2)$ does not poles at $q^2=0$.

In this way, to obtain an interpretation relation from RHS of relations (5.66) and (5.71), let us analyze the SV and AP-amplitudes in the limit in kinematical point. We have that $\lim_{q^2\to 0}(q^{\mu_1}T^{VS}_{\mu_1}) = 0$ is satisfied, since the $J_2 = i/4\pi Z_0^{(-1)}$ where the function $Z_0^{(-1)}$ in this point is given by

$$Z_0^{(-1)}(0, m_2, m_1) = \frac{1}{(m_1^2 - m_2^2)} \log \frac{m_2^2}{m_1^2}. \tag{5.72}$$

From Eq (5.66) and the explicit result (5.7), follows

$$\begin{aligned} (m_2 - m_1)T^{SS}\Big|_{q^2=0} &= 2(m_2 - m_1)I_{\log}(\lambda^2) + (i/2\pi)\left[m_1 \log(m_1^2/\lambda^2) - m_2 \log(m_2^2/\lambda^2)\right] && (5.73) \\ &= -\left[T^S(1) - T^S(2)\right], && (5.74) \end{aligned}$$

therefore $q^{\mu_1}T^{VS}_{\mu_1}\Big|_{q^2=0} = 0$. The low-energy theorem for $T^{AP}_{\mu_1}$ is also fulfilled because the same operations leads us to

$$(m_1 + m_2)\, T^{PP}\Big|_{q^2=0} = -\left[T^S(1) + T^S(2)\right]. \tag{5.75}$$

We saw that the one-point functions were indispensable for satisfying the deduced kinematical implication based on the tensor structure for amplitude with one Lorentz index. That is the opposite of the situation for amplitudes with two indices. The reason for the need for scalar one-point functions can be understood by analyzing the canonical structure of WIs for multiple masses. There, the meaning of these terms finds a justification.

5.2 Ward Identities: Two Masses

Here we will argue why the scalar one-point functions are part of WIs from one-index two-point functions. We take free fields that generate our amplitudes, of particular interest to our purposes, obeying the equal-time anticommutation relation

$$\left\{\psi_i^\alpha(y), \psi_j^{\dagger\kappa}(x)\right\} = \delta_{ij}\delta^{\alpha\kappa}\delta(\mathbf{x} - \mathbf{y}), \tag{5.76}$$

where i and j refer to different species of fermions (ψ_1 and ψ_2), all other anticommutators are null. Fermionic densities, defined as a set of bilinear in the fermions, are

$$J^{\Gamma_i} = \bar{\psi}_2\Gamma_i\psi_1, \quad \text{and} \quad J^{\Gamma_i\dagger} = \bar{\psi}_1(\gamma_0\Gamma_i^\dagger\gamma_0)\psi_2, \tag{5.77}$$

where Γ_i belong to set of the vertices given by (2.2). Explicitly we have

$$V^\mu = \left(\bar{\psi}_2\gamma^\mu\psi_1\right), \quad A_*^\mu = \left(\bar{\psi}_2\gamma_*\gamma^\mu\psi_1\right), \quad S = \left(\bar{\psi}_2\psi_1\right), \quad P = \left(\bar{\psi}_2\gamma_*\psi_1\right).$$

The adjoints yield the same matrices $\gamma_0\Gamma_i^\dagger\gamma_0 = \Gamma_i$ with the exception of pseudo-scalar one $\gamma_0\gamma_*\gamma_0 = -\gamma_*$. We adopted a different notation here to avoid confusion with J-integrals. Two-point functions can be seen in position space as

$$T^{\Gamma_1\Gamma_2}(x - y) = \mathrm{tr}\left[\Gamma_1 S_F(x - y, m_1)\,\Gamma_2 S_F(y - x, m_2)\right] = -\left\langle J^{\Gamma_1}(x)\, J^{\Gamma_2\dagger}(y)\right\rangle. \tag{5.78}$$

The minus sign occurs because Wick contraction yields i times our propagator definition, and $\langle\cdot\rangle = \langle 0\,|\mathcal{T}\,\{\cdot\}|\,0\rangle$ is an abbreviation for a time-ordered product. We recovered the letter for the Feynman propagator to not mistake it for scalar density.

To clarify the WIs for two-point functions with one-index, we use Dirac equations,

$$\gamma^{\mu}\partial_{\mu}\psi_1 = -im_1\psi_1; \quad \left(\partial_{\mu}\bar{\psi}_2\right)\gamma^{\mu} = im_2\bar{\psi}_2. \tag{5.79}$$

Through them, we obtain that the vector and axial currents satisfy

$$\partial_{\mu}V^{\mu} = +i\left(m_2 - m_1\right)S\left(x\right) = i\left(m_2 - m_1\right)\bar{\psi}_2\psi_1 \tag{5.80}$$

$$\partial_{\mu}A^{\mu} = -i\left(m_2 + m_1\right)P\left(x\right) = -i\left(m_2 + m_1\right)\bar{\psi}_2\gamma_*\psi. \tag{5.81}$$

The next step is to notice that when we perform space-time derivatives in the time ordering for densities carrying Lorentz indices, equal-time commutators will appear; to them, we will use the identity

$$[AB, CD] = -AC\left\{B, D\right\} + A\left\{B, C\right\}D - C\left\{D, A\right\}B + \left\{C, A\right\}DB. \tag{5.82}$$

Necessary formal commutators arise to time components, but in general, we will have

$$\begin{aligned}\left[J^{\Gamma_1}\left(x\right), J^{\Gamma_2\dagger}\left(y\right)\right]_{x^0=y^0} &= \left[\bar{\psi}_2\left(x\right)\Gamma^1\psi_1\left(x\right), \bar{\psi}_1\left(y\right)\Gamma^2\psi_2\left(y\right)\right] \qquad (5.83)\\ &= \left[\bar{\psi}_2\left(x\right)\Gamma^1\gamma^0\Gamma^2\psi_2\left(y\right) - \bar{\psi}_1\left(y\right)\Gamma^2\gamma^0\Gamma^1\psi_1\left(x\right)\right]\delta^2\left(x-y\right).\end{aligned}$$

The commutators necessary to point out the differences between symmetry relations of two and one indices two-point functions (satisfied for VS and AP amplitudes) are

$$\left[V_0\left(x\right), V^{\nu\dagger}\left(y\right)\right] = \left[\bar{\psi}_2\left(x\right)\gamma^{\nu}\psi_2\left(y\right) - \bar{\psi}_1\left(y\right)\gamma^{\nu}\psi_1\left(x\right)\right]\delta^2\left(x-y\right) \tag{5.84}$$

$$\left[A_0\left(x\right), V^{\nu\dagger}\left(y\right)\right] = \left[\bar{\psi}_2\left(x\right)\gamma_*\gamma^{\nu}\psi_2\left(y\right) - \bar{\psi}_1\left(y\right)\gamma_*\gamma^{\nu}\psi_1\left(x\right)\right]\delta^2\left(x-y\right) \tag{5.85}$$

$$\left[V_0\left(x\right), S^{\dagger}\left(y\right)\right] = \left[\bar{\psi}_2\left(x\right)\psi_2\left(y\right) - \bar{\psi}_1\left(y\right)\psi_1\left(x\right)\right]\delta^2\left(x-y\right) \tag{5.86}$$

$$\left[A_0\left(x\right), P^{\dagger}\left(y\right)\right] = \left[-\bar{\psi}_2\left(x\right)\psi_2\left(y\right) - \bar{\psi}_1\left(y\right)\psi_1\left(x\right)\right]\delta^2\left(x-y\right), \tag{5.87}$$

all evaluated in $x_0 = y_0$. Observe that densities in LHS carry two distinct masses, and the RHS bilinears appear with only one mass, though the two terms carry a distinct mass.

Taking the derivative of VV, using the motion's equation to the currents, and observing the commutator at equal times (5.84), we get the formal result

$$\begin{aligned}\partial_x^{\mu}\left\langle V_{\mu}\left(x\right)V_{\nu}^{\dagger}\left(y\right)\right\rangle &= i\left(m_2 - m_1\right)\left\langle S\left(x\right)V_{\nu}^{\dagger}\left(y\right)\right\rangle \qquad (5.88)\\ &\quad + \left[\left\langle\bar{\psi}_2\left(x\right)\gamma_{\nu}\psi_2\left(y\right)\right\rangle - \left\langle\bar{\psi}_1\left(y\right)\gamma_{\nu}\psi_1\left(x\right)\right\rangle\right]\delta^2\left(x-y\right),\end{aligned}$$

where $\partial_x^{\mu} = \partial/\partial x_{\mu}$. The Ward identity for equal masses came from cancellation in the last line since the terms become equal, and we are ignoring Schwinger's terms. As for two masses, it arises from Lorentz symmetry that implies the vanishing of one-point vector function individually, e.g., $\left\langle 0\left|\bar{\psi}_1(x)\gamma^\nu\psi_1(y)\right|0\right\rangle = 0$. It is understood by using the generator of translations in a vector operator $O^\mu(x)$,

$$\langle 0\left|O^\mu(x)\right|0\rangle = \left\langle 0\left|e^{-iP\cdot x}O^\mu(0)\,e^{iP\cdot x}\right|0\right\rangle = \langle 0\left|O^\mu(0)\right|0\rangle = 0. \tag{5.89}$$

Furthermore, because o Lorentz symmetry, such a constant vector must vanish. Note that this constraint may not be valid perturbatively. Putting aside that, the proposed WI is

$$\partial_x^\mu \left\langle V_\mu(x)\,V_\nu^\dagger(y)\right\rangle = i\,(m_2 - m_1)\left\langle S(x)\,V_\nu^\dagger(y)\right\rangle. \tag{5.90}$$

In it, only the contribution of motion's equations plays a part; additionally, if the correlator involves one axial and one vector current, the argument for vanishing the one-point amplitudes in (5.85) is the same.

The situation is quite different for VS and AP functions; symmetry constraints pass

$$\begin{aligned}\partial_\mu^x \left\langle V^\mu(x)\,S^\dagger(y)\right\rangle &= i\,(m_2 - m_1)\left\langle S(x)\,S^\dagger(y)\right\rangle \\ &\quad + \left\langle \bar{\psi}_2(x)\,\psi_2(y) - \bar{\psi}_1(y)\,\psi_1(x)\right\rangle \delta^2(x-y),\end{aligned} \tag{5.91}$$

where the commutator $\left[V_0(x), S^\dagger(y)\right] =$ (5.86) generates one-point scalar functions that formally cancel each other for equal masses, but in that case, the VS-amplitude is null. Nonetheless, in AP (or PA), they appear in a non-canceling way

$$\begin{aligned}\partial_\mu^x \left\langle A^\mu(x)\,P^\dagger(y)\right\rangle &= -i\,(m_1 + m_2)\left\langle P(x)\,P^\dagger(y)\right\rangle \\ &\quad - \left\langle \bar{\psi}_2(x)\,\psi_2(y) + \bar{\psi}_1(y)\,\psi_1(x)\right\rangle \delta^2(x-y).\end{aligned} \tag{5.92}$$

The commutator yields a sum, not a cancellation, for equal masses. So the canonical commutator terms appear and may not be zero due to other symmetry arguments.

As in the two masses scenario, the scalar one-point functions are not removed from expression to Ward identities and are an integral part of them. For one species of fermions, the commutator of vector (and axial) densities being zero is a particular phenomenon; this term comes from canonical algebra. Their eliminations are to be accounted for by additional arguments, e.g., Lorentz invariance. Such statements are not present against scalar densities that, in turn, guarantee a low-energy theorem to the VS and AP amplitudes.

To visualize consequences of this reasoning line and connect it with calculated expression, let us remind that Wick contractions yield i times our definition of the propagator,

$$\left\langle 0\left|T\psi^\alpha(x)\,\bar{\psi}^\kappa(y)\right|0\right\rangle = iS_F^{\alpha\kappa}(x-y, m_i). \tag{5.93}$$

Therefore, Fourier transforming the two-point functions (5.78),

$$T^{\Gamma_1\Gamma_2}(q) = \int \mathrm{d}^2 z e^{-iq\cdot z}\left[T^{\Gamma_1\Gamma_2}(z)\right] = -\int \mathrm{d}^2 z e^{-iq\cdot z}\left\langle J^{\Gamma_1}(x)\,J^{\Gamma_2\dagger}(y)\right\rangle \tag{5.94}$$

$$= \int \frac{\mathrm{d}^2 k}{(2\pi)^2}\mathrm{tr}\left[\Gamma_1 S_F(k+k_1, m_1)\,\Gamma_2 S_F(k+k_2, m_2)\right], \tag{5.95}$$

where $z = x - y$ and $k_2 - k_1 = q$. In the case of double-vector VV (5.10) and VS, we may write the motion's equation, and the commutation relations furnish the formal equations

$$\begin{aligned}\partial_z^\mu T_{\mu\nu}^{VV}(z) &= i(m_2 - m_1) T_\nu^{SV}(z) \\ &\quad + i\text{tr}[\gamma_\nu S_F(z, m_1)]\,\delta^2(z) - i\text{tr}[\gamma_\nu S_F(-z, m_2)]\,\delta^2(z), \end{aligned} \tag{5.96}$$

$$\begin{aligned}\partial_z^\mu T_\mu^{VS}(z) &= i(m_2 - m_1) T^{SS}(z) \\ &\quad + i\text{tr}[S_F(z, m_1)]\,\delta^2(z) - i\text{tr}[S_F(-z, m_2)]\,\delta^2(z), \end{aligned} \tag{5.97}$$

whose Fourier transform returns an expression where we do not neglect any term,

$$\begin{aligned} q^\mu T_{\mu\nu}^{VV}(k_1, k_2) &= (m_2 - m_1) T_\nu^{SV} \\ &\quad + \int \frac{\mathrm{d}^2 k}{(2\pi)^2} \{\text{tr}[\gamma^\nu S_F(K_1, m_1)] - \text{tr}[\gamma^\nu S_F(K_2, m_2)]\} \end{aligned} \tag{5.98}$$

$$\begin{aligned} q^\mu T_\mu^{VS}(k_1, k_2) &= (m_2 - m_1) T^{SS} \\ &\quad + \int \frac{\mathrm{d}^2 k}{(2\pi)^2} \{\text{tr}[S_F(K_1, m_1)] - \text{tr}[S_F(K_2, m_2)]\}. \end{aligned} \tag{5.99}$$

Recapitulating the facts, the parts from the time component of the commutator of currents with vector and axial currents formally cancel for one species of massive fermions. We got a WI whose contribution comes only from motion equations. On the other hand, for two masses, formal Lorentz invariance requires the vector and axial one-point functions to vanish as well, and thus they are not part of the WI. Indeed using our strategy, we saw in momentum space that they become pure surface-term that can be made zero. Additionally, the anomalies of the odd amplitudes are related to the impossibility of the formal/canonical WI being realized, which we establish as a consequence of a Low energy implication from a finite function; see the next section where that point is discussed and the relation with the linearity of integration.

In contrast, the commutator of the time component of the currents with scalar densities, or pseudo-scalar ones, giving rise to scalar one-point functions, besides the term coming from the motion's equations, is not necessarily zero. The point is that when the masses are equal, that difference of amplitudes vanishes in pairs for SV and sum for AP. They do not cancel in any situation for distinct masses and can not be zero because they are not a constant function of their mass parameters.

One way to see the difference between the two situations is to take into account that for even dimension, there is a matrix such that $C\gamma_\mu C^{-1} = -\gamma_\mu^T$, the charge conjugation matrix. This matrix implies a behavior to the vertexes, viz.,

$$C\left[1, \gamma_*, \gamma_\mu, \gamma_*\gamma_\mu\right] C^{-1} = [1, -\gamma_*^T, -\gamma_\mu^T, -\left(\gamma_*\gamma_\mu\right)^T]. \tag{5.100}$$

It is direct to see that the propagator obeys $CS_F(K_i, m_i)C^{-1} = S_F^T(-K_i, m_i)$. Applying it to the definition of one-point function, we have

$$t^{\Gamma_1} = \text{tr}[\Gamma_1 S_F(K_i, m_i)] = \text{tr}[C\Gamma_1 C^{-1} C S_F(K_i, m_i) C^{-1}]. \tag{5.101}$$

Using the trace properties and as well the relation for matrices

$$\mathrm{tr}\left(B_1^T \cdots B_n^T\right) = \mathrm{tr}\left(B_n \cdots B_1\right)^T = \mathrm{tr}\left(B_n \cdots B_1\right), \tag{5.102}$$

we may write from general considerations established above

$$t^{\Gamma_1} = \mathrm{tr}[\left(C\Gamma_1 C^{-1}\right)^T S_F\left(-K_i, m_i\right)].$$

At this point, note that there is a sign change to the pseudo-scalar, vector, and axial vertices. Then integrating the result above, we have

$$T^{\Gamma_1} = -\int \frac{\mathrm{d}^2 k}{(2\pi)^2} \mathrm{tr}[\Gamma_1 S_F\left(-k - k_i, m_i\right)]. \tag{5.103}$$

Reflecting on the integration variable and shifting, as the hypothesis, we get

$$T^{\Gamma_1}\left(k_i\right) = -\int \frac{\mathrm{d}^2 k}{(2\pi)^2} \mathrm{tr}[\Gamma_1 S_F\left(K_i, m_i\right)] = -T^{\Gamma_1}\left(k_i\right). \tag{5.104}$$

That implies that axial and vector one-point functions must vanish identically, as $T^P = 0$ already in the trace level. As trivial as it may appear, this is not a direct consequence of Feynman's rules; the possibility of shifting is coded in the intrinsic surface term present in the amplitudes, which is why the T^V, T^A are only surface terms. Nevertheless, it does not mean these parts in the amplitudes could not be non-zero and violate WIs.

For instance, the scalar function may have surface terms in 4D, but it is not obliged to be identically zero by translational invariance. In that case, the above equation picks up a positive sign. Those $T^S\left(k_i, m_i\right)$ amplitudes show a masse dependence through a logarithm. Since they are proportional to the basic divergent object, taking its derivative,

$$\frac{\partial I_{\log}\left(m_i^2\right)}{\partial m_i^2} = -\frac{i}{4\pi}\frac{1}{m_i^2}. \tag{5.105}$$

The integration picks up an arbitrary constant $I_{\log}\left(m_i^2\right) = (i4\pi)^{-1}\log\left(m_i^2/\lambda_0^2\right)$ that could help with cancellations; however, in combinations, this is not possible, see

$$I_{\log}(m_i^2) - I_{\log}(m_j^2) = -\frac{i}{4\pi}\log\frac{m_i^2}{m_j^2}. \tag{5.106}$$

However, the scalar-one cancels each other for equal masses when they arise from a commutator of vector currents. When the masses are unequal, there is no reason for them to disappear in the perturbative expression. They are integral parts of WI and necessary for their consistency. The low-energy theorem derived for them requires that part to occur

$$q^\mu T_\mu^{VS} = q^2 F\left(q^2\right) = 0. \tag{5.107}$$

Next, in addition to the paper [41], we will have to present the construction of a low-energy theorem, ultimately responsible for violations associated with the chiral anomaly in the odd amplitude where the vector current as the axial are not classically conserved.

5.3 Low-Energy Theorem and RAGFs

As observed, WIs to AV-versions can not both simultaneously hold. Firstly, vanishing the surface term eliminates the one-point functions; however, it implies linearity breaking, and an additional constant can not get rid of by any other choice. On the other hand, if the non-zero value corresponding to the maintenance of RAGFs (linearity) is chosen, axial one-point functions violate WIs in any case. In the scenario where the surface term could be arbitrary through some device or interpretation, the violation does not give up. To understand this state of affairs, we have resorted to an explanation only utilizing properties that are immune to choices and do not privilege one symmetry over another: *the kinematical behavior of PV function.*

We return to the last claims of the Chapter (4), assuming the general tensor for odd amplitudes (4.5). In 2D, the amplitude has Feynman integrals of power counting zero, one of which is a tensor integral. These types of integrals, in any dimension, indeed own surface terms, notwithstanding the coefficient of them only depending on the difference of routings; they are intrinsic to Feynman diagrams, not only when the power counting is linear. These features must be considered when stating general theorems about kinematical properties and their relations to the symmetry content of amplitudes coming from Feynman's rules. In 4D, we will have a more complex scenario: the surface terms appear with ambiguous combinations of routing sums, see Sections (6.2) and (6.3).

Only external momenta imply that preserving divergent content intact follows an expression to general tensor structure that accounts for the presence of surface terms because, in the last instance, they contribute a coefficient proportional to the metric,

$$F_{\mu_1\mu_2} = \varepsilon_{\mu_1\mu_2}F_1 + \varepsilon_{\mu_1\nu}q^{\nu}q_{\mu_2}F_2 + \varepsilon_{\mu_2\nu}q^{\nu}q_{\mu_1}F_3. \tag{5.108}$$

The path often trailed to study symmetry violations is to perform contractions and use some symmetry constraints to derive implications over others. Nonetheless, we shall derive a device that prescinds from the choice of some, *a priori*, selected symmetry. Performing contractions and identifying two invariant functions constructed with form factors F_i, viz.,

$$q^{\mu_1}F_{\mu_1\mu_2} =: \varepsilon_{\mu_2\nu}q^{\nu}V_1\left(q^2\right) \tag{5.109}$$

$$q^{\mu_2}F_{\mu_1\mu_2} =: \varepsilon_{\mu_1\nu}q^{\nu}V_2\left(q^2\right). \tag{5.110}$$

We got two equations that are strict and intrinsic consequences of tensor properties. If we sum them, F_1 drops, and an independent equation emerges

$$V_1\left(q^2\right) + V_2\left(q^2\right) = q^2\left(F_3 + F_2\right). \tag{5.111}$$

For F_2 and F_3 sufficiently regular in the point $q^2 = 0$ this equation becomes

$$V_1(0) + V_2(0) = 0. \tag{5.112}$$

From it, being aware of its generality, we establish some computational-free conclusions. First, suppose the general tensor is chosen to correspond with the axial-vector amplitude and function of two masses, i.e., $F_{\mu\nu} = T^{AV}_{\mu\nu}$. In that case, we may inquire about expected amplitudes related to the hypothesis of WIs.

The systematization of 2pt, 1st-rank amplitude arising from contraction q^{μ_i} starts with

$$(q^{\mu_i} T^{\Gamma_{12}}_{\mu_{12}})^{2\text{pt}} = \varepsilon_{\mu_k \nu} q^{\nu} \Omega^{(2\text{pt})}_i, \qquad \{i,k\} = \{1,3\}, \; k \neq i. \tag{5.113}$$

That is a form to compare standard identifications with consequences of tensor structure in the LHS. It denotes the 2pt functions (finite) coming from the i-th contraction. They can be zero to some contractions, e.g., vector contraction for equal masses. Particularly,

$$\varepsilon_{\mu_i \nu} q^{\nu} \Omega^{PV} = -(m_1 + m_2) T^{PV}_{\mu_i} \tag{5.114}$$

$$\varepsilon_{\mu_i \nu} q^{\nu} \Omega^{AS} = +(m_2 - m_1) T^{AS}_{\mu_i}, \tag{5.115}$$

given by (5.14), (5.15). The vector and scalar integrals (3.63)-(3.64) enable to write

$$\Omega^{AS} = \frac{i}{2\pi}\left[\left(m_2^2 - m_1^2\right) Z_1^{(-1)} - (m_2 - m_1) m_1 Z_0^{(-1)}\right] \tag{5.116}$$

$$\Omega^{PV} = \frac{i}{2\pi}\left[\left(m_2^2 - m_1^2\right) Z_1^{(-1)} + (m_1 + m_2) m_1 Z_0^{(-1)}\right]. \tag{5.117}$$

Summing them, we have from combination (3.31), a result independent of masses,

$$\left(\Omega^{AS} + \Omega^{PV}\right)(0) = -\frac{i}{\pi}\left[\left(m_1^2 - m_2^2\right) Z_1^{(-1)} - m_1^2 Z_0^{(-1)}\right]\Big|_{q^2=0} = -\frac{i}{\pi}. \tag{5.118}$$

A moment of reflection shows that anomalous amplitudes share this combination. As it is incompatible with the low-energy theorem, we derived a general parity-odd second-rank tensor of mass dimension zero. That is an inviolable property if it is free of kinematical singularities. We have anomalies in the vertices, which themselves can be arbitrary,

$$V_1(0) + V_2(0) = 0 \neq -\frac{i}{\pi} = \left(\Omega^{AS} + \Omega^{PV}\right)(0). \tag{5.119}$$

Hence, we at least can write $V_i(q^2) = \Omega_i(q^2) + \mathcal{A}_i$, where the additional parameter will be constrained by the equation above

$$\mathcal{A}_1 + \mathcal{A}_2 = \frac{i}{\pi}. \tag{5.120}$$

That represents the restriction of arbitrary anomalies in the axial and vector vertices. This kinematical implication has an important consequence over the RAGFs as well.

5.3.1 RAGFs: Linearity and Low-Energy Implications

The surface terms appear in explicit computations and are the only type of non-finite structures for the 2nd-rank amplitudes. Also, we have observed that they conditioned the

RAGFs. Nonetheless, we needed to establish in the absolute how they do it. Besides the exciting fact that versions one and two are the only independent possibilities, the answer to how this appears to be so must be constructed. Therefore, we explicit this intrinsic part of perturbative amplitudes; first, we split the general representation in

$$F_{\mu_1\mu_2} = F^{\Delta}_{\mu_1\mu_2} + \hat{F}_{\mu_1\mu_2}, \tag{5.121}$$

where $\hat{F}_{\mu_1\mu_2}$ encodes the finite parts. The term $F^{\Delta}_{\mu_1\mu_2}$ stands for the most general combination of surface terms, given by the equation

$$F^{\Delta}_{\mu_1\mu_2} = a\varepsilon_{\mu_1\nu}\Delta^{\nu}_{2\mu_2} + b\varepsilon_{\mu_2\nu}\Delta^{\nu}_{2\mu_1} + c\varepsilon_{\mu_1\mu_2}\Delta^{\nu}_{2\nu}.$$

Since there is a linear relation in such tensor due to the vanishing of 3rd-rank complete antisymmetric tensor in $2D$, $\varepsilon_{[\mu_1\mu_2}\Delta^{\nu}_{2\nu]} = 0$, we have a redefinition $a_1 = (a+c)$ and $a_2 = (b-c)$ of the coefficients. Henceforth, the general structure assumes the form

$$\begin{aligned} F_{\mu_1\mu_2} &= a_1\varepsilon_{\mu_1\nu}\Delta^{\nu}_{2\mu_2} + a_2\varepsilon_{\mu_2\nu}\Delta^{\nu}_{2\mu_1} \\ &\quad +\varepsilon_{\mu_1\mu_2}\hat{F}_1 + \varepsilon_{\mu_1\nu}q^{\nu}q_{\mu_2}\hat{F}_2 + \varepsilon_{\mu_2\nu}q^{\nu}q_{\mu_1}\hat{F}_3. \end{aligned} \tag{5.122}$$

The equation that represents the satisfaction of RAGFs can be systematized through

$$q^{\mu_i}T^{\Gamma_{12}}_{\mu_{12}} = T^{A}_{(-)\mu_k} + \varepsilon_{\mu_k\nu}\Omega_i. \tag{5.123}$$

Remember the notation for the one-point differences (5.18). The condition of linearity of integration is embodied in the following equations when performing the contractions,

$$q^{\mu_1}F_{\mu_1\mu_2} = a_1q^{\mu_1}\varepsilon_{\mu_1\nu}\Delta^{\nu}_{2\mu_2} + a_2\varepsilon_{\mu_2\nu}q^{\mu_1}\Delta^{\nu}_{2\mu_1} + \varepsilon_{\mu_2\nu}q^{\nu}(q^2\hat{F}_3 - \hat{F}_1) \tag{5.124}$$

$$q^{\mu_2}F_{\mu_1\mu_2} = a_1\varepsilon_{\mu_1\nu}q^{\mu_2}\Delta^{\nu}_{2\mu_2} + a_2q^{\mu_2}\varepsilon_{\mu_2\nu}\Delta^{\nu}_{2\mu_1} + \varepsilon_{\mu_1\nu}q^{\nu}(q^2\hat{F}_2 + \hat{F}_1). \tag{5.125}$$

We rearrange their indices and recognize the one-point functions

$$q^{\mu_1}F_{\mu_1\mu_2} = -\frac{1}{2}(a_1+a_2)\,T^{A}_{(-)\mu_2} + \varepsilon_{\mu_2\nu}q^{\nu}(q^2\hat{F}_3 - \hat{F}_1 - a_1\Delta^{\alpha}_{2\alpha}) \tag{5.126}$$

$$q^{\mu_2}F_{\mu_1\mu_2} = -\frac{1}{2}(a_1+a_2)\,T^{A}_{(-)\mu_1} + \varepsilon_{\mu_1\nu}q^{\nu}(q^2\hat{F}_2 + \hat{F}_1 - a_2\Delta^{\alpha}_{2\alpha}). \tag{5.127}$$

The RAGFs require for the first terms $a_1 + a_2 = -2$, and the other part must comply with the 2pt functions, Ω^{PV} and Ω^{AS}, which means

$$\Omega^{PV} = q^2\hat{F}_3 - \hat{F}_1 - a_1\Delta^{\alpha}_{2\alpha} \tag{5.128}$$

$$\Omega^{AS} = q^2\hat{F}_2 + \hat{F}_1 - a_2\Delta^{\alpha}_{2\alpha}. \tag{5.129}$$

Eliminating $\hat{F}_1$ and considering the first condition $a_1 + a_2 = -2$, we obtain

$$2\Delta^{\alpha}_{2\alpha} = \Omega^{PV} + \Omega^{AS} - q^2\hat{F}_2 - q^2\hat{F}_3. \tag{5.130}$$

In the point $q^2 = 0$ follows the low-energy implication of the finite amplitudes over the integration linearity (RAGFs)

$$2\Delta_{2\alpha}^{\alpha} = \Omega^{PV}(0) + \Omega^{AS}(0) = -\frac{i}{\pi}. \tag{5.131}$$

Consequences: The coefficients a_1 and a_2 may be arbitrary, but once one is selected to satisfy one RAGF in automatic form, the other must be zero. This unique solution signifies that most RAGFs found without conditions are achieved by the basic versions we have defined. This fact is independent of explicit computations through the traces of four Dirac matrices and continues to happen in four dimensions. Another consequence is that the satisfaction of all RAGFs is conditioned through kinematical features of finite functions that require a non-zero and specific amount value to the surface terms, implying that shifts in the integration variable and linearity of integration are incompatible. The T_{μ}^{A} functions depend on the routings, and their subtraction is zero if shifts are possible; only their difference is a function of the external momentum. This aspect is peculiar to this dimension; nonetheless, the restrictions from low-energy implications are precisely mirrored in four dimensions. Simultaneously satisfaction of RAGFs and translational invariance in momentum space is prohibited by the low-energy behavior of finite functions.

Chapter 6

Four-Dimensional Three-Point Functions

The analysis developed in the physical dimension focuses on odd amplitudes that are rank-3 tensors, namely AVV, VAV, VVA, and AAA. Their mathematical structures follow the same features seen in two dimensions. They depend on the trace involving six Dirac matrices plus the chiral one, whose computation yields products between the Levi-Civita symbol and metric tensor. After the integration, that generates expressions that differ in their dependence on surface terms and finite parts. We want to verify these prospects by evaluating the triangles' basic versions[1]. Once these resources are clear, we study how symmetries, linearity of integration, and uniqueness manifest.

From Eqs. (2.9) and (2.11), integrated three-point amplitudes are denoted through capital letters $T^{\Gamma_1\Gamma_2\Gamma_3}$ and exhibit the integrand

$$t^{\Gamma_1\Gamma_2\Gamma_3} = \mathrm{tr}\left[\Gamma_1 S\left(1\right)\Gamma_2 S\left(2\right)\Gamma_3 S\left(3\right)\right]. \tag{6.1}$$

Thus, after replacing vertex operators and disregarding vanishing traces, 3rd-order amplitudes assume the forms

$$t^{AVV}_{\mu_{123}} = [K^{\nu_{123}}_{123}\mathrm{tr}(\gamma_{*\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3}) + m^2\mathrm{tr}(\gamma_{*\mu_1\mu_2\mu_3\nu_1})(K^{\nu_1}_1 - K^{\nu_1}_2 + K^{\nu_1}_3)]\frac{1}{D_{123}} \tag{6.2}$$

$$t^{VAV}_{\mu_{123}} = [K^{\nu_{123}}_{123}\mathrm{tr}(\gamma_{*\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3}) + m^2\mathrm{tr}(\gamma_{*\mu_1\mu_2\mu_3\nu_1})(K^{\nu_1}_1 + K^{\nu_1}_2 - K^{\nu_1}_3)]\frac{1}{D_{123}} \tag{6.3}$$

$$t^{VVA}_{\mu_{123}} = [K^{\nu_{123}}_{123}\mathrm{tr}(\gamma_{*\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3}) - m^2\mathrm{tr}(\gamma_{*\mu_1\mu_2\mu_3\nu_1})(K^{\nu_1}_1 - K^{\nu_1}_2 - K^{\nu_1}_3)]\frac{1}{D_{123}} \tag{6.4}$$

$$t^{AAA}_{\mu_{123}} = [K^{\nu_{123}}_{123}\mathrm{tr}(\gamma_{*\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3}) - m^2\mathrm{tr}(\gamma_{*\mu_1\mu_2\mu_3\nu_1})(K^{\nu_1}_1 + K^{\nu_1}_2 + K^{\nu_1}_3)]\frac{1}{D_{123}}, \tag{6.5}$$

where we recall the conventions $K^{\nu_{123}}_{123} = K^{\nu_1}_1 K^{\nu_2}_2 K^{\nu_3}_3$ and $D_{123} = D_1 D_2 D_3$.

Although the trace involving four Dirac matrices plus the chiral one is univocal, different expressions are attributed to the leading trace when considering identities (2.5).

[1] To this aim, we compute twenty-four triangles of rank-one. Twelve parity-even triangles: VPP, ASP, VSS, and their permutations. Twelve parity-odd tensors: ASS, APP, VPS, and their permutations. Besides, we identify three standard tensors in a similar fashion for two dimensions.

Since Appendix (A.1) shows that forms achieved through definition $\gamma_* = i\varepsilon_{\nu_{1234}}\gamma^{\nu_{1234}}/4!$ are enough to compound any other, our starting point is on their structure

$$\begin{aligned}(4i)^{-1}\,\mathrm{tr}(\gamma_{*abcdef}) \;&=\; +g_{ab}\varepsilon_{cdef} + g_{ad}\varepsilon_{bcef} + g_{af}\varepsilon_{bcde}\\ &\quad +g_{bc}\varepsilon_{adef} + g_{cd}\varepsilon_{abef} + g_{cf}\varepsilon_{abde}\\ &\quad +g_{be}\varepsilon_{acdf} + g_{de}\varepsilon_{abcf} + g_{ef}\varepsilon_{abcd}\\ &\quad -g_{bd}\varepsilon_{acef} - g_{df}\varepsilon_{abce} - g_{bf}\varepsilon_{acde}\\ &\quad -g_{ac}\varepsilon_{bdef} - g_{ce}\varepsilon_{abdf} - g_{ae}\varepsilon_{bcdf}.\end{aligned} \tag{6.6}$$

There are three basic versions, each corresponding to replacing the chiral matrix near a specific vertex operator. We introduce a numeric label to distinguish them:

$$[\mathrm{tr}(\gamma_{*\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3})]_1 = [\mathrm{tr}(\gamma_{*\mu_2\nu_2\mu_3\nu_3\mu_1\nu_1})]_2 = [\mathrm{tr}(\gamma_{*\mu_3\nu_3\mu_1\nu_1\mu_2\nu_2})]_3. \tag{6.7}$$

They arise when setting the index configuration in the trace above (6.6), differing in the signs of terms. We cast their contraction with $K_{123}^{\nu_{123}}$ in the sequence. Their integration leads to three not (automatically) equivalent expressions for each triangle.

$$\begin{aligned}[K_{123}^{\nu_{123}}\mathrm{tr}(\gamma_{*\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3})]_1 \;&=\; -4i\varepsilon_{\mu_{23}\nu_{12}}[K_{1\mu_1}K_{23}^{\nu_{12}} - K_{2\mu_1}K_{13}^{\nu_{12}} + K_{3\mu_1}K_{12}^{\nu_{12}}]\\ &\quad -4i\varepsilon_{\mu_{13}\nu_{12}}[K_{1\mu_2}K_{23}^{\nu_{12}} + K_{2\mu_2}K_{13}^{\nu_{12}} - K_{3\mu_2}K_{12}^{\nu_{12}}]\\ &\quad +4i\varepsilon_{\mu_{12}\nu_{12}}[K_{1\mu_3}K_{23}^{\nu_{12}} - K_{2\mu_3}K_{13}^{\nu_{12}} - K_{3\mu_3}K_{12}^{\nu_{12}}]\\ &\quad -4i\varepsilon_{\mu_{123}\nu_1}[K_1^{\nu_1}(K_2\cdot K_3) - K_2^{\nu_1}(K_1\cdot K_3) + K_3^{\nu_1}(K_1\cdot K_2)]\\ &\quad +4i[-g_{\mu_{12}}\varepsilon_{\mu_3\nu_{123}} - g_{\mu_{23}}\varepsilon_{\mu_1\nu_{123}} + g_{\mu_{13}}\varepsilon_{\mu_2\nu_{123}}]K_{123}^{\nu_{123}}\end{aligned} \tag{6.8}$$

$$\begin{aligned}[K_{123}^{\nu_{123}}\mathrm{tr}(\gamma_{*\mu_2\nu_2\mu_3\nu_3\mu_1\nu_1})]_2 \;&=\; +4i\varepsilon_{\mu_{13}\nu_{12}}[K_{1\mu_2}K_{23}^{\nu_{12}} - K_{2\mu_2}K_{13}^{\nu_{12}} + K_{3\mu_2}K_{12}^{\nu_{12}}]\\ &\quad -4i\varepsilon_{\mu_{12}\nu_{12}}[K_{1\mu_3}K_{23}^{\nu_{23}} + K_{2\mu_3}K_{13}^{\nu_{13}} + K_{3\mu_3}K_{12}^{\nu_{12}}]\\ &\quad -4i\varepsilon_{\mu_{23}\nu_{12}}[K_{1\mu_1}K_{23}^{\nu_{23}} + K_{2\mu_1}K_{13}^{\nu_{13}} - K_{3\mu_1}K_{12}^{\nu_{12}}]\\ &\quad -4i\varepsilon_{\mu_{123}\nu_1}[K_1^{\nu_1}(K_2\cdot K_3) + K_2^{\nu_1}(K_1\cdot K_3) - K_3^{\nu_1}(K_1\cdot K_2)]\\ &\quad +4i[g_{\mu_{12}}\varepsilon_{\mu_3\nu_{123}} - g_{\mu_{13}}\varepsilon_{\mu_2\nu_{123}} - g_{\mu_{23}}\varepsilon_{\mu_1\nu_{123}}]K_{123}^{\nu_{123}}\end{aligned} \tag{6.9}$$

$$\begin{aligned}[K_{123}^{\nu_{123}}\mathrm{tr}(\gamma_{*\mu_3\nu_3\mu_1\nu_1\mu_2\nu_2})]_3 \;&=\; -4i\varepsilon_{\mu_{12}\nu_{12}}[K_{1\mu_3}K_{23}^{\nu_{12}} - K_{2\mu_3}K_{13}^{\nu_{12}} + K_{3\mu_3}K_{12}^{\nu_{12}}]\\ &\quad -4i\varepsilon_{\mu_{23}\nu_{12}}[K_{1\mu_1}K_{23}^{\nu_{12}} - K_{2\mu_1}K_{13}^{\nu_{12}} - K_{3\mu_1}K_{12}^{\nu_{12}}]\\ &\quad -4i\varepsilon_{\mu_{13}\nu_{12}}[K_{1\mu_2}K_{23}^{\nu_{12}} + K_{2\mu_2}K_{13}^{\nu_{12}} + K_{3\mu_2}K_{12}^{\nu_{12}}]\\ &\quad +4i\varepsilon_{\mu_{123}\nu_1}[K_1^{\nu_1}(K_2\cdot K_3) - K_2^{\nu_1}(K_1\cdot K_3) - K_3^{\nu_1}(K_1\cdot K_2)]\\ &\quad +4i[-g_{\mu_{12}}\varepsilon_{\mu_3\nu_{123}} - g_{\mu_{13}}\varepsilon_{\mu_2\nu_{123}} + g_{\mu_{23}}\varepsilon_{\mu_1\nu_{123}}]K_{123}^{\nu_{123}}\end{aligned} \tag{6.10}$$

Analogously to two-dimensional calculations, our next task consists of organizing and integrating the complete expressions. As the three first rows of the above equations are similar to the object (4.24), we define the tensors

$$\varepsilon_{\mu_{ab}\nu_{12}}t_{\mu_c}^{\nu_{12}(s_1s_2)} = \varepsilon_{\mu_{ab}\nu_{12}}\left(K_{1\mu_c}K_{23}^{\nu_{12}} + s_1K_{2\mu_c}K_{13}^{\nu_{12}} + s_2K_{3\mu_c}K_{12}^{\nu_{12}}\right)\frac{1}{D_{123}} \tag{6.11}$$

where $s_i = \pm 1$. We rewrite this equation using $K_i = K_j + p_{ij}$ and $\varepsilon_{\mu_{ab}\nu_{12}} K_{ij}^{\nu_{12}} = \varepsilon_{\mu_{ab}\nu_{12}} p_{ji}^{\nu_2} K_i^{\nu_1}$ to achieve the structures introduced in Section (3.3):

$$\begin{aligned}\varepsilon_{\mu_{ab}\nu_{12}} t_{\mu_c}^{\nu_{12}(s_1 s_2)} &= \varepsilon_{\mu_{ab}\nu_{12}} [(1+s_1) p_{31}^{\nu_2} - (1-s_2) p_{21}^{\nu_2}] K_1^{\nu_1} K_{1\mu_c} \frac{1}{D_{123}} \\ &\quad + \varepsilon_{\mu_{ab}\nu_{12}} [p_{21}^{\nu_1} p_{32}^{\nu_2} K_{1\mu_c} + (s_1 p_{21\mu_c} p_{31}^{\nu_2} + s_2 p_{31\mu_c} p_{21}^{\nu_2}) K_1^{\nu_1}] \frac{1}{D_{123}}.\end{aligned} \tag{6.12}$$

Hence, final expressions arise directly by replacing vector and tensor Feynman integrals from Subsection (3.3.2). Although four sign configurations are available, the expression taking $s_1 = -1$ and $s_2 = 1$ cancels out. That is straightforward for the first row, but a closer look at the composition of the following integral is necessary to analyze the second:

$$\overline{J}_3^\mu = J_3^\mu = i\,(4\pi)^{-2} \left[-p_{21}^\mu Z_{10}^{(-1)}(p_{21}, p_{31}) - p_{31}^\mu Z_{01}^{(-1)}(p_{21}, p_{31}) \right]. \tag{6.13}$$

Since it is proportional to external momenta, it leads to symmetric tensors that vanish when contracted with Levi-Civita symbol. We cast all sign configurations in the sequence:

$$2\varepsilon_{\mu_{ab}\nu_{12}} T_{\mu_c}^{\nu_{12}(-+)} = 2\varepsilon_{\mu_{ab}\nu_{12}} [p_{21}^{\nu_1} p_{32}^{\nu_2} J_{3\mu_c} + (-p_{21\mu_c} p_{31}^{\nu_2} + p_{31\mu_c} p_{21}^{\nu_2}) J_3^{\nu_1}] \equiv 0, \tag{6.14}$$

$$\begin{aligned}2\varepsilon_{\mu_{ab}\nu_{12}} T_{\mu_c}^{\nu_{12}(+-)} &= 4\varepsilon_{\mu_{ab}\nu_{12}} [p_{31}^{\nu_2} (J_{3\mu_c}^{\nu_1} + p_{21\mu_c} J_3^{\nu_1}) - p_{21}^{\nu_2} (J_{3\mu_c}^{\nu_1} + p_{31\mu_c} J_3^{\nu_1})] \\ &\quad + (\varepsilon_{\mu_{ab}\nu_{12}} p_{32}^{\nu_2} \Delta_{3\mu_c}^{\nu_1} + \varepsilon_{\mu_{abc}\nu_1} p_{32}^{\nu_1} I_{\log}),\end{aligned} \tag{6.15}$$

$$\begin{aligned}2\varepsilon_{\mu_{ab}\nu_{12}} T_{\mu_c}^{\nu_{12}(--)} &= -4\varepsilon_{\mu_{ab}\nu_{12}} p_{21}^{\nu_2} (J_{3\mu_c}^{\nu_1} + p_{31\mu_c} J_3^{\nu_1}) \\ &\quad - (\varepsilon_{\mu_{ab}\nu_{12}} p_{21}^{\nu_2} \Delta_{3\mu_c}^{\nu_1} + \varepsilon_{\mu_{abc}\nu_1} p_{21}^{\nu_1} I_{\log}),\end{aligned} \tag{6.16}$$

$$\begin{aligned}2\varepsilon_{\mu_{ab}\nu_{12}} T_{\mu_c}^{\nu_{12}(++)} &= +4\varepsilon_{\mu_{ab}\nu_{12}} p_{31}^{\nu_2} (J_{3\mu_c}^{\nu_1} + p_{21\mu_c} J_3^{\nu_1}) \\ &\quad + (\varepsilon_{\mu_{ab}\nu_{12}} p_{31}^{\nu_2} \Delta_{3\mu_c}^{\nu_1} + \varepsilon_{\mu_{abc}\nu_1} p_{31}^{\nu_1} I_{\log}).\end{aligned} \tag{6.17}$$

Different tensor contributions appear for each trace version from (6.8)-(6.10). Thus, after disregarding the vanishing contribution, we identify the corresponding combinations

$$C_{1\mu_{123}} = -\varepsilon_{\mu_{13}\nu_{12}} T_{\mu_2}^{\nu_{12}(+-)} + \varepsilon_{\mu_{12}\nu_{12}} T_{\mu_3}^{\nu_{12}(--)} \tag{6.18}$$

$$C_{2\mu_{123}} = -\varepsilon_{\mu_{12}\nu_{12}} T_{\mu_3}^{\nu_{12}(++)} - \varepsilon_{\mu_{23}\nu_{12}} T_{\mu_1}^{\nu_{12}(+-)} \tag{6.19}$$

$$C_{3\mu_{123}} = -\varepsilon_{\mu_{23}\nu_{12}} T_{\mu_1}^{\nu_{12}(--)} - \varepsilon_{\mu_{13}\nu_{12}} T_{\mu_2}^{\nu_{12}(++)}. \tag{6.20}$$

The sampling of indexes reflects the absence of the index μ_i of the vertex Γ_i in the sign tensors of the $C_{i\mu_{123}}$, enabling the anticipation of violations of either WIs or RAGFs. That occurs because this specific index appears in the tensor $\varepsilon_{\mu_{ab}\nu_{12}} T_{\mu_i}^{\nu_{12}(-,+)}$, which is finite and identically zero, present in each of the above expressions before integration.

Let us return to the last row of Eqs. (6.8)-(6.10), which corresponds to 1st-order odd triangles. The precise identifications among the possibilities occur when replacing the vertex configurations in the general integrand (6.1); however, all of them are proportional to ASS amplitude:

$$t_{\mu_i}^{ASS} = 4i\varepsilon_{\mu_i\nu_{123}} K_{123}^{\nu_{123}} \frac{1}{D_{123}} = 4i\varepsilon_{\mu_i\nu_{123}} p_{21}^{\nu_2} p_{31}^{\nu_3} K_1^{\nu_1} \frac{1}{D_{123}}. \tag{6.21}$$

We already performed some simplifications through the same resources from the tensor discussion (beginning of the previous paragraph). After integration, this function depends on the Feynman integral $J_3^{\nu_1}$. Since this object is a finite tensor proportional to external momenta p_{ij}, the contraction with the Levi-Civita symbol necessarily vanishes

$$T_{\mu_i}^{ASS} = 4i\varepsilon_{\mu_i\nu_{123}} p_{21}^{\nu_2} p_{31}^{\nu_3} J_3^{\nu_1} = 0. \tag{6.22}$$

For this reason, we omit this class of amplitudes from the final triangles.

We left the fourth line of (6.8)-(6.10) for last since bilinears get summed with mass terms from the remaining trace. Each investigated case leads to a subamplitude identified after comparing vertex arrangements in (6.1). This result is general: besides $C_{i\mu_{123}}$ tensors, different rank-1 even subamplitudes appear inside each version of rank-3 odd amplitudes. Table 6.1 accounts for all of these possibilities, while Appendix (E) presents explicit expressions for subamplitudes. Let us consider the first version of AVV to illustrate. After combining mass terms from Eq. (6.2) with bilinears from Eq. (6.8), we find the VPP subamplitude

$$\text{sub}(t_{\mu_{123}}^{AVV})_1 = i\varepsilon_{\mu_{123}\nu_1}(t^{VPP})^{\nu_1}. \tag{6.23}$$

The integrand of this correlator has the structure

$$(t^{VPP})^{\nu_1} = \text{tr}[\gamma^{\nu_1} S(1)\,\gamma_* S(2)\,\gamma_* S(3)] = 4(-K_1^{\nu_1} S_{23} + K_2^{\nu_1} S_{13} - K_3^{\nu_1} S_{12})\frac{1}{D_{123}}, \tag{6.24}$$

where the combination $S_{ij} = K_i \cdot K_j - m^2$ comes from definition (4.23). After reducing the denominator, we perform the integration

$$\begin{aligned}(T^{VPP})^{\nu_1} \;=\;& 2[P_{31}^{\nu_2}\Delta_{3\nu_2}^{\nu_1} + (p_{21}^{\nu_1} - p_{32}^{\nu_1})I_{\log}] - 4\,(p_{21}\cdot p_{32})\,J_3^{\nu_1} \\ & +2[(p_{31}^{\nu_1}p_{21}^2 - p_{21}^{\nu_1}p_{31}^2)J_3 + p_{21}^{\nu_1}J_2\,(p_{21}) - p_{32}^{\nu_1}J_2\,(p_{32})].\end{aligned} \tag{6.25}$$

Table 6.1: Even sub-amplitudes related to each version of 3rd-order odd amplitudes.

Version/Type	AVV	VAV	VVA	AAA
1	$+VPP$	$+ASP$	$-APS$	$-VSS$
2	$-SAP$	$+PVP$	$+PAS$	$-SVS$
3	$+SPA$	$-PSA$	$+PPV$	$-SSV$

Since all pieces are known, compounding triangle amplitudes is possible. For instance, the i-th version of the AVV arises as a combination involving the C_i-tensor and the corresponding vector subamplitude. Thus, consulting Table 6.1 leads to the following associations

$$(T_{\mu_{123}}^{AVV})_1 = 4iC_{1\mu_{123}} + i\varepsilon_{\mu_{123}\nu_1}(T^{VPP})^{\nu_1}, \tag{6.26}$$

$$(T_{\mu_{123}}^{AVV})_2 = 4iC_{2\mu_{123}} - i\varepsilon_{\mu_{123}\nu_1}(T^{SAP})^{\nu_1}, \tag{6.27}$$

$$(T_{\mu_{123}}^{AVV})_3 = 4iC_{3\mu_{123}} + i\varepsilon_{\mu_{123}\nu_1}(T^{SPA})^{\nu_1}. \tag{6.28}$$

The generalization for VAV, VVA, and AAA is straightforward:

$$(T^{\Gamma_1\Gamma_2\Gamma_3}_{\mu_{123}})_i = 4iC_{i,\mu_{123}} \pm i\varepsilon_{\mu_{123}\nu_1}\,(\text{Corresponding sub-amplitude})^{\nu_1}\,. \tag{6.29}$$

We still want to detail some important points about these amplitudes. To illustrate this subject, we use tools developed in this section to build up the first version of AVV,

$$\begin{aligned}(T^{AVV}_{\mu_{123}})_1 \;=\;& S_{1\mu_{123}} - 8i\varepsilon_{\mu_{12}\nu_{12}} p^{\nu_2}_{21}(J^{\nu_1}_{3\mu_3} + p_{31\mu_3} J^{\nu_1}_{3}) \\ & -8i\varepsilon_{\mu_{13}\nu_{12}}[p^{\nu_2}_{31}(J^{\nu_1}_{3\mu_2} + p_{21\mu_2} J^{\nu_1}_{3}) - p^{\nu_2}_{21}(J^{\nu_1}_{3\mu_2} + p_{31\mu_2} J^{\nu_1}_{3})] \\ & -4i\varepsilon_{\mu_{123}\nu_1}(p_{21}\cdot p_{32})J^{\nu_1}_3 + 2i\varepsilon_{\mu_{123}\nu_1}[(p^{\nu_1}_{31}p^2_{21} - p^{\nu_1}_{21}p^2_{31})]J_3 \\ & +2i\varepsilon_{\mu_{123}\nu_1}[p^{\nu_1}_{21}J_2\,(p_{21}) - p^{\nu_1}_{32}J_2\,(p_{32})].\end{aligned} \tag{6.30}$$

The divergent part of the tensor (6.18) comes from Eqs. (6.15) and (6.16) as

$$4iC_{1\mu_{123}} = -2i[\varepsilon_{\mu_{13}\nu_{12}} p^{\nu_2}_{32}\Delta^{\nu_1}_{3\mu_2} + \varepsilon_{\mu_{12}\nu_{12}} p^{\nu_2}_{21}\Delta^{\nu_1}_{3\mu_3} + \varepsilon_{\mu_{123}\nu_1}(p^{\nu_1}_{21} - p^{\nu_1}_{32})I_{\log}].$$

When combined with the VPP subamplitude, we acknowledge the exact cancellation of the object $I_{\log}$ as it occurs for all investigated versions. Thus, surface terms compound the whole structure of divergences

$$S_{1\mu_{123}} = -2i(\varepsilon_{\mu_{13}\nu_{12}} p^{\nu_2}_{32}\Delta^{\nu_1}_{3\mu_2} + \varepsilon_{\mu_{12}\nu_{12}} p^{\nu_2}_{21}\Delta^{\nu_1}_{3\mu_3}) + 2i\varepsilon_{\mu_{123}\nu_1} P^{\nu_2}_{31}\Delta^{\nu_1}_{3\nu_2}. \tag{6.31}$$

Moreover, contributions from vector subamplitudes exhibit arbitrary momenta $P_{ij} = k_i + k_j$ as coefficients. We stress that the divergent content is shared; the first version of amplitudes AVV, VAV, VVA, and AAA contains the same structure (6.31). That is a feature of the specific version and not on the vertex content of the diagram. For later use, we define the other sets of surface terms

$$\begin{aligned} S_{2\mu_{123}} &= -2i(\varepsilon_{\mu_{12}\nu_{12}} p^{\nu_2}_{31}\Delta^{\nu_1}_{3\mu_3} + \varepsilon_{\mu_{23}\nu_{12}} p^{\nu_2}_{32}\Delta^{\nu_1}_{3\mu_1}) + 2i\varepsilon_{\mu_{123}\nu_1} P^{\nu_2}_{21}\Delta^{\nu_1}_{3\nu_2}, & (6.32)\\ S_{3\mu_{123}} &= -2i(\varepsilon_{\mu_{13}\nu_{12}} p^{\nu_2}_{31}\Delta^{\nu_1}_{3\mu_2} - \varepsilon_{\mu_{23}\nu_{12}} p^{\nu_2}_{21}\Delta^{\nu_1}_{3\mu_1}) + 2i\varepsilon_{\mu_{123}\nu_1} P^{\nu_2}_{32}\Delta^{\nu_1}_{3\nu_2}. & (6.33)\end{aligned}$$

That concludes the preliminary discussion on rank-3 triangles, so investigating RAGFs is possible. That is the subject of the following sections.

6.1 Relations Among Green Functions and Uniqueness

The next step is to perform momenta contractions that lead to RAGFs following the recipes in (2.14) and (2.17). Although they are algebraic identities at the integrand level, their satisfaction is not automatic after integration. In parallel to what we saw in the two-dimensional case, possibilities for Dirac traces and values of surface terms have important implications for this analysis.

$$\begin{aligned} p_{31}^{\mu_1}t_{\mu_{123}}^{AVV} &= t_{\mu_{32}}^{AV}(1,2)-t_{\mu_{23}}^{AV}(2,3)-2mt_{\mu_{23}}^{PVV} \\ p_{21}^{\mu_2}t_{\mu_{123}}^{AVV} &= t_{\mu_{13}}^{AV}(1,3)-t_{\mu_{13}}^{AV}(2,3) \\ p_{32}^{\mu_3}t_{\mu_{123}}^{AVV} &= t_{\mu_{12}}^{AV}(1,2)-t_{\mu_{12}}^{AV}(1,3) \end{aligned} \tag{6.34}$$

$$\begin{aligned} p_{31}^{\mu_1}t_{\mu_{123}}^{VAV} &= t_{\mu_{23}}^{AV}(2,1)-t_{\mu_{23}}^{AV}(2,3) \\ p_{21}^{\mu_2}t_{\mu_{123}}^{VAV} &= t_{\mu_{31}}^{AV}(3,1)-t_{\mu_{13}}^{AV}(2,3)+2mt_{\mu_{13}}^{VPV} \\ p_{32}^{\mu_3}t_{\mu_{123}}^{VAV} &= t_{\mu_{21}}^{AV}(2,1)-t_{\mu_{21}}^{AV}(3,1) \end{aligned} \tag{6.35}$$

$$\begin{aligned} p_{31}^{\mu_1}t_{\mu_{123}}^{VVA} &= t_{\mu_{32}}^{AV}(1,2)-t_{\mu_{32}}^{AV}(3,2) \\ p_{21}^{\mu_2}t_{\mu_{123}}^{VVA} &= t_{\mu_{31}}^{AV}(3,1)-t_{\mu_{31}}^{AV}(3,2) \\ p_{32}^{\mu_3}t_{\mu_{123}}^{VVA} &= t_{\mu_{12}}^{AV}(1,2)-t_{\mu_{21}}^{AV}(3,1)+2mt_{\mu_{12}}^{VVP} \end{aligned} \tag{6.36}$$

$$\begin{aligned} p_{31}^{\mu_1}t_{\mu_{123}}^{AAA} &= t_{\mu_{23}}^{AV}(2,1)-t_{\mu_{32}}^{AV}(3,2)-2mt_{\mu_{23}}^{PAA} \\ p_{21}^{\mu_2}t_{\mu_{123}}^{AAA} &= t_{\mu_{13}}^{AV}(1,3)-t_{\mu_{31}}^{AV}(3,2)+2mt_{\mu_{13}}^{APA} \\ p_{32}^{\mu_3}t_{\mu_{123}}^{AAA} &= t_{\mu_{21}}^{AV}(2,1)-t_{\mu_{12}}^{AV}(1,3)+2mt_{\mu_{12}}^{AAP} \end{aligned} \tag{6.37}$$

Let us introduce the structures that emerged within the relations above. First, the RHS's three-point functions are finite tensors external momenta dependent. That is transparent due to their connection with finite Feynman integrals introduced in Subsection (3.2.2), so we only remove the overbar notation from corresponding tensors $\bar{J}_3^{\nu_1}=J_3^{\nu_1}$ and $\bar{J}_3=J_3$. We have for single axial triangles

$$-2mT_{\mu_{23}}^{PVV} = \varepsilon_{\mu_{23}\nu_{12}}p_{21}^{\nu_1}p_{32}^{\nu_2}(8im^2J_3), \tag{6.38}$$

$$2mT_{\mu_{13}}^{VPV} = \varepsilon_{\mu_{13}\nu_{12}}p_{21}^{\nu_1}p_{32}^{\nu_2}(8im^2J_3), \tag{6.39}$$

$$2mT_{\mu_{12}}^{VVP} = \varepsilon_{\mu_{12}\nu_{12}}p_{21}^{\nu_1}p_{32}^{\nu_2}(-8im^2J_3), \tag{6.40}$$

while momenta contractions for the triple axial triangle lead to

$$-2mT_{\mu_{23}}^{PAA} = \varepsilon_{\mu_{23}\nu_{12}}p_{31}^{\nu_2}[8im^2(2J_3^{\nu_1}+p_{21}^{\nu_1}J_3)], \tag{6.41}$$

$$2mT_{\mu_{13}}^{APA} = \varepsilon_{\mu_{13}\nu_{12}}p_{21}^{\nu_2}[-8im^2(2J_3^{\nu_1}+p_{31}^{\nu_1}J_3)], \tag{6.42}$$

$$2mT_{\mu_{12}}^{AAP} = \varepsilon_{\mu_{12}\nu_{12}}p_{32}^{\nu_2}[8im^2(2J_3^{\nu_1}+p_{21}^{\nu_1}J_3)]. \tag{6.43}$$

These amplitudes have a low-energy behavior that we aim to explore in connection with RAGFs in Sections (6.2) and (6.3). Since they depend on functions $Z_{n_1n_2}^{(-1)}$ (3.40) through the scalar three-point integral $J_3=i(4\pi)^{-2}Z_{00}^{(-1)}$ and the vector one (6.13). We use (3.42) to determine the behavior of these tensors when all bilinears in their momenta are zero:

$$-2mT_{\mu_{23}}^{PVV}\Big|_0 = \frac{1}{(2\pi)^2}; \quad 2mT_{\mu_{13}}^{VPV}\Big|_0 = \frac{1}{(2\pi)^2}; \quad 2mT_{\mu_{12}}^{VVP}\Big|_0 = -\frac{1}{(2\pi)^2}; \tag{6.44}$$

$$-2mT_{\mu_{23}}^{PAA}\Big|_0 = \frac{1}{3(2\pi)^2}; \quad 2mT_{\mu_{13}}^{APA}\Big|_0 = \frac{1}{3(2\pi)^2}; \quad 2mT_{\mu_{12}}^{AAP}\Big|_0 = -\frac{1}{3(2\pi)^2}. \tag{6.45}$$

Each term above is multiplied by the corresponding tensor $\varepsilon_{\mu_{kl}\nu_{12}}p_{21}^{\nu_1}p_{32}^{\nu_2}$ with $k<l$.

Second, the other structures that appeared in the RAGFs are AV functions, which are proportional to two-point vector integrals. Using the result (3.82), we achieve

$$T_{\mu_{ij}}^{AV}(a,b) = -4i\varepsilon_{\mu_i\mu_j\nu_1\nu_2}p_{ba}^{\nu_2}\bar{J}_2^{\nu_1}(a,b) = 2i\varepsilon_{\mu_i\mu_j\nu_1\nu_2}p_{ba}^{\nu_2}P_{ab}^{\nu_3}\Delta_{3\nu_3}^{\nu_1}. \tag{6.46}$$

As contributions (exclusively) on the external momentum cancel out in the contraction, they are pure surface terms proportional to arbitrary label combinations. After replacing the adequate labels (k_a and k_b), combinations seen in the RAGFs above arise:

$$T_{\mu_{32}}^{AV}(1,2) - T_{\mu_{23}}^{AV}(2,3) = -2i\varepsilon_{\mu_{23}\nu_{12}}\left(p_{21}^{\nu_2}P_{12}^{\nu_3} + p_{32}^{\nu_2}P_{32}^{\nu_3}\right)\Delta_{3\nu_3}^{\nu_1} \tag{6.47}$$

$$T_{\mu_{13}}^{AV}(1,3) - T_{\mu_{13}}^{AV}(2,3) = -2i\varepsilon_{\mu_{13}\nu_{12}}\left(p_{32}^{\nu_2}P_{32}^{\nu_3} - p_{31}^{\nu_2}P_{31}^{\nu_3}\right)\Delta_{3\nu_3}^{\nu_1} \tag{6.48}$$

$$T_{\mu_{12}}^{AV}(1,2) - T_{\mu_{12}}^{AV}(1,3) = -2i\varepsilon_{\mu_{12}\nu_{12}}\left(p_{31}^{\nu_2}P_{31}^{\nu_3} - p_{21}^{\nu_2}P_{21}^{\nu_3}\right)\Delta_{3\nu_3}^{\nu_1}. \tag{6.49}$$

We stress that these forms depend only on the vertex contraction and not specific amplitude (AVV, VAV, VVA, and AAA). That occurs because there is a sign change in the AV when permuting the position of free indexes (see $\varepsilon_{\mu_i\mu_j\nu_1\nu_2}$) or changing the role of routings (see $p_{ba}^{\nu_2}P_{ab}^{\nu_3}$).

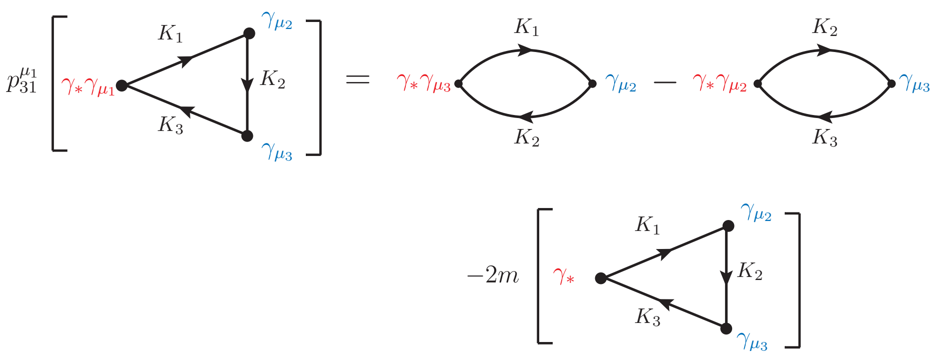

Figure 6.1: The RAGF established for the contraction with momenta $q_{31}^{\mu_1}T_{\mu_{123}}^{AVV}$.

To verify RAGFs, we must contract external momenta with the explicit forms of amplitudes. Observe the finite contributions displayed in the example (6.30) to clarify operations involving finite contributions. These results use well-defined relations involving finite quantities. After contracting with momenta, some terms vanish due to the Levi-Civita symbol. Then, we manipulate the remaining terms using tools developed in Subsection (3.3.2). The procedure involves reducing J-tensors to identify finite 2nd-order amplitudes or achieve some cancellations. The referred reductions are for tensor integrals

$$2p_{21}^{\nu_2}J_{3\nu_2}^{\nu_1} = -p_{21}^2J_3^{\nu_1} + J_2^{\nu_1}(p_{31}) + J_2^{\nu_1}(p_{32}) + p_{31}^{\nu_1}J_2(p_{32}), \tag{6.50}$$

$$2p_{31}^{\nu_2}J_{3\nu_2}^{\nu_1} = -p_{31}^2J_3^{\nu_1} + J_2^{\nu_1}(p_{21}) + J_2^{\nu_1}(p_{32}) + p_{31}^{\nu_1}J_2(p_{32}), \tag{6.51}$$

$$2J_{3\nu_1}^{\nu_1} = 2m^2J_3 + 2J_2(p_{32}) + i(4\pi)^{-2}, \tag{6.52}$$

and vector integrals

$$2p_{21\nu_1}J_3^{\nu_1} = -p_{21}^2 J_3 + J_2(p_{31}) - J_2(p_{32}), \quad (6.53)$$
$$2p_{31\nu_1}J_3^{\nu_1} = -p_{31}^2 J_3 + J_2(p_{21}) - J_2(p_{32}). \quad (6.54)$$

Although some reductions arise directly, other occurrences require further algebraic manipulations. This circumstance manifests in cases where a J-tensor couples to the Levi-Civita symbol so that rearranging indexes is necessary to find momenta contractions. For vector integrals, we consider the identity $\varepsilon_{[\mu_a\mu_b\nu_1\nu_2 p_{\nu_3]}}J_3^{\nu_1} = 0$ to achieve the formula[2]

$$2\varepsilon_{\mu_{ab}\nu_{12}}\left[p_{21}^{\nu_2}(p_{ij}\cdot p_{31}) - p_{31}^{\nu_2}(p_{ij}\cdot p_{21})\right]J_3^{\nu_1} = -\varepsilon_{\mu_{ab}\nu_{23}}p_{21}^{\nu_2}p_{31}^{\nu_3}\left[2p_{ij\nu_1}J_3^{\nu_1}\right]. \quad (6.55)$$

Similarly, we use $\varepsilon_{[\mu_a\nu_1\nu_2\nu_3}J_{3\mu_c]}^{\nu_1} = 0$ to reorganize terms involving the tensor integral

$$\begin{aligned} & 2\varepsilon_{\mu_b\nu_{123}}p_{21}^{\nu_2}p_{31}^{\nu_3}J_{3\mu_a}^{\nu_1} - 2\varepsilon_{\mu_a\nu_{123}}p_{21}^{\nu_2}p_{31}^{\nu_3}J_{3\mu_b}^{\nu_1} \\ = \; & \varepsilon_{\mu_{ab}\nu_{13}}p_{31}^{\nu_3}\left[2p_{21}^{\nu_2}J_{3\nu_2}^{\nu_1}\right] - \varepsilon_{\mu_{ab}\nu_{12}}p_{21}^{\nu_2}\left[2p_{31}^{\nu_3}J_{3\nu_3}^{\nu_1}\right] - \varepsilon_{\mu_{ab}\nu_{23}}p_{21}^{\nu_2}p_{31}^{\nu_3}\left[2J_{3\nu_1}^{\nu_1}\right]. \end{aligned} \quad (6.56)$$

In the amplitudes, we have two structures: standard tensors $C_{i\mu_{123}}$ (6.18)-(6.20) and subamplitudes. The tensors are common to the amplitudes versions and are comprised of the sign tensors (6.14)-(6.17). To illustrate the operations necessary for the RAGFs, let us take the case

$$C_{1\mu_{123}}^{\text{finite}} = -2\varepsilon_{\mu_{13}\nu_{12}}[p_{31}^{\nu_2}(J_{3\mu_2}^{\nu_1} + p_{21\mu_2}J_3^{\nu_1}) - p_{21}^{\nu_2}(J_{3\mu_2}^{\nu_1} + p_{31\mu_2}J_3^{\nu_1})] \quad (6.57)$$
$$-2\varepsilon_{\mu_{12}\nu_{12}}p_{21}^{\nu_2}(J_{3\mu_3}^{\nu_1} + p_{31\mu_3}J_3^{\nu_1}). \quad (6.58)$$

The first term in parenthesis cancels when contracting with $p_{31}^{\mu_1}$, the remaining terms are

$$p_{31}^{\mu_1}C_{1\mu_{123}}^{\text{finite}} = -2[\varepsilon_{\mu_3\nu_{123}}p_{21}^{\nu_2}p_{31}^{\nu_3}J_{3\mu_2}^{\nu_1} - \varepsilon_{\mu_2\nu_{123}}p_{21}^{\nu_2}p_{31}^{\nu_3}J_{3\mu_3}^{\nu_1}]. \quad (6.59)$$

Then, we employ the identity (6.56) to permute indexes and perform reductions. That accomplishes our objective; furthermore, this rearrangement implies the presence of Eq. (6.52), and that brings two additional contributions: one proportional to squared mass and a numeric factor. That differs from contractions $p_{21}^{\mu_2}$ and $p_{32}^{\mu_3}$, where reductions of tensor integrals are immediate, and it is only necessary to use (6.55). The behavior of different contractions is not associated with vertex content but with amplitude version.

$$\begin{aligned} p_{31}^{\mu_1}C_{1\mu_{123}}^{\text{finite}} &= \varepsilon_{\mu_{23}\nu_1\nu_2}\{(p_{31}^{\nu_2}p_{21}^2 - p_{21}^{\nu_2}p_{31}^2)J_3^{\nu_1} \\ &\quad + p_{21}^{\nu_1}p_{31}^{\nu_2}[2m^2J_3 + i(4\pi)^{-2} + J_2(p_{32})]\} \end{aligned} \quad (6.60)$$
$$p_{21}^{\mu_2}C_{1\mu_{123}}^{\text{finite}} = \frac{1}{2}\varepsilon_{\mu_{13}\nu_{12}}p_{32}^{\nu_2}\left[2p_{21}^2(J_3^{\nu_1} + p_{21}^{\nu_1}J_3) - p_{21}^{\nu_1}J_2(p_{31})\right] \quad (6.61)$$
$$p_{32}^{\mu_3}C_{1\mu_{123}}^{\text{finite}} = \frac{1}{2}\varepsilon_{\mu_{12}\nu_1\nu_2}p_{21}^{\nu_2}[-2p_{32}^2J_3^{\nu_1} - p_{31}^{\nu_1}J_2(p_{31})] \quad (6.62)$$

[2] Two terms like $p_a\varepsilon_b\nu_{123}p_{21}^{\nu_2}p_{31}^{\nu_3}J_3^{\nu_1}$ cancel due to triple contraction.

$$p_{31}^{\mu_1} C_{2\mu_{123}}^{\text{finite}} = \frac{1}{2}\varepsilon_{\mu_{23}\nu_{12}} p_{32}^{\nu_2}[2p_{31}^2 (J_3^{\nu_1} + p_{21}^{\nu_1} J_3) - p_{21}^{\nu_1} J_2 (p_{21})] \tag{6.63}$$
$$p_{21}^{\mu_2} C_{2\mu_{123}}^{\text{finite}} = \varepsilon_{\mu_{13}\nu_{12}}\{(p_{31}^{\nu_2} p_{21}^2 - p_{21}^{\nu_2} p_{31}^2) J_3^{\nu_1} + p_{21}^{\nu_1} p_{31}^{\nu_2}[2m^2 J_3 + i(4\pi)^{-2} + J_2(p_{32})]\} \tag{6.64}$$
$$p_{32}^{\mu_3} C_{2\mu_{123}}^{\text{finite}} = \frac{1}{2}\varepsilon_{\mu_{12}\nu_{12}}[2p_{31}^{\nu_2} p_{32}^2 J_3^{\nu_1} + p_{21}^{\nu_1} p_{31}^{\nu_2} J_2 (p_{21})] \tag{6.65}$$

$$p_{31}^{\mu_1} C_{3\mu_{123}}^{\text{finite}} = \frac{1}{2}\varepsilon_{\mu_{23}\nu_{12}} p_{21}^{\nu_2}[2p_{31}^2 J_3^{\nu_1} + p_{31}^{\nu_1} J_2(p_{32})] \tag{6.66}$$
$$p_{21}^{\mu_2} C_{3\mu_{123}}^{\text{finite}} = \frac{1}{2}\varepsilon_{\mu_{13}\nu_{12}} p_{31}^{\nu_2}[-2p_{21}^2 J_3^{\nu_1} - p_{21}^{\nu_1} J_2(p_{32})] \tag{6.67}$$
$$p_{32}^{\mu_3} C_{3\mu_{123}}^{\text{finite}} = \varepsilon_{\mu_{12}\nu_{12}}\{\left(p_{21}^{\nu_2} p_{31}^2 - p_{31}^{\nu_2} p_{21}^2\right) J_3^{\nu_1} - p_{21}^{\nu_1} p_{31}^{\nu_2}[2m^2 J_3 + i(4\pi)^{-2} + J_2(p_{32})]\}. \tag{6.68}$$

We have to sum contributions from the subamplitudes to complete finite-parts results. That requires the same resources discussed above, but only vector integrals remain, and again we use Eq. (6.55) to reduce these integrals to scalar ones. Terms proportional to the squared mass arise from a part of the common tensors and subamplitudes. They cancel in all vector-vertex contractions and combine into the expected finite functions for all axial-vertex contractions (6.38)-(6.43). Lastly, regardless of the specific amplitude, the additional term $i\,(4\pi)^{-2}$ arises when the contracted index μ_i matches the i-th version.

To complete the RAGFs analysis, we recall Eqs. (6.31)-(6.33). In the set of surface terms $S_{i\mu_{123}}$, the index μ_i appears only in the Levi-Civita tensor and not in $\Delta_{3\mu\nu}$. Hence, contracting other indexes leads to the expected differences (6.47)-(6.49). Regardless of the particular triangle amplitude, identifications are automatic whenever contractions with $S_{i\mu_{123}}$ consider the index μ_j with $i \neq j$. On the other hand, when the contracted index corresponds to the vertex that defines the version $(i = j)$, the contraction between $p_{31}^{\mu_1}$ and $S_{1\mu_{123}}$ does not produce the required index configuration since we do not find momenta contractions with surface terms required to identify AV functions. Thus, in parallel to the procedure for 2nd-order J-tensors, indexes are reorganized through the identity

$$\varepsilon_{\mu_1\mu_3\nu_1\nu_2}\Delta_{3\mu_2}^{\nu_1} - \varepsilon_{\mu_1\mu_2\nu_1\nu_2}\Delta_{3\mu_3}^{\nu_1} = \varepsilon_{\mu_2\mu_3\nu_1\nu_2}\Delta_{3\mu_1}^{\nu_1} + \varepsilon_{\mu_1\mu_2\mu_3\nu_1}\Delta_{3\nu_2}^{\nu_1} - \varepsilon_{\mu_1\mu_2\mu_3\nu_2}\Delta_{3\nu_1}^{\nu_1}. \tag{6.69}$$

After organizing the momenta by $p_{ij} = P_{ir} - P_{jr}$, these operations yield (6.70). Besides the expected contributions, note the presence of an additional term on the trace $\Delta_{3\nu}^{\nu}$ resembling what occurred for the finite part.

$$p_{31}^{\mu_1} S_{1\mu_{123}} = -2i\varepsilon_{\mu_{23}\nu_{12}}\left(p_{21}^{\nu_2} P_{12}^{\nu_3} + p_{32}^{\nu_2} P_{32}^{\nu_3}\right)\Delta_{3\nu_3}^{\nu_1} + 2i\varepsilon_{\mu_2\mu_3\nu_2\nu_3} p_{21}^{\nu_2} p_{31}^{\nu_3}\Delta_{3\nu_1}^{\nu_1} \tag{6.70}$$
$$p_{21}^{\mu_2} S_{1\mu_{123}} = -2i\varepsilon_{\mu_{13}\nu_{12}}(p_{32}^{\nu_2} P_{32}^{\nu_3} - p_{31}^{\nu_2} P_{31}^{\nu_3})\Delta_{3\nu_3}^{\nu_1} \tag{6.71}$$
$$p_{32}^{\mu_3} S_{1\mu_{123}} = -2i\varepsilon_{\mu_{12}\nu_{12}}\left(p_{31}^{\nu_2} P_{31}^{\nu_3} - p_{21}^{\nu_2} P_{21}^{\nu_3}\right)\Delta_{3\nu_3}^{\nu_1} \tag{6.72}$$

$$p_{31}^{\mu_1} S_{2\mu_{123}} = -2i\varepsilon_{\mu_{23}\nu_{12}}\left(p_{21}^{\nu_2} P_{12}^{\nu_3} + p_{32}^{\nu_2} P_{32}^{\nu_3}\right)\Delta_{3\nu_3}^{\nu_1} \tag{6.73}$$
$$p_{21}^{\mu_2} S_{2\mu_{123}} = -2i\varepsilon_{\mu_{13}\nu_{12}}\left(p_{32}^{\nu_2} P_{32}^{\nu_3} - p_{31}^{\nu_2} P_{31}^{\nu_3}\right)\Delta_{3\nu_3}^{\nu_1} + 2i\varepsilon_{\mu_1\mu_3\nu_2\nu_3} p_{21}^{\nu_2} p_{31}^{\nu_3}\Delta_{3\nu_1}^{\nu_1} \tag{6.74}$$
$$p_{32}^{\mu_3} S_{2\mu_{123}} = -2i\varepsilon_{\mu_{12}\nu_{12}}\left(p_{31}^{\nu_2} P_{31}^{\nu_3} - p_{21}^{\nu_2} P_{21}^{\nu_3}\right)\Delta_{3\nu_3}^{\nu_1} \tag{6.75}$$

$$p_{31}^{\mu_1} S_{3\mu_{123}} = -2i\varepsilon_{\mu_{23}\nu_{12}} \left(p_{21}^{\nu_2} P_{12}^{\nu_3} + p_{32}^{\nu_2} P_{32}^{\nu_3}\right) \Delta_{3\nu_3}^{\nu_1} \quad (6.76)$$
$$p_{21}^{\mu_2} S_{3\mu_{123}} = -2i\varepsilon_{\mu_{13}\nu_{12}} \left(p_{32}^{\nu_2} P_{32}^{\nu_3} - p_{31}^{\nu_2} P_{31}^{\nu_3}\right) \Delta_{3\nu_3}^{\nu_1} \quad (6.77)$$
$$p_{32}^{\mu_3} S_{3\mu_{123}} = -2i\varepsilon_{\mu_{12}\nu_{12}} \left(p_{31}^{\nu_2} P_{31}^{\nu_3} - p_{21}^{\nu_2} P_{21}^{\nu_3}\right) \Delta_{3\nu_3}^{\nu_1} - 2i\varepsilon_{\mu_1\mu_2\nu_2\nu_3} p_{21}^{\nu_2} p_{31}^{\nu_3} \Delta_{3\nu_1}^{\nu_1} \quad (6.78)$$

With these properties in hands, we establish RAGFs for the explicit $(T_{\mu_{123}}^{AVV})_1$, see (6.26) to illustrate how to proceed in any case. The axial contraction comes from reducing the common tensor in Eq. (6.60) plus the nonzero terms from subamplitude (6.25)

$$i\varepsilon_{\mu_{123}\nu_1} p_{31}^{\mu_1} (T^{VPP})^{\nu_1} = 2i\varepsilon_{\mu_{23}\nu_{12}} p_{31}^{\nu_2} \left\{ 2(p_{21}\cdot p_{32}) J_3^{\nu_1} + p_{21}^{\nu_1} [p_{31}^2 J_3 - J_2(p_{21}) - J_2(p_{32})] \right\}.$$

At this stage, we have when summing both contributions

$$\begin{aligned} p_{31}^{\mu_1} (T_{\mu_{123}}^{AVV})_1 &= p_{31}^{\mu_1} S_{1\mu_{123}} + 4i\varepsilon_{\mu_{23}\nu_{12}} p_{21}^{\nu_1} p_{31}^{\nu_2} [2m^2 J_3 + i\,(4\pi)^{-2}] \\ &\quad + 4i\varepsilon_{\mu_{23}\nu_{12}} \left[p_{31}^{\nu_2} (p_{21}\cdot p_{31}) - p_{21}^{\nu_2} p_{31}^2\right] J_3^{\nu_1} \\ &\quad + 2i\varepsilon_{\mu_{23}\nu_{12}} p_{21}^{\nu_1} p_{31}^{\nu_2} [p_{31}^2 J_3 + J_2(p_{32}) - J_2(p_{21})]. \end{aligned} \quad (6.79)$$

To find reductions in terms like the second row, we use (6.55) to identify the needed contraction and obtain a cancellation

$$p_{31}^{\mu_1} (T_{\mu_{123}}^{AVV})_1 = p_{31}^{\mu_1} S_{1\mu_{123}} + 4i\varepsilon_{\mu_{23}\nu_{12}} p_{21}^{\nu_1} p_{31}^{\nu_2} [2m^2 J_3 + i\,(4\pi)^{-2}]. \quad (6.80)$$

After contracting surface terms using (6.70) and identifying the PVV (6.38), we write

$$p_{31}^{\mu_1} (T_{\mu_{123}}^{AVV})_1 = T_{\mu_{32}}^{AV}(1,2) - T_{\mu_{23}}^{AV}(2,3) - 2mT_{\mu_{23}}^{PVV} + \underline{2i\varepsilon_{\mu_{23}\nu_{12}} p_{21}^{\nu_1} p_{31}^{\nu_2} [\Delta_{3\alpha}^{\alpha} + 2i\,(4\pi)^{-2}]}. \quad (6.81)$$

Similarly, RAGFs coming from vector vertices use (6.61)-(6.62) for the common tensor and identity (6.55). They imply the vanishing of finite parts, while the remaining parts correspond to AV differences:

$$p_{21}^{\mu_2} (T_{\mu_{123}}^{AVV})_1 = p_{21}^{\mu_2} S_{1\mu_{123}} = T_{\mu_{13}}^{AV}(1,3) - T_{\mu_{13}}^{AV}(2,3) \quad (6.82)$$
$$p_{32}^{\mu_3} (T_{\mu_{123}}^{AVV})_1 = p_{32}^{\mu_3} S_{1\mu_{123}} = T_{\mu_{12}}^{AV}(1,2) - T_{\mu_{12}}^{AV}(1,3). \quad (6.83)$$

This pattern repeats for the first version of the other amplitudes (VAV, VVA, and AAA). Whereas the contraction with first vertex exhibits the additional term, the other RAGFs are satisfied without conditions. The pattern changes to the second and third versions, for they show the violating term in the second and third vertex independent of its nature: axial or vector vertex.

Following the developed steps, equations below subsume all potentially offending terms, which emerge in momentum contractions where the version is defined. We adopt the notation to the routing differences $q_1 = p_{31}$, $q_2 = p_{21}$, and $q_3 = p_{32}$ to mark a convention for first, second, and third vertices. The notation has already appeared in Figure 2.1 for the general diagram. In addition, the symbol $\Gamma_{123} \equiv \Gamma_1\Gamma_2\Gamma_3$ is an abbreviation for all combinations of vertices $\Gamma_i \in \{A, V\}$ we are investigating.

$$\begin{aligned} q_1^{\mu_1}(T^{\Gamma_{123}}_{\mu_{123}})_1^{\text{viol}} &= +2i\varepsilon_{\mu_{23}\nu_{12}} q_2^{\nu_1} q_3^{\nu_2} [\Delta^{\alpha}_{3\alpha} + 2i\,(4\pi)^{-2}] \\ q_2^{\mu_2}(T^{\Gamma_{123}}_{\mu_{123}})_2^{\text{viol}} &= +2i\varepsilon_{\mu_{13}\nu_{12}} q_2^{\nu_1} q_3^{\nu_2} [\Delta^{\alpha}_{3\alpha} + 2i\,(4\pi)^{-2}] \\ q_3^{\mu_3}(T^{\Gamma_{123}}_{\mu_{123}})_3^{\text{viol}} &= -2i\varepsilon_{\mu_{12}\nu_{12}} q_2^{\nu_1} q_3^{\nu_2} [\Delta^{\alpha}_{3\alpha} + 2i\,(4\pi)^{-2}]. \end{aligned} \tag{6.84}$$

The other vertices (to each version) have their RAGFs identically satisfied. To visualize this violation pattern, we offer the schematic graph in Figure 6.2.

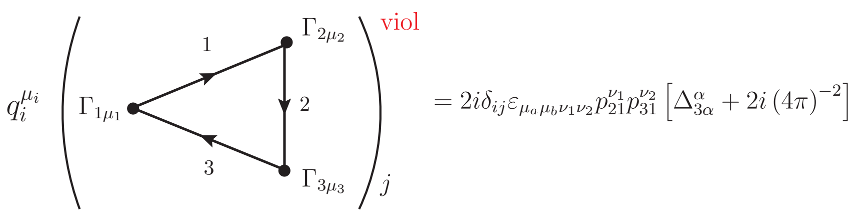

Figure 6.2: The violation factor of the RAGF established for the contraction with momenta $q_i^{\mu_1}$.

RAGFs are not automatic as they require further explorations regarding values accessible to surface terms, meaning they only apply under the constraint

$$\Delta^{\alpha}_{3\alpha} = -\frac{2i}{(4\pi)^2}. \tag{6.85}$$

From another perspective, if these relations apply identically, we could satisfy all Ward identities by nullifying surface terms (this works channel by channel). That is not the case because it requires conflicting interpretations of surface terms: zero for the momentum-space translational invariance and nonzero for the linearity of integration. Thence, these properties do not hold simultaneously. General tensor properties and the low-energy behavior of PVV-PAA and permutations show these conclusions are inescapable in Section (6.3). That is independent of any possible trace.

Once the RAGFs are clear, we would like to deepen the discussion about different versions of amplitudes. The investigated integrands are well-defined tensors and obey $(t^{\Gamma_{123}}_{\mu_{123}})_i = (t^{\Gamma_{123}}_{\mu_{123}})_j$. Even if we separate expressions in finite and divergent sectors without commitment to the divergences, after integration, the sampling of indexes makes the results of finite parts and tensor surface terms different. We highlight differences among the three main versions to elucidate this point:

$$(T^{\Gamma_{123}}_{\mu_{123}})_1 - (T^{\Gamma_{123}}_{\mu_{123}})_2 = +2i\varepsilon_{\mu_{123}\nu_1} p_{32}^{\nu_1} [\Delta^{\alpha}_{3\alpha} + 2i\,(4\pi)^{-2}], \tag{6.86}$$

$$(T^{\Gamma_{123}}_{\mu_{123}})_1 - (T^{\Gamma_{123}}_{\mu_{123}})_3 = -2i\varepsilon_{\mu_{123}\nu_1} p_{21}^{\nu_1} [\Delta^{\alpha}_{3\alpha} + 2i\,(4\pi)^{-2}], \tag{6.87}$$

$$(T^{\Gamma_{123}}_{\mu_{123}})_2 - (T^{\Gamma_{123}}_{\mu_{123}})_3 = -2i\varepsilon_{\mu_{123}\nu_1} p_{31}^{\nu_1} [\Delta^{\alpha}_{3\alpha} + 2i\,(4\pi)^{-2}]. \tag{6.88}$$

After subtracting two versions, we reorganized indexes to identify reductions of finite functions and recognize the same potentially violating term acknowledged in (6.84). At this point, we define the meaning of uniqueness adopted within this investigation: any possible form to compute the same expression returns the same result. Canceling the RHS of these equations would be required to achieve this property. That only happens when adopting the same prescription seen above $\Delta_{3\alpha}^{\alpha} = -2i\,(4\pi)^{-2}$. This notion of uniqueness implies that an amplitude does not depend on Dirac traces. Nevertheless, unlike in the two-dimensional context, the nonzero surface terms required by this notion allow dependence on ambiguous combinations of arbitrary internal momenta. In this sense, there is no unique expression in the external momenta.

The trace of six matrices is the unique place where the amplitude versions differ. Achieving traces different from those starting this argumentation is possible through other identities involving the chiral matrix, Eq. (2.5). Nonetheless, as detailed in Appendix (A.1), versions that are linear combinations of them arise. Observe the form

$$[T_{\mu_{123}}^{\Gamma_{123}}]_{i;j} = [(T_{\mu_{123}}^{\Gamma_{123}})_i + (T_{\mu_{123}}^{\Gamma_{123}})_j]/2, \tag{6.89}$$

which manifests potentially violating terms in RAGFs for both vertices Γ_i and Γ_j. The three independent combinations (setting i and j) are enough to reproduce any expressions achieved through the referred identities. That justifies taking $(T_{\mu_{123}}^{\Gamma_{123}})_i$ as the basic versions; moreover, *they have the maximum number of RAGFs identically satisfied*, see Section (6.3). For instance, the expression associated with the substitution

$$\gamma_*\gamma_{\mu_i\nu_i\mu_{i+1}} = i\varepsilon_{\mu_i\nu_i\mu_{i+1}\nu}\gamma^{\nu} + \gamma_*(g_{\nu_i\mu_{i+1}}\gamma_{\mu_i} - g_{\mu_i\mu_{i+1}}\gamma_{\nu_i} + g_{\mu_i\nu_i}\gamma_{\mu_{i+1}}) \tag{6.90}$$

has an integrand differing from $[T_{\mu_{123}}^{\Gamma_{123}}]_{i;i+1}$[3] in terms that have finite and identically vanishing integrals (6.14) and (6.22). Using this identity or combining traces of basic versions before integration makes expressions exhibit the same terms when integrated, divergent and finite parts. As another example, employing the identity $\gamma_*\gamma_{\mu_i} = \varepsilon_{\mu_i\nu_{123}}\gamma^{\nu_{123}}/3!$ expresses the trace through ten monomials. Even without some index configurations, the integrated expression coincides with the i-th version. That means the chiral matrix definition has no special role compared to other identities.

With these facts in mind, we define linear combinations that reproduce any possible expression with the building-block versions

$$[T_{\mu_{123}}^{\Gamma_{123}}]_{\{r_1r_2r_3\}} = \frac{1}{r_1+r_2+r_3}\sum_{i=1}^{3} r_i(T_{\mu_{123}}^{\Gamma_{123}})_i, \tag{6.91}$$

where $r_1+r_2+r_3 \neq 0$. They have equivalent integrands as it occurs for combinations (6.89). This general form compiles all involved arbitrariness, accounting for any choices regarding routings or Dirac traces. From this formula, assuming zero surface terms after the integration, we identify an infinity set of amplitudes that violate RAGFs by arbitrary amounts. That is useful for obtaining different violation values in the literature, e.g., [73].

[3]Note that when $i=3$ the notation means $[T_{\mu_{123}}^{\Gamma_{123}}]_{3,1}$, or $\gamma_*\gamma_{\mu_2\nu_2\mu_1}$ in the identity used.

We have shown how traces and surface terms interfere with the investigated tensors' linearity of integration and uniqueness. In the subsequent subsections, we demonstrate that these properties are unavoidable since conditions for RAGFs arise without explicit computations of the primary amplitudes.

6.2 A Low-Energy Theorem and its Relation with Ward Identities

This section proposes a structure depending only on external momenta to formulate a low-energy implication for a tensor representing three-point amplitudes. That does not mean we ignore the possible presence of ambiguous routing combinations because these terms can be transformed into linear covariant combinations of physical momenta. The structure is a general 3rd-order tensor having odd parity:

$$\begin{aligned} F_{\mu_{123}} &= \varepsilon_{\mu_{123}\nu}(q_2^{\nu}F_1 + q_3^{\nu}F_2) + \varepsilon_{\mu_{12}\nu_{12}}q_2^{\nu_1}q_3^{\nu_2}(q_{2\mu_3}G_1 + q_{3\mu_3}G_2) \\ &\quad + \varepsilon_{\mu_{13}\nu_{12}}q_2^{\nu_1}q_3^{\nu_2}\left(q_{2\mu_2}G_3 + q_{3\mu_2}G_4\right) + \varepsilon_{\mu_{23}\nu_{12}}q_2^{\nu_1}q_3^{\nu_2}(q_{2\mu_1}G_5 + q_{3\mu_1}G_6). \end{aligned} \tag{6.92}$$

That is a function of two variables: the incoming external momenta q_2 and q_3 associated with vertices Γ_2 and Γ_3. Conservation sets the relation $q_1 = q_2 + q_3$ with the outcoming momentum of the vertex Γ_1.

After performing the momenta contractions, one identifies the arrangements $q_i^{\mu_i}F_{\mu_{123}} = \varepsilon_{\mu_{kl}\nu_{12}}q_2^{\nu_1}q_3^{\nu_2}V_i$ with $k < l \neq i$. These operations lead to three functions written regarding form factors of the general tensor

$$V_1 = -F_1 + F_2 + (q_1 \cdot q_2)\, G_5 + (q_1 \cdot q_3)\, G_6, \tag{6.93}$$

$$V_2 = -F_2 + q_2^2 G_3 + (q_2 \cdot q_3)\, G_4, \tag{6.94}$$

$$V_3 = -F_1 + q_3^2 G_2 + (q_2 \cdot q_3)\, G_1. \tag{6.95}$$

At the kinematical point where all bilinears are zero $q_i \cdot q_j = 0$, if G_i are regular or at most discontinuous, we have the relations

$$V_1(0) = F_2 - F_1, \quad V_2(0) = -F_2, \quad V_3(0) = -F_1.$$

From the steps above, we derive the following equation among invariants

$$V_1(0) + V_2(0) - V_3(0) = 0. \tag{6.96}$$

This relation contains information about symmetries or their violations at the zero limit, even if no particular symmetry is needed for its deduction. That occurs because it represents a constraint over three-point structures arising in the RHS of proposed WIs.

To illustrate this resource, suppose that the axial contraction with the AVV connects to the amplitude coming from the pseudo-scalar density

$$\varepsilon_{\mu_{23}\nu_{12}}q_2^{\nu_1}q_3^{\nu_2}V_1(0) = -2mT_{\mu_{23}}^{PVV}(0) =: \varepsilon_{\mu_{23}\nu_{12}}q_2^{\nu_1}q_3^{\nu_2}\Omega_1^{PVV}(0)\,, \tag{6.97}$$

with the behavior (6.44) leading to the value for the first invariant $V_1(0) = 1/(2\pi)^2$. Since the constraint above prevents the simultaneous vanishing of both other invariants $V_2(0) = V_3(0) = 0$, at least one vector WI is violated. On the other hand, supposing that both vector WIs apply implies violating the axial one. That occurs because parameters defining the considered tensor and regularity require the existence of an additional term $V_1(0) = 1/(2\pi)^2 + \mathcal{A}$, the anomaly. Thus, $\mathcal{A} = -\Omega_1^{PVV}(0)$, relating a property of the finite amplitude and the symmetry content of a rank-3 amplitude. Satisfying the symmetry at this point does not guarantee invariance for all points; however, its violation at zero implies symmetry violation.

That is the starting point of the violation pattern in anomalous amplitudes. Numerical values presented above for invariants V_i at zero represent the preservation of corresponding WIs. Nevertheless, their co-occurrence implies a violation of the linear-algebra type solution (6.96). No tensor, independent of its origin, can connect to the PVV and simultaneously have vanishing contractions with momenta $q_2^{\mu_2}$ and $q_3^{\mu_3}$. Whenever an axial-vertex contraction is connected to an amplitude coming from the pseudo-scalar density (anomalously or not), there will be an anomaly in at least one of the vertices; the same conclusion stands for other diagrams. These facts are known; however, the form we raise is general. The low-energy theorem invoking vector WIs is only one of the solutions, as in Section (4.2) of [37]. The built equation is an exclusive and inviolable consequence of properties assumed to the 3rd-order tensor, and symmetry violations occur when the RHS terms of WIs do not behave accordingly.

The explicit computation of perturbative expressions corroborates these assertions. Moreover, the RAGFs furnish an exact connection among ultraviolet and infrared features of amplitudes, namely $\Omega_1^{PVV}(0) = 2i\Delta_{3\alpha}^{\alpha}$. That is the requirement for linearity seen after evaluating the RAGFs, and it will be derived in the next subsection. There, we assume the form $V_i = \Omega_i + \mathcal{A}_i$ and demonstrate the implication

$$\Omega_1(0) + \Omega_2(0) - \Omega_3(0) = (2\pi)^{-2}, \tag{6.98}$$

where we suppress superindexes in Ω_i coming from finite functions (e.g., PVV-PAA), see (6.100). The equation above holds even to classically non-conserved vector currents or amplitudes with three arbitrary masses running in the loop. Albeit rank-2 amplitudes of multiple masses are complicated functions of these masses, the relation at the point zero is ever the finite constant above.

Independently of divergent aspects, the last equation is incompatible with (6.96); therefore, characterizing violations for rank-3 triangles under the form (6.92). Hence, anomalous terms coming from different vertices $\mathcal{A}_i$ obey the general constraint

$$\mathcal{A}_1 + \mathcal{A}_2 - \mathcal{A}_3 = -(2\pi)^{-2}, \tag{6.99}$$

This equation shows that the value of axial anomaly is unique by preserving two vector

WIs. Likewise, any explicit tensor[4] having WIs violated by any quantity obeys this equation if $\mathcal{A}_i$ relates to finite amplitudes from Feynman's rules. The crossed channel of finite amplitudes brings a multiplicative factor 2 in the last couple of equations.

It is possible to anticipate restrictions over surface terms based on the general dependence that 3rd-order tensors have on such terms and preserving the independence and arbitrariness of internal momenta sums. That is achieved through the connection with AV functions via integration linearity. In the next section, this reasoning leads to the proposition $\Omega_1^{PVV}(0) = 2i\Delta_{3\alpha}^{\alpha}$ and Eq. (6.98).

6.3 RAGFs and Kinematical Behavior of Amplitudes

In Section (6.1), we performed explicit calculations related to different amplitude versions. When satisfying all RAGFs, a condition connecting the surface term with a finite contribution emerged in at least one of the relations (6.85). This condition appeared without explicitly calculating surface terms, inferring it from potentially violating terms. Furthermore, these additional terms arise in RAGFs associated with the vertex that defines the version (6.84). Here, we will show generality how the constraints based on linearity are obtained by carefully analyzing the most general tensor structure of 3pt-amplitudes without using any specific traces. The meaning of the basic version emerges as the one that automatically satisfies the most possible RAGFs but not all. Also, we will consider that when the contractions are done, a set of results is generated that can only be restricted by linearity for arbitrary and independent internal momenta. Such a condition shows how the finite amplitudes in the RHS of the RAGFs determine the surface terms.

From the explicit calculation, we can write the general equation for linearity as

$$q_i^{\mu_i} T_{\mu_{123}}^{\Gamma_{123}} = T_{i(-)\mu_{kl}}^{AV} + \varepsilon_{\mu_{kl}\nu_{12}} q_2^{\nu_1} q_3^{\nu_2} \Omega_i, \tag{6.100}$$

the ordering of indexes is always by $k < l \neq i$. The first term of the RHS is the differences (6.47)-(6.49). The second one has the invariants corresponding to the rank-2 amplitudes in RAGFs. Note that some are zero to vertices of specific diagrams. Expressing the three independent differences of AV functions in terms of P_{ij}, we have

$$T_{1(-)\mu_{23}}^{AV} = -2i\varepsilon_{\mu_{23}\nu_{12}} \left[-P_{21}^{\nu_2} P_{32}^{\nu_3} + P_{31}^{\nu_2} \left(P_{32}^{\nu_3} - P_{21}^{\nu_3} \right) + P_{32}^{\nu_2} P_{21}^{\nu_3} \right] \Delta_{3\nu_3}^{\nu_1} \tag{6.101}$$

$$T_{2(-)\mu_{13}}^{AV} = -2i\varepsilon_{\mu_{13}\nu_{12}} \left[+P_{21}^{\nu_2} \left(P_{31}^{\nu_3} - P_{32}^{\nu_3} \right) + P_{31}^{\nu_2} P_{32}^{\nu_3} - P_{32}^{\nu_2} P_{31}^{\nu_3} \right] \Delta_{3\nu_3}^{\nu_1} \tag{6.102}$$

$$T_{3(-)\mu_{12}}^{AV} = -2i\varepsilon_{\mu_{12}\nu_{12}} \left[-P_{21}^{\nu_2} P_{31}^{\nu_3} + P_{31}^{\nu_2} P_{21}^{\nu_3} + P_{32}^{\nu_2} \left(P_{31}^{\nu_3} - P_{21}^{\nu_3} \right) \right] \Delta_{3\nu_3}^{\nu_1}. \tag{6.103}$$

[4] This tensor can be obtained via regularization or not. See the approach of G. Scharf ([65]) in Section 5.1, using causal perturbation theory. The analogous to PVV is not computed until the very end. Instead, the authors study analogous differences between the contraction of AVV and the PVV without Feynman diagrams.

The notation $T^{AV}_{i(-)}$ is used to remember it came from the RAGF where we contracted with q^{μ_i} in the integrand. These equations preserve the arbitrary label for the internal lines and the value of the surface term and do not depend on the traces used because there is no ambiguity in expressing the trace of four Dirac matrices and a chiral one.

Due to the tensor integral of power counting zero e vector with power counting one, it must be expected from the expression to depend on surface term with physical as well ambiguous momenta. On the other hand, the routings present are not obliged to be written as external momenta, as we assumed in the previous section. The general tensor must consider that the perturbative amplitudes are a function of the six variables: the sums and differences of routings; the last ones are restricted by momentum conservation, notwithstanding the sums are arbitrary, reducing for five variables. In turn, with the sums, we generate the differences; thereby, the number of variables is three. Nevertheless, the summation of routings appears multiplied necessarily and only by surface terms.

Since central amplitudes are linear-diverging tensors, they have mass one and depend on the arbitrary momenta and surface terms, as q_i vectors are differences of the routings k_i (but not the opposite), we replace the former with the latter. Then using the combinations $P_{ij} = k_i + k_j$, the most general tensor of these variables under the stated conditions is

$$\begin{aligned} F^{\Delta}_{\mu_{123}} = &+\varepsilon_{\mu_{23}\nu_{12}} \left(a_{11}P_{21} + a_{12}P_{31} + a_{13}P_{32}\right)^{\nu_2} \Delta^{\nu_1}_{3\mu_1} \\ &+\varepsilon_{\mu_{13}\nu_{12}} \left(a_{21}P_{21} + a_{22}P_{31} + a_{23}P_{32}\right)^{\nu_2} \Delta^{\nu_1}_{3\mu_2} \\ &+\varepsilon_{\mu_{12}\nu_{12}} \left(a_{31}P_{21} + a_{32}P_{31} + a_{33}P_{32}\right)^{\nu_2} \Delta^{\nu_1}_{3\mu_3} \\ &+\varepsilon_{\mu_{123}\nu_1} \left(b_1P_{21} + b_2P_{31} + b_3P_{32}\right)^{\nu_2} \Delta^{\nu_1}_{3\nu_2}. \end{aligned} \tag{6.104}$$

Finite parts are handled separately. The a_{ij} and b_j are twelve arbitrary constants that summarize all the freedom of such tensor: Function of three variables of the diagram routings, rank, parity, and power counting. The j captures the P momenta in the order (P_{21}, P_{31}, P_{32}), and the index i links to the index μ_i associated with the vertex in the amplitudes $T^{\Gamma_{123}}_{\mu_{123}}$. Contracting (6.104) with the routing differences, for this tensor to be related to the AV tensors, we used the identity[5] $\varepsilon_{[\mu_1\mu_2\mu_3\nu_1}\Delta^{\nu_2}_{3\nu_2]} = 0$ to cast the tensor. That reduces, without losing information, the number of arbitrary parameters.

Now the question is: Performing the three contractions with the vertices momenta, is it possible to identify all of them with the two-point functions without additional conditions? That means they must be simultaneously valid for any value of the surface term. The answer is no, as we show that requiring two RAGF satisfied without conditions over surface term determines all coefficients a_{ij} and b_i. The other relation belongs to an incompatible solution for these coefficients. We will see as the finite amplitudes condition the satisfaction of all RAGFs.

[5] These structures have indices of surface terms contracted with the coefficient and the epsilon tensor and no trace of the surface term, by example, $q_2^{\mu_2} T^{AVV}_{\mu_{123}} = T^{AV}_{\mu_{13}}(1,3) - T^{AV}_{\mu_{13}}(2,3)$.

Beginning by contracting $F^{\Delta}_{\mu_{123}}$ with $q_1^{\mu_1} = p_{31}^{\mu_1}$, we have the expression

$$
\begin{aligned}
p_{31}^{\mu_1} F^{\Delta}_{\mu_{123}} = & +\varepsilon_{\mu_3\nu_{123}}[-(a_{21}+a_{23})P_{21}^{\nu_2}P_{32}^{\nu_3} + a_{22}P_{31}^{\nu_2}(P_{21}^{\nu_3} - P_{32}^{\nu_3})]\Delta^{\nu_1}_{3\mu_2} \\
& +\varepsilon_{\mu_2\nu_{123}}[-(a_{31}+a_{33})P_{21}^{\nu_2}P_{32}^{\nu_3} + a_{32}P_{31}^{\nu_2}(P_{21}^{\nu_3} - P_{32}^{\nu_3})]\Delta^{\nu_1}_{3\mu_3} \\
& +\varepsilon_{\mu_{23}\nu_{12}}[-(a_{11}-b_1)P_{21}^{\nu_2}P_{21}^{\nu_3} + (a_{13}-b_3)P_{32}^{\nu_2}P_{32}^{\nu_3}]\Delta^{\nu_1}_{3\nu_3} \\
& +\varepsilon_{\mu_{23}\nu_{12}}[+(a_{11}+b_3)P_{21}^{\nu_2}P_{32}^{\nu_3} - a_{12}P_{31}^{\nu_2}(P_{21}^{\nu_3} - P_{32}^{\nu_3})]\Delta^{\nu_1}_{3\nu_3} \\
& +\varepsilon_{\mu_{23}\nu_{12}}[-(a_{13}+b_1)P_{32}^{\nu_2}P_{21}^{\nu_3} + b_2(P_{21}^{\nu_2} - P_{32}^{\nu_2})P_{31}^{\nu_3}]\Delta^{\nu_1}_{3\nu_3}.
\end{aligned} \tag{6.105}
$$

From the first two rows, $\boldsymbol{a}_2 = (-a_{23}, 0, a_{23})$ and $\boldsymbol{a}_3 = (-a_{33}, 0, a_{33})$, the remaining compared with $T^{AV}_{1(-)\mu_{23}}$, we have $a_{11} + b_3 = 2i$; $a_{12} = -2i$; $a_{13} + b_1 = 2i$; $b_2 = 0$; and $b_3 = 2i - b_1$. In vector notation, the full solution is

$$
\begin{pmatrix} \boldsymbol{b} \\ \boldsymbol{a}_1 \\ \boldsymbol{a}_2 \\ \boldsymbol{a}_3 \end{pmatrix}_1 = \begin{pmatrix} b_1 & 0 & 2i - b_1 \\ b_1 & -2i & 2i - b_1 \\ -a_{23} & 0 & a_{23} \\ -a_{33} & 0 & a_{33} \end{pmatrix}. \tag{6.106}
$$

Note the reduction from twelve parameters to just three $\{a_{23}, a_{33}, b_1\}$ by requiring just one of the relations to be satisfied. Repeating the analysis to $q^{\mu_2} F^{\Delta}_{\mu_{123}}$ with $q_2^{\mu_2} = p_{21}^{\mu_2}$ and forming the system of linear equation by comparing with (6.48), follows the solution

$$
\begin{pmatrix} \boldsymbol{b} \\ \boldsymbol{a}_1 \\ \boldsymbol{a}_2 \\ \boldsymbol{a}_3 \end{pmatrix}_2 = \begin{pmatrix} 0 & b_2 & 2i - b_2 \\ 0 & -a_{13} & a_{13} \\ 2i & -b_2 & b_2 - 2i \\ 0 & -a_{33} & a_{33} \end{pmatrix}, \tag{6.107}
$$

for the RAGF in the second vertex. The conditions for $q_3^{\mu_3} F^{\Delta}_{\mu_{123}} = T^{AV}_{3(-)\mu_{12}}$ with $q_3^{\mu_3} = p_{32}^{\mu_3}$, follows that the solution to the automatic satisfaction of the RAGF is

$$
\begin{pmatrix} \boldsymbol{b} \\ \boldsymbol{a}_1 \\ \boldsymbol{a}_2 \\ \boldsymbol{a}_3 \end{pmatrix}_3 = \begin{pmatrix} b_1 & 2i - b_1 & 0 \\ a_{11} & -a_{11} & 0 \\ a_{21} & -a_{21} & 0 \\ b_1 & 2i - b_1 & -2i \end{pmatrix}. \tag{6.108}
$$

The intersection of (6.106) and (6.107), the ones that automatically satisfy the RAGFs coming from the contraction with $q_1^{\mu_1}$ and $q_2^{\mu_2}$, leads to a unique solution with $b_1 = 0$, $b_2 = 0$, $b_3 = 2i$, and all the other coefficients are also determined. Replacing in the tensor,

$$
(F^{\Delta}_{\mu_{123}})_{12} = -2i[\varepsilon_{\mu_{23}\nu_{12}}(P_{32}^{\nu_2} - P_{31}^{\nu_2})\Delta^{\nu_1}_{3\mu_1} + \varepsilon_{\mu_{13}\nu_{12}}(P_{21}^{\nu_2} - P_{32}^{\nu_2})\Delta^{\nu_1}_{3\mu_2} + \varepsilon_{\mu_{123}\nu_1}P_{32}^{\nu_2}\Delta^{\nu_1}_{3\nu_2}], \tag{6.109}
$$

where $p_{ij} = P_{il} - P_{jl}$. Sub-index ij in $(F^{\Delta}_{\mu_{123}})_{ij}$ stands for the vertices where the RAGFs are satisfied without further assumptions. As the relations above depend on three parameters and are compatible in pairs, the coefficients solution is unique once one pair of two RAGFs is determined. Complementary contraction is always an incompatible solution; coefficients are different for each solution $(F^{\Delta}_{\mu_{123}})_{ij}$. The pair solutions for at most two

RAGFs identically satisfied correspond to the amplitudes versions computed explicitly. See (6.31)-(6.33), namely

$$(F^{\Delta}_{\mu_{123}})_{23} = S_{1\mu_{123}}; \quad (F^{\Delta}_{\mu_{123}})_{13} = S_{2\mu_{123}}; \quad (F^{\Delta}_{\mu_{123}})_{12} = S_{3\mu_{123}},$$

the trace of the surface term separates from the difference of AV in one of the contractions.

Consequences: With this derivation in hand, we draw a similar conclusion to the one stated in the Subsection (6.2). The value at zero of PVV had consequences over symmetries. Here this amplitude will establish a connection between linearity in the RAGFs and the low-energy behavior of the same PVV.

For this, we have to read this result in light of form factors in (6.92), taken as the *finite parts*. Choosing the solution satisfying the RAGFs in vertices two and three

$$T^{\Gamma_{123}}_{\mu_{123}} = F_{\mu_{123}} + (F^{\Delta}_{\mu_{123}})_{23} = F_{\mu_{123}} + S_{1\mu_{123}}, \tag{6.110}$$

to any vertices combination. Let Ω_i represent the finite scalar invariants of 2nd-order tensors from RAGFs; writing the equations of the hypothesis of satisfaction, (6.100),

$$\begin{aligned} q_1^{\mu_1} T^{\Gamma_{123}}_{\mu_{123}} - T^{AV}_{1(-)\mu_{23}} &= \varepsilon_{\mu_{23}\nu_{12}} q_2^{\nu_1} q_3^{\nu_2} \Omega_1 = \varepsilon_{\mu_{23}\nu_{12}} q_2^{\nu_1} q_3^{\nu_2} (V_1 + 2i\Delta^{\alpha}_{3\alpha}) & (6.111)\\ q_2^{\mu_2} T^{\Gamma_{123}}_{\mu_{123}} - T^{AV}_{2(-)\mu_{13}} &= \varepsilon_{\mu_{13}\nu_{12}} q_2^{\nu_1} q_3^{\nu_2} \Omega_2 = \varepsilon_{\mu_{13}\nu_{12}} q_2^{\nu_1} q_3^{\nu_2} V_2 & (6.112)\\ q_3^{\mu_3} T^{\Gamma_{123}}_{\mu_{123}} - T^{AV}_{3(-)\mu_{12}} &= \varepsilon_{\mu_{12}\nu_{12}} q_2^{\nu_1} q_3^{\nu_2} \Omega_3 = \varepsilon_{\mu_{12}\nu_{12}} q_2^{\nu_1} q_3^{\nu_2} V_3. & (6.113) \end{aligned}$$

Using the previous results, we see that the trace of the surface term must be put together with the finite part of the first contraction due to the Eq. (6.70),

$$q_1^{\mu_1} S_{1\mu_{123}} = T^{AV}_{1(-)\mu_{23}} + \varepsilon_{\mu_{23}\nu_{23}} q_2^{\nu_2} q_3^{\nu_3} (2i\Delta^{\nu_1}_{3\nu_1}). \tag{6.114}$$

We wrote the AV structures on LHS to focus on the non-trivial part of the relations. We get the final condition: $V_1 + 2i\Delta^{\alpha}_{3\alpha} = \Omega_1$; $V_2 = \Omega_2$; and $V_3 = \Omega_3$. Observing the formulas

$$\begin{aligned} V_1 &= -F_1 + F_2 + q_2^2 G_5 + q_3^2 G_6 + (q_2 \cdot q_3)(G_5 + G_6) & (6.115)\\ V_2 &= -F_2 + q_2^2 G_3 + (q_2 \cdot q_3) G_4 & (6.116)\\ V_3 &= -F_1 + q_3^2 G_2 + (q_2 \cdot q_3) G_1. & (6.117) \end{aligned}$$

It is possible to eliminate the F_i form factors to reach at

$$\begin{aligned} 2i\Delta^{\alpha}_{3\alpha} + \Omega_3 - \Omega_2 - \Omega_1 &= -q_2^2 (G_3 + G_5) + q_3^2 (G_2 - G_6) & (6.118)\\ &\quad + (q_2 \cdot q_3)(G_1 - G_4 - G_5 - G_6). \end{aligned}$$

Under the condition that G_i functions are regular at zero[6], follows

$$2i\Delta^{\alpha}_{3\alpha} = \Omega_1(0) + \Omega_2(0) - \Omega_3(0). \tag{6.119}$$

[6] The functions $Z^{(0)}_{nm}$, $Z^{(-1)}_{nm}$, $Z^{(0)}_{n}$ that comprise the finite part of any of these amplitudes do not have kinematical singularities at the point $q_i \cdot q_j = 0$.

The equation is true irrespective of the choice of which relation is satisfied without restriction. Suppose one starts with a version with $S_{2\mu_{123}}$ that satisfies the RAGFs in the first and third vertex. To this tensor, the term $\Delta^{\alpha}_{3\alpha}$ will appear in $q_2^{\mu_2} S_{2\mu_{123}}$, see Eq. (6.74). From $V_1 = \Omega_1$ and $V_3 = \Omega_3$, and trading the F_1 and F_2 by G_i plus finite functions, again in zero, we retrieve the previous result. That is a proper relation between a low-energy property and surface terms stated in the former section in (6.2). The hypotheses were a tensor with two RAGFs satisfied without restriction, connected to AVdifferences and PVV/PAA-like amplitudes. From that, the zero value of rank-2 amplitudes bound the third RAGF. It is always possible to achieve these hypotheses in explicit computations.

When assessing $\Omega_i(0)$, see (6.44), $\Omega^{PVV} = \Omega^{VPV} = -\Omega^{VVP} = (2\pi)^{-2}$, we find out

$$\Omega_1(0) + \Omega_2(0) - \Omega_3(0) = (2\pi)^{-2}, \tag{6.120}$$

Notice that for the AVV, VAV, and VVA, two of the Ω_i are zero to each amplitude, which means the result above represents three situations. The same happens to the AAA triangle. In this case, the three contractions of the same amplitude relate to PAA, APA, and AAP. Combining the constants cast in Eq. (6.45), we have

$$\Omega_1^{PAA}(0) + \Omega_2^{APA}(0) - \Omega_3^{AAP}(0) = (2\pi)^{-2}. \tag{6.121}$$

Since the AV differences depend only on the contractions with the momenta, but the correlators with the P density are distinct, it could be that distinct diagrams would require different numerical values to the surface term, despite that one always find

$$\text{RAGF} \Leftrightarrow 2\Delta^{\alpha}_{3\alpha} = -i(2\pi)^{-2}. \tag{6.122}$$

Constraint remains for amplitudes where three distinct masses run in the internal lines.

Let us consider an example of this scenario for the AVV. The propagator's indexes now account for the masses too, $S(a) = (\not{K}_a - m_a)^{-1}$. Using the standard identity $[\not{p}_{ij} = S^{-1}(i) - S^{-1}(j) + (m_i - m_j)]$ to derive the RAGFs expressed in Eqs. (6.34), the terms associated with the three-point functions are now

$$-(m_1 + m_3)T^{PVV}_{\mu_{23}}; \quad (m_2 - m_1)T^{ASV}_{\mu_{13}}; \text{ and } \quad (m_3 - m_2)T^{AVS}_{\mu_{12}},$$

coming from verteces Γ_1, Γ_2, and Γ_3 respectively. In this scenario, vector currents are not classically conserved. However, ASV, AVS, and PVV will not comply the Eq. (6.96), and their relations are identical to the ones (6.98). For the three-point rank-2 amplitudes,

$$\begin{aligned}
T^{PVV}_{\mu_{23}} &= \varepsilon_{\mu_{23}\nu_{12}} p_{21}^{\nu_1} p_{32}^{\nu_2} [(m_1 - m_2) Z^{(-1)}_{10} + (m_1 - m_3) Z^{(-1)}_{01} - m_1 Z^{(-1)}_{00}] \\
T^{ASV}_{\mu_{13}} &= \varepsilon_{\mu_{13}\nu_{12}} p_{21}^{\nu_1} p_{32}^{\nu_2} [(m_1 + m_2) Z^{(-1)}_{10} + (m_1 + m_3) Z^{(-1)}_{01} - m_1 Z^{(-1)}_{00}] \\
T^{AVS}_{\mu_{12}} &= \varepsilon_{\mu_{12}\nu_{12}} p_{21}^{\nu_1} p_{32}^{\nu_2} [(m_2 - m_1) Z^{(-1)}_{10} - (m_3 + m_1) Z^{(-1)}_{01} + m_1 Z^{(-1)}_{00}],
\end{aligned}$$

it is possible to identify the form factor through the relation

$$\begin{aligned} \varepsilon_{\mu_{23}\nu_{12}} p_{21}^{\nu_1} p_{32}^{\nu_2} \Omega_1^{PVV} &= -(m_1+m_3) T_{\mu_{23}}^{PVV} \\ \varepsilon_{\mu_{13}\nu_{12}} p_{21}^{\nu_1} p_{32}^{\nu_2} \Omega_2^{ASV} &= +(m_2-m_1) T_{\mu_{13}}^{ASV} \\ \varepsilon_{\mu_{12}\nu_{12}} p_{21}^{\nu_1} p_{32}^{\nu_2} \Omega_3^{AVS} &= +(m_3-m_2) T_{\mu_{12}}^{AVS}. \end{aligned}$$

By combining them as done in the other cases, we have

$$\Omega_1^{PVV} + \Omega_2^{ASV} - \Omega_3^{AVS} = 2(2\pi)^{-2}[(m_1^2-m_2^2)Z_{10}^{(-1)} + (m_1^2-m_3^2)Z_{01}^{(-1)} - m_1^2 Z_{00}^{(-1)}].$$

Since in the definition, the Q polynomial for distinct masses[7], hence the relation is

$$\left[(m_1^2-m_2^2)Z_{10}^{(-1)} + (m_1^2-m_3^2)Z_{01}^{(-1)} - m_1^2 Z_{00}^{(-1)}\right]_{q_i\cdot q_j=0} = 1/2. \tag{6.123}$$

Finally, in the limit studied follows $(\Omega_1^{PVV} + \Omega_2^{ASV} - \Omega_3^{AVS})|_0 = (2\pi)^{-2}$. The integrals with various masses are laborious, but integrating all these functions explicitly in the limit under consideration follows the result.

The kinematical limits of all rank-2 amplitudes are incompatible with the satisfaction of all Ward identities since they ask for additional constants to be compatible with the tensor structure of rank-3 amplitudes, as already established in the 2D. Although these claims are implicit in the discussion of these tensors, often, the focus is the regularization properties. In this way, when we write the internal momenta as covariant combinations (non-covariant combinations amount to Lorentz violations), we must have

$$\begin{aligned} & \left[V_1^{AVV}(m_1,m_2,m_3) - V_2^{AVV}(m_1,m_2,m_3) - V_3^{AVV}(m_1,m_2,m_3)\right](0) \\ = \; & (\Omega_1^{PVV} + \Omega_2^{ASV} - \Omega_3^{AVS})|_0 + \left(\mathcal{A}_1^{AVV} - \mathcal{A}_2^{AVV} - \mathcal{A}_3^{AVV}\right) = 0. \end{aligned}$$

That means we can not simultaneously make all $\mathcal{A}_i = 0$ by reasons unrelated to divergences. Utilizing this equation to study the symmetries, we have the scenario. If eventually is not found symmetry violation in that point, it does not mean they could not be in other points. However, finding a problem in zero implies a violation.

6.4 General Parameters to the Violations[8]

Summarizing the last sections: (i) Integration linearity holds if and only if the surface terms are nonzero (6.122). Simultaneously the results are independent of Dirac traces

[7] To arbitrary masses, the Feynman polynomial for the function involved in this derivation reads

$$Q = q_1^2 x_1(1-x_1) + q_2^2 x_2(1-x_2) - 2q_1\cdot q_2 x_1 x_2 + \left(m_1^2-m_2^2\right)x_1 + \left(m_1^2-m_3^2\right)x_2 - m_1^2.$$

And the function is given by

$$Z_{rs}^{(-1)} = \int_0^1 \mathrm{d}x_1 \int_0^{1-x_1} \mathrm{d}x_2 \frac{x_1^r x_2^s}{Q\left(q_i^2, m_1^2, m_2^2, m_3^2\right)}.$$

In the kinematical point the the polynomial assumes the form $Q(0) = \left(m_1^2-m_2^2\right)x_1 + \left(m_1^2-m_3^2\right)x_2 - m_1^2$.

[8] Throughout this section, we factored out three-point rank-two finite amplitudes from the discussion.

for the same value, which saves linearity. (ii) Since some surface-terms coefficients are ambiguous combinations of the routings, we must make choices for them. iii) From (ii), if a procedure nullifies that terms, the linearity is violated by $\sim \pm(2\pi)^{-2}$; see these results in (6.84). There is an equilibrium between routing and trace ambiguities organized by the surface term's value. Let us see the parameter space for this competition.

Combining versions that save the most RAGFs with no condition on the surface term[10],

$$[t^{\Gamma_{123}}_{\mu_{123}}]_{\{r_1 r_2 r_3\}} = \frac{1}{R}[r_1(t^{\Gamma_{123}}_{\mu_{123}})_1 + r_2(t^{\Gamma_{123}}_{\mu_{123}})_2 + r_3(t^{\Gamma_{123}}_{\mu_{123}})_3], \tag{6.124}$$

where $R = r_1 + r_2 + r_3 \neq 0$. As discussed at the end of Section (6.1), they are identical before integration. However, when $\Delta_{3\mu\nu} = 0$, they become an infinity set of different tensors. In particular, they reproduce any tensor through our strategy using any identity for the chiral matrix. For zero surface terms, their symmetry violations are in the i-th vertex and get a factor of r_i/R, satisfying the equation determined to its anomalies (6.99) due to kinematic properties of finite amplitudes.

If we have considered the surface term as an arbitrary parameter given by a constant c_1, equal to one for the satisfaction of RAGFs or zero for the momentum-space translational invariance. Parametrizing internal lines by choosing any of the sums $P_{ij} = k_i + k_j$, we have $P_{31} = c_2 q_2 + c_3 q_3 \to P_{21} = c_2 q_2 + (c_3 - 1)\, q_3$, and $P_{32} = (c_2 + 1)\, q_2 + c_3 q_3$, with

$$2\Delta_{3\mu_{12}} = -ic_1 (4\pi)^{-2} g_{\mu_{12}}, \tag{6.125}$$

the AV functions, see Section (6.1), Eqs. (6.47)-(6.49), are written as function of c_1, c_2,and c_3, and also violations of RAGFs, Eqs. (6.84).Those parameters express any possible values to the contractions of basic versions. With the caveat that only in the contraction of i-th version with $q_i^{\mu_i}$, both the two-point functions and the linearity-breaking term contributes. For this version, the contraction with q_j, $j \neq i$, only AV's contribute.

Modulus finite amplitudes, the combination defined in Eq. (6.124) has the properties

$$\begin{aligned} q_1^{\mu_1}[T^{\Gamma_{123}}_{\mu_{123}}]_{\{r_1 r_2 r_3\}} &= \varepsilon_{\mu_{23}\nu_{12}} q_2^{\nu_1} q_3^{\nu_2} \{[4R\,(2\pi)^2]^{-1}[4r_1\,(c_1 - 1) + Rc_1\,(c_3 - c_2 - 2)]\} \\ &= \varepsilon_{\mu_{23}\nu_{12}} q_2^{\nu_1} q_3^{\nu_2} \mathcal{A}_1 \end{aligned} \tag{6.126}$$

$$\begin{aligned} q_2^{\mu_2}[T^{\Gamma_{123}}_{\mu_{123}}]_{\{r_1 r_2 r_3\}} &= \varepsilon_{\mu_{13}\nu_{12}} q_2^{\nu_1} q_3^{\nu_2} \{[4R\,(2\pi)^2]^{-1}[4r_2\,(c_1 - 1) - Rc_1\,(c_3 + 1)]\} \\ &= \varepsilon_{\mu_{13}\nu_{12}} q_2^{\nu_1} q_3^{\nu_2} \mathcal{A}_2 \end{aligned} \tag{6.127}$$

$$\begin{aligned} q_3^{\mu_3}[T^{\Gamma_{123}}_{\mu_{123}}]_{\{r_1 r_2 r_3\}} &= \varepsilon_{\mu_{12}\nu_{12}} q_2^{\nu_1} q_3^{\nu_2} \{[4R\,(2\pi)^2]^{-1}[4r_3\,(1 - c_1) - Rc_1\,(c_2 - 1)]\} \\ &= \varepsilon_{\mu_{12}\nu_{12}} q_2^{\nu_1} q_3^{\nu_2} \mathcal{A}_3. \end{aligned} \tag{6.128}$$

Parameters combination implies $\mathcal{A}_1 = \mathcal{A}_3 - \mathcal{A}_2 - (2\pi)^{-2}$, decreasing the number of independent variables for two. So when we have numerical amounts of two violations, no matter the path leading them, the third arises without ambiguity. Derived in the previous sections based only on finite functions and when the internal momenta as covariant functions of external ones.

[10] This claim is independent of explicit computations performed in the previous section.

If $c_1 = 1$, there is no dependence in r_i, we have the unique solution that satisfies linearity but is not a function of the external momenta. If $c_1 = 0$, there will be no dependence in c_2 and c_3, and the tensors are functions of the external momenta but not unique. These parameters are the full range of possibilities. The crossed diagrams add more parameters to the discussion but have the same behavior: linearity break, ambiguities, and symmetries violation. The crucial factor is the kinematic behavior of finite functions that code amplitudes for pseudo-scalar density. In the massless limit, this aspect falls in the values to the residue of poles of form factors, which are regular in the massive case. Breaking linearity has a function in divergent amplitudes that corroborates with the low-energy value of finite amplitude PV^n in dimension $d = 2n$. If it does not occur, shifts in the integration variable are allowed by removing surface terms. Hence the AV functions through (6.96) relate the V_i, and the finite amplitudes would have to be zero at the point where the bilinears vanish.

The situation happens when integrating an identically zero tensor; it is obtained a nonzero result. Take the identity for the integrand of the Feynman integral $\bar{J}_{3\mu\nu}$,

$$[K_1^{\mu_5}(\varepsilon_{\mu_{5123}} K_{1\mu_4} + \varepsilon_{\mu_{4512}} K_{1\mu_3} + \varepsilon_{\mu_{3451}} K_{1\mu_2} + \varepsilon_{\mu_{2345}} K_{1\mu_1}) + \varepsilon_{\mu_{1234}} m^2] \frac{1}{D_{123}} = -\varepsilon_{\mu_{1234}} \frac{1}{D_{23}}$$

the equation comes from $\varepsilon_{[\mu_{1234}} K_{1\mu_5]} = 0$, multiplying by $K_1^{\mu_5}/D_{123}$, and using $K_1^2 = D_1 + m^2$. When integrated, the identity is only valid for just one surface-term value. The critical step arises when we separate the finite and divergent parts, explicitly

$$\begin{aligned}
\bar{J}_2\,(2,3) &= J_2\,(p_{32}) + I_{\log} \\
\bar{J}_3\,(1,2,3) &= J_3\,(p_{21}, p_{31}) \\
\bar{J}_{3\mu\nu}\,(1,2,3) &= J_{3\mu\nu}\,(p_{21}, p_{31}) + (\Delta_{3\mu\nu} + g_{\mu\nu} I_{\log})\,/4, \\
J_{3\alpha}^{\alpha}\,(p_{21}, p_{31}) &= m^2 J_3\,(p_{21}, p_{31}) + J_2\,(p_{32}) + i[2\,(4\pi)^2]^{-1}.
\end{aligned}$$

This step is performed using $\varepsilon_{[\mu_{1235}} \Delta^{\mu_5}_{3\mu_5]} = 0$ and $\varepsilon_{[\mu_{1235}} J^{\mu_5}_{3\mu_5]} = 0$. Then, the initial identity gets transformed in a condition to the linearity breaking $\varepsilon_{\mu_{1234}}[\Delta^{\mu_5}_{3\mu_5} + 2i/\,(4\pi)^2] = 0$. Now, the identity for the surface term is consistent to any value, constrained only by $\Delta_{3\mu\nu} = [g^{\alpha\beta}\Delta_{3\alpha\beta}]/4$, however the same is not true to the bare integral $\bar{J}_{3\mu\nu}$. The identity is respected if and only if $\Delta^{\alpha}_{3\alpha} = -2i/(4\pi)^2$, derived without explicitly manipulating divergent integrals. As a part of the Feynman integrals, the satisfaction of the Schouten identity to any surface-term value is not enough to make it valid for the entire integrals. We used the results of Section (3.3.2).

We must mention that the violation by an evanescent term that occurs in dimensional methods[12] does not affect linearity breaking. The finite value we demonstrate to be necessary is not a function of the dimension, and it corresponds to the low-energy limit of the integral J_3. No limiting process can change that value and, if not adopted, violates the linearity and uniqueness of these perturbative amplitudes.

[12] See [74][75] for this type of view.

Chapter 7

Gravitational Perturbative Amplitudes

The quantization of fermionic fields is according to the canonical rules of Quantum Field Theory. To introduce these fields in a curved space, we associate to space-time a Lorentz manifold, in which each point has a plane space tangent to it. The connection between the two spaces is through vielbein fields defined by

$$g_{\mu\nu}(x) = \eta_{ab} e^a_\mu(x) e^b_\nu(x) \tag{7.1}$$

$$\eta_{ab} = \mathrm{diag}(1, -1-1-1) \tag{7.2}$$

$$e^\mu_a e^b_\mu = \delta^b_a; \qquad e^\mu_a e^a_\nu = \delta^\mu_\nu. \tag{7.3}$$

These fields work in such a way as to transform the coordinate basis into an orthonormal basis. Through that basis, it is possible to introduce locally the Clifford algebra whose representations the spinor field can be defined. The algebra acquires a local character,

$$\gamma^\mu(x) := e^\mu_a(x)\gamma^a \tag{7.4}$$

$$\{\gamma_\mu(x), \gamma_\nu(x)\} = 2g_{\mu\nu}(x) \tag{7.5}$$

$$\{\gamma^a, \gamma^b\} = 2\eta^{ab} \tag{7.6}$$

$$\gamma_{[ab]} = \frac{1}{2}[\gamma_a, \gamma_b]; \tag{7.7}$$

the last term $\gamma_{[ab]}/2$ corresponds to the spinor generator to the Lorentz group.

In this way, we will introduce a covariant generalization of the equations formulated in flat spacetime to introduce fermions coupled to a spacetime with arbitrary metrics. The action S must be invariant by Lorentz transformations and general transformations of coordinates. We start by considering the flat-space real Lagrangian

$$\mathcal{L} = \frac{1}{2}[i\bar{\psi}\gamma^\mu\partial_\mu\psi - i\left(\partial_\mu\bar{\psi}\right)\gamma^\mu\psi] = \frac{1}{2}[i\bar{\psi}\gamma^\mu\partial_\mu\psi + \left(i\bar{\psi}\gamma^\mu\partial_\mu\psi\right)^\dagger], \tag{7.8}$$

and replace the covariant for the flat-spacetime metric in coordinate and orthonormal frame ∂_μ (cartesian one) by the spinor covariant derivative in an arbitrary coordinate frame (but still flat geometry), we have

$$\nabla_\mu \psi := \partial_\mu \psi + \frac{1}{4}\omega_\mu{}^{ab}\gamma_{[ab]}\psi; \qquad \text{and} \qquad \nabla_\mu \bar{\psi} := \partial_\mu \bar{\psi} - \frac{1}{4}\bar{\psi}\omega_\mu{}^{ab}\gamma_{[ab]}. \tag{7.9}$$

We used $\gamma_0\gamma^\dagger_{[ab]}\gamma_0 = -\gamma_{[ab]}$ in defining last equation; $\omega_\mu{}^{ab}$ are components of metric-compatible spin connection

$$\omega^{ab}{}_\mu = \omega^a{}_{c\mu}\eta^{cb} \tag{7.10}$$

$$\omega^a{}_{c\mu} = e^a{}_\nu \partial_\mu e_c{}^\nu + e_c{}^\nu e^a{}_\lambda \Gamma^\lambda_{\mu\nu}, \tag{7.11}$$

being $\Gamma^\lambda_{\mu\nu}$ the components of the connection in the coordinate basis. Then, we allow the metric to correspond to a curved background geometry, and thereby, the fermion propagation will be classically given by

$$S = \int_{\mathcal{M}} \mathrm{d}^2 x e\left(x\right) \frac{i}{2}\left[\bar{\psi}\gamma^\mu \nabla_\mu \psi - \left(\nabla_\mu \bar{\psi}\right)\gamma^\mu \psi\right], \tag{7.12}$$

where we introduced the scalar density $e\left(x\right) = \sqrt{\left|g\left(x\right)\right|}$ in the volume 2-form $\mathrm{d}V = \sqrt{\left|g\left(x\right)\right|}\mathrm{d}x_1\mathrm{d}x_2$, $g\left(x\right) = \det g_{\mu\nu}$, and modulus is due to the Lorentz signature.

The extremization of action yields the motion's equations: $\nabla_\mu \psi = 0$ and $\nabla_\mu \bar{\psi} = 0$. Additionally, in 2D, the term coupling to the spin-connection drops out from the action

$$\frac{1}{4}\omega_\mu{}^{ab} e^\mu_c \bar{\psi}\{\gamma^c, \gamma_{[ab]}\}\psi = 0, \tag{7.13}$$

due to the in this dimension $\gamma_{[ab]} = -i\gamma_* \varepsilon_{ab}$ and $\{\gamma^c, \gamma_*\} = 0$. Therefore, we adopt Weyl fermions henceforth, and the action simplifies to

$$S = \frac{i}{2}\int_{\mathcal{M}} \mathrm{d}^2 x e\left(x\right) e^\mu_a [\bar{\psi}\gamma^a \overleftrightarrow{\partial}_\mu P_\pm \psi], \tag{7.14}$$

where the chiral projectors are given by $P_\pm = \left(1 \pm \gamma_*\right)/2$, being that the chiral matrix (2.3) is $\gamma_* = \varepsilon_{ab}\gamma^a\gamma^b$ and the 'flat' Levi-Cevita symbol is normalized by $\varepsilon^{01} = 1$ (it is a tensor density with world indices).

The gravitational field appears only as a background field, without being necessarily quantized and without associated dynamics. Then, we consider the approximation expanding in powers of $h_{\mu\nu}$ around the Minkowski metric

$$g_{\mu\nu} = \eta_{\mu\nu} + \kappa h_{\mu\nu} \tag{7.15}$$

$$g^{\mu\nu} = \eta^{\mu\nu} - \kappa h_{\mu\nu} + \mathcal{O}\left(\kappa^2\right). \tag{7.16}$$

$$e^a_\mu = \delta^a_\mu + \frac{1}{2}\kappa h^a_\mu; \qquad e^\mu_a = \delta^\mu_a - \frac{1}{2}\kappa h^\mu_a; \qquad e = 1 + \frac{1}{2}\kappa h^\mu_\mu. \tag{7.17}$$

We may expand in $e\left(x\right)$ and inverse vielbein e^μ_a independently; in this way, we would get

$$\frac{i}{2}e\left(x\right) e^\mu_a [\bar{\psi}\gamma^a \overleftrightarrow{\partial}_\mu \psi] = \frac{i}{2}\bar{\psi}\gamma^\mu \overleftrightarrow{\partial}_\mu \psi - \frac{1}{2}h^{\mu\nu}\left[\frac{i}{4}\bar{\psi}\gamma_{(\nu}\overleftrightarrow{\partial}_{\mu)}\psi - \frac{i}{2}\eta_{\mu\nu}\bar{\psi}\overleftrightarrow{\not{\partial}}\psi\right] + \mathcal{O}\left(h^2\right), \tag{7.18}$$

where $\overleftrightarrow{\partial\!\!\!/} = \gamma^{\rho}\overleftrightarrow{\partial}_{\rho}$. The energy-momentum tensor, in this linearized approximation, reads

$$T'_{\mu\nu} = \frac{i}{4}(\bar{\psi}\gamma_{\nu}\overleftrightarrow{\partial}_{\mu}\psi + \bar{\psi}\gamma_{\mu}\overleftrightarrow{\partial}_{\nu}\psi) - \frac{i}{2}\eta_{\mu\nu}\bar{\psi}\overleftrightarrow{\partial\!\!\!/}\psi. \tag{7.19}$$

Alternatively, we can absorb the $e = \sqrt{|g|}$ into a redefinition of $e^{1/2}\psi = \Psi$ see Bonara et al. ([76]) in Appendix B of that reference. Therefore, we have

$$S = \frac{i}{2}\int \mathrm{d}^2x e^{\mu}_{a}(\bar{\Psi}\gamma^{a}\overleftrightarrow{\partial}_{\mu}\Psi) = \frac{i}{2}\int \mathrm{d}^2x[\bar{\Psi}\gamma^{\mu}\overleftrightarrow{\partial}_{\mu}\Psi + \mathcal{L}_{\text{int}}(h,\Psi) + \mathcal{O}\left(h^2\right)]. \tag{7.20}$$

In this way, the interaction Lagrangian $\mathcal{L}_{\text{int}}$ is still defined as

$$\mathcal{L}_{\text{int}}(h,\Psi) = -\frac{1}{2}h^{\mu\nu}\left[\frac{i}{4}(\bar{\Psi}e_{a\mu}\gamma^{a}\overleftrightarrow{\partial}_{\nu}P_{\pm}\Psi + \bar{\Psi}e_{a\nu}\gamma^{a}\overleftrightarrow{\partial}_{\mu}P_{\pm}\Psi)\right] = -\frac{1}{2}h^{\mu\nu}T_{\mu\nu}. \tag{7.21}$$

Then the linearized approximation of the energy-momentum tensor definition follows as

$$T_{\mu\nu} = \frac{i}{4}\bar{\Psi}\gamma_{(\mu}\overleftrightarrow{\partial}_{\nu)}P_{\pm}\Psi. \tag{7.22}$$

From interaction Lagrangian follows the Feynman rules that will be used in this work. The two-point gravitational amplitude is

$$T^{G}_{\mu\nu\alpha\beta}(q) = i\int \mathrm{d}^2x e^{iq\cdot x}\left\langle 0\right| T\left[T_{\mu\nu}(x), T_{\alpha\beta}(0)\right]\left|0\right\rangle. \tag{7.23}$$

Moreover, the vertices of the perturbative amplitudes relative to the interaction between the graviton and a fermion-antifermion pair are

$$\Gamma^{G}_{\mu\nu} = -\frac{i}{4}[\gamma_{\mu}(K_1+K_2)_{\nu} + \gamma_{\nu}(K_1+K_2)_{\mu}]P_{\pm}. \tag{7.24}$$

At the trace level, the gravitational amplitude of our interest is, see the figure 7.1,

$$t^{G}_{\mu\nu\alpha\beta} = \mathrm{tr}[\Gamma^{G}_{\mu\nu}S(1)\,\Gamma^{G}_{\alpha\beta}S(2)]. \tag{7.25}$$

After integration, we will call $T^{G}_{\mu\nu\alpha\beta}$. The total amplitude with massive propagators is

$$\begin{aligned}(i64)\,T^{G}_{\mu\nu\alpha\beta}(q) \quad = \quad & \int \frac{\mathrm{d}^2k}{(2\pi)^2}\mathrm{tr}[(1\pm\gamma_*)\gamma_{(\mu}(K_1+K_2)_{\nu)}S(1) \\ & \times(1\pm\gamma_*)\gamma_{(\alpha}(K_1+K_2)_{\beta)}S(2)].\end{aligned} \tag{7.26}$$

We recall that the fermionic propagator is given by (2.7).

We will offer some layers of notations to devise an organizational scheme to deal with this amplitude, as our approach presents multiple characteristics and complexities. For the first one, let us break it down into four basic permutations, given by

$$t^{G}_{\mu\nu\alpha\beta} = -\frac{i}{64}\left(\hat{t}^{G}_{\mu\nu\alpha\beta} + \hat{t}^{G}_{\mu\nu\beta\alpha} + \hat{t}^{G}_{\nu\mu\alpha\beta} + \hat{t}^{G}_{\nu\mu\beta\alpha}\right). \tag{7.27}$$

The structures presented above can be identified as

$$\hat{t}^{G}_{\mu\nu\alpha\beta} = (K_{1\nu}+K_{2\nu})(K_{1\beta}+K_{2\beta})\mathrm{tr}[(1\pm\gamma_*)\gamma_{\mu}S(1)\,(1\pm\gamma_*)\gamma_{\alpha}S(2)]. \tag{7.28}$$

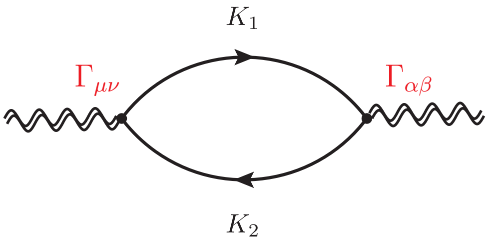

Figure 7.1: The diagram for two-point function of the the linearized energy momentum tensor.

The other three tensors come from the permutation $\mu \leftrightarrow \alpha$, followed by $\nu \leftrightarrow \beta$.

Any computational element developed to this permutation can be mirrored in the others. Second step: expanding the products like $(1 \pm \gamma_*)\gamma_\mu$, we identify the integrand of typical fermionic amplitudes as the one explored in the previous chapters. Explicitly

$$\hat{t}^{G}_{\mu\nu\alpha\beta} = (K_{1\nu} + K_{2\nu})(K_{1\beta} + K_{2\beta}) \left[t^{VV}_{\mu\alpha} + t^{AA}_{\mu\alpha} \pm t^{AV}_{\mu\alpha} \pm t^{VA}_{\mu\alpha} \right]. \tag{7.29}$$

When integrated, we recognize another element in this decomposition layer, allowing us to write the basic permutation for the structure below

$$\mathcal{T}^{\Gamma_1\Gamma_2}_{\mu\alpha\nu\beta} = \int \frac{\mathrm{d}^2 k}{(2\pi)^2} (K_1 + K_2)_\nu (K_1 + K_2)_\beta \, [t^{\Gamma_1\Gamma_2}_{\mu\alpha}(k_1, k_2)], \tag{7.30}$$

where the vertices are $\Gamma_i \in \{1, \gamma_*, \gamma_\mu, \gamma_*\gamma_\mu\}$, see (2.2). This last equation will be constructed explicitly in the next chapter since it comprises even more fundamental components. To cast these components, we observe that $K_2 - K_1 = q \to K_1 + K_2 = 2K_1 + q$.

Furthermore, expanding the Eq. (7.30) we write this combination

$$\mathcal{T}^{\Gamma_{12}}_{\mu\alpha\nu\beta} = 4T^{\Gamma_{12}}_{\mu\alpha;\nu\beta} + 2q_\nu T^{\Gamma_{12}}_{\mu\alpha;\beta} + 2q_\beta T^{\Gamma_{12}}_{\mu\alpha;\nu} + q_\nu q_\beta T^{\Gamma_{12}}_{\mu\alpha}. \tag{7.31}$$

We must define what we mean by $T^{\Gamma_{12}}_{\mu\alpha;\nu\beta}$, $T^{\Gamma_{12}}_{\mu\alpha;\beta}$, and $T^{\Gamma_{12}}_{\mu\alpha;\nu}$, which we call derivative amplitudes for the sake of simplicity. As an example, we have

$$T^{VV}_{\mu\alpha;\nu\beta} = \int \frac{\mathrm{d}^2 k}{(2\pi)^2} t^{VV}_{\mu\alpha;\nu\beta} = \int \frac{\mathrm{d}^2 k}{(2\pi)^2} K_{1\nu} K_{1\beta} \left(t^{VV}_{\mu\alpha} \right). \tag{7.32}$$

Derivative two-point amplitudes are defined even to Γ_i that do not carry Lorentz indexes,

$$T^{\Gamma_1\Gamma_2}_{-;\alpha_1} = \int \frac{\mathrm{d}^2 k}{(2\pi)^2} t^{\Gamma_1\Gamma_2}_{-;\alpha_1} = \int \frac{\mathrm{d}^2 k}{(2\pi)^2} K_{1\alpha_1} \mathrm{tr}[\Gamma_1 S(1) \Gamma_2 S(2)] \tag{7.33}$$

$$T^{\Gamma_1\Gamma_2}_{-;\alpha_1\alpha_2} = \int \frac{\mathrm{d}^2 k}{(2\pi)^2} t^{\Gamma_1\Gamma_2}_{-;\alpha_1\alpha_2} = \int \frac{\mathrm{d}^2 k}{(2\pi)^2} K_{1\alpha_1} K_{1\alpha_2} \mathrm{tr}[\Gamma_1 S(1) \Gamma_2 S(2)]. \tag{7.34}$$

When vertices to the matrix Γ_i have Lorentz indices, the notation will carry such indices in the position we left a blank space. Indexes α_i attached to factor K_1 are derivative indexes. The $\mathcal{T}^{\Gamma_{12}}_{\mu\nu\alpha\beta}$ contain only a subset of general amplitudes we have defined in our last layer. Typical amplitudes associated with $T^{\Gamma_1\Gamma_2}$ are the ones investigated in Chapter (4). On the other hand, amplitudes as (7.33)-(7.34) carrying derivative indices are the new ingredients to comprise two-point functions of the energy-momentum tensor.

To illustrate the notation, let us take a derivative amplitude that is not part of the permutations $\mathcal{T}^{\Gamma_{12}}_{\mu\nu\alpha\beta}$, by example selecting $\Gamma_1 = S$ and $\Gamma_2 = V$, we have

$$t^{SV}_{\mu;\nu} = K_{1\nu} t^{SV}_{\mu} = K_{1\nu} \mathrm{tr}[S\,(1)\,\gamma_{\mu}\,(1)]. \tag{7.35}$$

Note that the index ν appearing after the semicolon is a derivative index. It may happen that integration, through our technique, returns an expression symmetric in the indices, being this amplitude an example $T^{SV}_{\mu;\nu} = T^{SV}_{\nu;\mu}$ as we will see. Besides these comments, introducing these general definitions is crucial because they are all related through RAGFs.

Relations relevant to this chapter arise from two types of momentum contraction and traces, e.g., $g^{\mu\alpha} t^{VV}_{\mu\nu;\alpha\beta} = m t^{SV}_{\nu;\beta} + t^{V}_{\nu;\beta}\,(k_2)$. The one-point functions are part of the set:

$$t^{\Gamma_1} = \mathrm{tr}\,[\Gamma_1 S\,(k_i)]\,; \tag{7.36}$$

$$t^{\Gamma_1}_{\_;\alpha_1} = K_{1\alpha_1} \mathrm{tr}\,[\Gamma_1 S\,(k_i)]\,; \tag{7.37}$$

$$t^{\Gamma_1}_{\_;\alpha_1\alpha_2} = K_{1\alpha_1} K_{1\alpha_2} \mathrm{tr}\,[\Gamma_1 S\,(k_i)]\,. \tag{7.38}$$

Amplitudes t^{Γ_1} and their integrals are the ones used for RAGF investigations, fully developed in Chapter (4). When integrated, they get a capital letter also.

To systematically analyze $\hat{t}^{G}_{\mu\nu\alpha\beta}$, we split it in even and odd tensors: amplitudes with two vector vertices, called VV, and two axial vertices, called AA, are even, and amplitudes with composite vertices, AV and VA, are odd. For this permutation of indices, we get

$$\hat{T}^{G}_{\mu\nu\alpha\beta} = \hat{T}^{V}_{\mu\nu\alpha\beta} + \hat{T}^{A}_{\mu\nu\alpha\beta}, \tag{7.39}$$

where each of the sectors above has the following combination of amplitudes,

$$\hat{T}^{V}_{\mu\nu\alpha\beta} = \mathcal{T}^{VV}_{\mu\alpha\nu\beta} + \mathcal{T}^{AA}_{\mu\alpha\nu\beta} \tag{7.40}$$

$$\hat{T}^{A}_{\mu\nu\alpha\beta} = \pm\left(\mathcal{T}^{AV}_{\mu\alpha\nu\beta} + \mathcal{T}^{VA}_{\mu\alpha\nu\beta}\right). \tag{7.41}$$

The disposition of indices can be a trick to avoid confusion. Observe the indexes in $\hat{T}^{V}_{\mu\nu\alpha\beta}$, we chose the sequence $\mu\nu\alpha\beta$ since they come from T^{G}, however in $\mathcal{T}^{VV}_{\mu\alpha;\nu\beta}$ the disposition emphasizes that the last two indices correspond to derivative type, what is quite helpful in the calculations. The basic permutations above ($\mathcal{T}^{\Gamma_1\Gamma_2}_{\mu\alpha\nu\beta}$) are shown here to make clear the expansion in terms of derivatives structures

$$\begin{aligned}
\mathcal{T}^{VV}_{\mu\alpha\nu\beta} &= 2\left(2T^{VV}_{\mu\alpha;\nu\beta}+q_\nu T^{VV}_{\mu\alpha;\beta}\right)+q_\beta\left(2T^{VV}_{\mu\alpha;\nu}+q_\nu T^{VV}_{\mu\alpha}\right) && (7.42)\\
\mathcal{T}^{AA}_{\mu\alpha\nu\beta} &= 2\left(2T^{AA}_{\mu\alpha;\nu\beta}+q_\nu T^{AA}_{\mu\alpha;\beta}\right)+q_\beta\left(2T^{AA}_{\mu\alpha;\nu}+q_\nu T^{AA}_{\mu\alpha}\right) && (7.43)\\
\mathcal{T}^{AV}_{\mu\alpha\nu\beta} &= 2\left(2T^{AV}_{\mu\alpha;\nu\beta}+q_\nu T^{AV}_{\mu\alpha;\beta}\right)+q_\beta\left(2T^{AV}_{\mu\alpha;\nu}+q_\nu T^{AV}_{\mu\alpha}\right) && (7.44)\\
\mathcal{T}^{VA}_{\mu\alpha\nu\beta} &= 2\left(2T^{VA}_{\mu\alpha;\nu\beta}+q_\nu T^{VA}_{\mu\alpha;\beta}\right)+q_\beta\left(2T^{VA}_{\mu\alpha;\nu}+q_\nu T^{VA}_{\mu\alpha}\right). && (7.45)
\end{aligned}$$

Summing the four permutations, we get

$$\begin{aligned}
\mathcal{T}^{V}_{\mu\nu\alpha\beta} &= \hat{T}^{V}_{\mu\nu\alpha\beta}+\hat{T}^{V}_{\alpha\nu\mu\beta}+\hat{T}^{V}_{\mu\beta\alpha\nu}+\hat{T}^{V}_{\alpha\beta\mu\nu} && (7.46)\\
\mathcal{T}^{A}_{\mu\nu\alpha\beta} &= \hat{T}^{A}_{\mu\nu\alpha\beta}+\hat{T}^{A}_{\alpha\nu\mu\beta}+\hat{T}^{A}_{\mu\beta\alpha\nu}+\hat{T}^{A}_{\alpha\beta\mu\nu}. && (7.47)
\end{aligned}$$

Finally inserting in the definition it was given above (7.27), we have

$$T^{G}_{\mu_1\mu_2\sigma_1\sigma_2}=-\frac{i}{64}\{[\mathcal{T}^{\mathrm{V}}_{\mu_{12}\sigma_{12}}]+[\mathcal{T}^{\mathrm{A}}_{\mu_{12}\sigma_{12}}]\}. \quad (7.48)$$

From these elaborations, we can identify that we have already exposed the amplitudes with two Lorentz indices $T^{\Gamma_1\Gamma_2}_{\mu\nu}$ in the Chapter (4). So our task boils down to calculating only typical fermionic amplitudes with three and four indices as the following sequence.

Ward Identities: The symmetries role is crucial for understanding a QFT because we have an anomaly in quantum theory when there is a symmetry violation of the action or the classical conservation law. However, in some cases, we can avoid these anomalies by imposing severe restrictions on the physical content of the approach. In this section, we will establish symmetries and general restrictions that will guide the consistency of the method and the interpretation of the presence of anomalies.

Classically, the energy-momentum tensor defined in (7.21) has symmetry properties, $T_{\mu\nu}=T_{\nu\mu}$, current conservation, $\nabla^\mu T_{\mu\nu}=0$, and null trace, $T^\mu_\mu=0$, see [61]. These would lead us to the identities for the green function defined in (7.23)

$$\begin{aligned}
T^{G}_{\mu\nu\alpha\beta}(q) &= T^{G}_{\nu\mu\alpha\beta}(q); && (7.49)\\
q^\mu T^{G}_{\mu\nu\alpha\beta}(q) &= 0; && (7.50)\\
g^{\mu\alpha} T^{G}_{\mu\nu\alpha\beta}(q) &= 0. && (7.51)
\end{aligned}$$

However, the literature shows gravitation as a gauge theory. Therefore these canonical identities are not necessarily satisfied. We will have an Einstein anomaly in the violation of general coordinate transformations (diffeomorphisms) and Lorentz anomalies that imply an antisymmetric part in the first equation above. In the case of conformal transformations (Weyl transformations) violations, we will have a Weyl anomaly.

In the context of Einstein and Weyl invariances, we obtain consistency tests before the symmetry analysis. They arise when we perform $q^\mu T^{G}_{\mu\nu\alpha\beta}$ and $g^{\mu\nu}T^{G}_{\mu\nu\alpha\beta}$ to their integrands and obtain relations (based on integration linearity) among the set of structures defined above, i.e., through RAGFs. Since decomposition (7.39) can be done, writing a basic permutation of gravitational amplitude, in terms of amplitudes with vertices analogous to those of vector and axial currents, these can be studied individually, as they will present well-defined relations among them. Their complete introduction and detailed verification occur in the Section to even amplitudes (7.3) and (7.4).

7.1 VV-AA: Even Amplitudes

We aim to determine all components that integrate gravitational amplitudes while assuming no choice in intermediate steps. In this way, it is possible to systematize all odd amplitudes in terms of even amplitudes VV's: their divergent properties are functions of divergent parts from VV-amplitudes, and their finite parts gain an additional term proportional to the mass squared. So, we will focus on this amplitude, finding a set of definitions that makes their discussion viable. Otherwise, it would be too long due to the number of surface terms within the *IReg strategy.* From here on, all the time, metric symbol $g_{\mu\nu}$ means flat metric $g_{\mu\nu} = \eta_{\mu\nu}$.

As we saw in (4.42), the expression for the amplitude VV is given by

$$T^{VV}_{\mu_1\mu_2} = 2\Delta_{2\mu_1\mu_2} + \theta_{\mu_1\mu_2}\left(4m^2 J_2 + i/\pi\right).$$

That also can be written in closed form by Feynman integrals basis, see Section (3.3),

$$T^{VV}_{\mu_1\mu_2} = \mathcal{D}^{VV}_{\mu_1\mu_2} + 4J_{2\mu_1\mu_2} + 2q_{(\mu_1}J_{2\mu_2)} + g_{\mu_1\mu_2}q^2 J_2. \tag{7.52}$$

Finite parts come from definitions J_2, $J_{2\mu_i}$, and $J_{2\mu_1\mu_2}$ as combinations of $Z^{(n)}_k$. As for the divergent part, we collect all divergent terms and combine them in the definition

$$\mathcal{D}^{VV}_{\mu_1\mu_2} = 2\Delta_{2\mu_1\mu_2}. \tag{7.53}$$

Amplitudes with additional factors $K_{1\alpha_i}$ follow the operations of those without derivative indices. The effect of this factor is to produce an algebraic structure similar to J-integrals but with higher tensor degrees. From previous definitions,

$$t^{VV}_{\mu_1\mu_2;\alpha_1} = K_{1\alpha_1}t^{VV}_{\mu_1\mu_2} \tag{7.54}$$
$$t^{VV}_{\mu_1\mu_2;\alpha_1\alpha_2} = K_{1\alpha_1}K_{1\alpha_2}t^{VV}_{\mu_1\mu_2}. \tag{7.55}$$

Therefore, amplitudes will have a greater degree of divergence, implying that finite and divergent parts are more complex and lengthier. These are expressed in Section (3.3).

Expressions appear as a standard tensor plus a PP amplitude; see (4.31), thus

$$t^{VV}_{\mu_{12};\alpha_1} = 2t^{(+)}_{\mu_1\mu_2;\alpha_1} + g_{\mu_1\mu_2}t^{PP}_{\alpha_1} \tag{7.56}$$
$$t^{VV}_{\mu_{12};\alpha_1\alpha_2} = 2t^{(+)}_{\mu_1\mu_2;\alpha_1\alpha_2} + g_{\mu_1\mu_2}t^{PP}_{\alpha_1\alpha_2}. \tag{7.57}$$

Tensors $t^{(+)}_{\mu_1\mu_2;\alpha_1}$ and $t^{(+)}_{\mu_1\mu_2;\alpha_1\alpha_2}$ appearing above are particular cases of general tensors:

$$t^{(s_1)}_{\mu_{12};\alpha_1} = K_{1\alpha_1}\left(K_{1\mu_1}K_{2\mu_2}+s_1K_{1\mu_2}K_{2\mu_1}\right)\frac{1}{D_{12}} \tag{7.58}$$

$$t^{(s_1)}_{\mu_{12};\alpha_{12}} = K_{1\alpha_1}t^{(s_1)}_{\mu_{12};\alpha_1}; \tag{7.59}$$

where $s_1=\pm$, see (4.24). For example, in the tensor of 3rd-order, two cases assume are

$$t^{(+)}_{\mu_{12};\alpha_1} = 2\frac{K_{1\alpha_1}K_{1\mu_1}K_{1\mu_2}}{D_{12}}+\frac{q_{(\mu_1}K_{1\mu_2)}K_{1\alpha_1}}{D_{12}} \tag{7.60}$$

$$t^{(-)}_{\mu_{12};\alpha_1} = \frac{q_{[\mu_2}K_{1\mu_1]}K_{1\alpha_1}}{D_{12}}. \tag{7.61}$$

Moreover, 4th-rank ones naturally get one more $K_{1\alpha_2}$ factor. To PP amplitude (4.21), we add a $K_{1\alpha_1}$ according to our definitions

$$t^{PP}_{\alpha_1} = K_{1\alpha_1}\left(q^2\frac{1}{D_{12}}-\frac{1}{D_1}-\frac{1}{D_2}\right) = K_{1\alpha_1}t^{PP}, \tag{7.62}$$

and with two indices $t^{PP}_{\alpha_1\alpha_2}=K_{1\alpha_2}t^{PP}_{\alpha_1}$.

Integrating (7.56) using (7.60) and (7.62), derivative VV with three indices become

$$\begin{aligned} T^{VV}_{\mu_{12};\alpha_1} = {} & 4\bar{J}_{2\mu_{12}\alpha_1}+2q_{(\mu_1}\bar{J}_{2\mu_2)\alpha_1}+g_{\mu_1\mu_2}q^2J_{2\alpha_1} \\ & -g_{\mu_{12}}[\bar{J}_{1\alpha_1}(k_2)+\bar{J}_{1\alpha_1}(k_1)]+g_{\mu_{12}}q_{\alpha_1}\bar{J}_1(k_2)\,. \end{aligned} \tag{7.63}$$

For VV with four indices (7.57), we have when integrating the tensor (7.59) and $K_{1\alpha_2}t^{PP}_{\alpha_1}$,

$$\begin{aligned} T^{VV}_{\mu_{12};\alpha_{12}} = {} & 4\bar{J}_{2\mu_{12}\alpha_{12}}+2q_{(\mu_1}\bar{J}_{2\mu_2)\alpha_{12}}+g_{\mu_{12}}q^2\bar{J}_{2\alpha_{12}} \\ & +g_{\mu_{12}}\left[\bar{J}_{1\alpha_{12}}(k_2)-\bar{J}_{1\alpha_{12}}(k_1)\right]-g_{\mu_{12}}\left[q_{(\alpha_1}\bar{J}_{1\alpha_2)}(k_2)-q_{\alpha_{12}}\bar{J}_1(k_2)\right]. \end{aligned} \tag{7.64}$$

Additional terms in the k_2 that appear in the J's (with only one propagator) come from the translations $K_1=K_2-q$ used to define functions in Section (3.1). Remember that barred J's have finite and divergent parts.

To simplify the exposition of finite and divergent parts from equations (3.64), (3.66), (3.71), and (3.73), it is possible to write the results as

$$T^{VV}_{\mu_{12};\alpha_1} = 4J_{2\mu_{12}\alpha_1}+2q_{(\mu_1}J_{2\mu_2)\alpha_1}+g_{\mu_{12}}q^2J_{2\alpha_1}+\mathcal{D}^{VV}_{\mu_{12};\alpha_1} \tag{7.65}$$

$$T^{VV}_{\mu_{12};\alpha_{12}} = 4J_{2\mu_{12}\alpha_{12}}+2q_{(\mu_1}J_{2\mu_2)\alpha_{12}}+g_{\mu_{12}}q^2J_{2\alpha_{12}}+\mathcal{D}^{VV}_{\mu_{12};\alpha_{12}}. \tag{7.66}$$

In these cases, all divergent terms of integrals and define the 3rd-order tensor

$$\mathcal{D}^{VV}_{\mu_{12};\alpha_1} = -P^{\nu_1}W_{3\mu_{12}\alpha_1\nu_1}+P_{(\mu_1}\Delta_{2\mu_2\alpha_1)}+g_{\mu_{12}}P^{\nu_1}\Delta_{2\alpha_1\nu_1}-q_{\alpha_1}\Delta_{2\mu_{12}}, \tag{7.67}$$

and the 4th-order tensor

$$\begin{aligned}
\mathcal{D}^{VV}_{\mu_{12};\alpha_{12}} &= +(W_{2\mu_{12}\alpha_{12}} - g_{\mu_{12}}\Delta_{1\alpha_{12}}) + g_{\mu_1(\alpha_1}g_{\alpha_2)\mu_2}I_{\text{quad}} + \frac{\Omega_{\mu_{12}\alpha_{12}}}{6q^2}I_{\log} \\
&+\frac{1}{12}(3P^{\nu_{12}} + q^{\nu_{12}})W_{4\mu_{12}\alpha_{12}\nu_{12}} - \frac{1}{4}(P^{\nu_{12}} + q^{\nu_{12}})g_{\mu_{12}}W_{3\alpha_{12}\nu_{12}} \\
&-\frac{1}{2}P^{\nu_1}\left(P_{\mu_1}W_{3\mu_2\alpha_{12}\nu_1} + P_{\mu_2}W_{3\mu_1\alpha_{12}\nu_1}\right) + \frac{1}{2}P^{\nu_1}g_{\mu_{12}}(P-q)_{(\alpha_1}\Delta_{2\alpha_2)\nu_1} \\
&-\frac{1}{4}(P^2+q^2)W_{3\mu_{12}\alpha_{12}} - \frac{1}{2}P^{\nu_1}(P-q)_{(\alpha_1}W_{3\alpha_2)\mu_{12}\nu_1} \\
&+\frac{1}{4}[2(\theta_{\mu_{12}} + P_{\mu_{12}}) + g_{\mu_{12}}(P^2+q^2)]\Delta_{2\alpha_{12}} + \frac{1}{2}(P-q)_{\alpha_1}(P-q)_{\alpha_2}\Delta_{2\mu_{12}} \\
&+\frac{1}{2}P_{\mu_2}(P-q)_{(\alpha_1}\Delta_{2\alpha_2)\mu_1} + \frac{1}{2}P_{\mu_1}(P-q)_{(\alpha_1}\Delta_{2\alpha_2)\mu_2}.
\end{aligned} \tag{7.68}$$

We use definition of projectors $\theta_{\mu_{12}}$ and $\Omega_{\mu_{12}\alpha_{12}}$ as

$$\theta_{\mu_{12}} = g_{\mu_1\mu_2}q^2 - q_{\mu_1}q_{\mu_2} \tag{7.69}$$

$$\Omega_{\mu_{12}\alpha_{12}}(q) = 2\theta_{\mu_{12}}\theta_{\alpha_{12}} - \left(\theta_{\mu_1\alpha_1}\theta_{\mu_2\alpha_2} + \theta_{\mu_1\alpha_2}\theta_{\mu_2\alpha_1}\right); \tag{7.70}$$

both are transverse; additionally, Ω is traceless in all its indices. Note that here the projector $\theta_{\mu\nu}$ *is not dimensionless* as in Chapter (4); it has mass dimension two.

The finite part also can be expressed from explicit functions plus $\mathcal{D}$-tensor

$$T^{VV}_{\mu_{12};\alpha_1} = \frac{i}{2\pi}q_{\alpha_1}\theta_{\mu_{12}}(Z_2^{(-1)} - Z_1^{(-1)}) + \mathcal{D}^{VV}_{\mu_{12};\alpha_1}. \tag{7.71}$$

The four-index amplitude is more complicated but can be written in the projectors

$$\begin{aligned}
T^{VV}_{\mu_{12};\alpha_{12}} &= \frac{i}{4\pi}\frac{1}{q^2}\left[-\Omega_{\mu_{12}\alpha_{12}}(2Z_2^{(0)} - Z_1^{(0)}) + 2\theta_{\mu_{12}}\theta_{\alpha_{12}}(3Z_2^{(0)} - 2Z_1^{(0)})\right] \\
&-\frac{i}{4\pi}q_{\alpha_{12}}\theta_{\mu_{12}}(Z_2^{(-1)} - Z_1^{(-1)}) + \mathcal{D}^{VV}_{\mu_{12};\alpha_{12}}.
\end{aligned} \tag{7.72}$$

It is possible to maintain closed form in J's, as we will see in RAGF, through reductions as in Section (3.2). In this way, we find leading amplitudes as a substructure of $T^{VV}_{\mu_{12}}$,

$$T^{VV}_{\mu_{12};\alpha_1} = -\frac{1}{2}q_{\alpha_1}T^{VV}_{\mu_{12}} + \mathcal{D}^{VV}_{\mu_{12};\alpha_1} + \frac{1}{2}q_{\alpha_1}\mathcal{D}^{VV}_{\mu_{12}}. \tag{7.73}$$

Moreover, the same is true for the 4th-order amplitude

$$\begin{aligned}
T^{VV}_{\mu_{12};\alpha_{12}} &= \frac{1}{4}q_{\alpha_{12}}T^{VV}_{\mu_{12}} + \mathcal{D}^{VV}_{\mu_{12};\alpha_{12}} - \frac{1}{4}q_{\alpha_{12}}\mathcal{D}^{VV}_{\mu_{12}} \\
&+\frac{i}{4\pi}\frac{1}{q^2}\left[-\Omega_{\mu_{12}\alpha_{12}}(2Z_2^{(0)} - Z_1^{(0)}) + 2\theta_{\mu_{12}}\theta_{\alpha_{12}}(3Z_2^{(0)} - 2Z_1^{(0)})\right].
\end{aligned} \tag{7.74}$$

The next amplitude to be calculated is the AA. Like VV, this amplitude will contribute to the even sector of the gravitational amplitude in (7.40). From the chapter on equal masses, after traces, we have expressed it exactly as (4.32). However, writing this result in terms of amplitude $t^{VV}_{\mu_{12}}$ plus a scalar function proportional to the metric, is feasible

$$t^{AA}_{\mu_{12}} = t^{VV}_{\mu_{12}} - 4m^2 g_{\mu_{12}}\frac{1}{D_{12}}. \tag{7.75}$$

This form allows us to write equations directly from definitions for derivative amplitudes

$$t^{AA}_{\mu_{12};\alpha_1} = t^{VV}_{\mu_{12};\alpha_1} - 4m^2 g_{\mu_{12}} \frac{K_{1\alpha_1}}{D_{12}} \tag{7.76}$$

$$t^{AA}_{\mu_{12};\alpha_{12}} = t^{VV}_{\mu_{12};\alpha_{12}} - 4m^2 g_{\mu_{12}} \frac{K_{1\alpha_1}K_{1\alpha_2}}{D_{12}}, \tag{7.77}$$

and their integrals

$$T^{AA}_{\mu_{12}} = T^{VV}_{\mu_{12}} - 4m^2 g_{\mu_{12}} J_2 \tag{7.78}$$

$$T^{AA}_{\mu_{12};\alpha_1} = T^{VV}_{\mu_{12};\alpha_1} - 4m^2 g_{\mu_{12}} J_{2\alpha_1} \tag{7.79}$$

$$T^{AA}_{\mu_{12};\alpha_{12}} = T^{VV}_{\mu_{12};\alpha_{12}} - 4m^2 g_{\mu_{12}} \bar{J}_{2\alpha_{12}}. \tag{7.80}$$

Additional contributions of massive terms present in this amplitude are worth noting. For the divergent part, only the amplitude with four indices has a non-zero term in $\bar{J}_{2\alpha_{12}}$, see (3.66). Integrals appearing in the amplitudes of fewer indices contribute only to the finite part. However, the 4th-rank amplitude has an additional contribution as a surface term and $I_{\log}$. The final result is identical to that obtained from the first form presented.

7.2 AV-VA: Odd amplitudes

We will calculate all odd parts of gravitational amplitude. As seen in (4.33), we wrote two-index functions in terms of even ones using general identity for $2D$, $\gamma_*\gamma_{\mu_1} = -\varepsilon_{\mu_1\nu_1}\gamma^{\nu_1}$, present in (4.15). For higher-rank amplitudes, traces operate in the same way but add indices to the integrals:

$$(T^{AV}_{\mu_{12};\alpha_1})_1 = -\varepsilon_{\mu_1}{}^{\nu_1} T^{VV}_{\nu_1\mu_2;\alpha_1}; \qquad (T^{AV}_{\mu_{12};\alpha_1})_2 = -\varepsilon_{\mu_2}{}^{\nu_1} T^{AA}_{\mu_1\nu_1;\alpha_1} \tag{7.81}$$

$$(T^{AV}_{\mu_{12};\alpha_{12}})_1 = -\varepsilon_{\mu_1}{}^{\nu_1} T^{VV}_{\nu_1\mu_2;\alpha_{12}}; \qquad (T^{AV}_{\mu_{12};\alpha_{12}})_2 = -\varepsilon_{\mu_2}{}^{\nu_1} T^{AA}_{\mu_1\nu_1;\alpha_{12}}. \tag{7.82}$$

To complete odd amplitudes, we cast the analogous VA equations:

$$(T^{VA}_{\mu_{12};\alpha_1})_1 = -\varepsilon_{\mu_1}{}^{\nu_1} T^{AA}_{\nu_1\mu_2;\alpha_1} \quad (T^{VA}_{\mu_{12};\alpha_1})_2 = -\varepsilon_{\mu_2}{}^{\nu_1} T^{VV}_{\mu_1\nu_1;\alpha_1}; \tag{7.83}$$

$$(T^{VA}_{\mu_{12};\alpha_{12}})_1 = -\varepsilon_{\mu_1}{}^{\nu_1} T^{AA}_{\nu_1\mu_2;\alpha_{12}} \quad (T^{VA}_{\mu_{12};\alpha_{12}})_2 = -\varepsilon_{\mu_2}{}^{\nu_1} T^{VV}_{\mu_1\nu_1;\alpha_{12}}. \tag{7.84}$$

The same considerations can be made when using the chiral matrix definition (4.14) directly in the Dirac traces. By considering expressions for amplitudes with additional terms, as in (4.27) and (4.28), for amplitudes with derivative vertices, we have

$$(t^{AV}_{\mu_{12};\alpha_1})_1 = -\varepsilon_{\mu_1}{}^{\nu_1} t^{VV}_{\nu_1\mu_2;\alpha_1} + 2\varepsilon_{\mu_2}{}^{\nu_1} t^{(-)}_{\mu_1\nu_1;\alpha_1} + g_{\mu_{12}} t^{SP}_{\alpha_1} \tag{7.85}$$

$$(t^{AV}_{\mu_{12};\alpha_1})_2 = -\varepsilon_{\mu_2}{}^{\nu_1} t^{AA}_{\mu_1\nu_1;\alpha_1} + 2\varepsilon_{\mu_1}{}^{\nu_1} t^{(-)}_{\nu_1\mu_2;\alpha_1} - g_{\mu_{12}} t^{SP}_{\alpha_1} \tag{7.86}$$

$$(t^{AV}_{\mu_{12};\alpha_{12}})_1 = -\varepsilon_{\mu_1}{}^{\nu_1} t^{VV}_{\nu_1\mu_2;\alpha_{12}} + 2\varepsilon_{\mu_2}{}^{\nu_1} t^{(-)}_{\mu_1\nu_1;\alpha_{12}} + g_{\mu_{12}} t^{SP}_{\alpha_{12}} \tag{7.87}$$

$$(t^{AV}_{\mu_{12};\alpha_{12}})_2 = -\varepsilon_{\mu_2}{}^{\nu_1} t^{AA}_{\mu_1\nu_1;\alpha_{12}} + 2\varepsilon_{\mu_1}{}^{\nu_1} t^{(-)}_{\nu_1\mu_2;\alpha_{12}} - g_{\mu_{12}} t^{SP}_{\alpha_{12}}. \tag{7.88}$$

Additional terms combine and cancel out when integrated, so the equations above reduce to those given in (7.81)-(7.82). Let us demonstrate this fact, using the definition (7.58) to $t^{(-)}$ at the beginning of the last section. Thus we have

$$2\varepsilon_{\mu_2}{}^{\nu_1}T^{(-)}_{\mu_1\nu_1;\alpha_1}+g_{\mu_{12}}T^{SP}_{\alpha_1}=2\varepsilon_{\mu_2\nu_1}q_{\mu_1}\bar{J}^{\nu_1}_{2\alpha_1}-2\varepsilon_{\mu_2\nu_1}q^{\nu_1}\bar{J}_{2\mu_1\alpha_1}+2g_{\mu_1\mu_2}\varepsilon_{\nu_1\nu_2}q^{\nu_2}\bar{J}^{\nu_1}_{2\alpha_1}. \quad (7.89)$$

We applied our definitions of J_2 integrals, and employed the identity below in the las term

$$\varepsilon_{\nu_1\nu_2}g_{\mu_2\mu_1}+\varepsilon_{\mu_2\nu_1}g_{\nu_2\mu_1}+\varepsilon_{\nu_2\mu_2}g_{\nu_1\mu_1}=0. \quad (7.90)$$

It is direct to observe the exact cancellation of the first two terms

$$2\varepsilon_{\mu_2}{}^{\nu_1}T^{(-)}_{\mu_1\nu_1;\alpha_1}+g_{\mu_{12}}T^{SP}_{\alpha_1}=0. \quad (7.91)$$

That occurs independently of divergent content of $\bar{J}_{2\mu\nu}$. It is easy to see that the same happens to the analogous terms in the 4th-rank amplitude's version,

$$2\varepsilon_{\mu_2}{}^{\nu_1}T^{(-)}_{\mu_1\nu_1;\alpha_{12}}+g_{\mu_{12}}T^{SP}_{\alpha_1\alpha_2}=0. \quad (7.92)$$

Definitions for the VA computed with the definition of the chiral matrix were not present because the logic and result are the same. As for the relation between VV and AA amplitudes, we write from the integrand level

$$\begin{aligned}(T^{VA}_{\mu_{12};\alpha_1})_1 &= -\varepsilon_{\mu_1}{}^{\nu_1}T^{AA}_{\nu_1\mu_2;\alpha_1}=-\varepsilon_{\mu_1}{}^{\nu_1}(T^{VV}_{\nu_1\mu_2;\alpha_1}-4m^2g_{\nu_1\mu_2}J_{2\alpha_1})\\ &= (T^{AV}_{\mu_1\mu_2;\alpha_1})_1+4m^2\varepsilon_{\mu_1\mu_2}J_{2\alpha_1}.\end{aligned}$$

This relation is satisfied without any conditions. In general, we have

$$(T^{VA}_{\mu_{12}})_i = (T^{AV}_{\mu_{12}})_i+4m^2\varepsilon_{\mu_1\mu_2}J_2 \quad (7.93)$$

$$(T^{VA}_{\mu_{12};\alpha_1})_i = (T^{AV}_{\mu_{12};\alpha_1})_i+4m^2\varepsilon_{\mu_1\mu_2}J_{2\alpha_1} \quad (7.94)$$

$$(T^{VA}_{\mu_{12};\alpha_{12}})_i = (T^{AV}_{\mu_{12};\alpha_{12}})_i+4m^2\varepsilon_{\mu_1\mu_2}\bar{J}_{2\alpha_{12}}, \quad (7.95)$$

where the index $i=1,2$ is associated with versions, and the Eqs (7.93)-(7.95) will often be used to reduce manipulations required for the gravitational anomaly.

On the other hand, basic and independent versions one and two are only strictly equivalent with conditions. This fact was worked in Chapters (4) and (5), where a single mass and two masses in odd amplitudes were handled. Let us retrieve the explicitly computed result to establish general results to be used in the sequel

$$\begin{aligned}(T^{AV}_{\mu_{12}})_1-(T^{AV}_{\mu_{12}})_2 &= -2(\varepsilon_{\mu_1\nu}\Delta^{\nu}_{2\mu_2}-\varepsilon_{\mu_2\nu}\Delta^{\nu}_{2\mu_1}) \quad (7.96)\\ &\quad -(\varepsilon_{\mu_1\nu_1}\theta^{\nu}_{\mu_2}-\varepsilon_{\mu_2\nu_1}\theta^{\nu}_{\mu_1})\frac{1}{q^2}\left(4m^2J_2+i/\pi\right)+4\varepsilon_{\mu_1\mu_2}m^2J_2.\end{aligned}$$

We rearrange the finite part using $\varepsilon_{[\mu_1\nu}\theta^{\nu}_{\mu_2]}=0$ and surface terms $\varepsilon_{[\mu_1\nu}\Delta^{\nu}_{2\mu_2]}=0$,

$$\varepsilon_{\mu_1\nu}\Delta^{\nu}_{2\mu_2}+\varepsilon_{\mu_2\mu_1}\Delta^{\nu}_{2\nu}+\varepsilon_{\nu\mu_2}\Delta^{\nu}_{2\mu_1} = 0=\varepsilon_{[\mu_1\nu}\Delta^{\nu}_{2\mu_2]}, \quad (7.97)$$

$$\varepsilon_{\mu_1\nu}\theta^{\nu}_{\mu_2}+\varepsilon_{\mu_2\mu_1}\theta^{\nu}_{\nu}+\varepsilon_{\nu\mu_2}\theta^{\nu}_{\mu_1} = 0=\varepsilon_{[\mu_1\nu}\theta^{\nu}_{\mu_2]}; \quad (7.98)$$

hence, the difference between the two versions reduces to

$$(T^{AV}_{\mu_{12}})_1 - (T^{AV}_{\mu_{12}})_2 = -\varepsilon_{\mu_1\mu_2}\left(2\Delta^{\alpha}_{2\alpha} + i/\pi\right). \tag{7.99}$$

Here we clarify how this result can be written systematically. It boils down to using the definitions and caveat that each term present represents complete amplitudes,

$$T^{VV}_{\mu_1\mu_2} = 2\Delta_{2\mu_1\mu_2} + \frac{\theta_{\mu_1\mu_2}}{q^2}\left(4m^2 J_2 + i/\pi\right), \tag{7.100}$$

$$T^{AA}_{\mu_1\mu_2} = 2\Delta_{2\mu_1\mu_2} + \frac{\theta_{\mu_1\mu_2}}{q^2}\left(4m^2 J_2 + i/\pi\right) - g_{\mu_1\mu_2}\left(4m^2 J_2\right). \tag{7.101}$$

Using (4.33), the versions for AV-amplitudes arise

$$(T^{AV}_{\mu_{12}})_1 = -2\varepsilon_{\mu_1}{}^{\nu}\Delta_{2\mu_2\nu} - \frac{\varepsilon_{\mu_1\nu}\theta^{\nu}_{\mu_2}}{q^2}\left(4m^2 J_2 + i/\pi\right), \tag{7.102}$$

$$(T^{AV}_{\mu_{12}})_2 = -2\varepsilon_{\mu_2}{}^{\nu}\Delta_{2\mu_1\nu} - \frac{\varepsilon_{\mu_2\nu}\theta^{\nu}_{\mu_1}}{q^2}\left(4m^2 J_2 + i/\pi\right) - \varepsilon_{\mu_1\mu_2}\left(4m^2 J_2\right). \tag{7.103}$$

After writing the difference between them

$$(T^{AV}_{\mu_{12}})_1 - (T^{AV}_{\mu_{12}})_2 = -\varepsilon_{\mu_1}{}^{\nu_1}T^{VV}_{\nu_1\mu_2} + \varepsilon_{\mu_2}{}^{\nu_1}T^{AA}_{\mu_1\nu_1}, \tag{7.104}$$

we take into account identity among AA and VV (7.78):

$$(T^{AV}_{\mu_{12}})_1 - (T^{AV}_{\mu_{12}})_2 = -\varepsilon_{\mu_1}{}^{\nu_1}T^{VV}_{\nu_1\mu_2} + \varepsilon_{\mu_2}{}^{\nu_1}T^{VV}_{\mu_1\nu_1} - 4m^2\varepsilon_{\mu_2\mu_1}J_2. \tag{7.105}$$

Lastly, employ

$$\varepsilon_{[\mu_1\nu_1}\left(T^{VV}\right)^{\nu_1}{}_{\mu_2]} = 0 \Leftrightarrow -\varepsilon_{\mu_1}{}^{\nu_1}T^{VV}_{\nu_1\mu_2} + \varepsilon_{\mu_2}{}^{\nu_1}T^{VV}_{\mu_1\nu_1} = \varepsilon_{\mu_2\mu_1}\left(g^{\nu_{12}}T^{VV}_{\nu_1\nu_2}\right),$$

to reach an expression equivalent to work term by term on the amplitude,

$$(T^{AV}_{\mu_{12}})_1 - (T^{AV}_{\mu_{12}})_2 = \varepsilon_{\mu_2\mu_1}\left(g^{\nu_{12}}T^{VV}_{\nu_1\nu_2} - 4m^2 J_2\right) =: \Upsilon. \tag{7.106}$$

With the help of explicit expression, follows

$$\Upsilon = 2\Delta^{\rho}_{2\rho} + \left(4m^2 J_2 + i/\pi\right) - 4m^2 J_2 = 2\Delta^{\rho}_{2\rho} + i/\pi. \tag{7.107}$$

As it must be, this condition is equal to that deduced to the equivalence of basic (4.52). For these amplitudes, the equality among independent expressions is obtained through any possible way to employ the trace of four gamma matrices and a chiral one.

It is a direct task to identify this condition to higher-rank amplitudes, namely,

$$(T^{AV}_{\mu_{12}})_1 - (T^{AV}_{\mu_{12}})_2 = \varepsilon_{\mu_2\mu_1}\Upsilon \tag{7.108}$$

$$(T^{AV}_{\mu_{12};\alpha_1})_1 - (T^{AV}_{\mu_{12};\alpha_1})_2 = \varepsilon_{\mu_2\mu_1}\Upsilon_{\alpha_1} \tag{7.109}$$

$$(T^{AV}_{\mu_{12};\alpha_{12}})_1 - (T^{AV}_{\mu_{12};\alpha_{12}})_2 = \varepsilon_{\mu_2\mu_1}\Upsilon_{\alpha_{12}}. \tag{7.110}$$

Due to the relevance of these terms, we present the following definition

$$\begin{aligned}\Upsilon &= \left(g^{\nu_{12}}T^{VV}_{\nu_1\nu_2} - 4m^2 J_2\right) &(7.111)\\ \Upsilon_{\alpha_1} &= \left(g^{\nu_{12}}T^{VV}_{\nu_1\nu_2;\alpha_1} - 4m^2 J_{2\alpha_1}\right) &(7.112)\\ \Upsilon_{\alpha_{12}} &= \left(g^{\nu_{12}}T^{VV}_{\nu_1\nu_2;\alpha_{12}} - 4m^2 J_{2\alpha_{12}}\right). &(7.113)\end{aligned}$$

At the end of calculations, identities of this type must be used in surface terms and finite parts of amplitudes. This approach simplifies the conclusions that can be given by exposing hundreds of terms that build up some of these amplitudes, making that path prohibitively long to be exposed. The identities only express the vanishing of a complete antisymmetric tensor of degree three in two dimensions.

The last section exposed detailed results for finite and divergent parts of core component VV amplitudes that appear in RHS of (7.112) and (7.113). Thus, we take expressions (7.65) and (7.66) into account to write

$$\begin{aligned}g^{\nu_{12}}T^{VV}_{\nu_{12};\alpha_1} &= 4g^{\nu_{12}}J_{2\nu_{12}\alpha_1} + 2(2q^{\nu_1}J_{2\nu_1\alpha_1} + q^2 J_{2\alpha_1}) + g^{\nu_{12}}\mathcal{D}^{VV}_{\nu_{12};\alpha_1} &(7.114)\\ g^{\nu_{12}}T^{VV}_{\nu_{12};\alpha_{12}} &= 4g^{\nu_{12}}J_{2\nu_{12}\alpha_{12}} + 2(2q^{\nu_1}J_{2\nu_1\alpha_{12}} + q^2 J_{2\alpha_{12}}) + g^{\nu_{12}}\mathcal{D}^{VV}_{\nu_{12};\alpha_{12}}. &(7.115)\end{aligned}$$

Observe that J-functions comprise the entire finite part while $\mathcal{D}^{VV}$-tensor accounts for divergent terms. Therefore, these calculations require the traces

$$\begin{aligned}4g^{\nu_{12}}J_{2\nu_{12}\alpha_1} &= 4m^2 J_{2\alpha_1} - \frac{i}{2\pi}q_{\alpha_1} &(7.116)\\ 4g^{\nu_{12}}J_{2\nu_{12}\alpha_{12}} &= 4m^2 J_{2\alpha_{12}} - \frac{i}{12\pi}[\theta_{\alpha_1\alpha_2}(q) - 3q_{\alpha_1}q_{\alpha_2}], &(7.117)\end{aligned}$$

and relations coming from momentum contractions

$$\begin{aligned}2q^{\nu_1}J_{2\nu_1\alpha_1} + q^2 J_{2\alpha_1} &= 0 &(7.118)\\ 2q^{\nu_1}J_{2\nu_1\alpha_1\alpha_2} + q^2 J_{2\alpha_1\alpha_2} &= 0; &(7.119)\end{aligned}$$

results derived in Sections (3.2) and (3.3). Substituting in (7.114) and (7.115) yields

$$\begin{aligned}g^{\nu_{12}}T^{VV}_{\nu_{12};\alpha_1} &= 4m^2 J_{2\alpha_1} - \frac{i}{2\pi}q_{\alpha_1} + g^{\nu_{12}}\mathcal{D}^{VV}_{\nu_{12};\alpha_1} &(7.120)\\ g^{\nu_{12}}T^{VV}_{\nu_{12};\alpha_{12}} &= 4m^2 J_{2\alpha_{12}} - \frac{i}{12\pi}(\theta_{\alpha_1\alpha_2} - 3q_{\alpha_1}q_{\alpha_2}) + g^{\nu_{12}}\mathcal{D}^{VV}_{\nu_{12};\alpha_{12}}. &(7.121)\end{aligned}$$

The trace of $\mathcal{D}^{VV}$-tensor, their explicit forms from (7.67) and (7.68). For the one derivative index, the divergent terms have only logarithmic divergent surface terms

$$g^{\nu_{12}}\mathcal{D}^{VV}_{\nu_{12};\alpha_1} = -\frac{1}{2}P^{\nu_1}(2W^{\rho}_{3\rho\alpha_1\nu_1} - 8\Delta_{2\alpha_1\nu_1}) + P_{\alpha_1}\Delta^{\rho}_{2\rho} - q_{\alpha_1}\Delta^{\rho}_{2\rho}. \quad (7.122)$$

As for the trace of the two-derivative indices tensor, its divergent part is more complex. It presents a relation involving the trace of quadratically divergent objects, as seen in the first line of the following equation

$$g^{\nu_{12}}\mathcal{D}^{VV}_{\nu_{12};\alpha_{12}} = (W^{\rho}_{2\rho\alpha_{12}} - 2\Delta_{1\alpha_{12}}) + 2g_{\alpha_1\alpha_2}I_{\text{quad}} + \frac{1}{6p^2}g^{\mu_{12}}\Omega_{\mu_{12}\alpha_{12}}I_{\log} \tag{7.123}$$
$$+\frac{1}{36}(3P^{\nu_{12}} + q^{\nu_{12}})(3W^{\rho}_{4\rho\alpha_{12}\nu_{12}} - 18W_{3\alpha_{12}\nu_{12}})$$
$$-\frac{1}{4}P^{\nu_1}(P_{\alpha_2} - q_{\alpha_2})(2W^{\rho}_{3\rho\alpha_1\nu_1} - 8\Delta_{2\alpha_1\nu_1})$$
$$-\frac{1}{4}P^{\nu_1}(P_{\alpha_1} - q_{\alpha_1})(2W^{\rho}_{3\rho\alpha_2\nu_1} - 8\Delta_{2\alpha_2\nu_1})$$
$$-\frac{1}{8}(P^2 + q^2)(2W^{\rho}_{3\rho\alpha_{12}} - 8\Delta_{2\alpha_{12}})$$
$$+\frac{1}{2}(P_{\alpha_1} - q_{\alpha_1})(P_{\alpha_2} - q_{\alpha_2})\Delta^{\rho}_{2\rho}.$$

Identities involving $W^{\rho}_{4\rho\alpha_{12}\nu_{12}}$, $W^{\rho}_{3\rho\alpha_2\nu_1}$ and $\Delta^{\rho}_{2\rho}$ are a valuable way to write the results. They arise from taking the trace of W's and applying combinatorial analysis in their definition as linear expansions of surface terms, which was performed in Section (3.1), Eqs (3.24)-(3.25). They are

$$2W^{\rho}_{3\rho\mu_{12}} - 8\Delta_{2\mu_{12}} = [2(\Box^{\rho}_{3\rho\mu_{12}} - \Delta_{2\mu_{12}}) - g_{\mu_{12}}\Delta^{\rho}_{2\rho}] + 2g_{\mu_{12}}\Delta^{\rho}_{2\rho}, \tag{7.124}$$

$$3W^{\rho}_{4\rho\mu_{1234}} - 18W_{3\mu_{1234}} = [3\Sigma^{\rho}_{4\rho\mu_{1234}} - 8\Box_{3\mu_{1234}} - g_{(\mu_{12}}g_{\mu_{34})}\Delta^{\rho}_{2\rho}] \tag{7.125}$$
$$+g_{(\mu_{12}}[\Box^{\rho}_{3\rho\mu_{34})} - \Delta_{2\mu_{34})} - \frac{1}{2}g_{\mu_{34})}\Delta^{\rho}_{2\rho}] + 3g_{(\mu_{12}}g_{\mu_{34})}\Delta^{\rho}_{2\rho}.$$

The use of these relations will become apparent in the course of the investigation.

To get an explicit expression for terms that make versions of amplitudes distinct, see (7.112) and (7.113), we join the results $g^{\nu_{12}}\mathcal{D}^{VV}_{\nu_{12};\alpha_1}$ and $g^{\nu_{12}}\mathcal{D}^{VV}_{\nu_{12};\alpha_{12}}$ with finite part previously calculated, which allow us to write:

$$\Upsilon_{\alpha_1} = \left(g^{\nu_{12}}T^{VV}_{\nu_{12};\alpha_1} - 4m^2J_{2\alpha_1}\right) \tag{7.126}$$
$$= -\frac{1}{2}P^{\nu_1}[2(\Box^{\rho}_{3\rho\alpha_1\nu_1} - \Delta_{2\alpha_1\nu_1}) - g_{\alpha_1\nu_1}\Delta^{\rho}_{2\rho}] - q_{\alpha_1}(\Delta^{\rho}_{2\rho} + i/2\pi)$$

$$\Upsilon_{\alpha_1\alpha_2} = \left(g^{\nu_{12}}T^{VV}_{\nu_{12};\alpha_{12}} - 4m^2\bar{J}_{2\alpha_{12}}\right) \tag{7.127}$$
$$= -\frac{1}{6}\left(\theta_{\alpha_1\alpha_2} - 3q_{\alpha_1}q_{\alpha_2}\right)\left(\Delta^{\rho}_{2\rho} + i/2\pi\right)$$
$$+\frac{1}{36}(3P^{\nu_{12}} + q^{\nu_{12}})[3\Sigma^{\rho}_{4\rho\alpha_{12}\nu_{12}} - 8\Box_{3\alpha_{12}\nu_{12}} - g_{(\alpha_{12}}g_{\nu_{12})}\Delta^{\rho}_{2\rho}]$$
$$+\frac{1}{72}(3P^{\nu_{12}} + q^{\nu_{12}})g_{(\alpha_{12}}[2\Box^{\rho}_{3\nu_{12})\rho} - 2\Delta_{2\nu_{12})} - g_{\nu_{12})}\Delta^{\rho}_{2\rho}]$$
$$-\frac{1}{4}P^{\nu_1}\left(P_{\alpha_2} - q_{\alpha_2}\right)[2(\Box^{\rho}_{3\rho\alpha_1\nu_1} - \Delta_{2\alpha_1\nu_1}) - g_{\alpha_1\nu_1}\Delta^{\rho}_{2\rho}]$$
$$-\frac{1}{4}P^{\nu_1}\left(P_{\alpha_1} - q_{\alpha_1}\right)[2(\Box^{\rho}_{3\rho\alpha_2\nu_1} - \Delta_{2\alpha_2\nu_1}) - g_{\alpha_2\nu_1}\Delta^{\rho}_{2\rho}]$$
$$-\frac{1}{8}(P^2 + q^2)[2(\Box^{\rho}_{3\rho\alpha_{12}} - \Delta_{2\alpha_{12}}) - g_{\alpha_{12}}\Delta^{\rho}_{2\rho}]$$
$$+(W^{\rho}_{2\rho\alpha_{12}} - 2\Delta_{1\alpha_{12}}) + 2g_{\alpha_1\alpha_2}I_{\text{quad}} - 2m^2\left(\Delta_{2\alpha_{12}} + g_{\alpha_{12}}I_{\log}\right).$$

For the last relation, we defined the complete two-point tensor integral

$$\bar{J}_{2\alpha_{12}} = \frac{1}{2}\left(\Delta_{2\alpha_{12}} + g_{\alpha_{12}} I_{\log}\right) + J_2.$$

We already have all expressions that make up gravitational amplitude. However, we also need to know how they manifest in RAGFs, a subject we will address next. In a second step, we will analyze its consequences for symmetries of keeping these relations preserved and whether it is possible to determine them independently of amplitudes context.

7.3 Even Amplitudes: RAGFs

Now, we will explore RAGFs for even amplitudes. In Chapters (4) and (5), relations served as a bridge to establish how they operate in odd amplitudes since contractions related to vertex indices (called *internal indices*) are trivially satisfied. Beyond the relation

$$q^{\mu_1} t^{VV}_{\mu_{12}} = t^{V}_{\mu_2}\left(k_1\right) - t^{V}_{\mu_2}\left(k_2\right) = t^{V}_{(-)\mu_2}, \tag{7.128}$$

already verified in (4.53), we need relations for amplitudes derivative:

$$\begin{aligned} q^{\mu_1} t^{VV}_{\mu_{12};\alpha_1} &= t^{V}_{\mu_2;\alpha_1}\left(k_1\right) - t^{V}_{\mu_2;\alpha_1}\left(k_2\right) = t^{V}_{(-)\mu_2;\alpha_1} & (7.129)\\ q^{\mu_1} t^{VV}_{\mu_{12};\alpha_{12}} &= t^{V}_{\mu_2;\alpha_{12}}\left(k_1\right) - t^{V}_{\mu_2;\alpha_{12}}\left(k_2\right) = t^{V}_{(-)\mu_2;\alpha_{12}}, & (7.130)\end{aligned}$$

where $t^{V}_{(-)\mu_2;\alpha_1}$ and $t^{V}_{\mu_2;\alpha_{12}}\left(k_1\right)$ denotes the difference of vectorial one-point functions.

In addition to relations for internal indices, the contractions with derivative indices momentum (called *external indices*) also produce relations for the gravitational amplitudes. They are obtained using the following identity inside the Dirac trace

$$2q^{\alpha_1} K_{1\alpha_1} = [S^{-1}\left(K_2\right) \not{q} + \not{q} S^{-1}\left(K_1\right) + 2m\not{q} - q^2], \tag{7.131}$$

in two distinct positions: around the first or second vertex. For example, we apply it in front of the first vertex and split terms in the sum as

$$\begin{aligned} q^{\alpha_1} t^{VV}_{\mu_{12};\alpha_1} = & -\frac{1}{2} q^2 t^{VV}_{\mu_{12}} + m\mathrm{tr}\left[\not{q}\gamma_{\mu_1} S\left(K_1\right)\gamma_{\mu_2} S\left(K_2\right)\right] \\ & +\frac{1}{2}\mathrm{tr}\left[\not{q} S^{-1}\left(K_1\right)\gamma_{\mu_1} S\left(K_1\right)\gamma_{\mu_2} S\left(K_2\right)\right] \\ & +\frac{1}{2}\mathrm{tr}\left[S^{-1}\left(K_2\right)\not{q}\gamma_{\mu_1} S\left(K_1\right)\gamma_{\mu_2} S\left(K_2\right)\right]. \end{aligned} \tag{7.132}$$

Substituting in the second line of the relation above

$$S^{-1}\left(K_1\right)\gamma_{\mu_1} S\left(K_1\right) = 2K_{1\mu_1} S\left(K_1\right) - \gamma_{\mu_1} - 2m\gamma_{\mu_1} S\left(K_1\right),$$

the mass term was canceled. Applying $\not{q} = [S^{-1}\left(K_2\right) - S^{-1}\left(K_1\right)]$ leads to the difference of the one-point functions with derivative indices, $t^{V}_{(-)\mu_2;\mu_1}$. We use the (anti)-commutations among γ_{μ_i} and $\not{q}$ matrices,

$$2\gamma_{\mu_1}\gamma_{\mu_2} = \{\gamma_{\mu_1},\gamma_{\mu_2}\}+[\gamma_{\mu_1},\gamma_{\mu_2}] \quad (7.133)$$
$$[\gamma_{\mu_1},\gamma_{\mu_2}] = -2\varepsilon_{\mu_1\mu_2}\gamma_* \quad (7.134)$$
$$2g_{\mu_{12}}\not{q} = \not{q}\gamma_{\mu_2}\gamma_{\mu_1}+\gamma_{\mu_2}\gamma_{\mu_1}\not{q}. \quad (7.135)$$

The systematic procedure gives back the identity given by

$$\mathrm{tr}\left[\gamma_{\mu_2}\not{q}\gamma_{\mu_1}S(K_1)\right]-\mathrm{tr}\left[\not{q}\gamma_{\mu_1}\gamma_{\mu_2}S(K_2)\right]=q_{\mu_1}t^V_{(-)\mu_2}+q_{\mu_2}t^V_{(+)\mu_1}-g_{\mu_{12}}q^\nu t^V_{(+)\nu} \quad (7.136)$$

where the notation $t^V_{(+)\mu_2;\alpha_1}$, is associated with the sum of vectorial one-point functions

$$t^V_{(+)\mu_2;\alpha_1}=t^V_{\mu_2;\alpha_1}(k_1)+t^V_{\mu_2;\alpha_1}(k_2), \quad (7.137)$$

similarly to $t^V_{(+)\mu_1}$. The operations described above leads the relation for q^α contraction,

$$q^{\alpha_1}t^{VV}_{\mu_{12};\alpha_1}=-\frac{1}{2}q^2t^{VV}_{\mu_{12}}+t^V_{(-)\mu_2;\mu_1}+\frac{1}{2}q_{\mu_1}t^V_{(-)\mu_2}+\frac{1}{2}q_{\mu_2}t^V_{(+)\mu_1}-\frac{1}{2}g_{\mu_{12}}q^{\alpha_1}t^V_{(+)\alpha_1}. \quad (7.138)$$

Starting with the initial identity (7.131) close to the right of vertex γ_{μ_2}, the relation obtained is equal to the previous one after interchanging $\mu_1\leftrightarrow\mu_2$ on the RHS. The relation is the same for four indices amplitude, just adding a derived index on the amplitudes.

Amplitudes with derivative indices also account for trace identities, which will later be necessary to characterize Weyl anomalies. The result emerges directly by using $\not{K}_1=S^{-1}(k_1)+m$, so relations for the two amplitudes are given by

$$g^{\mu_1\alpha_1}t^{VV}_{\mu_{12};\alpha_1} = t^V_{\mu_2}(k_2)+mt^{SV}_{\mu_2} \quad (7.139)$$
$$g^{\mu_1\alpha_1}t^{VV}_{\mu_{12};\alpha_{12}} = t^V_{\mu_2;\alpha_2}(k_2)+mt^{SV}_{\mu_2;\alpha_2}. \quad (7.140)$$

These relations are symmetric for $\mu_1\leftrightarrow\mu_2$ exchanges.

To finalize the exposure of RAGFs for even amplitudes, we extend the procedure adopted for the VVs to AA. For the momentum contraction q^{μ_i} (internal contractions):

$$q^{\mu_1}t^{AA}_{\mu_{12};\alpha_1} = t^V_{\mu_2;\alpha_1}(k_1)-t^V_{\mu_2;\alpha_1}(k_2)-2mt^{PA}_{\mu_2;\alpha_1}=t^V_{(-)\mu_2;\alpha_1}-2mt^{PA}_{\mu_2;\alpha_1} \quad (7.141)$$
$$q^{\mu_1}t^{AA}_{\mu_{12};\alpha_{12}} = t^V_{\mu_2;\alpha_{12}}(k_1)-t^V_{\mu_2;\alpha_{12}}(k_2)-2mt^{PA}_{\mu_2;\alpha_{12}}=t^V_{(-)\mu_2;\alpha_{12}}-2mt^{PA}_{\mu_2;\alpha_{12}}. \quad (7.142)$$

For the momentum contraction q^{α_i} (external contractions):

$$q^{\alpha_1}t^{AA}_{\mu_{12};\alpha_1} = -\frac{1}{2}q^2t^{AA}_{\mu_{12}}+t^V_{(-)\mu_2;\mu_1}+\frac{1}{2}q_{\mu_1}t^V_{(-)\mu_2}+\frac{1}{2}q_{\mu_2}t^V_{(+)\mu_1}-\frac{1}{2}g_{\mu_{12}}q^{\alpha_1}t^V_{(+)\alpha_1} \quad (7.143)$$
$$+mg_{\mu_{12}}[t^S(k_2)-t^S(k_1)]+m\varepsilon_{\mu_{12}}[t^P(k_2)+t^P(k_1)].$$

The last line has two additional terms compared to (7.138). These terms do not contribute for three-index amplitudes; however, those with four indices have $[t^S_{\alpha_2}(k_2)-t^S_{\alpha_2}(k_1)]\neq 0$ when integrated. And for the trace Contractions:

$$g^{\mu_1\alpha_1}t^{AA}_{\mu_{12};\alpha_1} = t^V_{\mu_2}(k_2)+mt^{PA}_{\mu_2} \quad (7.144)$$
$$g^{\mu_1\alpha_1}t^{AA}_{\mu_{12};\alpha_{12}} = t^V_{\mu_2;\alpha_2}(k_2)+mt^{PA}_{\mu_2;\alpha_2}. \quad (7.145)$$

The following subsections pursue links between the finite and divergent parts that will guide us in studying even and odd parts of Einstein and Weyl anomalies. Some passages are detailed to explain that all mathematical operations carried out follow rigorously.

7.3.1 Internal contractions: $q^{\mu}T^{VV}_{\mu\nu;\sigma}$ and $q^{\mu}T^{VV}_{\mu\nu;\sigma\lambda}$

From detailed results for amplitudes, we can proceed to the verification of RAGFs, starting with those involving vertex-index contractions (7.129) and (7.130). We expect them to remain valid to ensure the linearity of integration operation:

$$q^{\mu_1}T^{VV}_{\mu_{12};\alpha_1} = T^{V}_{\mu_2;\alpha_1}(k_1) - T^{V}_{\mu_2;\alpha_1}(k_2) = T^{V}_{(-)\mu_2;\alpha_1} \quad (7.146)$$

$$q^{\mu_1}T^{VV}_{\mu_{12};\alpha_{12}} = T^{V}_{\mu_2;\alpha_{12}}(k_1) - T^{V}_{\mu_2;\alpha_{12}}(k_2) = T^{V}_{(-)\mu_2;\alpha_{12}}. \quad (7.147)$$

As they involve differences of vector one-point functions from (7.37) and (7.38), we need to calculate these values $\{T^{V}_{\mu_1;\alpha_1}(k_i);\quad T^{V}_{\mu_1;\alpha_{12}}(k_i)\}$. These amplitudes are expressed in terms of one-point integrals $\bar{J}_{1\mu_1}(k_i)$, $\bar{J}_{1\mu_{12}}(k_i)$ and $\bar{J}_{1\mu_{123}}(k_i)$ in Eq's (C.2)-(C.7) in Appendix (C). For the amplitudes with the label k_1 as the reference momentum, expressions follow directly from definitions used in the J-integrals, namely,

$$T^{V}_{\mu_1;\alpha_1}(k_1) = 2\bar{J}_{1\mu_1\alpha_1}(k_1) \quad (7.148)$$

$$T^{V}_{\mu_2;\alpha_{12}}(k_1) = 2\bar{J}_{1\mu_1\alpha_{12}}(k_1). \quad (7.149)$$

The amplitudes with the label k_2 momentum require the translation $K_1 \to k + k_2 - q$, just for convenience because J's functions were defined using this convention, so

$$T^{V}_{\mu_1;\alpha_1}(k_2) = 2\bar{J}_{1\mu_1\alpha_1}(k_2) - 2q_{\alpha_1}\bar{J}_{1\mu_1}(k_2) \quad (7.150)$$

$$T^{V}_{\mu_2;\alpha_{12}}(k_2) = 2\bar{J}_{1\mu_2\alpha_{12}}(k_2) - 2q_{(\alpha_1}\bar{J}_{1\alpha_2)\mu_2}(k_2) + 2q_{\alpha_{12}}\bar{J}_{1\mu_2}(k_2). \quad (7.151)$$

Differences of one-point vectorial functions with one and two derivative indices are

$$\begin{aligned} T^{V}_{(-)\mu_2;\alpha_1} = {} & -q^{\nu_2}P^{\nu_1}W_{3\alpha_1\mu_2\nu_{12}} + (P_{\mu_2}q^{\nu_1} + P^{\nu_1}q_{\mu_2})\Delta_{3\alpha_1\nu_1} \\ & + (P\cdot q)\,\Delta_{3\mu_2\alpha_1} + (P_{\alpha_1} - q_{\alpha_1})\,q^{\nu_1}\Delta_{2\mu_2\nu_1}, \end{aligned} \quad (7.152)$$

$$\begin{aligned} T^{V}_{(-)\mu_2;\alpha_{12}} = {} & q^{\nu_1}W_{2\mu_2\alpha_{12}\nu_1} - q_{\mu_2}\Delta_{1\alpha_{12}} + q_{(\alpha_1}g_{\alpha_2)\mu_2}I_{\text{quad}} \\ & +\frac{1}{12}[P^{(\nu_{12}}q^{\nu_3)} + q^{\nu_{123}}]W_{4\mu_2\alpha_{12}\nu_{123}} \\ & -\frac{1}{4}[2q^{\nu_1}P^{\nu_2}P_{\mu_2} + (P^{\nu_{12}} + q^{\nu_{12}})q_{\mu_2}]W_{3\alpha_{12}\nu_{12}} \\ & -\frac{1}{4}\left[2\,(P\cdot q)\,P^{\nu_1} + q^{\nu_1}(P^2 + q^2)\right]W_{3\mu_2\alpha_{12}\nu_1} \\ & -\frac{1}{2}q^{\nu_2}P^{\nu_1}(P-q)_{(\alpha_1}W_{3\alpha_2)\mu_2\nu_{12}}] \\ & +\frac{1}{4}\left[2(P\cdot q)P_{\mu_2} + (P^2+q^2)q_{\mu_2}\right]\Delta_{2\alpha_{12}} + \frac{1}{2}(P\cdot q)(P-q)_{(\alpha_1}\Delta_{2\ \alpha_2)\mu_2} \\ & +\frac{1}{2}(P_{\mu}q^{\nu_1} + q_{\mu}P^{\nu_1})(P-q)_{(\alpha_1}\Delta_{2\ \alpha_2)\nu_1} + \frac{1}{2}(P-q)_{\alpha_1}(P-q)_{\alpha_2}q^{\nu_1}\Delta_{2\mu_2\nu_1}. \end{aligned} \quad (7.153)$$

We will begin verifying relations obtained for even amplitudes. From the relation for 2nd-order VV amplitude in (4.53), the verification for derivative amplitudes follows the same procedure. We have for the 3rd and 4th-order VV amplitude

$$q^{\mu_1}T^{VV}_{\mu_{12};\alpha_1} = 2(2q^{\mu_1}J_{2\mu_{12}\alpha_1} + q^2 J_{2\mu_2\alpha_1}) + q_{\mu_2}(2q^{\mu_1}J_{2\mu_1\alpha_1} + q^2 J_{2\alpha_1}) + q^{\mu_1}\mathcal{D}^{VV}_{\mu_{12};\alpha_1} \quad (7.154)$$

$$q^{\mu_1}T^{VV}_{\mu_{12};\alpha_{12}} = 2(2q^{\mu_1}J_{2\mu_{12}\alpha_{12}} + q^2 J_{2\mu_2\alpha_{12}}) + q_{\mu_2}(2q^{\mu_1}J_{2\mu_1\alpha_{12}} + q^2 J_{2\alpha_{12}}) + q^{\mu_1}\mathcal{D}^{VV}_{\mu_{12};\alpha_{12}}. \quad (7.155)$$

from Section (3.3), which are of the same type used in establishing constraints over odd amplitudes (by example $2q^{\mu_1}J_{2\mu_{12}\alpha_1} = -q^2 J_{2\mu_2\alpha_1}$), we have that finite part vanishes and divergent factors $q^{\mu_1}\mathcal{D}^{VV}_{\mu_{12};\alpha_1}$ and $q^{\mu_1}\mathcal{D}^{VV}_{\mu_{12};\alpha_{12}}$ satisfy identically

$$q^{\mu_1}\mathcal{D}^{VV}_{\mu_{12};\alpha_1} = T^{V}_{(-)\mu_2;\alpha_1} \quad (7.156)$$

$$q^{\mu_1}\mathcal{D}^{VV}_{\mu_{12};\alpha_{12}} = T^{V}_{(-)\mu_2;\alpha_{12}}. \quad (7.157)$$

Due to the definitions of tensors $W_{4\mu_{123456}}$ and $W_{3\mu_{1234}}$, see Section (3.1) there are hundreds of surface terms in the last relation. Although it seems complicated to verify such equality, its satisfaction follows from the observation that each of the lines that we arrange for tensor $\mathcal{D}^{VV}_{\mu_{12};\alpha_{12}}$ in (7.68) will correspond to one of the lines expressed by difference $T^{V}_{(-)\mu_2;\alpha_{12}}$ in (7.153), when contracting with momentum. We facilitate these identifications by classifying surface terms, following criteria regarding the divergence degree, tensor rank, and contraction type. For example, it is necessary to note that index μ_1 becomes a contracted index, $3q^{\mu_1}P^{\nu_{12}}W_{4\mu_{12}\alpha_{12}\nu_{12}} = P^{(\nu_{12}}q^{\nu_3)}W_{4\mu_2\alpha_{12}\nu_{123}}$. As the tensor $W_{4\mu_{123456}}$ is fully symmetric, terms are identical, and so on for all others. Expanding W combinations in primary surface terms is not necessary. In this way, relations for amplitudes at the trace level incorporate integration linearity established in (7.147) and are satisfied without restriction on the divergent parts of expressions.

7.3.2 External Contractions: $q^{\sigma}T^{VV}_{\mu\nu;\sigma}$ and $q^{\sigma}T^{VV}_{\mu\nu;\sigma\lambda}$

We have one more momentum contraction to check regarding amplitudes $T^{VV}_{\mu_{12};\alpha_1}$ and $T^{VV}_{\mu_{12};\alpha_{12}}$: they are $q^{\alpha_1}T^{VV}_{\mu_{12};\alpha_1}$ and $q^{\alpha_1}T^{VV}_{\mu_{12};\alpha_{12}}$ from (7.138). It can be made by contracting the amplitude and identifying the function of the RHS. Nevertheless, we proceed through an alternative route, using manipulations to reorganize integrands of amplitudes. Effectively these indices exchange from Dirac matrices μ_i with indices from derivative factors α_i. In this way, if previously verified relations (7.147) are satisfied, they will also be satisfied since they come from Dirac traces. We will detail calculations for relations involving amplitude with a derivative index. At the end of the operations, we expect to obtain (7.138) integrated. We will extend this result to amplitude with two derivatives, drawing attention to their differences. These two amplitudes with exchange indexes will be the basis for calculating the relations for the other even and odd amplitudes.

From definition for the amplitude $t^{VV}_{\mu_{12}}$ and $t^{VV}_{\mu_{12};\alpha_1}$, see Eqs (4.24) and (7.56), we obtain

$$2t^{(+)}_{\mu_{12}} = t^{VV}_{\mu_{12}} - g_{\mu_{12}} t^{PP} \quad (7.158)$$
$$2t^{(+)}_{\mu_{12};\alpha_1} = t^{VV}_{\mu_{12};\alpha_1} - g_{\mu_{12}} t^{PP}_{\alpha_1}. \quad (7.159)$$

It can be noted that the role of indexes position in the second tensor is

$$2t^{(+)}_{\mu_{12};\alpha_1} = 2K_{1\alpha_1}(K_{1\mu_1}K_{2\mu_2} + K_{2\mu_1}K_{1\mu_2})\frac{1}{D_{12}} \quad (7.160)$$
$$2t^{(+)}_{\alpha_1\mu_2;\mu_1} = 2K_{1\mu_1}\left(K_{1\alpha_1}K_{2\mu_2} + K_{2\alpha_1}K_{1\mu_2}\right)\frac{1}{D_{12}}, \quad (7.161)$$

where the outside term in parentheses comes from derivative contribution. Manipulating the expression for $t^{(+)}_{\mu_{12};\alpha_1}$ using $K_2 = K_1 + q$ relate both tensors, changing the role of indices $\mu_1 \leftrightarrow \alpha_1$. We use the notation to represent the antisymmetry of indices [ ]:

$$t^{(+)}_{\mu_{12};\alpha_1} = t^{(+)}_{\alpha_1\mu_2;\mu_1} - q_{[\alpha_1}K_{1\mu_1]}K_{1\mu_2}\frac{1}{D_{12}}. \quad (7.162)$$

The tensors $t^{(+)}_{\mu_{12};\alpha_1}$ and $t^{(+)}_{\alpha_1\mu_2;\mu_1}$ differ by an additional tensor from translation of K_1 momentum. Expressing $t^{(+)}$ parts in terms of VV and PP amplitudes leads to

$$t^{VV}_{\mu_{12};\alpha_1} = t^{VV}_{\alpha_1\mu_2;\mu_1} + g_{\mu_{12}}t^{PP}_{\alpha_1} - g_{\alpha_1\mu_2}t^{PP}_{\mu_1} - 2q_{[\alpha_1}K_{1\mu_1]}K_{1\mu_2}\frac{1}{D_{12}}. \quad (7.163)$$

Furthermore, the last term with this equation also has a form in terms of VV-amplitude

$$2q_{[\alpha_1}K_{1\mu_1]}K_{1\mu_2}\frac{1}{D_{12}} = \frac{1}{2}(q_{\alpha_1}t^{VV}_{\mu_{12}} - q_{\mu_1}t^{VV}_{\alpha_1\mu_2}) - \frac{1}{2}q_{[\alpha_1}g_{\mu_1]\mu_2}t^{PP} - q_{\mu_2}q_{[\alpha_1}K_{1\mu_1]}\frac{1}{D_{12}}. \quad (7.164)$$

Starting from the definition of amplitude $T^{PP}_{\alpha_1}$ (7.62) and using $K_1 = K_2 - q$, the sum of one-point vector functions appears straightforwardly

$$t^{PP}_{\alpha_1} = q^2\frac{K_{1\alpha_1}}{D_{12}} + q_{\alpha_1}\frac{1}{D_2} - \frac{1}{2}[t^V_{\alpha_1}(k_1) + t^V_{\alpha_1}(k_2)]. \quad (7.165)$$

These observations, we obtain an identity representing the exchanging of indices that facilitate the study of this relation coming from contractions involving derivatives indices

$$t^{VV}_{\mu_{12};\alpha_1} = -\frac{1}{2}q_{\alpha_1}t^{VV}_{\mu_{12}} + t^{VV}_{\alpha_1\mu_2;\mu_1} + \frac{1}{2}q_{\mu_1}t^{VV}_{\alpha_1\mu_2} - \frac{1}{2}g_{\mu_{12}}t^{V}_{(+)\alpha_1} + \frac{1}{2}g_{\alpha_1\mu_2}t^{V}_{(+)\mu_1} + r_{\mu_{12};\alpha_1}. \quad (7.166)$$

Here, $r_{\mu_{12};\alpha_1}$ is a residual term anissymetric in μ_1 and α_1

$$r_{\mu_{12};\alpha_1} = \frac{1}{2}q_{[\alpha_1}g_{\mu_1]\mu_2}\left(q^2\frac{1}{D_{12}} + \frac{1}{D_2} - \frac{1}{D_1}\right) - \theta_{\mu_2[\alpha_1}K_{1\mu_1]}\frac{1}{D_{12}}, \quad (7.167)$$

whose integration yields

$$\begin{aligned} R_{\mu_{12};\alpha_1} = & \frac{1}{2}q_{[\alpha_1}g_{\mu_1]\mu_2}q^2J_2 - q^2 g_{\mu_2[\alpha_1}J_{2\mu_1]} + q_{\mu_2}q_{[\alpha_1}J_{2\mu_1]} \\ & + \frac{1}{2}q_{[\alpha_1}g_{\mu_1]\mu_2}q^2[\bar{J}_1(k_2) - \bar{J}_1(k_1)]. \end{aligned} \quad (7.168)$$

After substitutions of $J_{2\mu}$, J_2 and $\bar{J}_1(k_i)$, this term is null, $R_{\mu_{12};\alpha_1} = 0$. It is essential to mention that we carry out passive operations. Rearranging amplitude terms does not represent any operations performed on the original amplitude. The full expression is

$$T^{VV}_{\mu_{12};\alpha_1} = -\frac{1}{2}q_{\alpha_1}T^{VV}_{\mu_{12}} + T^{VV}_{\alpha_1\mu_2;\mu_1} + \frac{1}{2}q_{\mu_1}T^{VV}_{\alpha_1\mu_2} - \frac{1}{2}g_{\mu_{12}}T^{V}_{(+)\alpha_1} + \frac{1}{2}g_{\alpha_1\mu_2}T^{V}_{(+)\mu_1}. \tag{7.169}$$

Let us analyze q^{α_1} contractions. We have already verified that RAGF is satisfied with matrix indices. Thus, we have automatic satisfaction of contractions with α_1 index

$$q^{\alpha_1}T^{VV}_{\mu_{12};\alpha_1} = -\frac{1}{2}q^2T^{VV}_{\mu_{12}} + T^{V}_{(-)\mu_2;\mu_1} + \frac{1}{2}q_{\mu_1}T^{V}_{(-)\mu_2} + \frac{1}{2}q_{\mu_2}T^{V}_{(+)\mu_1} - \frac{1}{2}g_{\mu_{12}}q^{\alpha_1}T^{V}_{(+)\alpha_1}. \tag{7.170}$$

Adding one more factor $K_{1\alpha_2}$ in (7.166), the structure is the same as the previous one,

$$\begin{aligned} t^{VV}_{\mu_{12};\alpha_{12}} &= -\frac{1}{2}q_{\alpha_1}t^{VV}_{\mu_{12};\alpha_2} + t^{VV}_{\alpha_1\mu_2;\mu_1\alpha_2} + \frac{1}{2}q_{\mu_1}t^{VV}_{\alpha_1\mu_2;\alpha_2} \\ &\quad -\frac{1}{2}g_{\mu_{12}}t^{V}_{(+)\alpha_1;\alpha_2} + \frac{1}{2}g_{\alpha_1\mu_2}t^{V}_{(+)\mu_1;\alpha_2} + r_{\mu_{12};\alpha_{12}}. \end{aligned} \tag{7.171}$$

However, we need to analyze the effect on the term $r_{\mu_{12};\alpha_{12}} = K_{1\alpha_2}r_{\mu_{12};\alpha_1}$ from (7.167),

$$r_{\mu_{12};\alpha_{12}} = \frac{1}{2}q_{[\alpha_1}g_{\mu_1]\mu_2}K_{1\alpha_2}\left(q^2\frac{1}{D_{12}} + \frac{1}{D_2} - \frac{1}{D_1}\right) - \theta_{\mu_2[\alpha_1}K_{1\mu_1]}\frac{K_{1\alpha_2}}{D_{12}}. \tag{7.172}$$

After being integrated, the residual terms can be organized as

$$\begin{aligned} R_{\mu_{12};\alpha_{12}} &= \frac{1}{2}q_{[\alpha_1}g_{\mu_1]\mu_2}\left[q^2J_{2\alpha_2} - J_{1\alpha_2}(k_1) + J_{1\alpha_2}(k_2) - q_{\alpha_2}J_1(k_2)\right] - \theta_{\mu_2[\alpha_1}\bar{J}_{2\mu_1]\alpha_2} \\ &= \frac{1}{2}q_{[\alpha_1}g_{\mu_1]\mu_2}(q^2J_{2\alpha_2} - q^{\nu_1}\Delta_{2\alpha_2\nu_1} - q_{\alpha_2}I_{\log}) - \theta_{\mu_2[\alpha_1}\bar{J}_{2\mu_1]\alpha_2}. \end{aligned} \tag{7.173}$$

This term does not cancel itself when integrated; nonetheless, its contractions do not contribute to the relations

$$2q^{\alpha_1}R_{\mu_{12};\alpha_{12}} = \theta_{\mu_{12}}\left(2q^{\alpha_1}\bar{J}_{2\alpha_{12}} + q^2J_{2\alpha_2} - q^{\nu_1}\Delta_{2\alpha_2\nu_1} - q_{\alpha_2}I_{\log}\right) = 0. \tag{7.174}$$

Furthermore, we find the same outcome for the trace

$$g^{\mu_1\alpha_1}R_{\mu_{12};\alpha_{12}} = \theta^{\alpha_1}_{\mu_2}\bar{J}_{2\alpha_1\alpha_2} - \theta^{\alpha_1}_{\mu_2}\bar{J}_{2\alpha_1\alpha_2} = 0. \tag{7.175}$$

Thus, it will not contribute to any of the contractions that remain to be verified.

$$q^{\mu_1}R_{\mu_{12};\alpha_{12}} = 0 \quad q^{\mu_2}R_{\mu_{12};\alpha_{12}} = 0 \quad q^{\alpha_1}R_{\mu_{12};\alpha_{12}} = 0 \tag{7.176}$$

$$g^{\mu_2\alpha_2}R_{\mu_{12};\alpha_{12}} = 0 \quad g^{\mu_1\alpha_1}R_{\mu_{12};\alpha_{12}} = 0 \quad g^{\mu_1\mu_2}R_{\mu_{12};\alpha_{12}} = 0. \tag{7.177}$$

The complete expression is given by

$$\begin{aligned} T^{VV}_{\mu_{12};\alpha_{12}} &= -\frac{1}{2}q_{\alpha_1}T^{VV}_{\mu_{12};\alpha_2} + T^{VV}_{\alpha_1\mu_2;\mu_1\alpha_2} + \frac{1}{2}q_{\mu_1}T^{VV}_{\alpha_1\mu_2;\alpha_2} \\ &\quad -\frac{1}{2}g_{\mu_{12}}T^{V}_{(+)\alpha_1;\alpha_2} + \frac{1}{2}g_{\alpha_1\mu_2}T^{V}_{(+)\mu_1;\alpha_2} + R_{\mu_{12};\alpha_{12}}. \end{aligned} \tag{7.178}$$

Divergent parts are not restricted to any values. Contracting the equation and using (7.174), we have the relation (7.138) satisfied for this amplitude.

7.3.3 Metric Contractions: $g^{\mu\sigma}T^{VV}_{\mu\nu;\sigma}$ and $g^{\mu\sigma}T^{VV}_{\mu\nu;\sigma\lambda}$

Relations from metric contraction (7.139) and (7.140) can be rewritten as

$$g^{\mu_1\alpha_1}t^{VV}_{\mu_1\mu_2;\alpha_1} - t^{V}_{\mu_2}(k_2) = mt^{SV}_{\mu_2} \tag{7.179}$$

$$g^{\mu_1\alpha_1}t^{VV}_{\mu_1\mu_2;\alpha_{12}} - t^{V}_{\mu_2;\alpha_2}(k_2) = mt^{SV}_{\mu_2;\alpha_2}. \tag{7.180}$$

They can be reformulated based on what was discussed for contractions involving derivative indices—in this case, exchanging $\mu_1 \leftrightarrow \alpha_1$ to get the relation. That is also valid for the permutation $\mu_2 \leftrightarrow \alpha_1$ since two matrix indices μ's are symmetric, and for the second expression, the same is valid for indexes α's. We have to the integrated (7.170)

$$2g^{\mu_1\alpha_1}T^{VV}_{\mu_{12};\alpha_1} = g^{\nu_{12}}[2T^{VV}_{\nu_{12};\mu_2} + q_{\mu_2}T^{VV}_{\nu_{12}}] + [T^{V}_{(+)\mu_2} - q^{\mu_1}T^{VV}_{\mu_{12}}]. \tag{7.181}$$

The argument follows the previous case: if the relation (4.53) is valid, then

$$2[g^{\mu_1\alpha_1}T^{VV}_{\mu_{12};\alpha_1} - T^{V}_{\mu_2}(k_2)] = g^{\nu_{12}}[2T^{VV}_{\nu_{12};\mu_2} + q_{\mu_2}T^{VV}_{\nu_{12}}]. \tag{7.182}$$

Moreover, the contraction of (7.178) is condionated by satisfaction of (7.146), therefore

$$2[g^{\mu_1\alpha_1}T^{VV}_{\mu_{12};\alpha_{12}} - T^{V}_{\mu_2;\alpha_2}(k_2)] = g^{\nu_{12}}[2T^{VV}_{\nu_{12};\mu_2\alpha_2} + q_{\mu_2}T^{VV}_{\nu_{12};\alpha_2}]. \tag{7.183}$$

If we compare these expressions with the integrated ones (7.179) and (7.180), showing their equivalence is doable. In this way, the RHS can be written as

$$g^{\nu_{12}}[2T^{VV}_{\nu_{12};\mu_2} + q_{\mu_2}T^{VV}_{\nu_{12}}] = 2mT^{SV}_{\mu_2} \tag{7.184}$$

$$g^{\nu_{12}}[2T^{VV}_{\nu_{12};\mu_2\alpha_2} + q_{\mu_2}T^{VV}_{\nu_{12};\alpha_2}] = 2mT^{SV}_{\mu_2;\alpha_2}. \tag{7.185}$$

We need traces of the VV to verify if these relations are satisfied since divergent terms will be contained in traces of $\mathcal{D}^{VV}$-parts. Nonetheless, there is a path using exclusively Υ_{μ_2} and $\Upsilon_{\alpha_2\mu_2}$ that emerged in constraint of equivalence among odd amplitudes. Explicit forms of SV-amplitudes regard J_2's integrals; therefore, let us write the results

$$T^{SV}_{\mu_2} = 2m(2J_{2\mu_2} + q_{\mu_2}J_2) = 0. \tag{7.186}$$

Even if it is identically zero due to relations among finite integrals of equal masses, we will use its terms separately in the sequel. The other

$$T^{SV}_{\mu_2;\alpha_2} = 2m(2\bar{J}_{2\mu_2\alpha_2} + q_{\mu_2}J_{2\alpha_2}) = 2m(\Delta_{2\mu_2\alpha_2} + g_{\mu_2\alpha_2}I_{\log}) + 2m(2J_{2\mu_2\alpha_2} + q_{\mu_2}J_{2\alpha_2}). \tag{7.187}$$

Beginning with Eq. (7.184), we write

$$2(g^{\nu_{12}}T^{VV}_{\nu_{12};\mu_2} - 4m^2J_{2\mu_2}) + q_{\mu_2}(g^{\nu_{12}}T^{VV}_{\nu_{12}} - 4m^2J_2) = 0. \tag{7.188}$$

It is a matter of recognizing Υ-factors; consult their explicit expressions in Eqs. (7.107) and (7.126) to write $2\Upsilon_{\mu_2} + q_{\mu_2}\Upsilon = 0$. That is a condition for compliance with RAGF derived through the metric contraction. Extending this construction to Eq. (7.185),

$$2(g^{\nu_{12}}T^{VV}_{\nu_{12};\mu_2\alpha_2} - 4m^2\bar{J}_{2\mu_2\alpha_2}) + q_{\mu_2}(g^{\nu_{12}}T^{VV}_{\nu_{12};\alpha_2} - 4m^2J_{2\alpha_2}) = 0. \tag{7.189}$$

That means $2\Upsilon_{\mu_2\alpha_2} + q_{\mu_2}\Upsilon_{\alpha_2} = 0$ due to the definition already given, see (7.127) for the explicit expression of $\Upsilon_{\mu_2\alpha_2}$. Hence, metric RAGFs are not automatically satisfied also for even amplitudes. Owing derivations until this point, we can lay down the equations:

$$g^{\nu_{12}}[2T^{VV}_{\nu_{12};\mu_2} + q_{\mu_2}T^{VV}_{\nu_{12}}] = 2mT^{SV}_{\mu_2} + 2\Upsilon_{\mu_2} + q_{\mu_2}\Upsilon \tag{7.190}$$
$$g^{\nu_{12}}[2T^{VV}_{\nu_{12};\mu_2\alpha_2} + q_{\mu_2}T^{VV}_{\nu_{12};\alpha_2}] = 2mT^{SV}_{\mu_2;\alpha_2} + 2\Upsilon_{\mu_2\alpha_2} + q_{\mu_2}\Upsilon_{\alpha_2}. \tag{7.191}$$

Alternatively, we can express them in the way it was derived

$$2g^{\mu_1\alpha_1}T^{VV}_{\mu_{12};\alpha_1} = 2T^{V}_{\mu_2}(k_2) + 2mT^{SV}_{\mu_2} + \left(2\Upsilon_{\mu_2} + q_{\mu_2}\Upsilon\right). \tag{7.192}$$
$$2g^{\mu_1\alpha_1}T^{VV}_{\mu_{12};\alpha_{12}} = 2T^{V}_{\mu_2;\alpha_2}(k_2) + 2mT^{SV}_{\mu_2;\alpha_2} + (2\Upsilon_{\mu_2\alpha_2} + q_{\mu_2}\Upsilon_{\alpha_2}). \tag{7.193}$$

The vanishing of individual violating terms Υ is enough to satisfy these relations. This constraint preserves all RAGFs in all amplitudes; however, in (7.192), combinations of violating terms can be made zero without canceling each term. That is the only place this happens; they always arise individually in other relations. Two-index combination requires that terms cancel independently. Calling for the full results (7.107) and (7.126), it is clear that violating terms in three-indices relation

$$2\Upsilon_{\mu_2} + q_{\mu_2}\Upsilon = -P^{\nu_1}[2(\Box^{\rho}_{3\rho\mu_2\nu_1} - \Delta_{2\mu_2\nu_1}) - g_{\mu_2\nu_1}\Delta^{\rho}_{2\rho}]. \tag{7.194}$$

It can be restricted to zero without each component being zero independently. As a last comment, violating factors come from suitably complex functions of momenta, physical q, or ambiguous P. Nonetheless, they are local polynomials in these variables, which can be asserted from their expressions. The remaining appears in (7.127).

Discussing if violating terms are null and the consequences of this property is a crucial point of this investigation and what perspective we can establish from conditions for RAGF satisfaction in odd amplitudes context.

7.3.4 Internal Contractions: $q^{\mu}T^{AA}_{\mu\nu;\sigma}$ and $q^{\mu}T^{AA}_{\mu\nu;\sigma\lambda}$

We must analyze RAGF for two-point amplitudes with two axial vertexes to complete relations for even amplitudes; see (7.141) and (7.142). These relations differ from those associated with vector amplitudes by an additional term given by PA-amplitudes,

$$T^{PA}_{\mu_2} = 2mq_{\mu_2}J_2 \tag{7.195}$$
$$T^{PA}_{\mu_2;\alpha_1} = 2mq_{\mu_2}J_{2\alpha_1} \tag{7.196}$$
$$T^{PA}_{\mu_2;\alpha_{12}} = 2mq_{\mu_2}\bar{J}_{2\alpha_{12}}. \tag{7.197}$$

As they exactly match the additional terms through connection with the VV-amplitudes, we have when contracting the expressions (7.78)

$$q^{\mu_1}T^{AA}_{\mu_{12}} = q^{\mu_1}T^{VV}_{\mu_{12}} - 2mT^{PA}_{\mu_2} = T^{V}_{(-)\mu_2} - 2mT^{PA}_{\mu_2} \tag{7.198}$$
$$q^{\mu_1}T^{AA}_{\mu_{12};\alpha_1} = q^{\mu_1}T^{VV}_{\mu_{12};\alpha_1} - 2mT^{PA}_{\mu_2;\alpha_1} = T^{V}_{(-)\mu_2;\alpha_1} - 2mT^{PA}_{\mu_2;\alpha_1} \tag{7.199}$$
$$q^{\mu_1}T^{AA}_{\mu_{12};\alpha_{12}} = q^{\mu_1}T^{VV}_{\mu_{12};\alpha_{12}} - 2mT^{PA}_{\mu_2;\alpha_1\alpha_2} = T^{V}_{(-)\mu_2;\alpha_{12}} - 2mT^{PA}_{\mu_2;\alpha_1\alpha_2}. \tag{7.200}$$

We have unconditional RAGF, the satisfaction established for VV-amplitudes, followed by the satisfaction of these for AA-amplitudes.

7.3.5 External Contractions: $q^{\sigma}T^{AA}_{\mu\nu;\sigma}$ and $q^{\sigma}T^{AA}_{\mu\nu;\sigma\lambda}$

To extend the results obtained in (7.166) and (7.171) for AA-amplitudes, use the relation connecting even amplitudes (7.78), (7.79), and (7.80),

$$\begin{aligned} T^{AA}_{\mu_{12};\alpha_1} &= -\frac{1}{2}q_{\alpha_1}T^{AA}_{\mu_{12}} + T^{AA}_{\alpha_1\mu_1;\mu_2} + \frac{1}{2}q_{\mu_2}T^{AA}_{\alpha_1\mu_1} + \frac{1}{2}g_{\alpha_1\mu_1}T^{V}_{(+)\mu_2} - \frac{1}{2}g_{\mu_{12}}T^{V}_{(+)\alpha_1} \\ &+2m^2\left[g_{\mu_1\alpha_1}\left(2J_{2\mu_2}+q_{\mu_2}J_2\right) - g_{\mu_{12}}\left(2J_{2\alpha_1}+q_{\alpha_1}J_2\right)\right]. \end{aligned} \quad (7.201)$$

The combination $2J_{2\mu_2}+q_{\mu_2}J_2 = 0$ cancels out the last two terms. Using (7.198) and (7.199) allows us to show that the relation with indices α_i are also automatically satisfied:

$$\begin{aligned} q^{\alpha_1}T^{AA}_{\mu_{12};\alpha_1} &= -\frac{1}{2}q^2T^{AA}_{\mu_{12}} + T^{V}_{(-)\mu_2;\mu_1} + \frac{1}{2}q_{\mu_1}T^{V}_{(-)\mu_2} + \frac{1}{2}q_{\mu_2}T^{V}_{(+)\mu_1} \\ &-\frac{1}{2}g_{\mu_{12}}q^{\nu_1}T^{V}_{(+)\nu_1} + mg_{\mu_{12}}[T^{S}(k_2) - T^{S}(k_1)]. \end{aligned} \quad (7.202)$$

We have two additional terms corresponding to PA-amplitudes from RAGF with q^{μ_1} contraction. Using (7.195) and (7.196) is easy to see which combination

$$2T^{PA}_{\mu_1;\mu_2} + q_{\mu_2}T^{PA}_{\mu_1} = 2mq_{\mu_2}\left(2J_{2\mu_1}+q_{\mu_1}J_2\right) = 0.$$

We have the cancelation $T^{P}(k_i) = 0$, and the difference between one-point functions also vanishes $T^{S}(k_2) - T^{S}(k_1) = 0$, satisfying relation (7.143).

For the expression with four indices, we have

$$\begin{aligned} T^{AA}_{\mu_{12};\alpha_{12}} &= -\frac{1}{2}q_{\alpha_1}T^{AA}_{\mu_{12};\alpha_2} + T^{AA}_{\alpha_1\mu_2;\mu_1\alpha_2} + \frac{1}{2}q_{\mu_1}T^{AA}_{\alpha_1\mu_2;\alpha_2} \\ &+\frac{1}{2}g_{\alpha_1\mu_2}T^{V}_{(+)\mu_1;\alpha_2} - \frac{1}{2}g_{\mu_{12}}T^{V}_{(+)\alpha_1;\alpha_2} + R_{\mu_{21};\alpha_{21}} \\ &+2m^2\left[g_{\mu_2\alpha_1}\left(2\bar{J}_{2\mu_1\alpha_2}+q_{\mu_1}J_{2\alpha_2}\right) - g_{\mu_{12}}\left(2\bar{J}_{2\alpha_{12}}+q_{\alpha_1}J_{2\alpha_2}\right)\right], \end{aligned} \quad (7.203)$$

where $R_{\mu_{21};\alpha_{21}}$ is defined in (7.173). Eq. (7.174) shows that contracting the form above with q produces a null result. Considering the relations (7.199)-(7.200),

$$\begin{aligned} q^{\alpha_1}T^{AA}_{\mu_{12};\alpha_{12}} &= -\frac{1}{2}q^2T^{AA}_{\mu_{12};\alpha_2} + T^{V}_{(-)\mu_2;\mu_1\alpha_2} + \frac{1}{2}q_{\mu_1}T^{V}_{(-)\mu_2;\alpha_1} + \frac{1}{2}q_{\mu_2}T^{V}_{(+)\mu_1;\alpha_2} - \frac{1}{2}g_{\mu_{12}}q^{\nu_1}T^{V}_{(+)\nu_1;\alpha_2} \\ &-m\left[\left(2T^{PA}_{\mu_2;\mu_1\alpha_2}+q_{\mu_1}T^{PA}_{\mu_2;\alpha_1}\right) - 2mq_{\mu_2}\left(2\bar{J}_{2\mu_1\alpha_2}+q_{\mu_1}J_{2\alpha_2}\right)\right] \\ &-2m^2g_{\mu_{12}}\left(\Delta_{2\alpha_2\nu_1}+g_{\alpha_2\nu_1}I_{\log}\right) - 2m^2g_{\mu_{12}}\left(2q^{\alpha_1}J_{2\alpha_{12}}+q^2J_{2\alpha_2}\right). \end{aligned} \quad (7.204)$$

Using $2q^{\alpha_1}J_{2\alpha_{12}} = -q^2J_{2\alpha_2}$, last term is null. Still, identifying other null combinations

$$\left(2T^{PA}_{\mu_2;\mu_1\alpha_2}+q_{\mu_1}T^{PA}_{\mu_2;\alpha_2}\right) = 2mq_{\mu_2}\left(2\bar{J}_{2\mu_1\alpha_2}+q_{\mu_1}J_{2\alpha_2}\right). \quad (7.205)$$

The difference between one-point scalar functions, using (C.1), (C.2), and (C.3),

$$T_{\alpha}^{S}(k_2) - T_{\alpha}^{S}(k_1) = 2m[\bar{J}_{1\alpha}(k_2) - \bar{J}_{1\alpha}(k_1) - q_{\alpha}\bar{J}_{1}(k_2)] = -2mq^{\nu}(\Delta_{2\alpha\nu} + g_{\alpha\nu}I_{\log}). \quad (7.206)$$

The relation is satisfied directly, such that

$$\begin{aligned} q^{\alpha_1}T^{AA}_{\mu_{12};\alpha_{12}} &= -\frac{1}{2}q^2 T^{AA}_{\mu_{12};\alpha_2} + T^{V}_{(-)\mu_2;\mu_1\alpha_2} + \frac{1}{2}q_{\mu_1}T^{V}_{(-)\mu_2;\alpha_1} + \frac{1}{2}q_{\mu_2}T^{V}_{(+)\mu_1;\alpha_2} \\ &\quad -\frac{1}{2}g_{\mu_{12}}q^{\nu_1}T^{V}_{(+)\nu_1;\alpha_2} + mg_{\mu_{12}}[T^{S}_{\alpha_2}(k_2) - T^{S}_{\alpha_2}(k_1)]. \end{aligned} \quad (7.207)$$

7.3.6 Metric Contractions: $g^{\mu\sigma}T^{AA}_{\mu\nu;\sigma}$ and $g^{\mu\sigma}T^{AA}_{\mu\nu;\sigma\lambda}$

The same conditions as VV-amplitudes will constrain relations involving traces,

$$g^{\mu_1\alpha_1}T^{AA}_{\mu_{12};\alpha_1} = T^{V}_{\mu_2}(k_2) + mT^{PA}_{\mu_2} + \frac{1}{2}\left(2\Upsilon_{\mu_2} + q_{\mu_2}\Upsilon\right) \quad (7.208)$$

$$g^{\mu_1\alpha_1}T^{AA}_{\mu_{12};\alpha_{12}} = T^{V}_{\mu_2;\alpha_2}(k_2) + mT^{PA}_{\mu_2;\alpha_2} + \frac{1}{2}\left(2\Upsilon_{\mu_2\alpha_2} + q_{\mu_2}\Upsilon_{\alpha_2}\right). \quad (7.209)$$

Requiring that tensors calculated on (7.107), (7.126), and (7.127) being zero leads to its satisfaction. All relations deduced for even amplitudes are symmetric by exchanges $\mu_1 \leftrightarrow \mu_2$ and $\alpha_1 \leftrightarrow \alpha_2$. To make this part complete must be noticed that if we contract with the second index μ_2 and one derivative index, we get a superficially different expression; however, two-point amplitudes in the RHS obey $T^{AP} = -T^{PA}$.

For instance, to obtain (7.208) one may use (7.79),

$$2g^{\mu_1\alpha_1}T^{AA}_{\mu_{12};\alpha_1} = 2T^{V}_{\mu_2}(k_2) + 2m(T^{SV}_{\mu_2} - 4mJ_{2\mu_2}) + \left(2\Upsilon_{\mu_2} + q_{\mu_2}\Upsilon\right). \quad (7.210)$$

Furthermore, notice that the identity $(T^{SV}_{\mu_2} - 4mJ_{2\mu_2}) = 2mq_{\mu_2}J_2 = T^{PA}_{\mu_2}$ returns the first equation we showed. The deduction steps for two derivative indices are unchanged. One could also invoke Eq. (7.201) for trading between one derivative and one matrix index; thus, taking the trace, there will appear a RAGF to inner contractions (with matrix indices), which in turn are identically satisfied as demonstrated previously. Therefore, we employ that derivation in the equation below

$$2g^{\mu_1\alpha_1}T^{AA}_{\mu_{12};\alpha_1} = g^{\mu_1\alpha_1}(2T^{AA}_{\alpha_1\mu_1;\mu_2} + q_{\mu_2}T^{AA}_{\alpha_1\mu_1}) - q^{\mu_1}T^{AA}_{\mu_{12}} + T^{V}_{(+)\mu_2} + 4m^2(2J_{2\mu_2} + q_{\mu_2}J_2). \quad (7.211)$$

Reminding that $T^{SV}_{\mu_2} = 2m\left(2J_{2\mu_2} + q_{\mu_2}J_2\right)$, final expression assumes the form

$$2g^{\mu_1\alpha_1}T^{AA}_{\mu_{12};\alpha_1} = g^{\nu_{12}}(2T^{AA}_{\nu_{12};\mu_2} + q_{\mu_2}T^{AA}_{\nu_{12}}) + 2T^{V}_{\mu_2}(k_2) + 2mT^{PA}_{\mu_2} + 2mT^{SV}_{\mu_2}. \quad (7.212)$$

After that, we transform AA into VV on the LHS following (7.208).

We finished calculating all the amplitudes and RAGF of even amplitudes. The relations involving momentum with matrix indices and derivatives are all automatically satisfied. However, in the case of traces, we saw that two groups of amplitudes presented violations by the same terms.

7.4 Odd Amplitudes: RAGFs

For odd amplitudes AV-VA, internal contractions are different by the vertex character; specifying the contraction with the axial vertex is necessary. As we saw, these relations are not satisfied without restriction, and the presence of an anomalous term is due to the existence of a chiral anomaly at this vertex,

$$q^{\mu_1} t^{AV}_{\mu_{12};\alpha_1} = [t^{A}_{\mu_2;\alpha_1}(k_1) - t^{A}_{\mu_2;\alpha_1}(k_2)] - 2m t^{PV}_{\mu_2;\alpha_1} = t^{A}_{(-)\mu_2;\alpha_1} - 2m t^{PV}_{\mu_2;\alpha_1} \quad (7.213)$$
$$q^{\mu_1} t^{AV}_{\mu_{12};\alpha_{12}} = [t^{A}_{\mu_2;\alpha_{12}}(k_1) - t^{A}_{\mu_2;\alpha_{12}}(k_2)] - 2m t^{PV}_{\mu_2;\alpha_{12}} = t^{A}_{(-)\mu_2;\alpha_{12}} - 2m t^{PV}_{\mu_2;\alpha_{12}}, \quad (7.214)$$

where $t^{A}_{(-)\mu_2;\alpha_1}$ and $t^{A}_{(-)\mu_2;\alpha_{12}}$ are associated with difference of axial one-point function,

$$t^{A}_{(-)\mu_2;\alpha_1} = t^{A}_{\mu_2;\alpha_1}(k_1) - t^{A}_{\mu_2;\alpha_1}(k_2) \quad (7.215)$$
$$t^{A}_{(-)\mu_2;\alpha_{12}} = t^{A}_{\mu_2;\alpha_{12}}(k_1) - t^{A}_{\mu_2;\alpha_{12}}(k_2). \quad (7.216)$$

Relations for vectorial vertexes are given by

$$q^{\mu_2} t^{AV}_{\mu_{12};\alpha_1} = t^{A}_{\mu_1;\alpha_1}(k_1) - t^{A}_{\mu_1;\alpha_1}(k_2) = t^{A}_{(-)\mu_1;\alpha_1} \quad (7.217)$$
$$q^{\mu_2} t^{AV}_{\mu_{12};\alpha_{12}} = t^{A}_{\mu_1;\alpha_{12}}(k_1) - t^{A}_{\mu_1;\alpha_{12}}(k_2) = t^{A}_{(-)\mu_1;\alpha_{12}}. \quad (7.218)$$

Two identities can be constructed in external contractions, as explored in the even ones. If we insert the factor (7.131) next to the first vertex we will obtain

$$q^{\alpha_1} t^{AV}_{\mu_{12};\alpha_1} = -\frac{1}{2} q^2 t^{AV}_{\mu_{12}} + t^{A}_{(-)\mu_2;\mu_1} + \frac{1}{2} q_{\mu_1} t^{A}_{(-)\mu_2} + \frac{1}{2} q_{\mu_2} t^{A}_{(+)\mu_1} - \frac{1}{2} g_{\mu_{12}} q^{\alpha_1} t^{A}_{(+)\alpha_1} \quad (7.219)$$
$$+ m\varepsilon_{\mu_{12}}[t^{S}(k_2) - t^{S}(k_1)] + m g_{\mu_{12}}[t^{P}(k_2) + t^{P}(k_1)].$$

The notation $t^{A}_{(+)\mu_1}$ is associated with the sum of the axial one-point function, namely

$$t^{A}_{(+)\mu_1} = t^{A}_{\mu_1}(k_1) + t^{A}_{\mu_1}(k_2). \quad (7.220)$$

But if we use the same identity around the second vertex, the relations are

$$q^{\alpha_1} t^{AV}_{\mu_{12};\alpha_1} = -\frac{1}{2} q^2 t^{AV}_{\mu_{12}} + t^{A}_{(-)\mu_1;\mu_2} + \frac{1}{2} q_{\mu_2} t^{A}_{(-)\mu_1} + \frac{1}{2} q_{\mu_1} t^{A}_{(+)\mu_2} - \frac{1}{2} g_{\mu_{12}} q^{\rho} t^{A}_{(+)\rho}. \quad (7.221)$$

The same to the four-indexes amplitudes, adding one index more. In addition to the relations (7.219) having additional terms when compared to (7.221). The roles of indices μ_1 and μ_2 are different. We will see its consequences in the course of this investigation.

In contractions with the metric, the indices μ_1 and μ_2 give us different relations:

$$g^{\mu_1\alpha_1} t^{AV}_{\mu_{12};\alpha_1} = t^{A}_{\mu_2}(k_2) + m t^{PV}_{\mu_2} \quad (7.222)$$
$$g^{\mu_1\alpha_1} t^{AV}_{\mu_{12};\alpha_{12}} = t^{A}_{\mu_2;\alpha_2}(k_2) + m t^{PV}_{\mu_2;\alpha_2}. \quad (7.223)$$

$$g^{\mu_2\alpha_1} t^{AV}_{\mu_{12};\alpha_1} = t^{A}_{\mu_1}(k_2) + m t^{AS}_{\mu_1} \quad (7.224)$$
$$g^{\mu_2\alpha_2} t^{AV}_{\mu_{12};\alpha_{12}} = t^{A}_{\mu_1;\alpha_1}(k_2) + m t^{AS}_{\mu_1;\alpha_1}. \quad (7.225)$$

The relations for VA amplitudes are analogous and complementary.

These relations, it is possible to establish all relations that come from the contractions for the complete expression of Gravitational Amplitude, see (7.48). Their violations or satisfactions are closely related to the symmetries to be determined. From the view of our strategy, these relations establish a minimum consistency test of amplitudes after integration. In other words, if they are satisfied, the linearity of the integration operation is maintained. Since we expect that when we explicitly calculate an amplitude, whatever calculation procedure is used, the contraction of the final result with the external momentum for each amplitude vertex should reproduce the expected RAGF. Otherwise, we can establish some relations of amplitude violations.

As we have seen in sections for even amplitudes, relations with momenta contractions are unconditionally satisfied. It was not necessary to impose any condition regarding divergent content. However, the case is somewhat different for odd amplitudes. This relation type is not trivially satisfied. Furthermore, we will show that presence of terms (7.107), (7.126), and (7.127) violate different contractions depending on AV-versions.

7.4.1 Internal Contractions: $q^{\mu}T^{AV}_{\mu\nu;\sigma}$ and $q^{\mu}T^{AV}_{\mu\nu;\sigma\lambda}$ and $V \leftrightarrow A$

Derived in Chapter (4), we have that contraction with the axial vertex for the first version of AV-amplitudes in (4.57) is violated. The mechanism develops similarly for q^{μ_2} contraction; the index meets the index inside VV-amplitude and, through its identities, implies automatic preservation of RAGF,

$$q^{\mu_1}(T^{AV}_{\mu_{12}})_1 = T^{A}_{(-)\mu_2} - 2mT^{PV}_{\mu_2} + \varepsilon_{\mu_2\mu_1}q^{\mu_1}\Upsilon \tag{7.226}$$
$$q^{\mu_2}(T^{AV}_{\mu_{12}})_1 = T^{A}_{(-)\mu_1}. \tag{7.227}$$

The second version works oppositely and satisfies relations established for q^{μ_1}. Just because the AA automatically satisfies its RAGF, the relation for index μ_2 follows with an additional term, as expected. To see this, we use the link connecting versions and obtain

$$q^{\mu_1}(T^{AV}_{\mu_{12}})_2 = T^{A}_{(-)\mu_2} - 2mT^{PV}_{\mu_2} \tag{7.228}$$
$$q^{\mu_2}(T^{AV}_{\mu_{12}})_2 = T^{A}_{(-)\mu_1} + \varepsilon_{\mu_1\nu}q^{\nu}\Upsilon. \tag{7.229}$$

Hence, to this relation type and for amplitudes with derivative indices also, the RAGF coming from q^{μ_i} contraction is directly verified if a version is $j = i$ and needs manipulation in its indices given by relations among versions (7.108) if $i = j$. In the second case arises factors $\{\Upsilon, \Upsilon_{\alpha}, \Upsilon_{\alpha_1\alpha_2}\}$ that we developed as specific tensors connecting two basic versions.

Elements that we have elaborated on are enough to establish relations for both contractions q^{μ_i} and both versions $\{(AV)_i, (VA)_i\}$ and any number of derivative indices. To do this, first, we call attention to specific results $T^{PA}_{\mu} = -T^{AP}_{\mu}$ and $T^{PV}_{\mu} = -T^{VP}_{\mu}$. This result is valid irrespective of their finite character since they do not depend on the traces employed in their calculation. Therefore, they are also helpful for structures with more indices. The required results are listed below

$$-\varepsilon_{\mu_2}{}^{\nu_1}T^{PA}_{\nu_1} = T^{PV}_{\mu_2} = -\varepsilon_{\mu_2}{}^{\nu_1}(2mq_{\nu_1}J_2) \tag{7.230}$$
$$-\varepsilon_{\mu_2}{}^{\nu_1}T^{PA}_{\nu_1;\alpha_1} = T^{PV}_{\mu_2;\alpha_1} = -\varepsilon_{\mu_2}{}^{\nu_1}(2mq_{\nu_1}J_{2\alpha_1}) \tag{7.231}$$
$$-\varepsilon_{\mu_2}{}^{\nu_1}T^{PA}_{\nu_1;\alpha_{12}} = T^{PV}_{\mu_2;\alpha_{12}} = -\varepsilon_{\mu_2}{}^{\nu_1}(2mq_{\nu_1}\bar{J}_{2\alpha_{12}}). \tag{7.232}$$

General structures of RAGFs are obtained by explicitly calculating all amplitudes

$$q^{\mu_i}(T^{AV}_{\mu_{12}})_j = T^{A}_{(-)\mu_k} - \delta_{1,i}(2mT^{PV}_{\mu_2}) + \delta_{i,j}\left(\varepsilon_{\mu_k\nu}q^{\nu}\Upsilon\right) \tag{7.233}$$
$$q^{\mu_i}(T^{AV}_{\mu_{12};\alpha_1})_j = T^{A}_{(-)\mu_k;\alpha_1} - \delta_{1,i}(2mT^{PV}_{\mu_2;\alpha_1}) + \delta_{i,j}\left(\varepsilon_{\mu_k\nu}q^{\nu}\Upsilon_{\alpha_1}\right) \tag{7.234}$$
$$q^{\mu_i}(T^{AV}_{\mu_{12};\alpha_{12}})_j = T^{A}_{(-)\mu_k;\alpha_{12}} - \delta_{1,i}(2mT^{PV}_{\mu_2;\alpha_{12}}) + \delta_{i,j}\left(\varepsilon_{\mu_k\nu}q^{\nu}\Upsilon_{\alpha_{12}}\right), \tag{7.235}$$

$i,j,k = \{1,2\}$ with $k \neq i$, and δ_{ij} is Kronecker delta equal to one if $i = j$ and zero otherwise. The formulae encode when one contracts with q^{μ_i} the version $j = i$, i.e., with vertex index where the version was defined, there is a Υ-factor, not if there is no match $i \neq j$, δ_{ij} encodes these behaviors; it also captures if contraction has a PV function (see $\delta_{1,i}$). Note that when $i \neq j$, there is no constraint over surface terms; in complementary cases, constraints are to be studied. They happen over the same Υ-factors as even amplitude traces; however, not in combination as in Subsection (7.3.3).

To complete, we ought to remind condition-less relations among VA and AV-tensors:

$$\begin{aligned} T^{VA}_{\mu_{12}} &= T^{AV}_{\mu_{12}} + 4m^2\varepsilon_{\mu_1\mu_2}J_2 \\ T^{VA}_{\mu_{12};\alpha_1} &= T^{AV}_{\mu_{12};\alpha_1} + 4m^2\varepsilon_{\mu_1\mu_2}J_{2\alpha_1} \\ T^{VA}_{\mu_{12};\alpha_{12}} &= T^{AV}_{\mu_{12};\alpha_{12}} + 4m^2\varepsilon_{\mu_1\mu_2}\bar{J}_{2\alpha_{12}}. \end{aligned} \tag{7.236}$$

As they are valid for any version, we did not use indices. Despite this, we could also study the unicity relations $(T^{VA}_{\mu_{12}})_2 - (T^{VA}_{\mu_{12}})_1 = -\varepsilon_{\mu_{21}}\Upsilon$, and so on for higher rank. In parallel to previous deductions, we can cast the pattern of contractions related to the RAGFs explicitly and in a systematic form as AV versions:

$$q^{\mu_i}(T^{VA}_{\mu_{12}})_j = T^{A}_{(-)\mu_k} + \delta_{2,i}(2mT^{VP}_{\mu_2}) + \delta_{i,j}\left(\varepsilon_{\mu_k\nu}q^{\nu}\Upsilon\right) \tag{7.237}$$
$$q^{\mu_i}(T^{VA}_{\mu_{12};\alpha_1})_j = T^{A}_{(-)\mu_k;\alpha_1} + \delta_{2,i}(2mT^{VP}_{\mu_2;\alpha_1}) + \delta_{i,j}\left(\varepsilon_{\mu_k\nu}q^{\nu}\Upsilon_{\alpha_1}\right) \tag{7.238}$$
$$q^{\mu_i}(T^{VA}_{\mu_{12};\alpha_{12}})_j = T^{A}_{(-)\mu_k;\alpha_{12}} + \delta_{2,i}(2mT^{VP}_{\mu_2;\alpha_{12}}) + \delta_{i,j}\left(\varepsilon_{\mu_k\nu}q^{\nu}\Upsilon_{\alpha_{12}}\right). \tag{7.239}$$

It is worth noticing that $\delta_{2,i}$ makes precise VP functions appear in q^{μ_2}-relations. Once more, this is a summary of the results; an important point is the appearance of conditioning factors in relations corresponding to the vertices around those we used the chiral matrix definition. As demonstrated in sections, that is equivalent to substituting (4.15).

7.4.2 External Contractions: $q^\sigma T^{AV}_{\mu\nu;\sigma}$ and $q^\sigma T^{AV}_{\mu\nu;\sigma\lambda}$ and $V \leftrightarrow A$

Treating relations involving derivative indices as we did for the even case is possible. The VV amplitudes can be manipulated and written through the identities (7.166) and (7.171); when we exchange any derivative index for a matrix index,

$$t^{VV}_{\mu_{12};\alpha_1} = -\frac{1}{2}q_{\alpha_1}t^{VV}_{\mu_{12}} + t^{VV}_{\alpha_1\mu_2;\mu_1} + \frac{1}{2}q_{\mu_1}t^{VV}_{\alpha_1\mu_2} + \frac{1}{2}g_{\alpha_1\mu_2}t^{V}_{(+)\mu_1} - \frac{1}{2}g_{\mu_{12}}t^{V}_{(+)\alpha_1} + r_{\mu_{12};\alpha_1}.$$

The exchange effect is equally valid for $\alpha_1 \leftrightarrow \mu_2$, resulting $\mu_1 \leftrightarrow \mu_2$ in the equation above.

We can get relations for odd amplitudes obtained of VV-amplitudes. Appropriately exchanging indices and multiplying by tensor $-\varepsilon_{\mu_i}^{\ \nu_1}$ leads us to unconditional identities

$$\begin{aligned}(T^{AV}_{\mu_{12};\alpha_1})_1 &= -\frac{1}{2}q_{\alpha_1}(T^{AV}_{\mu_{12}})_1 + (T^{AV}_{\mu_1\alpha_1;\mu_2})_1 + \frac{1}{2}q_{\mu_2}(T^{AV}_{\mu_1\alpha_1})_1 \\ &\quad +\frac{1}{2}\varepsilon_{\mu_1\mu_2}T^{V}_{(+)\alpha_1} - \frac{1}{2}\varepsilon_{\mu_1\alpha_1}T^{V}_{(+)\mu_2}\end{aligned} \tag{7.240}$$

$$\begin{aligned}(T^{VA}_{\mu_{12};\alpha_1})_2 &= -\frac{1}{2}q_{\alpha_1}(T^{VA}_{\mu_{12}})_2 + (T^{VA}_{\alpha_1\mu_2;\mu_1})_2 + \frac{1}{2}q_{\mu_1}(T^{VA}_{\alpha_1\mu_2})_2 \\ &\quad -\frac{1}{2}\varepsilon_{\mu_1\mu_2}T^{V}_{(+)\alpha_1} - \frac{1}{2}\varepsilon_{\mu_2\alpha_1}T^{V}_{(+)\mu_1}.\end{aligned} \tag{7.241}$$

It is necessary to remember the versions of amplitudes in terms of AA (7.201) and (7.203). Follow the other identities satisfied by odd amplitudes,

$$\begin{aligned}(T^{AV}_{\mu_{12};\alpha_1})_2 &= -\frac{1}{2}q_{\alpha_1}(T^{AV}_{\mu_{12}})_2 + (T^{AV}_{\alpha_1\mu_2;\mu_1})_2 + \frac{1}{2}q_{\mu_1}(T^{AV}_{\alpha_1\mu_2})_2 \\ &\quad -\frac{1}{2}\varepsilon_{\mu_1\mu_2}T^{V}_{(+)\alpha_1} - \frac{1}{2}\varepsilon_{\mu_2\alpha_1}T^{V}_{(+)\mu_1}\end{aligned} \tag{7.242}$$

$$\begin{aligned}(T^{VA}_{\mu_{12};\alpha_1})_1 &= -\frac{1}{2}q_{\alpha_1}(T^{VA}_{\mu_{12}})_1 + (T^{VA}_{\mu_1\alpha_1;\mu_2})_1 + \frac{1}{2}q_{\mu_2}(T^{VA}_{\mu_1\alpha_1})_1 \\ &\quad +\frac{1}{2}\varepsilon_{\mu_1\mu_2}T^{V}_{(+)\alpha_1} - \frac{1}{2}\varepsilon_{\mu_1\alpha_1}T^{V}_{(+)\mu_2}.\end{aligned} \tag{7.243}$$

By construction, we will see that these identities will always be satisfied. Starting to analyze this trajectory by the first version. From expression (7.240), we have

$$\begin{aligned}q^{\alpha_1}(T^{AV}_{\mu_{12};\alpha_1})_1 &= -\frac{1}{2}q^2(T^{AV}_{\mu_{12}})_1 + q^{\alpha_1}(T^{AV}_{\mu_1\alpha_1;\mu_2})_1 + \frac{1}{2}q_{\mu_2}[q^{\alpha_1}(T^{AV}_{\mu_1\alpha_1})_1] \\ &\quad +\frac{1}{2}\varepsilon_{\mu_1\mu_2}q^{\alpha_1}T^{V}_{(+)\alpha_1} - \frac{1}{2}\varepsilon_{\mu_1\alpha_1}q^{\alpha_1}T^{V}_{(+)\mu_2}.\end{aligned} \tag{7.244}$$

Identifying relations with internal indices that are satisfied for version one yields

$$\begin{aligned}q^{\alpha_1}(T^{AV}_{\mu_{12};\alpha_1})_1 &= -\frac{1}{2}q^2(T^{AV}_{\mu_{12}})_1 + T^{A}_{(-)\mu_1;\mu_2} + \frac{1}{2}q_{\mu_2}T^{A}_{(-)\mu_1} \\ &\quad +\frac{1}{2}\varepsilon_{\mu_1\mu_2}q^{\alpha_1}T^{V}_{(+)\alpha_1} - \frac{1}{2}\varepsilon_{\mu_1\alpha_1}q^{\alpha_1}T^{V}_{(+)\mu_2}.\end{aligned} \tag{7.245}$$

As in the last line, there is no direct identification of one-point vectorial functions with axial ones. We need to use the Schouten identity just like

$$[\varepsilon_{\mu_{12}} q^{\nu_2} - q^{\nu_1}\varepsilon_{\mu_1\nu_1}\delta^{\nu_2}_{\mu_2}]T^{V}_{(+)\nu_2} = q_{\mu_1}T^{A}_{(+)\mu_2} - g_{\mu_{12}}q^{\nu_1}T^{A}_{(+)\nu_1}. \quad (7.246)$$

Thus, replacing in equation above, we obtain

$$\begin{aligned} q^{\alpha_1}(T^{AV}_{\mu_{12};\alpha_1})_1 &= -\frac{1}{2}q^2(T^{AV}_{\mu_{12}})_1 + T^{A}_{(-)\mu_1;\mu_2} + \frac{1}{2}q_{\mu_2}T^{A}_{(-)\mu_1} \\ &\quad -\frac{1}{2}g_{\mu_{12}}q^{\nu_1}T^{A}_{(+)\nu_1} + \frac{1}{2}q_{\mu_1}T^{A}_{(+)\mu_2}. \end{aligned} \quad (7.247)$$

That is the relation obtained around the second vertex (7.221). The reason for satisfaction is that index replaced by α_i always appears as the one amplitude version, and q^{α_i} is always complimentary. In the case of $q^{\alpha_1}(T^{AV}_{\mu_1\alpha_1;\mu_2})_1$ and $q^{\alpha_1}(T^{AV}_{\mu_1\alpha_1})_1$, the RAGF for vectorial indices are automatically satisfied. The same happens contraction for $(T^{VA}_{\mu_{12};\alpha_1})_1$: contractions with axial indices are satisfied, and the additional finite part cancels out

$$-m(2T^{PV}_{\mu_2;\mu_1} + q_{\mu_1}T^{PV}_{\mu_2}) = 2m^2\varepsilon_{\mu_2\nu_1}q^{\nu_1}(2J_{2\mu_1} + q_{\mu_1}J_2) = 0. \quad (7.248)$$

So, we have the RAGF satisfied around the second vertex.

Violations occur precisely in relations established around the vertex associated with version: first vertex, thus first version, second vertex, second version. For example, the same manipulations lead to

$$\begin{aligned} q^{\alpha_1}(T^{AV}_{\mu_{12};\alpha_1})_2 &= -\frac{1}{2}q^2(T^{AV}_{\mu_{12}})_2 + T^{A}_{(-)\mu_2;\mu_1} + \frac{1}{2}q_{\mu_1}T^{A}_{(-)\mu_2} \\ &\quad +\frac{1}{2}[\varepsilon_{\mu_{21}}q^{\nu_2} - \varepsilon_{\mu_2\nu_1}\delta^{\nu_2}_{\mu_1}q^{\nu_1}]T^{V}_{(+)\nu_2} + m(2T^{PV}_{\mu_2;\mu_1} + q_{\mu_1}T^{PV}_{\mu_2}). \end{aligned} \quad (7.249)$$

Applying Schouten identity in the last line and canceling out additional finite parts,

$$\begin{aligned} q^{\alpha_1}(T^{AV}_{\mu_{12};\alpha_1})_2 &= -\frac{1}{2}q^2(T^{AV}_{\mu_{12}})_2 + T^{A}_{(-)\mu_2;\mu_1} + \frac{1}{2}q_{\mu_1}T^{A}_{(-)\mu_2} \\ &\quad +\frac{1}{2}q_{\mu_2}T^{A}_{(+)\mu_1} - \frac{1}{2}g_{\mu_{12}}q^{\nu_1}T^{A}_{(+)\nu_1}. \end{aligned} \quad (7.250)$$

It satisfies the relation deduced around the first vertex (7.219) but does not satisfy the relation deduced around the second (7.221). Remembering that massive terms do not contribute because they are null for these amplitudes.

Taking advantage of equations (7.108) and (7.109) incorporate uniqueness conditions and invariably connect them, we will have the possible violating term:

$$q^{\alpha_1}(T^{AV}_{\mu_{12};\alpha_1})_1\Big|^{\text{viol}} = -\frac{1}{2}\varepsilon_{\mu_{12}}q^{\alpha_1}\left(2\Upsilon_{\alpha_1} + q_{\alpha_1}\Upsilon\right).$$

However, let us consider that the expression obtained around the second vertex is valid (7.221). The same type of violation will be present in the second version, and the first will be automatically satisfied.

For amplitude with two derivative factors, the calculation follows equation (7.203),

$$\begin{aligned}(T^{AV}_{\mu_{12};\alpha_{12}})_2 &= -\frac{1}{2}q_{\alpha_1}(T^{AV}_{\mu_{12};\alpha_2})_2 + (T^{AV}_{\alpha_1\mu_2;\mu_1\alpha_2})_2 + \frac{1}{2}q_{\mu_1}(T^{AV}_{\alpha_1\mu_2;\alpha_2})_2 \\ &\quad -\frac{1}{2}\varepsilon_{\mu_2\alpha_1}T^{V}_{(+)\mu_1;\alpha_2} - \frac{1}{2}\varepsilon_{\mu_1\mu_2}T^{V}_{(+)\alpha_1;\alpha_2} - \varepsilon_{\mu_2}{}^{\nu_1}R_{\mu_1\nu_1;\alpha_{12}} \\ &\quad -2m^2\varepsilon_{\mu_2\alpha_1}(2\bar{J}_{2\mu_1\alpha_2} + q_{\mu_1}J_{2\alpha_2}) - 2m^2\varepsilon_{\mu_{12}}(2\bar{J}_{2\alpha_1\alpha_2} + q_{\alpha_1}J_{2\alpha_2}), \end{aligned} \tag{7.251}$$

where $R_{\mu_1\nu_1;\alpha_{12}}$ is defined in (7.173) and null by contraction. It is simple to show that version one, using (7.110), the possible violating term is given by

$$q^{\alpha_1}(T^{AV}_{\mu_{12};\alpha_{12}})_1\Big|^{\text{viol}} = -\frac{1}{2}\varepsilon_{\mu_{12}}q^{\nu}\left(2\Upsilon_{\alpha_2\nu} + q_\nu\Upsilon_{\alpha_2}\right). \tag{7.252}$$

The same analysis leads to similar conclusions for the second version of amplitudes if the relation around the second vertex is the reference.

7.4.3 Metric Contractions: $g^{\mu\sigma}T^{AV}_{\mu\nu;\sigma}$ and $g^{\mu\sigma}T^{AV}_{\mu\nu;\sigma\lambda}$

Finally, the last relation we need to calculate. Once again, we will make use of relations through a reorganization of terms that can be seen from

$$\begin{aligned}(T^{AV}_{\mu_{12};\alpha_1})_1 &= -\frac{1}{2}q_{\alpha_1}(T^{AV}_{\mu_{12}})_1 + (T^{AV}_{\mu_1\alpha_1;\mu_2})_1 + \frac{1}{2}q_{\mu_2}(T^{AV}_{\mu_1\alpha_1})_1 \\ &\quad +\frac{1}{2}\varepsilon_{\mu_1\mu_2}T^{V}_{(+)\alpha_1} - \frac{1}{2}\varepsilon_{\mu_1\alpha_1}T^{V}_{(+)\mu_2}.\end{aligned} \tag{7.253}$$

Starting by contracting the expression above with $g^{\mu_1\alpha_1}$,

$$g^{\mu_1\alpha_1}(T^{AV}_{\mu_{12};\alpha_1})_1 = \frac{1}{2}[-q^{\mu_1}(T^{AV}_{\mu_{12}})_1 + T^{A}_{(+)\mu_2}] + \frac{1}{2}g^{\mu_1\alpha_1}[2(T^{AV}_{\mu_1\alpha_1;\mu_2})_1 + q_{\mu_2}(T^{AV}_{\mu_1\alpha_1})_1]. \tag{7.254}$$

At this point, it is straightforward to note that the AV amplitude can be written as

$$g^{\mu_1\alpha_1}[2(T^{AV}_{\mu_1\alpha_1;\mu_2})_1 + q_{\mu_2}(T^{AV}_{\mu_1\alpha_1})_1] = -\varepsilon^{\alpha_1\nu_1}[2T^{VV}_{\nu_1\alpha_1;\mu_2} + q_{\mu_2}T^{VV}_{\nu_1\alpha_1}] = 0. \tag{7.255}$$

It is canceled because complete VV-amplitudes are symmetric in its first indices, finite and non-finite parts. Using (7.226) for $q^{\mu_1}(T^{AV}_{\mu_{12}})_1$, where appear $\Upsilon = (2\Delta^{\rho}_{2\rho} + i/\pi)$, follows

$$g^{\mu_1\alpha_1}(T^{AV}_{\mu_{12};\alpha_1})_1 = mT^{PV}_{\mu_2} + T^{A}_{\mu_2}(k_2) - \frac{1}{2}\varepsilon_{\mu_2\nu}q^{\nu}\Upsilon. \tag{7.256}$$

For $g^{\mu_1\alpha_1}(T^{AV}_{\mu_{12};\alpha_1})_2$ we also find this relation conditioned. Using the equation that connects two versions (7.108) and (7.109), we obtain the desired relation

$$g^{\mu_1\alpha_1}(T^{AV}_{\mu_{12};\alpha_1})_2 = mT^{PV}_{\mu_2} + T^{A}_{\mu_2}(k_2) - \frac{1}{2}\varepsilon_{\mu_2\nu}\left(2\Upsilon^{\nu} + q^{\nu}\Upsilon\right). \tag{7.257}$$

An alternative way to extract this information, valid whenever the index of inner vertices is not the one used to define the version, is to invoke the equation derived from VV or

AA functions and multiply them by an adequate tensor. Explicitly, we multiplied the equation below by $-\varepsilon_{\mu_2}{}^{\nu}$,

$$2g^{\mu_1\alpha_1}T^{AA}_{\mu_1\nu;\alpha_1} - 2T^{V}_{\nu}(k_2) = 2mT^{PA}_{\nu} + (2\Upsilon_{\nu} + q_{\nu}\Upsilon). \tag{7.258}$$

By definition, it follows

$$2g^{\mu_1\alpha_1}(T^{AV}_{\mu_1\mu_2;\alpha_1})_2 = 2mT^{PV}_{\mu_2} + 2T^{A}_{\mu_2}(k_2) - \varepsilon_{\mu_2\nu}(2\Upsilon^{\nu} + q^{\nu}\Upsilon). \tag{7.259}$$

Contracting with $g^{\mu_2\alpha_1}$, the application of equation (7.242) leads to

$$2g^{\mu_2\alpha_1}(T^{AV}_{\mu_{12};\alpha_1})_2 = -q^{\mu_2}(T^{AV}_{\mu_{12}})_2 + T^{A}_{(+)\mu_1} + g^{\nu_2\nu_1}[2(T^{AV}_{\nu_{12};\mu_1})_2 + q_{\mu_1}(T^{AV}_{\nu_{12}})_2].$$

The last line drops out by index symmetry in the AA-amplitudes. Then using (**??**) and finite piece $T^{AS}_{\mu_1} = 0$, follows

$$g^{\mu_2\alpha_1}(T^{AV}_{\mu_{12};\alpha_1})_2 = mT^{AS}_{\mu_1} + T^{A}_{\mu_1}(k_2) - \frac{1}{2}\varepsilon_{\mu_1\nu}q^{\nu}\Upsilon \tag{7.260}$$

For version one, we also find this relation violated

$$g^{\mu_2\alpha_1}(T^{AV}_{\mu_{12};\alpha_1})_1 = mT^{AS}_{\mu_1} + T^{A}_{\mu_1}(k_2) - \frac{1}{2}\varepsilon_{\mu_1\nu}(2\Upsilon^{\nu} + q^{\nu}\Upsilon). \tag{7.261}$$

The 4th-rank amplitudes with two external indices are easily obtained following the same steps. Thus, we have the list of equations below,

$$g^{\mu_i\alpha_1}(T^{AV}_{\mu_{12};\alpha_1})_j = T^{A}_{\mu_k}(k_2) + m(\delta_{i,1}T^{PV}_{\mu_k} + \delta_{i,2}T^{AS}_{\mu_k}) - \frac{1}{2}\varepsilon_{\mu_k\nu}[q^{\nu}\Upsilon + 2(1-\delta_{i,j})\Upsilon^{\nu}] \tag{7.262}$$

$$g^{\mu_i\alpha_1}(T^{AV}_{\mu_{12};\alpha_{12}})_j = T^{A}_{\mu_k;\alpha_2}(k_2) + m(\delta_{i,1}T^{PV}_{\mu_k;\alpha_2} + \delta_{i,2}T^{AS}_{\mu_k;\alpha_2}) - \frac{1}{2}\varepsilon_{\mu_k\nu}\left[q^{\nu}\Upsilon_{\alpha_2} + 2(1-\delta_{i,j})\Upsilon^{\nu}_{\alpha_2}\right], \tag{7.263}$$

where $\{i, j, k\} = \{1, 2\}$, $k \neq i$. The Kronecker delta guarantees that only correct terms appear in each equation; note that they reproduce all the previous equations. Additionally, for the VA amplitude, we have

$$g^{\mu_i\alpha_1}(T^{VA}_{\mu_{12};\alpha_1})_j = T^{A}_{\mu_k}(k_2) + m(\delta_{i,1}T^{AS}_{\mu_2} - \delta_{i,2}T^{VP}_{\mu_1}) - \frac{1}{2}\varepsilon_{\mu_k\nu}[q^{\nu}\Upsilon + 2(1-\delta_{i,j})\Upsilon^{\nu}] \tag{7.264}$$

$$g^{\mu_i\alpha_1}(T^{VA}_{\mu_{12};\alpha_{12}})_j = T^{A}_{\mu_k;\alpha_2}(k_2) + m(\delta_{i,1}T^{AS}_{\mu_2;\alpha_2} - \delta_{i,2}T^{VP}_{\mu_1;\alpha_1}) - \frac{1}{2}\varepsilon_{\mu_k\nu}\left[q^{\nu}\Upsilon_{\alpha_2} + 2(1-\delta_{i,j})\Upsilon^{\nu}_{\alpha_2}\right], \tag{7.265}$$

where $T^{AS}_{\mu_1;\alpha_1} = -2m\varepsilon_{\mu_1}{}^{\nu_1}(2\bar{J}_{2\alpha_1\nu_1} + q_{\nu_1}J_{2\alpha_1}) = T^{SA}_{\mu_1;\alpha_1}$.

We have seen in this chapter that terms that may violate the RAGFs are local polynomials in P and q momenta. These violating terms have values determined from the set (7.107), (7.126), and (7.127). We will see that choosing to save the linearity of integration operation, manifested in the satisfaction of RAGF, will force us to establish finite values for surface terms present in amplitudes. From now on, we will analyze the results' consequences and their implications for Einstein and Weyl anomalies.

Chapter 8

Gravitational Anomalies

This chapter will list the formulas and general results developed in the previous chapter as a form of organization. They are used in the sequence to track the possible violating terms of the RAGFs that appear when we combine the core elements in the permutations contributing to the full two-point functions of the energy-momentum tensor. As we will adopt the following set of indices $\left\langle T_{\mu_1\mu_2}(x)\, T_{\alpha_1\alpha_2}(0)\right\rangle$ to the energy-momentum tensors in the correlator, see Eq. (7.48), we will have

$$T^{G}_{\mu_1\mu_2\alpha_1\alpha_2} = -\frac{i}{64}\{[\mathcal{T}^{V}_{\mu_{12}\alpha_{12}}] + [\mathcal{T}^{A}_{\mu_{12}\alpha_{12}}]\}. \tag{8.1}$$

Hence, the formulas from the previous deductions have indices for even and odd amplitudes arranged according to the sequence below

$$\begin{aligned} \hat{T}^{V}_{\mu_1\mu_2\alpha_1\alpha_2} &= \mathcal{T}^{VV}_{\mu_1\alpha_1\mu_2\alpha_2} + \mathcal{T}^{AA}_{\mu_1\alpha_1\mu_2\alpha_2} \\ \hat{T}^{A}_{\mu_1\mu_2\alpha_1\alpha_2} &= \mathcal{T}^{AV}_{\mu_1\alpha_1\mu_2\alpha_2} + \mathcal{T}^{VA}_{\mu_1\alpha_1\mu_2\alpha_2}. \end{aligned}$$

The sum of permutations $\mu_1 \leftrightarrow \mu_2$ and from the result $\alpha_1 \leftrightarrow \alpha_2$ deliver the vector and axial part of the gravitational amplitude.

$$[\mathcal{T}^{V}_{\mu_{12}\alpha_{12}}] = [\hat{T}^{V}_{\mu_1\mu_2\alpha_1\alpha_2}] + [\hat{T}^{V}_{\mu_2\mu_1\alpha_1\alpha_2}] + [\hat{T}^{V}_{\mu_1\mu_2\alpha_2\alpha_1}] + [\hat{T}^{V}_{\mu_2\mu_1\alpha_2\alpha_1}] \tag{8.2}$$

$$[\mathcal{T}^{A}_{\mu_{12}\alpha_{12}}] = [\hat{T}^{A}_{\mu_1\mu_2\alpha_1\alpha_2}] + [\hat{T}^{A}_{\mu_2\mu_1\alpha_1\alpha_2}] + [\hat{T}^{A}_{\mu_1\mu_2\alpha_2\alpha_1}] + [\hat{T}^{A}_{\mu_2\mu_1\alpha_2\alpha_1}]. \tag{8.3}$$

Basic Permutations $\mathcal{T}^{\Gamma_{12}}_{\mu\alpha\sigma\rho}$: As elaborated at the beginning of the previous chapter, the next task after computing all the equations satisfied to the amplitudes is to explore the basic permutations. Through their definition, we expanded our definitions for derivative amplitudes accordingly. We have

$$\mathcal{T}^{\Gamma_1\Gamma_2}_{\mu_1\alpha_1\mu_2\alpha_2} = 4T^{\Gamma_1\Gamma_2}_{\mu_1\alpha_1;\mu_2\alpha_2} + 2q_{\mu_2}T^{\Gamma_1\Gamma_2}_{\mu_1\alpha_1;\alpha_2} + q_{\alpha_2}(2T^{\Gamma_1\Gamma_2}_{\mu_1\alpha_1;\mu_2} + q_{\mu_2}T^{\Gamma_1\Gamma_2}_{\mu_1\alpha_1}). \tag{8.4}$$

We must call attention to two features of the notation: The placement of indices in $\hat{T}^{V}_{\mu_1\mu_2\alpha_1\alpha_2}$ is chosen to mirror the ones from $T^{G}_{\mu_1\mu_2\alpha_1\alpha_2}$, however in $\mathcal{T}^{VV}_{\mu_1\alpha_1\mu_2\alpha_2}$ the disposition emphasizes that the last two indices correspond to derivative type. This attitude is helpful in the calculations to distinguish their origin, either as the matrix or derivative indices. Another point in the calligraphic letter $\mathcal{T}^{\Gamma_{12}}_{\mu\alpha\sigma\rho}$ is to contrast the 4th-rank derivative amplitude that comes with a semi-colon and the basic permutation involves four terms.

The basic permutations regarding derivatives structures were listed in (7.42)-(7.45). We resume them with the indexes:

$$\begin{aligned}
\mathcal{T}^{VV}_{\mu_1\alpha_1\mu_2\alpha_2} &= 2(2T^{VV}_{\mu_1\alpha_1;\mu_2\alpha_2} + q_{\mu_2}T^{VV}_{\mu_1\alpha_1;\alpha_2}) + q_{\alpha_2}(2T^{VV}_{\mu_1\alpha_1;\mu_2} + q_{\mu_2}T^{VV}_{\mu_1\alpha_1}) && (8.5)\\
\mathcal{T}^{AA}_{\mu_1\alpha_1\mu_2\alpha_2} &= 2(2T^{AA}_{\mu_1\alpha_1;\mu_2\alpha_2} + q_{\mu_2}T^{AA}_{\mu_1\alpha_1;\alpha_2}) + q_{\alpha_2}(2T^{AA}_{\mu_1\alpha_1;\mu_2} + q_{\mu_2}T^{AA}_{\mu_1\alpha_1}) && (8.6)\\
\mathcal{T}^{AV}_{\mu_1\alpha_1\mu_2\alpha_2} &= 2(2T^{AV}_{\mu_1\alpha_1;\mu_2\alpha_2} + q_{\mu_2}T^{AV}_{\mu_1\alpha_1;\alpha_2}) + q_{\alpha_2}(2T^{AV}_{\mu_1\alpha_1;\mu_2} + q_{\mu_2}T^{AV}_{\mu_1\alpha_1}) && (8.7)\\
\mathcal{T}^{VA}_{\mu_1\alpha_1\mu_2\alpha_2} &= 2(2T^{VA}_{\mu_1\alpha_1;\mu_2\alpha_2} + q_{\mu_2}T^{VA}_{\mu_1\alpha_1;\alpha_2}) + q_{\alpha_2}(2T^{VA}_{\mu_1\alpha_1;\mu_2} + q_{\mu_2}T^{VA}_{\mu_1\alpha_1}). && (8.8)
\end{aligned}$$

In the RAGFs, combinations of the basic derivative amplitudes from momenta and metric contractions often arise:

$$\begin{aligned}
\mathcal{B}_{\alpha_1;\alpha_2} &= \int \frac{\mathrm{d}^2 k}{(2\pi)^2} (K_1 + K_2)_{\alpha_2} \left[t^{V}_{\alpha_1}(k_1) + t^{V}_{\alpha_1}(k_2)\right] && (8.9)\\
\mathcal{S}^{\Gamma_1}_{(-)\alpha_1;\mu_2\alpha_2} &= \int \frac{\mathrm{d}^2 k}{(2\pi)^2} (K_1 + K_2)_{\mu_2} (K_1 + K_2)_{\alpha_2} [t^{\Gamma_1}_{\alpha_1}(k_1) - t^{\Gamma_1}_{\alpha_1}(k_2)], && (8.10)
\end{aligned}$$

where $\Gamma_1 = \{V, A\}$. By projecting $K_2 = K_1 + q$, we may decompose them in

$$\begin{aligned}
\mathcal{B}_{\alpha_1;\alpha_2} &= 2T^{V}_{(+)\alpha_1;\alpha_2} + q_{\alpha_2}T^{V}_{(+)\alpha_1} && (8.11)\\
\mathcal{S}^{\Gamma_1}_{(-)\alpha;\rho\sigma} &= 4T^{\Gamma_1}_{(-)\alpha;\rho\sigma} + 2q_\sigma T^{\Gamma_1}_{(-)\alpha;\rho} + 2q_\rho T^{\Gamma_1}_{(-)\alpha;\sigma} + q_\rho q_\sigma T^{\Gamma_1}_{(-)\alpha}, && (8.12)
\end{aligned}$$

being careful to remind that $T^{\Gamma_1}_{(\pm)}$ stands for the difference or sum of one-point functions

$$T^{\Gamma_1}_{(\pm)} = T^{\Gamma_1}(k_1) \pm T^{\Gamma_1}(k_2).$$

For contractions with metric, only $\mathcal{B}_{\alpha_1;\alpha_2}$ arises; for momentum contraction in matrix indices, only $\mathcal{S}^{\Gamma_1}_{(-)\alpha_1;\mu_2\alpha_2}$ is present. In contrast, for derivatives indexes, there arises both.

In the course of the previous chapter, we dealt with a set of finite functions that are identically zero due to relations among the scalar and vector J_2-integrals of for equal masses (3.70) coming from the reduction for $Z_1^{(-1)}$ (3.33). Here we list them to make it easier to follow the next stages of derivations.

$$\begin{aligned}
T^{SV}_{\mu} &= +2m\,(2J_{2\mu} + q_\mu J_2) \equiv 0 && (8.13)\\
T^{SA}_{\mu} &= -2m\varepsilon_{\mu\nu}\,(2J_2^{\nu} + q^{\nu} J_2) \equiv 0 && (8.14)\\
2T^{PV}_{\mu;\alpha} + q_\alpha T^{PV}_{\mu} &= -2m\varepsilon_{\mu\nu}q^{\nu}\,(2J_{2\alpha} + q_\alpha J_2) \equiv 0 && (8.15)\\
2T^{PA}_{\mu;\alpha} + q_\alpha T^{PA}_{\mu} &= +2mq_\mu\,(2J_{2\alpha} + q_\alpha J_2) \equiv 0 && (8.16)
\end{aligned}$$

Amplitudes with non-negative power counting that we meet by studying the RHS of RAGFs are combinations of the set $\{SV, SA, PV, PA\}$ and contain one or two derivative indices. Among those amplitudes is a set of relevant identities fully used to systematize the final results.

$$T^{SV}_{\mu_2;\alpha_2} = T^{VS}_{\mu_2;\alpha_2} = 2m(2\bar{J}_{2\mu_2\alpha_2} + q_{\mu_2}J_{2\alpha_2}) \tag{8.17}$$

$$T^{SV}_{\mu_1;\mu_2} = 2m(\Delta_{2\mu_1\mu_2} + g_{\mu_1\mu_2}I_{\log}) - \frac{im}{2\pi}\theta_{\mu_1\mu_2}[2Z_2^{(-1)} - Z_1^{(-1)}] \tag{8.18}$$

$$T^{AS}_{\mu_1;\mu_2} = T^{SA}_{\mu_1;\mu_2} = -2m\varepsilon_{\mu_1\nu}(2\bar{J}^{\nu}_{2\mu_2} + q^{\nu}J_{2\mu_2}) \tag{8.19}$$

$$T^{AS}_{\mu_2;\alpha_2} = -\varepsilon_{\mu_2}{}^{\nu}T^{VS}_{\nu;\alpha_2} \tag{8.20}$$

$$2T^{PA}_{\alpha_1;\mu_2\alpha_2} + q_{\mu_2}T^{PA}_{\alpha_1;\alpha_2} = q_{\alpha_1}T^{SV}_{\mu_2;\alpha_2} \tag{8.21}$$

$$2T^{PV}_{\alpha_1;\mu_2\alpha_2} + q_{\mu_2}T^{PV}_{\alpha_1;\alpha_2} = -\varepsilon_{\alpha_1\nu}q^{\nu}T^{SV}_{\mu_2;\alpha_2} \tag{8.22}$$

All the 4th-order tensors corresponding to a VV-AA and AV-VA can be expressed as

$$\begin{aligned}\mathcal{T}^{VV}_{\mu_1\alpha_1\mu_2\alpha_2} = &+\frac{4\Omega_{\mu_1\alpha_1\mu_2\alpha_2}}{q^2}\left\{-\frac{i}{(4\pi)}[2Z_2^{(0)} - Z_1^{(0)}] + \frac{1}{6}I_{\log}\right\} \\ &+\frac{i}{(4\pi)}\frac{8\theta_{\mu_1\alpha_1}\theta_{\mu_2\alpha_2}}{q^2}[3Z_2^{(0)} - 2Z_1^{(0)}] + \mathcal{D}^{VV}_{\mu_1\alpha_1\mu_2\alpha_2},\end{aligned} \tag{8.23}$$

with attention to their finite parts.

To express the relations due to contractions with derivative indices we list the identities needed for the exchange indices and reduce the verification to the contractions with the matrix indices (coming from Γ_i):

$$\begin{aligned}2T^{VV}_{\mu_1\alpha_1;\mu_2} + q_{\mu_2}T^{VV}_{\mu_1\alpha_1} &= 2T^{VV}_{\mu_2\alpha_1;\mu_1} + q_{\mu_1}T^{VV}_{\mu_2\alpha_1} + g_{\mu_2\alpha_1}T^{V}_{(+)\mu_1} - g_{\mu_1\alpha_1}T^{V}_{(+)\mu_2} \\ 2T^{VV}_{\mu_1\alpha_1;\mu_2\alpha_2} + q_{\mu_2}T^{VV}_{\mu_1\alpha_1;\alpha_2} &= 2T^{VV}_{\mu_2\alpha_1;\mu_1\alpha_2} + q_{\mu_1}T^{VV}_{\mu_2\alpha_1;\alpha_2} + 2R_{\mu_1\alpha_1;\mu_2\alpha_2} \\ &\quad + g_{\mu_2\alpha_1}T^{V}_{(+)\mu_1;\alpha_2} - g_{\mu_1\alpha_1}T^{V}_{(+)\mu_2;\alpha_2}.\end{aligned}$$

Multiplying by two the second identity and summing both, we have an expression of basic permutation given by

$$\mathcal{T}^{VV}_{\mu_1\alpha_1\mu_2\alpha_2} = \mathcal{T}^{VV}_{\mu_2\alpha_1;\mu_1\alpha_2} + g_{\mu_2\alpha_1}\mathcal{B}_{\mu_1;\alpha_2} - g_{\mu_1\alpha_1}\mathcal{B}_{\mu_2;\alpha_2} + 4R_{\mu_1\alpha_1;\mu_2\alpha_2}.$$

Double axial amplitudes follows (7.78)-(7.80).

Odd amplitudes: The AV-amplitudes:

$$\begin{aligned}2(T^{AV}_{\mu_1\alpha_1;\mu_2})_1 + q_{\mu_2}(T^{AV}_{\mu_1\alpha_1})_1 &= 2(T^{AV}_{\mu_1\mu_2;\alpha_1})_1 + q_{\alpha_1}(T^{AV}_{\mu_1\mu_2})_1 + \varepsilon_{\mu_1\alpha_1}T^{V}_{(+)\mu_2} - \varepsilon_{\mu_1\mu_2}T^{V}_{(+)\alpha_1} \\ 2(T^{AV}_{\mu_1\alpha_1;\mu_2})_2 + q_{\mu_2}(T^{AV}_{\mu_1\alpha_1})_2 &= 2(T^{AV}_{\mu_2\alpha_1;\mu_1})_2 + q_{\mu_1}(T^{AV}_{\mu_2\alpha_1})_2 - \varepsilon_{\alpha_1\mu_2}T^{V}_{(+)\mu_1} + \varepsilon_{\alpha_1\mu_1}T^{V}_{(+)\mu_2}\end{aligned}$$

$$\begin{aligned}2(T^{AV}_{\mu_1\alpha_1;\mu_2\alpha_2})_1 + q_{\mu_2}(T^{AV}_{\mu_1\alpha_1;\alpha_2})_1 &= 2(T^{AV}_{\mu_1\mu_2;\alpha_1\alpha_2})_1 + q_{\alpha_1}(T^{AV}_{\mu_1\mu_2;\alpha_2})_1 - 2\varepsilon_{\mu_1}{}^{\nu}R_{\alpha_1\nu;\mu_2\alpha_2} \\ &\quad + \varepsilon_{\mu_1\alpha_1}T^{V}_{(+)\mu_2;\alpha_2} - \varepsilon_{\mu_1\mu_2}T^{V}_{(+)\alpha_1;\alpha_2} \\ 2(T^{AV}_{\mu_1\alpha_1;\alpha_2\mu_2})_2 + q_{\mu_2}(T^{AV}_{\mu_1\alpha_1;\alpha_2})_2 &= 2(T^{AV}_{\mu_2\alpha_1;\mu_1\alpha_2})_2 + q_{\mu_1}(T^{AV}_{\mu_2\alpha_1;\alpha_2})_2 - 2\varepsilon_{\alpha_1}{}^{\nu_1}R_{\mu_1\nu_1;\mu_2\alpha_2} \\ &\quad - \varepsilon_{\alpha_1\mu_2}T^{V}_{(+)\mu_1;\alpha_2} + \varepsilon_{\alpha_1\mu_1}T^{V}_{(+)\mu_2;\alpha_2} \\ &\quad - 2m[\varepsilon_{\alpha_1\mu_2}T^{SV}_{\mu_1;\alpha_2} - \varepsilon_{\alpha_1\mu_1}T^{SV}_{\alpha_2;\mu_2}]\end{aligned}$$

We have omitted the VA formulas because, as was seen in the previous chapter, they are perfectly retrievable from AV ones.

8.1 Table of RAGFs

Even Amplitudes:

$$q^{\mu_1}T^{VV}_{\mu_1\alpha_1} = T^{V}_{(+)\alpha_1} \quad (8.24)$$
$$2q^{\mu_1}T^{VV}_{\mu_1\alpha_1;\alpha_2} = 2T^{V}_{(+)\alpha_1;\alpha_2} \quad (8.25)$$
$$4q^{\mu_1}T^{VV}_{\mu_1\alpha_1;\mu_2\alpha_2} = 4T^{V}_{(+)\alpha_1;\alpha_2\mu_2} \quad (8.26)$$

$$2g^{\mu_{12}}T^{VV}_{\mu_1\alpha_1;\mu_2} = 2\left[mT^{SV}_{\alpha_1} + T^{V}_{\alpha_1}(k_2)\right] + (2\Upsilon_{\alpha_1} + q_{\alpha_1}\Upsilon) \quad (8.27)$$
$$2g^{\mu_{12}}T^{VV}_{\mu_1\alpha_1;\mu_2\alpha_2} = 2\left[mT^{SV}_{\alpha_1;\alpha_2} + T^{V}_{\alpha_1;\alpha_2}(k_2)\right] + (2\Upsilon_{\alpha_1\alpha_2} + q_{\alpha_1}\Upsilon_{\alpha_2}) \quad (8.28)$$

The contractions with $g^{\alpha_{12}}$ have the same results.

Odd amplitudes:

$$q^{\mu_1}(T^{AV}_{\mu_1\alpha_1})_1 = T^{A}_{(-)\alpha_1} - 2mT^{PV}_{\alpha_1} + \varepsilon_{\alpha_1\nu_1}q^{\nu_1}\Upsilon \quad (8.29)$$
$$2q^{\mu_1}(T^{AV}_{\mu_1\alpha_1;\mu_2})_1 = 2T^{A}_{(-)\alpha_1;\mu_2} - 4mT^{PV}_{\alpha_1;\mu_2} + 2\varepsilon_{\alpha_1\nu_1}q^{\nu_1}\Upsilon_{\mu_2} \quad (8.30)$$
$$4q^{\mu_1}(T^{AV}_{\mu_1\alpha_1;\mu_2\alpha_2})_1 = 4T^{A}_{(-)\alpha_1;\mu_2\alpha_2} - 8mT^{PV}_{\alpha_1;\mu_2\alpha_2} + 4\varepsilon_{\alpha_1\nu_1}q^{\nu_1}\Upsilon_{\mu_2\alpha_2}, \quad (8.31)$$

remmember that $T^{A}_{(-)\alpha_1} = [T^{A}_{\alpha_1}(k_1) - T^{A}_{\alpha_1}(k_2)]$. The other relations for q^{α_1}-contraction,

$$q^{\alpha_1}(T^{AV}_{\mu_1\alpha_1})_1 = T^{A}_{(-)\mu_1} \quad (8.32)$$
$$2q^{\alpha_1}(T^{AV}_{\mu_1\alpha_1;\mu_2})_1 = 2T^{A}_{(-)\mu_1;\mu_2} \quad (8.33)$$
$$4q^{\alpha_1}(T^{AV}_{\mu_1\alpha_1;\mu_2\alpha_2})_1 = 4T^{A}_{(-)\mu_1;\mu_2\alpha_2}. \quad (8.34)$$

Organizing the trace relations in the form they appear in this part:

$$2g^{\mu_1\mu_2}(T^{AV}_{\mu_1\alpha_1;\mu_2})_1 = 2mT^{PV}_{\alpha_1} + 2T^{A}_{\alpha_1}(k_2) - \varepsilon_{\alpha_1\nu}q^{\nu}\Upsilon \quad (8.35)$$
$$2g^{\mu_1\mu_2}(T^{AV}_{\mu_1\alpha_1;\mu_2\alpha_2})_1 = 2mT^{PV}_{\alpha_1;\alpha_2} + 2T^{A}_{\alpha_1;\alpha_2}(k_2) - \varepsilon_{\alpha_1\nu}q^{\nu}\Upsilon_{\alpha_2} \quad (8.36)$$
$$2g^{\mu_1\mu_2}(T^{AV}_{\mu_1\alpha_1;\mu_2})_2 = 2mT^{PV}_{\alpha_1} + 2T^{A}_{\alpha_1}(k_2) - \varepsilon_{\alpha_1\nu}(2\Upsilon^{\nu} + q^{\nu}\Upsilon) \quad (8.37)$$
$$2g^{\mu_1\mu_2}(T^{AV}_{\mu_1\alpha_1;\mu_2\alpha_2})_2 = 2mT^{PV}_{\alpha_1;\alpha_2} + 2T^{A}_{\alpha_1;\alpha_2}(k_2) - \varepsilon_{\alpha_1\nu}\left(2\Upsilon^{\nu}_{\alpha_2} + q^{\nu}\Upsilon_{\alpha_2}\right) \quad (8.38)$$

$$2g^{\alpha_1\alpha_2}(T^{AV}_{\mu_1\alpha_1;\alpha_2})_1 = -\varepsilon_{\mu_1\nu}(2\Upsilon^{\nu} + q^{\nu}\Upsilon) + 2mT^{AS}_{\mu_1} + 2T^{A}_{\mu_1}(k_2) \quad (8.39)$$
$$2g^{\alpha_1\alpha_2}(T^{AV}_{\mu_1\alpha_1;\mu_2\alpha_2})_1 = -\varepsilon_{\mu_1\nu}(2\Upsilon^{\nu}_{\mu_2} + q^{\nu}\Upsilon_{\mu_2}) + 2mT^{AS}_{\mu_1;\mu_2} + 2T^{A}_{\mu_1;\mu_2}(k_2) \quad (8.40)$$
$$2g^{\alpha_1\alpha_2}(T^{AV}_{\mu_1\alpha_1;\alpha_2})_2 = -\varepsilon_{\mu_1\nu}q^{\nu}\Upsilon + 2mT^{AS}_{\mu_1} + 2T^{A}_{\mu_1}(k_2) \quad (8.41)$$
$$2g^{\alpha_1\alpha_2}(T^{AV}_{\mu_1\alpha_1;\mu_2\alpha_2})_2 = -\varepsilon_{\mu_1\nu}q^{\nu}\Upsilon_{\mu_2} + 2mT^{AS}_{\mu_1;\mu_2} + 2T^{A}_{\mu_1;\mu_2}(k_2) \quad (8.42)$$

8.1.1 Even amplitudes: $(\mathcal{T}^{VV}_{\mu\alpha\sigma\rho})$ and $(\mathcal{T}^{AA}_{\mu\alpha\sigma\rho})$

From now on, we will systematically explore all the results from the amplitude combinations that effectively appear in the relations for the gravitational amplitude. Starting by (7.78)-(7.80) follows

$$\mathcal{T}^{AA}_{\mu_1\alpha_1\mu_2\alpha_2} = \mathcal{T}^{VV}_{\mu_1\alpha_1\mu_2\alpha_2} - 2mg_{\mu_1\alpha_1}(4\bar{J}_{2\mu_2\alpha_2} + 2q_{\alpha_2}J_{2\mu_2}), \quad (8.43)$$

The terms corresponding to the J-vector and J-scalar functions do not appear because their combination is null. The relation is given by

$$\mathcal{T}^{AA}_{\mu_1\alpha_1\mu_2\alpha_2} = \mathcal{T}^{VV}_{\mu_1\alpha_1\mu_2\alpha_2} - 4mg_{\mu_1\alpha_1}T^{SV}_{\mu_2;\alpha_2}. \tag{8.44}$$

That amounts to replacing double axial structures for the double vector diminishing the number of operations necessary to express the relevant results.

Internal Contractions

The contractions with internal indices for these amplitudes follow from the definition

$$q^{\mu_1}\mathcal{T}^{VV}_{\mu_1\alpha_1\mu_2\alpha_2} = 2q^{\mu_1}(2T^{VV}_{\mu_1\alpha_1;\mu_2\alpha_2} + q_{\mu_2}T^{VV}_{\mu_1\alpha_1;\alpha_2}) + q^{\mu_1}(2q_{\alpha_2}T^{VV}_{\mu_1\alpha_1;\mu_2} + q_{\alpha_2}q_{\mu_2}T^{VV}_{\mu_1\alpha_1}). \tag{8.45}$$

The index of q^{μ_1} hits only the matrix vertex of the amplitude, and the consequence is that only the difference of one-point functions appears, see (7.129), (7.146) and (7.147). Hence, employing our definition

$$\mathcal{S}^{V}_{(-)\alpha_1;\alpha_2\mu_2} = 2[2T^{V}_{(-)\alpha_1;\mu_2\alpha_2} + q_{\mu_2}T^{V}_{(-)\alpha_1;\alpha_2}] + q_{\alpha_2}[2T^{V}_{(-)\alpha_1;\mu_2} + q_{\alpha_2}T^{V}_{(-)\alpha_1}], \tag{8.46}$$

the equation obtained reads

$$q^{\mu_1}\mathcal{T}^{VV}_{\mu_1\alpha_1\mu_2\alpha_2} = \mathcal{S}^{V}_{(-)\alpha_1;\alpha_2\mu_2} = \mathcal{S}^{V}_{\alpha_1;\mu_2\alpha_2}(k_1) - \mathcal{S}^{V}_{\alpha_1;\mu_2\alpha_2}(k_2). \tag{8.47}$$

Note the symmetry in the indices corresponding to derivatives, $\mathcal{S}^{\Gamma_1}_{\alpha_1;\mu_2\alpha_2} = \mathcal{S}^{\Gamma_1}_{\alpha_1;\alpha_2\mu_2}$.

For the $\mathcal{T}^{AA}$, we could either use for its contraction the PA's as in (7.198)-(7.200),

$$q^{\mu_1}\mathcal{T}^{AA}_{\mu_1\alpha_1\mu_2\alpha_2} = -2m[4T^{PA}_{\alpha_1;\mu_2\alpha_2} + 2q_{\mu_2}T^{PA}_{\alpha_1;\alpha_2} + q_{\alpha_2}(2T^{PA}_{\alpha_1;\mu_2} + q_{\mu_2}T^{PA}_{\alpha_1})] + \mathcal{S}^{V}_{(-)\alpha_1;\mu_2\alpha_2} \tag{8.48}$$

which is their composition of RAGFs. Using the connection with $\mathcal{T}^{VV}$ (8.44), we have

$$q^{\mu_1}\mathcal{T}^{AA}_{\mu_1\alpha_1\mu_2\alpha_2} = \mathcal{S}^{V}_{(-)\alpha_1;\mu_2\alpha_2} - 4mq_{\alpha_1}T^{SV}_{\mu_2;\alpha_2}. \tag{8.49}$$

The PA amplitudes did not appear since they are related to derivative SV through (8.16) and (8.21). In this amplitude, if the operation is done in α_1, the RHS shows AP-structures, however, with the opposite sign. As $T^{AP}_{\mu} = -T^{PA}_{\mu}$ and so on for more indices, hence the results written in terms of $T^{SV}_{\mu_2;\alpha_2}$ amplitude have the same functional form.

External Contractions

Terms from relations involving the derivative indices organize in the tensor $\mathcal{B}_{\mu;\alpha}$ besides $S^{V}_{\alpha_1;\alpha_2\mu_1}$, see Eqs. (8.11) and (8.12). To see this, we combine the identities used to trade a derivative for a matrix index, as in (7.166) and (7.171). We have,

$$\mathcal{T}^{VV}_{\mu_1\alpha_1;\mu_2\alpha_2} = \mathcal{T}^{VV}_{\mu_2\alpha_1;\mu_1\alpha_2} + g_{\mu_2\alpha_1}\mathcal{B}_{\mu_1;\alpha_2} - g_{\mu_1\alpha_1}\mathcal{B}_{\mu_2;\alpha_2} + 2R_{\mu_1\alpha_1;\mu_2\alpha_2}.$$

Note the presence of $B_{\sigma;\rho}$ and the residual term, which always vanishes under contraction. Contracting with q^{μ_2}, the first term in the RHS, we have μ_2 index in the position of a matrix index, whose result we developed previously. Follows the compact result

$$q^{\mu_2}\mathcal{T}^{VV}_{\mu_1\alpha_1;\mu_2\alpha_2} = \mathcal{S}^{V}_{(-)\alpha_1;\mu_1\alpha_2} + q_{\alpha_1}\mathcal{B}_{\mu_1;\alpha_2} - g_{\mu_1\alpha_1}q^{\nu}\mathcal{B}_{\nu;\alpha_2}. \tag{8.50}$$

For the $\mathcal{T}^{AA}$, we substitute the equation (8.44) into the last one, what implies in

$$q^{\mu_2}\mathcal{T}^{AA}_{\mu_1\alpha_1\mu_2\alpha_2} = \mathcal{S}^{V}_{(-)\alpha_1;\mu_1\alpha_2} + q_{\alpha_1}\mathcal{B}_{\mu_1;\alpha_2} - g_{\mu_1\alpha_1}q^{\nu}\mathcal{B}_{\nu;\alpha_2} - 4mg_{\mu_1\alpha_1}T^{S}_{(-)\alpha_2}. \tag{8.51}$$

remember that $q^{\nu}T^{SV}_{\nu;\alpha_2} = T^{S}_{\alpha_2}(k_1) - T^{S}_{\alpha_2}(k_2) = T^{S}_{(-)\alpha_2}$. There does not exist any condition for the momentum RAGFs. A different scenario occurs to the metric RAGFs.

Metric Contractions

These relations combine the metric relations of the basic derivative amplitudes and the momentum relations for the matrix indices. Make explicit this property by

$$g^{\mu_{12}}\mathcal{T}^{VV}_{\mu_1\alpha_1\mu_2\alpha_2} = 2g^{\mu_{12}}(2T^{VV}_{\mu_1\alpha_1;\mu_2\alpha_2} + q_{\alpha_2}T^{VV}_{\mu_1\alpha_1;\mu_2}) + q^{\mu_1}(2T^{VV}_{\mu_1\alpha_1;\alpha_2} + q_{\alpha_2}T^{VV}_{\mu_1\alpha_1}). \tag{8.52}$$

The next stage is observing that momentum RAGFs in even amplitudes are automatically satisfied. Replacing them and e summing with the equations for metric contractions (7.192) and (7.193), we arrive at

$$\begin{aligned} g^{\mu_{12}}\mathcal{T}^{VV}_{\mu_1\alpha_1\mu_2\alpha_2} \;=\;& +4mT^{SV}_{\alpha_1;\alpha_2} + [2T^{V}_{(+)\alpha_1;\alpha_2} + q_{\alpha_2}T^{V}_{(+)\alpha_1}] \\ & +4\Upsilon_{\alpha_1\alpha_2} + 2q_{\alpha_1}\Upsilon_{\alpha_2} + 2q_{\alpha_2}\Upsilon_{\alpha_1} + q_{\alpha_2}q_{\alpha_1}\Upsilon, \end{aligned} \tag{8.53}$$

where we used the pattern that appears in one-point functions, $T^{V}_{(-)\alpha_1} + 2T^{V}_{\alpha_1}(k_2) = T^{V}_{(+)\alpha_1}$. We dropped the $T^{SV}_{\alpha_1} = 0$ term.

The conditioning factors $\{\Upsilon, \Upsilon_{\alpha_1}, \Upsilon_{\alpha_1\alpha_2}\}$ were combined in a fundamental tensor called uniqueness factor; it will encompass the conditions for satisfaction of all RAGFs as well the equivalence of the odd-amplitude versions. Because of its importance, we define it as

$$U_{\alpha_1\alpha_2} = 4\Upsilon_{\alpha_1\alpha_2} + 2q_{\alpha_1}\Upsilon_{\alpha_2} + 2q_{\alpha_2}\Upsilon_{\alpha_1} + q_{\alpha_2}q_{\alpha_1}\Upsilon. \tag{8.54}$$

The investigation of values assumed to this tensor and its connection to the finite part and surface terms will be developed soon. Thus, we have the compact expression

$$g^{\mu_{12}}\mathcal{T}^{VV}_{\mu_1\alpha_1\mu_2\alpha_2} = 4mT^{SV}_{\alpha_1;\alpha_2} + \mathcal{B}_{\alpha_1;\alpha_2} + U_{\alpha_1\alpha_2}. \tag{8.55}$$

The relations for $g^{\alpha_{12}}$-contraction are identical, changing the indices $\mu_{12} \leftrightarrow \alpha_{12}$.

Calculating directly or using the relation (8.44) between $\mathcal{T}^{AA}$ and $\mathcal{T}^{VV}$, follows

$$g^{\mu_{12}}\mathcal{T}^{AA}_{\mu_1\alpha_1\mu_2\alpha_2} = \mathcal{B}_{\alpha_1;\alpha_2} + U_{\alpha_1\alpha_2}, \tag{8.56}$$

$$g^{\alpha_{12}}\mathcal{T}^{AA}_{\mu_1\alpha_1\mu_2\alpha_2} = \mathcal{B}_{\mu_1;\mu_2} + U_{\mu_1\mu_2}. \tag{8.57}$$

Uniqueness factor: The definitions follow in (7.107), (7.126) and (7.127), thus

$$\begin{aligned} U_{\alpha_1\alpha_2} &= -\frac{1}{3}\theta_{\alpha_1\alpha_2}\Upsilon \\ &+\frac{1}{9}(3P^{\nu_{12}}+q^{\nu_{12}})\left[3\Sigma^{\rho}_{4\rho\alpha_{12}\nu_{12}}-8\Box_{3\alpha_{12}\nu_{12}}-g_{(\alpha_{12}}g_{\nu_{12})}\Delta^{\rho}_{2\rho}\right] \\ &+\frac{1}{18}(3P^{\nu_{12}}+q^{\nu_{12}})g_{(\alpha_{12}}[2\Box^{\rho}_{3\nu_{12})\rho}-2\Delta_{2\nu_{12})}-g_{\nu_{12})}\Delta^{\rho}_{2\rho}] \\ &-P_{\alpha_2}P^{\nu}[2(\Box^{\rho}_{3\rho\alpha_1\nu}-\Delta_{2\alpha_1\nu})-g_{\alpha_1\nu}\Delta^{\rho}_{2\rho}] \\ &-P_{\alpha_1}P^{\nu}[2(\Box^{\rho}_{3\rho\alpha_2\nu}-\Delta_{2\alpha_2\nu})-g_{\alpha_2\nu}\Delta^{\rho}_{2\rho}] \\ &-\frac{1}{2}(P^2+q^2)[2(\Box^{\rho}_{3\rho\alpha_{12}}-\Delta_{2\alpha_{12}})-g_{\alpha_{12}}\Delta^{\rho}_{2\rho}] \\ &+4[(W^{\rho}_{2\rho\alpha_{12}}-2\Delta_{1\alpha_{12}})+2g_{\alpha_1\alpha_2}I_{\text{quad}}-2m^2\left(\Delta_{2\alpha_{12}}+g_{\alpha_{12}}I_{\log}\right)]. \end{aligned} \tag{8.58}$$

8.1.2 Odd Amplitudes: $(\mathcal{T}^{AV}_{\mu\alpha\sigma\rho})$ and $(\mathcal{T}^{VA}_{\mu\alpha\sigma\rho})$

In this part, a series of considerations are in order. The decomposition in derivatives was taken to the most basic level; a set of possibilities from Dirac traces is fully exploited. We came out with two independent forms, version one and two, as we called them. Now, for any term of the basic permutation, an arbitrary version choice must be made because the choice of traces employed is arbitrary. Nonetheless, even if the analysis can be performed in the most general scenario, we will adopt the position of considering the uniform version, where $\mathcal{T}^{\Gamma_{12}}_{\mu\alpha\sigma\rho}$ is an odd tensor. Then we will have the notation

$$\begin{aligned} (\mathcal{T}^{AV}_{\mu_1\alpha_1\mu_2\alpha_2})_i &= 2[2(T^{AV}_{\mu_1\alpha_1;\mu_2\alpha_2})_i+2q_{\mu_2}(T^{AV}_{\mu_1\alpha_1;\alpha_2})_i]+ \\ &+q_{\alpha_2}[2(T^{AV}_{\mu_1\alpha_1;\mu_2})_i+q_{\mu_2}(T^{AV}_{\mu_1\alpha_1})_i] \end{aligned} \tag{8.59}$$

$$\begin{aligned} (\mathcal{T}^{VA}_{\mu_1\alpha_1\mu_2\alpha_2})_i &= 2[2(T^{VA}_{\mu_1\alpha_1;\mu_2\alpha_2})_i+q_{\mu_2}(T^{VA}_{\mu_1\alpha_1;\alpha_2})_i]+ \\ &+q_{\alpha_2}[2(T^{VA}_{\mu_1\alpha_1;\mu_2})_i+q_{\mu_2}(T^{VA}_{\mu_1\alpha_1})_i], \end{aligned} \tag{8.60}$$

with $i=1,2$. In this moment we may use the transition equations (7.108)-(7.110) to derive the relations among what we call basic permutations

$$(\mathcal{T}^{AV}_{\mu_1\alpha_1\mu_2\alpha_2})_2=(\mathcal{T}^{AV}_{\mu_1\alpha_1\mu_2\alpha_2})_1-\varepsilon_{\alpha_1\mu_1}\left(4\Upsilon_{\mu_2\alpha_2}+2q_{\mu_2}\Upsilon_{\alpha_2}+2q_{\alpha_2}\Upsilon_{\mu_2}+q_{\alpha_2}q_{\mu_2}\Upsilon\right). \tag{8.61}$$

In the RHS appear, the U-factor, making it simpler to express the uniqueness relation as

$$(\mathcal{T}^{AV}_{\mu_1\alpha_1\mu_2\alpha_2})_2=(\mathcal{T}^{AV}_{\mu_1\alpha_1\mu_2\alpha_2})_1-\varepsilon_{\alpha_1\mu_1}U_{\mu_2\alpha_2}. \tag{8.62}$$

Analogously the transition between AA-VV, the amplitude $\mathcal{T}^{VA}_{\mu_1\alpha_1\mu_2\alpha_2}$ can be written in term of $\mathcal{T}^{AV}_{\mu_1\alpha_1\mu_2\alpha_2}$, in a way independent of traces employed. See Eqs. (7.93)-(7.95) to derive the relation

$$(\mathcal{T}^{VA}_{\mu_1\alpha_1\mu_2\alpha_2})_i=(\mathcal{T}^{AV}_{\mu_1\alpha_1\mu_2\alpha_2})_i+4m^2\varepsilon_{\mu_1\alpha_1}\left[4\bar{J}_{2\mu_2\alpha_2}+2q_{\alpha_2}J_{2\mu_2}+q_{\mu_2}\left(2J_{2\alpha_2}+q_{\alpha_2}J_2\right)\right], \tag{8.63}$$

using (8.13)-(8.17) to identify the integrals as amplitudes, we obtain VA-AV connection

$$(\mathcal{T}^{VA}_{\mu_1\alpha_1\mu_2\alpha_2})_i=(\mathcal{T}^{AV}_{\mu_1\alpha_1\mu_2\alpha_2})_i+4m\varepsilon_{\mu_1\alpha_1}T^{SV}_{\mu_2;\alpha_2}. \tag{8.64}$$

This enables us to study only the versions $(\mathcal{T}^{AV})_1$.

8.1.3 Permutation's versions: $(\mathcal{T}^{AV}_{\mu\alpha\sigma\rho})_1$ and $(\mathcal{T}^{AV}_{\mu\alpha\sigma\rho})_2$

Momentum: Internal Contractions

To make apparent the notation's use, let us explore the internal contraction with $q^{\mu}(\mathcal{T}^{AV}_{\mu\alpha\sigma\rho})_i$. We begin with the definition (8.59) and the formulas generalized in (7.233)-(7.235). Notice that those relations turn up with Υ factors; summing the contributions,

$$\begin{aligned} q^{\mu_1}(\mathcal{T}^{AV}_{\mu_1\alpha_1\mu_2\alpha_2})_1 &= -2m[4T^{PV}_{\alpha_1;\mu_2\alpha_2}+2q_{\mu_2}T^{PV}_{\alpha_1;\alpha_2}+2q_{\alpha_2}T^{PV}_{\alpha_1;\mu_2}+q_{\alpha_2}q_{\mu_2}T^{PV}_{\alpha_1}] \\ &+\mathcal{S}^{A}_{\alpha_1;\alpha_2\mu_2}(k_1)-\mathcal{S}^{A}_{\alpha_1;\alpha_2\mu_2}(k_2)+\varepsilon_{\alpha_1\nu_1}q^{\nu_1}U_{\mu_2\alpha_2}. \end{aligned} \tag{8.65}$$

We gathered the one-point functions in our definition of $\mathcal{S}^{A}_{\alpha_1;\alpha_2\mu_2}$. The identities (8.22) and (8.15) involving the PV enables one to write the result

$$q^{\mu_1}(\mathcal{T}^{AV}_{\mu_1\alpha_1\mu_2\alpha_2})_1 = 4m\varepsilon_{\alpha_1\nu}q^{\nu}T^{SV}_{\mu_2;\alpha_2}+\mathcal{S}^{A}_{(-)\alpha_1;\alpha_2\mu_2}+\varepsilon_{\alpha_1\nu_1}q^{\nu_1}U_{\mu_2\alpha_2}. \tag{8.66}$$

For the contraction with q^{α_1}, the relations to the component amplitudes are identically satisfied. Hence there are no Υ factors, namely

$$q^{\alpha_1}(\mathcal{T}^{AV}_{\mu_1\alpha_1\mu_2\alpha_2})_1 = \mathcal{S}^{A}_{(-)\mu_1;\mu_2\alpha_2}=\mathcal{S}^{A}_{\mu_1;\mu_2\alpha_2}(k_1)-\mathcal{S}^{A}_{\mu_1;\mu_2\alpha_2}(k_2). \tag{8.67}$$

The other form of the basic permutation will readily comply with the equations

$$q^{\mu_1}(\mathcal{T}^{AV}_{\mu_1\alpha_1\mu_2\alpha_2})_2 = \mathcal{S}^{A}_{(-)\alpha_1;\alpha_2\mu_2}+4m\varepsilon_{\alpha_1\nu}q^{\nu}T^{SV}_{\mu_2;\alpha_2} \tag{8.68}$$

$$q^{\alpha_1}(\mathcal{T}^{AV}_{\mu_1\alpha_1\mu_2\alpha_2})_2 = \mathcal{S}^{A}_{(-)\mu_1;\mu_2\alpha_2}+\varepsilon_{\mu_1\nu}q^{\nu}U_{\mu_2\alpha_2}. \tag{8.69}$$

Momentum: External contractions

We have one identity automatically satisfied and one with U-factor. Beginning by

$$\begin{aligned} q^{\mu_2}(\mathcal{T}^{AV}_{\mu_1\alpha_1\mu_2\alpha_2})_1 &= 4q^{\mu_2}(T^{AV}_{\mu_1\alpha_1;\mu_2\alpha_2})_1+2q^2(T^{AV}_{\mu_1\alpha_1;\alpha_2})_1 \\ &+2q_{\alpha_2}q^{\mu_2}(T^{AV}_{\mu_1\alpha_1;\mu_2})_1+q_{\alpha_2}q^2(T^{AV}_{\mu_1\alpha_1})_1. \end{aligned} \tag{8.70}$$

The equation below can be written in compact form through the use of formulae developed before that do not require any new ingredient but careful application,

$$q^{\mu_2}(\mathcal{T}^{AV}_{\mu_1\alpha_1\mu_2\alpha_2})_1 = \mathcal{S}^{A}_{(-)\mu_1;\alpha_1\alpha_2}+\varepsilon_{\mu_1\alpha_1}q^{\nu}\mathcal{B}_{\nu;\alpha_2}-\varepsilon_{\mu_1\nu}q^{\nu}\mathcal{B}_{\alpha_1;\alpha_2}. \tag{8.71}$$

Making one more manipulation by using $\varepsilon_{[\mu_1\alpha_1}\mathcal{B}_{\nu];\alpha_2}=0$, follows the final form

$$q^{\mu_2}(\mathcal{T}^{AV}_{\mu_1\alpha_1\mu_2\alpha_2})_1 = \mathcal{S}^{A}_{(-)\mu_1;\alpha_1\alpha_2}-\varepsilon_{\alpha_1\nu}q^{\nu}\mathcal{B}_{\mu_1;\alpha_2}. \tag{8.72}$$

The version $(\mathcal{T}^{AV}_{\mu_1\alpha_1\mu_2\alpha_2})_2$ also have a relation which is satisfied by construction, namely,

$$q^{\mu_2}(\mathcal{T}^{AV}_{\mu_1\alpha_1\mu_2\alpha_2})_2 = \mathcal{S}^{A}_{(-)\alpha_1;\alpha_2\mu_1}-\varepsilon_{\mu_1\nu}q^{\nu}\mathcal{B}_{\alpha_1;\alpha_2}-4m\varepsilon_{\mu_1\alpha_1}\left[T^{S}_{\alpha_2}(k_1)-T^{S}_{\alpha_2}(k_2)\right]. \tag{8.73}$$

For this, we have observed the combination of two-point functions (8.15) and (8.22).

Now, the relations where arises the Υ factors came from the use of the equation that exists between the versions (8.62). They furnish

$$\begin{aligned} q^{\mu_2}(\mathcal{T}^{AV}_{\mu_1\alpha_1\mu_2\alpha_2})_1 &= \mathcal{S}^A_{(-)\alpha_1;\alpha_2\mu_1} - \varepsilon_{\mu_1\nu}q^\nu\mathcal{B}_{\alpha_1;\alpha_2} + \varepsilon_{\alpha_1\mu_1}q^\nu U_{\nu\alpha_2} - 4m\varepsilon_{\mu_1\alpha_1}T^S_{(-)\alpha_2} \quad (8.74)\\ q^{\mu_2}(\mathcal{T}^{AV}_{\mu_1\alpha_1\mu_2\alpha_2})_2 &= \mathcal{S}^A_{(-)\mu_1;\alpha_1\alpha_2} - \varepsilon_{\alpha_1\nu}q^\nu\mathcal{B}_{\mu_1;\alpha_2} - \varepsilon_{\alpha_1\mu_1}q^\nu U_{\nu\alpha_2}. \quad (8.75)\end{aligned}$$

Two forms obtained for these relations are equivalent. As we saw, we always kept intact all terms where the results could deviate. Therefore is straightforward to see that they ought to be equal. Moreover, the ones with violating terms are obtained by employing those free of U-term, using an identity again. Even so, if one desires to check such a statement explicitly, the path is reasonably long but feasible. Here we give the directions; start by using $\mathcal{S}^A_{\sigma;\alpha\rho} = -\varepsilon^\nu_\sigma\mathcal{S}^V_{\nu;\alpha\rho}$, then subtract the identities without U and with U,

$$\begin{aligned} q^{\mu_2}[(\mathcal{T}^{AV}_{\mu_1\alpha_1\mu_2\alpha_2})_1 - (\mathcal{T}^{AV}_{\mu_1\alpha_1\mu_2\alpha_2})_1] &= \varepsilon_{\alpha_1}{}^\nu\mathcal{S}^V_{(-)\nu;\alpha_2\mu_1} - \varepsilon_{\mu_1}{}^\nu\mathcal{S}^V_{(-)\nu;\alpha_{12}} - \varepsilon_{\alpha_1\mu_1}q^\nu U_{\nu\alpha_2} \quad (8.76)\\ &\quad -\varepsilon_{\alpha_1\nu}q^\nu\mathcal{B}_{\mu_1;\alpha_2} + \varepsilon_{\mu_1\nu}q^\nu\mathcal{B}_{\alpha_1;\alpha_2} - 4m\varepsilon_{\alpha_1\mu_1}T^S_{(-)\alpha_2},\end{aligned}$$

employing the identities $g^{\nu\rho}\mathcal{S}^V_{\nu;\rho[\alpha}\varepsilon_{\sigma\rho]} = 0$ and $\varepsilon_{[\alpha_1\nu}\mathcal{B}_{\mu_1];\alpha_2} = 0$, we obtain an expression where everything is known and whose summation cancels without any conditions,

$$\varepsilon_{\alpha_1\mu_1}\{g^{\nu\rho}\mathcal{S}^V_{(-)\nu;\rho\alpha_2} - q^\nu\mathcal{B}_{\nu;\alpha_2} - 4mT^S_{(-)\alpha_2} - q^\nu U_{\nu\alpha_2}\} \equiv 0. \quad (8.77)$$

Metric Contractions

We use the form $(\mathcal{T}^{AV})_i$ and perform the analysis for $g^{\mu_{12}}$ and $g^{\alpha_{12}}$. First, we have

$$\begin{aligned} g^{\mu_{12}}(\mathcal{T}^{AV}_{\mu_1\alpha_1\mu_2\alpha_2})_1 &= 4g^{\mu_{12}}(T^{AV}_{\mu_1\alpha_1;\mu_2\alpha_2})_1 + 2q^{\mu_1}(T^{AV}_{\mu_1\alpha_1;\alpha_2})_1 \quad (8.78)\\ &\quad +q_{\alpha_2}[2g^{\mu_{12}}(T^{AV}_{\mu_1\alpha_1;\mu_2})_1 + q^{\mu_1}(T^{AV}_{\mu_1\alpha_1})_1],\end{aligned}$$

then, recollecting the formulas for traces and gathering the contributions for momentum contractions, the PV functions from both sectors cancel each other and the conditioning Υ factors. The remaining T^A amplitudes arrange themselves as

$$g^{\mu_{12}}(\mathcal{T}^{AV}_{\mu_1\alpha_1\mu_2\alpha_2})_1 = 2T^A_{(+)\alpha_1;\alpha_2} + q_{\alpha_2}T^A_{(+)\alpha_1} = -\varepsilon_{\alpha_1}{}^\nu\mathcal{B}_{\nu;\alpha_2}.$$

These amplitudes are precisely related to T^V ones.

The equation satisfied by $g^{\alpha_{12}}(\mathcal{T}^{AV}_{\mu_1\alpha_1\mu_2\alpha_2})_1$ starts with

$$\begin{aligned} g^{\alpha_{12}}(\mathcal{T}^{AV}_{\mu_1\alpha_1\mu_2\alpha_2})_1 &= 4g^{\alpha_{12}}(T^{AV}_{\mu_1\alpha_1;\mu_2\alpha_2})_1 + 2q_{\mu_2}g^{\alpha_{12}}(T^{AV}_{\mu_1\alpha_1;\alpha_2})_1 \quad (8.79)\\ &\quad +2q^{\alpha_1}(T^{AV}_{\mu_1\alpha_1;\mu_2})_1 + q_{\mu_2}q^{\alpha_1}(T^{AV}_{\mu_1\alpha_1})_1.\end{aligned}$$

The first line is the only one with conditioning factors; the momentum contraction is identically satisfied because the relation appears for the second vertex (specifically a vector one) and in the first version. Lumping together all these considerations, we get

$$g^{\alpha_{12}}(\mathcal{T}^{AV}_{\mu_1\alpha_1\mu_2\alpha_2})_1 = 4mT^{AS}_{\mu_1;\mu_2} - \varepsilon_{\mu_1}{}^\nu\mathcal{B}_{\nu;\mu_2} - \varepsilon_{\mu_1\nu}U^\nu_{\mu_2}, \quad (8.80)$$

$U^{\nu}_{\mu_2}$ is a term common to all relations with a constraint. For version two,

$$g^{\mu_{12}}(\mathcal{T}^{AV}_{\mu_1\alpha_1\mu_2\alpha_2})_2 = -\varepsilon_{\alpha_1}{}^{\nu}\mathcal{B}_{\nu;\alpha_2} - \varepsilon_{\alpha_1}{}^{\nu_1}U_{\nu_1\alpha_2} \tag{8.81}$$
$$g^{\alpha_{12}}(\mathcal{T}^{AV}_{\mu_1\alpha_1\mu_2\alpha_2})_2 = 4mT^{AS}_{\mu_1;\mu_2} - \varepsilon_{\mu_1}{}^{\nu}\mathcal{B}_{\nu;\mu_2}. \tag{8.82}$$

Concerning VA as it can be expressed in AV terms without conditions from (8.64),

$$g^{\mu_{12}}(\mathcal{T}^{VA}_{\mu_1\alpha_1\mu_2\alpha_2})_1 = 4mT^{AS}_{\alpha_1;\alpha_2} - \varepsilon_{\alpha_1}{}^{\nu}B_{\nu;\alpha_2} \tag{8.83}$$
$$g^{\mu_{12}}(\mathcal{T}^{VA}_{\mu_1\alpha_1\mu_2\alpha_2})_2 = 4mT^{AS}_{\alpha_1;\alpha_2} - \varepsilon_{\alpha_1}{}^{\nu}\mathcal{B}_{\nu;\alpha_2} - \varepsilon_{\alpha_1}{}^{\nu_1}U_{\nu_1\alpha_2} \tag{8.84}$$
$$g^{\alpha_{12}}(\mathcal{T}^{VA}_{\mu_1\alpha_1\mu_2\alpha_2})_1 = -\varepsilon_{\mu_1}{}^{\nu}\mathcal{B}_{\nu;\mu_2} - \varepsilon_{\mu_1\nu}U^{\nu}_{\mu_2} \tag{8.85}$$
$$g^{\alpha_{12}}(\mathcal{T}^{VA}_{\mu_1\alpha_1\mu_2\alpha_2})_2 = -\varepsilon_{\mu_1}{}^{\nu}\mathcal{B}_{\nu;\mu_2}. \tag{8.86}$$

Different from momentum relations, when an index is the one that defines the version, then U-factor appears in the complementary contraction, $g^{\alpha_{12}}(\mathcal{T}^{AV}_{\mu_1\alpha_1\mu_2\alpha_2})_1 = -\varepsilon^{\nu}_{\mu_1}g^{\alpha_{12}}(\mathcal{T}^{VV}_{\nu\alpha_1\mu_2\alpha_2})$ shows a possible violation, as opposed to $q^{\alpha_1}(\mathcal{T}^{AV}_{\mu_1\alpha_1\mu_2\alpha_2})_1$ which is identically satisfied.

8.2 Summing all permutations: $[\hat{T}^V]$ and $[\hat{T}^A]_{ij}$

In preparation for summing all contributions, that will constitute the two-point function of the stress tensor, it is necessary to establish a point of view about the odd part. In the preceding expressions, we adopted a uniform version to $\{(\mathcal{T}^{AV}_{\mu_1\alpha_1\mu_2\alpha_2})_i; (\mathcal{T}^{VA}_{\mu_1\alpha_1\mu_2\alpha_2})_i\}$, signifying the same version of derivatives amplitudes were chosen. For the permutation $\mu_1 \leftrightarrow \mu_2$ and subsequently $\alpha_1 \leftrightarrow \alpha_2$, it is entirely free which combinations to use in this step. In this work, we will explore a subset of possibilities,

$$[\hat{T}^A_{\mu_1\mu_2\alpha_1\alpha_2}]_{ij} = (\mathcal{T}^{AV}_{\mu_1\alpha_1\mu_2\alpha_2})_i + (\mathcal{T}^{VA}_{\mu_1\alpha_1\mu_2\alpha_2})_j, \tag{8.87}$$

with $i, j = \{1, 2\}$, amounting to four combinations in principle. Permutations do not change this choice as it could be done.

The even sector works as $[\hat{T}^V_{\mu_1\mu_2\alpha_1\alpha_2}] = \mathcal{T}^{VV}_{\mu_1\alpha_1\mu_2\alpha_2} + \mathcal{T}^{AA}_{\mu_1\alpha_1\mu_2\alpha_2}$. To get the total contribution, it is necessary to sum the permutation $\mu_1 \leftrightarrow \mu_2$ and then $\alpha_1 \leftrightarrow \alpha_2$ of that result. In the even sector, we use (8.44) and to have the systematic formula

$$\begin{aligned}[\mathcal{T}^V_{\mu_{12}\alpha_{12}}] = {} & 2[\mathcal{T}^{VV}_{\mu_1\mu_2\alpha_1\alpha_2} + \mathcal{T}^{VV}_{\mu_2\mu_1\alpha_1\alpha_2} + \mathcal{T}^{VV}_{\mu_1\mu_2\alpha_2\alpha_1} + \mathcal{T}^{VV}_{\mu_2\mu_1\alpha_2\alpha_1}] \\ & -4m[g_{\mu_1\alpha_1}T^{SV}_{\mu_2;\alpha_2} + g_{\mu_2\alpha_1}T^{SV}_{\mu_1;\alpha_2} + g_{\mu_1\alpha_2}T^{SV}_{\mu_2;\alpha_1} + g_{\mu_2\alpha_2}T^{SV}_{\mu_1;\alpha_1}].\end{aligned} \tag{8.88}$$

For the odd sector, we go in search of a simplification in the operations; for that,

$$(\mathcal{T}^{VA}_{\mu_1\alpha_1\mu_2\alpha_2})_j = (\mathcal{T}^{AV}_{\mu_1\alpha_1\mu_2\alpha_2})_j + 4m\varepsilon_{\mu_1\alpha_1}T^{SV}_{\mu_2;\alpha_2} \tag{8.89}$$
$$(\mathcal{T}^{AV}_{\mu_1\alpha_1\mu_2\alpha_2})_i = (\mathcal{T}^{AV}_{\mu_1\alpha_1\mu_2\alpha_2})_1 - \delta_{i,2}\varepsilon_{\alpha_1\mu_1}U_{\mu_2\alpha_2}, \tag{8.90}$$

where $\delta_{i,2}$. Its function is to capture only version two, given that the second term is zero if it already has version one. The above equations allow us to write the result

$$[\hat{T}^A_{\mu_1\mu_2\alpha_1\alpha_2}]_{ij} = 2(\mathcal{T}^{AV}_{\mu_1\alpha_1\mu_2\alpha_2})_1 + 4m\varepsilon_{\mu_1\alpha_1}T^{SV}_{\mu_2;\alpha_2} - (\delta_{i,2} + \delta_{j,2})\,\varepsilon_{\alpha_1\mu_1}U_{\mu_2\alpha_2}. \tag{8.91}$$

These arguments have the consequence that it is also possible to write

$$\begin{aligned}[\mathcal{T}^A_{\mu_{12}\alpha_{12}}]_{ij} &= 2(\mathcal{T}^{AV}_{\mu_1\alpha_1\mu_2\alpha_2})_1 + 2(\mathcal{T}^{AV}_{\mu_2\alpha_1\mu_1\alpha_2})_1 + 2(\mathcal{T}^{AV}_{\mu_1\alpha_2\mu_2\alpha_1})_1 + 2(\mathcal{T}^{AV}_{\mu_2\alpha_2\mu_1\alpha_1})_1 \\ &+4m[\varepsilon_{\mu_1\alpha_1}T^{SV}_{\mu_2;\alpha_2} + \varepsilon_{\mu_2\alpha_1}T^{SV}_{\mu_1;\alpha_2} + \varepsilon_{\mu_1\alpha_2}T^{SV}_{\mu_2;\alpha_1} + \varepsilon_{\mu_2\alpha_2}T^{SV}_{\mu_1;\alpha_1}] \\ &-(\delta_{i,2}+\delta_{j,2})[\varepsilon_{\alpha_1\mu_1}U_{\mu_2\alpha_2} + \varepsilon_{\alpha_1\mu_2}U_{\mu_1\alpha_2} + \varepsilon_{\alpha_2\mu_1}U_{\mu_2\alpha_1} + \varepsilon_{\alpha_2\mu_2}U_{\mu_1\alpha_1}].\end{aligned} \quad (8.92)$$

In this way, we can sum the Eqs. (8.88) and (8.92) corresponding to the odd and the even part to obtain the two-point correlator of the stress tensor reads

$$T^G_{\mu_1\mu_2\alpha_1\alpha_2} = -\frac{i}{64}\{[\mathcal{T}^V_{\mu_{12}\alpha_{12}}] + [\mathcal{T}^A_{\mu_{12}\alpha_{12}}]_{ij}\}. \quad (8.93)$$

Now it is easy to organize all the contractions obtained by sector from this tensor.

8.2.1 Even Part

We must observe from the permutations sum $\mu_1 \leftrightarrow \mu_2$; that the index μ_1 occupies the positions in such a way that contraction with q^{μ_1} corresponds to the two types of momentum relations (in the matrix and derivative index positions). Hence we get

$$\begin{aligned}q^{\mu_1}[\hat{T}^V_{\mu_1\mu_2\alpha_1\alpha_2}] &= q^{\mu_1}\mathcal{T}^{VV}_{\mu_1\alpha_1\mu_2\alpha_2} + q^{\mu_1}\mathcal{T}^{AA}_{\mu_1\alpha_1\mu_2\alpha_2} = 2\mathcal{S}^V_{(-)\alpha_1;\alpha_2\mu_2} - 4mq_{\alpha_1}T^{SV}_{\mu_2;\alpha_2} \\ q^{\mu_1}[\hat{T}^V_{\mu_2\mu_1\alpha_1\alpha_2}] &= 2\mathcal{S}^V_{(-)\alpha_1;\alpha_2\mu_2} + 2q_{\alpha_1}\mathcal{B}_{\mu_2;\alpha_2} - 2g_{\mu_2\alpha_1}q^\nu\mathcal{B}_{\nu;\alpha_2} - 4mg_{\mu_2\alpha_1}T^S_{(-)\alpha_2}.\end{aligned} \quad (8.94)$$

Summing the permutation $\alpha_1 \leftrightarrow \alpha_2$ of these contributions symmetrize[1] the final expression in these last indices. The complete result of the vector part of gravitational amplitude is

$$\begin{aligned}q^{\mu_1}[\mathcal{T}^V_{\mu_{12}\alpha_{12}}] &= -4m[q_{\alpha_1}T^{SV}_{\mu_2;\alpha_2} + q_{\alpha_2}T^{SV}_{\mu_2;\alpha_1}] - 4m[g_{\mu_2\alpha_1}T^S_{(-)\alpha_2} + g_{\mu_2\alpha_2}T^S_{(-)\alpha_1}] \\ &+4\mathcal{S}^V_{(-)(\alpha_1;\alpha_2)\mu_2} + 2[q_{\alpha_1}\mathcal{B}_{\mu_2;\alpha_2} + q_{\alpha_2}\mathcal{B}_{\mu_2;\alpha_1} - g_{\mu_2\alpha_1}q^\nu\mathcal{B}_{\nu;\alpha_2} - g_{\mu_2\alpha_2}q^\nu\mathcal{B}_{\nu;\alpha_1}].\end{aligned} \quad (8.95)$$

Notably, the distinction of derivative or matrix indices gets dissolved in the complete expression. Due to this equation's symmetries and unique form, we do not show the other contractions, as they may be extracted simply by substituting the convenient indices.

The compilation of the identities involving the traces is given by

$$g^{\mu_{12}}[\hat{T}^V_{\mu_1\mu_2\alpha_1\alpha_2}] = g^{\mu_{12}}\mathcal{T}^{VV}_{\mu_1\alpha_1\mu_2\alpha_2} + g^{\mu_{12}}\mathcal{T}^{AA}_{\mu_1\alpha_1\mu_2\alpha_2} = 4mT^{SV}_{\alpha_1;\alpha_2} + 2\mathcal{B}_{\alpha_1;\alpha_2} + 2U_{\alpha_1\alpha_2}. \quad (8.96)$$

Noticing that the trace $g^{\mu_{12}}\hat{T}^V_{\mu_2\mu_1\alpha_1\alpha_2}$ is equal. The symmetrization brought about by $\alpha_1 \leftrightarrow \alpha_2$ furnishes the complete result

$$g^{\mu_{12}}[\mathcal{T}^V_{\mu_{12}\alpha_{12}}] = 8mT^{SV}_{(\alpha_1;\alpha_2)} + 4\mathcal{B}_{(\alpha_1;\alpha_2)} + 8U_{\alpha_1\alpha_2} \quad (8.97)$$

$$g^{\alpha_{12}}[\mathcal{T}^V_{\mu_{12}\alpha_{12}}] = 8mT^{SV}_{(\mu_1;\mu_2)} + 4\mathcal{B}_{(\mu_1;\mu_2)} + 8U_{\mu_1\mu_2}, \quad (8.98)$$

where identical arguments implies to the second equation.

[1]Our definition of symmetrization and unit coeffient: $\mathcal{S}^V_{(-)(\alpha_1;\alpha_2)\mu_2} = \mathcal{S}^V_{(-)\alpha_1;\alpha_2\mu_2} + \mathcal{S}^V_{(-)\alpha_2;\alpha_1\mu_2}$

8.2.2 Odd Part

To discuss the more intricate odd part in combinations seen in equation (8.92), we only need results for the basic permutation of version one. Nonetheless, different from the even sector, the odd part allows for an extensive set of possibilities whose contractions with q^{μ_1}, q^{μ_2}, q^{α_1}, and q^{α_2} may be, in principle, all unrelated. However, to our adopted representatives, only independent contractions with momentum are with q^{μ_1} and q^{α_1}.

To express the first relation, we recall that version one has a U-term when index μ_1 is in the first position $(\mathcal{T}^{AV}_{\mu_1\alpha_1\mu_2\alpha_2})_1$, but in permutation $(\mathcal{T}^{AV}_{\mu_2\alpha_1\mu_1\alpha_2})_1$ it corresponds to an external contraction that has two forms. Selecting a convenient expression follows

$$\begin{aligned} q^{\mu_1}[(\mathcal{T}^{AV}_{\mu_1\alpha_1\mu_2\alpha_2})_1 + (\mathcal{T}^{AV}_{\mu_2\alpha_1\mu_1\alpha_2})_1] &= 4m[\varepsilon_{\alpha_1\nu}q^\nu T^{SV}_{\mu_2;\alpha_2} - \varepsilon_{\mu_2\alpha_1}T^{S}_{(-)\alpha_2}] \\ &\quad +2\mathcal{S}^A_{(-)\alpha_1;\alpha_2\mu_2} - \varepsilon_{\mu_2\nu}q^\nu \mathcal{B}_{\alpha_1;\alpha_2} \\ &\quad +\varepsilon_{\alpha_1\mu_2}q^\nu U_{\nu\alpha_2} + \varepsilon_{\alpha_1\nu}q^\nu U_{\mu_2\alpha_2}. \end{aligned} \tag{8.99}$$

Finally, summing with the above equation the permutations in α_i, we arrive at

$$\begin{aligned} 2q^{\mu_1}[(\mathcal{T}^{AV}_{\mu_1\alpha_1\mu_2\alpha_2})_1 + \text{3-perm}] &= 8m[\varepsilon_{\alpha_1\nu}q^\nu T^{SV}_{\mu_2;\alpha_2} + \varepsilon_{\alpha_2\nu}q^\nu T^{SV}_{\mu_2;\alpha_1}] \\ &\quad -8m[\varepsilon_{\mu_2\alpha_1}T^S_{(-)\alpha_2} + \varepsilon_{\mu_2\alpha_2}T^S_{(-)\alpha_1}] \\ &\quad +4\mathcal{S}^A_{(-)(\alpha_1;\alpha_2)\mu_2} - 2\varepsilon_{\mu_2\nu}q^\nu \mathcal{B}_{(\alpha_1;\alpha_2)} \\ &\quad +2q^\nu\left(\varepsilon_{\alpha_1\mu_2}U_{\nu\alpha_2} + \varepsilon_{\alpha_2\mu_2}U_{\nu\alpha_1}\right) \\ &\quad +2q^\nu\left(\varepsilon_{\alpha_1\nu}U_{\mu_2\alpha_2} + \varepsilon_{\alpha_2\nu}U_{\mu_2\alpha_1}\right). \end{aligned} \tag{8.100}$$

Remaining contributions are easy to be dealt with

$$q^{\mu_1}[4m\varepsilon_{\mu_1\alpha_1}T^{SV}_{\mu_2;\alpha_2} - (\delta_{i,2}+\delta_{j,2})\,\varepsilon_{\alpha_1\mu_1}U_{\mu_2\alpha_2} + \text{3-perm}].$$

When added to the previous equation, it follows one of the important results of this section

$$\begin{aligned} q^{\mu_1}[\mathcal{T}^A_{\mu_{12}\alpha_{12}}]_{ij} &= 4m[\varepsilon_{\alpha_1\nu}q^\nu T^{SV}_{\mu_2;\alpha_2} + \varepsilon_{\alpha_2\nu}q^\nu T^{SV}_{\mu_2;\alpha_1}] \\ &\quad -4m[\varepsilon_{\mu_2\alpha_1}T^S_{(-)\alpha_2} + \varepsilon_{\mu_2\alpha_2}T^S_{(-)\alpha_1}] \\ &\quad +4\mathcal{S}^A_{(-)(\alpha_1;\alpha_2)\mu_2} - 2\varepsilon_{\mu_2\nu}q^\nu \mathcal{B}_{(\alpha_1;\alpha_2)} \\ &\quad -(2-\delta_{i,2}-\delta_{j,2})\,q^\nu(\varepsilon_{\mu_2\alpha_1}U_{\nu\alpha_2} + \varepsilon_{\mu_2\alpha_2}U_{\nu\alpha_1}) \\ &\quad +(2-\delta_{i,2}-\delta_{j,2})\,q^\nu(\varepsilon_{\alpha_1\nu}U_{\mu_2\alpha_2} + \varepsilon_{\alpha_2\nu}U_{\mu_2\alpha_1}). \end{aligned} \tag{8.101}$$

The results to q^{μ_2} come from permuting μ_2 by μ_1 because, among other things, they hit the contracted indices that become dummy ones in an equivalent position.

As concerning $q^{\alpha_1}[\mathcal{T}^A_{\mu_{12}\mu_{12}}]_{ij}$ contraction, we exploit the permutation

$$q^{\alpha_1}[(\mathcal{T}^{AV}_{\mu_1\alpha_1\mu_2\alpha_2})_1 + (\mathcal{T}^{AV}_{\mu_1\alpha_2\mu_2\alpha_1})_1] = 2\mathcal{S}^A_{(-)\mu_1;\mu_2\alpha_2} - \varepsilon_{\alpha_2\nu}q^\nu \mathcal{B}_{\mu_1;\mu_2}.$$

We are choosing formulas for the external contraction without U-term. The contraction in the second vertex of version one has an automatically satisfied RAGF using an appropriate form of relation with the derivative index is suitable. Adding the permutation in μ_i, we have a symmetrization of these indices. The last part of this derivation needs

$$q^{\alpha_1}[4m\varepsilon_{\mu_1\alpha_1}T^{SV}_{\alpha_2;\mu_2} - (\delta_{i,2}+\delta_{j,2})\,\varepsilon_{\alpha_1\mu_1}U_{\mu_2\alpha_2} + 3\text{-perm}].$$

They organize the final expression as

$$\begin{aligned} q^{\alpha_1}[\mathcal{T}^A_{\mu_{12}\alpha_{12}}]_{ij} &= +4m[\varepsilon_{\mu_1\nu}q^\nu T^{SV}_{\alpha_2;\mu_2} + \varepsilon_{\mu_2\nu}q^\nu T^{SV}_{\alpha_2;\mu_1}] \\ &\quad -4m[\varepsilon_{\alpha_2\mu_1}T^S_{(-)\mu_2} + \varepsilon_{\alpha_2\mu_2}T^S_{(-)\mu_1}] \\ &\quad +4\mathcal{S}^A_{(-)(\mu_1;\mu_2)\alpha_2} - 2\varepsilon_{\alpha_2\nu}q^\nu\mathcal{B}_{(\mu_1;\mu_2)} \\ &\quad -(\delta_{i,2}+\delta_{j,2})\,q^\nu[\varepsilon_{\alpha_2\mu_1}U_{\nu\mu_2} + \varepsilon_{\alpha_2\mu_2}U_{\nu\mu_1}] \\ &\quad +(\delta_{i,2}+\delta_{j,2})\,q^\nu[\varepsilon_{\mu_2\nu}U_{\mu_1\alpha_2} + \varepsilon_{\mu_1\nu}U_{\mu_2\alpha_2}]. \end{aligned} \tag{8.102}$$

The trace equation has interesting properties compared with momentum contraction: through analysis of basic permutation, conditioning factors appear in a complementary set of indexes. First, we have for the trace of the combination

$$2g^{\mu_{12}}[(\mathcal{T}^{AV}_{\mu_1\alpha_1\mu_2\alpha_2})_1 + (\mathcal{T}^{AV}_{\mu_2\alpha_1\mu_1\alpha_2})_1] = 4g^{\mu_{12}}(\mathcal{T}^{AV}_{\mu_1\alpha_1\mu_2\alpha_2})_1 = -4\varepsilon_{\alpha_1}{}^{\nu}\mathcal{B}_{\nu;\alpha_2}.$$

Summing up all terms with the exchange of indices $\alpha_1 \leftrightarrow \alpha_2$ with the remaining components leaves us with a final expression given by

$$\begin{aligned} g^{\mu_{12}}[\mathcal{T}^A_{\mu_{12}\alpha_{12}}]_{ij} &= 8mT^{SA}_{(\alpha_1;\alpha_2)} - 4\varepsilon_{\alpha_1}{}^{\nu}\mathcal{B}_{\nu;\alpha_2} - 4\varepsilon_{\alpha_2}{}^{\nu}\mathcal{B}_{\nu;\alpha_1} \\ &\quad -2\,(\delta_{i,2}+\delta_{j,2})\,(\varepsilon_{\alpha_1\nu}U^\nu_{\alpha_2} + \varepsilon_{\alpha_2\nu}U^\nu_{\alpha_1}). \end{aligned} \tag{8.103}$$

We utilized the relation $-\varepsilon_{\alpha_1}{}^{\nu}T^{SV}_{\nu;\alpha_2} = T^{SA}_{\alpha_1;\alpha_2}$. So remember, version one is automatically satisfied. However, U-contribution came from the equation between versions one and two.

Another trace independent is with $g^{\alpha_{12}}$; the conditioning factors coming from

$$\begin{aligned} 2g^{\alpha_{12}}[(\mathcal{T}^{AV}_{\mu_1\alpha_1\mu_2\alpha_2})_1 + (\mathcal{T}^{AV}_{\mu_1\alpha_2\mu_2\alpha_1})_1] &= 4g^{\alpha_{12}}(\mathcal{T}^{AV}_{\mu_1\alpha_1\mu_2\alpha_2})_1 && (8.104) \\ &= 16mT^{AS}_{\mu_1;\mu_2} - 4\varepsilon_{\mu_1}{}^{\nu}\mathcal{B}_{\nu;\mu_2} - 4\varepsilon_{\mu_1\nu}U^\nu_{\mu_2}. && (8.105) \end{aligned}$$

Thus, symmetrizing in μ_i and adding the remaining contributions, we arrive at

$$\begin{aligned} g^{\alpha_{12}}[\mathcal{T}^A_{\mu_{12}\alpha_{12}}]_{ij} &= 8mT^{AS}_{(\mu_1;\mu_2)} - 4\varepsilon_{\mu_1}{}^{\nu}\mathcal{B}_{\nu;\mu_2} - 4\varepsilon_{\mu_2}{}^{\nu}\mathcal{B}_{\nu;\mu_1} \\ &\quad -2[2-(\delta_{i,2}+\delta_{j,2})](\varepsilon_{\mu_1\nu}U^\nu_{\mu_2} + \varepsilon_{\mu_2\nu}U^\nu_{\mu_1}). \end{aligned} \tag{8.106}$$

The only difference is for the coefficients of violating terms. One immediate consequence is the existence of operations with the Dirac traces and surface terms where such terms do not arise. That is thoroughly argued in the next part, where the surface terms in these expressions are investigated. After that, the Weyl and Einstein anomalies are discussed.

To illustrate how they look like when everything is put together, see a trace relation associated do the Weyl anomaly:

$$\begin{aligned}(64i)\, g^{\alpha_{12}}[T^{G}_{\mu_{12}\alpha_{12}}]_{ij} &= 8mT^{SV}_{(\mu_1;\mu_2)} + 8mT^{AS}_{(\mu_1;\mu_2)} + 4\mathcal{B}_{(\mu_1;\mu_2)} - 4\varepsilon_{\mu_1}{}^{\nu}\mathcal{B}_{\nu;\mu_2} - 4\varepsilon_{\mu_2}{}^{\nu}\mathcal{B}_{\nu;\mu_1} \\ &\quad -2[2-(\delta_{i,2}+\delta_{j,2})](\varepsilon_{\mu_1\nu}U^{\nu}_{\mu_2} + \varepsilon_{\mu_2\nu}U^{\nu}_{\mu_1}) + 8U_{\mu_1\mu_2} \qquad (8.107) \\ (64i)\, g^{\mu_{12}}[T^{G}_{\mu_{12}\alpha_{12}}]_{ij} &= 8mT^{SV}_{(\alpha_1;\alpha_2)} + 8mT^{SA}_{(\alpha_1;\alpha_2)} + 4\mathcal{B}_{(\alpha_1;\alpha_2)} - 4\varepsilon_{\alpha_1}{}^{\nu}\mathcal{B}_{\nu;\alpha_2} - 4\varepsilon_{\alpha_2}{}^{\nu}\mathcal{B}_{\nu;\alpha_1} \\ &\quad -2\left(\delta_{i,2}+\delta_{j,2}\right)\left(\varepsilon_{\alpha_1\nu}U^{\nu}_{\alpha_2} + \varepsilon_{\alpha_2\nu}U^{\nu}_{\alpha_1}\right) + 8U_{\alpha_1\alpha_2}.\end{aligned}$$

And a momentum equation related to the Einstein anomaly:

$$\begin{aligned}(64i)\, q^{\mu_1}[T^{G}_{\mu_{12}\alpha_{12}}]_{ij} &= 4m[\varepsilon_{\alpha_1\nu}q^{\nu}T^{SV}_{\mu_2;\alpha_2} + \varepsilon_{\alpha_2\nu}q^{\nu}T^{SV}_{\mu_2;\alpha_1} - q_{\alpha_1}T^{SV}_{\mu_2;\alpha_2} - q_{\alpha_2}T^{SV}_{\mu_2;\alpha_1}] \\ &\quad -4m[\varepsilon_{\mu_2\alpha_1}T^{S}_{(-)\alpha_2} + \varepsilon_{\mu_2\alpha_2}T^{S}_{(-)\alpha_1} + g_{\mu_2\alpha_1}T^{S}_{(-)\alpha_2} + g_{\mu_2\alpha_2}T^{S}_{(-)\alpha_1}] \\ &\quad +4\mathcal{S}^{A}_{(-)(\alpha_1;\alpha_2)\mu_2} + 4\mathcal{S}^{V}_{(-)(\alpha_1;\alpha_2)\mu_2} - 2\varepsilon_{\mu_2\nu}q^{\nu}\mathcal{B}_{(\alpha_1;\alpha_2)} \qquad (8.108) \\ &\quad +2[q_{\alpha_1}\mathcal{B}_{\mu_2;\alpha_2} + q_{\alpha_2}\mathcal{B}_{\mu_2;\alpha_1} - g_{\mu_2\alpha_1}q^{\nu}\mathcal{B}_{\nu;\alpha_2} - g_{\mu_2\alpha_2}q^{\nu}\mathcal{B}_{\nu;\alpha_1}] \\ &\quad -\left(2-\delta_{i,2}-\delta_{j,2}\right)q^{\nu}(\varepsilon_{\mu_2\alpha_1}U_{\nu\alpha_2} + \varepsilon_{\mu_2\alpha_2}U_{\nu\alpha_1}) \\ &\quad +\left(2-\delta_{i,2}-\delta_{j,2}\right)q^{\nu}(\varepsilon_{\alpha_1\nu}U_{\mu_2\alpha_2} + \varepsilon_{\alpha_2\nu}U_{\mu_2\alpha_1}).\end{aligned}$$

8.3 Constraints: The Matter of RAGFs Satisfaction

RAGFs for derivative amplitudes as a whole require that $\{\Upsilon, \Upsilon_{\alpha_1}, \Upsilon_{\alpha_1\alpha_2}\} = 0$ holds independently. We already composed them into $U_{\alpha_1\alpha_2}$, which arises in the final form of gravitational amplitude. We will recover their explicit expression by simplifying the investigation but with some notation to relevant structures. Combinations of surface terms, which we carefully introduced and managed since the first chapter, are given by

$$\Xi^{(a)}_{\mu_1\mu_2} = [2(\square^{\rho}_{3\rho\mu_1\mu_2} - \Delta_{2\mu_1\mu_2}) - g_{\mu_1\mu_2}\Delta^{\rho}_{2\rho}] = a g_{\mu_1\mu_2} \qquad (8.109)$$

$$\Xi^{(b)}_{\mu_1\mu_2\mu_3\mu_4} = \left[3\Sigma^{\rho}_{4\rho\mu_{1234}} - 8\square_{3\mu_{1234}} - g_{(\mu_1\mu_2}g_{\mu_3\mu_4)}\Delta^{\rho}_{2\rho}\right] = b g_{(\mu_1\mu_2}g_{\mu_3\mu_4)} \qquad (8.110)$$

$$\Xi^{\rho\text{quad}}_{\rho a_1\alpha_2} = (W^{\rho}_{2\rho\alpha_{12}} - 2\Delta_{1\alpha_{12}}) + 2g_{\alpha_1\alpha_2}I_{\text{quad}} - 2m^2\left(\Delta_{2\alpha_{12}} + g_{\alpha_{12}}I_{\log}\right) \qquad (8.111)$$

The importance of this attitude is two-fold: one, it reduces the size of expressions, and two, if bilinears are reduced in the integrand, these tensors become convergent surface terms that identically vanish; see Appendix (F.1). Moreover, their integrands are typical of 4D integrals. On the other hand, all the following analyses do not use such an operation.

Evoking Eqs. (7.107), (7.126) and (7.127), we have the set

$$\begin{aligned}\Upsilon &= 2\Delta^{\rho}_{2\rho} + i/\pi \qquad (8.112) \\ \Upsilon_{\alpha_1} &= -\frac{1}{2}P^{\nu_1}\Xi^{(a)}_{\alpha_1\nu} - \frac{1}{2}q_{\alpha_1}\Upsilon \\ \Upsilon_{\alpha_1\alpha_2} &= -\frac{1}{12}\left(\theta_{\alpha_{12}} - 3q_{\alpha_{12}}\right)\Upsilon - \frac{1}{4}P^{\nu}[(P_{\alpha_2} - q_{\alpha_2})\,\Xi^{(a)}_{\alpha_1\nu} + (P_{\alpha_1} - q_{\alpha_1})\,\Xi^{(a)}_{\alpha_2\nu}] + \Xi^{\text{quad}}_{a_{12}} \\ &\quad + \frac{1}{72}(3P^{\nu_{12}} + q^{\nu_{12}})g_{(\alpha_{12}}\Xi^{(a)}_{\nu_{12})} - \frac{1}{8}(P^2+q^2)\Xi^{(a)}_{\alpha_{12}} + \frac{1}{36}(3P^{\nu_{12}} + q^{\nu_{12}})\Xi^{(b)}_{\alpha_{12}\nu_{12}}.\end{aligned}$$

Now, as the variables $\{P; q\}$ or the routings $\{k_1; k_2\}$ are linearly independent, only solution for their vanishing is $\Upsilon = 0$, $\Xi^{(a)}_{\mu_1\mu_2} = 0$, and $\Xi^{(b)}_{\mu_1\mu_2\mu_3\mu_4} = 0$. For quadratic terms, we have

$$\Xi^{\text{quad}}_{a_{12}} = (W^{\rho}_{2\rho\alpha_{12}} - 2\Delta_{1\alpha_{12}}) + 2g_{\alpha_1\alpha_2} I_{\text{quad}} - 2m^2 \left(\Delta_{2\alpha_{12}} + g_{\alpha_{12}} I_{\log}\right) = 0.$$

This happens because if $\Upsilon = 0$ and $\Upsilon_{\alpha_1} = 0 \Rightarrow \Xi^{(a)}_{\alpha_1\nu} = 0$, that substituted in $\Upsilon_{\alpha_1\alpha_2}$ oblige other terms to vanish. If one takes $\Upsilon_{\alpha_1\alpha_2}$ alone, it has crossed terms $q_{\alpha_i} P^{\nu}$ that requires its coefficient $\Xi^{(a)}_{\alpha_1\nu}$ to be zero and the term $P^{\nu_{12}} \Xi^{(b)}_{\alpha_{12}\nu_{12}}$ in the only remnant of arbitrary P-variable, hence this tensor will have to be zero and subsequently $\Upsilon = 0$ as well. In any case, we have conditions stated. Additionally, the condition $\Upsilon_{\mu_2} = 0$ alone would be the same since for arbitrary P and q, both terms, $\Xi^{(a)}_{\alpha_1\nu}$ and Υ, must vanish.

In the last statement, we have the exception of the places whose violating terms sum into $2\Upsilon_{\alpha_1} + q_{\alpha_1}\Upsilon$, that occur exactly for combinations $[2T^{\Gamma_{12}}_{\mu_{12};\alpha_1} + q_{\alpha_1} T^{\Gamma_{12}}_{\mu_{12}}]$. However, if finite, this combination ought to vanish. Why? Because in 2D for vértices $\Gamma_i = \{\gamma_\mu, \gamma_* \gamma_\mu\}$ the charge conjugation C matrix implies $C\Gamma_i C^{-1} = -\Gamma_i^T$ and for the propagator

$$CS\left(K_i\right) C^{-1} = (C\not{K}_i C^{-1} + m)/D_i = S^T\left(-K_i\right). \tag{8.113}$$

Expliciting the structure $[2T^{\Gamma_{12}}_{\mu_{12};\alpha_1} + q_{\alpha_1} T^{\Gamma_{12}}_{\mu_{12}}]$ can be written as

$$[2T^{\Gamma_{12}}_{\mu_{12};\alpha_1} + q_{\alpha_1} T^{\Gamma_{12}}_{\mu_{12}}] = \int \frac{\mathrm{d}^2 k}{(2\pi)^2} \left(K_1 + K_2\right)_{\alpha_1} \operatorname{tr}[\Gamma_1 S\left(K_1\right) \Gamma_2 S\left(K_2\right)] \tag{8.114}$$

$$= \int \frac{\mathrm{d}^2 k}{(2\pi)^2} \left(K_1 + K_2\right)_{\alpha_1} t^{\Gamma_1\Gamma_2}, \tag{8.115}$$

where integrand $t^{\Gamma_1\Gamma_2}$ s the function without derivative index. It readily obeys

$$t^{\Gamma_1\Gamma_2} = \operatorname{tr}\{[C\Gamma_1 C^{-1}][CS\left(K_1\right) C^{-1}][C\Gamma_2 C^{-1}][CS\left(K_2\right) C^{-1}]\} \tag{8.116}$$

$$= (-1)^2 \operatorname{tr}[S\left(-K_2\right) \Gamma_2 S\left(-K_1\right) \Gamma_1]^T \tag{8.117}$$

$$= \operatorname{tr}[\Gamma_1 S\left(-k - k_2\right) \Gamma_2 S\left(-k - k_1\right)]. \tag{8.118}$$

Under integration, reflecting the integration variable $k \to -k$ after shifting it by $k \to k + k_1 + k_2$, the arguments of $t^{\Gamma_1\Gamma_2}$ return to their starting configuration. However, the factor $(K_1 + K_2)$ picks up a minus sign $-(K_1 + K_2)$, and the derivative vertex behaves like it had negative parity. These steps are valid as hypotheses; observe that at the beginning that we mentioned, if finite, we can do the operations listed. Therefore, we would get

$$2T^{\Gamma_{12}}_{\mu_{12};\alpha_1} + q_{\alpha_1} T^{\Gamma_{12}}_{\mu_{12}} = (-1)\,[T^{\Gamma_{12}}_{\mu_{12};\alpha_1} + q_{\alpha_1} T^{\Gamma_{12}}_{\mu_{12}}]. \tag{8.119}$$

If shifts can be done, the result must vanish. As the surface terms violate this hypothesis, the non-polynomial sector of the finite part disappears, which depends on external momentum $q = k_2 - k_1$. The leftover part, in general, is a local polynomial in q and P momenta and surface terms, with a degree up to power counting of amplitude.

That fact naturally can be checked in their explicit forms, where no shift of the loop momentum was performed. For instance, see the combination above between VV's,

$$\begin{aligned} 2T^{VV}_{\mu_{12};\alpha_1} + q_{\alpha_1} T^{VV}_{\mu_{12}} &= \mathcal{D}^{VV}_{\mu_{12};\alpha_1} + q_{\alpha_1}\mathcal{D}^{VV}_{\mu_{12}} \\ &= -2P^{\nu_1} W_{3\mu_{12}\alpha_1\nu_1} + 2P_{(\mu_1}\Delta_{2\mu_2\alpha_1)} + 2g_{\mu_{12}}P^{\nu_1}\Delta_{2\alpha_1\nu_1}. \end{aligned} \quad (8.120)$$

That happens to odd amplitudes and also in its two basic modalities. Without derivatives, the finite functions $T^{SV}_{\mu} = 0$ and $T^{AS}_{\mu} = 0$ have a vertex that picks a minus sign (V e A, respectively). We always expressed one part in the basic permutation the way we did because the most complex part, finite ones, drops from calculations. For this subset of amplitudes, the violating terms either are not present, as in

$$g^{\mu_1\alpha_1}[2(T^{AV}_{\mu_{12};\alpha_1})_1 + q_{\alpha_1}(T^{AV}_{\mu_{12}})_1] = T^{A}_{\alpha_1}(k_1) + T^{A}_{\alpha_1}(k_2). \quad (8.121)$$

Alternatively, they are present and appear in the form

$$g^{\mu_1\alpha_1}[2T^{VV}_{\mu_{12};\alpha_1} + q_{\alpha_1}T^{VV}_{\mu_{12}}] = T^{V}_{\mu_2}(k_1) + T^{V}_{\mu_2}(k_2) + 2mT^{SV}_{\mu_2} + (2\Upsilon_{\mu_2} + q_{\mu_2}\Upsilon), \quad (8.122)$$

where $2\Upsilon_{\mu_2} + q_{\mu_2}\Upsilon = -P^{\nu_1}\Xi^{(a)}_{\mu_2\nu}$ happens to vanish either for surface terms corresponding to RAGFs satisfied or with zero value.

Therefore, back to the analysis, the constraints $(\Upsilon, \Upsilon_{\alpha_1}, \Upsilon_{\alpha_1\alpha_2}) = 0$, in addition to satisfying all RAGFs imply in defined values for the tensors (7.107), (8.109) and (8.110)

$$\Upsilon = 2\Delta^{\rho}_{2\rho} + i/\pi \quad (8.123)$$

$$\Xi^{(a)}_{\alpha_1\nu} = 2\Box^{\rho}_{3\rho\alpha_1\nu} - 2g_{\alpha_1\nu}\Delta^{\rho}_{2\rho} = 0 \quad (8.124)$$

$$\Xi^{(a)}_{\alpha_{12}\nu_{12}} = 3\Sigma^{\rho}_{4\rho\alpha_{12}\nu_{12}} - 3g_{(\alpha_1\alpha_2}g_{\nu_{12})}\Delta^{\rho}_{2\rho} = 0. \quad (8.125)$$

That choice, in turn, allows us to organize a ladder of restrictions on surface terms:

$$\Box^{\rho}_{3\rho\alpha_1\nu_1} = g_{\alpha_1\nu_1}\Delta^{\rho}_{2\rho} \quad (8.126)$$

$$\Box^{\rho}_{3\rho\alpha_1\nu_1} = cg^{\nu_{23}}g_{(\alpha_1\nu_1}g_{\nu_{23})} = 4cg_{\alpha_1\nu_1} \quad (8.127)$$

$$\Box_{3\alpha_{12}\nu_{12}} = \frac{1}{4}g_{(\alpha_{12}}g_{\nu_{12})}\Delta^{\rho}_{2\rho}. \quad (8.128)$$

Notice that we adopted an utterly symmetric definition of surface terms. As they are dimensionless, we got to determine their coefficients. The fourth order will be given by

$$\Sigma^{\rho}_{4\rho\alpha_{12}\nu_{12}} = g_{(\alpha_1\alpha_2}g_{\nu_{12})}\Delta^{\rho}_{2\rho} \quad (8.129)$$

$$\Sigma^{\rho}_{4\rho\alpha_{12}\nu_{12}} = dg^{\nu_{23}}g_{(\alpha_1\alpha_2}g_{\nu_{12}}g_{\nu_{23})} = 6dg_{(\alpha_1\alpha_2}g_{\nu_{12})} \quad (8.130)$$

$$\Sigma_{4\alpha_{12}\nu_{12}\nu_{34}} = \frac{1}{6}g_{(\alpha_1\alpha_2}g_{\nu_{12}}g_{\nu_{34})}\Delta^{\rho}_{2\rho}, \quad (8.131)$$

As the trace is $2\Delta^{\rho}_{2\rho} = -i/\pi$, see (8.123), there arise the values to the surface terms. Only the concepts of the RAGFs and unicity are enough to determine the other values,

$$\Delta_{2\mu\nu} = -\frac{ig_{\mu\nu}}{4\pi} \tag{8.132}$$

$$\Box_{3\alpha_{12}\nu_{12}} = -\frac{ig_{(\alpha_{12}}g_{\nu_{12})}}{8\pi} \tag{8.133}$$

$$\Sigma_{4\alpha_{12}\nu_{12}\nu_{34}} = -\frac{i}{12\pi}g_{(\alpha_1\alpha_2}g_{\nu_{12}}g_{\nu_{34})}. \tag{8.134}$$

However, if the attitude towards the undetermined parts were to preserve translational invariance in momentum space. The interpretation given to this tensor should be

$$\Box_{3\alpha_{12}\nu_{12}} = \Delta_{2\mu\nu} = \Sigma_{4\alpha_{12}\nu_{12}\nu_{34}} = 0,$$

In this way, we have the complementary consequence in the tensors,

$$\Upsilon = \frac{i}{\pi}; \quad \Upsilon_{\alpha_1} = -\frac{1}{2}q_{\alpha_1}\Upsilon; \quad \Upsilon_{\alpha_1} = -\frac{1}{12}\left(\theta_{\alpha_1\alpha_2} - 3q_{\alpha_1}q_{\alpha_2}\right)\Upsilon. \tag{8.135}$$

And, about the U-tensor, if the vanishing surface terms, we break integration linearity by

$$U_{\alpha_1\alpha_2} = -\frac{1}{3}\left(\frac{i}{\pi}\right)\theta_{\alpha_1\alpha_2}. \tag{8.136}$$

In parallel, if RAGFs hold or the odd amplitudes are unique or independent of intermediary steps of the calculation, e.g., Dirac traces used. Using the results to Υ in this scenario, we have $U_{\alpha_1\alpha_2} = 0$. To clarify that conditions are exactly equal for the U-factor since the crossed term qP drops out, it may be possible that other linear combinations of Ξ's could cancel the RAGF's violator.

Once more, the explicit expression for U, in terms of (8.109) and (8.110), is

$$\begin{aligned} U_{\alpha_1\alpha_2} = & -\frac{1}{3}\theta_{\alpha_1\alpha_2}\Upsilon + \frac{1}{18}q^{\nu_{12}}[2\Xi_{2\alpha_{12}\nu_{12}} + g_{(\alpha_{12}}\Xi_{1\nu_1\nu_2)}] - \frac{1}{2}q^2\Xi_{1\alpha_1\alpha_2} + 4\Xi^{\text{quad}}_{a_1\alpha_2} \\ & +\frac{1}{6}P^{\nu_{12}}[2\Xi^{(b)}_{2\alpha_{12}\nu_{12}} + g_{(\alpha_{12}}\Xi^{(a)}_{1\nu_1\nu_2)}] - \frac{1}{2}P^2\Xi_{1\alpha_1\alpha_2} - P_{\alpha_2}P^{\nu_1}\Xi_{1\alpha_1\nu_1} - P_{\alpha_1}P^{\nu_1}\Xi_{1\alpha_2\nu_1}. \end{aligned} \tag{8.137}$$

Expanding in its coefficients and using the arbitrary internal momenta, we get

$$\begin{aligned} U_{\alpha_1\alpha_2} = & +\frac{1}{9}\left(4b - 5a - 3\Upsilon\right)g_{\alpha_1\alpha_2}\left(k_1^2 + k_2^2\right) \\ & +\frac{1}{9}\left(4b + 4a + 6\Upsilon\right)g_{\alpha_1\alpha_2}\left(k_1 \cdot k_2\right) \\ & +\frac{1}{9}\left(8b - 10a + 3\Upsilon\right)\left(k_{1\alpha_1}k_{1\alpha_2} + k_{2\alpha_1}k_{2\alpha_2}\right) \\ & +\frac{1}{9}\left(4b - 14a - 3\Upsilon\right)\left(k_{1\alpha_1}k_{2\alpha_2} + k_{2\alpha_1}k_{1\alpha_2}\right) = 0. \end{aligned} \tag{8.138, 8.139}$$

As each row corresponds to linearly independent tensors, the only solution to the system is $a = b = \Upsilon = 0$. That is the unique solution we have discussed so far.

To deep down into the reasons, as demonstrated in the Appendix (F.1), if one accepts a natural reduction in the integrand, it leads to, by example,

$$\Xi^{(b)}_{\mu_1\mu_2\mu_3\mu_4} = \left[3\Sigma^{\rho}_{4\rho\mu_1\mu_2\mu_3\mu_4} - 8\Box_{3\mu_1\mu_2\mu_3\mu_4} - g_{(\mu_1\mu_2}g_{\mu_3\mu_4)}\Delta^{\rho}_{2\rho}\right] \quad (8.140)$$

$$= m^2 \int \frac{\mathrm{d}^2 k}{(2\pi)^2}\left\{\sum_{i=1}^{4}\frac{\partial}{\partial k^{\mu_i}}\frac{-6k_{\mu_1\cdots\hat{\mu}_i\cdots\mu_4}}{D^3_\lambda} - g_{(\mu_3\mu_4}\frac{\partial}{\partial k^{\mu_5}}\frac{k_{\mu_6)}}{D^2_\lambda}\right\} = 0. \quad (8.141)$$

Hence, this corresponds to a convergent integral that vanishes. Nevertheless, we established this result based on the RAGFs without this manipulation.

It is worthwhile to call attention to that $\Upsilon = i/\pi$-factor emerged in the description of the chiral anomaly (from $T^{AV}_{\mu_{12}}$). It uses methods that allow variable integration shifts,

$$q^{\mu_1}(T^{AV}_{\mu_1\mu_2})_1 = -2mT^{PV}_{\mu_2} + \varepsilon_{\mu_2\nu}q^{\nu}\Upsilon \text{ and } q^{\mu_2}(T^{AV}_{\mu_1\mu_2})_2 = \varepsilon_{\mu_1\nu}q^{\nu}\Upsilon; \quad (8.142)$$

while the other Ward Identities are fulfilled in and equal to zero.

The combination of the quadratic surface terms $\Xi^{\text{quad}}_{\alpha_1\alpha_2}$ may be organized in the form

$$\Xi^{\text{quad}}_{\alpha_1\alpha_2} = (W^{\rho}_{2\rho\alpha_{12}} - 2\Delta_{1\alpha_{12}}) + 2g_{\alpha_1\alpha_2}I_{\text{quad}} - 2m^2\left(\Delta_{2\alpha_{12}} + g_{\alpha_{12}}I_{\log}\right) \quad (8.143)$$

$$= \int \frac{\mathrm{d}^2 k}{(2\pi)^2}\left(\frac{4\left(k^2-m^2\right)k_{\alpha_{12}}}{D^2_\lambda} - \frac{4k_{\alpha_{12}}}{D_\lambda}\right) = 0. \quad (8.144)$$

We chose the mass parameter such that $D_\lambda = k^2 - m^2$. There are three arguments, reducing bilinear in the integrand of the last line yields an exact cancellation, or in the massless limit since it is proportional to the mass that goes to zero. Thirdly, some prescriptions make this term zero in various analytic regularization methods.

8.4 Einstein and Weyl Anomalies

We now turn to anomalies; we must take the massless limit. First, looking into the results of contractions, for instance, q^{μ_1}-contraction of the vector part (8.95), axial part (8.101), or with the metric (8.97) or (8.103). There are terms proportional to the mass: the two and one-point functions with mass as coefficient go to zero in this limit:

$$4mT^{SV}_{\mu_1;\mu_2} = 8m^2\left(\Delta_{2\mu_1\mu_2} + g_{\mu_1\mu_2}I_{\log}\right) - \frac{i}{\pi}2m^2\theta_{\mu_1\mu_2}[2Z^{(-1)}_2 - Z^{(-1)}_1] \quad (8.145)$$

$$4mT^{S}_{(-)\mu_2} = 8m^2 q^{\nu}\left(\Delta_{2\nu\mu_2} + g_{\nu\mu_2}I_{\log}\right). \quad (8.146)$$

The last line can also be seen through $q^{\nu}T^{SV}_{\nu;\mu_2} = T^{S}_{(-)\mu_2}$. Thereby $\lim_{m^2\to 0} 4mT^{SV}_{\mu_1;\mu_2} = 0$ and $\lim_{m^2\to 0} 4mT^{S}_{(+)\alpha_2} = 0$. Furthermore, in this way, we have only the vector and axial one-point functions and the RAGFs violating factor $U_{\alpha\beta}$.

For these terms that remain, we consider two scenarios: One that derives from the preservation of WI for $T^{VV}_{\alpha\beta}$ and $T^{AA}_{\alpha\beta}$, which requires vanishing of surface terms and preserves momentum-space translational invariance. The other scenario exploited is when surface terms are finite and determined by the constraint of RAGFs.

$$\begin{aligned}(64i)\, q^{\alpha_1}[T^{G}_{\mu_{12}\alpha_{12}}]_{ij} &= 4\mathcal{S}^{A}_{(-)(\mu_1;\mu_2)\alpha_2} + 4\mathcal{S}^{V}_{(-)(\mu_1;\mu_2)\alpha_2} - 2\varepsilon_{\alpha_2\nu}q^{\nu}\mathcal{B}_{(\mu_1;\mu_2)} \\ &+2[q_{\mu_1}\mathcal{B}_{\alpha_2;\mu_2} + q_{\mu_2}\mathcal{B}_{\alpha_2;\mu_1} - g_{\alpha_2\mu_1}q^{\nu}\mathcal{B}_{\nu;\mu_2} - g_{\alpha_2\mu_2}q^{\nu}\mathcal{B}_{\nu;\mu_1}] \\ &-(\delta_{i,2}+\delta_{j,2})\, q^{\nu}[\varepsilon_{\alpha_2\mu_1}U_{\nu\mu_2} + \varepsilon_{\alpha_2\mu_2}U_{\nu\mu_1}] \\ &+(\delta_{i,2}+\delta_{j,2})\, q^{\nu}[\varepsilon_{\mu_2\nu}U_{\mu_1\alpha_2} + \varepsilon_{\mu_1\nu}U_{\mu_2\alpha_2}]\end{aligned} \tag{8.147}$$

$$\begin{aligned}(64i)\, q^{\mu_1}[T^{G}_{\mu_{12}\alpha_{12}}]_{ij} &= 4\mathcal{S}^{A}_{(-)(\alpha_1;\alpha_2)\mu_2} + 4\mathcal{S}^{V}_{(-)(\alpha_1;\alpha_2)\mu_2} - 2\varepsilon_{\mu_2\nu}q^{\nu}\mathcal{B}_{(\alpha_1;\alpha_2)} \\ &+2[q_{\alpha_1}\mathcal{B}_{\mu_2;\alpha_2} + q_{\alpha_2}\mathcal{B}_{\mu_2;\alpha_1} - g_{\mu_2\alpha_1}q^{\nu}\mathcal{B}_{\nu;\alpha_2} - g_{\mu_2\alpha_2}q^{\nu}\mathcal{B}_{\nu;\alpha_1}] \\ &-(2-\delta_{i,2}-\delta_{j,2})\, q^{\nu}(\varepsilon_{\mu_2\alpha_1}U_{\nu\alpha_2} + \varepsilon_{\mu_2\alpha_2}U_{\nu\alpha_1}) \\ &+(2-\delta_{i,2}-\delta_{j,2})\, q^{\nu}(\varepsilon_{\alpha_1\nu}U_{\mu_2\alpha_2} + \varepsilon_{\alpha_2\nu}U_{\mu_2\alpha_1}).\end{aligned} \tag{8.148}$$

$$\begin{aligned}(64i)\, g^{\alpha_{12}}[T^{G}_{\mu_{12}\alpha_{12}}]_{ij} &= 4\mathcal{B}_{(\mu_1;\mu_2)} - 4\varepsilon_{\mu_1}{}^{\nu}\mathcal{B}_{\nu;\mu_2} - 4\varepsilon_{\mu_2}{}^{\nu}\mathcal{B}_{\nu;\mu_1} \\ &-2[2-(\delta_{i,2}+\delta_{j,2})](\varepsilon_{\mu_1\nu}U^{\nu}_{\mu_2} + \varepsilon_{\mu_2\nu}U^{\nu}_{\mu_1}) + 8U_{\mu_1\mu_2}\end{aligned} \tag{8.149}$$

$$\begin{aligned}(64i)\, g^{\mu_{12}}[T^{G}_{\mu_{12}\alpha_{12}}]_{ij} &= 4\mathcal{B}_{(\alpha_1;\alpha_2)} - 4\varepsilon_{\alpha_1}{}^{\nu}\mathcal{B}_{\nu;\alpha_2} - 4\varepsilon_{\alpha_2}{}^{\nu}\mathcal{B}_{\nu;\alpha_1} \\ &-2\,(\delta_{i,2}+\delta_{j,2})\,(\varepsilon_{\alpha_1\nu}U^{\nu}_{\alpha_2} + \varepsilon_{\alpha_2\nu}U^{\nu}_{\alpha_1}) + 8U_{\alpha_1\alpha_2}\end{aligned} \tag{8.150}$$

8.4.1 Vanishing Surface Terms: Violating RAGFs

In the first scenario investigated, we adopt the interpretation of the surfaces as

$$\Delta_{2\mu\nu} = 0; \Box_{3\alpha_{12}\nu_{12}} = 0; \Sigma_{4\alpha_{12}\nu_{12}\nu_{34}} = 0.$$

In the massless limit, dropping out the quadratic structures as they are proportional to the mass is possible. The condition implies $W_4 = W_3 = 0$ as well because these tensors are defined as a linear combination of the previous ones (3.17-3.16). In tandem, this restriction sets the result to the sum and differences of one-point functions $\mathcal{S}^{V}_{(-)} = \mathcal{S}^{A}_{(-)} = \mathcal{B} = 0$. The present interpretation for surface terms violates RAGFs, the amount which the U-factor gives shown in the previous section, see (8.136). We recover its value

$$U_{\alpha\mu} = -\frac{1}{3}\theta_{\alpha\mu}\Upsilon = -\frac{1}{3}\left(\frac{i}{\pi}\right)\theta_{\alpha\mu}. \tag{8.151}$$

Einstein Anomaly: They could appear in the vector and axial sectors; however, in the current setting, the vector part vanishes. For this symmetry, we only need to evaluate the results for one index, namely,

$$q^{\mu_1}[\mathcal{T}^{V}_{\mu_{12}\alpha_{12}}] = 4\mathcal{S}^{V}_{(-)(\alpha_1;\alpha_2)\mu_2} + 2[q_{\alpha_1}\mathcal{B}_{\mu_2;\alpha_2} + q_{\alpha_2}\mathcal{B}_{\mu_2;\alpha_1} - g_{\mu_2\alpha_1}q^{\nu}\mathcal{B}_{\nu;\alpha_2} - g_{\mu_2\alpha_2}q^{\nu}\mathcal{B}_{\nu;\alpha_1}] = 0.$$

That is an interesting consequence of this perspective; however, it breaks integration linearity if even and odd amplitudes should have a uniform mathematical treatment. The other equations to be discussed get contributions from the axial part and are

$$q^{\alpha_1}[T^G_{\mu_{12}\alpha_{12}}]_{ij} = -\frac{1}{96}\left(\frac{1}{\pi}\right)(\delta_{i,2}+\delta_{j,2})\,\varepsilon_{\mu_1\nu}q^\nu\theta_{\mu_2\alpha_2} \quad (8.152)$$

$$q^{\mu_1}[T^G_{\mu_{12}\alpha_{12}}]_{ij} = -\frac{1}{96}\left(\frac{1}{\pi}\right)(2-\delta_{i,2}-\delta_{j,2})\,\varepsilon_{\alpha_1\nu}q^\nu\theta_{\mu_2\alpha_2}, \quad (8.153)$$

where was used the identity $\varepsilon_{\alpha_2\nu}q^\nu\theta_{\mu_2\alpha_1} = \varepsilon_{\alpha_1\nu}q^\nu\theta_{\mu_2\alpha_2}$.

It exhibits a richer structure because, for null surface terms, the axial sector reveals a dependence on the version of trace with the chiral and four Dirac matrices that are employed. After integration, the identities valid for the integrand are transformed by the present interpretation in different tensors. It implies that intermediary operations lead to many possibilities, some of which are present above. The breaking of linearity makes the versions unequal as the simpler $T^{AV}_{\mu\nu}$. The version $ij = \{11, 22\}$ only has anomalies in one set of indexes, μ_i or α_i. A table of results can clarify these statements:

$$\left\{\begin{array}{ll} q^{\alpha_1}[T^G_{\mu_{12}\alpha_{12}}]_{11} = 0 & q^{\alpha_1}[T^G_{\mu_{12}\alpha_{12}}]_{22} = -\frac{1}{48}\left(\frac{1}{\pi}\right)\varepsilon_{\mu_1\nu}q^\nu\theta_{\mu_2\alpha_2} \\ q^{\mu_1}[T^G_{\mu_{12}\alpha_{12}}]_{11} = -\frac{1}{48}\left(\frac{1}{\pi}\right)\varepsilon_{\alpha_1\nu}q^\nu\theta_{\mu_2\alpha_2} & q^{\mu_1}[T^G_{\mu_{12}\alpha_{12}}]_{22} = 0 \end{array}\right\}$$

In the case of $ij = \{12, 21\}$, the mixed versions of the anomaly appear equally distributed and are half of the other versions:

$$\left\{\begin{array}{ll} q^{\alpha_1}[T^G_{\mu_{12}\alpha_{12}}]_{12} = -\frac{1}{96}\left(\frac{1}{\pi}\right)\varepsilon_{\mu_1\nu}q^\nu\theta_{\mu_2\alpha_2} & q^{\alpha_1}[T^G_{\mu_{12}\alpha_{12}}]_{21} = -\frac{1}{96}\left(\frac{1}{\pi}\right)\varepsilon_{\mu_1\nu}q^\nu\theta_{\mu_2\alpha_2} \\ q^{\mu_1}[T^G_{\mu_{12}\alpha_{12}}]_{12} = -\frac{1}{96}\left(\frac{1}{\pi}\right)\varepsilon_{\alpha_1\nu}q^\nu\theta_{\mu_2\alpha_2} & q^{\mu_1}[T^G_{\mu_{12}\alpha_{12}}]_{21} = -\frac{1}{96}\left(\frac{1}{\pi}\right)\varepsilon_{\alpha_1\nu}q^\nu\theta_{\mu_2\alpha_2} \end{array}\right\}$$

The results above are the common finding in the literature. In other words, we have options for expressing the AV/VA functions in terms of the even VV/AA amplitudes.

Weyl Anomaly: In the scenario of RAGFs violations, we get

$$g^{\alpha_{12}}[T^G_{\mu_{12}\alpha_{12}}]_{ij} = -\frac{1}{96\pi}\left[4\theta_{\mu_1\mu_2} - [2-(\delta_{i,2}+\delta_{j,2})](\varepsilon_{\mu_1\nu}\theta^\nu_{\mu_2} + \varepsilon_{\mu_2\nu}\theta^\nu_{\mu_1})\right] \quad (8.154)$$

$$g^{\mu_{12}}[T^G_{\mu_{12}\alpha_{12}}]_{ij} = -\frac{1}{96\pi}\left[4\theta_{\alpha_1\alpha_2} - (\delta_{i,2}+\delta_{j,2})(\varepsilon_{\alpha_1\nu}\theta^\nu_{\alpha_2} + \varepsilon_{\alpha_2\nu}\theta^\nu_{\alpha_1})\right]. \quad (8.155)$$

As the equations are not unique, the odd part of Weyl anomaly is absent in some versions,

$$g^{\alpha_{12}}[T^G_{\mu_{12}\alpha_{12}}]_{11} = -\frac{1}{48\pi}[2\theta_{\mu_1\mu_2} - (\varepsilon_{\mu_1\nu}\theta^\nu_{\mu_2} + \varepsilon_{\mu_2\nu}\theta^\nu_{\mu_1})] \quad (8.156)$$

$$g^{\mu_{12}}[T^G_{\mu_{12}\alpha_{12}}]_{11} = -\frac{1}{24\pi}\theta_{\alpha_1\alpha_2} \quad (8.157)$$

$$g^{\alpha_{12}}[T^G_{\mu_{12}\alpha_{12}}]_{22} = -\frac{1}{24\pi}\theta_{\mu_1\mu_2} \quad (8.158)$$

$$g^{\mu_{12}}[T^G_{\mu_{12}\alpha_{12}}]_{22} = -\frac{1}{48\pi}[2\theta_{\alpha_1\alpha_2} - (\varepsilon_{\alpha_1\nu}\theta^\nu_{\alpha_2} + \varepsilon_{\alpha_2\nu}\theta^\nu_{\alpha_1})]. \quad (8.159)$$

Note that the above equation expresses the possibility of not having anomalies in one energy-momentum tensor occurring when that version has an Einstein anomaly. The mixed versions show the same amount of violation in all contractions

$$g^{\alpha_{12}}[T^G_{\mu_{12}\alpha_{12}}]_{12} = -\frac{1}{96\pi}\left[4\theta_{\mu_1\mu_2} - (\varepsilon_{\mu_1\nu}\theta^{\nu}_{\mu_2} + \varepsilon_{\mu_2\nu}\theta^{\nu}_{\mu_1})\right] \quad (8.160)$$

$$g^{\mu_{12}}[T^G_{\mu_{12}\alpha_{12}}]_{12} = -\frac{1}{96\pi}\left[4\theta_{\alpha_1\alpha_2} - (\varepsilon_{\alpha_1\nu}\theta^{\nu}_{\alpha_2} + \varepsilon_{\alpha_2\nu}\theta^{\nu}_{\alpha_1})\right] \quad (8.161)$$

$$g^{\mu_{12}}[T^G_{\mu_{12}\alpha_{12}}]_{21} = -\frac{1}{96\pi}\left[4\theta_{\alpha_1\alpha_2} - (\varepsilon_{\alpha_1\nu}\theta^{\nu}_{\alpha_2} + \varepsilon_{\alpha_2\nu}\theta^{\nu}_{\alpha_1})\right] \quad (8.162)$$

$$g^{\alpha_{12}}[T^G_{\mu_{12}\alpha_{12}}]_{21} = -\frac{1}{96\pi}\left[4\theta_{\mu_1\mu_2} - (\varepsilon_{\mu_1\nu}\theta^{\nu}_{\mu_2} + \varepsilon_{\mu_2\nu}\theta^{\nu}_{\mu_1})\right]. \quad (8.163)$$

They show Einstein anomalies in all contractions as well.

For the sake of commentary, we rederived the finite part of the U-factor. The finite part of the basic permutation may be written as

$$\mathcal{T}^{VV}_{\mu_1\alpha_1\mu_2\alpha_2} = \left(\frac{i}{\pi}\right)\frac{1}{q^2}\left\{2\theta_{\mu_1\alpha_1}\theta_{\mu_2\alpha_2}[3Z_2^{(0)} - 2Z_1^{(0)}] - \Omega_{\mu_1\alpha_1\mu_2\alpha_2}[2Z_2^{(0)} - Z_1^{(0)}]\right\}. \quad (8.164)$$

The finite part of the U-factor comes from the equation below

$$U_{\alpha_2\mu_2} = (g^{\mu_1\alpha_1}\mathcal{T}^{VV}_{\mu_1\alpha_1\mu_2\alpha_2} - 4mT^{SV}_{\mu_2;\alpha_2}) \quad (8.165)$$

$$= \frac{2i}{\pi}\theta_{\mu_2\alpha_2}\{[3Z_2^{(0)} - 2Z_1^{(0)}] + m^2[2Z_2^{(-1)} - Z_1^{(-1)}]\} = -(i/3\pi)\theta_{\mu_2\alpha_2}. \quad (8.166)$$

For the last equation, we have used the reductions above

$$3Z_2^{(0)} - 2Z_1^{(0)} = -\frac{m^2}{q^2}Z_0^{(0)} - \frac{1}{6}; \qquad Z_0^{(0)} = q^2[2Z_2^{(-1)} - Z_1^{(-1)}].$$

8.4.2 Finite Surface Terms: RAGFs satisfied

Summarizing: In this scenario to be investigated, we adopt the interpretation of surfaces as finite and their values determined by RAGFs, (8.132)-(8.134). Thus, all relations are satisfied, and odd amplitudes become unique and independent of the trace prescription. However, now the one-point functions take finite values while $U = 0$.

The finite violating terms in the momentum contractions: to derive this term in general, we remind that $q^a\mathcal{T}^{VV}_{abcd} = \mathcal{S}^V_{(-)b;cd}$, where $\mathcal{S}^V_{(-)b;cd}$ is the difference of combining the vectorial one-point functions defined in (8.12). In the massless limit, the explicit contribution of the surface term can be arranged as

$$\begin{aligned}\mathcal{S}^V_{(-)b;cd} = &+P^{\nu_{12}}q^{\nu_3}W_{4bcd\nu_{123}} \\ &-2P^{\nu_1}q^{\nu_2}(P_bW_{3cd\nu_{12}} + P_dW_{3bc\nu_{12}} + P_cW_{3bd\nu_{12}}) - q_bP^{\nu_{12}}W_{3cd\nu_{12}} \\ &+2P^{\nu_1}\left[-(P\cdot q)W_{3bcd\nu_1} + q_b(P_d\Delta_{2c\nu_1} + P_c\Delta_{2d\nu_1})\right] \\ &+q^{\nu_1}\left[-P^2W_{3bcd\nu_1} + 2(P_bP_c\Delta_{2d\nu_1} + P_bP_d\Delta_{2c\nu_1} + P_cP_d\Delta_{2b\nu_1})\right] \\ &+2(P\cdot q)(P_b\Delta_{2cd} + P_c\Delta_{2bd} + P_d\Delta_{2bc}) + q_bP^2\Delta_{2cd} \\ &+\frac{1}{3}q^{\nu_{12}}q^{\nu_3}W_{4bcd\nu_{123}} - q_bq^{\nu_{12}}W_{3cd\nu_{12}} - q^2q^{\nu_1}W_{3bcd\nu_1} + q_bq^2\Delta_{2cd}.\end{aligned} \quad (8.167)$$

Here we are using Latin letters in order to make index replacement operational. The combination of surface terms defined in (3.9), (3.16) and 3.17 assuming the values

$$W_{4abcd\nu_{12}} = -\frac{i}{4\pi}\frac{11}{6}g_{(ab}g_{cd}g_{\nu_{12})}; \quad W_{3abcd} = -\frac{i}{4\pi}\frac{3}{2}g_{(ab}g_{cd)}; \quad W_{2ab} = \Delta_{2ab} = -\frac{i}{4\pi}g_{ab}.$$

And for the basic permutation as well, it is reasonable to get

$$\begin{aligned} -i\left(4\pi\right)\mathcal{T}^{VV}_{abcd} &= -\frac{1}{3}g_{(ab}g_{cd)}P^2 + \frac{1}{2}g_{ab}g_{cd}P^2 + \frac{1}{3}P_{(a}P_b g_{cd)} - P_d P_c g_{ab} \\ &+\frac{8}{9}g_{(ab}g_{cd)}q^2 - \frac{11}{9}q_{(a}q_b g_{cd)} + 3g_{ab}q_c q_d - \frac{3}{2}g_{ab}g_{cd}q^2 + 2g_{cd}q_a q_b. \end{aligned} \tag{8.168}$$

where the symmetrization of the notation follows (the same for $q_{(a}q_b g_{cd)}$),

$$P_{(a}P_b g_{cd)} = P_a P_b g_{cd} + P_a P_c g_{bd} + P_a P_d g_{bc} + P_b P_c g_{ad} + P_b P_d g_{ac} + P_c P_d g_{ab}. \tag{8.169}$$

Now, we admit a covariant parameterization of the ambiguous momentum concerning the external one. As an example, we have

$$P_\mu = \left(k_{1\mu} + k_{2\mu}\right) = \chi q_\mu. \tag{8.170}$$

Therefore one of the terms in the RAGFs can be expressed as

$$\begin{aligned} \mathcal{S}^V_{(-)b;cd} &= \frac{i}{(4\pi)}\frac{\chi^2}{2}q_b\left(\theta_{cd} + q_c q_d\right) + \frac{i}{2\left(4\pi\right)}q_b\theta_{cd} \\ &+\frac{i}{6\left(4\pi\right)}\left[-2\left[q_d\theta_{bc} + q_c\theta_{bd} + q_b\theta_{cd}\right] - 7q_b q_c q_d\right], \end{aligned} \tag{8.171}$$

inside the full contractions we get symmetrizations $\mathcal{S}^V_{(b;c)d}$.

The factor that appears in the trace relations, defined (8.11), is developed in the form

$$\mathcal{B}_{\alpha_1;\alpha_2} = 2T^V_{(+)\alpha_1;\alpha_2} + q_{\alpha_2}T^V_{(+)\alpha_1} \tag{8.172}$$

$$\begin{aligned} &= 4\left(\Delta_{1\alpha_1\alpha_2} + g_{\alpha_1\alpha_2}I_{\text{quad}}\right) + 2q_{\alpha_2}q^{\nu_1}\Delta_{2\alpha_1\nu_1} \\ &+P^{\nu_{12}}W_{3\alpha_1\alpha_2\nu_{12}} - P^2\Delta_{2\alpha_1\alpha_2} - 2P^{\nu_1}\left(P_{\alpha_1}\Delta_{2\alpha_2\nu_1} + P_{\alpha_2}\Delta_{2\alpha_1\nu_1}\right) \\ &+q^{\nu_{12}}W_{3\alpha_1\alpha_2\nu_{12}} - q^2\Delta_{2\alpha_1\alpha_2} - 2q^{\nu_1}\left(q_{\alpha_1}\Delta_{2\alpha_2\nu_1} + q_{\alpha_2}\Delta_{2\alpha_1\nu_1}\right). \end{aligned} \tag{8.173}$$

In the symmetric limit (massless limit) and using the parametrization (8.170), we have

$$-i\left(4\pi\right)\mathcal{B}_{(\alpha_1;\alpha_2)} = -\chi^2\left(\theta_{\alpha_1\alpha_2} - q_{\alpha_2}q_{\alpha_1}\right) - \left(\theta_{\alpha_1\alpha_2} + 3q_{\alpha_2}q_{\alpha_1}\right) \tag{8.174}$$

$$-i\left(4\pi\right)q^{\alpha_1}\mathcal{B}_{\alpha_1;\alpha_2} = \frac{\left(\chi^2 - 3\right)}{2}q_{\alpha_2}q^2. \tag{8.175}$$

Axial combinations $\mathcal{S}^A_{a;bc} = -\varepsilon_a{}^\nu\mathcal{S}^V_{a;bc}$, symmetrizing these terms as in the final result

$$\begin{aligned} \mathcal{S}^A_{(-)(a;b)c} &= -\frac{\chi^2}{2}\varepsilon_{a\nu}[2q^\nu\theta_{bc} - q_c\theta^\nu_b + q_b q_c q^\nu] \\ &+\frac{1}{6}\varepsilon_{a\nu}[-5q_c\theta^\nu_b + 4q_b\theta^\nu_c - 2q^\nu\theta_{bc} + 5q_b q_c q^\nu]. \end{aligned} \tag{8.176}$$

Einstein Anomaly: The total contribution for the odd sector where we can isolate one term that corresponds to the version $[\mathcal{T}^{A}_{\mu_{12}\alpha_{12}}]_{12}$,

$$q^{\mu_1}[\mathcal{T}^{A}_{\mu_{12}\alpha_{12}}] = -\frac{i\varepsilon_{\alpha_1\nu}}{12\pi}\{8q^{\nu}\theta_{\alpha_2\mu_2}+(6\chi^2-10)[q^{\nu}\theta_{\alpha_2\mu_2}-q_{\mu_2}\theta^{\nu}_{\alpha_2}-q_{\alpha_2}\theta^{\nu}_{\mu_2}+q_{\alpha_2}q_{\mu_2}q^{\nu}]\}, \quad (8.177)$$

therefore the choice $\chi^2 = 5/3$ can recover that value. Despite that, there is a choice of routings that can reproduce the values for a specific version when surface terms are made null; the even part does not show such a possibility, as can be seen in

$$q^{\mu_1}[\mathcal{T}^{V}_{\mu_{12}\alpha_{12}}] = \frac{i}{6\pi}\{(6\chi^2-10)q_{\alpha_1}q_{\alpha_2}q_{\mu_2}+2(q_{\alpha_1}\theta_{\alpha_2\mu_2}+q_{\alpha_2}\theta_{\alpha_1\mu_2}-2q_{\mu_2}\theta_{\alpha_1\alpha_2}-2q_{\alpha_1}q_{\alpha_2}q_{\mu_2})\}. \quad (8.178)$$

This presents us with two features: it is impossible to use any choice of routings to eliminate the anomaly, and the choice that makes the axial part with a standard value implies in the equation above,

$$q^{\mu_1}[\mathcal{T}^{V}_{\mu_{12}\alpha_{12}}] = \frac{i}{3\pi}(q_{\alpha_1}\theta_{\alpha_2\mu_2}+q_{\alpha_2}\theta_{\alpha_1\mu_2}-2q_{\mu_2}\theta_{\alpha_1\alpha_2}-2q_{\alpha_1}q_{\alpha_2}q_{\mu_2}). \quad (8.179)$$

Summing the Eqs. (8.177) and (8.179), the gravitational amplitude independent of the Dirac trace becomes

$$\begin{aligned} q^{\mu_1}T^{G}_{\mu_1\mu_2\alpha_1\alpha_2} &= -\frac{1}{96\pi}\varepsilon_{\alpha_1\nu}q^{\nu}\theta_{\alpha_2\mu_2}+\frac{1}{192\pi}(q_{\alpha_1}\theta_{\alpha_2\mu_2}+q_{\alpha_2}\theta_{\alpha_1\mu_2}-2q_{\mu_2}\theta_{\alpha_1\alpha_2}-2q_{\alpha_1}q_{\alpha_2}q_{\mu_2}) \\ &+\frac{(3\chi^2-5)}{384\pi}\{2q_{\alpha_1}q_{\alpha_2}q_{\mu_2}+\varepsilon_{\alpha_1}{}^{\nu}\left(q_{\mu_2}\theta_{\nu\alpha_2}+q_{\alpha_2}\theta_{\nu\mu_2}-q_{\nu}\theta_{\alpha_2\mu_2}-q_{\alpha_2}q_{\mu_2}q_{\nu}\right)\}. \end{aligned}$$

The vector part is irremovable through choices that are intrinsic elements of Feynman's diagrammatic computation of this correlator.

Weyl Anomaly: The odd part of this symmetry violation arises from tensor $\mathcal{B}_{\sigma;\rho}$,

$$g^{\mu_{12}}[\mathcal{T}^{A}_{\mu_{12}\alpha_{12}}] = -4\varepsilon_{\alpha_1}{}^{\nu}\mathcal{B}_{\nu;\alpha_2}-4\varepsilon_{\alpha_2}{}^{\nu}\mathcal{B}_{\nu;\alpha_1} \quad (8.180)$$

$$g^{\alpha_{12}}[\mathcal{T}^{A}_{\mu_{12}\alpha_{12}}] = -4\varepsilon_{\mu_1}{}^{\nu}\mathcal{B}_{\nu;\mu_2}-4\varepsilon_{\mu_2}{}^{\nu}\mathcal{B}_{\nu;\mu_1}. \quad (8.181)$$

Simple manipulation of indices yields the expressions

$$g^{\mu_{12}}[\mathcal{T}^{A}_{\mu_{12}\alpha_{12}}] = -\frac{i}{\pi}\left(\chi^2-1\right)q^{\nu}\left(\varepsilon_{\alpha_1\nu}q_{\alpha_2}+\varepsilon_{\alpha_2\nu}q_{\alpha_1}\right), \quad (8.182)$$

and analogously for the other trace. The odd part of the Weyl anomaly can be removed, but this does not happen to the even part. If the parameter χ is chosen to make the Einstein anomaly with the standard form, we obtain an equivalent result as

$$g^{\mu_{12}}[\mathcal{T}^{A}_{\mu_{12}\alpha_{12}}] = -\frac{2i}{3\pi}q^{\nu}(\varepsilon_{\alpha_1\nu}q_{\alpha_2}+\varepsilon_{\alpha_2\nu}q_{\alpha_1})-\frac{i}{3\pi}(3\chi^2-5)q^{\nu}(\varepsilon_{\alpha_1\nu}q_{\alpha_2}+\varepsilon_{\alpha_2\nu}q_{\alpha_1}). \quad (8.183)$$

Since that constraint is given by $\chi^2 = 5/3$.

Through the same line of reasoning, we obtain the even part

$$g^{\mu_{12}}[\mathcal{T}^{V}_{\mu_{12}\alpha_{12}}] = 4\mathcal{B}_{(\alpha_1;\alpha_2)} = -\frac{i}{\pi}\chi^2\left(\theta_{\alpha_1\alpha_2}-q_{\alpha_2}q_{\alpha_1}\right)-\frac{i}{\pi}\left(\theta_{\alpha_1\alpha_2}+3q_{\alpha_2}q_{\alpha_1}\right), \quad (8.184)$$

similar to the other set of indices. However, now the constraint which reproduced the standard result to the odd part furnishes a different expression to the Weyl anomaly of the even part, namely,

$$g^{\mu_{12}}[\mathcal{T}^{V}_{\mu_{12}\alpha_{12}}] = -\frac{4i}{3\pi}\left(2\theta_{\alpha_1\alpha_2} + q_{\alpha_2}q_{\alpha_1}\right) - \frac{i}{6\pi}\left(6\chi^2 - 10\right)\left(\theta_{\alpha_1\alpha_2} - q_{\alpha_2}q_{\alpha_1}\right). \tag{8.185}$$

Therefore, the total routing-dependent trace anomaly is given by

$$\begin{aligned} g^{\mu_{12}}[T^{G}_{\mu_{12}\alpha_{12}}] &= -\frac{1}{96\pi}q^{\nu}\left(\varepsilon_{\alpha_1\nu}q_{\alpha_2} + \varepsilon_{\alpha_2\nu}q_{\alpha_1}\right) - \frac{1}{48\pi}\left(2\theta_{\alpha_1\alpha_2} + q_{\alpha_2}q_{\alpha_1}\right) \\ &\quad -\frac{1}{192\pi}\left(3\chi^2 - 5\right)\left[q^{\nu}\left(\varepsilon_{\alpha_1\nu}q_{\alpha_2} + \varepsilon_{\alpha_2\nu}q_{\alpha_1}\right) + \left(\theta_{\alpha_1\alpha_2} - q_{\alpha_2}q_{\alpha_1}\right)\right]. \end{aligned} \tag{8.186}$$

In this context, where the integration linearity is maintained, and intermediary operations on the Dirac traces have no effect, we have the finiteness of the relevant surface terms as the constraint. However, this also implies violations of the energy-momentum tensor symmetries and the break of translational invariance (in momentum space, at least). To keep Ward identities, which crucially depend on translational invariance, the attitude often adopted is, by some regularization, to remove the surface terms. The algebraic consequence is to spoil the RAGFs to odd-tensor amplitudes, deduced without making any shifts whose unique hypothesis is the linearity of integration. Equivalently, the uniqueness of these amplitudes is lost as they come from the Feynman rules, thus opening the room for multiple expressions that violate the symmetries under study anyway. Only a subset of these possibilities is visualized in the literature.

Chapter 9

Final Remarks and Perspectives

We performed a detailed probe of a significant number of pseudo-tensor diagrams that correspond to anomalous amplitudes in two and four dimensions, following a strategy to cope with the divergences introduced in the thesis of O.A. Battistel. We apply this procedure to the bubbles (the gravitational case is discussed in the sequel) and triangles with power counting logarithmic and linear, respectively. The finite ones get integrated after splitting off and organizing the divergent parts without further action. In this point, the scalar objects $I_{\log}^{(2n)}$ exactly cancel, letting the final result as a sum of finite tensors and surface terms, $\Delta_{n+1;\mu_{12}}^{(2n)}$. This recipe relies on the principle of the linearity of integration.

The role of that aspect emerges in the odd amplitudes in even dimensions; see the e-print ([40]). Contracting with the external momenta follows RAGFs that, after integration, incorporate the linearity of integration. For the relevant two and three-point functions in the respective dimensions, we wrote the equations (because they are not automatically valid) representing that property as

$$\begin{aligned} q^{\mu_i} T_{\mu_{12}}^{(2D)\Gamma_1\Gamma_2} &= T_{i(-)\mu_a}^{(2D)A} + \varepsilon_{\mu_a \nu}\Omega_i^{(2pt)}, \quad i,a = \{1,2\}, i \neq a \\ q_i^{\mu_i} T_{\mu_{123}}^{(4D)\Gamma_1\Gamma_2\Gamma_3} &= T_{i(-)\mu_{ab}}^{(4D)AV} + \varepsilon_{\mu_{ab}\nu_{12}} q_2^{\nu_1} q_3^{\nu_2}\Omega_i^{(3pt)}, \quad i,a,b = \{1,2,3\}, i \neq a < b \end{aligned} \quad (9.1)$$

where the vertices $\Gamma_i \in (V;A) = (\gamma_\mu;\gamma_*\gamma_\mu)$ and the notation $T_{(-)}^{(2D)A}$, $T_{i(-)}^{(4D)AV}$ means the actual differences that appear in (5.18) and (6.47-6.49). The explicit surface terms read

$$T_{\mu}^{(2D)A}(k_i) = 2\varepsilon_{\mu\alpha} k_i^{\nu} \Delta_{2\nu}^{(2)\alpha} \quad (9.2)$$

$$T_{\mu\nu}^{(4D)AV}(k_i,k_j) = 2i\varepsilon_{\mu\nu\alpha\sigma}(k_j - k_i)^{\sigma}(k_i + k_j)^{\gamma}\Delta_{3\gamma}^{(4)\alpha}. \quad (9.3)$$

Let us start with four dimensions and then back to two. There, if the three equations for the RAGFs (9.1) hold at the same time and the vanishing of $T_{\mu\nu}^{(4D)AV}$ functions, or their difference, were possible, then that would allow the vector and partial axial symmetry to hold simultaneously. That signifies we can make shifts and thus have momentum-space translational invariance since the only hypothesis necessary to prove $T_{\mu\nu}^{(4)AV} = -T_{\mu\nu}^{(4)AV} = 0$ is this symmetry. However, such structures depend on the unphysical and arbitrary sum of routings and are proportional to surface terms that can violate translational symmetry.

If we were only searching to cancel that terms, it would be seen that choosing routings is not possible since we should have $P_{31} = P_{21} = P_{32} = 0 \to q_i = 0$. A partial solution is to make the surface term zero, then recover that symmetry.

Nevertheless, low-energy theorems demonstrated in Section (6.2) showed that a tensor with the characteristics of AVV, for example, a function of the external momenta related to PVV tensor, must satisfy, in this case, $p_{31}^{\mu_1} T_{\mu_{123}}^{AVV}|_0 = 0 \neq -2mT_{\mu_{23}}^{PVV}|_0$. That is impossible since the finite PVV does not behave like that. In general, we demonstrated that assuming the most general tensor (when written in terms of the physical momenta), without resorting to a specific symmetry, we got to have

$$q_i^{\mu_i} T_{\mu_{123}}^{(4D)\Gamma_{123}} = \varepsilon_{\mu_{ab}\nu_{12}} q_2^{\nu_1} q_3^{\nu_2} V_i \to \left.(V_1 + V_2 - V_3)\right|_0 = 0. \tag{9.4}$$

On the other hand, computing the three-point form factors Ω_i from the amplitudes PVV, PAA, and for amplitudes AVS and ASV with three different masses, we find

$$\Omega_1(0) + \Omega_2(0) - \Omega_3(0) = 1/(2\pi)^2. \tag{9.5}$$

Thus, if the linearity of integration and translational symmetry were simultaneously valid, we should have $V_i = \Omega_i$. Therefore, the two last and independent equations above would be in contradiction. We can say that the low-energy behavior of finite functions precludes these two properties from living together. Writing $V_i = \Omega_i + \mathcal{A}_i$, we have a constraint over the anomalies $\mathcal{A}_i$ by finite functions, stating that once two of them are fixed, the third is unambiguously determined. At this point, we have that integration linearity can not hold for any value of the surface term, in particular, not for the vanishing one.

All the tensors we investigated show independent combinations of routings, surface terms, and the ε-tensor. We took these elements as hypotheses and general as allowed, not writing the internal through external momenta since the former can also be non-covariant. Thus, by knowing the RHS of the relations, we lay down: it is impossible without additional conditions to satisfy all the RAGF. In other words, they are not valid for any value of the surface term, see Section (6.3). The satisfaction of all the RAGFs makes the low-energy limit above (9.5) the value and the reason why the surface term can not vanish; see the derivation of the equation in (6.119), as integration linearity requires

$$2i\Delta_{3\rho}^{(4)\rho} = 1/(2\pi)^2 = \Omega_1(0) + \Omega_2(0) - \Omega_3(0). \tag{9.6}$$

For this reason, we demonstrated that translational symmetry and linearity of integration are incompatible properties for these perturbative amplitudes. Furthermore, the same derivations clarify the nomenclature and choice of the versions; they are the expressions that automatically satisfy as many RAGFs as possible.

Returning to two dimensions: In this scenario, the 2pt functions do not show linearly divergent integrals that are the assumed source of the symmetry violations. However, they show power-counting zero and tensor integrals with intrinsic surface terms, though the coefficients are the physical momenta. In reality, in context with the one-point axial amplitudes $T^{A}_{\mu}(k_i)$, we have linear power counting integrals, and their shift invariance takes place in the discussion when establishing WIs. The constraints on the differences $T^{A}_{(-)\mu} = -\varepsilon_{\mu}{}^{\alpha} T^{V}_{(-)\alpha} = 2\varepsilon_{\mu\alpha}(k_1^{\nu} - k_2^{\nu})\Delta^{(2)\alpha}_{2\nu}$ are formally necessary for the WIs for even and odd amplitudes (VV-AA and AV-VA), but we cannot choose the arbitrary momenta as $k_1 = k_2 = 0$ since this implies the physical one is $q = 0 = k_2 - k_1$, we must have $\Delta^{(2)}_{2\mu\nu} = 0$. Nonetheless, this is a premature conclusion once we know that we must have both RAGFs and vanishing of surface terms. The non-concomitant presence of these properties is due to the kinematical implications below that we also showed without resorting to a particular symmetry, and for two masses,

$$q^{\mu_i} T^{(2D)AV}_{\mu_{12}} = \varepsilon_{\mu_a \nu} q^{\nu} V_i \to (V_1 + V_2)|_0 = 0. \tag{9.7}$$

The kinematical theorem is incompatible with the low-energy limit of finite functions

$$\Omega^{PV}_1(0) + \Omega^{AS}_1(0) = -i/\pi. \tag{9.8}$$

Hence, the V_1 and V_2 functions are inevitably of the form $V_i = \Omega_i + \mathcal{A}_i$, with $\mathcal{A}_1 + \mathcal{A}_2 = i/\pi$.

Moreover, considering the surface terms for the expression to the general tensor, an analogous condition is derived through the constraint of algebraic property encoded by the RAGFs, viz.,

$$2\Delta^{(2)\alpha}_{2\alpha} = -i/\pi = \Omega^{PV}(0) + \Omega^{AS}(0). \tag{9.9}$$

This constraint also makes the amplitudes unique concerning the Dirac traces used. To four dimensions, this turns the amplitudes quantities subject to routing choices. In contrast, to two dimensions, satisfying RAGFs leads to Dirac-trace independent expressions that only depend on the physical momentum.

The feature of Dirac traces appearing in all the treated amplitudes and the analogous ones for $2n$ dimensions arises for the trace of $2n+2$ Dirac matrices and an odd number of the chiral matrices. An assortment of expressions is available when one writes the tensor representing that trace, differing by the number of monomials and their signs, plus what subset of its Lorentz indexes appear. Those expressions are equivalent under the condition that surface terms have a value corresponding to the low-energy limit of finite-functions combination (9.5) in $4D$ or (9.8) in $2D$.

Adopting the *zero value* follows a set of expressions to each amplitude that may keep at most two RAGFs in $4D$ or one in $2D$. These expressions can be obtained either applying the definition of γ_*, in some position along the trace or using the identity below in the adjacent position of matrix γ_{μ_i},

$$(2n) \quad : \quad \gamma_* \gamma_{\mu_i} = \frac{i^{n+1}}{(2n-1)!} \varepsilon_{\mu_i \nu_2 \cdots \nu_{2n}} \gamma^{\nu_2 \cdots \nu_{2n}} \tag{9.10}$$

$$(2D) \quad : \quad \gamma_* \gamma_{\mu_i} = -\varepsilon_{\mu_i \nu_1} \gamma^{\nu_1} \quad \text{and} \quad (4D) : \gamma_* \gamma_{\mu_i} = \varepsilon_{\mu_i \nu_{123}} \gamma^{\nu_{123}}/6. \tag{9.11}$$

Thus, the tensors calculated for the amplitudes will correspond to the versions defined as the main ingredients of the investigation. They violate the RAGF for the vertex corresponding to γ_{μ_i}, and the WI gets violated in the same vertex. Two aspects must be noticed: (i) To have all the indices present, or to use the definition of the chiral matrix, is not exceptional since identities (above ones) yield fewer terms and deliver the same integrated expressions. (ii) The specialty of these identities is that they furnish the maximum number of RAGF automatically satisfied; hence the last RAGF can not be met because we would be violating a low-energy implication (9.6) in $4D$ and (9.9) in $2D$.

To sum up, adopting null surface terms makes the amplitudes depend on the traces used. The Schouten identity inside the integral that connects the integrands ceases to make it in the final integrated results. Ultimately, this breaks the linearity of integration and violates the RAGFs. Different formulae for the traces do not deliver identical tensors. The main elements involved in the versions were that they correspond to the same integrand; for instance, in $2D$ $(t^{AV}_{\mu_{12}})_1 = (t^{AV}_{\mu_{12}})_2$. However, after being integrated separately, we find their subtraction as

$$(T^{(2D)AV}_{\mu_{12}})_1 - (T^{(2D)AV}_{\mu_{12}})_2 = 2\varepsilon_{\mu_2\mu_1}(2\Delta^{(2)\rho}_{2\rho} + i/\pi). \tag{9.12}$$

Following the same argument, we build up the combination

$$(t^{(2D)AV}_{\mu_{12}})_1 = (t^{(2D)AV}_{\mu_{12}})_2 = \frac{1}{r_1 + r_2}[r_1(t^{(2D)AV}_{\mu_{12}})_1 + r_2(t^{(2D)AV}_{\mu_{12}})_2], \tag{9.13}$$

with $r_1 + r_2 \neq 0$ and otherwise arbitrary numbers; thus, after integration and adoption of $\Delta^{(2)}_{2\mu\nu} = 0$ we may write any other expression, in particular, the version $(T^{(2D)AV}_{\mu_{12}})_3$ discussed in Chapter (4) which is the linear combination above with $r_1 = r_2 = 1$. In that chapter, it was used one of the identities satisfied by the antisymmetric products of Dirac matrices, viz., $\gamma_*\gamma_{[\mu_1\mu_2]} = -\varepsilon_{\mu_1\mu_2}$. In general, not only 2D, all expressions obtainable utilizing those identities are a linear combination of the basic versions. Once more because they satisfy the most RAGFs as possible. With this algorithm in mind, we can build, if desired, the content one needs, by example,

$$(T^{AAA}_{\mu_{123}})_{\{1,1,1\}} = \frac{1}{3}[(T^{AAA}_{\mu_{123}})_1 + (T^{AAA}_{\mu_{123}})_2 + (T^{AAA}_{\mu_{123}})_3] \tag{9.14}$$

has one-third of the anomaly in $(T^{AVV}_{\mu_{123}})_1$, for each vertex.

About uniqueness, some definition is necessary. A criterion that makes the amplitudes unique in a universal sense is impossible since they are divergent quantities. After renormalization, they become dependent on an arbitrary mass scale. We employed the definition: One expression coming from the Feynman rules is unique if, for all intrinsic arbitrariness in intermediary algebraic manipulations, as Dirac traces and arbitrary routings, the final result is the same. This concept definition is well defined in the odd and non-derivative amplitudes studied in $2D$ because we got an expression depending on the external momentum and independent from Dirac traces. To the amplitudes investigated in $4D$, the 'unique' answer is a function of the routings taken as independent variables. Meaning one does not have a unique amplitude of the external momenta.

As for rules, it makes the surface terms zero as done in even amplitudes and by an intelligent choice of Dirac trace to obtain the symmetry content. Notwithstanding, if RAGFs are respected, turning amplitudes unique functions of their routings, this enables one to recover the symmetry content by choice of the remaining ambiguities for the momenta labels k_i, except 2D; this can be done in all even dimensions to the tensors like $T^{(2n)A^{2r+1}V^{n-2r}}_{\mu_1\cdots\mu_{n+1}}; r \leq [n/2]$.

Gravitation: The situation changes drastically when the power counting is higher than linear. For quadratic divergent gravitational amplitude, by preserving the RAGFs, we have the finiteness of the relevant surface terms as the constraint; see (8.132,8.133 and 8.134). Thus, it follows a unique form independent of manipulations in the Dirac algebra but ambiguous in what refers to the routing of the diagram. The results, in this scenario, for the Weyl anomaly is

$$\begin{aligned} \mathcal{W}_{\alpha_1\alpha_2} \quad : &= g^{\mu_{12}} T^{G}_{\mu_{12}\alpha_{12}} = -\frac{1}{96\pi} q^{\nu}\left(\varepsilon_{\alpha_1\nu} q_{\alpha_2} + \varepsilon_{\alpha_2\nu} q_{\alpha_1}\right) - \frac{1}{24\pi}\theta_{\alpha_1\alpha_2} - \frac{1}{48\pi} q_{\alpha_2} q_{\alpha_1} \\ & -\frac{1}{192\pi}\left(3\chi^2 - 5\right)\left[q^{\nu}\left(\varepsilon_{\alpha_1\nu} q_{\alpha_2} + \varepsilon_{\alpha_2\nu} q_{\alpha_1}\right) + \left(\theta_{\alpha_1\alpha_2} - q_{\alpha_2} q_{\alpha_1}\right)\right]. \end{aligned} \tag{9.15}$$

Furthermore, for the Einstein anomaly, we have the expression above

$$\begin{aligned} \mathcal{E}_{\mu_2\alpha_1\alpha_2} \quad : &= q^{\mu_1} T^{G}_{\mu_1\mu_2\alpha_1\alpha_2} = -\frac{1}{96\pi}\varepsilon_{\alpha_1\nu} q^{\nu}\theta_{\alpha_2\mu_2} + \\ & +\frac{1}{192\pi}\left(q_{\alpha_1}\theta_{\alpha_2\mu_2} + q_{\alpha_2}\theta_{\alpha_1\mu_2} - 2q_{\mu_2}\theta_{\alpha_1\alpha_2} - 2q_{\alpha_1}q_{\alpha_2}q_{\mu_2}\right) \\ & +\frac{(3\chi^2-5)}{384\pi}\{2q_{\alpha_1}q_{\alpha_2}q_{\mu_2} + \varepsilon_{\alpha_1}{}^{\nu}\left(q_{\mu_2}\theta_{\nu\alpha_2} + q_{\alpha_2}\theta_{\nu\mu_2} - q_{\nu}\theta_{\alpha_2\mu_2} - q_{\alpha_2}q_{\mu_2}q_{\nu}\right)\}. \end{aligned} \tag{9.16}$$

The first terms of each expression correspond to the ones in Bertlmann and Kohlprath [61, 62]. The result shows that apart from the question of the origin of the additional terms as trivial anomalies and which actions generate them. They are the product of preserving algebraic operations determined without resorting to a specific evaluation of divergent integrals, even though the representation of surface terms appears in this fashion.

Distinctly from the chiral anomalies, and in a certain sense similar to the vacuum polarization tensor of 4D quantum electrodynamics, the symmetry content (or violation thereof) can not be recovered by choice of the arbitrary internal momenta $k_1 + k_2 = \chi q$, at least for the even part (we restrict ourselves to covariant choices). The odd part allows this for the parameter $\chi^2 = 5/3$, namely

$$\mathcal{W}_{\alpha_{12}}\big|_{\chi^2=5/3} = -\frac{1}{96\pi}\left[q^{\nu}\left(\varepsilon_{\alpha_1\nu}q_{\alpha_2} + \varepsilon_{\alpha_2\nu}q_{\alpha_1}\right) + 4\theta_{\alpha_{12}} + 2q_{\alpha_{12}}\right] \tag{9.17}$$

$$\mathcal{E}_{\mu_2\alpha_{12}}\big|_{\chi^2=5/3} = -\frac{1}{192\pi}\left[2\varepsilon_{\alpha_1\nu}q^{\nu}\theta_{\mu_2\alpha_2} - q_{(\alpha_1}\theta_{\alpha_2)\mu_2} + 2q_{\mu_2}\theta_{\alpha_{12}} + 2q_{\mu_2}q_{\alpha_{12}}\right]. \tag{9.18}$$

There is no choice of χ which eliminates the vector part of the Einstein anomaly for finite surface terms, nor the vector part of the Weyl one. The only possibility to eliminate the even part of Einstein's anomaly is to spoil the linearity of integration and turn off the surface terms. This attitude brings a complex set of possibilities in the axial sector to be discussed in the sequel. The axial part of the Weyl anomaly can be eliminated by adopting $\chi^2 = 1$. However, we did not explore the aspect, which is interesting since adding the Bardeen-Zumino polynomial in the stress tensor to change the consistent anomaly in the covariant one, the odd part disappears; see the book of Bertlmann [11], pg. 541 or the paper cited previously.

Turning to the scenario where surface terms vanish and thus freeing the even part of the Einstein anomaly, the odd part, constituted of multiple terms, allows the exploration of the traces in each component. It is a choice available once the algebraic properties of the amplitudes are broken. In this thesis, we restricted to simplifications where the expressions to each of the four permutations $(\mu_1 \leftrightarrow \mu_2)\leftrightarrow(\alpha_1 \leftrightarrow \alpha_2)$ in the expansion that follows have the same version for each term.

$$\mathcal{T}^{AV}_{\mu_1\alpha_1\mu_2\alpha_2} = 4T^{AV}_{\mu_1\alpha_1;\mu_2\alpha_2} + 2q_{\mu_2}T^{AV}_{\mu_1\alpha_1;\alpha_2} + 2q_{\alpha_2}T^{AV}_{\mu_1\alpha_1;\mu_2} + q_{\alpha_2}q_{\mu_2}T^{AV}_{\mu_1\alpha_1}. \tag{9.19}$$

We allowed other trace choices only for the partner $\mathcal{T}^{VA}$, uniformly in its terms. We do not impose a priori symmetries in the indices, exploiting just the freedom of the versions. Those symmetries are preserved once the RAGFs are so, e.g., $T^G_{\mu_1\mu_2\alpha_1\alpha_2} = T^G_{\alpha_1\alpha_2\mu_1\mu_2}$. In making the selections stated, we arrive at a phenomenon already observed in the chiral counterparts: the anomalies can migrate from contraction to contraction. The compact formula for the Einstein anomalies becomes

$$\begin{aligned}
\mathcal{E}^{ij}_{\mu_1\mu_2\alpha_r} &= -\frac{1}{96\pi}\left(\delta_{i,2}+\delta_{j,2}\right)\varepsilon_{\mu_1\nu}q^{\nu}\theta_{\mu_2\alpha_r} && (9.20)\\
\mathcal{E}^{ij}_{\mu_r\alpha_1\alpha_2} &= -\frac{1}{96\pi}\left(2-\delta_{i,2}-\delta_{j,2}\right)\varepsilon_{\alpha_1\nu}q^{\nu}\theta_{\alpha_2\mu_r}. && (9.21)
\end{aligned}$$

They come from the contraction with $q^{\alpha_1;\alpha_2}$ and $q^{\mu_1;\mu_2}$, being that upper-indices in $\mathcal{E}^{ij}$ assumes 1 or 2 values. The Weyl ones are

$$\begin{aligned}
\mathcal{W}^{ij}_{\mu_1\mu_2} &= -\frac{1}{24\pi}\theta_{\mu_1\mu_2} - \frac{1}{96\pi}(2-\delta_{i,2}-\delta_{j,2})q^{\nu}(\varepsilon_{\mu_1\nu}q_{\mu_2}+\varepsilon_{\mu_2\nu}q_{\mu_1}) && (9.22)\\
\mathcal{W}^{ij}_{\alpha_1\alpha_2} &= -\frac{1}{24\pi}\theta_{\alpha_1\alpha_2} - \frac{1}{96\pi}\left(\delta_{i,2}+\delta_{j,2}\right)q^{\nu}(\varepsilon_{\alpha_1\nu}q_{\alpha_2}+\varepsilon_{\alpha_2\nu}q_{\alpha_1}). && (9.23)
\end{aligned}$$

Notice that when the Einstein anomaly (odd part) drops out in one group of indices, the Weyl anomaly does so in the complementary set, occurring when $i = j$. In the combinations $ij = 12$ or $ij = 21$, none are zero and equal to half of the results for the non-vanishing parts of $ij = 11$ or $ij = 22$. The mixed versions have coefficients equal to the ones in Bertlmann [61], which is one particular result of our analysis.

Ultimately, the expression (9.19) above admits independent choices for each term. As a consequence, the factor $\Upsilon, \Upsilon_\alpha, \Upsilon_{\alpha\mu}$ (7.107-7.126,7.127) do not combine into the $U_{\alpha\mu}$-factor, and the other projector aside $\theta_{\mu\alpha}$ ($\omega_{\mu\alpha} = q_\mu q_\alpha$) would arise with a proliferation of coefficients. This scenario is allowed for once the surface terms are interpreted as quantities that vanish. This element leads to expressions that exhibit Lorentz anomaly. We deviated from this anomaly once the same version was used when summing the basic permutations. Another interesting point is to study a low-energy theorem in the gravitational setting, as done for the chiral anomalies. Research along these lines is underway.

As a final comment, the possibility of final and compact expressions that preserve all the features of the computation is mainly due to the use of a definition of the surface terms of rank four $\square_{3\mu\nu\alpha\rho}$, and six $\Sigma_{4\mu\nu\alpha\rho\sigma\lambda}$, which are explicitly total symmetric in the Lorentz indices. In addition, their compilation into terms that may break the algebraic RAGFs, the objects $\{\Upsilon, \Upsilon_{\alpha}, \Upsilon_{\alpha\mu}\}$. In particular, we call attention to the scalar one, $\Upsilon = 2\Delta^{\rho}_{2\rho} + i/\pi$, which in the last instance, determines the satisfaction or not of all RAGFs for the energy-momentum two-point function. It is precisely the same one that appears in the 2D chiral anomaly. The extension of these protocols to four dimensions facilitates the investigations underway associated with trace anomalies closely related to the recent publications in Bonora [20] and [77]. The RAGFs will become exceedingly complicated; as an example, we have

$$\begin{aligned}
& 2p_{31}^{\alpha}T^{AVV}_{\mu_{123};\alpha} + p_{31}^{2}T^{AVV}_{\mu_{123}} \\
= \; & -2m[T^{\tilde{T}V}_{\mu_{31}\mu_{2}}(1,2) + T^{\tilde{T}V}_{\mu_{12}\mu_{3}}(2,3)] \\
& +i[\varepsilon_{\mu_{12}}{}^{\nu_{12}}p_{31\nu_{2}}T^{VV}_{\nu_{1}\mu_{3}}(2,3)] - i[\varepsilon_{\mu_{13}}{}^{\nu_{12}}p_{31\nu_{2}}T^{VV}_{\nu_{1}\mu_{2}}(1,2)] \\
& +2[T^{AV}_{\mu_{32};\mu_{1}}(1,2) - T^{AV}_{\mu_{23};\mu_{1}}(2,3)] \\
& -p_{31}^{\nu_{1}}[g_{\mu_{1}\mu_{2}}T^{AV}_{\nu_{1}\mu_{3}}(2,3) + g_{\mu_{1}\mu_{3}}T^{AV}_{\nu_{1}\mu_{2}}(1,2)] \\
& +[p_{31\mu_{1}}T^{AV}_{\mu_{32}}(1,2) + p_{31\mu_{3}}T^{AV}_{\mu_{12}}(1,2) - p_{31\mu_{1}}T^{AV}_{\mu_{23}}(2,3) + p_{31\mu_{2}}T^{AV}_{\mu_{13}}(2,3)],
\end{aligned}$$

where even arises a pseudo-tensor vertex $\tilde{T} = \gamma_{*}\gamma_{[\mu\nu]}$. Nonetheless, by the systematization developed in this thesis such task becomes feasible as well.

Appendix A

Dirac Matrices and Traces

Lets us introduce the Clifford algebra representation in terms of matrices $\left\{\gamma_{\mu_1}, \gamma_{\mu_2}\right\} = 2g_{\mu_{12}}\mathbf{1}$, the dimension of irreducible representations are $\dim(\gamma) = 2^{[d/2]} \times 2^{[d/2]}$, and the basic traces are

$$\begin{aligned} \mathrm{tr}\left(\gamma^{\mu}\right) &= 0 && \text{(A.1)} \\ \mathrm{tr}\left\{\gamma_{\alpha}, \gamma_{\beta}\right\} &= 2g_{\alpha\beta}\mathrm{tr}\left(\mathbf{1}_{2^n\times 2^n}\right). && \text{(A.2)} \end{aligned}$$

For the two dimensional representation, we have:

$$\begin{aligned} \gamma_0 &= \sigma_2; \gamma_1 = i\sigma_1; \gamma_3 = \sigma_3 && \text{(A.3)} \\ \gamma_0 &= \sigma_1; \gamma_1 = i\sigma_2; \gamma_3 = -\sigma_3. \end{aligned}$$

For even dimensions, $d = 2n$, there is a matrix given by

$$\gamma_* := i^{n-1}\gamma_0\gamma_1\cdots\gamma_{2n-1} = \frac{i^{n-1}}{(2n)!}\varepsilon_{\nu_1\cdots\nu_{2n}}\gamma^{\nu_1\cdots\nu_{2n}} \tag{A.4}$$

that obeys $\left\{\gamma_*, \gamma_\mu\right\} = 0$, with $\varepsilon_{012\cdots d-1} = -1$. For four matrices, we have the trace

$$\mathrm{tr}\left(\gamma^{\mu_1\cdots\mu_4}\right) = \mathrm{tr}\left(\mathbf{1}_{2^n\times 2^n}\right)\left(g^{\mu_1\mu_2}g^{\mu_3\mu_4} - g^{\mu_1\mu_3}g^{\mu_2\mu_4} + g^{\mu_1\mu_4}g^{\mu_2\mu_3}\right), \tag{A.5}$$

the general formula is

$$\mathrm{tr}\left(\gamma_{\mu_1\cdots\mu_{2n}}\right) = \sum_{i=2}^{2n}\left(-1\right)^i g_{\mu_1\mu_i}\mathrm{tr}\left(\gamma_{\mu_1\cdots\hat{\mu}_i\cdots\mu_{2n}}\right). \tag{A.6}$$

The first non-zero trace with the chiral matrix in any even dimension is given by

$$\mathrm{tr}\left(\gamma_*\gamma_{\mu_1}\gamma_{\mu_2}\cdots\gamma_{\mu_{2n}}\right) = 2^n i^{n-1}\left(-1\right)^n\varepsilon_{\mu_{12\ldots}\mu_{2n}}, \tag{A.7}$$

for $d = 2n$ to the string of $2n+2$ gamma matrices plus γ_* using its definition follows the formula

$$\mathrm{tr}(\gamma_*\gamma_{a_1a_2\cdots a_{2n+1}a_{2n+2}}) = 2^n i^{3n-1}\sum_{k=1}^{2n+1}\sum_{j=k+1}^{2n+2}\left(-1\right)^{j+k+1}g_{a_ka_j}\varepsilon_{a_1\cdots\hat{a}_k\cdots\hat{a}_j\cdots(2n+2)}, \tag{A.8}$$

where we have used the abbreviation $\gamma_{a_1a_2\cdots a_{2n+1}a_{2n+2}} = \prod_{j=1}^{2b+2}\gamma_{a_j}$. The Latin index ought to be substituted to whatever configuration of Lorentz indices is scrutinized.

A.1 Traces of a String of Six Gamma and the Chiral Matrix

One uses the following identities to insert the Levi-Civita tensor in traces with the chiral matrix

$$\gamma_* \gamma_{[\mu_1 \cdots \mu_r]} = \frac{i^{n-1+r(r+1)}}{(2n-r)!} \varepsilon_{\mu_1 \cdots \mu_r \nu_{r+1} \cdots \nu_{2n}} \gamma^{[\nu_{r+1} \cdots \nu_{2n}]},$$

where the notation $\gamma_{[\mu_1 \cdots \mu_r]}$ indicates antisymmetrized products of gammas and the investigated dimension is $2n = 4$. This appendix uses this resource to achieve different trace expressions and explore their relations.

Trace using the definition $\gamma_* = i\varepsilon_{\nu_1\nu_2\nu_3\nu_4}\gamma^{\nu_1\nu_2\nu_3\nu_4}/4!$ **-** The three leading positions to substitute the definition are around vertices Γ_1, Γ_2, and Γ_3. Even if that brings six options, the same integrated expressions arise regardless of replacing at the left or right. Thus, we cast the possibilities in the sequence

$$\begin{aligned} t_1 &= \mathrm{tr}(\gamma_* \gamma_{\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3}) = i\varepsilon^{\alpha_1\alpha_2\alpha_3\alpha_4}\mathrm{tr}(\gamma_{\alpha_1\alpha_2\alpha_3\alpha_4}\gamma_{\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3})/4! \\ &= +g_{\mu_1\nu_1}\varepsilon_{\mu_2\nu_2\mu_3\nu_3} - g_{\mu_1\mu_2}\varepsilon_{\nu_1\nu_2\mu_3\nu_3} + g_{\mu_1\nu_2}\varepsilon_{\nu_1\mu_2\mu_3\nu_3} - g_{\mu_1\mu_3}\varepsilon_{\nu_1\mu_2\nu_2\nu_3} + g_{\mu_1\nu_3}\varepsilon_{\nu_1\mu_2\nu_2\mu_3} \\ &\quad +g_{\nu_1\mu_2}\varepsilon_{\mu_1\nu_2\mu_3\nu_3} - g_{\nu_1\nu_2}\varepsilon_{\mu_1\mu_2\mu_3\nu_3} + g_{\nu_1\mu_3}\varepsilon_{\mu_1\mu_2\nu_2\nu_3} - g_{\nu_1\nu_3}\varepsilon_{\mu_1\mu_2\nu_2\mu_3} + g_{\mu_2\nu_2}\varepsilon_{\mu_1\nu_1\mu_3\nu_3} \\ &\quad -g_{\mu_2\mu_3}\varepsilon_{\mu_1\nu_1\nu_2\nu_3} + g_{\mu_2\nu_3}\varepsilon_{\mu_1\nu_1\nu_2\mu_3} + g_{\nu_2\mu_3}\varepsilon_{\mu_1\nu_1\mu_2\nu_3} - g_{\nu_2\nu_3}\varepsilon_{\mu_1\nu_1\mu_2\mu_3} + g_{\mu_3\nu_3}\varepsilon_{\mu_1\nu_1\mu_2\nu_2}, \end{aligned}$$

$$\begin{aligned} t_2 &= \mathrm{tr}(\gamma_{\mu_1\nu_1}\gamma_* \gamma_{\mu_2\nu_2\mu_3\nu_3}) = i\varepsilon^{\alpha_1\alpha_2\alpha_3\alpha_4}\mathrm{tr}(\gamma_{\mu_1\nu_1}\gamma_{\alpha_1\alpha_2\alpha_3\alpha_4}\gamma_{\mu_2\nu_2\mu_3\nu_3})/4! \\ &= +g_{\mu_1\nu_1}\varepsilon_{\mu_2\nu_2\mu_3\nu_3} + g_{\mu_1\mu_2}\varepsilon_{\nu_1\nu_2\mu_3\nu_3} - g_{\mu_1\nu_2}\varepsilon_{\mu_2\nu_2\mu_3\nu_3} + g_{\mu_1\mu_3}\varepsilon_{\nu_1\mu_2\nu_2\nu_3} - g_{\mu_1\nu_3}\varepsilon_{\nu_1\mu_2\nu_2\mu_3} \\ &\quad -g_{\nu_1\mu_2}\varepsilon_{\mu_1\nu_2\mu_3\nu_3} + g_{\nu_1\nu_2}\varepsilon_{\mu_1\mu_2\mu_3\nu_3} - g_{\nu_1\mu_3}\varepsilon_{\mu_1\mu_2\nu_2\nu_3} + g_{\nu_1\nu_3}\varepsilon_{\mu_1\mu_2\nu_2\mu_3} + g_{\mu_2\nu_2}\varepsilon_{\mu_1\nu_1\mu_3\nu_3} \\ &\quad -g_{\mu_2\mu_3}\varepsilon_{\mu_1\nu_1\nu_2\nu_3} + g_{\mu_2\nu_3}\varepsilon_{\mu_1\nu_1\nu_2\mu_3} + g_{\nu_2\mu_3}\varepsilon_{\mu_1\nu_1\mu_2\nu_3} - g_{\nu_2\nu_3}\varepsilon_{\mu_1\nu_1\mu_2\mu_3} + g_{\mu_3\nu_3}\varepsilon_{\mu_1\nu_1\mu_2\nu_2}, \end{aligned}$$

$$\begin{aligned} t_3 &= \mathrm{tr}(\gamma_{\mu_1\nu_1\mu_2\nu_2}\gamma_* \gamma_{\mu_3\nu_3}) = i\varepsilon^{\alpha_1\alpha_2\alpha_3\alpha_4}\mathrm{tr}(\gamma_{\mu_1\nu_1\mu_2\nu_2}\gamma_{\alpha_1\alpha_2\alpha_3\alpha_4}\gamma_{\mu_3\nu_3})/4! \\ &= +g_{\mu_1\nu_1}\varepsilon_{\mu_2\nu_2\mu_3\nu_3} - g_{\mu_1\mu_2}\varepsilon_{\nu_1\nu_2\mu_3\nu_3} + g_{\mu_1\nu_2}\varepsilon_{\mu_2\nu_2\mu_3\nu_3} + g_{\mu_1\mu_3}\varepsilon_{\nu_1\mu_2\nu_2\nu_3} - g_{\mu_1\nu_3}\varepsilon_{\nu_1\mu_2\nu_2\mu_3} \\ &\quad +g_{\nu_1\mu_2}\varepsilon_{\mu_1\nu_2\mu_3\nu_3} - g_{\nu_1\nu_2}\varepsilon_{\mu_1\mu_2\mu_3\nu_3} - g_{\nu_1\mu_3}\varepsilon_{\mu_1\mu_2\nu_2\nu_3} + g_{\nu_1\nu_3}\varepsilon_{\mu_1\mu_2\nu_2\mu_3} + g_{\mu_2\nu_2}\varepsilon_{\mu_1\nu_1\mu_3\nu_3} \\ &\quad +g_{\mu_2\mu_3}\varepsilon_{\mu_1\nu_1\nu_2\nu_3} - g_{\mu_2\nu_3}\varepsilon_{\mu_1\nu_1\nu_2\mu_3} - g_{\nu_2\mu_3}\varepsilon_{\mu_1\nu_1\mu_2\nu_3} + g_{\nu_2\nu_3}\varepsilon_{\mu_1\nu_1\mu_2\mu_3} + g_{\mu_3\nu_3}\varepsilon_{\mu_1\nu_1\mu_2\nu_2}, \end{aligned}$$

where we omit the global factor $4i$. Since each expression contains fifteen monomials featuring all index configurations, different signs are the unique distinguishing factor among them. That is also the reason why references often name them symmetric or democratic [28, 73, 50].

These (main) versions play fundamental roles in this investigation as they are enough to obtain any other result. If we use any other identity constructed with the equations involving the antisymmetric products the trace expressions relate directly to them or their combinations $t_{ij} = (t_i + t_j)/2$ only using sums and no other operation. Consequently, any expression attributed to the investigated triangles is a linear combination of those detailed in the main body of this work. All of them produce the mentioned relations, so we cast some at the end of this appendix.

$$\begin{aligned} t_{12} &= -g_{\mu_1\nu_1}\varepsilon_{\mu_2\mu_3\nu_2\nu_3} - g_{\mu_2\nu_2}\varepsilon_{\mu_1\mu_3\nu_1\nu_3} + g_{\mu_2\nu_3}\varepsilon_{\mu_1\mu_3\nu_1\nu_2} \\ &\quad -g_{\nu_2\mu_3}\varepsilon_{\mu_1\mu_2\nu_1\nu_3} - g_{\mu_3\nu_3}\varepsilon_{\mu_1\mu_2\nu_1\nu_2} - g_{\mu_2\mu_3}\varepsilon_{\mu_1\nu_1\nu_2\nu_3} - g_{\nu_2\nu_3}\varepsilon_{\mu_1\mu_2\mu_3\nu_1}, \end{aligned}$$

$$\begin{aligned} t_{13} &= -g_{\mu_3\nu_3}\varepsilon_{\mu_1\mu_2\nu_1\nu_2} - g_{\mu_1\nu_1}\varepsilon_{\mu_2\mu_3\nu_2\nu_3} + g_{\mu_1\nu_2}\varepsilon_{\mu_2\mu_3\nu_1\nu_3} \\ &\quad -g_{\nu_1\mu_2}\varepsilon_{\mu_1\mu_3\nu_2\nu_3} - g_{\mu_2\nu_2}\varepsilon_{\mu_1\mu_3\nu_1\nu_3} - g_{\mu_1\mu_2}\varepsilon_{\mu_3\nu_1\nu_2\nu_3} - g_{\nu_1\nu_2}\varepsilon_{\mu_1\mu_2\mu_3\nu_3}, \end{aligned}$$

$$\begin{aligned} t_{23} &= -g_{\mu_2\nu_2}\varepsilon_{\mu_1\mu_3\nu_1\nu_3} - g_{\mu_1\nu_1}\varepsilon_{\mu_2\mu_3\nu_2\nu_3} + g_{\mu_1\nu_3}\varepsilon_{\mu_2\mu_3\nu_1\nu_2} \\ &\quad -g_{\nu_1\mu_3}\varepsilon_{\mu_1\mu_2\nu_2\nu_3} - g_{\mu_3\nu_3}\varepsilon_{\mu_1\mu_2\nu_1\nu_2} - g_{\mu_1\mu_3}\varepsilon_{\mu_2\nu_1\nu_2\nu_3} - g_{\nu_1\nu_3}\varepsilon_{\mu_1\mu_2\mu_3\nu_2}, \end{aligned}$$

Trace using $\gamma_*\gamma_a = -i\varepsilon_{a\nu_1\nu_2\nu_3}\gamma^{\nu_1\nu_2\nu_3}/3!$ - After using this identity for the chiral matrix and the first gamma, we write this trace through ten monomials.

$$\eta_1(a) = \mathrm{tr}\left(\gamma_*\gamma_{abcdef}\right) = -i\varepsilon_a{}^{\nu_1\nu_2\nu_3}\mathrm{tr}\left(\gamma_{\nu_1\nu_2\nu_3}\gamma_{bcdef}\right)/6$$

$$\begin{aligned} \eta_1(a) &= g_{bc}\varepsilon_{adef} - g_{bd}\varepsilon_{acef} + g_{be}\varepsilon_{acdf} - g_{bf}\varepsilon_{acde} + g_{cd}\varepsilon_{abef} \\ &\quad -g_{ce}\varepsilon_{abdf} + g_{cf}\varepsilon_{abde} + g_{de}\varepsilon_{abcf} + g_{ef}\varepsilon_{abcd} - g_{df}\varepsilon_{abce} \end{aligned}$$

Trace using $\gamma_*\gamma_{[ab]} = -i\varepsilon_{ab\nu_1\nu_2}\gamma^{\nu_1\nu_2}/2!$ - This case requires expressing the ordinary product in terms of the antisymmetrized one. We find seven monomials after taking the traces.

$$\gamma_*\gamma_{ab} = -\frac{1}{2}i\varepsilon_{ab\nu_1\nu_2}\gamma^{\nu_1\nu_1} + g_{ab}\gamma_*$$

$$\begin{aligned} \eta_2(ab) = \mathrm{tr}\left(\gamma_*\gamma_{abcdef}\right) &= g_{ab}\varepsilon_{cdef} + g_{cd}\varepsilon_{abef} - g_{ce}\varepsilon_{abdf} + g_{cf}\varepsilon_{abde} \\ &\quad +g_{de}\varepsilon_{abcf} - g_{df}\varepsilon_{abce} + g_{ef}\varepsilon_{abcd} \end{aligned}$$

Trace using $\gamma_*\gamma_{[abc]} = i\varepsilon_{abc\nu}\gamma^\nu$ - Following a similar procedure we find six monomials.

$$\gamma_*\gamma_{abc} = i\varepsilon_{abc\nu}\gamma^\nu + \gamma_*\left(g_{bc}\gamma_a - g_{ac}\gamma_b + g_{ab}\gamma_c\right)$$

$$\eta_3(abc) = \mathrm{tr}\left(\gamma_*\gamma_{abcdef}\right) = g_{ab}\varepsilon_{cdef} - g_{ac}\varepsilon_{bdef} + g_{bc}\varepsilon_{adef} + g_{de}\varepsilon_{abcf} - g_{df}\varepsilon_{abce} + g_{ef}\varepsilon_{abcd}$$

Trace using $\gamma_*\gamma_{[abcd]} = i\varepsilon_{abcd}$ - This case also generates seven monomials.

$$\begin{aligned} \gamma_*\gamma_{abcd} &= i\varepsilon_{abcd}\mathbf{1} + g_{ab}\gamma_*\gamma_{[cd]} - g_{ac}\gamma_*\gamma_{[bd]} + g_{ad}\gamma_*\gamma_{[bc]} \\ &\quad +g_{bc}\gamma_*\gamma_{[ad]} - g_{bd}\gamma_*\gamma_{[ac]} + g_{cd}\gamma_*\gamma_{[ab]} + \left(g_{ab}g_{cd} - g_{ac}g_{bd} + g_{ad}g_{bc}\right)\gamma_* \end{aligned}$$

$$\begin{aligned} \eta_4(abcd) = \mathrm{tr}\left(\gamma_*\gamma_{abcdef}\right) &= g_{ab}\varepsilon_{cdef} - g_{ac}\varepsilon_{bdef} + g_{ad}\varepsilon_{bcef} + g_{bc}\varepsilon_{adef} \\ &\quad -g_{bd}\varepsilon_{acef} + g_{cd}\varepsilon_{abef} + g_{ef}\varepsilon_{abcd} \end{aligned}$$

Interconnection among formulas: When computing the difference between two integrated versions of the same amplitude, we acknowledge two situations. First, it cancels out identically as their integrands are precisely equal, for example:

$$[t_{12} - \eta_2 (\mu_1 \nu_1)] = 0, \qquad [t_{23} - \eta_4 (\mu_3 \nu_3 \mu_1 \nu_1)] = 0.$$

Second, it vanishes in the integration because the explicit computation corresponds to finite null integrals embodied into the $t^{(-+)}$ tensor (6.14) and the ASS amplitude (6.22). Some examples are:

$$\begin{aligned}
[t_1 - \eta_1 (\mu_1)] \frac{K_{123}^{\nu_{123}}}{D_{123}} &= \varepsilon_{\mu_2\mu_3\nu_1\nu_2} t_{\mu_1}^{(-+)\nu_{12}} - g_{\mu_1\mu_3} t_{\mu_2}^{ASS} + g_{\mu_1\mu_2} t_{\mu_3}^{ASS}, \\
[t_{12} + \eta_2 (\nu_1 \mu_2)] \frac{K_{123}^{\nu_{123}}}{D_{123}} &= -\varepsilon_{\mu_2\mu_3\nu_1\nu_2} t_{\mu_1}^{(-+)\nu_{12}} + \varepsilon_{\mu_1\mu_3\nu_1\nu_2} t_{\mu_2}^{(-+)\nu_{12}} - g_{\mu_2\mu_3} t_{\mu_1}^{ASS} + g_{\mu_1\mu_3} t_{\mu_2}^{ASS}, \\
[t_{13} + \eta_4 (\nu_1 \mu_2 \nu_2 \mu_3)] \frac{K_{123}^{\nu_{123}}}{D_{123}} &= -\varepsilon_{\mu_2\mu_3\nu_1\nu_2} t_{\mu_1}^{(-+)\nu_{12}} - \varepsilon_{\mu_1\mu_2\nu_1\nu_2} t_{\mu_3}^{(-+)\nu_{12}} + g_{\mu_2\mu_3} t_{\mu_1}^{ASS} - g_{\mu_1\mu_2} t_{\mu_3}^{ASS}.
\end{aligned}$$

$$\begin{aligned}
[t_{12} - \eta_3 (\mu_1 \nu_1 \mu_2)] \frac{K_{123}^{\nu_{123}}}{D_{123}} &= -g_{\mu_2\mu_3} t_{\mu_1}^{ASS} + \varepsilon_{\mu_{13}\nu_{12}} t_{\mu_2}^{(-+)\nu_{12}} + g_{\mu_1\mu_2} t_{\mu_3}^{ASS}, \\
[t_{23} - \eta_3 (\mu_2 \nu_2 \mu_3)] \frac{K_{123}^{\nu_{123}}}{D_{123}} &= -g_{\mu_3\mu_1} t_{\mu_2}^{ASS} - \varepsilon_{\mu_{12}\nu_{12}} t_{\mu_3}^{(-+)\nu_{12}} + g_{\mu_2\mu_3} t_{\mu_1}^{ASS} \\
[t_{31} - \eta_3 (\mu_3 \nu_3 \mu_1)] \frac{K_{123}^{\nu_{123}}}{D_{123}} &= -g_{\mu_1\mu_2} t_{\mu_3}^{ASS} - \varepsilon_{\mu_{23}\nu_{12}} t_{\mu_1}^{(-+)\nu_{12}} + g_{\mu_3\mu_1} t_{\mu_2}^{ASS}
\end{aligned}$$

We showed the forms that identically correspond here, not that all differences are finite and vanishing. For example, the form obtained from t_{12} is not identical without conditions to any t_i.

Appendix B

Feynman Integrals

B.1 Feynman's parametrization

Any integral that is explicitly evaluated in this work is well defined. To operate, we combine the denominators that appear using Feynman parametrization. The functions that occur after they have been split through the formula (3.4) share the form

$$\frac{1}{D_\lambda^N D_1 ... D_n}. \tag{B.1}$$

They can be combined as

$$\frac{1}{D_\lambda^N D_1 ... D_n} = (N)_n \int_0^1 \mathrm{d}x_1 \cdots \int_0^{1-x_1-...-x_{n-1}} \mathrm{d}x_n \frac{(1 - x_1 - \cdots x_N)^{N-1}}{\left[\sum_{i=1}^n (D_i - D_\lambda) x_i + D_\lambda\right]^{n+N}}, \tag{B.2}$$

where $(N)_n$ is the Pochhammer symbol $(N)_n = \Gamma(N+n)/\Gamma(N)$. It is a direct task by induction to show that

$$\begin{aligned} \sum_{i=1}^n (D_i - D_\lambda) x_i + D_\lambda &= k^2 - \lambda^2 + \sum_{i=1}^n \left(2k \cdot k_i + k_i^2\right) x_i + \sum_{i=1}^n \left(\lambda^2 - m_i^2\right) x_i \\ &= \left(k + \sum_{i=1}^n k_i x_i\right)^2 + Q\left(\left\{k_i, m_i^2\right\}; \lambda^2\right), \end{aligned} \tag{B.3}$$

where we define the Q polynomial

$$Q\left(\left\{k_i, m_i^2\right\}; \lambda^2\right) = \sum_{i=1}^n k_i^2 x_i (1 - x_i) - 2 \sum_{j>i} (k_i \cdot k_j) x_i x_j + \sum_{i=1}^n \left(\lambda^2 - m_i^2\right) x_i - \lambda^2. \tag{B.4}$$

After integrating into the momentum k, we have a function of Q whose integral over adequate parameter delivers the integrals used in work. As of the finite functions, they appear as

$$\frac{1}{D_1 ... D_n} = \Gamma(n) \int_0^1 \mathrm{d}x_1 \cdots \int_0^{1-x_1-...-x_{n-2}} \mathrm{d}x_{n-1} \frac{1}{\left[\sum_{i=1}^{n-1} (D_i - D_1) x_i + D_1\right]^n}. \tag{B.5}$$

An example to illustrate this is the finite integral

$$I_2 = \int \frac{\mathrm{d}^2 k}{(2\pi)^2} \frac{1}{D_{12}} \tag{B.6}$$

the explicit D_i are

$$D_1 = (k + k_1)^2 - m_1^2 \tag{B.7}$$

$$D_2 = (k + k_2)^2 - m_2^2 \tag{B.8}$$

thus we identify

$$(D_2 - D_1)\,x + D_1 = k^2 + 2k\cdot[(k_2 - k_1)\,x + k_1] + \left(k_2^2 - k_1^2\right)x + k_1^2 + \left(m_1^2 - m_2^2\right)x - m_1^2 \tag{B.9}$$

$$= [k + (k_2 - k_1)\,x + k_1]^2 + (k_2 - k_1)^2\,x\,(1 - x) + \left(m_1^2 - m_2^2\right)x - m_1^2 \tag{B.10}$$

and with $q = k_2 - k_1$ the Q polynomial

$$Q = q^2 x\,(1 - x) + \left(m_1^2 - m_2^2\right)x - m_1^2. \tag{B.11}$$

When integrating the translation in the k variable

$$k \to k - [(k_2 - k_1)\,x + k_1] \tag{B.12}$$

allows us to write the integral as

$$J_2 = \int_0^1 \mathrm{d}z \int \frac{\mathrm{d}^2 k}{(2\pi)^2} \frac{1}{(k^2 + Q)^2}. \tag{B.13}$$

The next step is integration in the momentum, where the next section derives the necessary formulae.

B.2 The $J^{(2)}_{2\mu\nu}$ Integral

For non-negative power counting integrals, we must split them using the identity (3.4). Let us illustrate the type of operations needed to integrate such integrals using as an example the fundamental tensor integral with arbitrary masses in two dimensions

$$\bar{J}_2^{(2)\mu\nu} = \int \frac{\mathrm{d}^2 k}{(2\pi)^2} \frac{K_1^\mu K_1^\nu}{D_{12}}. \tag{B.14}$$

Its integrand is decomposed in the form

$$\frac{K_1^\mu K_1^\nu}{D_{12}} = \frac{K_1^\mu K_1^\nu}{D_\lambda^2} - \frac{K_1^\mu K_1^\nu A_2}{D_\lambda^2 D_2} - \frac{K_1^\mu K_1^\nu A_1}{D_\lambda D_{12}}. \tag{B.15}$$

Then, the following integrals are required to perform

$$\bar{J}_2^{(2)\mu\nu} = \int \frac{\mathrm{d}^2 k}{(2\pi)^2} \frac{K_1^\mu K_1^\nu}{D_{12}} = \int \frac{\mathrm{d}^2 k}{(2\pi)^2} \left\{ \frac{K_1^\mu K_1^\nu}{D_\lambda^2} - \frac{K_1^\mu K_1^\nu A_2}{D_\lambda^2 D_2} - \frac{K_1^\mu K_1^\nu A_1}{D_\lambda D_{12}} \right\} \tag{B.16}$$

$$= \int \frac{\mathrm{d}^2 k}{(2\pi)^2} \frac{K_1^\mu K_1^\nu}{D_\lambda^2} - F_b^{\mu\nu} - F_a^{\mu\nu}. \tag{B.17}$$

The final answer will be expressed as functional in $Q = q^2 x(1-x) + (m_1^2 - m_2^2)x - m_1^2$.

To start with, we combine the denominators with Feynman parametrization for $F_a^{\mu\nu}$

$$\frac{K_1^\mu K_1^\nu A_1}{D_\lambda D_{12}} = 2\int_0^1 \mathrm{d}x_1 \int_0^{1-x_1} \mathrm{d}x_2 \frac{1}{\left[(D_2 - D_\lambda)x_1 + (D_1 - D_\lambda)x_2 + D_\lambda\right]^3}. \tag{B.18}$$

Integrating into the loop momentum and making the shift $k \to k - (k_2 x_1 + k_1 x_2)$, we reach to

$$F_a^{\mu\nu} = \int \frac{\mathrm{d}^2 k}{(2\pi)^2} \frac{K_1^\mu K_1^\nu A_1}{D_\lambda D_{12}} = 2\int_0^1 \mathrm{d}x_1 \int_0^{1-x_1} \mathrm{d}x_2 \int \frac{\mathrm{d}^2 k}{(2\pi)^2} \frac{\left[K_1^\mu K_1^\nu A_1\right]_{k-(k_2x_1+k_1x_2)}}{(k^2+Q)^3}, \tag{B.19}$$

where the Q polynomial is given by

$$\begin{aligned} Q(k_2, k_1, x_1, x_2) &= k_2^2 x_1(1-x_1) + k_1^2 x_2(1-x_2) - 2(k_2 \cdot k_1)x_1 x_2 \\ &\quad + \left(\lambda^2 - m_2^2\right)x_1 + \left(\lambda^2 - m_1^2\right)x_2 - \lambda^2. \end{aligned} \tag{B.20}$$

The integration limits satisfies

$$Q(x_1, 1-x_2) = q^2 x_1(1-x_1) + \left(m_1^2 - m_2^2\right)x_1 - m_1^2 \tag{B.21}$$

$$Q(x_1, 0) = k_2^2 x_1(1-x_1) + \left(\lambda^2 - m_2^2\right)x_1 - \lambda^2. \tag{B.22}$$

Recovering definition of $A_i = 2k \cdot k_i + k_i^2 + \lambda^2 - m_i^2$. After shifting, it assumes the form

$$(A_1)_{k-(k_2x_1+k_1x_2)} = (2k \cdot k_1) + \frac{\partial Q}{\partial x_2}. \tag{B.23}$$

This feature will always happen to some A_i, which means one factor becomes a sum of a bilinear and a derivative about the last integration parameter. The next stage is to make partial integrations until all derivatives are consumed.

For the vector K_1 that we used as reference (although any other could be chosen) in definitions of the integral, under shifting, it turns into $(K_1)_{k-(k_2x_1+k_1x_2)} = k-(k_2x_1 + k_1x_2 - k_1)$. Moreover, in order to simplify and organize, we define

$$L = (k_2 x_1 + k_1 x_2 - k_1) \tag{B.24}$$

$$L(x_1, 1-x_1) = (k_2 - k_1)x_1 = qx \tag{B.25}$$

$$L(x_1, 0) = L_0 = (k_2 x_1 - k_1) \tag{B.26}$$

Gathering all the elements, we are left with this expression to integrate

$$F_a^{\mu\nu} = 2\int_0^1 \mathrm{d}x_1 \int_0^{1-x_1} \mathrm{d}x_2 \int \frac{\mathrm{d}^2 k}{(2\pi)^2} \left\{ \left[2k \cdot k_1 + \frac{\partial Q}{\partial x_2}\right] \frac{(k-L)^\mu (k-L)^\nu}{(k^2+Q)^3} \right\}. \tag{B.27}$$

At this point, we use the results that are elaborated in the sequel, namely

$$\int \frac{\mathrm{d}^2 k}{(2\pi)^2} \frac{1}{(k^2+Q)^3} = \frac{i}{4\pi} \frac{1}{2Q^2} \tag{B.28}$$

$$\int \frac{\mathrm{d}^2 k}{(2\pi)^2} \frac{k^\mu k^\nu}{(k^2+Q)^3} = \frac{i}{4\pi} \frac{1}{2} g^{\mu\nu} \frac{1}{2Q}, \tag{B.29}$$

odd integrals drop from the expression, and we get

$$F_a^{\mu\nu} = \frac{i}{4\pi}\int_0^1 \mathrm{d}x_1 \int_0^{1-x_1} \mathrm{d}x_2 \left[-\left(k_1^\mu L^\nu + k_1^\nu L^\mu\right)\frac{1}{Q} + \frac{1}{2}g^{\mu\nu}\frac{\partial Q}{\partial x_2}\frac{1}{Q} + L^\mu L^\nu \frac{\partial Q}{\partial x_2}\frac{1}{Q^2}\right]. \tag{B.30}$$

Integrating by parts, we find a total derivative

$$F_a^{\mu\nu} = \frac{i}{4\pi}\int_0^1 \mathrm{d}x_1 \int_0^{1-x_1} \mathrm{d}x_2 \frac{\partial}{\partial x_2}\left[\frac{1}{2}g^{\mu\nu}\log\frac{Q}{-\lambda^2} - L^\mu L^\nu \frac{1}{Q}\right] \tag{B.31}$$

that gives us

$$\begin{aligned} F_a^{\mu\nu} &= \frac{i}{4\pi}\int_0^1 \mathrm{d}x_1 \left[\frac{1}{2}g^{\mu\nu}\log\frac{Q\left(x_1,1-x_1\right)}{-\lambda^2} - q^\mu q^\nu \frac{x^2}{Q\left(x_1,1-x_1\right)}\right] \\ &\quad -\frac{i}{4\pi}\int_0^1 \mathrm{d}x_1 \left[\frac{1}{2}g^{\mu\nu}\log\frac{Q\left(x_1,0\right)}{-\lambda^2} - L_0^\mu L_0^\nu \frac{1}{Q\left(x_1,0\right)}\right] \end{aligned} \tag{B.32}$$

recalling that

$$Q\left(x_1, 1-x_2\right) = q^2 x_1\left(1-x_1\right) + \left(m_1^2 - m_2^2\right)x_1 - m_1^2 \tag{B.33}$$

$$Q\left(x_1, 0\right) = k_2^2 x_1\left(1-x_1\right) + \left(\lambda^2 - m_2^2\right)x_1 - \lambda^2. \tag{B.34}$$

The other integral is easily expressed in the form

$$F_b^{\mu\nu} = \frac{i}{4\pi}\int_0^1 \mathrm{d}x_1 \left(1-x_1\right)\left[-\left(k_2^\mu L_0^\nu + k_2^\nu L_0^\mu\right)\frac{1}{Q} + \frac{1}{2}g^{\mu\nu}\frac{1}{Q}\frac{\partial Q}{\partial x_1} + L_0^\mu L_0^\nu \frac{1}{Q^2}\frac{\partial Q}{\partial x_1}\right]. \tag{B.35}$$

Here the argument of polynomial is $Q\left(x_1, 0\right)$. Thus, partial integration follows

$$F_b^{\mu\nu} = \frac{i}{4\pi}\int_0^1 \mathrm{d}x_1 \left[\frac{1}{2}g^{\mu\nu}\log\frac{Q\left(x_1,0\right)}{-\lambda^2} - \frac{L_0^\mu L_0^\nu}{Q\left(x_1,0\right)}\right] + \frac{i}{4\pi}\frac{k_1^\mu k_1^\nu}{\left(-\lambda^2\right)}, \tag{B.36}$$

again taking into account that $L\left(x_1,0\right) = L_0 = \left(k_2 x_1 - k_1\right)$.

Finally, summing both contributions $F_a^{\mu\nu}$ and $F_b^{\mu\nu}$, plus a external-momentum independent finite piece

$$\int \frac{\mathrm{d}^2 k}{\left(2\pi\right)^2}\frac{k_1^\mu k_1^\nu}{D_\lambda^2} = \frac{i}{4\pi}\frac{k_1^\mu k_1^\nu}{\left(-\lambda^2\right)}, \tag{B.37}$$

follows the complete integration of finite parts. The organization of tensor integral for general masses give us the result

$$\bar{J}_2^{\mu\nu} = \frac{1}{2}\left[\Delta_2^{\mu\nu} + g^{\mu\nu} I_{\log}\left(\lambda^2\right)\right] + \frac{i}{4\pi}\left[-\frac{1}{2}g^{\mu\nu}Z_0^{(0)} + q^\mu q^\nu Z_2^{(-1)}\right]. \tag{B.38}$$

Any other integral in this thesis can be obtained with the computational elements illustrated here.

B.3 Integration in the loop momentum

After Feynman parametrization, all integrals assume the form of the rational functions

$$\int \frac{\mathrm{d}^n k}{(2\pi)^n} \frac{(1, k_\mu, k_{\mu\nu}, k_{\mu\nu\rho}, \ldots)}{(k^2+Q)^\alpha}. \tag{B.39}$$

To solve the integral, we start with the form

$$I(k, M, n) = \int \frac{\mathrm{d}^n k}{(2\pi)^n} \frac{1}{(k^2 - M^2)^\alpha} = \int \frac{\mathrm{d}^n k}{(2\pi)^n} \frac{1}{(k^2 - 2k\cdot q + Q)^\alpha}, \tag{B.40}$$

where $2\alpha > n$ and $M^2 = q^2 - Q$. The auxiliary variable q helps to develop the tensor integrals. The integration measure $\mathrm{d}^n k = \mathrm{d}^{n-1}k\mathrm{d}k_0$. The square the momentum loop $k^2 = k_0^2 - \mathbf{k}^2$; and $\mathbf{k}^2 = \sum_{i=1}^{n-1} k_i^2$. The integral (B.40) only

$$I(Q, n) = \int \frac{\mathrm{d}^n k}{(2\pi)^n} \frac{1}{(k^2 - M^2)^\alpha} = \int \frac{\mathrm{d}^{n-1} k}{(2\pi)^n} \left[\int_{-\infty}^{+\infty} \mathrm{d}k_0 f(k_0)\right] \tag{B.41}$$

$$f(k_0) = \left[k_0^2 - (\sqrt{\mathbf{k}^2 + M^2} - i\varepsilon)^2\right]^{-\alpha}, \tag{B.42}$$

$$f(k_0) \sim \frac{1}{k_0^{2\alpha}}, \text{ as } k_0^2 \to \infty. \tag{B.43}$$

The poles and prescription coming from Feynman propagators

$$k_0^2 = \sqrt{\mathbf{k}^2 + M^2} - i\varepsilon \tag{B.44}$$

$$k_0^2 = -\sqrt{\mathbf{k}^2 + M^2} + i\varepsilon. \tag{B.45}$$

To compute the integral, we extend the integration for $k_0 \in \mathbb{C}$ and consider the following contour $C = C_1 + C_2 + C_3 + C_4$ in the figure below Then take the integral over

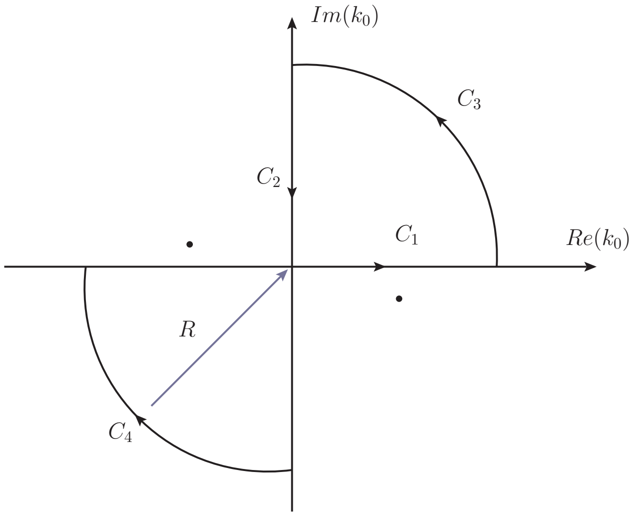

Figure B.1: Contour of integration

that contour

$$F_C(\mathbf{k}^2, M^2) = \int_C \mathrm{d}k_0 f(k_0) = \left(\int_{C_1} + \int_{C_2} + \int_{C_3} + \int_{C_4}\right) \mathrm{d}k_0 f(k_0) = 0 \tag{B.46}$$

since there are no poles inside the closed path of integration. We write the integral as $F_C = F_1+F_2+F_3+F_4$; the semi-circle contributions vanish in the limit $\lim_{R\to\infty}(F_{C_3}+F_{C_4})=0$. The reminder contribution gives the desired relation

$$\lim_{R\to\infty} F_{C_1} = -\lim_{R\to\infty} F_{C_2} \to \int_{-\infty}^{\infty} \mathrm{d}k_0 f(k_0) = -\int_{+i\infty}^{-i\infty} \mathrm{d}k_0 f(k_0). \tag{B.47}$$

Changing the integration variable in the last integral over the imaginary axis by adopting $k_0 = ik_0'$, we may write

$$I(Q,n) = \int \frac{\mathrm{d}^{n-1}k}{(2\pi)^{n-1}} \int_{-\infty}^{\infty} \mathrm{d}k_0 \frac{1}{(k_0^2-\mathbf{k}^2-M^2)^{\alpha}} = \int \frac{\mathrm{d}^{n-1}k}{(2\pi)^{n-1}} \int_{-\infty}^{\infty} \mathrm{d}k_0' \frac{i}{(-k_0'^2-\mathbf{k}^2-M^2)^{\alpha}} \tag{B.48}$$

and effectively we have an euclidean signature ($k'^2 := k_0'^2 + \sum_{i=1}^{n-1} k_i^2$) to perform the integral

$$I(Q,n) = i(-1)^{\alpha} \int \frac{\mathrm{d}^n k'}{(2\pi)^n} \frac{1}{(k'^2+M^2)^{\alpha}}. \tag{B.49}$$

Now we introduce spherical coordinates to these variables and split the radius and solid angle integrations

$$I(Q,n) = \frac{i(-1)^{\alpha}}{(2\pi)^n} \int_{S^{n-1}} \mathrm{d}\Omega \int_0^{\infty} \mathrm{d}r r^{n-1} \frac{1}{(r^2+M^2)^{\alpha}}. \tag{B.50}$$

The solid angle furnish

$$\frac{1}{(2\pi)^n} \int_{S^{n-1}} \mathrm{d}\Omega = \frac{2}{(4\pi)^{n/2}\Gamma\left(\frac{n}{2}\right)} \tag{B.51}$$

and simple manipulations bring the form

$$I(Q,n) = \frac{2i(-1)^{\alpha}}{(4\pi)^{n/2}\Gamma\left(\frac{n}{2}\right)} \frac{1}{2M^{2(\alpha-n/2)}} \int_0^{\infty} \mathrm{d}\left(r'^2\right)\left(r'^2\right)^{(n-2)/2}\left(r'^2+1\right)^{-\alpha}. \tag{B.52}$$

Another variables change $r'^2 = (1-y)/y \to \mathrm{d}(r'^2) = -\mathrm{d}y 1/y^2$, the Beta function is

$$B\left(\alpha-\frac{n}{2},\frac{n}{2}\right) = \int_0^1 \mathrm{d}y y^{(\alpha-n/2)-1}(1-y)^{n/2-1}.$$

We have

$$I(Q,n) = \frac{i(-1)^{\alpha}}{(4\pi)^{n/2}\Gamma\left(\frac{n}{2}\right)M^{2(\alpha-n/2)}} \int_0^1 \mathrm{d}y y^{(\alpha-n/2)-1}(1-y)^{n/2-1} \tag{B.53}$$

$$= \frac{i(-1)^{\alpha}}{(4\pi)^{n/2}} \frac{\Gamma\left(\alpha-\frac{n}{2}\right)}{\Gamma(\alpha)M^{2(\alpha-n/2)}}, \tag{B.54}$$

thus, from $M^2 = q^2 - Q$ follows

$$I(Q,n) = \frac{i(-1)^{\alpha}}{(4\pi)^{n/2}} \frac{\Gamma\left(\alpha-\frac{n}{2}\right)}{\Gamma(\alpha)(q^2-Q)^{\alpha-n/2}}. \tag{B.55}$$

Now taking derivatives concerning the variable q on both sides and shifting the parameters $\alpha \to \alpha - 1$ in the form

$$I(Q,n) = \frac{i}{(-4\pi)^{n/2}} \frac{\Gamma(\alpha - n/2)}{\Gamma(\alpha)(Q-q^2)^{(\alpha-n/2)}}, \tag{B.56}$$

the explicit derivative is

$$\frac{\partial I}{\partial q^{\mu_1}} = \frac{i}{(-4\pi)^{n/2}} \frac{2q_{\mu_1}\Gamma(\alpha - n/2 + 1)}{\Gamma(\alpha)(Q-q^2)^{\alpha+1-n/2}} = \int \frac{d^n k}{(2\pi)^n} \frac{2\alpha k_{\mu_1}}{(k^2 - 2k\cdot q + Q)^{\alpha+1}}, \tag{B.57}$$

follows the relation

$$\int \frac{d^n k}{(2\pi)^n} \frac{k_{\mu_1}}{(k^2 - 2k\cdot q + Q)^{\alpha}} = \frac{i}{(-4\pi)^{n/2}} q_{\mu_1} \frac{\Gamma(\alpha - n/2)}{\Gamma(\alpha)(Q-q^2)^{\alpha-n/2}}. \tag{B.58}$$

Recursively

$$\int \frac{d^n k}{(2\pi)^n} \frac{k_{\mu_2}k_{\mu_1}}{(k^2+Q)^{\alpha}} = \frac{i}{(-4\pi)^{n/2}} \frac{1}{2} g_{\mu_{12}} \frac{\Gamma(\alpha - n/2 - 1)}{\Gamma(\alpha) Q^{\alpha-n/2-1}}. \tag{B.59}$$

From the formulae presented, it is possible to obtain a general result, adopting $n = 2\omega$, which reads

$$\int \frac{d^{2\omega} k}{(2\pi)^{2\omega}} \frac{k_{\mu_1}\cdots k_{\mu_{2l+1}}}{(k^2+Q)^{\alpha}} = 0 \tag{B.60}$$

$$\int \frac{d^{2\omega} k}{(2\pi)^{2\omega}} \frac{k_{\mu_1}\cdots k_{\mu_{2l}}}{(k^2+Q)^{\alpha}} = \frac{i}{(4\pi)^{\omega}} \frac{1}{2^l} g_{(\mu_1\mu_2}\cdots g_{\mu_{2l-1},\mu_{2l})} \frac{\Gamma(\alpha-\omega-l)}{\Gamma(\alpha) Q^{\alpha-\omega-l}} \tag{B.61}$$

It is interesting to note that these results imply in the properties:

$$\int \frac{d^{2\omega} k}{(2\pi)^{2\omega}} f(k^2)\, k_\mu = 0 \tag{B.62}$$

$$\int \frac{d^{2\omega} k}{(2\pi)^{2\omega}} k_\mu k_\nu f(k^2) = \frac{g_{\mu\nu}}{2\omega} \int \frac{d^{2\omega} k}{(2\pi)^{2\omega}} k^2 f(k^2) \tag{B.63}$$

$$\int \frac{d^{2\omega} k}{(2\pi)^{2\omega}} k_{\mu\nu\alpha\beta} f(k^2) = \frac{(g_{\mu\nu\alpha\beta} + g_{\mu\alpha\nu\beta} + g_{\mu\beta\nu\alpha})}{4(\omega+1)} \int \frac{d^{2\omega} k}{(2\pi)^{2\omega}} k^4 f(k^2). \tag{B.64}$$

Appendix C

The One point Integrals in Two Dimensions

After performing the Dirac traces present in the definitions we established for the perturbative amplitudes in (7.33) and (7.34), their integrals naturally decompose in Feynman integrals that we define in the equations (3.52) and (3.53). The calculations follow the IReg method by applying the separation identity (3.4) on the divergent integrals. The finite part is integrated and projected in definitions (3.27) and (3.28). The residual divergent part is projected onto divergent objects of the set, expressed in (3.5) and their relations in the session (3.1).

We start with integrals that have only one propagator. These have only divergent structures. The finite parts after separating the labels, cancel out when they are integrated.

Integral J_1 : by power-counting this integral has a superficial degree of divergence is logarithmic

$$\bar{J}_1\left(k_i\right) = I_{\log} \tag{C.1}$$

From the next integral, it is necessary to specify the k_1 and k_2 labels of the integral.

Integral $J_{1\mu_1}$: superficial degree of divergence is linear

$$2\bar{J}_{1\mu_1}\left(k_1\right) = -\left(P-q\right)^{\nu_1}\Delta_{2\mu_1\nu_1} \tag{C.2}$$

$$2\bar{J}_{1\mu_1}\left(k_2\right) = -\left(P+q\right)^{\nu_1}\Delta_{2\mu_1\nu_1} \tag{C.3}$$

Integral $J_{1\mu_{12}}$: superficial degree of divergence is quadratic

$$\begin{aligned}\bar{J}_{1\mu_{12}}\left(k_1\right) &= \frac{1}{2}\left(\Delta_{1\mu_{12}}+g_{\mu_{12}}I_{\text{quad}}\right)-\frac{1}{8}\left(P-q\right)^2\Delta_{2\mu_{12}} \\ &+\frac{1}{8}\left(P-q\right)^{\nu_1}\left[\left(P-q\right)^{\nu_2}W_{3\mu_{12}\nu_{12}}-2\left(P-q\right)_{(\mu_1}\Delta_{2\mu_2)\nu_1}\right]\end{aligned} \tag{C.4}$$

$$\bar{J}_{1\mu_{12}}(k_2) = \frac{1}{2}\left(\Delta_{1\mu_{12}} + g_{\mu_{12}} I_{\mathrm{quad}}\right) - \frac{1}{8}(P+q)^2 \Delta_{2\mu_{12}} + \frac{1}{8}(P+q)^{\nu_1}\left[(P+q)^{\nu_2} W_{3\mu_{12}\nu_{12}} - 2(P+q)_{(\mu_1}\Delta_{2\mu_2)\nu_1}\right] \tag{C.5}$$

Integral $J_{1\mu_{123}}$: superficial degree of divergence is cubic and are the integrals with the highest power-counting

$$\begin{aligned} J_{1\mu_{123}}(k_1) = & -\frac{1}{4}(P-q)^{\nu_1} W_{2\mu_{123}\nu_1} + \frac{1}{4}(P-q)_{(\mu_1}\Delta_{1\mu_{23})} \\ & -\frac{1}{48}(P-q)^{\nu_1}(P-q)^{\nu_2}(P-q)^{\nu_3} W_{4\mu_{123}\nu_{123}} \\ & +\frac{1}{16}(P-q)^{\nu_1}(P-q)^{\nu_2}(P-q)_{(\mu_1} W_{3\mu_{23})\nu_{12}} \\ & +\frac{1}{16}(P-q)^2\left[(P+p)^{\nu_1} W_{3\mu_{123}\nu_1} - (P-q)_{(\mu_1}\Delta_{2\mu_{23})}\right] \\ & -\frac{1}{8}(P-q)^{\nu_1}(P-q)_{(\mu_1}(P-q)_{\mu_2}\Delta_{2\mu_3)\nu_1} \end{aligned} \tag{C.6}$$

$$\begin{aligned} J_{1\mu_{123}}(k_2) = & -\frac{1}{4}(P+q)^{\nu_1} W_{2\mu_{123}\nu_1} + \frac{1}{4}(P+q)_{(\mu_1}\Delta_{1\mu_{23})} \\ & -\frac{1}{48}(P+q)^{\nu_1}(P+q)^{\nu_2}(P+q)^{\nu_3} W_{4\mu_{123}\nu_{123}} \\ & +\frac{1}{16}(P+q)^{\nu_1}(P+q)^{\nu_2}(P+q)_{(\mu_1} W_{3\mu_{23})\nu_{12}} \\ & +\frac{1}{16}(P+q)^2\left[(P+q)^{\nu_1} W_{3\mu_{123}\nu_1} - (P+q)_{(\mu_1}\Delta_{2\mu_{23})}\right] \\ & -\frac{1}{8}(P+q)^{\nu_1}(P+q)_{(\mu_1}(P+q)_{\mu_2}\Delta_{2\mu_3)\nu_1}. \end{aligned} \tag{C.7}$$

For instance, we calculated the $J_1^{\mu_1\mu_2}(k_i)$. The complete expression:

$$\begin{aligned} J_1^{\mu_1\mu_2}(k_i) = & \int \frac{\mathrm{d}^2 k}{(2\pi)^2}\frac{k^{\mu_1}k^{\mu_2}}{D_i} \\ & + \int \frac{\mathrm{d}^2 k}{(2\pi)^2}\left(k_i^{\mu_1}k^{\mu_2} + k_i^{\mu_2}k^{\mu_1}\right)\frac{1}{D_i} \\ & + k_i^{\mu_1}k_i^{\mu_2}\int \frac{\mathrm{d}^2 k}{(2\pi)^2}\frac{1}{D_i}. \end{aligned} \tag{C.8}$$

Using the expansions for two first integral above, we have

$$\left[\frac{k^{\mu_1}k^{\mu_2}}{D_i}\right]_{\mathrm{even}} = \frac{k^{\mu_1}k^{\mu_2}}{D_\lambda} - k_i^2\frac{k^{\mu_1}k^{\mu_2}}{D_\lambda^2} + 4k_{i\nu_{12}}\frac{k^{\mu_1}k^{\mu_2}k^{\nu_1}k^{\nu_2}}{D_\lambda^3} \tag{C.9}$$

$$\left[\frac{k^{\mu_1}}{D_i}\right]_{\mathrm{even}} = -2k_{i\nu_1}\frac{k^{\mu_1}k^{\nu_1}}{D_\lambda^2}. \tag{C.10}$$

So the expanded integral is given by

$$J_1^{\mu_1\mu_2}(k_i) = \int \frac{\mathrm{d}^2k}{(2\pi)^2}\left(\frac{k^{\mu_1}k^{\mu_2}}{D_\lambda} - k_i^2\frac{k^{\mu_1}k^{\mu_2}}{D_\lambda^2} + 4k_{i\nu_1}k_{i\nu_2}\frac{k^{\mu_1}k^{\mu_2}k^{\nu_1}k^{\nu_2}}{D_\lambda^3}\right) \tag{C.11}$$
$$-2k_{i\nu_1}\int \frac{\mathrm{d}^2k}{(2\pi)^2}\left(k_i^{\mu_1}\frac{k^{\mu_2}k^{\nu_1}}{D_\lambda^2} + k_i^{\mu_2}\frac{k^{\mu_1}k^{\nu_1}}{D_\lambda^2}\right)$$
$$+k_i^{\mu_1}k_i^{\mu_2}\int \frac{\mathrm{d}^2k}{(2\pi)^2}\frac{1}{D_i}.$$

Identifying the divergent objects in Section (3.1)

$$\int \frac{\mathrm{d}^2k}{(2\pi)^2}\frac{8k^{\mu_1\mu_2\nu_1\nu_2}}{D_\lambda^3} = W_3^{\mu_1\mu_2\nu_1\nu_2} + g^{\mu_1\mu_2\nu_1\nu_2}I_{\log}$$
$$W_3^{\mu_1\mu_2\nu_1\nu_2} = \Box_3^{\mu_1\mu_2\nu_1\nu_2} + \frac{1}{2}g^{(\mu_1\nu_1}\Delta_2^{\mu_2\nu_2)}$$
$$\int \frac{\mathrm{d}^2k}{(2\pi)^2}\frac{2k^{\mu_1\mu_2}}{D_\lambda^2} = \Delta_2^{\mu_1\mu_2} + g^{\mu_1\mu_2}I_{\log}$$
$$\int \frac{\mathrm{d}^2k}{(2\pi)^2}\frac{2k^{\mu_1\mu_2}}{D_\lambda} = \Delta_1^{\mu_1\mu_2} + g^{\mu_1\mu_2}I_{\mathrm{quad}}.$$

Substituting in (C.11), we can see the scalars $I_{\log}$ cancel and remains the final expression

$$J_1^{\mu_1\mu_2}(k_i) = \frac{1}{2}\left[\Delta_1^{\mu_1\mu_2} + g^{\mu_1\mu_2}I_{\mathrm{quad}}\right] + \frac{1}{2}k_{i\nu_{12}}W_3^{\mu_1\mu_2\nu_1\nu_2} \tag{C.12}$$
$$-k_i^{\mu_1}k_{i\nu_1}\Delta_2^{\mu_2\nu_1} - k_i^{\mu_2}k_{i\nu_1}\Delta_2^{\mu_1\nu_1} - \frac{1}{2}k_i^2\Delta_2^{\mu_1\mu_2}.$$

The expression above can be written as (C.4) and C.5 replacing the routing k_i by $2k_1 = (P-q)$ or $2k_2 = (P+q)$.

Appendix D

Function $Z_k^{(-1)}\left(q^2, m_1^2, m_2^2\right)$

As we saw throughout the text, it is sometimes interesting to consider explicit forms of these functions due to their importance in discussing some important aspects of amplitudes. So we consider the following function

$$Z_k^{(-1)}\left(q^2, m_1^2, m_2^2\right) \equiv \int_0^1 dz \frac{z^k}{Q\left(q^2, m_1^2, m_2^2\right)},$$

where $Q = q^2 z\left(1-z\right) + \left(m_1^2 - m_2^2\right) z - m_1^2$ is the polynomial form of denominator. Since all the functions $Z_k^{(-1)}$ can be put in terms of the functions $Z_0^{(-1)}$, we will consider in this appendix the calculation explicitly only of the function, defined by

$$Z_0^{(-1)}\left(q^2, m_1^2, m_2^2\right) = \int_0^1 \frac{1}{Q}. \tag{D.1}$$

One way to integrate is to write the polynomial present in the denominator through its roots. We do

$$Q = -q^2\left[z^2 - \frac{1}{q^2}\left(q^2 + m_1^2 - m_2^2\right)z + \frac{m_1^2}{q^2}\right] = -q^2\left(z-\alpha\right)\left(z-\beta\right). \tag{D.2}$$

Where the roots of the polynomial are α and β given by

$$\alpha = \frac{\left(q^2 + m_1^2 - m_2^2\right) + \sqrt{\left(q^2 + m_1^2 - m_2^2\right)^2 - 4m_1^2 q^2}}{2q^2}; \tag{D.3}$$

$$\beta = \frac{q^2 + m_1^2 - m_2^2 - \sqrt{\left(q^2 + m_1^2 - m_2^2\right)^2 - 4m_1^2 q^2}}{2q^2}, \tag{D.4}$$

where α and β satisfy the following relations:

$$\alpha + \beta = \frac{\left(q^2 + m_1^2 - m_2^2\right)}{q^2}; \qquad \alpha\beta = \frac{m_1^2}{q^2} \tag{D.5}$$

$$\alpha - \beta = \frac{\sqrt{\left(q^2 + m_1^2 - m_2^2\right)^2 - 4m_1^2 q^2}}{q^2}. \tag{D.6}$$

Rewriting Eq. (D.1) as

$$Z_0^{(-1)} = -\frac{1}{q^2}\int_0^1 \frac{1}{(z-\alpha)(z-\beta)} = -\frac{1}{q^2}\frac{1}{\alpha-\beta}\int_0^1 dz\left[\frac{1}{(z-\alpha)} - \frac{1}{(z-\beta)}\right]. \tag{D.7}$$

Using the passage

$$\int_0^1 dz\frac{1}{(z-\alpha)} = \ln(1-\alpha) - \ln(-\alpha) = \ln\left(\frac{\alpha-1}{\alpha}\right), \tag{D.8}$$

we will have

$$Z_0^{(-1)} = -\frac{1}{q^2}\frac{1}{\alpha-\beta}\left\{\ln\left(\frac{\alpha-1}{\alpha}\right) - \ln\left(\frac{\beta-1}{\beta}\right)\right\} = \frac{1}{q^2}\frac{1}{\alpha-\beta}\left[\ln\left(\frac{\alpha-1}{\alpha}\right)\left(\frac{\beta}{\beta-1}\right)\right]. \tag{D.9}$$

From that, we can write the explicit form for the function $Z_0^{(-1)}$,

$$Z_0^{(-1)} = \frac{1}{\sqrt{(q^2+m_1^2-m_2^2)^2 - 4m_1^2q^2}}\ln\left[\frac{(m_1^2+m_2^2-q^2) + \sqrt{(q^2+m_1^2-m_2^2)^2 - 4m_1^2q^2}}{(m_1^2+m_2^2-q^2) - \sqrt{(q^2+m_1^2-m_2^2)^2 - 4m_1^2q^2}}\right] \tag{D.10}$$

In the kinematical limit, where $q^2 \ll 1$, we have the result

$$Z_0^{(-1)} = \frac{1}{(m_1^2-m_2^2)}\ln\left[\frac{(m_1^2-m_2^2)+(m_1^2-m_2^2)}{(m_1^2-m_2^2)-(m_1^2-m_2^2)}\right]. \tag{D.11}$$

Appendix E

Subamplitudes

We cast vector subamplitudes in this appendix. They are ordered following the amplitudes that originate them (AVV, VAV, VVA, and AAA) and then grouped according to the version. That emphasizes patterns attributed to each version and additional terms depending on the squared mass.

First version:

$$\left(t^{VPP}\right)^{\nu_1} = \left[-K_1^{\nu_1} S_{23} + K_2^{\nu_1} S_{13} - K_3^{\nu_1} S_{12}\right] \frac{1}{D_{123}} \tag{E.1}$$

$$\left(t^{ASP}\right)^{\nu_1} = \left[-K^{\nu_1} S_{23} + K_2^{\nu_1}\left(S_{13}+2m^2\right) - K_3^{\nu_1}\left(S_{12}+2m^2\right)\right] \frac{1}{D_{123}} \tag{E.2}$$

$$\left(t^{APS}\right)^{\nu_1} = \left[K_1^{\nu_1}\left(S_{23}+2m^2\right) - K_2^{\nu_1}\left(S_{13}+2m^2\right) + K_3^{\nu_1} S_{12}\right] \frac{1}{D_{123}} \tag{E.3}$$

$$\left(t^{VSS}\right)^{\nu_1} = \left[K_1^{\nu_1}\left(S_{23}+2m^2\right) - K_2^{\nu_1} S_{13} + K_3^{\nu_1}\left(S_{12}+2m^2\right)\right] \frac{1}{D_{123}} \tag{E.4}$$

$$\begin{aligned} (T^{VPP})^{\nu_1} &= 2\left[P_{31}^{\nu_2}\Delta_{3\nu_2}^{\nu_1} + (p_{21}^{\nu_1} - p_{32}^{\nu_1}) I_{\log}\right] - 4\left(p_{21}\cdot p_{32}\right) J_3^{\nu_1} \\ &\quad +2\left[(p_{31}^{\nu_1} p_{21}^2 - p_{21}^{\nu_1} p_{31}^2) J_3 + p_{21}^{\nu_1} J_2\left(p_{21}\right) - p_{32}^{\nu_1} J_2\left(p_{32}\right)\right] \end{aligned} \tag{E.5}$$

$$\begin{aligned} \left(T^{ASP}\right)^{\nu_1} &= 2\left[P_{31}^{\nu_2}\Delta_{3\nu_2}^{\nu_1} + (p_{21}^{\nu_1} - p_{32}^{\nu_1})\, I_{\log}\right] - 4\left(p_{21}\cdot p_{32}\right) J_3^{\nu_1} \\ &\quad +2\left[\left(p_{31}^{\nu_1} p_{21}^2 - p_{21}^{\nu_1} p_{31}^2 - 4m^2 p_{32}^{\nu_1}\right) J_3 + p_{21}^{\nu_1} J_2\left(p_{21}\right) - p_{32}^{\nu_1} J_2\left(p_{32}\right)\right] \end{aligned} \tag{E.6}$$

$$\begin{aligned} -\left(T^{APS}\right)^{\nu_1} &= 2\left[P_{31}^{\nu_2}\Delta_{3\nu_2}^{\nu_1} + (p_{21}^{\nu_1} - p_{32}^{\nu_1}) I_{\log}\right] - 4\left(p_{21}\cdot p_{32}\right) J_3^{\nu_1} \\ &\quad +2\left[\left(p_{31}^{\nu_1} p_{21}^2 - p_{21}^{\nu_1} p_{31}^2 + 4m^2 p_{21}^{\nu_1}\right) J_3 + p_{21}^{\nu_1} J_2\left(p_{21}\right) - p_{32}^{\nu_1} J_2\left(p_{32}\right)\right] \end{aligned} \tag{E.7}$$

$$\begin{aligned} -\left(T^{VSS}\right)^{\nu_1} &= 2\left[P_{31}^{\nu_2}\Delta_{3\nu_2}^{\nu_1} + (p_{21}^{\nu_1} - p_{32}^{\nu_1}) I_{\log}\right] - 4\left(p_{21}\cdot p_{32} + 4m^2\right) J_3^{\nu_1} \\ &\quad +2\left[\left(p_{31}^{\nu_1} p_{21}^2 - p_{21}^{\nu_1} p_{31}^2 - 4m^2 p_{31}^{\nu_1}\right) J_3 + p_{21}^{\nu_1} J_2\left(p_{21}\right) - p_{32}^{\nu_1} J_2\left(p_{32}\right)\right] \end{aligned} \tag{E.8}$$

Second version:

$$\left(t^{SAP}\right)^{\nu_1} = \left[K_1^{\nu_1} S_{23} + K_2^{\nu_1}\left(S_{13}+2m^2\right) - K_3^{\nu_1}\left(S_{12}+2m^2\right)\right]\frac{1}{D_{123}} \tag{E.9}$$

$$\left(t^{PVP}\right)^{\nu_1} = \left[-K_1^{\nu_1} S_{23} - K_2^{\nu_1} S_{13} + K_3^{\nu_1} S_{12}\right]\frac{1}{D_{123}} \tag{E.10}$$

$$\left(t^{PAS}\right)^{\nu_1} = -\left[K_1^{\nu_1}\left(S_{23}+2m^2\right) + K_2^{\nu_1} S_{13} - K_3^{\nu_1}\left(S_{12}+2m^2\right)\right]\frac{1}{D_{123}} \tag{E.11}$$

$$\left(t^{SVS}\right)^{\nu_1} = \left[K_1^{\nu_1}\left(S_{23}+2m^2\right) + K_2^{\nu_1}\left(S_{13}+2m^2\right) - K_3^{\nu_1} S_{12}\right]\frac{1}{D_{123}} \tag{E.12}$$

$$\begin{aligned} -\left(T^{SAP}\right)^{\nu_1} &= 2\left[P_{21}^{\nu_2}\Delta_{3\nu_2}^{\nu_1} + \left(p_{32}^{\nu_1}+p_{31}^{\nu_1}\right) I_{\log}\right] + 4\left(p_{32}\cdot p_{31}\right) J_3^{\nu_1} \\ &\quad +2\left[\left(p_{21}^{\nu_1}p_{31}^2 - p_{31}^{\nu_1}p_{21}^2 + 4m^2 p_{32}^{\nu_1}\right) J_3 + p_{32}^{\nu_1} J_2\left(p_{32}\right) + p_{31}^{\nu_1} J_2\left(p_{31}\right)\right] \end{aligned} \tag{E.13}$$

$$\begin{aligned} \left(T^{PVP}\right)^{\nu_1} &= 2\left[P_{21}^{\nu_2}\Delta_{3\nu_2}^{\nu_1} + \left(p_{32}^{\nu_1}+p_{31}^{\nu_1}\right) I_{\log}\right] + 4\left(p_{32}\cdot p_{31}\right) J_3^{\nu_1} \\ &\quad +2\left[\left(p_{21}^{\nu_1}p_{31}^2 - p_{31}^{\nu_1}p_{21}^2\right) J_3 + p_{32}^{\nu_1} J_2\left(p_{32}\right) + p_{31}^{\nu_1} J_2\left(p_{31}\right)\right] \end{aligned} \tag{E.14}$$

$$\begin{aligned} \left(T^{PAS}\right)^{\nu_1} &= 2\left[P_{21}^{\nu_2}\Delta_{3\nu_2}^{\nu_1} + \left(p_{32}^{\nu_1}+p_{31}^{\nu_1}\right) I_{\log}\right] + 4\left(p_{32}\cdot p_{31}\right) J_3^{\nu_1} \\ &\quad +2\left[\left(p_{21}^{\nu_1}p_{31}^2 - p_{31}^{\nu_1}p_{21}^2 + 4m^2 p_{31}^{\nu_1}\right) J_3 + p_{32}^{\nu_1} J_2\left(p_{32}\right) + p_{31}^{\nu_1} J_2\left(p_{31}\right)\right] \end{aligned} \tag{E.15}$$

$$\begin{aligned} -\left(T^{SVS}\right)^{\nu_1} &= 2\left[P_{21}^{\nu_2}\Delta_{3\nu_2}^{\nu_1} + \left(p_{32}^{\nu_1}+p_{31}^{\nu_1}\right) I_{\log}\right] + 4\left(p_{32}\cdot p_{31} - 4m^2\right) J_3^{\nu_1} \\ &\quad +2\left[\left(p_{21}^{\nu_1}p_{31}^2 - p_{31}^{\nu_1}p_{21}^2 - 4m^2 p_{21}^{\nu_1}\right) J_3 + p_{32}^{\nu_1} J_2\left(p_{32}\right) + p_{31}^{\nu_1} J_2\left(p_{31}\right)\right] \end{aligned} \tag{E.16}$$

Third version:

$$\left(t^{SPA}\right)^{\nu_1} = \left[K_1^{\nu_1}\left(S_{23}+2m^2\right) - K_2^{\nu_1}\left(S_{13}+2m^2\right) - K_3^{\nu_1} S_{12}\right]\frac{1}{D_{123}} \tag{E.17}$$

$$\left(t^{PSA}\right)^{\nu_1} = \left[-K_1^{\nu_1}\left(S_{23}+2m^2\right) + K_2^{\nu_1} S_{13} + K_3^{\nu_1}\left(S_{12}+2m^2\right)\right]\frac{1}{D_{123}} \tag{E.18}$$

$$\left(t^{PPV}\right)^{\nu_1} = -\left[-K_1^{\nu_1} S_{23} + K_2^{\nu_1} S_{13} + K_3^{\nu_1} S_{12}\right]\frac{1}{D_{123}} \tag{E.19}$$

$$\left(t^{SSV}\right)^{\nu_1} = \left[-K_1^{\nu_1} S_{23} + K_2^{\nu_1}\left(S_{13}+2m^2\right) + K_3^{\nu_1}\left(S_{12}+2m^2\right)\right]\frac{1}{D_{123}} \tag{E.20}$$

$$\begin{aligned} \left(T^{SPA}\right)^{\nu_1} &= 2\left[P_{32}^{\nu_2}\Delta_{3\nu_2}^{\nu_1} - \left(p_{21}^{\nu_1}+p_{31}^{\nu_1}\right) I_{\log}\right] + 4\left(p_{21}\cdot p_{31}\right) J_3^{\nu_1} \\ &\quad +2\left[\left(p_{31}^{\nu_1}p_{21}^2 + p_{21}^{\nu_1}p_{31}^2 - 4m^2 p_{21}^{\nu_1}\right) J_3 - p_{21}^{\nu_1} J_2\left(p_{21}\right) - p_{31}^{\nu_1} J_2\left(p_{31}\right)\right] \end{aligned} \tag{E.21}$$

$$\begin{aligned} -\left(T^{PSA}\right)^{\nu_1} &= 2\left[P_{32}^{\nu_2}\Delta_{3\nu_2}^{\nu_1} - \left(p_{21}^{\nu_1}+p_{31}^{\nu_1}\right) I_{\log}\right] + 4\left(p_{21}\cdot p_{31}\right) J_3^{\nu_1} \\ &\quad +2\left[\left(p_{31}^{\nu_1}p_{21}^2 + p_{21}^{\nu_1}p_{31}^2 - 4m^2 p_{31}^{\nu_1}\right) J_3 - p_{21}^{\nu_1} J_2\left(p_{21}\right) - p_{31}^{\nu_1} J_2\left(p_{31}\right)\right] \end{aligned} \tag{E.22}$$

$$\begin{aligned} \left(T^{PPV}\right)^{\nu_1} &= 2\left[P_{32}^{\nu_2}\Delta_{3\nu_2}^{\nu_1} - \left(p_{21}^{\nu_1}+p_{31}^{\nu_1}\right) I_{\log}\right] + 4\left(p_{21}\cdot p_{31}\right) J_3^{\nu_1} \\ &\quad +2\left[\left(p_{31}^{\nu_1}p_{21}^2 + p_{21}^{\nu_1}p_{31}^2\right) J_3 - p_{21}^{\nu_1} J_2\left(p_{21}\right) - p_{31}^{\nu_1} J_2\left(p_{31}\right)\right] \end{aligned} \tag{E.23}$$

$$\begin{aligned} -\left(T^{SSV}\right)^{\nu_1} &= 2\left[P_{32}^{\nu_2}\Delta_{3\nu_2}^{\nu_1} - \left(p_{21}^{\nu_1}+p_{31}^{\nu_1}\right) I_{\log}\right] + 4\left(p_{21}\cdot p_{31} - 4m^2\right) J_3^{\nu_1} \\ &\quad +2\left[\left(p_{31}^{\nu_1}p_{21}^2 + p_{21}^{\nu_1}p_{31}^2 - 4m^2\left(p_{21}^{\nu_1}+p_{31}^{\nu_1}\right)\right) J_3 - p_{21}^{\nu_1} J_2\left(p_{21}\right) - p_{31}^{\nu_1} J_2\left(p_{31}\right)\right] \end{aligned} \tag{E.24}$$

Appendix F

Surface Terms

The surface terms used in this work appear in a totally symmetrical way in the indices, for the first time treated from the point of view of the IReg strategy. The meaning of the notation used is

$$g_{(\mu_{12}}g_{\mu_{34})} = g_{\mu_{12}}g_{\mu_{34}} + g_{\mu_{13}}g_{\mu_{24}} + g_{\mu_{14}}g_{\mu_{23}}. \tag{F.1}$$

For instance, in the case of permutations involving six indices as the product of the metrics by the logarithmically divergent object $\Delta_{2\mu\nu}$, we have forty-five terms given by,

$$\begin{aligned} & g_{(\mu_{12}}g_{\mu_{34}} \\ = \; & \Delta_{2\mu_{12}}g_{(\mu_{34}}g_{\mu_{56})} + \Delta_{2\mu_{13}}g_{(\mu_{24}}g_{\mu_{56})} + \Delta_{2\mu_{14}}g_{(\mu_{23}}g_{\mu_{56})} + \Delta_{2\mu_{15}}g_{(\mu_{23}}g_{\mu_{46})} + \Delta_{2\mu_{16}}g_{(\mu_{23}}g_{\mu_{45})} \\ & + \Delta_{2\mu_{23}}g_{(\mu_{14}}g_{\mu_{56})} + \Delta_{2\mu_{24}}g_{(\mu_{13}}g_{\mu_{56})} + \Delta_{2\mu_{25}}g_{(\mu_{13}}g_{\mu_{46})} + \Delta_{2\mu_{26}}g_{(\mu_{13}}g_{\mu_{45})} \\ & + \Delta_{2\mu_{34}}g_{(\mu_{12}}g_{\mu_{56})} + \Delta_{2\mu_{35}}g_{(\mu_{12}}g_{\mu_{46})} + \Delta_{2\mu_{36}}g_{(\mu_{12}}g_{\mu_{45})} \\ & + \Delta_{2\mu_{45}}g_{(\mu_{12}}g_{\mu_{36})} + \Delta_{2\mu_{46}}g_{(\mu_{12}}g_{\mu_{35})} \\ & + \Delta_{2\mu_{56}}g_{(\mu_{12}}g_{\mu_{34})}. \end{aligned} \tag{F.2}$$

This can be written succinctly as

$$g_{(\mu_{12}}g_{\mu_{34}}\Delta_{2\mu_{56})} = \sum_{i_2>i_1=1}^{5} \Delta_{2i_1i_2}g_{(i_3i_4}g_{i_5i_6)} \text{ with } i_n \neq i_m \tag{F.3}$$

where i_n denotes μ_{i_n}. For the box terms we may also write

$$g_{(\mu_1\mu_2}\Box_{3\mu_{3456})} = \sum_{i_2>i_1=1}^{5} g_{\mu_{i_1}\mu_{i_2}}\Box_{3\mu_{i_3}\mu_{i_4}\mu_{i_5}\mu_{i_6}}. \tag{F.4}$$

F.1 Uniqueness Factor: Combination of the violating terms

As we saw, surface terms violate several symmetry relations. However, if the relations are satisfied, relations between surface terms emerge for their traces and the finite part.

Through the strategy (3), we saw that all the divergent objects were organized into standardized objects as to their tensor degree and power counting. We have

$$\begin{aligned}\Xi^{(a)}_{\nu_{23}} &= \left[2\Box^{\rho}_{3\rho\nu_{23}} - 2\Delta_{2\nu_{23}} - g_{\nu_{23}}\Delta^{\rho}_{2\rho}\right] &\text{(F.5)}\\ \Xi^{(b)}_{\alpha_{12}\nu_{23}} &= \left[3\Sigma^{\rho}_{4\rho\alpha_{12}\nu_{23}} - 8\Box_{3\alpha_{12}\nu_{23}} - g_{\alpha_{12}\nu_{23}}\Delta^{\rho}_{2\rho}\right] &\text{(F.6)}\\ \Xi^{\text{quad}}_{\alpha_1\alpha_2} &= \left[W^{\rho}_{2\rho\alpha_1\alpha_2} - 2\Delta_{1\alpha_{12}} + 2g_{\alpha_{12}}I_{\text{quad}} - 2m^2\left(\Delta_{2\alpha_{12}} + g_{\alpha_{12}}I_{\log}\right)\right]. &\text{(F.7)}\end{aligned}$$

In this way, this organization allows us to write the U-factor as

$$\begin{aligned}U_{\alpha_1\alpha_2} &= -\frac{1}{3}\theta_{\alpha_1\alpha_2}\left(2\Delta^{\rho}_{2\rho} + i/\pi\right) + \frac{1}{9}\left(3P^{\nu_2}P^{\nu_3} + q^{\nu_{23}}\right)\Xi^{b}_{\alpha_1\alpha_2\nu_{23}} &\text{(F.8)}\\ &+\frac{1}{18}\left(3P^{\nu_2}P^{\nu_3} + q^{\nu_{23}}\right)g_{(\alpha_1\alpha_2}\Xi^{a}_{\nu_2\nu_3)}\\ &-\frac{1}{2}\left(P^2+q^2\right)\Xi^{a}_{\alpha_{12}} - P^{\nu_1}P_{(\alpha_2}\Xi^{a}_{\alpha_1)\nu_1} + 4\Xi^{\text{quad}}\end{aligned}$$

The uniqueness factor that arises in the basic permutations

$$U_{\alpha_2\nu_1} = \left(4\Upsilon_{\nu_1\alpha_2} + 2q_{\nu_1}\Upsilon_{\alpha_2} + 2q_{\alpha_2}\Upsilon_{\nu_1} + q_{\alpha_2}q_{\nu_1}\Upsilon\right), \qquad \text{(F.9)}$$

its explicit expression reads

$$\begin{aligned}U_{\alpha_2\nu_1} &= -\frac{1}{3}\theta_{\nu_1\alpha_2}\left(2\Delta^{\rho}_{2\rho} + i/\pi\right) &\text{(F.10)}\\ &+\frac{1}{9}\left(3P^{\nu_2}P^{\nu_3} + q^{\nu_{23}}\right)\left[3\Sigma^{\rho}_{4\rho\nu_1\alpha_2\nu_{23}} - 8\Box_{3\nu_1\alpha_2\nu_{23}} - g_{\nu_1\alpha_2\nu_{23}}\Delta^{\rho}_{2\rho}\right]\\ &+\frac{1}{18}\left(3P^{\nu_2}P^{\nu_3} + q^{\nu_{23}}\right)\left[g_{(\alpha_1\alpha_2}\left(2\Box^{\rho}_{3\rho\nu_{23})} - 2\Delta_{2\nu_{23})} - g_{\nu_{23})}\Delta^{\rho}_{2\rho}\right)\right]\\ &-\frac{1}{2}\left(P^2+q^2\right)\left[2\left(\Box^{\rho}_{3\rho\nu_1\alpha_2} - \Delta_{2\nu_1\alpha_2}\right) - g_{\nu_1\alpha_2}\Delta^{\rho}_{2\rho}\right]\\ &-P_{\alpha_2}P^{\nu_2}\left[2\left(\Box^{\rho}_{3\rho\nu_1\nu_2} - \Delta_{2\nu_1\nu_2}\right) - g_{\nu_1\nu_2}\Delta^{\rho}_{2\rho}\right]\\ &-P_{\nu_1}P^{\nu_2}\left[2\left(\Box^{\rho}_{3\rho\alpha_2\nu_2} - \Delta_{2\alpha_2\nu_2}\right) - g_{\alpha_2\nu_2}\Delta^{\rho}_{2\rho}\right]\\ &+4\left[W^{\rho}_{2\rho\nu_1\alpha_2} - 2\Delta_{1\nu_1\alpha_2} + 2g_{\nu_1\alpha_2}I_{\text{quad}} - 2m^2\left(\Delta_{2\nu_1\alpha_2} + g_{\nu_1\alpha_2}I_{\log}\right)\right].\end{aligned}$$

In the massless limit and independent of unique or vanishing surface terms

$$U_{\alpha_2\nu_1} = -\frac{1}{3}\theta_{\nu_1\alpha_2}\left(2\Delta^{\rho}_{2\rho} + \frac{i}{\pi}\right) = -\frac{1}{3}\theta_{\nu_1\alpha_2}\Upsilon \qquad \text{(F.11)}$$

$$\begin{aligned}U_{\alpha_1\alpha_2} &= -\frac{1}{3}\theta_{\alpha_1\alpha_2}\Upsilon + \frac{1}{9}\left(3P^{\nu_{12}} + q^{\nu_{12}}\right)\Xi^{(b)}_{\alpha_1\alpha_2\nu_{12}} - P^{\nu_1}P_{(\alpha_2}\Xi^{(a)}_{\alpha_1)\nu_1} &\text{(F.12)}\\ &+\frac{1}{18}\left(3P^{\nu_{12}} + q^{\nu_{12}}\right)g_{(\alpha_1\alpha_2}\Xi^{(a)}_{1\nu_{12})} - \frac{1}{2}\left(P^2+q^2\right)\Xi^{(a)}_{\alpha_{12}} + 4\Xi^{\text{quad}},\end{aligned}$$

where the definitions

$$\begin{aligned}\Xi^{\text{quad}}_{\alpha_{12}} &= \Box^{\rho}_{2\rho\alpha_{12}} + \frac{1}{2}g_{\alpha_{12}}\Delta^{\rho}_{1\rho} + \Delta_{1\alpha_{12}} + 2g_{\alpha_1\alpha_2}I_{\text{quad}} - 2m^2\left(\Delta_{2\alpha_{12}} + g_{\alpha_{12}}I_{\log}\right) &\text{(F.13)}\\ &= \Box^{\rho}_{2\rho\alpha_{12}} + \int\frac{\mathrm{d}^2k}{(2\pi)^2}\left[\frac{g_{\alpha_{12}}k^2}{D_\lambda} + \frac{2k_{\alpha_{12}}}{D_\lambda}\right] - 2m^2\left(\Delta_{2\alpha_{12}} + g_{\alpha_{12}}I_{\log}\right) &\text{(F.14)}\end{aligned}$$

$$\int \frac{\mathrm{d}^2k}{(2\pi)^2} \frac{2k_{\alpha_{12}}}{D_\lambda^2} = \Delta_{2\alpha_{12}} + g_{\alpha_{12}} I_{\log} \tag{F.15}$$

$$\int \frac{\mathrm{d}^2k}{(2\pi)^2} \frac{2k_{\alpha_{12}}}{D_\lambda} = \Delta_{1\alpha_{12}} + g_{\alpha_1\alpha_2} I_{\text{quad}} \tag{F.16}$$

$$\frac{1}{2} g_{\alpha_{12}} \Delta_{1\rho}^{\rho} + \Delta_{1\alpha_{12}} + 2g_{\alpha_1\alpha_2} I_{\text{quad}} = \int \frac{\mathrm{d}^2k}{(2\pi)^2} \left(g_{\alpha_{12}} k^2 + 2k_{\alpha_{12}}\right) \frac{1}{D_\lambda} \tag{F.17}$$

$$\Box_{2\rho\alpha_{12}}^{\rho} = \int \frac{\mathrm{d}^2k}{(2\pi)^2} \left[4k^2 k_{\alpha_{12}} - 6k_{\alpha_{12}} D_\lambda - g_{\alpha_{12}} k^2 D_\lambda\right] \frac{1}{D_\lambda^2}. \tag{F.18}$$

Where the quadratic form can be made null as

$$\Xi_{\alpha_{12}}^{\text{quad}} = \int \frac{\mathrm{d}^2k}{(2\pi)^2} \left[4\left(k^2 - m^2\right) - 4D_\lambda\right] \frac{k_{\alpha_{12}}}{D_\lambda^2} = 0. \tag{F.19}$$

F.2 Bilinears reductions and the accessible values to the uniqueness factor

Observing the expressions

$$\Xi_{\nu_{23}}^{(a)} = 2\int \frac{\mathrm{d}^2k}{(2\pi)^2} \left\{ \left[\frac{8k^2 k_{\nu_{23}}}{D_\lambda^3} - \frac{g_{\nu_{23}} k^2 + 6k_{\nu_{23}}}{D_\lambda^2}\right] - \left[\frac{2k_{\nu_{23}}}{D_\lambda^2} - \frac{g_{\nu_{23}}}{D_\lambda}\right] - g_{\nu_{23}} \left[\frac{k^2}{D_\lambda^2} - \frac{1}{D_\lambda}\right] \right\}. \tag{F.20}$$

If it is linear and bilinears are reduced, follow the solid resu

$$\Xi_{\nu_{23}}^{(a)} = 4m^2 \int \frac{\mathrm{d}^2k}{(2\pi)^2} \left[\frac{4k_{\nu_{23}}}{D_\lambda^3} - g_{\nu_{23}} \frac{1}{D_\lambda^2}\right] = -4m^2 \int \frac{\mathrm{d}^2k}{(2\pi)^2} \frac{\partial}{\partial k^{\nu_3}} \frac{k_{\nu_2}}{D_\lambda^2} \equiv 0. \tag{F.21}$$

The last passage involves defining a surface term that appears in 4D. Here it is finite and indisputably zero.

As the higher rank term, they appear in the violations of RAGFs and unicity of odd amplitudes

$$\Xi_{\mu_{1234}}^{(b)} = \left[3\Sigma_{4\rho\mu_{1234}}^{\rho} - 8\Box_{3\mu_{1234}} - g_{\mu_{1234}} \Delta_{2\rho}^{\rho}\right], \tag{F.22}$$

we will have for the first term

$$3g^{\mu_{12}} \Sigma_{4\mu_{123456}} = \int \frac{\mathrm{d}^2k}{(2\pi)^2} \left[\frac{144k^2 k_{\mu_{3456}}}{D_\lambda^4} - \frac{8\left[10k_{\mu_{3456}} + k^2 g_{(\mu_{34}} k_{\mu_{56})}\right]}{D_\lambda^3}\right], \tag{F.23}$$

using the formula $g^{\mu_{12}} g_{(\mu_{12}} k_{\mu_{3456})} = 10k_{\mu_{3456}} + k^2 g_{(\mu_{34}} k_{\mu_{56})}$ and the definitions

$$8\Box_{3\mu_{3456}} = \int \frac{\mathrm{d}^2k}{(2\pi)^2} \left[\frac{64k_{\mu_{3456}}}{D_\lambda^3} - \frac{8g_{(\mu_{34}} k_{\mu_{56})}}{D_\lambda^2}\right] \tag{F.24}$$

$$g_{(\mu_{12}} g_{\mu_{34})} \Delta_{2\rho}^{\rho} = g_{(\mu_{12}} g_{\mu_{34})} \int \frac{\mathrm{d}^2k}{(2\pi)^2} \left[\frac{2k^2}{D_\lambda^2} - \frac{2}{D_\lambda}\right], \tag{F.25}$$

it is obtained the result

$$\begin{aligned}&\left[3g^{\mu_{12}}\Sigma_{4\mu_{123456}}-8\Box_{3\mu_{3456}}-g_{(\mu_{12}}g_{\mu_{34})}\Delta_{2\rho}^{\rho}\right] \\ =&\int\frac{\mathrm{d}^2k}{(2\pi)^2}\left\{\left[\frac{144k_{\mu_{3456}}}{D_\lambda^3}-\frac{8g_{(\mu_{34}}k_{\mu_{56})}}{D_\lambda^2}-\frac{2g_{(\mu_{12}}g_{\mu_{34})}}{D_\lambda}\right]\frac{k^2}{D_\lambda}\right. \\ &\left.-\left[\frac{144k_{\mu_{3456}}}{D_\lambda^3}-\frac{8g_{(\mu_{34}}k_{\mu_{56})}}{D_\lambda^2}-\frac{2g_{(\mu_{12}}g_{\mu_{34})}}{D_\lambda}\right]\right\}.\end{aligned} \tag{F.26}$$

Reducing bilinears by adding and subtracting the mass makes obtaining the identity

$$\frac{k^2}{k^2-m^2}=1+\frac{m^2}{k^2-m^2}. \tag{F.27}$$

We reach at

$$\begin{aligned}&\left[3g^{\mu_{12}}\Sigma_{4\mu_{123456}}-8\Box_{3\mu_{3456}}-g_{(\mu_{12}}g_{\mu_{34})}\Delta_{2\rho}^{\rho}\right] \\ =&\ m^2\int\frac{\mathrm{d}^2k}{(2\pi)^2}\left\{\frac{144k_{\mu_{3456}}}{D_\lambda^4}-\frac{8g_{(\mu_{34}}k_{\mu_{56})}}{D_\lambda^3}-\frac{2g_{(\mu_{12}}g_{\mu_{34})}}{D_\lambda^2}\right\}.\end{aligned} \tag{F.28}$$

Mass terms do not vanish identically; what remains are precisely convergent surface terms

$$\begin{aligned}&\left[3g^{\mu_{12}}\Sigma_{4\mu_{123456}}-8\Box_{3\mu_{3456}}-g_{(\mu_{12}}g_{\mu_{34})}\Delta_{2\rho}^{\rho}\right] \\ =&\ m^2\int\frac{\mathrm{d}^2k}{(2\pi)^2}\left\{12\left[\frac{12k_{\mu_{3456}}}{D_\lambda^4}-\frac{g_{(\mu_{34}}k_{\mu_{56})}}{D_\lambda^3}\right]+4\frac{g_{(\mu_{34}}k_{\mu_{56})}}{D_\lambda^3}-\frac{g_{(\mu_{12}}g_{\mu_{34})}}{D_\lambda^2}-\frac{g_{(\mu_{12}}g_{\mu_{34})}}{D_\lambda^2}\right\},\end{aligned} \tag{F.29}$$

these terms own integrands that are typical of four dimensions. Integrating in 2D they are precisely zero

$$\Delta_{3;\mu_{ij}}=-\int\frac{\mathrm{d}^2k}{(2\pi)^2}\frac{\partial}{\partial k^{\mu_i}}\frac{k_{\mu_j}}{D_\lambda^2}=\int\frac{\mathrm{d}^2k}{(2\pi)^2}\left[\frac{4k_{\mu_{ij}}}{D_\lambda^3}-\frac{g_{\mu_{ij}}}{D_\lambda^2}\right]\equiv 0 \tag{F.30}$$

$$\Box_{4;\mu_{3456}}=-\frac{1}{2}\sum_{i=1}^{4}\int\frac{\mathrm{d}^2k}{(2\pi)^2}\frac{\partial}{\partial k^{\mu_i}}\frac{k_{\mu_1\cdots\hat{\mu}_i\cdots\mu_4}}{D_\lambda^3}=\int\frac{\mathrm{d}^2k}{(2\pi)^2}\left[\frac{12k_{\mu_{3456}}}{D_\lambda^4}-\frac{g_{(\mu_{34}}k_{\mu_{56})}}{D_\lambda^3}\right]\equiv 0 \tag{F.31}$$

thereby

$$\left[3g^{\mu_{12}}\Sigma_{4\mu_{123456}}-8\Box_{3\mu_{3456}}-g_{(\mu_{12}}g_{\mu_{34})}\Delta_{2\rho}^{\rho}\right]=m^2\left[12\Box_{4\mu_{3456}}+g_{(\mu_{34}}\Delta_{3\mu_{56})}\right]=0 \tag{F.32}$$

if the total derivative character of the expression is desired, we can also write in the form

$$\begin{aligned}&\left[3g^{\mu_{12}}\Sigma_{4\mu_{123456}}-8\Box_{3\mu_{3456}}-g_{(\mu_{12}}g_{\mu_{34})}\Delta_{2\rho}^{\rho}\right] \\ =&\ m^2\int\frac{\mathrm{d}^2k}{(2\pi)^2}\left\{-6\sum_{i=1}^{4}\frac{\partial}{\partial k^{\mu_i}}\frac{k_{\mu_1\cdots\hat{\mu}_i\cdots\mu_4}}{D_\lambda^3}-g_{(\mu_3\mu_4}\frac{\partial}{\partial k^{\mu_5}}\frac{k_{\mu_6)}}{D_\lambda^2}\right\}.\end{aligned} \tag{F.33}$$

Quadratic term in the Uniqueness factor: We assume bilinear reduction this term cancels identically independent from the definition of the quadratic scalar

$$U_{\alpha_{12}}^{\text{quad}}=\int\frac{\mathrm{d}^2k}{(2\pi)^2}\left\{\frac{16k^2k_{\alpha_{12}}}{D_\lambda^2}-\frac{16k_{\alpha_{12}}}{D_\lambda}\left[1-m^2\frac{1}{D_\lambda}\right]\right\} \tag{F.34}$$

in other words $U_{\alpha_{12}}^{\text{quad}}=0$.

Bibliography

[1] Fukuda, H.; Miyamoto, Y. On the γ-decay of neutral meson. *Prog. Theor. Phys.* **1949,** 4, 347-357.

[2] Steinberger, J. On the Use of Subtraction Fields and the Lifetimes of Some Types of Meson Decay. *Phys. Rev.* **1949**, 77, 1180-1186.

[3] Schwinger, J. On Gauge Invariance and Vacuum Polarization. *Phys. Rev.* **1951,** 82, 664-679.

[4] Rosenberg, L. Electromagnetic Interactions of Neutrinos. *Phys. Rev.* **1963**, 129, 2786.

[5] Johnson, K. γ_5 Invariance. *Phys. Lett.* **1963**, 5, 253.

[6] Adler, S. L. Axial-Vector vertex in spinor electrodynamics. *Phys. Rev.* **1969,** 177, 2426-2438.

[7] Bardeen, W. A. Anomalous Ward identities in spinor field theories. *Phys. Rev.* **1969**, 184, 1848-1857.

[8] Bell, J. S.; Jackiw, R. A PCAC puzzle: $\pi^0 \rightarrow \gamma\gamma$ in the σ-model. *Nuovo Cim. A* **1969,** 60, 47-61.

[9] Bastianelli, F.; Van Nieuwenhuizen, P. *Path Integrals and Anomalies in Curved Space,* 1st ed.; Cambridge University Press: Cambridge: Great Britain, 2006.

[10] Vassilevich, D.V. Heat kernel expansion: user's manual. *Physics Reports* **2003**, 388, 279-360.

[11] Bertlmann, R. A. *Anomalies in Quantum Field Theory*, 1st ed.; Oxford University Press Inc: New York, United States, 1996.

[12] Lee, W. B.; Justin, Z. J. Spontaneously broken gauge symmetries II. Perturbation theory and renormalization. *Phys. Rev. D* **1972**, 4, 3137-3155.

[13] Gross, D. J.; Jackiw, R. Effect of anomalies on quasi-renormalizable theories. *Phys. Rev. D* **1972**, 6, 477-493.

[14] Delbourgo, R.; Salam, A. The Gravitational Correction to PCAC. *Physics Letters* **1972,** 40B, 381-382.

[15] Kimura,T. Divergence of Axial-Vector Current in the Gravitational Field. *Progress of Theoretical Physics*, **1969**, 42, 1191-1205.

[16] Alvarez-Gaumé, L.; Witten, E. Gravitational Anomalies. *Nucl. Phy. B.* **1983**, 234, 269-330.

[17] Capper, D. M.; Duff M.J. Trace Anomalies in Dimensional Regularization. *Nuovo Cim.* **1974,** 23A, 173-183.

[18] Duff, M. J. Twenty years of the Weyl anomaly. *Classical and Quantum Gravity* **1994**,11,1387-1403.

[19] Bonora, L.; Giaccari, S.; Souza B. L. de. Trace anomalies in chiral theories revisited . JHEP **2014,** 07, 117.

[20] Bonora, L.; Pereira, A. D.; Souza B. L. de. Regularization of energy-momentum tensor correlators and parity-odd terms. JHEP **2015**, 06**,** 024.

[21] Bonora, L.; Cvitan, M.; Prester, P. D.; Pereira, A. D.; Giaccari, S.; Štemberga, T. Axial gravity, massless fermions and trace anomalies. *Eur. J. Phys. C.* **2017**, 77, 511.

[22] Bardeen, W. A.; Zumino, B. Consistent and Covariant Anomalies in Gauge and Gravitational Theories. *Nucl. Phy. B.* **1984,** 244, 421-453.

[23] Jackiw, R.; Johnson, K. Anomalies of the axial-vector current. *Phys. Rev. D* **1969**, 182, 1459-1469.

[24] Bollini, C. G.; Giambiagi, J. J. Dimensional renormalization: The number of dimensions as a regularizing parameter. *Phys. Lett. B* **1972**, 40**,** 566.

[25] 't Hooft, G.; Veltman, M. Regularization and renormalization of gauge fields. *Nucl. Phys. B* **1972**, 44**,** 189.

[26] Slavnov, A. A. Invariant regularization of gauge theories. *Theor. Math. Phys.* **1972**, 13, 1064-1066.

[27] Bakeyev, T. D.; Slavnov, A. A. Higher covariant derivative regularization revisited. *Mod. Phys. Lett. A* **1996**, 11, 1539-1554.

[28] Águila, F. del.; Pérez-Victoria, M. Differential Renormalization of Gauge Theories. *Acta Physica Polonica B.* **1998,** 28, 2857-2863.

[29] Pittau, R. A four-dimensional approach to quantum field theories. *JHEP* **2012,** 11, 024.

[30] Gnendiger, C.; Signer, A.; Stöckinger, D.; Broggio, A.; Cherchiglia, A. L.; Driencourt-Mangin, F.; Fazio, A. R.; Hiller, B.; Mastrolia, P.; Peraro, T.; Pittau, R.; Pruna, G. M.; Rodrigo, G.; Sampaio, M.; Sborlini, G.; Bobadilla, W. J. T.; Tramontano, F.; Ulrich, Y.; Visconti, A. To d, or not to d: recent developments and comparisons of regularization schemes. *Eur. Phys. J. C* **2017**, 77, 471.

[31] Breitenlohner, P.; Maison D. Dimensional renormalization and the action Principle. *Commun. math. Phys.* **1977**, 52, 11–38.

[32] Jegerlehner, F. Facts of life with γ_5. *Eur. Phys. J. C* **2001,** 18**,** 673-679.

[33] Tsai, Er-C. Gauge invariant treatment of γ_5 in the scheme of 't Hooft and Veltman. *Phys. Rev. D* **2011**, 83, 025020.

[34] Tsai, Er-C. Maintaining gauge symmetry in renormalizing chiral gauge theories. *Phys. Rev. D* **2011**, 83, 065011.

[35] Ferrari, R. Managing γ_5 in Dimensional Regularization II: the Trace with more γ_5's. *Int. J. Theor. Phys.* **2017,** 56, 691-705.

[36] Bruque, A. M.; Cherchiglia, A. L.; Pérez-Victoria, M. Dimensional regularization vs methods in fixed dimension with and without γ_5. *JHEP* **2018**, 08 109.

[37] Treiman, S. B.; Jackiw, R.; Zumino, B.; Witten, E. *Current algebra and anomalies.* Princeton University Press: Princeton, New Jersey, 1985.

[38] Cheng, T.P.; Li, L.F. *Gauge theory of elementary particle physics, 1st ed.;* Oxford University Press: Oxford, UK, 1984.

[39] Sterman, G. *An Introduction to Quantum Field Theory*, 1st ed.; Cambridge University Press: Cambridge, Great Britain, 1993; pp. 94–98.

[40] Ebani, L.; Girardi, T. J; Thuorst, J. F. Symmetries in one loop solutions: The AV, AVV, and AVVV diagrams, from 2D, 4D, and 6D dimensions and the role of breaking integration linearity. *arXiv* **2022,** arXiv:2212.03309.

[41] Battistel, O. A.; Dallabona,G.; Fonseca, M. V.; Ebani, L. Can Really Regularized Amplitudes Be Obtained as Consistent with Their Expected Symmetry Properties?. *Journal of Modern Physics* **2018**, 9, 1153-1178.

[42] Battistel, O. A. A new strategy for manipulations e calculations involving divergences in QFT. Ph.D. Thesis, Universidade Federal de Minas Gerais, Belo Horizonte, Minas Gerais, Brazil, 1999.

[43] Battistel, O. A.; Fonseca, M. V. S.; Dallabona, G. Anomalies in finite amplitudes: Two-dimensional single axial-vector triangle. *Phys. Rev. D* **2012**, 85, 085007.

[44] Battistel, O. A.; Traboussy, F.; Dallabona, G. Anomalies in finite amplitudes: Two-dimensional single and triple axial-vector triangles. *Int. J. Mod. Phys. A* **2018** 33**,** 1850136.

[45] Battistel, O. A.; Dallabona, G. From arbitrariness to ambiguities in the evaluation of perturbative physical amplitudes and their symmetry relations. *Phys. Rev. D* **2002***,* 65, 125017.

[46] Battistel, O. A.; Dallabona, G. Anomalies dismissed of ambiguities and the neutral pion decay. *J. Phys. G: Nucl. Part. Phys.* **2002*,*** 28, 2539.

[47] Fonseca, M. V. S.; Dallabona, G.; Battistel, O. A. Perturbative calculations in space time having extra dimensions: The 6D single axial box anomaly. *Int. J. Mod. Phys. A* **2014**, 29, 1450168.

[48] Fonseca, M. V. S.; Girardi, T. J.; Dallabona, G.; Battistel, O. A. Ambiguities and symmetry relations in five-dimensional perturbative calculations: The explicit evaluation of the QED_5 vacuum polarization tensor. *Int. J. Mod. Phys. A* **2013,** 28, 1350135-1350160.

[49] Battistel, O. A.; Dallabona, G. Consistency and universality in odd and even dimensional space time QFT perturbative calculations. *Int. J. Mod. Phys. A* **2014**, 29, 1450068.

[50] Viglioni, A. C. D.; Cherchiglia, A. L.; Vieira, A. R.; Hiller, B.; Sampaio, M. γ_5 algebra ambiguities in Feynman amplitudes: Momentum routing invariance and anomalies in $D = 4$ and $D = 2$. *Phys. Rev. D* **2016,** 94, 065023.

[51] Vieira, A. R.; Cherchiglia, A. L.; Sampaio, M. Momentum routing invariance in extended QED: Assuring gauge invariance beyond tree level. *Phys. Rev. D* **2016**, 93, 025029.

[52] Ferreira, L. C.; Cherchiglia, A. L.; Hiller, B.; Sampaio, M.; Nemes, M. C. Momentum routing invariance in Feynman diagrams and quantum symmetry breakings, *Phys. Rev. D* **2012**, 86, 025016.

[53] Porto, J. S.; Vieira, A. R.; Cherchiglia, A. L.; Sampaio, M. Hiller, B. On the Bose symmetry and the left- and right-chiral anomalies. *Eur. Phys. J. C*, **2018,** 78, 160.

[54] Battistel, O. A.; Dallabona, G. A systematization for one-loop 4D Feynman integrals. *Eur. Phys. J. C* **2006**, 45, 721.

[55] Battistel, O. A.; Dallabona, G. A Systematization for One-Loop 4D Feynman Integrals-Different Species of Massive Fields. *Journal of Modern Physics*, **2012**, 3, 1408-1449.

[56] Sun, Y.; Chang H-R. One loop integrals reduction. *Chinese Physics C* **2012, 36,** 1055-1064.

[57] Langouche, F. Gravitational and Lorentz anomalies in two dimensions. *Phys. Lett.* **1984,** 148B, 93-98.

[58] Leutwyler, H.; Mallik, S. Gravitational anomalies. *Z. Phys. C* **1986**, 33, 205-226.

[59] Hwang, D. S. Gauge and gravitational anomalies in two dimensions. *Phys. Rev. D* **1987,** 35, 1268-1279.

[60] Berger, T. Fermions in two (1+1)-dimensional anomalous gauge theories: The chiral Schwinger model and the chiral quantum gravity. *DESY* **1990**, 90-084.

[61] Bertlmann R. A.; Kohlprath, E. Two-Dimensional Gravitational Anomalies, Schwinger Terms, and Dispersion Relations. *Annals of Physics* **2001,** 288, 137-163.

[62] Bertlmann R. A.; Kohlprath, E. Gravitational anomalies in a dispersive approach. *Nucl. Phy. B* **2001,** 96 293-298.

[63] Wit, B. de.; Smith, J. *Field theory in particle physics. Volume 1.* 1st ed.; North-Holland Physics Publishing: Amsterdam: The Netherlands, 1986; pp. 457–466.

[64] Scharf, G. *Finite Quantum Electrodynamics: The Causal Approach*, 3rd ed.; Dover Publications Inc: Mineloa, New York, 2014.

[65] Aste, A.; Arx, C. von; Scharf, G. Regularization in quantum field theory from the causal point of view. *Progress in Particle and Nuclear Physics.* **2010**, 64, 61–119.

[66] Epstein, H.; Glaser, V. The role of locality in perturbation theory. *Ann. IHP, Phys. théor.* **1973**, 19, 211-295.

[67] Aste, A. Two-Loop Diagrams in Causal Perturbation Theory. *Ann. Phys. (N. Y.)* **1997**, 257, 158–204.

[68] Aste, A.; Trautmann, D. Finite calculation of divergent self-energy diagrams. *Can. J. Phys.* **2003**, 8, 1433–1445.

[69] Aste, A. Finite Field Theories and Causality. *PoS LC2008* **2008**, 001.

[70] Battistel, O. A.; Mota, A. L.; Nemes,M. C. Consistency Conditions for 4-D Regularizations. *Mod. Phys.Lett. A* **1998**, 13, 1597-1610.

[71] Battistel,O. A.; Nemes,M. C. Consistency in regularizations of the gauged NJL model at the one loop level. *Phys. Rev. D* **1999**, 59, 055010.

[72] Battistel, O. A. From arbitrariness to anomalies in two-dimensional perturbative calculations. *J. Phys. G: Nucl. Part. Phys.* **2004,**30, 543–564.

[73] Ma, Y-L.; Wu, Y-L.; Anomaly and Anomaly-Free Treatment of QFT's Based on Symmetry-Preserving Loop Regularization. *Int. J. Mod. Phys. A* **2006,** 21, 6383-6456.

[74] Elias, V.; McKeon, G.; Mann, R. B. VVA Triangle graph ambiguities in four and N dimensions. *Nucl. Phys. B* **1983,** 229, 487-498.

[75] Chowdhury, A. M.; McKeon, D. G. C.; Mann, R. B. Ambiguities of the chiral-anomaly graph in higher dimensions. *Phys. Rev. D* **1986,** 33, 3090.

[76] Bonora, L.; Cvitan, M.; Prester, P. D.; Lima de Souza, B.; Smolic, I. Massive fermion model in 3d and higher spin currents. JHEP **2016**, 05, 072.

[77] Bonora, L. Perturbative and Non-Pertrubative Trace Anomalies. *Symmetry* **2021**, 13, 1292.