DISORDERED SYSTEMS: FROM THE REPLICA METHOD TO THE DISTRIBUTIONAL ZETA-FUNCTION APPROACH

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ABSTRACT

We introduce a new mathematical rigorous technique using replica fields for computing the average free energy of disordered systems with quenched randomness. The basic tool of this technique is the distributional zeta function from which we can write the average free energy of a system as the sum of two contributions. The first one is a series in which all the integer moments of the partition function of the model contribute. The the second one can be made as small as desired. As applications of this approach we study three models.

Although the main focus in this thesis is to study some disordered models under the magnifying glass of that alternative technique, we start discussing finite-size effects in a disordered $\lambda\varphi^4$ model defined in a d-dimensional Euclidean space using the replica approach, the scalar field is coupled to a quenched random field and satisfies periodic boundary conditions along one dimension. In this scenario, we examine finite-size effects in the one-loop approximation for $d = 3$ and $d = 4$. We show that in both cases there is a critical length where the system develops a second-order phase transition. Additionally, by considering the composite field operator method we obtain that the size-dependent squared mass is a decrescent function.

In the framework of the distributional zeta function approach, at the beginning we consider the field theory formulation for directed polymers and interfaces in presence of quenched disorder. We obtain similar results to those of the replica method with the replica anzats. After this, we follow discussing a disordered d-dimensional Euclidean $\lambda\varphi^4$ model where the scalar field is coupled to an external random field. In each replica partition function, or the moments of the partition function, we show the emergence of the spontaneous symmetry breaking mechanism. Besides, we discuss finite temperature effects considering periodic boundary condition in one Euclidean dimension, in the low temperature regime we prove the existence of N instantons in the model.

Finally, we study a disordered $\lambda\varphi^4 + \rho\varphi^6$ Landau-Ginzburg model defined in a d-dimensional Euclidean space. We prove that the average free energy as expressed in the distributional zeta function approach can be used to represent a system with multiple ground states with different order parameters. For low temperatures, we show the presence of metastable equilibrium states in some replica fields for a range of values of the physical parameters. To end, going beyond the mean-field approximation, the one-loop renormalization of this model is performed in the leading order replica partition function.

RESUMO

Nesta tese apresenta-se uma técnica matematicamente rigorosa para calcular a energia livre media de sistemas com desordem *quenched*, cujo elemento fundamental é o que os proponentes denominam função zeta distribucional. A partir desse novo objeto matemático é possível escrever a energia livre media desse tipo de sistemas como a soma de duas contribuições, a primeira é representada por uma serie conformada por todos e cada um dos momentos da função de partição, a outra, por sua vez, pode considerar-se desprezível. Com o intuito de analisar as consequências da implementação dessa nova abordagem aqui estudamos três modelos diferentes de sistemas desordenados na formulação continua.

Embora o foco principal da tese seja estudar modelos com desordem sob a lente da técnica acima referida, inicialmente usamos o método de réplicas para discutir efeitos de tamanho finito no modelo $\lambda\varphi^4$ definido num espaçõ Euclideano d-dimensional onde o campo escalar satisfaz condições de contorno periódicas e é acoplado linearmente a um campo externo desordenado. Assim, examinando os efeitos de tamanho finito usando à aproximação a um *loop* para $d = 3$ e $d = 4$ mostramos que em ambos os casos existe um comprimento crítico onde o sistema sofre uma transição de fase de segunda ordem. De modo similar, o uso do método do operador de campo composto indica que o quadrado da masa renormalizada decresce em função do radio de compactificação.

No que se refere à função zeta distribucional, primeiramente consideramos a formulação de campo da teoria de polímeros e interfaces em presença de uma desordem *quenched*. Os resultados que nós obtemos sÃ£o similares ao resultados proporcionados pelo método de réplicas sem quebra de simetria de réplicas. Em seguida, discutimos o modelo $\lambda\varphi^4$ definido num espaçõ Euclideano d-dimensional onde o campo escalar é acoplado linearmente a um campo externo desordenado. A este respeito, a função de partição de cada conjunto de replicas, ou equivalentemente os momentos da função de partição, exibem o mecanismo de quebra espontânea de simetria. Ao discutir efeitos de temperatura finita nesse modelo através da impocisão de condições de contorno periódicas sobre campo ao longo de uma das dimensões Euclideanas, encontramos a existência de N *instantons* no regime de baixas temperaturas.

Finalmente, estudamos o modelo de Landau-Ginzburg $\lambda\varphi^4 + \rho\varphi^6$ definido num espaçõ Euclideano d-dimensional. Aqui mostramos que a expressão da energia livre media derivada da função zeta distribucional pode ser usada para representar um sistema com múltiplos estados fundamentais, cada um com parâmetro de ordem diferente.

Em baixas temperaturas, mostramos a presença de estados de equilíbrio metaestável para alguns grupos de réplicas, dependendo dos valores de certos parâmetros físicos. No final, com o objetivo de ir além da aproximação de campo médio, a renormalização do modelo a um *loop* é realizada.

PUBLICATIONS

The conceptual scheme and figures in this theses have already introduced in the following articles:

- R. Acosta Diaz and N. F. Svaiter. Finite-size effects in disordered $\lambda\varphi^4$ model. *International of Modern Physics B*, Vol 30, 1650207 (2016).
- R. Acosta Diaz, C.D. Rodriguez-Camargo and N. F. Svaiter. Free energy of polymers and interfaces in random media. $arXiv$: 1609.07084 (2017).
- R. Acosta Diaz, G. Menezes, N. F. Svaiter and C.A.D. Zarro. Spontaneous symmetry breaking in replica field theory. *Physical Review D*, Vol 96, 065012 (2017).
- R. Acosta, G. Krein, N. F. Svaiter and C. A. D. Zarro. Disordered $\lambda\varphi^4 + \rho\varphi^6$ Landau-Ginzburg model. *Physical Review D* Vol 97, 065017 (2018).

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1 INTRODUCTION

Disordered systems have been investigated for decades in statistical mechanics [1, 2, 3, 4], condensed matter [5] and even in gravitational physics [6, 7]. Spin glasses [8], disordered electronic systems [9] and directed polymers in random media [10, 11] are well known examples of such systems. Another important example of this sort of system it is the random field Ising model which was introduced in the literature by Y. Imra and S. Ma in the 70's [12]. Its Hamiltonian is analogous to the one of the classical Ising model but allowing a quenched random magnetic field interacting with each spin in the lattice. This model can be used, for instance, to represent binary fluids confined in porous media where the pore surfaces couple differently to the two components of a phase-separating mixture [13, 14].

The presence of disorder may significantly affect the critical behaviour of different systems in statistical mechanics [15, 16]. Consequently, the properties of the phase transitions in different models including the random field Ising model have been a question of debate for a long time. In this context, many researcher have discussed this model using statistical field theory methods, see for example Refs. [17, 18, 19, 20]. In terms of the language belonging to the Euclidean formulation of a field theory the generating functional of Schwinger functions can be interpreted as the partition function of a d-dimensional classical statistical field theory where the zero temperature limit corresponds to the usual classical partition function with infinite volume. Finite temperature effects in the generating functional of Schwinger functions corresponds to finite size effects in one direction for the classical partition function. Hence, instead of dealing with discrete elements, the random field Ising model can be analysed using a continuous description which is obtained in the limit of zero lattice spacing. In this case, the Hamiltonian of the discrete model can be replaced by an effective Hamiltonian corresponding to the Landau-Ginzburg model where the order parameter is a continuous field coupled to a quenched random field.

In the presence of disorder, the configurations of the classical ground state of the field are defined by a saddle-point equation whose solution depends on the particular realizations of the randomness degrees of freedom. This implies the existence of several local minima which make very difficult to implement a perturbative approach in a straightforward way. An established procedure to circumvented this situation is to average some extensive thermodynamic quantity with respect to the probability distribution of the disorder. For quenched disorder, one is mainly interested in averaging the free energy over the random field, which amounts to averaging the logarithm of the partition function. To do this, the well known technique is the replica trick. Here, n statistically independent copies (*replicas*) of the system are introduced but at the end of the procedure we have to take the limit $n \to 0$ which forces us to try to understand the behaviour of the permutation group of zero elements. Moreover, special procedures are sometimes required for obtaining sound physical results. For example, the average free energy in the Sherrington-Kirkpatrick model [21], which is the infinite-range version of the Edwards-Anderson model [22], exhibits a negative entropy at low temperatures, assuming a replica symmetric solution [23, 24]. The scheme of replica symmetry breaking was introduced to avoid this unphysical result [25, 26]. A step further was introduced by Parisi with another type of replica symmetry breaking solution, by choosing a suitable ultrametric parametrization of the replica matrix in the computation of the average free energy. This scheme describes many stable states with ultrametric structure in the phase space, in which the low-temperature phase consists of infinitely many pure thermodynamic states [27, 28, 29, 30].

Despite the success in the application of the replica method in disordered systems, some authors consider that a mathematical rigorous derivation to support this procedure is still lacking [31, 32, 33, 34, 35]. It is therefore natural to ask whether there exists a mathematically rigorous method, based on the use of replicas, for computing the average free energy of systems with quenched disorder. In this sense, in the Ref. [36] V. Dotsenko considered an alternative approach, where the summation of all integer moments of the partition function is used to evaluate the average free energy of the random energy model [37, 38]. Also, a replica calculation using only the integer moments of the partition function have been considered in Ref. [39]. Some alternative approaches to the replica method are the TAP appraoch (Thouless, Anderson and Palmer) [40, 2], the dynamical approach [41, 4] and the cavity method [42, 43].

On the other hand, Svaiter and Svaiter defined a new mathematical object associated with systems with quenched disorder, namely, a complex function which, due to its similarities with zeta-functions, was called the distributional zeta-function [44]. This terminology is in light with a probabilistic approach, since they introduced a probability distribution to define this zeta-function. They showed that the derivative of this function at the origin yields an expression for the quenched average free energy of a system. Therefore, here we will present this alternative approach and we study some models in order to discuss the physical consequences of its adoption to investigate disordered systems.

This thesis is organized as follows. In chapter 2 we will discuss generalities about disordered systems. The difference between annealed and quenched disorder will be made. Here, we will see what the replica method consists of and why it is necessary. The spin glass and the random field Ising model, the usual examples of systems with quenched disorder, will be presented. The distributional zeta function method will be introduced in this chapter.

Despite the aim of this thesis is to apply the distributional zeta function method and to analyse the consequences of it, in chapter 3 we discuss, in the framework of the replica method, the phase transition in a classical statistical system with disorder, being more specific, the $\lambda\varphi^4$ model in a d-dimensional Euclidean space where the scalar field, which satisfies periodic boundary conditions in one dimension, is coupled linearly to an external random field.

As a first application of the distributional zeta function method, in chapter 4 we will investigate the average free energy of directed polymers and interfaces in random media.

In chapter 5, we will study a d-dimensional Euclidean $\lambda\varphi^4$ model in the presence of a disorder field linearly coupled with the scalar field. A connection between spontaneous symmetry breaking mechanism and the structure of the replica space will be manifest in the context of the distributional zeta function method.

We will employ the distributional zeta function method in chapter 6 to explore the free energy landscape of the d-dimensional disordered Landau-Ginzburg $\lambda\varphi^4 + \rho\varphi^6$ model, where de disorder appears as mass contribution. It will showed that the average free energy represents a system with multiple ground states with different order parameters.

Finally, we will present some conclusions and perspectives in chapter 7. Appendices A, B and C are a brief presentation about Gaussian integrals, the Landau-Ginzburg model and the composite operator respectively.

2 DISORDERED SYSTEMS

The success of statistical mechanics in describing macroscopic properties of a large number of systems in condensed matter, is basically based on the translational invariance characterizing the structure of these. Initially, the studies carried out in this area gave emphasis to idealized homogeneous systems, such as localized spins and electron liquids in ideal impurity-free crystals, phase transitions and ordered states of homogeneous matter, and so on. Despite the lack of understanding of some systems that posses such characteristics, in a general way this trend continued until these types of systems were quite well understood as it is currently [56]. However, in real systems there are degrees of freedom corresponding to inhomogeneities or impurities that break the symmetries, leading consequently to the emergence of new properties which are much difficult to understand [57].

Here, we discuss generalities about systems with quenched disorder, we introduce briefly the spin glasses and the random filed Ising models as well as the replica method to treat with such systems. Additionally, we present the distributional zeta-function approach as an alternative to compute the average free energy in systems with quenched disorder.

2.1 ANNEALED AND QUENCHED DISORDER

Annealed disorder

In systems with annealed disorder the time scale of fluctuations of the degrees of freedom associated to impurities is much smaller than the measurement time over the system, they fluctuate on short time scales. Consequently, the disorder is in thermal equilibrium with the another degrees of freedom of the system, so that, both kind of variables (disordered and not disordered) are set on the same footing. If one wants to study the statistical properties of the whole system, it is necessary to compute the partition function by summing over all configurations of the original components (that are used to describe the pure system) and the impurities.

As an example, let us think about a pure piece of ferromagnetic material which is heated to its melting temperature and then cooled very slowly until to crystallize after adding some impurities. In this case, the impurities will be in thermal equilibrium with those degrees of freedom contemplated to describe the system without disorder, being then possible to use the Gibbs distribution to model the impurities

[58, 59]. In a general way, the partition function Z of the system can be written as

$$Z = \mathrm{Tr}_{(h,s)} e^{-\beta H(h,s)}, \tag{2.1}$$

where h and s stand for the impurity and the spin (or any other) degrees of freedom respectively, H is the Hamiltonian of the system, β is the inverse temperature, i.e. $1/T$[1]. The fact that the impurities variables are in thermal equilibrium with the other degrees of freedom of the system allows to treat them in the same way, which makes the study of systems with annealed disorder to be quiet simple. One can introduce an effective Hamiltonian H_{eff} by tracing over the impurity degrees of freedom h

$$e^{-\beta H_{eff}(S)} \equiv \mathrm{Tr}_{(h)} e^{-\beta H(h,s)}, \tag{2.2}$$

obtaining a disorder-free system with modified parameters, where the partition function becomes

$$Z = \mathrm{Tr}_{(s)} e^{-\beta H_{eff}(s)}. \tag{2.3}$$

As we can see, the thermodynamic properties of this class of system are obtained from the partition function which is traced over the impurity degrees of freedom in the same way that we trace over the thermal variables, remaining therefore within the context of the usual statistical mechanics. In spite of this, the annealed disorder has considerable effects on systems in which it appears, see for example the Refs. [60, 61, 62, 63, 64].

Quenched disorder

The degrees of freedom related to impurities in systems with quenched disorder evolve in different time scale from those characterizing the "clean" system. They fluctuate much more slowly, giving then the notion of remaining fixed or "frozen" in relation to the others, i.e. the disorder is static. Within an experimental context, the time scale of the observation (or measurement) is much smaller than the dynamical time scale of the impurities. If we take some experimental sample, the respective disordered variables assume a well defined (though unknown) time independent vale. This implies that each realization of the disorder corresponds to a unique realization of the random variables of the system, while its distribution describes the fluctuations between different realizations. Due to the nature of the quenched randomness, the impurity degrees of freedom are not in thermal equilibrium with the other degrees of freedom of the system, each disorder configuration of the system will be different from one another. In this case, the impurities are not allowed to move, the network structure

1 The units are chosen such that the Boltzmann constant $k = 1$.

is fixed, and the interactions are established. Therefore, in order to study the thermodynamic properties of the model, the partition function is determined for the specific disorder realizations, and only the state functions are averaged over distinct distributions of the impurities. According to this, the partition function

$$Z(h) = \mathrm{Tr}_{(s)} e^{-\beta H(h,s)}, \tag{2.4}$$

will depend on all of the impurities variables h and therefore the calculation of all thermodynamic quantities of the system become more difficult since it is not possible to proceed as in the case of the annealed disorder.

The distinction among the characteristics of the dynamical degrees of freedom from the disordered variables has important consequences when we want to define the thermodynamics properties of a system, since the thermal averages are not equivalent to averages over the disorder, as in the case of annealed disorder. Ideally, it is necessary to calculate quantities that determine the equilibrium of the system by computing averages over the Boltzmann measure. However, due to the presence of the disorder, it is only possible to evaluate quantities which are averaged also over the disorder distribution, leading us to think about what extended averaged quantities describe a single system.

Self-averaging

A disordered system has observables which vary when we go from one sample to another. More precisely, when we determine the outcomes of a sequence of experiments on a given observable we have in general different results from each another. Occur that extensive quantities, as the free energy, are interesting due to they have an essential attribute, to be *self-averaging* in the thermodynamics limit [65, 3]. Loosely speaking, this means that they take the same value for each realization of the disorder, which has a finite probability. In this case, sample to sample fluctuations are vanishing as the volume of the system is sent to infinity and the average value coincides with the typical one, i.e. the one assumed in a unique realization of the system. Differently, variables that are not self-averaging may fluctuate widely from one realization to another and when average over the disorder are done, some configurations with zero probability may give finite contributions.

The argument for averaging extensive quantities was initially given by Brout [66]. Let us think in a system which we can consider very large and divide it up into a large number of subsystems in such a way that each one is macroscopic and they have different set of disordered variables. If we assume the coupling between subsystem to be negligible, then the value of any extensible variable (normalized per unit volume or per degree of freedom) for the whole system is equal to the average of the values of this quantity over the subsystems. If the original system is quite huge, the number of subsystem is big enough such that the result of averaging in them will be different by only a small amount from the result of performing a complete average over all the realization of the disorder. One large system gives the same results for densities of extensive quantities as a configurational average.

Different ways to see the self-averaging have been consider. For instance, the fact that the free energy converges almost everywhere to the limiting quenched average free energy has been proved for spin systems with short-range and long-range interactions [67, 68]. For discussion about self-averaging in the statistical mechanics of lattice models see the Ref. [69]. See also Refs. [70, 71, 72] for discussions of self-averaging in phenomena related to polymers.

The average free energy

The free energy density $F(h)$ for a given disorder realization h is defined as

$$\begin{aligned} F(h) &= -\frac{1}{\beta V} \ln Z(h) \\ &= -\frac{1}{\beta V} \ln \operatorname{Tr}_{(s)} e^{-\beta H(h,s)}. \end{aligned} \tag{2.5}$$

where $Z(h)$ is the partition function of the model. The average over the probability distribution of the disorder is given by

$$\begin{aligned} F &= \mathbb{E}[F(h)] \\ &= \int dh P(h) F(h) \\ &= -\frac{1}{\beta V} \mathbb{E}[\ln Z(h)]. \end{aligned} \tag{2.6}$$

where $\mathbb{E}[\cdot]$ stands for the average over the disorder distribution. In the above expression it is necessary to perform the average of a logarithm which is not an easy task to do and quite unusual in statistical mechanics. The fact of having to deal with this situation is a direct consequence of the quenched nature of the disorder, which force us to average extensive quantities like the free energy rather than the partition function. Because of this, the Eq. (2.6) is customarily referred to as a *quenched average*. It is important to draw attention to the appearing of two different averages in the equation in mention keeping a given order. First, the thermodynamics average over the Boltzmann measure, which is used to obtain $F(h)$, followed then by the average over the disorder.

The probability distribution

Taking into account that we have no way to determine the values of the random variables for its different realizations, it is necessary then describe them through some probability distribution. It is assumed that the degrees of freedom characterizing the disorder exhibit no-long range correlations, therefore the probability distribution associated corresponds to a Gaussian, i.e,

$$P(h) = p \exp\left[-\frac{1}{2\sigma}\int d^d x \Big(h(x)\Big)^2\right], \tag{2.7}$$

where p is a normalization factor and the quantity σ is a small positive parameter that reflects the strength of the disorder. In this case we have a delta correlated random field where the related second moment or two-point correlation function is given by

$$\mathbb{E}[h(x)h(y)] = \sigma\delta^d(x-y). \tag{2.8}$$

As is characteristic for Gaussian distributions, we have that $\mathbb{E}[h(x)] = 0$. Another option for the probability distribution for the disorder is

$$P(h) = p \exp\left[-\frac{1}{2}\int d^d x d^d y h(x) V^{-1}(x-y) h(y)\right], \tag{2.9}$$

The respective two-point correlation function is

$$\mathbb{E}[h(x)h(y)] = V(x-y). \tag{2.10}$$

2.2 REPLICA METHOD

To compute the quenched average of the free energy defined by Eq. (2.6) is quite problematic because it does not exist a formal treatment to proceed. An indirect way to handle with this expression is to make use of the so-called *replica method* proposed by Edwards and Anderson to study the transition point observed experimentally in the susceptibility of dilute magnetic alloys [22]. This method depends entirely on a well-known elementary property of the logarithm function,

$$\ln z = \lim_{n\to 0}\frac{z^n - 1}{n}, \tag{2.11}$$

which allows us to write

$$\begin{aligned} F &= -\frac{1}{\beta V}\mathbb{E}[\ln Z(h)] \\ &= -\frac{1}{\beta V}\lim_{n\to 0}\left(\frac{\mathbb{E}[Z^n]-1}{n}\right) \\ &= -\frac{1}{\beta V}\lim_{n\to 0}\frac{\mathbb{E}[Z^n]}{n} \end{aligned} \tag{2.12}$$

As a result, the averaging of the logarithm is reduced to the computation of the average of Z^n. For an integer n, this approach implies that we need to construct the product of partition functions of n identical and non interacting copies or *replicas* of the original system and to evaluate the configurational average of the disorder before taking the limit $n \to 0$. Following this script, we can write

$$\begin{aligned} Z^n &= \mathrm{Tr}_{\left(s^{(a)}\right)} e^{-\beta \sum_{a=1}^{n} H\left(h, s^{(a)}\right)} \\ &= \sum_{\left(s^{(1)}\right)} \sum_{\left(s^{(2)}\right)} \cdots \sum_{\left(s^{(n)}\right)} e^{-\beta \sum_{a=1}^{n} H\left(h, s^{(a)}\right)}. \end{aligned} \tag{2.13}$$

where the superscript (a) is the replica index which denotes each replica and goes from 1 to n.

Observing Eq. (2.13) a conceptual difficulty can be noted. The index (a) is an integer number that must be sent to zero in the replica limit to keep the agreement with the Eq. (2.11) where n is a real number. This is a problematic situation resolved by explicitly writing the dependence on n of the replica partition function Z^n in such a way that it can be regarded as a continuous parameter [22]. This can be understood performing an analytic continuation on n, as integer, to the real line. This procedure is not mathematically rigorous and therefore it has been the source of many debates and much criticism over the years, as well as the motivation to propose the alternative method presented in this thesis. Despite this, it seems that the results obtained with this method are physical and therefore it has been assumed that the replica trick appropriately incorporates physical elements of the different systems where it is used. Nevertheless, it is a very appealing and widely accepted method which is deeply engrained and indispensable in the theory of glassy and other disordered systems systems.

2.3 SPIN GLASS MODEL

Spin glasses are disordered magnetic systems in which the interactions between the magnetic moments, the spins, are in conflict with each other due to some frozen-in structural disorder. As a consequence no regular long-range order ferromagnetic (or anti-ferromagnetic) state can be established. The theoretical description for this kind of system began with the initial proposal of Edwards and Anderson for whom the essential physics of spin glasses does not depend of the details of their microscopic interactions but rather in the competition between quenched ferromagnetic and anti-ferromagnetic interactions [22]. This disorder can arise from randomness in the structure or in the magnetic behaviour of the material.

The Edwards-Anderson model

The mathematical model introduced by Edward and Anderson in order to describe spin-glasses behaviour can be considered as the simplest generalization of the widely studied Ising model used to describe ferromagnets [73]. The Hamiltonian of a spin glass system composed by N spins S_i located at the sites i of a regular lattice can be written as[2]

$$\mathbf{H}(S_i; J_{ij}) = -\sum_{ij} J_{ij} S_i S_j - \sum_{i} h_i S_i, \tag{2.14}$$

where h_i is a magnetic field interacting locally with the spins and the sum is taken over all pairs (ij) of nearest neighbours spins. The interaction constant J_{ij} between the spins located at positions i and j are independent random variables defined by a Gaussian distribution $P(J_{ij})$ with mean zero value and variance $\frac{1}{N}$, that is[3]

$$P(J_{ij}) = \sqrt{\frac{N}{2\pi}} e^{-\frac{N}{2}(J_{ij})^2}. \tag{2.15}$$

The values $J_{ij} > 0$ and $J_{ij} < 0$ correspond to ferromagnetic and anti-ferromagnetic interactions respectively.

Frustration

The Edwards-Anderson model reproduces the two inherent properties that characterize spin glasses systems, namely, *quenched disorder* and *frustration*. Once the concept of quenched disorder has been understood, it is then necessary to discuss what is the notion of frustration. For this, take us into consideration an arrangement of three spins with interactions J_{12}, J_{23} and J_{13} between them. For the sake of simplicity, we assume that these interactions differ only in their signs, being equal in intensity. That being the case, it is possible to find two

2 Although the initial theoretical scheme to study spin glasses systems was proposed by Edwards and Anderson, the respective Hamiltonian was first written explicitly by Sherrington and Southern [74].

3 Here, we are considering the average value for the interaction constant being zero, *i.e.* $\langle J_{ij} \rangle = 0$. It is also possible take into consideration some value non null for this quantity, which it does not lead to any change in the system, simply we must to subtract such value from J_{ij} in the argument of the exponential in Eq. (2.15).

essentially different situations corresponding to the ground state. The ground state of this system is unique when all three interactions J_{12}, J_{23} and J_{13} are positive or when two of them are negative whereas the other one is positive[4]. On the other hand, the ground state would be degenerate if the product of these interactions is negative, which occurs when one of the interactions is negative, being the other two positive, or all three interactions are negative. We have that going from spin to spin the orientation of one of them is not satisfied with respect to the interactions with its neighbours. More generally, it is easy to see that if we take a closed spin chain C with an arbitrary number n of coupling, not all can be satisfied if the product of the spin-spin interactions along the chain is negative,

$$\prod_C J_{12}J_{23}...J_{n1} < 0 \quad \longrightarrow \quad \textit{frustation}. \tag{2.16}$$

So, any real lattice in two or more dimensions will have a complicated network of inter- penetrating frustrated loops making this matter a topic of current discussion.

The Sherrington-Kirkpatrick model

The Sherrington-Kirkpatrik model is the long-ranged version of the Edwards-Anderson to describe spin glass systems at low temperature [21]. It was proposed as a mean-field model, where all spins interact with each other. The Sherrington-Kirkpatrik Hamiltonian is written basically as in Eq. (2.14), that is

$$\mathbf{H}(S_i; J_{ij}) = -\sum_{ij} J_{ij} S_i S_j - \sum_i h_i S_i, \tag{2.17}$$

but the sum $\sum_{ij}$ run over all distinct pairs of spin, $N(N-1)/2$ of them. The fact that each spins interact with all the other make that the space structure (dimensionality, type of lattice, etc.) of this model is irrelevant for its properties. The space here is just the set of N sites in which the Ising spins are placed, and all these spins, in a sense, could be considered as the nearest neighbours. In the thermodynamic limit ($N \to \infty$) such a structure can be interpreted as the infinite dimensional lattice, making this property then that the mean-field approach be exact.

2.4 THE RANDOM FIELD ISING MODEL

The Random Field Ising Model (RFIM), initially introduced by Larkin in the early 70's [75] and later studied by Imry and Ma [76], is one of the most important and simplest models of great relevance for both

4 The uniqueness of the ground state in this case is valid except for the global change of signs of all the spins.

theoretical and experimental studies in statistical physics of systems with quenched disorder, being in a certain way complementary to the spin glass model. In this model we have the presence of a random external magnetic field which antagonizes the ordering induced by the ferromagnetic spin-spin interaction. The Hamiltonian of the RFIM of a system of N spins S_i located at points of a lattice is given by

$$\mathcal{H} = -\sum_{(i,j)} J_{ij} S_i S_j - \sum_i h_i S_j \tag{2.18}$$

where J_{ij} represent positive non-random interactions between the spins, typically restricted to nearest neighbour pairs (i,j) only, and the fields h_i are independent quenched random variables drawn with a Gaussian distribution

$$P(h_i) = \frac{1}{\sqrt{2\pi\sigma}} e^{-\frac{h_i^2}{2\sigma}}, \tag{2.19}$$

for which we have

$$\mathbb{E}(h_i) = 0, \quad \mathbb{E}(h_i h_j) = \sigma\delta_{ij}. \tag{2.20}$$

The symbol σ is the variance of the distribution and characterizes the strength of the disorder. In this sort of system there is competition between the long range order and the random ordering field, since the neighbouring spins tend to align parallel while the applied external field tries to fix each spin according to the sign of the local field.

In general, the Hamiltonian in Eq. (2.18) describes more or less any solid state system that has a transition with two degenerate ordered states, and contains frozen impurities that locally break the symmetry between these two states [16]. It is always important to keep in mind that due to the presence of the quenched disorder, two different kinds of average are present, the thermal average over the Boltzmann measure and the quenched average over the disorder distribution.

For a pure magnetic system the ordering of spins results from a competition between the energy interaction and the entropy. In one dimension, the entropy dominates over the energy except at absolute zero temperature and therefore spins are disordered at any finite temperature[5]. On other hand, in all dimensions $d > 1$ exist a ordered ferromagnetic phase for the case of the pure Ising model without external magnetic field. A second order phase transition take place at a given critical temperature T_c, below which the energy dominates over entropy and a state of long-range magnetic order can be established in such a way that the system has a separation between a high temperature paramagnetic phase from a low temperature ferromagnetic one. In most cases, thermal fluctuations break up the spin ordering more easily in lower dimensions. This leads to the notion of the lower

5 This may not hold if spin couplings are long ranged.

critical dimension, defined as the dimension above which an ordered phase is stable at finite temperature. Clearly, the critical dimension for the pure Ising model is $d = 1$. See Ref. [77] for a complete discussion about this matter.

The presence of a random external magnetic affect the ordering associated of the ferromagnetic exchange interactions. At low temperature the leading competition occurs between this kind energy, which contribute to the appearing of the long-ranged order, and the random field, which has a propensity to eliminate such order. The thermal fluctuations becomes less essential so that the critical behaviour at a possible phase transition would be ruled by the fixed point at zero temperature. When the strength of the disorder is sufficiently large compared to the coupling between spins, the random field energy is the dominant term in Eq. (2.18) leading to the system be completely disordered. This is mainly because the spins will be oriented according to their local fields h_i and turns therefore uncorrelated, regardless of dimension. In the opposite instance, that is for very weak random fields, the ferromagnetic ground-state becomes indeed unstable, one awaits that the transition temperature presents a decreasing behaviour as the disorder strength increases. Qualitatively, then the phase diagram show a paramagnetic phase for large disorder and temperature, and a ferromagnetic phase in the contrary situation. See Refs. [3, 1, 4].

At low enough dimensions the action of a weak disorder can interfere with the creation of the long-ranged ferromagnetic ordered phase. A quite strong argument has been given by Imry and Ma to explain how a robust random field can destroy a predominantly ferromagnetic environment [12]. Imagine a domain of size L in a ferromagnetic region. Reversing the spins inside this region require an energy cost due to the exchange interactions E_{exc} proportional to the domain wall area, therefore

$$E_{exc} \sim JL^{d-1}, \tag{2.21}$$

where d is the dimension of the physical system. For low temperatures and vanishing random fields, this is not favourable so that there will be no spontaneous formation of domains. The situations, however, is different if there are random fields present. There is an energy gain due the interaction with this fields in such a way that the corresponding energy E_{RF} is

$$E_{RF}^2 \sim \sigma L^d. \tag{2.22}$$

The total energy is then written as

$$E(L) \approx JL^{d-1} - \sqrt{\sigma L^d}, \tag{2.23}$$

and the fluctuations of h_i will always destroy the ferromagnetic state if

$$\frac{d}{2} > d - 1 \qquad \Rightarrow \qquad d < 2. \tag{2.24}$$

In $d = 2$ there is also an instability and therefore it does not exist any phase transition [78, 79]. For $d \geqslant 3$ there is a phase transition [80, 81].

Random field in the scalar Landau-Ginzburg model

A continuous description for the random field Ising model (a discrete model), generalized to m- component spins, is given by the Landau-Ginzburg model where the spin variables are replaced by multicomponent fields $\varphi_\mu(x)$, smooth functions in an Euclidean d-dimensional space, coupled to an external quenched random field $h_\mu(x)$. The respective Hamiltonian is written as [3, 4]

$$\begin{aligned}\mathcal{H}(\varphi; h) = \int d^d x \bigg[\frac{1}{2}\varphi_\mu(x)\Big(-\Delta + m_0^2\Big)\varphi_\mu(x) + \frac{\lambda}{4!}\Big(\varphi_\mu(x)\varphi_\mu(x)\Big)^2 \\ - h_\mu(x)\varphi_\mu(x)\bigg].\end{aligned} \tag{2.25}$$

where $\mu(1, ..., m)$ are the components of the fields, m_0 and λ are the bar mass and the bar coupling of the model respectively. The symbol Δ denotes the Laplacian in $\mathbb{R}^d$. From now on and without loss of generality we consider a disordered scalar field Landau-Ginzburg model, i.e, $\mu = 1$. Thus, the partition function, which has dependence on the disorder, is given by

$$Z(h) = \int [d\varphi] e^{-\mathcal{H}(\varphi, \mu)}. \tag{2.26}$$

Just to recall, in this expression $[d\varphi]$ is a formal Lebesgue measure. The random variables characterizing the disorder are defined by a Gaussian distribution as in the usual random field Ising model, namely

$$P(h) \sim \exp\left[-\frac{1}{2\,\sigma}\int d^d x \Big(h(x)\Big)^2\right]. \tag{2.27}$$

In this case, we have also a delta correlated random field where the two-point correlation function is

$$\mathbb{E}[h(x)h(y)] = \sigma\delta^d(x - y). \tag{2.28}$$

As a step to follow, it is necessary then to compute the h-dependent free energy $F(h)$ which, as previously said, is defined by

$$F(h) = -\frac{1}{\beta} \ln Z(h). \tag{2.29}$$

The ground state configuration for the system corresponds to the values of the field that minimize the free energy, usually defined by the saddle-point equation that in the presence of a quenched random field reads

$$(-\Delta + m_0^2)\,\varphi(x) + \frac{\lambda}{3!}\varphi^3(x) = h(x). \tag{2.30}$$

The solutions of the above equation depend on particular configurations of the quenched fields. The perturbation theory is an inappropriate procedure to be used in systems where the disorder defines a large number of local minima in the energy landscape, as occurs in this case [82]. One way to circumvent this problem consists in computing the average free energy over the disorder. According to this, we have

$$F = \int [dh] P(h) F(h). \tag{2.31}$$

Here, we are averaging the free energy $F(h)$ over all realizations of the random function $h(x)$ with the Gaussian distribution $P(h)$ given in Eq. (2.27). In order to implement the replica approach, we must consider the product of the partition functions Z^n of n identical and independent replicas, which corresponds to

$$Z^n = \int \prod_{i=1}^{n} [d\varphi_i] \exp\left\{ -\sum_{j=1}^{n} \int d^d x \left[\frac{1}{2}\varphi_j(x)\left(-\Delta + m_0^2\right)\varphi_j(x) + \frac{\lambda}{4!}\varphi_j^4(x) - h_j(x)\varphi_j(x) \right] \right\}. \tag{2.32}$$

Integrating over the disorder distribution we obtain the replica partition function Z_n, where

$$\begin{aligned} Z_n =& \mathbb{E}[Z^n] \\ =& \int [dh] \prod_{i=1}^{n} [d\varphi_i] \exp\left[\sum_{j=1}^{n} \int d^d x \left(-\frac{1}{2\sigma} h^2(x) + h(x)\varphi_j(x) \right) \right] \\ & \times \exp\left[-\sum_{j=1}^{n} \int d^d x \left(\frac{1}{2}\varphi_j(x)(-\Delta + m_0^2)\varphi_j(x) + \frac{\lambda}{4!}\varphi_j^4(x) \right) \right] \end{aligned} \tag{2.33}$$

As it can be seen immediately the integral with respect to the random variable h can be perform easily by means of a Gaussian integration, resulting in the following

$$Z_n = \int \prod_{i=1}^{n} [d\varphi_i] e^{-\mathcal{H}_{eff}(\varphi_i)}, \tag{2.34}$$

with

$$\mathcal{H}_{eff}(\varphi_i) = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\int d^d x \varphi_i(x)\left[(-\Delta + m_0^2)\delta_{ij} - \sigma\right]\varphi_j(y) + \frac{\lambda}{4!}\sum_{i=1}^{n}\int d^d x \varphi_i^4(x). \quad (2.35)$$

The saddle-point equations of the n replicas read

$$\left(-\Delta + m_0^2\right)\varphi_i(x) + \frac{\lambda}{3!}\varphi_i^3(x) = \sigma\sum_{j=1}^{n}\varphi_j(x). \quad (2.36)$$

If we take all the replicas the same, i.e. $\varphi_i(x) = \varphi(x)$, as it would be natural to think then the set of equations in Eq. (2.36) are reduced to

$$\left(-\Delta + \left(m_0^2 - n\sigma\right)\right)\varphi(x) + \frac{\lambda}{3!}\varphi^3(x) = 0, \quad (2.37)$$

so that by taking the limit $n \to 0$ this equation defines the ground state of a system without disorder, which has the trivial solution $\varphi(x) = 0$ for $m^2 > 0$. To obtain any non-trivial solution of Eq. (2.36) involving randomness, the fields $\varphi_i(x)$ in different replicas cannot be equal, in other words, the symmetry among replicas must be broken [83]. We will not go into this matter.

For convenience, instead of working in a coordinate space it is interesting to give a treatment in momentum space. Performing a Fourier transform in $\mathcal{H}_{eff}$, Eq. (2.35), we get

$$\mathcal{H}_{eff}(\varphi_i) = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\int \frac{d^d p}{(2\pi)^d}\varphi_i(p)\left[G_0\right]_{ij}^{-1}\varphi_j(-p) + \frac{\lambda}{4!}\sum_{i=1}^{n}\varphi_i^4, \quad (2.38)$$

where, in the quadratic part of $\mathcal{H}_{eff}(\varphi_i)$, the factor $\left[G_0\right]_{ij}^{-1}$ represents the inverse of the two-point correlation function in the tree-level approximation given by

$$\left[G_0\right]_{ij}^{-1}(p) = (p^2 + m_0^2)\delta_{ij} - \sigma. \quad (2.39)$$

To get the respective two-point correlation functions we have to invert this expression using the projector operators $(P_T)_{ij}$ and $(P_L)_{ij}$ defined respectively as

$$(P_T)_{ij} = \delta_{ij} - \frac{1}{n} \quad (2.40)$$

and

$$(P_L)_{ij} = \frac{1}{n}. \quad (2.41)$$

Expressing $\left[G_0\right]_{ij}^{-1}$ in terms of such operators, we write

$$\left[G_0\right]_{ij}^{-1}(p) = \left(p^2 + m_0^2\right)\left(\delta_{ij} - \frac{1}{n}\right) + \left(p^2 + m_0^2 - n\sigma\right)\frac{1}{n}. \tag{2.42}$$

Now, inverting this expression we obtain

$$\begin{aligned}\left[G_0\right]_{ij}(p) =& \frac{1}{p^2 + m_0^2}\left(\delta_{ij} - \frac{1}{n}\right) + \frac{1}{p^2 + m_0^2 - n\sigma}\left(\frac{1}{n}\right) \\ =& \frac{\delta_{ij}}{\left(p^2 + m_0^2\right)} + \frac{\sigma}{\left(p^2 + m_0^2\right)\left(p^2 + m_0^2 - n\sigma\right)}.\end{aligned} \tag{2.43}$$

As it was pointed out by De Dominicis and Giardina [4], the first term corresponds to the bare contribution to the connected two-point correlation function in absence of the random field while the second term is the contribution to the disconnected two-point correlation function which becomes connected after averaging in the random variable.

2.5 THE DISTRIBUTIONAL ZETA-FUNCTION METHOD

The central idea of the replica method is analytically to continue the replicated system to $n \longrightarrow 0$. Although great efforts have been made to give it support formally, such procedure is still quite controversial from a mathematical point of view. On the other hand, it should be noted that the replica analyticity can break down in some models, as well as the arising of problems in some disordered systems when the computing physical quantities in such limit [32, 33].

The purpose of this section is to present an alternative method to calculate the average free energy of a system with quenched disorder using still the notion of replicas.

The average free energy in the distributional zeta-function approach

Svaiter and Svaiter proposed an alternative procedure to compute the average free energy for quenched disorder by means which they called the *distributional zeta-function* [44]. Following these authors, let $\mathcal{M}$ be a d-dimensional Euclidean manifold, $C^\infty(\mathcal{M}, \mathbb{R})$ a space of smooth functions or scalar fields $\varphi(x)$ on $\mathcal{M}$, and the map $S : C^\infty(\mathcal{M}, \mathbb{R}) \to \mathbb{R}$ an action functional. To obtain the expression for the average free energy in this scenario, we consider the Euclidean action functional of a scalar field with $\lambda\varphi^4$ interaction and the disorder degrees of freedom h linearly coupled to the field $\varphi(x)$. This is given by

$$S(\varphi, h) = \int d^d x \left[\frac{1}{2}\varphi(x)\left(-\Delta + m_0^2\right)\varphi(x) + \frac{\lambda}{4!}\varphi^4(x) - h(x)\varphi(x)\right]. \tag{2.44}$$

Now, if $(X, \mathcal{A}, \mu)$ is a measure space and $f : X \to (0, \infty)$ is measurable, the generalized ζ-function is defined as

$$\zeta_{\mu,f}(s) = \int_{\Omega} f(x)^{-s} d\mu(x), \tag{2.45}$$

for $s \in \mathbb{C}$ such that $f^{-s} \in L^1(\mu)$. The factor $f^{-s} = \exp(-s \log(f))$ in the integrand is obtained using the principal branch of the logarithm. The actual scenario encompasses some well-known instances of zeta-functions for $f(x) = x$. For example, if $X = \mathbb{N}$ and μ represent a counting measure, we retrieve the Riemann zeta-function [84, 85]. Instead, if μ counts only the prime numbers, we retrieve the prime zeta-function [86, 87]. On another hand, if $X = \mathbb{R}$ and μ counts the eigenvalues of an elliptic operator with their respective multiplicity, we obtain the spectral zeta-function [88]. Further extending this formalism for the case where $f = Z(h)$ and $d\mu = [dh]P(h)$, Svaiter and Svaiter defined the distributional zeta-function $\Phi(s)$ as

$$\Phi(s) = \int [dh]P(h) \frac{1}{Z(h)^s}. \tag{2.46}$$

Since the part of the action that does not involve the disorder is even (a function of φ) then

$$Z(h) = Z(-h) = \frac{Z(h) + Z(-h)}{2}, \tag{2.47}$$

which is equivalent to

$$Z(h) = \int [d\varphi] \cosh \left[\int d^d x h(x) \varphi(x) \right] e^{-S(\varphi,0)}. \tag{2.48}$$

As consequence, $Z(h) \geqslant Z(0)$ and

$$\int [dh]P(h) \left| \frac{1}{Z(h)^s} \right| \leqslant \int d[h]P(h) \frac{1}{Z(0)^{\mathrm{Re}[s]}} = \frac{1}{Z(0)^{\mathrm{Re}[s]}} < \infty, \tag{2.49}$$

for $\mathrm{Re}[s] \geqslant 0$. This proves that the integral in Eq. (2.46) converges and then is well defined in the half-complex plane $\mathrm{Re}[s] \geqslant 0$. This implies that Φ is also well defined in the same region without resorting to analytic continuations. Now, taking into account that [6]

$$-\frac{d}{ds} Z(h)^{-s}|_{s=0^+} = \ln Z(h), \tag{2.50}$$

we can write

$$\begin{aligned} F &= -\int [dh]P(h) \frac{d}{ds} \left[\frac{1}{Z(h)^s} \right]_{s=0^+} \\ &= -\frac{d}{ds} \Phi(s) \Big|_{s=0^+}. \end{aligned} \tag{2.51}$$

6 $\frac{df}{ds}\big|_{s=0^+}$ stands for $\lim_{s\to 0^+} \frac{f(s)-f(0)}{s}$, whenever this limit exists.

The second equality is justified by the fact that $Z(h) \geqslant Z(0)$ and the application of the Lebesgue's dominated convergence theorem, if we interpret $d[h]P(h)$ as a measure. Observe that we obtained an analytic expression for F which, contrary to the standard replica method, *does not involve derivation of the (integer) moments of the partition function*. In order to proceed, we use the Euler's integral representation for the gamma function

$$\Gamma(s) = \int_0^\infty dt e^{-t} t^{s-1}. \tag{2.52}$$

Then, replacing t with $Z(h)t$ in this expression we can write

$$\frac{1}{Z(h)^s} = \frac{1}{\Gamma(s)} \int_0^\infty dt e^{-Z(h)t} t^{s-1}, \tag{2.53}$$

for $\mathrm{Re}[s] > 0$ and $Z(h) > 0$. Substituting Eq. (2.53) in Eq. (2.46) we get

$$\Phi(s) = \frac{1}{\Gamma(s)} \int [dh] P(h) \int_0^\infty dt e^{-Z(h)t} t^{s-1}, \tag{2.54}$$

which, as we know, is defined for $\mathrm{Re}[s] \geqslant 0$. To compute the derivative of the above equation at $s = 0$ we assume the commutativity between the disorder average, differentiation and integration when necessary. According to this, it follows that the average free energy F can be written as

$$F = -\frac{d}{ds} \left[\frac{1}{\Gamma(s)} \int [dh] P(h) \int_0^\infty dt e^{-Z(h)t} t^{s-1} \right]_{s=0^+}. \tag{2.55}$$

To continue, taking an arbitrary and positive real number a we write $\Phi = \Phi_1 + \Phi_2$, where

$$\Phi_1(s) = \frac{1}{\Gamma(s)} \int [dh] P(h) \int_0^a dt e^{-Z(h)t} t^{s-1} \tag{2.56}$$

and

$$\Phi_2(s) = \frac{1}{\Gamma(s)} \int [dh] P(h) \int_a^\infty dt e^{-Z(h)t} t^{s-1}. \tag{2.57}$$

so that the average free energy is rewritten as the sum of two terms,

$$F = -\frac{d}{ds} \Phi_1(s) \Big|_{s=0^+} - \frac{d}{ds} \Phi_2(s) \Big|_{s=0^+}. \tag{2.58}$$

The integral on t in the Eq. (2.57) defines an analytic function in the whole complex plane. On the other hand, we can use the series representation for the exponential in the Eq. (2.56) and write

$$\Phi_1(s) = \frac{1}{\Gamma(s)} \int [dh] P(h) \int_0^a dt\, t^{s-1} \sum_{k=0}^\infty \frac{(-1)^k}{k!} \big(Z(h)t\big)^k. \tag{2.59}$$

The series converges uniformly (for all h) so that, we can exchange the order of the integration on t and the summation operation to obtain

$$\begin{aligned}\Phi_1(s) =& \frac{1}{\Gamma(s)}\int [dh]P(h)\sum_{k=0}^{\infty}\frac{\left(-Z(h)\right)^k}{k!}\int_0^a dt\, t^{s+k-1} \\ =& \int [dh]P(h)\left[\frac{1}{\Gamma(s)}\sum_{k=0}^{\infty}\frac{(-1)^k a^{k+s}}{k!(k+s)}Z(h)^k\right]. \qquad (2.60)\end{aligned}$$

The first term of the series, which corresponds to $k=0$, has a singularity at $s=0$ but it can be removed since $\Gamma(s)s=\Gamma(s+1)$. In this way, we have

$$\begin{aligned}\Phi_1(s) =& \int [dh]P(h)\left[\frac{a^s}{\Gamma(s+1)}+\frac{1}{\Gamma(s)}\sum_{k=1}^{\infty}(-1)^k\frac{a^{k+s}}{k!(k+s)}Z(h)^k\right] \\ =& \frac{a^s}{\Gamma(s+1)}+\frac{1}{\Gamma(s)}\sum_{k=1}^{\infty}(-1)^k\frac{a^{k+s}}{k!(k+s)}\mathbb{E}[Z^k]. \qquad (2.61)\end{aligned}$$

This expression is valid for $\mathrm{Re}[s] \geqslant 0$. Differentiating Φ_1 as given in the Eq. (2.61) we obtain

$$-\frac{d}{ds}\Phi_1(s)\big|_{s=0^+} = \sum_{k=1}^{\infty}\frac{(-1)^{k+1}a^k}{k!k}\mathbb{E}[Z^k]+f(a), \qquad (2.62)$$

where we considered that the function $\Gamma(s)$ has a pole at $s=0$ with residue 1. The term $f(a)$ is given by

$$\begin{aligned}f(a) =& -\frac{d}{ds}\left[\frac{a^s}{\Gamma(s+1)}\right]_{s=0} \qquad (2.63)\\ =& -\left(\ln a+\gamma\right). \qquad (2.64)\end{aligned}$$

Here, γ is the Euler's constant $0.577\ldots$. Taking the derivative of Φ_2 in the Eq. (2.57) we obtain

$$\begin{aligned}-\frac{d}{ds}\Phi_2(s)\big|_{s=0} =& -\int [dh]P(h)\int_a^{\infty}\frac{dt}{t}e^{-Z(h)t} \\ =& R(a). \qquad (2.65)\end{aligned}$$

At this point, it is important to note that the contribution to the average free energy coming from $\Phi_1(s)$ is written as a series where all the integers moments of the partition function appear, Eq. (2.62). The situation is different for the contribution due to $\Phi_2(s)$, Eq. 2.65, where we do not have an explicit form for it. Nevertheless, we can show how to bound such contribution. Therefore, using again the inequality $Z(h) \geqslant Z(0)$, we obtain the bound

$$\left| -\frac{d}{ds}\Phi_2(s)\big|_{s=0} \right| \leqslant \int [dh] P(h) \int_a^\infty \frac{dt}{t} e^{-Z(0)t}$$
$$\leqslant \frac{1}{Z(0)a} e^{-Z(0)a}. \tag{2.66}$$

To conclude, using the distributional zeta-function, Eq. (2.46), we are able to represent the average free energy as

$$F_q = \sum_{k=1}^{\infty} \frac{(-1)^{k+1} a^k}{k!k} \mathbb{E}[Z^k] - \big(\ln a + \gamma\big) + R(a), \tag{2.67}$$

which $R(a)$ satisfies the condition

$$|R(a)| \leqslant \frac{1}{Z(0)a} e^{-Z(0)a}. \tag{2.68}$$

Observe that we cannot take the limit $a \to \infty$, because in this case the above series becomes meaningless. However, the contribution of $R(a)$ to the free energy can be made as small as desired by taking a large enough. Observe that unlike the usual procedure in the replica method where we need to take the limit to zero replicas, in this alternative framework the expression for the average free energy written as a series, where all the replicas contribute, plus another term that can not be represented in terms of replica partition functions. Consequently, it is not necessary to understand the properties of the permutation group S_k (the permutation group of k elements) when $k \to 0$.

3 FINITE-SIZE EFFECTS IN DISORDERED $\lambda\varphi^4$ MODEL

In classical statistical mechanics finite-size systems have at least one dimension with finite length [89]. In the framework of a scalar Euclidean quantum field theory, this it is the same as the field satisfying periodic boundary conditions, *i.e.* the respective spatial dimension is compactified. In the usual case, this leads to the emergence of a size-dependent renormalized mass for very small radius in the $\lambda\varphi^4$ model, the system presents finite correlation length [90, 91, 92, 93, 94]. This mechanism has been also studied in system with disorder, for example, in the Sherington-Kirkpatrick model [95, 96]. More recently, the study of finite-size corrections to the free energy density in disordered spin systems on random regular graphs was carried out in Refs. [97, 98].

In this chapter we are studying a scalar field theory with a $\lambda\varphi^4$ self-interaction defined in a d-dimensional Euclidean space in the presence of a disordered external field coupled linearly to the scalar field φ. Additionally, we assume finite-size effects by compacting of one dimension where the respective radius is L.

3.1 FINITE-SIZE EFFECTS AND DISORDER

One way to study the structure of the phase transition of the random field Ising model is to adopt the scalar field Landau-Ginzburg Hamiltonian density. The Hamiltonian of the system, introduced in section (2.4), is defined as

$$H = \int d^d x \left[\frac{1}{2}(\nabla\varphi)^2 + \frac{1}{2} m_0 \varphi^2(x) + \frac{1}{4!}\lambda\varphi^4(x) - h(x)\varphi(x) \right], \qquad (3.1)$$

where m_0) is a quantity depending on the temperature and the disordered field $h(x)$ is characterized by a Gaussian distribution, Eq. (2.27). The quantity m_0 drives the phase transition in the model. We will show that finite size effects act as temperature effects in this classical system with disorder. Therefore, it is possible to obtain a second order phase transition in one finite-size disordered $\lambda\varphi^4$ model when the renormalized mass of the model will acquire finite size corrections.

Periodic boundary condition in one space dimension makes this field theory being defined in $S^1 \times \mathbb{R}^{d-1}$ with the Euclidean topology of a field theory at finite temperature. For systems without disorder and described by scalar fields at thermal equilibrium with a reservoir are similar to systems with one compactified dimension, so finite-size effects are totally equivalent to finite temperature field theories where $\beta = L$ [99]. For systems with disorder degrees of freedom, this is not true anymore. In the case of finite temperature field theory, the disorder degrees of freedom are uncorrelated in space but correlated in the Euclidean time direction. In consequence, it is necessary to use the two-point correlation function in the form

$$\mathbb{E}[h(x)h(x')] = \sigma V(\tau - \tau')\delta^{d-1}(x - x'). \tag{3.2}$$

The anisotropy of the disorder in the Euclidean time direction shows that a quantum field theory with disorder degrees of freedom and finite temperature effects are not equivalent to finite size effects. In our case, the option is to consider

$$\mathbb{E}[h(x)h(x')] = \sigma\delta^{d}(x - x'). \tag{3.3}$$

which corresponds with the Gaussian distribution that defines the disorder in the d- dimensional Euclidean space.

3.2 ONE-LOOP APPROXIMATION AND NON-PERTURBATIVE RESULTS

In order to find the finite-size correction to the mass of the model we are considering the fact that all Feynman rules are the same as in the usual case, except that the momentum-space integrals over one component is replaced by a sum over discrete frequencies. For the case of Bose fields with periodic boundary conditions we must perform the replacement [90]

$$\int \frac{d^d p}{(2\pi)^d} f(p) \to \frac{1}{L}\int \frac{d^{d-1}p}{(2\pi)^{d-1}} \sum_{n=-\infty}^{\infty} f(\frac{2n\pi}{L}, \vec{p}\,), \tag{3.4}$$

where L is the radius of the compactified dimension of the system and the vector $\vec{p} = (p^2, p^3, .., p^d)$. The two-point correlation function in the one-loop approximation can be written as [47, 48, 4]

$$[G_0]_{lm}(x - y, L) =$$
$$\mu^{4-d} \sum_{ij} \int d^d z\, [G_0]_{li}(x - z, L)[G_0]_{ij}(z - z, L)[G_0]_{jm}(z - y, L). \tag{3.5}$$

where μ is a regularized parameter. The functions $[G_0]_{li}(x - z, L)$ and $[G_0]_{jm}(z - y, L)$ are singular at $x = z$ and $y = z$ but the singularities are integrable. The only contribution to the divergences is coming from $[G_0]_{ij}(z - z, L)$. Let us study the quantities that appear in the definition of $[G_0]_{ij}(z - z, L)$. In our case, the two-point correlation function of the model is given by Eq. (2.43). In the framework of the replica approach, which we are considering here, the limit $n \to 0$ is took. So, the two-correlation function is written as

$$[G_0]_{ij}(p) = \frac{\delta_{ij}}{(p^2 + m_0^2)} + \frac{\sigma}{(p^2 + m_0^2)^2}. \quad (3.6)$$

At one-loop approximation the correction to the unrenormalized mass comes from of the first order correction in λ of the two-point correlation function. Therefore, the unrenormalized size-dependent squared mass for the model is given by

$$m^2(L) = m_0^2 + \delta m_1^2(L) + \delta m_2^2(L), \quad (3.7)$$

where

$$\delta m_1^2(L) = \frac{\lambda}{2L} \int \frac{d^{d-1}p}{(2\pi)^{d-1}} \sum_{n=-\infty}^{\infty} \frac{1}{\left[\left(\frac{2\pi n}{L}\right)^2 + \vec{p}^{\,2} + m_0^2\right]}, \quad (3.8)$$

and

$$\delta m_2^2(L) = \frac{\lambda\sigma}{2L} \int \frac{d^{d-1}p}{(2\pi)^{d-1}} \sum_{n=-\infty}^{\infty} \frac{1}{\left[\left(\frac{2\pi n}{L}\right)^2 + \vec{p}^{\,2} + m_0^2\right]^2}. \quad (3.9)$$

The integrals in $\delta m_1^2(L)$ and $\delta m_2^2(L)$ can be calculated using dimensional regularization [100, 101, 102, 103, 104]. In the following we are using a mix between dimensional and regularization. Then, it is well known

$$\int d^d q \frac{1}{(q^2 + a^2)^s} = \frac{\pi^{\frac{d}{2}}}{\Gamma(s)} \Gamma\left(s - \frac{d}{2}\right) \frac{1}{(a^2)^{s-\frac{d}{2}}}. \quad (3.10)$$

where $\Gamma(x)$ is the Gamma function. Using Eq. (3.10) in Eq. (5.40) and Eq. (5.41) we obtain

$$\delta m_1^2(L) = \frac{\lambda}{2L} \frac{1}{(2\sqrt{\pi})^{d-1}} \Gamma\left(\frac{3-d}{2}\right) \sum_{n=-\infty}^{\infty} \frac{1}{\left[\left(\frac{2\pi n}{L}\right)^2 + m_0^2\right]^{\frac{3-d}{2}}} \quad (3.11)$$

and

$$\delta m_2^2(L) = \frac{\lambda\sigma}{2L} \frac{1}{(2\sqrt{\pi})^{d-1}} \Gamma\left(\frac{5-d}{2}\right) \sum_{n=-\infty}^{\infty} \frac{1}{\left[\left(\frac{2\pi n}{L}\right)^2 + m_0^2\right]^{\frac{5-d}{2}}}. \quad (3.12)$$

Note that since we are using dimensional regularization there is implicit in the definition of the coupling constant a factor μ^{4-d}. After using dimensional regularization we have to analytically extend the modified Epstein zeta function. A well suited representation for the analytic extension is given in Ref. [105]. This function is defined as

$$E(s,a) = \sum_{n=-\infty}^{\infty} \frac{1}{(n^2+a^2)^s}, \tag{3.13}$$

which converges absolutely and uniformly for $\mathrm{Re}(s) > \frac{1}{2}$. A useful representation of the analytic extension of this function is

$$E(s,a) = \frac{\sqrt{\pi}}{\Gamma(s)a^{2s-1}}\left[\Gamma\left(s-\frac{1}{2}\right) + 4\sum_{n=1}^{\infty}(n\pi a)^{s-\frac{1}{2}}K_{s-\frac{1}{2}}(2\pi n a)\right], \tag{3.14}$$

where $K_\nu(z)$ is the modified Bessel function of second kind. The gamma function $\Gamma(z)$ is a meromorphic function of the complex variable z with simple poles at the points $z = 0, -1, -2, \ldots$ In the neighbourhood of its poles, $z = -n$ for $n = 0, 1, 2, \ldots$, this function has a representation given by

$$\Gamma(z) = \frac{(-1)^n}{n!}\frac{1}{(z+n)} + \Omega(z+n), \tag{3.15}$$

where $\Omega(z+n)$ stands for the regular part of the analytic extension of $\Gamma(z)$. Both expression has a L-independent polar part plus a L-dependent analytic correction. The mass counterterm, i.e., the principal part of the Laurent series of the analytic regularized quantity, is L-independent. We are using a modified minimal subtraction renormalization scheme. In conclusion, to deal with the ultraviolet divergences of the one-loop two-point correlation function we use dimensional regularization to deal with the $(d-1)$ dimensional integrals and analytic regularization to deal with the frequency sums. We get finally that the L-dependent renormalized squared mass is given by

$$m^2(L) = m_0^2 + \lambda f_1(L, m_0) - \lambda\sigma f_2(L, m_0), \tag{3.16}$$

where $f_1(L, m_0)$ and $f_2(L, m_0)$ are given respectively by

$$f_1(L, m_0) = \frac{1}{(2\pi)^{d/2}}\sum_{n=1}^{\infty}\left(\frac{m_0}{nL}\right)^{\frac{d}{2}-1} K_{\frac{d}{2}-1}(m_0 n L), \tag{3.17}$$

and

$$f_2(L, m_0) = \frac{3}{2(2\pi)^{d/2}}\sum_{n=1}^{\infty}\left(\frac{m_0}{nL}\right)^{\frac{d}{2}-2} K_{\frac{d}{2}-2}(m_0 n L). \tag{3.18}$$

As we can see in Figs. (1) and (2), for some values of $\sigma \neq 0$ there is a critical size L_c in the cases of $d = 3$ and $d = 4$, where the L-dependent renormalized squared mass, at zero temperature, can vanishes. Here, the system develops a second-order phase transition. consequently, the system presents long-range correlations with power-law decay.

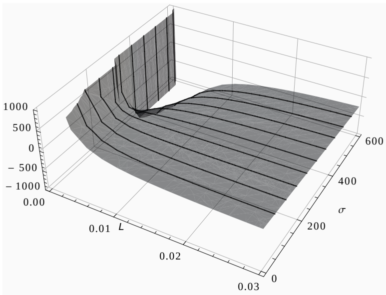

Figure 1: Renormalized squared mass in $d = 3$, as a function of the disorder parameter σ and the radius of the compactified dimension L in the one-loop approximation ($m_0 = 10$).

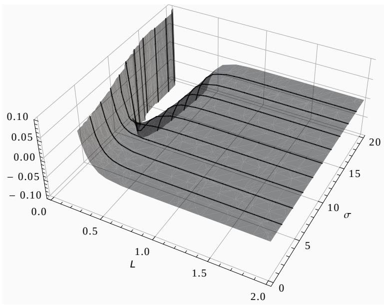

Figure 2: Renormalized squared mass in $d = 4$, as a function of the disorder parameter σ and the radius of the compactified dimension L in the one-loop approximation ($m_0 = 10$).

We can obtain nonperturbative results using the composite operator formalism or the Dyson-Schwinger equations [106, 107, 108]. See the appendix C. One way to implement the ressumation procedure is to write

$$m^2(L) = \delta m_1^2(L) + \delta m_2^2(L), \tag{3.19}$$

where

$$\delta m_1^2(L) = \frac{\lambda}{2L} \int \frac{d^{d-1}p}{(2\pi)^{d-1}} \sum_{n=-\infty}^{\infty} \frac{1}{\left[\left(\frac{2\pi n}{L}\right)^2 + \vec{p}^2 + m^2(L)\right]}, \tag{3.20}$$

and

$$\delta m_2^2(L) = \frac{\lambda\sigma}{2L} \int \frac{d^{d-1}p}{(2\pi)^{d-1}} \sum_{n=-\infty}^{\infty} \frac{1}{\left[\left(\frac{2\pi n}{L}\right)^2 + \vec{p}^2 + m^2(L)\right]^2}. \tag{3.21}$$

Considering again the dimensional regularization and the analytic regularization procedure we obtain that the gap equation that defines the size-dependent renormalized squared mass is given by

$$m^2(L) = \lambda f_1(L, m(L)) - \lambda\sigma f_2(L, m(L)), \tag{3.22}$$

where $f_1(L, m(L))$ and $f_2(L, m(L))$ are given respectively by

$$f_1(L, m(L)) = \frac{1}{(2\pi)^{d/2}} \sum_{n=1}^{\infty} \left(\frac{m(L)}{nL}\right)^{\frac{d}{2}-1} K_{\frac{d}{2}-1}(m(L)nL), \tag{3.23}$$

and

$$f_2(L, m(L)) = \frac{3}{2(2\pi)^{d/2}} \sum_{n=1}^{\infty} \left(\frac{m(L)}{nL}\right)^{\frac{d}{2}-2} K_{\frac{d}{2}-2}(m(L)nL). \tag{3.24}$$

We obtained, Figs (3) and (4) that for some σ the renormalized squared mass is a monotonically decrescent function of the length L. When $m^2(L) = 0$, this system presents long-range correlation with power-law decay.

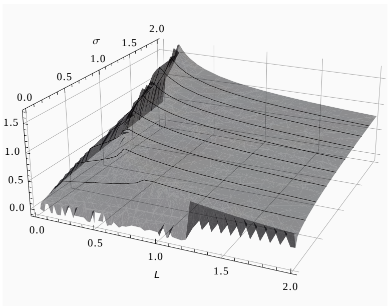

Figure 3: Renormalized squared mass in $d = 3$, as a function of the disorder parameter σ and the radius of the compactified dimension L, using the composite operator method ($m_0 = 10$).

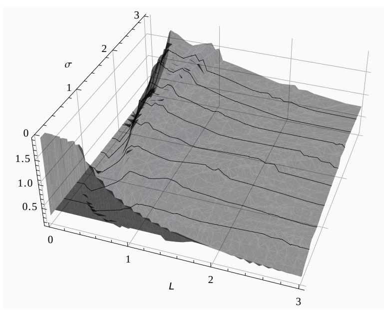

Figure 4: Renormalized squared mass in $d = 3$, as a function of the disorder parameter σ and the radius of the compactified dimension L, using the composite operator method ($m_0 = 10$).

Finally, direct application of field theoretical methods in the study of critical phenomena has a long story. In the calculations of the critical exponents of different models one finds non-integrable singularities in the integrals. In the critical region, it appears infrared divergences. One way to get around this problem is to enclose the system in a finite box and treat the zero mode properly. The behavior of the renormalized squared mass (size-dependent) in $d = 3$ and $d = 4$ is monotonically decrescent function of L.

4 FREE ENERGY OF POLYMERS AND INTERFACES IN RANDOM MEDIA

The statistical mechanics of random surfaces and membranes make possible to explore new ideas at the time that these systems began to be studied, since different mathematical tools were required to investigate them [109]. One of the simplest examples is the case of polymers, macromolecules with string-like shape [110, 111, 112]. These objects display certain universal properties that do not depend on their microscopic structure. For example, a complex fluid has complex physical and mechanical response to external fields. In the presence of external fields, these objects can behave like a liquid or a solid, concerning its dynamical properties, so that the viscoelasticity is an example of such behaviour. In the 70's functional methods with the saddle-point approximation formed the basis of a self-consistent field theory in the polymer physics. Using field theory methods it was shown that certain models of polymers can be discussed by a classical field theory [113, 114, 115]. Moreover, using the Cole-Hopf transformation an important stochastic non-linear partial differential equation (the KPZ equation) modelling surface and interface growing processes can be mapped into the equilibrium statistical mechanics of directed polymers [116, 117, 118].

4.1 THE FIELD THEORY IN $d = 1$ WITH QUENCHED DISORDER

The theory of polymers has attracted the attention of the community of physicists since the middle of the 1960's, when Edwards introduced the Hamiltonian to describe self-avoiding polymer chains [119]. The theoretical success of this formulation then allowed the development of another approaches considered more difficult to address before. Edwards formulated the self-avoiding polymer problem in a continuous model from which methods related to the path integral formulation of quantum mechanics were used. Generalizations to more complex polymer systems were perform in such formalism [120, 11]. The polymer field theory has been also influenced by the progress of the theory of phase transitions and the use of the scaling concepts [113, 121]. Some papers discussing the polymer field theory are the Refs. [122, 123, 124]

The Hamiltonian of a directed polymer of length L in a continuous approximation can be written as [11]

$$H(\varphi, \nu) = \int_0^L dx \left[\frac{c}{2} \left(\frac{d\varphi}{dx} \right)^2 + \nu\left(\varphi(x), x\right) \right], \tag{4.1}$$

where c is the linear tension of the polymer, x is the longitudinal coordinate $(0 \leqslant x \leqslant L)$ and $\varphi(x)$ is the transverse displacement of a polymer with respect to a horizontal straight line. The term $\nu\left(\varphi(x), x\right)$ represents the quenched disordered potential of the model. For simplicity we assume that the displacements of the polymer occur in one direction only.

In the literature we can find different proposed probability distributions associated to the disorder [109, 111, 125]. Here, we consider the more widely used, i.e. the Gaussian distribution. Then, the potential $\nu(\varphi(x), x)$ comes to be a Gaussian random variable. Therefore

$$\mathbb{E}\big[\nu(\varphi, x)\big] = 0 \tag{4.2}$$

and

$$\mathbb{E}\big[\nu(\varphi, x)\nu(\varphi', x')\big] = 2V(\varphi - \varphi')\delta(x - x'), \tag{4.3}$$

where $V(\varphi - \varphi')$ stands for the correlation function of the model. An important quantity in the description of directed polymer is the called wandering exponent ξ. This is defined through the expression

$$\mathbb{E}\big[\langle(\varphi(L) - \varphi(0))^2\rangle\big] \propto L^{2\xi}, \tag{4.4}$$

which gives the respective mean squared displacement of the field φ. From the Hamiltonian given by the Eq. (4.1), we have that the partition function of the model can be written as

$$Z(L, y; \nu) = \int_{\varphi(0)=0}^{\varphi(L)=y} [d\varphi] \exp\big[-\beta H(\nu, \varphi)\big], \tag{4.5}$$

To compute the average free energy we can to use the replica method as is done in the literature. Instead, we will use the distributional zeta-function method where this quantity is defined by the Eq. (2.67). To proceed, we need to obtain the k-th power of the partition function Z^k. This can be written as

$$[Z(L, y; \nu)]^k = \int_{\varphi_i(0)=0}^{\varphi_i(L)=y} \prod_{i=1}^{k} [d\varphi_i] \exp\left[-\beta \sum_{a=1}^{k} H(\varphi_a, \nu)\right]. \tag{4.6}$$

Averaging $[Z(L, y; \nu)]^k$ over the disorder, we have that the k-th replica partition function is given by

$$\mathbb{E}[Z^k] = \int_{\varphi_i(0)=0}^{\varphi_i(L)=y} \prod_{i=1}^{k} [d\varphi_i] \exp\big[-\beta H_{eff}(\varphi_a)\big], \tag{4.7}$$

where the effective Hamiltonian $H_{eff}(\varphi_a)$ is

$$H_{eff}(\varphi_a) = \int_0^L dx \left[\frac{c}{2} \sum_{a=1}^{k} \left(\frac{d\varphi_a}{dx} \right)^2 - \beta \sum_{a,b=1}^{k} V\big(\varphi_a(x) - \varphi_b(x)\big) \right]. \tag{4.8}$$

Since our interest is to have a soluble model, we follow the usual choice as in the the replica method, i.e. we consider $V(\varphi - \varphi')$ as given by [126]

$$V(\varphi - \varphi') = V_0 - \frac{1}{2} u (\varphi - \varphi')^2, \tag{4.9}$$

After integrating by parts, we can write

$$H_{eff} = H_{eff}^{(1)} + H_{eff}^{(2)} \tag{4.10}$$

where

$$H_{eff}^{(1)}(\varphi_a) = \frac{1}{2} \int_0^L dx \sum_{a,b=1}^{k} \varphi_a(x) \left[\left(-c \frac{d^2}{dx^2} + \beta u \right) \delta_{ab} - \beta u \right] \varphi_b(x), \tag{4.11}$$

and

$$H_{eff}^{(2)} = \int_0^L dx \, \beta V_0. \tag{4.12}$$

Studying the replica field theory for the problem of fluctuating manifold in a quenched random potential, Mézard, Parisi and others introduced a mass term in the effective Hamiltonian in order to regularize the model [127, 128]. Indeed, in the high temperature limit, i.e. $\beta \to 0$, the operator $(-\frac{c}{2} \frac{d^2}{dx^2} + \beta u)$ has the zero eigenvalue and therefore is not invertible. To circumvent this problem, we are following the same idea introducing the term

$$\frac{1}{2} \int_0^L dx \left[\delta_{ab} \omega^2 \varphi_a(x) \varphi_b(x) \right], \tag{4.13}$$

in the effective Hamiltonian H_{eff}. Disregarding $H_{eff}^{(2)}$ since it is a constant, we have

$$H_{eff}(\varphi_a; \omega) = \frac{1}{2} \int_0^L dx \sum_{a,b=1}^{k} \varphi_a(x) \left[\left(-c \frac{d^2}{dx^2} + \omega^2 + \beta u \right) \delta_{ab} - \beta u \right] \varphi_b(x). \tag{4.14}$$

We would like to stress that this Hamiltonian does not represent more the original model used to describe directed polymers, only in the limit $\omega \to 0$ we recover it. Writing

$$D_{ab} = \left[\left(-c\frac{d^2}{dx^2} + \omega^2 + \beta u\right)\delta_{ab} - \beta u\right], \tag{4.15}$$

the replica partition function $\mathbb{E}[Z^k]$ reads

$$\mathbb{E}[Z^k] = \int_{\varphi_i(0)=0}^{\varphi_i(L)=y} \prod_{i=1}^{k}[d\varphi_i] \exp\left[-\frac{\beta}{2}\sum_{a,b=1}^{k}\int_0^L dx \varphi_a(x) D_{ab} \varphi_b(x)\right]. \tag{4.16}$$

From the above equation we obtain the average free energy of the system. Now, focusing on the respective series and considering term by term, the saddle-point equations for each replica partition function are

$$\left[-c\frac{d^2}{dx^2} + \omega^2 + \beta u\right]\varphi_a(x) = \beta u \sum_{b=1}^{k}\varphi_b(x). \tag{4.17}$$

In each integer moment of the partition function we must have $\varphi_i(x) = \varphi(x)$. This is the unique solution for the problem of the structure in replica space compatible with the distributional zeta-function method unlike the replica approach where the symmetry replica is broken [129]. For equal replicas, the saddle-point equations become

$$\left[-c\frac{d^2}{dx^2} + \omega^2 + \beta u(1-k)\right]\varphi(x) = 0. \tag{4.18}$$

As it was mentioned before, in the standard replica approach the limit with the number of replicas goes to zero leads to the saddle-point equation of the system without disorder. The situation is quite different in the framework of the distributional zeta-function approach in this infrared regularized model. The condition $\omega^2 + (1-k)\beta u \geqslant 0$ must be satisfied to have a physical theory. Consider a generic term of the series for the average free energy where the corresponding replica partition function is $\mathbb{E}[Z^l]$. Defining k_c as

$$k_c = \left\lfloor \frac{\omega^2}{\beta u} + 1 \right\rfloor, \tag{4.19}$$

where $\lfloor x \rfloor$ means the integer part of x, the structure of the fields in each replica partition function is given by

$$\begin{cases} \varphi_i^{(l)}(x) = \varphi(x), & \text{for } l = 1, 2, ..., k_c, \\ \varphi_i^{(l)}(x) = 0, & \text{for } l > k_c. \end{cases} \tag{4.20}$$

Note that the superscript l stands for the l-th replica partition function. Only in the high-temperature limit ($\beta \to 0$) all the integer moments of the partition function contribute to the average free energy. For finite temperature, we must have only a finite number of terms in the series representation to the average free energy. In this case, according to Eq. (4.20), the leading contribution for the average free energy is given by

$$F_q = \frac{1}{\beta}\sum_{k=1}^{k_c}\frac{(-1)^k a^k}{k!k}\,\mathbb{E}\,Z^k. \qquad (4.21)$$

For $\omega \neq 0$ and large $k_c = N$ we have the large-N approximation for a Gaussian field theory. It is interesting to point out that the limit $\omega \to 0$ where the situation for the polymer is recovered the distributional zeta-function method gives $k_c = 1$ and the infrared divergences appear. The system is described by a replica field theory where the dimension of the order parameter is one. In the conventional replica method, both cases, namely, the infrared regularized theory and the model without any regularization are described by replica field theories where the dimension of the order parameter is zero. For Gaussian models defined in the continuous limit, the distributional zeta-function method and the conventional replica approach give the same results for the the wandering exponent. This will be clarified in the next section.

4.2 FIELD THEORY FOR INTERFACES IN RANDOM MEDIA

In this section, we study an interface, that is, a d-dimensional manifold with internal points, $x \in \mathbb{R}^d$, embedded in an external D-dimensional space with position vector $\vec{r}(x) \in \mathbb{R}^D$. Specifically, we are considering a d-dimensional manifold in a $D = d + N$ dimensional space which we can described in terms of a set of transverse coordinates, where N is the number of transverse dimensions. We are interested in an interface in a quenched random potential, the case where $N = 1$. The Hamiltonian that defines this class of system can be written as [127, 129]

$$H(\varphi, \nu) = \int d^d x\left[\frac{\sigma}{2}\left|\nabla\varphi(x)\right|^2 + \nu\big(\varphi(x), x\big)\right], \qquad (4.22)$$

where σ is the domain wall stiffness and $\nu\,(\varphi(x), x)$ is the quenched random potential of the model. Following Mezard and Parisi and also Cugliandolo *et al.* [127, 128], let us introduce a $\frac{1}{2}\omega^2\varphi^2(x)$ contribution, which constrain the manifold to fluctuate in a restricted region of the embedding space. The regularized Hamiltonian becomes

$$H(\varphi, \nu) = \frac{1}{2}\int d^d x\Big[\varphi(x)\left(-\sigma\Delta + \omega^2\right)\varphi(x) + \nu\big(\varphi(x), x\big)\Big]. \qquad (4.23)$$

The probability distribution associated to the random potential is again a Gaussian distribution so that it has zero mean and correlator

$$\mathbb{E}\left[\nu(\varphi,x)\nu(\varphi',x')\right] = 2V(\varphi-\varphi')\delta^d(x-x'), \tag{4.24}$$

In order to obtain the average free energy of the model we need again to compute the integer moments $\mathbb{E}[Z^k]$ of the partition function. The (integer) k-th power of the partition function is Z^k and after integrating over the disorder we obtain the replica partition function[1]

$$\mathbb{E}[Z^k] = \int\prod_{i=1}^{k}[d\varphi_i]\exp\left[-\beta H_{eff}(\varphi_a)\right], \tag{4.25}$$

where

$$\begin{aligned} H_{eff}(\varphi_a) = \frac{1}{2}\int d^dx \Bigg[&\sum_{i=a}^{k}\varphi_a(x)\left(-\sigma\Delta+\omega^2\right)\varphi_a(x) \\ &-\beta\sum_{a,b=1}^{k}V\Big(\varphi_a(x)-\varphi_b(x)\Big)\Bigg] \end{aligned} \tag{4.26}$$

Again, as in the case of the polymers, to proceed we must use some treatable approximation for $V(\varphi_a-\varphi_b)$. Following Balents and Fisher [130], we consider

$$V(\varphi_a-\varphi_b) = \sum_m \frac{1}{m!}V_m(\varphi_a-\varphi_b)^{2m}. \tag{4.27}$$

Taking $m=2$ or equivalently assuming that $V_1>0$, $V_2>0$ and $V_m=0$ for $m\geqslant 3$, we are going beyond to the Gaussian approximation. According to this, the potential $V(\varphi_a-\varphi_b)$ reads

$$V(\varphi_a-\varphi_b) = V_0 - \frac{1}{2}u_1(\varphi_a-\varphi_b)^2 - \frac{1}{4}u_2(\varphi_a-\varphi_b)^4. \tag{4.28}$$

In this case, the k-th replica partition function is given by

$$\mathbb{E}[Z^k] = \int\prod_{i=1}^{k}[d\varphi_i]\exp\left[-\beta H_{eff}(\varphi_i)\right], \tag{4.29}$$

where the effective Hamiltonian can be written as

$$H_{eff} = H_{eff}^{(0)} + H_{eff}^{(1)}. \tag{4.30}$$

In the above equation the Gaussian contribution $H_{eff}^{(0)}$ is written as

1 The procedure is exactly the same as in the case of the polymer performed in the previous section.

$$H^{(0)}_{eff} = \frac{1}{2}\sum_{a,b=1}^{k}\int d^d x \varphi_a(x)\Big[(-\sigma\Delta + \omega_0^2 + \beta u_1)\delta_{ab} - \beta u_1\Big]\varphi_b(x), \tag{4.31}$$

and the non-Gaussian contribution $H^{(1)}_{eff}$ as

$$H^{(1)}_{eff} = \frac{\beta u_2}{2}\sum_{a,b=1}^{k}\int d^d x\Big[\frac{1}{4}\varphi_a^4(x) + \frac{1}{4}\varphi_b^4(x) - \varphi_a^3(x)\varphi_b(x) + \frac{3}{2}\varphi_a^2(x)\varphi_b^2(x) - \varphi_a(x)\varphi_b^3(x)\Big]. \tag{4.32}$$

At this point, we can study the structure of the replica space form the the saddle-point equations. Assuming the replica symmetry ansatz, $\varphi_a(x) = \varphi(x)$, for all replica fields in each replica partition function, we have that the saddle-point equation reads

$$\Big[-\sigma\Delta + \omega_0^2 + (1-k)\beta u_1\Big]\varphi(x) = 0. \tag{4.33}$$

The condition $\omega_0^2 + (1-k)\beta u_1 \geqslant 0$ must be satisfied to have a physical theory. So, let us define a critical $k_c = \left\lfloor \frac{\omega_0^2}{\beta u_1} + 1 \right\rfloor$ considering again a generic term of the series that defines the average free energy, Eq. (2.67). For any replica partition function $\mathbb{E}[Z^l]$ the only choice in the replica space is compatible with us method is

$$\begin{cases} \varphi_i^{(l)}(x) = \varphi(x) & \text{for } l = 1, 2, ..., k_c \\ \varphi_i^{(l)}(x) = 0 & \text{for } l > k_c, \end{cases} \tag{4.34}$$

In this scenario and for an a very large the average free energy becomes

$$F_q = \frac{1}{\beta}\frac{(-1)^{k_c} a^{k_c}}{k_c! k_c}\mathbb{E}[Z^{k_c}]. \tag{4.35}$$

Using $k = k_c$, with the choice of the replica space, we obtain that the effective Hamiltonian can be written in the simple form

$$H_{eff}(\varphi_i) = \frac{1}{2}\sum_{a,b=1}^{k}\int d^d x \int d^d y \varphi_a(x) D_{ab}(x-y)\varphi_b(y), \tag{4.36}$$

where for simplicity we are using $u_1 = u$ and $D_{ab}(x-y)$ reads

$$D_{ij}(x-y) = \Big[(-\sigma\Delta + \omega^2 + \beta u)\delta_{ij} - \beta u\Big]\delta^d(x-y). \tag{4.37}$$

Therefore, the quantity $\mathbb{E}[Z^k]$ in the Eq. (4.35) is given by

$$\mathbb{E}[Z^k] = \int \prod_{i=1}^{k} [d\varphi_i] \exp\left[-\frac{\beta}{2} \sum_{a,b=1}^{k} \int d^d x \int d^d y \varphi_a(x) D_{ab}(x-y) \varphi_b(y)\right]. \tag{4.38}$$

In the replica symmetric ansatz framework, the dominant term of the series that represents the average free energy can be viewed as an Euclidean field theory for a k-component scalar field. Defining the k-vector field $\Phi(x)$ with the components $\varphi_1(x), \varphi_2(x), ..., \varphi_k(x)$ we can write the effective Hamiltonian as

$$H_{eff}(\Phi) = \frac{1}{2} \int d^d x \int d^d y \Phi^T(x) D(x-y;k) \Phi(y), \tag{4.39}$$

where $\Phi^T(x)$ stands for the transpose of the k-vector $\Phi(x)$. In view of Eqs. (4.37) and (4.38), the kernel $D(x-y;k)$ is

$$D(x-y;k) = \delta^d(x-y)\left[(-\sigma\Delta + \omega^2 + \beta u)\mathbb{I}_k - \beta u \mathbb{M}_k\right], \tag{4.40}$$

where $\mathbb{I}_k$ is the k-dimensional identity matrix and $\mathbb{M}_k$ is the square k-dimensional matrix with all elements 1 [131].

Now, performing a Fourier transform on the effective Hamiltonian we get

$$H_{eff}(\varphi_a) = \frac{1}{2} \sum_{a,b=1}^{k} \int \frac{d^d p}{(2\pi)^d} \varphi_a p) [G_0]_{ij}^{-1}(p) \varphi_j(-p), \tag{4.41}$$

where $[G_0]_{ij}^{-1}$ is the inverse of the two-point correlation function

$$[G_0]_{ij}^{-1}(p) = (\sigma p^2 + \omega^2 + \beta u)\delta_{ij} - \beta u. \tag{4.42}$$

Using the projector operators, Eqs. (2.40)-(2.41,) we can write the two-point correlation function $[G_0]_{ij}(p)$ as

$$[G_0]_{ij}(p) = \frac{\delta_{ij}}{(\sigma p^2 + \omega^2 + \beta u)} + \frac{\beta u}{(\sigma p^2 + \omega^2 + \beta u)(\sigma p^2 + \omega^2 + \beta u(1-k))}. \tag{4.43}$$

Taking $k = 1$ in the above expression, it reduces to the two-point correlation function in the usual replica approach when the number of replicas goes to zero [127].

To proceed, let us study the two-point correlation function. First write

$$[G_0]_{lm}(x-y) = [G_0]_{lm}^{(1)}(x-y) + [G_0]_{lm}^{(2)}(x-y), \tag{4.44}$$

with

$$[G_0]^{(1)}_{lm}(x-y) = \frac{\delta_{lm}}{(2\pi)^d} \int d^d q \frac{e^{i(x-y)q}}{(\sigma q^2 + \omega^2 + \beta u)} \tag{4.45}$$

and $[G_0]^{(2)}_{lm}(x-y) = [G_0]^{(2)}(x-y)$ where

$$[G_0]^{(2)}(x-y) = \frac{1}{(2\pi)^d} \beta u \int d^d q \frac{e^{i(x-y)q}}{[\sigma q^2 + \omega^2 + \beta u][\sigma q^2 + \omega^2 + \beta u(1-k)]}. \tag{4.46}$$

Now, we are able to discuss the functional form of $[G_0]^{(1)}_{lm}(x-y)$ and $[G_0]^{(2)}_{lm}(x-y)$. Using the results of the calculations presented in the next section we get

$$[G_0]^{(1)}_{lm}(r) = \frac{\delta_{lm}}{(2\pi)^{\frac{d}{2}} r^{\frac{d-2}{2}} \sigma^{\frac{d+2}{4}}} (\omega^2 + \beta u)^{\frac{d-2}{4}} K_{\frac{d}{2}-1}\left(r\sqrt{\sigma^{-1}(\omega^2 + \beta u)}\right). \tag{4.47}$$

Also, we can write that

$$[G_0]^{(2)}(r) = \frac{1}{(2\pi)^{\frac{d}{2}} r^{\frac{d-2}{2}} \sigma^{\frac{d+2}{4}} k} \times \left[-(\omega^2 + \beta u)^{\frac{d-2}{4}} K_{\frac{d}{2}-1}\left(r\sqrt{\sigma^{-1}(\omega^2 + \beta u)}\right) + (\omega^2 + \beta u(1-k))^{\frac{d-2}{4}} K_{\frac{d}{2}-1}\left(r\sqrt{\sigma^{-1}(\omega^2 + \beta u(1-k))}\right)\right]. \tag{4.48}$$

For the case which we are interested in, one must take the limit of $\omega \to 0$, since $\omega_0^2 + (1-k)\beta u_1 \geqslant 0$, only the contribution from $k = 1$ survives. In this case we get a situation quite similar to the standard replica method where the replica symmetry ansatz produces a saddle-point equation for a system without disorder. Nevertheless, rather than the standard replica method, where the vector dimension of the order parameter is zero, in the distributional zeta-function method, we show that this system must be described by an order parameter with dimension one. From Eqs. (4.47) and (4.48) we have

$$[G_0](r) = \frac{1}{4(\pi\sigma)^{\frac{d}{2}}} \Gamma\left(\frac{d-2}{2}\right) \frac{1}{r^{d-2}}. \tag{4.49}$$

We can introduce a normalized generating functional with the normalization factor $(\det' D)^{-1/2}$ where the prime sign means that the contribution of the zero mode must be omitted. The wandering exponent describes the growth of the transverse fluctuations of the manifold as function of the distances. This defined by Eq. (4.4). In this situation, where the limit $\omega \to 0$ is taken, we obtain from Eq. (4.43) the wandering exponent given by $\xi = \frac{4-d}{2}$ as was discussed by Parisi and Mézard in Ref. [127] for the replica symmetric solution. It is important to stress that the replica symmetry breaking solution is absent in our scenario.

4.3 THE TWO-POINT CORRELATION FUNCTION IN d-DIMENSIONS

In this section we show the calculations performed to obtain the two-point correlation function of the random manifold considered in the previous section. As starting point, let us define $[G_0]^{(1)}(x-y)$ where $\delta_{lm}[G_0]^{(1)}(x-y) = [G_0]^{(1)}_{lm}(x-y)$. Therefore, we have

$$[G_0]^{(1)}(x-y) = \frac{1}{(2\pi)^d \sigma} \int d^d q \frac{e^{i(x-y)q}}{\left(q^2 + \sigma^{-1}(\omega^2 + \beta u)\right)}. \tag{4.50}$$

To perform the $d-$dimensional integral in the Eq. (4.50), let us work in a $d-$dimensional polar coordinate system

$$\begin{aligned} q^1 &= q\cos(\theta_1) \\ q^2 &= q\cos(\theta_2)\sin(\theta_1) \\ &\vdots \\ q^{d-1} &= q\cos(\theta_{d-1})\sin(\theta_{d-2})\ldots\sin(\theta_1) \\ q^d &= q\sin(\theta_{d-1})\sin(\theta_{d-2})\ldots\sin(\theta_1), \end{aligned} \tag{4.51}$$

where $q^2 = \left(q_1^2 + q_2^2 + \cdots + q_d^2\right)$ and the volume element $d^d q$ turns into $d^d q = q^{d-1} dq d\Omega_d$. The correlation function $[G_0]^{(1)}(x-y)$ is written then as

$$[G_0]^{(1)}(r) = \frac{1}{(2\pi)^d \sigma} \int d\Omega_d \int_0^\infty dq q^{d-1} \frac{e^{iqr\cos\theta_1}}{\left[q^2 + \sigma^{-1}(\omega^2 + \beta u)\right]}, \tag{4.52}$$

where $r = |x-y|$ and $\int d\Omega_d$ is given by

$$\int d\Omega_d = \int_0^\pi d\theta_1 (\sin\theta_1)^{d-2} \int_0^\pi d\theta_2 (\sin\theta_2)^{d-3} \ldots \int_0^\pi d\theta_{d-2} (\sin\theta_{d-2}) \int_0^{2\pi} d\theta_{d-1}. \tag{4.53}$$

Since the exponential in the Eq. (4.52) depends of the angle θ_1 we write

$$[G_0]^{(1)}(r) = \frac{1}{(2\pi)^d \sigma} \int d\Omega_{d-1} \int_0^\pi d\theta_1 (\sin\theta_1)^{d-2} \int_0^\infty dq\, q^{d-1} \frac{e^{iqr\cos\theta_1}}{\left[q^2 + \sigma^{-1}(\omega^2 + \beta u)\right]}, \tag{4.54}$$

where $\int d\Omega_{d-1}$ is given by

$$\int d\Omega_{d-1} = \int_0^{\pi} d\theta_2 (\sin\theta_2)^{d-3} \dots \int_0^{\pi} d\theta_{d-2} (\sin\theta_{d-2}) \int_0^{2\pi} d\theta_{d-1} = \frac{2\pi^{\frac{d-1}{2}}}{\Gamma(\frac{d-1}{2})}. \tag{4.55}$$

Thus, the correlation function $[G_0]^{(1)}(r)$ can be written as

$$[G_0]^{(1)}(r) = \frac{1}{\pi(2\sqrt{\pi})^{d-1}\Gamma(\frac{d-1}{2})\sigma} \times \int_0^{\infty} dq\, q^{d-1} \int_0^{\pi} d\theta_1 (\sin\theta_1)^{d-2} \frac{e^{iqr\cos\theta_1}}{\left[q^2 + \sigma^{-1}(\omega^2 + \beta u)\right]}. \tag{4.56}$$

Considering the expression

$$\int_0^{\pi} d\theta\, e^{i\beta\cos\theta} (\sin\theta)^{2\nu} = \sqrt{\pi}\left(\frac{2}{\beta}\right)^{\nu} \Gamma\left(\nu + \frac{1}{2}\right) J_{\nu}(\beta), \tag{4.57}$$

where $J_{\nu}(\beta)$ is the Bessel function of the first kind [132, 133], we have

$$[G_0]^{(1)}(r) = \frac{1}{(2\pi)^{\frac{d}{2}} r^{\frac{d-2}{2}} \sigma} \int_0^{\infty} dq \frac{q^{\frac{d}{2}}}{\left[q^2 + \sigma^{-1}(\omega^2 + \beta u)\right]} J_{\frac{d}{2}-1}(qr). \tag{4.58}$$

Now, to solve the integral in the above equation we will consider the following identity

$$\int_0^{\infty} dx \frac{x^{\mu+1}}{(x^2 + a^2)^{\nu+1}} J_{\mu}(bx) = \frac{a^{\mu-\nu} b^{\nu}}{2^{\nu}\Gamma(\nu+1)} K_{\mu-\nu}(ab), \tag{4.59}$$

which is valid for $a > 0$, $b > 0$ and $-1 < Re(\mu) < Re(2\nu + \frac{3}{2})$. We have then

$$[G_0]^{(1)}(r) = \frac{1}{(2\pi)^{\frac{d}{2}} \sigma} \left[\frac{\sqrt{\sigma^{-1}(\omega^2 + \beta u)}}{r}\right]^{\frac{d}{2}-1} K_{\frac{d}{2}-1}\left(r\sqrt{\sigma^{-1}(\omega^2 + \beta u)}\right), \tag{4.60}$$

which is restricted to $0 < d < 5$, where $K_{\nu}(z)$ is the modified Bessel function of second kind. Next, defining $a_1(\sigma, \omega, \beta u) = \sigma^{-1}(\beta u + \omega^2)$ and $a_2(\sigma, \omega, \beta u; k) = \sigma^{-1}(\omega^2 + \beta u(1-k))$, we have that $[G_0]^{(2)}(x - y; k)$ can be written as

$$[G_0]^{(2)}(x-y) = \frac{1}{(2\pi)^d}\frac{\beta u}{\sigma^2}\times \int d^d q \frac{e^{i(x-y)q}}{\left[q^2 + a_1(\sigma,\omega,\beta u)\right]\left[q^2 + a_2(\sigma,\omega,\beta u;k)\right]}. \tag{4.61}$$

Working once again in a d-dimensional polar coordinate system and performing the integrals in the angular coordinates as was did above, we have

$$[G_0]^{(2)}(r) = \frac{1}{(2\pi)^{\frac{d}{2}} r^{\frac{d-2}{2}}}\frac{\beta u}{\sigma^2}\times \int_0^\infty dq \frac{q^{\frac{d}{2}}}{\left[q^2 + a_1(\sigma,\omega,\beta u)\right]\left[q^2 + a_2(\sigma,\omega,\beta u;k)\right]} J_{\frac{d}{2}-1}(qr). \tag{4.62}$$

Using partial fractions we can rewrite the integrand in the last expression in such a way that

$$[G_0]^{(2)}(r;k) = \frac{1}{(2\pi)^{\frac{d}{2}} r^{\frac{d-2}{2}}}\frac{1}{k\sigma}\left[-\int_0^\infty dq \frac{q^{\frac{d}{2}}}{\left(q^2 + a_1(\sigma,\omega,\beta u)\right)} J_{\frac{d}{2}-1}(qr) + \int_0^\infty dq \frac{q^{\frac{d}{2}}}{\left(q^2 + a_2(\sigma,\omega,\beta u;k)\right)} J_{\frac{d}{2}-1}(qr)\right]. \tag{4.63}$$

We have then two integrals in the form that to the Eq. (4.59). Therefore

$$[G_0]^{(2)}(r;k) = \frac{1}{(2\pi)^{\frac{d}{2}} r^{\frac{d-2}{2}}}\frac{1}{k\sigma^{\frac{d+2}{4}}}\times \left[-\left(\beta u + \omega^2\right)^{\frac{d-2}{4}} K_{\frac{d}{2}-1}\left(r\sqrt{\sigma^{-1}(\beta u + \omega^2)}\right) + \left(\omega^2 + \beta u(1-k)\right)^{\frac{d-2}{4}} K_{\frac{d}{2}-1}\left(r\sqrt{\sigma^{-1}(\omega^2 + \beta u(1-k))}\right)\right], \tag{4.64}$$

which is also restricted to $0 < d < 5$.

5 SPONTANEOUS SYMMETRY BREAKING IN REPLICA FIELD THEORY

In this chapter we discuss the relationship that exists between the spontaneous symmetry breaking mechanism and the replica symmetry ansatz in a disordered scalar model in the framework of the distributional zeta-function method. Specifically, we are interested in studying a d-dimensional Euclidean Landau-Ginzburg model in the presence of a external disorder field linearly coupled with a scalar field, i.e the continuous version of the random field Ising model. We discuss the physical consequences that arise with the adoption of such approach to evaluate the average free energy. Additionally, we examine the finite temperature effects in the model.

5.1 SPONTANEOUS SYMMETRY BREAKING IN REPLICA FIELD THEORY

Let us assume an Euclidean d-dimensional $\lambda\varphi^4$ model in the presence of a disorder field $h(x)$ with Gaussian probability distribution, the random field Ising model, where the disordered functional integral $Z(h)$ is given by

$$Z(h) = \int [d\varphi] e^{\left[-S + \int d^d x\, h(x)\varphi(x)\right]}. \tag{5.1}$$

As we discussed before, in the above equation the Euclidean action S corresponds to

$$S(\varphi) = \int d^d x \left[\frac{1}{2}(\partial\varphi)^2 + \frac{1}{2}m_0^2\,\varphi^2(x) + \frac{\lambda_0}{4!}\varphi^4(x)\right]. \tag{5.2}$$

We want to stress that our starting point is the semiclassical (tree) approximation where the coupling constant λ_0 and the mass m_0 are renormalized quantities. For a disorder field linearly coupled with the scalar field, the ground state configurations are defined by the saddle-point equation, where its solutions depend on the particular configurations of the disorder. The saddle-point equation for a particular realization of the disorder is given by Eq. (2.30). We must remember that the situation with several local minima, which is observed in such expression, precludes the realization of a perturbative expansion in a straightforward way. In the replica method, after integrating out the disorder field the saddle-point equation for n replicas is given by Eq. (2.36). In the zeta-distributional method the saddle-point in a generic replica partition function, $\mathbb{E}[Z^k]$, is exactly the same but instead, we have the corresponding k replicas. We have

$$\left(-\Delta + m_0^2\right)\varphi_i(x) + \frac{\lambda_0}{3!}\varphi_i^3(x) = \sigma\sum_{j=1}^{k}\varphi_j(x). \tag{5.3}$$

In the distributional zeta-function method, the average free energy is written as a series of the integer moments of the partition function of the model. Therefore, the only choice in each replica partition function is the replica symmetric ansatz, *i.e.* all replica fields must be equal in each replica partition function, $\varphi_i(x) = \varphi_j(x)$. This choice then implies that the saddle-point equations read

$$\left(-\Delta + m_0^2 - k\sigma\right)\varphi_i(x) + \frac{\lambda_0}{3!}\varphi_i^3(x) = 0. \tag{5.4}$$

In principle, observe that in this approach one must take into account all replica partition functions contributing to the average free energy, *i.e.* all values of k must be considered. In the following, in order to proceed, we are assuming $m_0^2 > 0$. Let us define a critical k_c such that $k_c = \lfloor m_0^2/\sigma \rfloor$ where $\lfloor x \rfloor$ means the integer part of x. For $m_0^2 \geqslant \sigma$, in a generic replica partition function, $m_0^2 - k\sigma \geqslant 0$ is satisfied as $k \leqslant k_c$. In such a case, each replica field fluctuates around the zero value, the stable equilibrium state. One must notice an interesting fact here, the effective mass of the replica fields in different replica partition functions are not equal. This situation is quite different when the contribution to the average free energy comes from the replica partition functions where $k > k_c$. From Eq. (5.4), all of these replica fields fluctuate around the zero value which is not an equilibrium state anymore. In the framework of field operators, this means that the vacuum expectation value of such fields does not vanish. This is exactly the scenario that a spontaneous symmetry breaking emerges.

Before continuing, we would like to summarize the main differences between the consequences of our formalism and the standard replica method. In the standard replica method, in the replica partition function, we must take the limit $k \to 0$. After choosing the replica symmetric ansatz, the saddle- point equation reduces to the standard model without disorder. In our formalism, for each replica field theory, investigating the saddle-point equations and imposing the replica symmetric ansatz we obtain a critical k_c. We can now ask what assumptions we must use to circumvent the above mentioned problem.

The point that we wish to stress is that due to replica fields for replica partition functions such that $k > k_c$, a spontaneous symmetry breaking mechanism occurs. To proceed, let us investigate some choices in the replica space. An interesting question is whether there are different choices for replica symmetry breaking. Consider a generic term of the series given by Eq. (2.63) with replica partition function given by $\mathbb{E}\,[Z^{\,l}]$. One choice in the structure of the fields in each replica partition function is given by

$$\begin{cases} \varphi_i^{(l)}(x) = \varphi(x) & \text{for } l = 1, 2, ..., k_c \\ \varphi_i^{(l)}(x) = 0 & \text{for } l > k_c, \end{cases} \tag{5.5}$$

where for the sake of simplicity we still employ the same notation for the field. However, the effect of this choice may represent a very constraining truncation for the series representation of the average free energy given by Eq. (2.67). Indeed, as discussed previously, this choice in replica space is not consistent with the distributional zeta-function method. In order to take into account more terms in this series, we consider $N > k_c$, where $m_0^2 - k\sigma < 0$, for $N > k > k_c$. To proceed, we must study in each replica partition function the vacuum structure that emerges in our scenario. In this situation we must consider the following structure of the replica space

$$\begin{cases} \varphi_i^{(l)}(x) = \varphi(x) & \text{for } l = 1, \cdots, k_c \text{ and } i = 1, \cdots, l \\ \varphi_i^{(l)}(x) = \phi(x) + v & \text{for } l = k_c + 1, \cdots, N \text{ and } i = 1, \cdots, l \\ \varphi_i^{(l)}(x) = 0 & \text{for } l > N, \end{cases} \tag{5.6}$$

where

$$v = \left(\frac{6(\sigma N - m_0^2)}{\lambda_0} \right)^{1/2}. \tag{5.7}$$

In terms of these new shifted fields, we get a positive mass squared with new self-interaction vertices ϕ^3 and ϕ^4. There is a spontaneous symmetry breaking for a finite N. We are interested in the case with large-N, which will be discussed in the following sections. This structure in replica space, defined by Eq. (5.6), also with the large-N limit is quite natural and it is the only choice compatible with the method developed in Ref. [44]. In conclusion, our arguments stated here show the uniqueness of the solution for the problem of the structure in replica space. Notice that all replica fields are the same in each replica partition function. This is not true anymore for different replica partition functions. Being more precise, in the scenario constructed by the replica method, the breaking of replica symmetry in a unique replica partition function occurs by choosing different replica fields. In the distributional zeta-function method, *a priori*, all replica fields are the same in each replica partition function. Since the replica fields of different replica partition function are different, we also call it replica symmetry breaking.

In the following, we are using the structure of replica space given by Eq. (5.6). With this choice, the dominant contribution to the average free energy can be written as

$$F(a) = \sum_{k=1}^{N} \frac{(-1)^k a^k}{k!k} \mathbb{E}\,[Z^k]. \tag{5.8}$$

Notice that this series representation has two kinds of replica partition functions. For $k \leqslant k_c$, $\mathbb{E}\,[Z^k]$ is given by Eqs. (2.34) and (2.35). For $k_c < k \leqslant N$, the replica partition function $\mathbb{E}\,[Z^k]$ is

$$\mathbb{E}[Z^k] = \int \prod_{j=1}^{k} [d\phi_j] \exp\Big[-S_{\text{eff}}\left(\phi_j\right)\Big], \tag{5.9}$$

where $S_{\text{eff}}\left(\phi_j\right)$ is given by

$$\begin{aligned} S_{\text{eff}}\left(\phi_i\right) =& \frac{1}{2}\sum_{i,j=1}^{k} \int d^dx d^dy \phi_i(x) C_{ij}(x-y)\phi_j(y) \\ &+ \frac{\lambda_0 \nu}{3!}\sum_{i=1}^{k} \int d^dx\, \phi_i^3(x) + \frac{\lambda_0}{4!}\sum_{i=1}^{k} \int d^dx\, \phi_i^4(x), \end{aligned} \tag{5.10}$$

where the operator $C_{ij}(x-y)$ is

$$C_{ij}(x-y) = \Big[(-\Delta + 3\sigma N - 2m_0^2)\delta_{ij} - \sigma\Big]\delta^d(x-y). \tag{5.11}$$

The Eq. (5.10) it is the first important result of our approach, *i.e.* the spontaneous symmetry breaking in the disorder scenario. As in the case without spontaneous symmetry breaking, performing a Fourier transform from the quadratic part, in the Eq. (5.10), we can again identify the inverse of the two-point correlation function of the replica field theory which is now given by

$$[G_0]_{ij}^{-1} = \left(p^2 + 3\sigma N - 2m_0^2\right)\delta_{ij} - \sigma. \tag{5.12}$$

Using the projector operators we can write the correlation function $[G_0]_{ij}$ as

$$\begin{aligned} \left[G_0\right]_{ij}(p) =& \frac{\delta_{ij}}{\left(p^2 + 3\sigma N - 2m_0^2\right)} \\ &+ \frac{\sigma}{\left(p^2 + 3\sigma N - 2m_0^2\right)\left(p^2 + \sigma(3N-k) - 2m_0^2\right)}. \end{aligned} \tag{5.13}$$

It is important to point out that the main difference between the usual situation considered in the literature and the scenario discussed by us is that Goldstone bosons appear when there are a breaking of a continuous symmetry. There are no Goldstone bosons in the model, since we are breaking a discrete symmetry. This issue will be clarified in Sec. (5.2). As we discussed before, for a large enough, the leading term in the series representation defined by Eq. (5.8) is given by $k = N$. In this situation the replica partition function, $\mathbb{E}\left[Z^N\right]$, for $m_0^2 \geqslant \sigma N$, all the replica fields are oscillating around the trivial vacuum. For $m_0^2 < \sigma N$, all the replica fields now oscillate around the non-trivial vacuum. In this case, the replica partition function reads

$$\mathbb{E}[Z^N] = \int \prod_{i=1}^{N} [d\phi_i] \exp\left[-S_{\text{eff}}(\phi_i)\right], \tag{5.14}$$

where the effective action $S_{\text{eff}}(\phi_i)$ is given by

$$\begin{aligned} S_{\text{eff}}(\phi_i) = \frac{1}{2} \sum_{i,j=1}^{N} \int d^d x \int d^d y \; \phi_i(x) C_{ij}(x-y) \phi_j(y) \\ + \frac{\lambda_0 \nu}{3!} \sum_{i=1}^{N} \int d^d x \, \phi_i^3(x) + \frac{\lambda_0}{4!} \sum_{i=1}^{N} \int d^d x \, \phi_i^4(x), \end{aligned} \tag{5.15}$$

and $C_{ij}(x-y)$ is given by Eq. (5.11).

Let us summarize our results. The leading contribution for the free energy consists in a series in which all the replica partition functions contribute. The subtle issue here is that as we perform an expansion in the integer moments of the partition function, we choose the structure in the replica space with the most symmetric case, namely all replica fields are the same in each replica partition function. All of the above discussion leads us to the large-N scenario in replica field theory. Notice that instead of having one 't Hooft coupling, which means that $g_0 = \lambda_0 N$ is finite although $N \to \infty$ and $\lambda_0 \to 0$, we also have another 't Hooft coupling, $f_0 = \sigma N$ which is finite although $N \to \infty$ and $\sigma \to 0$ (weak disorder). Here, we are interested in the situation where the disorder is weak. In this context we have just established a path to clarify the relationship between two hitherto unconnected results. It is known that for $d > 6$, the critical region in the random field Ising model can be described using the mean-field exponents [134]. In turn, in the $O(N)$ symmetric field theory of any real scalar fields with interaction $\lambda_0 \left(\varphi_i^2 \varphi_i^2\right)^2/4!$, the $1/N$ expansion for $d > 6$ is not useful [135]. Hence the $1/N$ expansion is efficient when disorder affects the critical region in the random field Ising model in a non-trivial way. We interpret this connection as a consequence of approaching quenched disorder in a large-N scenario in replica field theory. In any case, despite the above remark we assert that all calculations can be carried out irrespective of the space dimensions; in particular for $d < 6$ one may resort to a large-N expansion.

5.2 TEMPERATURE EFFECTS IN THE REPLICA FIELD THEORY

The aim of this section is to discuss temperature effects in the replica field theory defined by Eqs. (5.14) and (5.15). As we discussed before, when the quantum fluctuations are replaced by the thermodynamic ones, the model studied is the continuous version of the random field Ising model in a d-dimensional space. In order to describe the phase transition in this model we follow the Landau-Ginzburg phenomenological approach where, for a system without disorder, the mass squared depends on the reduced temperature, defined by $t = (T - T_c)/T_c$, where T is the temperature and T_c is the critical temperature of the system. To go beyond the tree-level approximation with quantum fluctuations we assume that the fields in the replica field theory satisfy periodic boundary condition in Euclidean time.

Landau-Ginzburg approach in replica field theory

The model considered in this work is also the continuous version of the random field Ising model in a d-dimensional space, where the dependence from the temperature is concentrated in m_0^2. In the following, we continue to use the semiclassical (tree) approximation. For a system without disorder at sufficiently high temperatures there is no spontaneous symmetry breaking, where the system presents a $\mathbb{Z}_2$ symmetry. On the other hand, in the low temperature regime ($T < T_c$), we have a spontaneous symmetry breaking, i.e., the $\mathbb{Z}_2$-symmetry is broken.

In this disordered system, this situation is more involved, since the average free energy is written as a series defined by Eq. (5.8). Inspired in the above situation, we will assume that m_0^2 depends on the temperature, it is not positive definite and is a monotonically increasing function on the temperature. For simplicity, let us assume that the disorder is weak and fixed. Before taking the large-N limit, one has three interesting cases, with two temperatures, $T_c^{(1)}$ and $T_c^{(2)}$.

I. For temperatures such that $m_0^2 \geqslant \sigma N$, all the replica fields in the replica partition functions in Eq. (5.8) oscillate around the trivial vacuum $\varphi = 0$. In this case, for a very large a, the average free energy is written as

$$F(a) = \sum_{k=1}^{N} \frac{(-1)^k a^k}{k!k} \mathbb{E}^{(1)}[Z^k], \tag{5.16}$$

where the replica partition functions $\mathbb{E}^{(1)}[Z^k]$ are

$$\mathbb{E}^{(1)}[Z^k] = \int \prod_{i=1}^{k} [d\varphi_i] \exp\Big(-S_{\text{eff}}^{(1)}(\varphi_i)\Big). \tag{5.17}$$

The effective action $S^{(1)}_{\text{eff}}(\varphi_i)$ is given by

$$\begin{aligned} S^{(1)}_{\text{eff}}(\varphi_i) &= \int d^d x \Bigg[\sum_{i=1}^{k} \Big(\frac{1}{2} \varphi_i(x) \big(-\Delta + m_0^2\big) \varphi_i(x) \\ &+ \frac{g_0}{4!N} \varphi_i^4(x) \Big) - \frac{f_0}{2N} \sum_{i,j=1}^{k} \varphi_i(x) \varphi_j(x) \Bigg]. \end{aligned} \tag{5.18}$$

In the large-N limit, such that $a \gg N$, the leading term of the series of the average free energy is given by the replica partition function with N fields φ_i. Hence, we have the symmetry $[\mathbb{Z}_2 \times \mathbb{Z}_2 \cdots \times \mathbb{Z}_2]$ for N replica fields. The temperature $T_c^{(1)}$ occurs when $m_0^2 = N\sigma$. Below this temperature, $[\mathbb{Z}_2 \times \mathbb{Z}_2 \cdots \times \mathbb{Z}_2]$ symmetry is broken.

II. For $\sigma N > m_0^2 \geqslant \sigma$, the temperature decreases. Before taking the large-N limit, all the replica fields of some replica partition functions oscillate around the non-trivial vacuum, and all the replica fields of the remaining replica partition functions oscillate around $\phi = 0$. Defining $k_c(T) = \lfloor m_0^2(T)/\sigma \rfloor$, we can write the series representation of the average free energy in the Landau-Ginzburg approach as

$$\begin{aligned} F(a) &= \sum_{k=1}^{k_c(T)} \frac{(-1)^k a^k}{k!k} \mathbb{E}^{(1)}[Z^k] \\ &+ \sum_{k=k_c(T)+1}^{N} \frac{(-1)^k a^k}{k!k} \mathbb{E}^{(2)}[Z^k], \end{aligned} \tag{5.19}$$

where $\mathbb{E}^{(1)}[Z^k]$ is given by Eq. (5.17) and

$$\mathbb{E}^{(2)}[Z^k] = \int \prod_{i=1}^{k} [d\phi_j] \exp\Big(-S^{(2)}_{\text{eff}}(\phi_j)\Big). \tag{5.20}$$

The effective action $S^{(2)}_{\text{eff}}(\phi_i)$ is written as

$$\begin{aligned} S^{(2)}_{\text{eff}}(\phi_i) &= \int d^d x \Bigg[\sum_{i=1}^{k} \Big(\frac{1}{2} \phi_i(x) \big(-\Delta + 3f_0 - 2m_0^2\big) \phi_i(x) \\ &+ \Big(\frac{f_0 g_0}{3!N}\Big)^{\frac{1}{2}} \Big(1 - \frac{m_0^2}{f_0}\Big)^{\frac{1}{2}} \phi_i^3(x) + \frac{g_0}{4!N} \phi_i^4(x) \Big) \\ &- \frac{f_0}{2N} \sum_{i,j=1}^{k} \phi_i(x) \phi_j(x) \Bigg]. \end{aligned} \tag{5.21}$$

Therefore, in this region one has two types of replica partition functions in the series representation of the average free energy. In the large-N approximation, using again that $a \gg N$, the average free energy is described by a unique replica partition function with all replica fields oscillating around the non-trivial vacuum.

III. For $m_0^2 < \sigma$, all the replica fields in replica partition functions are oscillating around the non-trivial vacuum. The temperature $T_c^{(2)}$ is given by $m_0^2 = \sigma$. The average free energy describing this case is given by

$$F(a) = \sum_{k=1}^{N} \frac{(-1)^k a^k}{k!k} \mathbb{E}^{(2)}[Z^k], \tag{5.22}$$

where $\mathbb{E}^{(2)}[Z^k]$ is given by Eq. (5.20).

For $a \gg N$, a very large N limit consists in taking the leading term of the series, which is given by a unique replica partition function with N replica fields ϕ_i. This situation is equivalent to the $\mathbb{Z}_2$-broken symmetry for a system without disorder. The symmetry $[\mathbb{Z}_2 \times \mathbb{Z}_2 \cdots \times \mathbb{Z}_2]$ for N replica fields remains broken.

In summary, in the disordered system, before taking the large-N approximation, there are two temperatures, $T_c^{(1)}$ and $T_c^{(2)}$. Above $T_c^{(1)}$ the average free energy is written by a series of replica partition functions where in all of them the replica fields are oscillating around the trivial vacuum. Below $T_c^{(1)}$ and above $T_c^{(2)}$ the average free energy is defined by two kinds of replica partition functions with replica fields φ_i and ϕ_i respectively. In the large-N limit, as only the leading term is considered, one has that all the replica fields of this leading replica partition function are oscillating around the non-trivial vacuum. Below $T_c^{(2)}$ all the replica partition functions that define the average free energy are composed by ϕ_i fields. In the large-N regime, as one is forced to consider the leading replica partition function, one has only one phase transition temperature, *i.e.* $T_c^{(1)}$.

Finite size effects in the the replica field theory

Here we are investigating temperature effects in a disordered $\lambda\varphi^4$ model defined in a d-dimensional Euclidean space going beyond the tree-level approximation. We assume that the fields in the replica field theory satisfy periodic boundary condition in Euclidean time and that $k_c < 1$, where we have spontaneous symmetry breaking.

Periodic boundary condition in Euclidean time implies that this replica field theory is defined in $S^1 \times \mathbb{R}^{d-1}$ with the Euclidean topology for a field theory at finite temperature [136, 137]. We consider the system defined in a space with periodic boundary conditions in Euclidean time using the following non-trivial replica structure given by Eq. (5.6). In this situation, the momentum-space integrals over one component is replaced by a sum over discrete frequencies. Let us define the radius of the compactified dimension of the system by $\beta = T^{-1}$, where T is the temperature of the system.

Let us calculate the one-loop correction to renormalized mass. We have two types of loop-corrections, one from the ϕ^4 vertex, which is written as

$$[G^{(4)}]_{lm}(x-y,\beta) = \sum_{i=1}^{N} \int d^d z\, [G_0]_{li}(x-z,\beta)[G_0]_{ii}(z-z,\beta)[G_0]_{im}(z-y,\beta), \tag{5.23}$$

and, another contribution, from two ϕ^3 vertices

$$[G^{(3)}]_{lm}(x-y,\beta) = \tag{5.24}$$

$$\sum_{ij=1}^{N} \int d^d z d^d z' [G_0]_{li}(x-z,\beta)[G_0]^2_{ij}(z-z',\beta)[G_0]_{jm}(z'-y,\beta). \tag{5.25}$$

To compute the renormalized mass, we must study the amputated correlation function in replica space. At the one-loop approximation, defining $M_0^2 = 3\sigma N - 2m_0^2$, the renormalized temperature-dependent mass squared can be written as

$$m_R^2(M_0,\beta,\sigma) = m_1^2(M_0,\beta,\sigma) + m_2^2(M_0,\beta,\sigma) \tag{5.26}$$

where

$$m_1^2(M_0,\beta,\sigma) = M_0^2 + \frac{\lambda_0}{2} \sum_{k=1}^{N} \Big[f_1(M_0,\beta;k) + f_2(M_0,\beta,\sigma;k) \Big] \tag{5.27}$$

and

$$m_2^2(M_0,\beta,\sigma) = \sum_{k=1}^{N} \Big[m_a^2(M_0,\beta;k) + m_b^2(M_0,\beta,\sigma;k) + m_c^2(M_0,\beta,\sigma;k) \Big]. \tag{5.28}$$

All these quantities are discussed in Sec. (5.4). For a very large N in $d = 4$ the temperature dependent renormalized mass squared can be written as

$$m_R^2(\beta) = M_0^2 + \frac{\lambda_0 N}{4\pi^2}\left[\sqrt{\frac{\pi}{2}}\sum_{n=1}^{\infty}\sqrt{\frac{M_0}{(n\beta)^3}}e^{-n\beta M_0} + \sqrt{\pi}\left(\frac{1}{\sqrt{2}}-\frac{1}{\sigma}\right)\sum_{n=1}^{\infty}\frac{1}{\sqrt{n\beta M_0}}e^{-n\beta M_0}\right]. \tag{5.29}$$

For the large-N limit the thermal mass correction in the one-loop approximation is given by Eq. (5.29). Notice that in the above equation there is a term proportional to σ^{-1}, a non-perturbative effect produced by the disorder. In turn, for a weak disorder parameter $\sigma \ll 1$ and for sufficiently small temperatures, $\beta \gg 1$, the last term dominates over the second and the third. In this case it is easy to see that there is a specific temperature in which the renormalized mass squared goes to zero. One says that the system of large-N replica fields presents a phase transition at such a critical temperature. One way to proceed is to use the gap equation to obtain non-perturbative results.

In the next section we use again the mean-field descriptions for phase transitions. We will restrict our attention to the regime of very low temperatures, investigating a different perturbative expansion for the replica field theory.

5.3 REPLICA INSTANTONS IN THE LARGE-N APPROXIMATION

The aim of this section is to show the presence of instantons (real or complex) in the model at some range of temperatures. At this point, let us introduce an external source $J_i(x)$ in replica space linearly coupled with each replica field. From Eqs. (5.14) and (5.15), we are able to define the generating functional of all correlation functions for a large-N Euclidean replica field theory as $\mathbb{E}[Z^N(J)] = \mathcal{Z}(J)$. Hence it is possible to define the generating functional of connected correlation functions and also the generating functional of one-particle irreducible correlations (vertex functions) in the theory. From the effective action it is possible to find the effective potential of this theory. This is a natural tool to investigate the vacuum structure of the field theory.

However, in the following, we are going to discuss a different perturbative expansion. Let us define $R(x-y) = \sigma\delta^d(x-y)$ and at the large-N limit we must have a fixed $f_0 = \sigma N$ as we discussed before. We write the replica partition function $\mathcal{Z}(J)$ as a functional differential operator acting on a modified replica partition function without the interaction between the replicas that we call $Q_0(J)$. This is a good representation for $\mathcal{Z}(J)$ in the weak disorder limit, and also for $m_0^2 < \sigma N$. The representation for the replica partition function, in the presence of an external source, is similar to the strong-coupling expansion in field theory. We have

$$\mathcal{Z}(J) = \exp\left[-\frac{1}{2}\sum_{i,j=1}^{N}\int d^dx\, d^dy \frac{\delta}{\delta J_i(x)} R(x-y) \frac{\delta}{\delta J_j(y)}\right] Q_0(J), \quad (5.30)$$

where $Q_0(J)$ is given by

$$Q_0(J) = \int \prod_{j=1}^{N} [d\phi_j] \exp\left[-S_{\text{eff}}^{(0)}(\phi_j, J)\right]. \quad (5.31)$$

In the above equation, taking the large-N limit, $S_{\text{eff}}^{(0)}(\phi_i, J)$ is defined as

$$S_{\text{eff}}^{(0)}(\phi_i, J) = \sum_{i=1}^{N}\int d^dx \left[\frac{1}{2}\phi_i(x)\left(-\Delta + 3f_0 - 2m_0^2\right)\phi_i(x)\right. \quad (5.32)$$

$$\left. + \left(\frac{f_0 g_0}{3!N}\right)^{\frac{1}{2}} \left(1 - \frac{m_0^2}{f_0}\right)^{\frac{1}{2}} \phi_i^3(x) + \frac{g_0}{4!N}\phi_i^4(x) + J_i(x)\phi_i(x)\right]. \quad (5.33)$$

The action defined by the above equation describes a large-N replica field theory with two fixed parameters g_0 and f_0. Notice that all the ultraviolet divergences of this model are fixed by Eqs. (5.31) and (5.33). It is possible to go beyond the tree-level approximation. Working with the bare quantities, and introducing the renormalization constants Z_ϕ, Z_g and Z_m one is able to renormalize the model for $d \leqslant 4$. This is the standard procedure. All the divergences of this theory can be eliminated by a wave function, coupling constant and mass renormalization procedure. In practice, performing the perturbative expansion defined by Eq. (5.30) it is not difficult. For instance, the two-point correlation function is defined as

$$\langle\phi_i(x)\phi_j(y)\rangle = \left.\frac{\delta^2 \mathcal{Z}(J)}{\delta J_i(x)\delta J_j(y)}\right|_{J_i=J_j=0}. \quad (5.34)$$

In the following we are interested to go in another direction. We would like to investigate the vacuum structure in the first term of the Eq. (5.30). For each replica field, we can define the following potential $U(\phi)$

$$U(\phi) = \frac{1}{2}(3f_0 - 2m_0^2)\phi^2 + \frac{\lambda_0 \nu}{3!}\phi^3 + \frac{\lambda_0}{4!}\phi^4, \quad (5.35)$$

where $\nu = \sqrt{6(f_0 - m_0^2)/\lambda_0}$. The false and the true vacuum states $\phi_\pm$ are given by

$$\phi_\pm = -\frac{3\nu}{2} \pm 3\sqrt{-\frac{f_0}{2\lambda_0} - \frac{m_0^2}{6\lambda_0}}. \quad (5.36)$$

Therefore, we obtained the following interesting result: there are instantons in our model. For $f_0 > m_0^2 > -3f_0$, the system develops a spontaneous symmetry breaking in the replica partition function. In this case, all N instantons are complex. On the other hand, for $m_0^2 < -3f_0$ we get a similar situation as before, however all the N instantons are real [50].

Let us briefly discuss the decay rate for one replica field in this case of real instantons. Since we would like to discuss such problems exactly as in the bounce problem in quantum mechanics let us define an Euclidean time τ such that $\phi(x) \equiv \phi(\tau, \vec{x})$. We have a false vacuum in the infinite past and we come back to it in the infinite future

$$\phi(\tau, \vec{x}) \to \phi_+, \qquad \tau \to \pm\infty. \quad (5.37)$$

In order to have a finite action for the bounce, we also need to go to the vacuum value at spatial infinity. Hence we have

$$\phi(\tau, \vec{x}) \to \phi_+, \qquad |\vec{x}| \to \pm\infty. \quad (5.38)$$

As discussed in the literature the picture is a formation of bubbles in the middle of the false vacuum. Actually, asymptotically in Euclidean space the replica configuration is in the false vacuum. A different state appears in the core of the bubble. The probability of decay can be calculated. There is a standard procedure to find the decay rate in a scalar theory [138, 139]. One interesting unsolved problem is the phase diagram of liquids in a random porous media. For strong coupling between the fluid and the porous media, for such confined fluids, the random field Ising model is used to describe such systems. These systems can develop a second or a first-order phase transition. Since bubble nucleation is a first order phase transition, we expect that our approach reveals a route to investigate such systems.

5.4 THE RENORMALIZED MASS IN ONE-LOOP APPROXIMATION

The aim of this section is to discuss the temperature-dependent renormalized mass in one-loop approximation considered in Sec. (5.2). We consider the system at finite temperature, i.e., with periodic boundary conditions in Euclidean time using the non-trivial structure in the replica space given by Eq. (5.6).In this situation, the momentum-space integrals over one component is replaced by a sum over discrete frequencies. For the case of Bose fields we must perform the replacement

$$\int \frac{d^d p}{(2\pi)^d} f(p) \to \frac{1}{\beta} \int \frac{d^{d-1} p}{(2\pi)^{d-1}} \sum_{n=-\infty}^{\infty} f\left(\frac{2n\pi}{\beta}, \mathbf{p}\right), \tag{5.39}$$

where β is the radius of the compactified dimension of the system. This field theory on $S^1 \times \mathbb{R}^{d-1}$ has the Euclidean topology of a field theory at finite temperature. From the two-point Schwinger function we have to calculate

$$f^{(1)}_{ij}(M_0, \beta) = \frac{\delta_{ij}}{\beta} \int \frac{d^{d-1} p}{(2\pi)^{d-1}} \sum_{n=-\infty}^{\infty} \frac{1}{\left[\left(\frac{2\pi n}{\beta}\right)^2 + \mathbf{p}^2 + M_0^2\right]} \tag{5.40}$$

and

$$f^{(2)}(M_0, \beta, \sigma) = \frac{\sigma}{\beta} \int \frac{d^{d-1} p}{(2\pi)^{d-1}} \times$$
$$\sum_{n=-\infty}^{\infty} \frac{1}{\left[\left(\frac{2\pi n}{\beta}\right)^2 + \mathbf{p}^2 + M_0^2\right]\left[\left(\frac{2\pi n}{\beta}\right)^2 + \mathbf{p}^2 + M_0^2 - k\sigma\right]}, \tag{5.41}$$

where $\mathbf{p} = (p^2, p^3, .., p^d)$. The integral $f^{(1)}_{ij}(M_0, \beta)$ can be calculated using dimensional regularization, Eq (3.10). We obtain

$$f^{(1)}_{ij}(M_0, \beta) = \frac{\delta_{ij}}{2\beta} \frac{1}{(2\sqrt{\pi})^{d-1}} \Gamma\left(\frac{3-d}{2}\right) \sum_{n=-\infty}^{\infty} \frac{1}{\left[\left(\frac{2\pi n}{\beta}\right)^2 + M_0^2\right]^{\frac{3-d}{2}}}. \tag{5.42}$$

After using dimensional regularization, we have to analytically extend the modified Epstein zeta function, Eq. (3.14). Using a modified minimal subtraction renormalization scheme and taking $f_1(M_0, \beta; k) = \delta^{ij} f^{(1)}_{ij}(M_0, \beta; k)$ we get

$$f_1(M_0, \beta; k) = \frac{k}{(2\pi)^{d/2}} \sum_{n=1}^{\infty} \left(\frac{M_0}{n\beta}\right)^{\frac{d}{2}-1} K_{\frac{d}{2}-1}(n\beta M_0). \tag{5.43}$$

Let us discuss $f^{(2)}(\sigma, M_0, \beta; k)$. We have

$$f^{(2)}(M_0, \beta, \sigma; k) = \frac{\sigma}{\beta} r(d) \int dq\, q^{d-2} \times$$
$$\sum_{n=-\infty}^{\infty} \frac{1}{\left[\left(\frac{2\pi n}{\beta}\right)^2 + q^2 + M_0^2\right]\left[\left(\frac{2\pi n}{\beta}\right)^2 + q^2 + M_0^2 - k\sigma\right]}, \tag{5.44}$$

where

$$r(d) = \frac{2\pi^{(d-2)/2}}{\Gamma\left(\frac{d-2}{2}\right)}, \tag{5.45}$$

is an analytic function in d. Let us use the following integral

$$\int_0^\infty dx \frac{x^{\mu-1}}{(x^2+\alpha)(x^2+\gamma)} = \frac{\pi}{2}\frac{\gamma^{(\frac{\mu}{2}-1)} - \alpha^{(\frac{\mu}{2}-1)}}{\alpha-\gamma} \csc\left(\frac{\pi\mu}{2}\right). \tag{5.46}$$

Defining

$$q(d) = \frac{\pi}{2} r(d) \csc\left[\frac{\pi}{2}(d-1)\right], \tag{5.47}$$

we can write $f^{(2)}(\sigma, M_0, \beta, k)$ as

$$f^{(2)}(M_0, \beta, \sigma; k) = f^{(21)}(M_0, \beta; k) + f^{(22)}(M_0, \beta, \sigma; k), \tag{5.48}$$

where

$$f^{(21)}(M_0, \beta; k) = -\frac{q(d)}{\beta k} \sum_{n=-\infty}^{\infty} \frac{1}{\left[\left(\frac{2\pi n}{\beta}\right)^2 + M_0^2\right]^{\frac{3-d}{2}}} \tag{5.49}$$

and

$$f^{(22)}(M_0, \beta, \sigma; k) = \frac{q(d)}{\beta k} \sum_{n=-\infty}^{\infty} \frac{1}{\left[\left(\frac{2\pi n}{\beta}\right)^2 + M_0^2 - k\sigma\right]^{\frac{3-d}{2}}}. \tag{5.50}$$

Using again the definition of the Epstein zeta function and $c(d)$ given by

$$c(d) = \frac{\pi^{(3d-6)/2}}{2^{(3-d)}\Gamma\left(\frac{d-2}{2}\right)} \csc\left[\frac{\pi}{2}(d-1)\right], \tag{5.51}$$

we can write $f^{(21)}(M_0, \beta; k)$ and $f^{(22)}(\sigma, M_0, \beta, k)$ respectively as

$$f^{(21)}(M_0, \beta; k) = -\frac{c(d)}{k} \beta^{2-d} E\left(\frac{3-d}{2}, \frac{M_0\beta}{2\pi}\right) \tag{5.52}$$

and

$$f^{(22)}(M_0, \beta, \sigma; k) = \frac{c(d)}{k} \beta^{2-d} E\left(\frac{3-d}{2}, \frac{\beta}{2\pi}\sqrt{M_0^2 - k\sigma}\right). \tag{5.53}$$

We will use once again the analytic representation of the function $E(s, a)$ and the modified minimal subtraction renormalization scheme. Defining

$$g_1(M_0; d, k) = 2\pi^{\frac{d-3}{2}} \frac{\Gamma(\frac{d-1}{2})}{\Gamma(\frac{d-2}{2})} \frac{M_0^{d-2}}{k} \tag{5.54}$$

and

$$g_2(M_0; d, k) = 2\pi^{\frac{d-1}{2}} \frac{\Gamma(\frac{d-1}{2})}{\Gamma(\frac{d-2}{2})} \frac{(M_0^2 - k\sigma)^{\frac{d-2}{2}}}{k} \tag{5.55}$$

we can write the Eq. (5.52) and Eq. (5.53) as

$$f^{(21)}(M_0, \beta; k) = -g_1 \sum_{n=0}^{\infty} (nM_0\beta)^{\frac{3-d}{2}} K_{\frac{3-d}{2}}(n\beta M_0) \tag{5.56}$$

and

$$f^{(22)}(M_0, \beta, \sigma; k) = g_2 \sum_{n=0}^{\infty} \left(n\beta\sqrt{M_0^2 - k\sigma}\right)^{\frac{3-d}{2}} K_{\frac{3-d}{2}}\left(n\beta\sqrt{M_0^2 - k\sigma}\right). \tag{5.57}$$

The renormalized temperature-dependent mass squared $m_1^2(M_0, \beta, \sigma; k)$ can be written as

$$m_1^2(M_0, \beta, \sigma; k) = M_0^2 + \lambda \sum_{k=1}^{N} \left[f_1(M_0, \beta; k) + f^{(21)}(M_0, \beta; k) + f^{(22)}(M_0, \beta, \sigma; k) \right]. \tag{5.58}$$

Let us defined $m_a^2(M_0, \beta; k)$, $m_b^2(M_0, \beta, \sigma; k)$ and $m_c^2(M_0, \beta, \sigma; k)$ such that the contribution given by $m_2^2(M_0, \beta, \sigma; k)$ is written as

$$m_2^2(M_0, \beta, \sigma) = \sum_{k=1}^{N} \Big[m_a^2(M_0, \beta; k) + m_b^2(M_0, \beta, \sigma; k) + m_c^2(M_0, \beta, \sigma; k) \Big]. \tag{5.59}$$

We have

$$m_a^2(M_0, \beta; k) = \frac{k}{\beta} \int \frac{d^{d-1}p}{(2\pi)^{d-1}} \sum_{n=-\infty}^{\infty} \frac{1}{\left[\left(\frac{2\pi n}{\beta}\right)^2 + \mathbf{p}^2 + M_0^2 \right]^2}, \tag{5.60}$$

$$m_b^2(M_0, \beta, \sigma; k) = \frac{k\sigma}{\beta} \int \frac{d^{d-1}p}{(2\pi)^{d-1}} \times \sum_{n=-\infty}^{\infty} \frac{1}{\left[\left(\frac{2\pi n}{\beta}\right)^2 + \mathbf{p}^2 + M_0^2 \right]^2 \left[\left(\frac{2\pi n}{\beta}\right)^2 + \mathbf{p}^2 + M_0^2 - k\sigma \right]} \tag{5.61}$$

and finally

$$m_c^2(M_0,\beta,\sigma;k) = \frac{\sigma^2}{\beta}\int \frac{d^{d-1}p}{(2\pi)^{d-1}}\times$$
$$\sum_{n=-\infty}^{\infty} \frac{1}{\left[\left(\frac{2\pi n}{\beta}\right)^2 + \mathbf{p}^2 + M_0^2\right]^2\left[\left(\frac{2\pi n}{\beta}\right)^2 + \mathbf{p}^2 + M_0^2 - k\sigma\right]^2}. \quad (5.62)$$

After using dimensional regularization and considering the analytical extension for the Epstein function we can write $m_a^2(M_0, L; k)$ as

$$m_a^2(M_0,\beta;k) = \frac{M_0^{d-4}k}{(2\pi)^{d/2}}\sum_{n=1}^{\infty}(n\beta M_0)^{\frac{4-d}{2}}K_{\frac{4-d}{2}}(n\beta M_0). \quad (5.63)$$

To solve the integral in m_b^2 and m_c^2, we can use the Feynman parametrization

$$\frac{1}{a^s b^l} = \frac{\Gamma(s+l)}{\Gamma(s)\Gamma(l)}\int_0^1 dx \frac{x^{s-1}(1-x)^{l-1}}{[ax+b(1-x)]^{s+l}}, \quad (5.64)$$

to write the respective integrands in an adequate form. After this and using the expression

$$\int \frac{d^d q}{(2\pi)^d}\frac{(q^2)^a}{(q^2+A)^b} = \frac{\Gamma(b-a-d/2)\Gamma(a+d/2)}{(4\pi)^{\frac{d}{2}}\Gamma(b)\Gamma(d/2)}A^{-(b-a-d/2)}, \quad (5.65)$$

and defining

$$h_1(d) = \frac{1}{(2\pi)^{\frac{d}{2}}(d-3)(d-5)}\Gamma\left(\frac{7-d}{2}\right) \quad (5.66)$$

and

$$h_2(d) = \frac{1}{(2\pi)^{\frac{d}{2}}(d-3)(d-5)(d-7)}\Gamma\left(\frac{9-d}{2}\right) \quad (5.67)$$

such contributions are given by

$$m_b^2(M_0,\beta,\sigma;k) = h_1(d)\left[\frac{\sqrt{8}(d-3)}{\Gamma\left(\frac{5-d}{2}\right)}\frac{kM_0^{d-4}}{\sigma}\sum_{n=1}^{\infty}\left(n\beta M_0\right)^{\frac{4-d}{2}}K_{\frac{4-d}{2}}(n\beta M_0)\right. \quad (5.68)$$
$$-\frac{16}{\Gamma\left(\frac{3-d}{2}\right)}\frac{M_0^{d-2}}{\sigma^2}\sum_{n=1}^{\infty}\left(n\beta M_0\right)^{\frac{2-d}{2}}K_{\frac{2-d}{2}}(n\beta M_0)$$
$$\left.+\frac{16}{\Gamma\left(\frac{3-d}{2}\right)}\frac{(M_0^2-k\sigma)^{\frac{d-2}{2}}}{\sigma^2}\sum_{n=1}^{\infty}\left(n\beta\sqrt{M_0^2-k\sigma}\right)^{\frac{2-d}{2}}K_{\frac{2-d}{2}}\left(n\beta\sqrt{M_0^2-k\sigma}\right)\right]$$

and

$$
\begin{aligned}
m_c^2(M_0,\beta,\sigma;k) = h_2(d)\Bigg[& \frac{\sqrt{32}(d-3)}{\Gamma\left(\frac{5-d}{2}\right)}\frac{M_0^{d-4}}{k^2}\sum_{n=1}^{\infty}\left(n\beta M_0\right)^{\frac{4-d}{2}}K_{\frac{4-d}{2}}(n\beta M_0) \\
& - \frac{32}{\Gamma\left(\frac{3-d}{2}\right)}\frac{M_0^{d-2}}{k^3\sigma}\sum_{n=1}^{\infty}\left(n\beta M_0\right)^{\frac{2-d}{2}}K_{\frac{2-d}{2}}(n\beta M_0) \\
& + \frac{2(d-3)}{\Gamma\left(\frac{7-d}{2}\right)}\frac{M_0^{d-6}}{k^2}\sum_{n=1}^{\infty}\left(n\beta M_0\right)^{\frac{6-d}{2}}K_{\frac{6-d}{2}}(n\beta M_0) \\
& + \frac{32}{\Gamma\left(\frac{3-d}{2}\right)}\frac{(M_0^2-k\sigma)^{\frac{d-2}{2}}}{k^3\sigma}\sum_{n=1}^{\infty}\left(n\beta\sqrt{M_0^2-k\sigma}\right)^{\frac{2-d}{2}}K_{\frac{2-d}{2}}\left(n\beta\sqrt{M_0^2-k\sigma}\right)\Bigg].
\end{aligned}
\tag{5.69}
$$

6 DISORDERED $\lambda\varphi^4 + \rho\varphi^6$ LANDAU-GINZBURG MODEL

The aim of this chapter is to emphasize the differences and similarities between the distributional zeta function method and the conventional replica method in the study of disordered systems. Here, we employ our approach to explore the free-energy landscape of the d-dimensional disordered Landau-Ginzburg $\lambda\varphi^4 + \rho\varphi^6$ model. First we write the dominant contribution to the average free energy as a series of the replica partition functions of the model. Next, the structure of the replica space is investigated using the saddle-point equations obtained from each replica field theory. Assuming the replica symmetric ansatz, we prove that the average free energy represents a system with multiple ground states with different order parameters. This situation is similar to the free energy of the spin-glass phase obtained in the Sherrington-Kirkpatrick model, in the replica symmetry breaking scenario. Also, for low temperatures we show the presence of metastable equilibrium states for some replica fields in a range of values of the physical parameters. One way to describe the spin glass behavior in the low temperature region in the model is to interpret that some terms of the series representation for the average free energy describes inhomogeneous domains. At low temperatures, some terms of the series may represent macroscopic regions in space with a given order parameter. Finally, we perform the one-loop renormalization of this model.

6.1 RANDOM TEMPERATURE LANDAU-GINZBURG MODEL

Here, we are interested in discussing the random temperature d-dimensional Landau-Ginzburg model. In the Landau-Ginzburg Hamiltonian, if λ_0 and m_0^2 are regular functions of the temperature, a random contribution $\delta m_0^2(x)$ added to m_0^2 can be considered as a local perturbation in the temperature. In this case the Hamiltonian of the model is given by

$$H(\varphi, \delta m_0^2) = \int d^d x \left[\frac{1}{2}\varphi(x)\Big(-\triangle + m_0^2 - \delta m_0^2(x)\Big)\varphi(x) + \frac{\lambda_0}{4}\varphi^4(x) + \frac{\rho_0}{6}\varphi^6(x) \right]. \tag{6.1}$$

The φ^6 contribution in the interaction Hamiltonian must be introduced to obtain a Hamiltonian bounded from below, as necessary to correctly describe the critical properties of the model. Brézin and Dominicis [140], studying a random field model, showed that new interactions should be considered. This term is related to the tricritical phenomenon [141, 142].

The local minima in the Hamiltonian are the configurations of the scalar field that satisfy the saddle-point equations where the solutions depend on the particular configuration of the random mass[1]. The existence of a large number of metastable states in many disordered systems and the loss of translational invariance makes the traditional pertubative expansion formalism quite problematic. As was discussed before, averaging the free energy over the disorder field allows us to implement a perturbative approach in a straightforward way.

The n-point correlation function associated with a disordered system

Let us briefly discuss the n-point correlation function associated with a disordered system. The disorder generating functional for one realization of the disorder is given by

$$Z(\delta m_0^2; j) = \int_{\partial\Omega} [d\varphi] \exp\left[-H(\varphi, \delta m_0^2) + \int d^d x j(x)\varphi(x)\right], \tag{6.2}$$

where a fictitious source, $j(x)$, is introduced. The n-point correlation function for one realization of disorder reads

$$\langle\varphi(x_1)\cdots\varphi(x_n)\rangle_{\delta m_0^2} = \frac{1}{Z(\delta m_0^2)} \int [d\varphi] \prod_{i=1}^{n} \varphi(x_i) \exp\left[-H(\varphi, \delta m_0^2)\right], \tag{6.3}$$

where the disordered functional integral $Z(\delta m_0^2) = Z(\delta m_0^2, j)|_{j=0}$. As in the pure system, one can define a generating functional for one disorder realization, $W_1(\delta m_0^2; j) = \ln Z(\delta m_0^2; j)$. Now, we can define a disorder-averaged correlation function as following

$$\mathbb{E}\left[\langle\varphi(x_1)\cdots\varphi(x_n)\rangle_{\delta m_0^2}\right] = \int [d\delta m_0^2] P\left(\delta m_0^2\right) \langle\varphi(x_1)\cdots\varphi(x_n)\rangle_{\delta m_0^2}, \tag{6.4}$$

where the probability distribution of the disorder is written as $[d\delta m_0^2]P\left(\delta m_0^2\right)$ where $P\left(\delta m_0^2\right)$ is Gaussian, that is, it is given by

$$P(\delta m_0^2) = p_0 \exp\left[-\frac{1}{4\sigma}\int d^d x \left(\delta m_0^2(x)\right)^2\right]. \tag{6.5}$$

1 The terms “random mass" and “random temperature"; “false vacuum" and “metastable equilibrium state"; “true vacuum" and “stable equilibrium state" are used interchangeably throughout the text.

As we know, in this case we have a delta correlated disorder field, i.e.

$$\mathbb{E}[\delta m_0^2(x)\delta m_0^2(y)] = \sigma\delta^d(x-y). \tag{6.6}$$

A relevant quantity is the disorder-averaged generating functional $W_2(j) = \mathbb{E}[W_1(\delta m_0^2; j)]$, written as

$$W_2(j) = \int [d\delta m_0^2] P(\delta m_0^2) \ln Z(\delta m_0^2; j). \tag{6.7}$$

Taking the functional derivative of $W_2(j)$ with respect to $j(x)$, we get

$$\left.\frac{\delta W_2(j)}{\delta j(x)}\right|_{j=0} = \int [d\delta m_0^2] P(\delta m_0^2) \left[\frac{1}{Z(h;j)} \frac{\delta Z(h;j)}{\delta j(x)}\right]\Bigg|_{j=0}. \tag{6.8}$$

Since $\langle\varphi(x)\rangle_{\delta m_0^2}$ is the expectation value of the field for a given configuration of the disorder in the Euclidean field theory with random mass, the above quantity is the averaged normalized expectation value of the field. Taking two functional derivatives of $W_2(j)$ with respect to $j(x)$, we get

$$\begin{aligned}\left.\frac{\delta^2 W_2(j)}{\delta j(x_1)\delta j(x_2)}\right|_{j(x_i)=0} &= \mathbb{E}\left[\langle\varphi(x_1)\varphi(x_2)\rangle_{\delta m_0^2}\right] \\ &\quad - \mathbb{E}\left[\langle\varphi(x_1)\rangle_{\delta m_0^2}\langle\varphi(x_2)\rangle_{\delta m_0^2}\right].\end{aligned} \tag{6.9}$$

Ccontrary to the pure system case, one finds that

$$\mathbb{E}\left[\langle\varphi(x_1)\rangle_{\delta m_0^2}\langle\varphi(x_2)\rangle_{\delta m_0^2}\right] \neq \mathbb{E}\left[\langle\varphi(x_1)\rangle_{\delta m_0^2}\right]\mathbb{E}\left[\langle\varphi(x_2)\rangle_{\delta m_0^2}\right] \tag{6.10}$$

This fact shows that the taking functional derivatives of $W(j)$ does not lead to connected correlation functions. Indeed, in the disordered system, there is a multivalley structure spoiling the usual perturbative approach, in which the field is expanded around only one minimum [143, 144]. One way to tackle this problem is to use the method of spectral zeta-function, which a global approach, i.e., we do not rely upon only one specific minimum. Our aim is to compute

$$W_2(j)|_{j=0} = \int [d\delta m_0^2] P(\delta m_0^2) \ln Z(\delta m_0^2). \tag{6.11}$$

which defines the average free energy of the system.

6.2 THE GLASSY-LIKE PHASE IN THE DISORDERED MODEL

In this section we will discuss the glassy-like phase in the disordered model. From the series representation of the average free energy we have to calculate the integer moments of the partition function $\mathbb{E}\left[Z^k\right]$. Using the probability distribution for the disorder and the Hamiltonian of the model, this quantity is given by

$$\mathbb{E}\left[Z^k\right] = \int \prod_{i=1}^{k} [d\varphi_i]\, e^{-H_{eff}(\varphi_i)}, \tag{6.12}$$

where the effective Hamiltonian $H_{eff}(\varphi_i)$ is written as

$$H_{eff}(\varphi_i) = \int d^d x \left[\frac{1}{2} \sum_{i=1}^{k} \varphi_i(x) \left(-\Delta + m_0^2\right) \varphi_i(x) + \frac{1}{4} \sum_{i,j=1}^{k} g_{ij}\, \varphi_i^2(x) \varphi_j^2(x) + \frac{\rho_0}{6} \sum_{i=1}^{k} \varphi_i^6(x) \right], \tag{6.13}$$

where the replica symmetric coupling constants g_{ij} are given by $g_{ij} = (\lambda_0 \delta_{ij} - \sigma)$. The saddle-point equations derived from each replica partition function read

$$\left(-\triangle + m_0^2\right) \varphi_i(x) + \lambda_0 \varphi_i^3(x) + \rho_0 \varphi_i^5(x) - \sigma \varphi_i(x) \sum_{j=1}^{k} \varphi_j^2(x) = 0. \tag{6.14}$$

Using the replica symmetric ansatz, $\varphi_i(x) = \varphi_j(x)$, the above equation becomes

$$\left(-\triangle + m_0^2\right) \varphi_i(x) + (\lambda_0 - k\sigma)\, \varphi_i^3(x) + \rho_0 \varphi_i^5(x) = 0. \tag{6.15}$$

In the replica method, using the simplest possible replica symmetric ansatz in each replica partition function, we obtain the saddle-point equations of systems without disorder. The replica symmetry breaking scheme was introduced to take into account the presence of many different local minima in the disorder Hamiltonian of the original model. For instance, a manifestation of this replica symmetry breaking appears in a ferromagnetic system with random spin bonds. There is a low-temperature regime with frustrated spin domains nucleated in a ferromagnetic background [145]. As we will see, it is possible to obtain a structure with different order parameters in the scenario constructed by the distributional zeta function method, where we are not following the standard replica symmetry breaking arguments. Indeed, consider a generic term of the series given by Eq. (2.67) with replica partition function given by $\mathbb{E}\left[Z^l\right]$ — see also Eqs. (6.12) and (6.13). We are lead to the following choice in the structure of the fields in each replica partition function

$$\begin{cases} \varphi_i^{(l)}(x) = \varphi^{(l)}(x) & \text{for } l = 1, 2, \ldots, N \\ \varphi_i^{(l)}(x) = 0 & \text{for } l > N, \end{cases} \tag{6.16}$$

where for the sake of simplicity we still employ the same notation for the field. Therefore the average free energy becomes

$$F = \frac{1}{\beta} \sum_{k=1}^{N} \frac{(-1)^k a^k}{k!\, k} \mathbb{E}\left[Z^k\right] + \cdots . \tag{6.17}$$

In Eq. (2.67), the free energy is independent of a. However the entire approach relies on the fact a can be chosen large enough so that $R(a)$ can be neglected in practice. In this case, the free energy is described by a series which is a-dependent. As we will see, this series is able to describe a system with multiple ground-states with different order parameters. A whole class of amorphous systems will be described changing this dimensionless parameter.

To proceed, the mean-field theory corresponds to a saddle-point approximation in each replica partition function. A perturbative approach give us the fluctuation corrections to mean-field theory. Hence, to implement a perturbative scheme, it is necessary to investigate fluctuations around the mean-field equations. Imposing the replica symmetric ansatz the replica partition function and the effective Hamiltonian for each replica partition function reads

$$\mathbb{E}\left[Z^k\right] = \frac{1}{k!} \int \prod_{i=1}^{k} \left[d\varphi_i^{(k)}\right] e^{-H_{eff}\left(\varphi_i^{(k)}\right)}, \tag{6.18}$$

and

$$H_{eff}\left(\varphi_i^{(k)}\right) = \int d^d x \sum_{i=1}^{k} \left[\frac{1}{2}\varphi_i^{(k)}(x)\left(-\Delta + m_0^2\right)\varphi_i^{(k)}(x) + \frac{1}{4}(\lambda_0 - k\sigma)\left(\varphi_i^{(k)}(x)\right)^4 + \frac{\rho_0}{6}\left(\varphi_i^{(k)}(x)\right)^6\right]. \tag{6.19}$$

Note that a $\frac{1}{k!}$ factor was absorbed in $\mathbb{E}\left[Z^k\right]$, which can be interpreted to represent an ensemble of k-identical replica fields. Also, the fields in each replica partition function are different since each field has the quartic coefficient $(\lambda_0 - k\sigma)$. Up to now, we have followed the approach developed in the previous chapters, where we have considered only the leading term in the series representation for the averaged free energy. However, in order to access the glassy-like phases that characterize a disordered system, we have to consider the contributions of all terms in the series given by Eq. (2.67). In the following we show that each term in the series in Eq. (2.67) describes a field theory with different order parameters. Therefore a single order parameter is insufficient to describe the low temperature phase of the disordered system.

Here we are following the discussion for the tricritical phenomenon presented in the Ref. [146]. Let us define a critical k_c for each temperature given by

$$k_c = \left\lfloor \frac{\lambda_0(T)}{\sigma} - \frac{4}{\sigma}\sqrt{\frac{m_0^2(T)\rho_0}{3}} \right\rfloor, \tag{6.20}$$

where $\lfloor x \rfloor$ means the integer part of x. Note that k_c is a function of σ, m_0, λ_0 and ρ_0. For simplicity, we consider the case where $m_0^2(T) > 0$. Possible functional forms for the squared mass and coupling constant are $m_0^2(T) = \mu^{2-\gamma}T^\gamma$ and $\lambda_0(T) = \mu^{d-4-\alpha}T^\alpha$, where μ, an arbitrary parameter, has mass dimensions. The region in the parameter space for which $k \leqslant k_c$ corresponds to the situation where metastability is absent, as the replica fields in each replica partition function fluctuate around zero value, stable equilibrium states. For $k > k_c$, the zero value for the replica fields is a metastable equilibrium state. For these replica partition functions there are first order phase transitions. The existence of domains with different order parameters can be mostly easily understood in an analogy with a dynamical phase transition induced by a deep temperature quenched [147]. Specifically, a system initially in a stable high-temperature equilibrium state will develop spatially inhomogeneous domains when quenched to sufficiently low temperatures. The dynamical evolution stops when the system reaches a new equilibrium state. The nature of the inhomogeneities depends on the equilibrium free energy landscape. In the present case, the inhomogeneities appear in the form of bubble nucleation due to the form of replica free energy in Eq. (6.19), which signals first-order phase transitions.

In the series representation for the free energy, each replica partition function is defined by a functional space where the replica fields are different. As already discussed, the contribution to the free energy that we are interested is a-dependent and, therefore, this structure with multiple ground- states with different order parameters depends on a, whose specific value depends on the physical system under consideration. This is a quite interesting situation where the structure of the vacuum states is modified by changing this dimensionless parameter. We claim that the series representation for the average free energy lends to natural interpretation of describing inhomogeneous systems. The average free energy, Eq. (6.17) can be written as

$$F = F_1 + \frac{1}{\beta}\sum_{k=k_c}^{N} \frac{(-1)^k a^k}{k}\mathbb{E}\left[Z^k\right] + \ldots, \tag{6.21}$$

where F_1 is the contribution to the average free energy for replica fields which oscillate around the true vacuum, i.e. $\varphi_0^{(k)} = 0$, for $k \leqslant k_c$. $\mathbb{E}\left[Z^k\right]$ is defined in Eqs. (6.18) and (6.19).

One interpretation for this series is that each term describes macroscopic homogeneous domains. Each domain $\Omega^{(k)}$ has at least one order parameter $\varphi_0^{(k)}$. An important question is the size of the domains in the model. The size of each domain is characterized by the correlation length $\xi^{(k)}$ that can be estimated, from the renormalized correlation functions. Therefore, in the next section we will perform the one-loop renormalization of the model.

6.3 ONE-LOOP RENORMALIZATION IN THE DISORDERED MODEL

To proceed we will go beyond the mean-field approximation by implementing the one-loop renormalization in this model. For the sake of simplicity, we consider the leading replica partition function. That partition function is described by a large-N Euclidean replica field theory as discussed in the last chapter. Notice that all the calculations can be computed in a generic replica partition function. The leading replica partition function is written as

$$\mathbb{E}\left[Z^N\right] = \frac{1}{N!}\int \prod_{i=1}^{N} [d\varphi_i]\, e^{-H_{eff}(\varphi_i)}, \tag{6.22}$$

where

$$H_{eff}(\varphi_i) = \int d^d x \sum_{i=1}^{N} \left[\frac{1}{2}\varphi_i(x)\left(-\Delta + m_0^2\right)\varphi_i(x) + \frac{1}{4}(\lambda_0 - N\sigma)\varphi_i^4(x) + \frac{\rho_0}{6}\varphi_i^6(x)\right], \tag{6.23}$$

where for simplicity, $\varphi_i^{(N)} = \varphi_i$. Let us define $g_0 = \lambda_0 - f_0$, where $f_0 = N\sigma$. We maintain f_0 fixed while $N \to \infty$ and $\sigma \to 0$. Since in the Landau-Ginzburg scenario λ_0 depends on the temperature, g_0 is not positive definite for sufficiently low temperatures. For simplicity we assume that m_0^2 is a positive quantity. In this situation we have N replicas with true and false vacua. Vacuum transitions in this theory with N replicas can be described in the following way. Lowering the temperature each replica field has a false vacuum and two degenerate true vacuum states. The transition from the false vacuum to the true one will nucleates bubbles of the true vacuum [138, 139, 148]. One way to proceed is to calculate the transition rates in the diluted instanton approximation. This is a standard calculation that can be found in the literature. Instead of this, our goal is to perform the one-loop renormalization of the model.

At this point, let us introduce an external source $J_i(x)$ in replica space linearly coupled with each replica. Considering only the leading term in the series representation for the average free energy, and absorbing the dimensionless quantity a in the functional measure, we are able to define the generating functional of all correlation functions for a large-N Euclidean field theory as $\mathbb{E}\left[Z^N(J)\right] = \mathcal{Z}(J)$. To proceed we are following the Ref. [149]. Accordingly this generating functional of all correlation functions of this Euclidean field theory is given by

$$\mathcal{Z}(J) = \frac{1}{N!}\int \prod_{i=1}^{N} [d\varphi_i]\, e^{-H_{eff}(\varphi_i)+\int d^d x \sum_{i=1}^{N} J_i\varphi_i}. \tag{6.24}$$

It is possible to define the generating functional of connected correlation functions $\mathcal{W}(\mathcal{J}) = \ln \mathcal{Z}(\mathcal{J})$. For simplicity we assume that we have one replica field. Since in the large-N approximation all the replica fields are equal, this procedure is identical for all the fields. The generating functional of one-particle irreducible correlations (vertex functions), $\Gamma[\overline{\phi}]$, is gotten by taking the Legendre transform of $\mathcal{W}(\mathcal{J})$

$$\Gamma[\overline{\phi}] + \mathcal{W}(\mathcal{J}) = \int d^{\,d}x \Big(\mathcal{J}(x)\overline{\phi}(x)\Big), \tag{6.25}$$

where

$$\overline{\phi}(x) = \frac{\delta\mathcal{W}(\mathcal{J})}{\delta\mathcal{J}}\bigg|_{\mathcal{J}=0}. \tag{6.26}$$

For the sake of completeness we will discuss the one-loop renormalization of the corresponding theory. First, a vertex expansion for the effective action is given by

$$\Gamma[\overline{\phi}] = \sum_{n=0}^{\infty}\frac{1}{n!}\int\prod_{i=1}^{n} d^{\,d}x_i\, \Gamma^{(n)}(x_1,...,x_n)\overline{\phi}(x_1)...\overline{\phi}(x_n), \tag{6.27}$$

where the expansion coefficients $\Gamma^{(n)}$ correspond to the one-particle irreducible (1PI) proper vertex. Writing the effective action in powers of momentum around the point where all external momenta vanish, we have

$$\Gamma[\overline{\phi}] = \int d^d x\, V(\overline{\phi}) + ... \tag{6.28}$$

The term $V(\overline{\phi})$ is called the effective potential which takes into account the fluctuation in the model. Let us define the Fourier transform of the one-particle irreducible (1PI) proper vertex. We get

$$\Gamma^{(n)}(x_1,...,x_n) = \frac{1}{(2\pi)^n}\int\prod_{i=1}^{n} d^{\,d}k_i (2\pi)^d \times$$
$$\delta(k_1+...+k_n)e^{i(k_1x_1+...+k_nx_n)}\tilde{\Gamma}^{(n)}(k_1,...,k_n). \tag{6.29}$$

Now, we assume that the field $\overline{\phi}(x) = \phi$, is uniform. This condition is similar to the diluted instanton approximation. In this case, we can write

$$\Gamma[\phi] = \int d^dx \sum_{n=1}^{\infty} \frac{1}{n!}\left[\tilde{\Gamma}^{(n)}(0,...,0)\phi^n + ...\right] \tag{6.30}$$

The effective potential can be written as

$$V(\phi) = \sum_n \frac{1}{n!}\tilde{\Gamma}^{(n)}(0,...,0)\phi^n. \tag{6.31}$$

From above discussion it is possible to write the effective potential for each replica field in the leading replica partition function as $V(\phi) = V_1(\phi) + V_2(\phi)$, where

$$\begin{aligned} V_1(\phi) =& \frac{1}{2}(m_0^2 + \delta m_0^2)\phi^2 + \frac{1}{4}(g_0 + \delta g_0)\phi^4 \\ &+ \frac{1}{4}(\rho_0 + \delta\rho_0)\phi^6 \end{aligned} \tag{6.32}$$

δm_0^2, δg_0 and $\delta\rho_0$ are the counterterms that have to be introduced to remove divergent terms, and

$$V_2(\phi) = \frac{1}{2}\int \frac{d^dp}{(2\pi)^d} \ln\left[1 + \frac{1}{p^2 + m_0^2}(3g_0\phi^2 + 5\rho_0\phi^4)\right]. \tag{6.33}$$

The calculation that we are presenting here is taking account the corrections due to the fluctuations around the saddle-point of each replica partition function. To proceed, we are interested to implement the one-loop renormalization in each replica field theory. The contribution to the effective potential given by $V_2(\phi)$ can be written as

$$V_2(\phi) = \sum_{s=1}^{\infty} \frac{(-1)^{s+1}}{2s}\left(3g_0\phi^2 + 5\rho_0\phi^4\right)^s I(s,d), \tag{6.34}$$

where $I(s,d)$ is given by

$$\begin{aligned} I(s,d) =& \int \frac{d^dq}{(2\pi)^d} \frac{1}{(q^2 + m_0^2)^s} \\ =& \frac{1}{(2\sqrt{\pi})^d} \frac{\Gamma(s - \frac{d}{2})}{\Gamma(s)} \left(m_0^2\right)^{\frac{d}{2} - s}. \end{aligned} \tag{6.35}$$

At this point, let us use an analytic regularization procedure that has been used in field theory [150] and also to obtain the renormalized vacuum energy of a quantum field in the presence of boundaries [151, 152, 153]. Using a well-known result that in the neighborhood of the pole $z = -n$ ($n = 0, 1, 2, ...$) and for $\varepsilon \to 0$, the Gamma function has the representation

$$\Gamma(-n + \varepsilon) = \frac{(-1)^n}{n!}\left[\frac{1}{\varepsilon} + \psi(n+1)\right], \tag{6.36}$$

where $\psi(n+1)$, the digamma function, is the regular part in the neighborhood of the pole, and using the renormalization conditions which are given by

$$
\begin{aligned}
\frac{d^2}{d\phi^2} V(\phi)|_{\phi=0} &= m_R^2, \\
\frac{d^4}{d\phi^4} V(\phi)|_{\phi=0} &= g_R, \\
\frac{d^6}{d\phi^6} V(\phi)|_{\phi=0} &= \rho_R,
\end{aligned}
\tag{6.37}
$$

we obtain renormalized physical quantities. Note that the normalization conditions are choosen in the metastable vacuum state. It is possible to choose another normalization condition, as for example in the true minimum of the effective potential. We would like to stress that all the renormalization conditions are equivalent after one establishes the correspondence between them [154]. For the sake of simplicity, we consider the case where $d = 4$:

$$m_R^2 = m_0^2 \left[1 - \frac{3 g_0 \psi(2)}{16\pi^2} \right], \tag{6.38}$$

$$g_R = 6 g_0 + \frac{9\psi(1)}{4\pi^2} g_0^2 - \frac{15\psi(2)}{4\pi^2} \rho_0 m_0^2, \tag{6.39}$$

$$\rho_R = \rho_0 \left[120 - \frac{675\psi(1)}{2\pi^2} g_0 \right] + \frac{369}{4\pi^2} \frac{g_0^3}{m_0^2}, \tag{6.40}$$

where $\psi(1) = -\gamma$ and $\psi(2) = -\gamma + 1$.

In conclusion, the distributional zeta function method allows us to write a series representation for the average free energy where each term is given by a replica partition function. In the leading order approximation we get only one replica partition function with N identical fields. In this case, it is sufficient to work with only one replica field. In order to renormalize this theory we use the effective potential approach. Combining an analytic regularization procedure to regularize the theory and the standard renormalization conditions we obtain a finite theory.

7 CONCLUSIONS AND PERSPECTIVES

7.1 CONCLUSIONS

Using the replica method, we studied finite-size effects in the one-loop approximation in the disordered random field $\lambda\varphi^4$ model defined in a d-dimensional Euclidean space. For a 3-dimensional and a 4-dimensional space with one compactified spatial dimension, we showed that there is a critical length where the system develop a second order phase transition. Moreover, considering the composite field operator method, non-perturbative results were obtained. In this case, the renormalized squared mass present a monotonically decrescent behaviour in function of the compactified radius L.

In other hand, we introduce an alternative approach, different of the replica method, in order to compute the average free energy of systems with quenched disorder. In this new framework, it was defined a distributional zeta function that leads to write the average free energy as series where all the integer moments of the partition function of the model contribute. The distributional zeta function technique is a mathematically rigorous method that uses the notion of replica fields and where each term of the series defines a replica field theory. A crucial point is that, in each replica partition function, all the replica fields must be equal, we have not replica symmetry breaking.

As an application of the distributional zeta function method, we studied the field theory of polymers and interfaces in random media. In this case, the series representation of the free energy is reduced to a single term which corresponds to the existence of a unique replica field. This situation is equivalent to that presented in the replica method where the limit of zero replicas is taken. In this context, the wandering exponent obtained by us is the same as that obtained using the replica method with the replica ansatz. Since the theory of polymers and interfaces in random media is Gaussian our results are not different from the results obtained in the replica trick framework. When the theory is not Gaussian we can go further than the replica method.

We studied also the disordered random field $\lambda\varphi^4$ model defined in a d-dimensional Euclidean space using the distributional zeta function method. Since all the replica fields must be equal, its number must be very large. This implies that we are in a large-N scenario. With respect to fluctuations around the saddle-point equations, we obtained two groups. In a group, a generic replica partition function with $k \leqslant k_c$, the replica fields fluctuate around the zero value which is a stable equilibrium state. In the other group of replica partition functions, the zero value of the fields does not describe stable equilibrium states. For replica partitions functions such that $k > k_c$ we must define shifted fields. Therefore, we established a connection between spontaneous symmetry breaking mechanism and the structure of the replica space in the disordered model. By investigating finite-size effects in the one-loop approximation, we showed that there must be a critical temperature where the renormalized mass is zero.

Also, following the Landau-Ginzburg approach and taking a generic replica partition function with N replica fields, we obtained three situations depending of the value of m_0^2. In the case where $m_0^2 \geqslant N\sigma$, all the N replica fields oscillates around the trivial vacuum. For $\sigma N > m_0^2 \geqslant \sigma$, the average free energy is defined by two kinds of replica partition functions, some with fields fluctuating around of the stable vacuum and others around of the non-trivial vacuum. For $m_0^2 < \sigma$, all the N replica fields in each replica partition function are oscillating around the non-trivial vacuum. Moreover, we obtained also real and complex instantons, depending of the value of m_0^2.

Considering the $\lambda\varphi^4 + \rho\varphi^6$ model with random mass we showed that the average free energy computed using the distributional zeta function represents a system with multiple ground states with different order parameters. For low temperatures, we can interpret this as the existence of metastable equilibrium states for some replica fields. So, we considered the possibility that the series representation for the average free energy describes inhomogeneous domains, i.e. macroscopic regions in a sample, with at least one proper characteristic order parameter.

7.2 PERSPECTIVES

In this thesis we applied the distributional zeta function to study the Landau-Ginzburg $\lambda\varphi^4$ and $\lambda\varphi^4 + \rho\varphi^6$ models with random external field and random mass respectively. Despite the results obtained by us which we considerer to be positive and help in the understanding of the disordered systems, we did not any analysis about the critical exponents of these models. So, it is necessary to go in this direction. We need to compute these quantities and compare with the already known results of the replica method.

We have to apply this new technique in many other systems in order to show its effectiveness and its possible limitations. In this sense, it is imperative to consider other models in the scenario of field theory and discrete formulations to compare the results with that obtained by the replica methods. Additionally, since we believe that the distributional zeta function is an appropriate method to study disordered system, we have to analyse the results obtained with other techniques, different to the replica approach, used also to study systems with disorder.

A

GAUSSIAN INTEGRATION

In this appendix we place the corresponding identities to Gaussian integrals in the case of a finite number of integration variables.

One-dimensional Gaussian integral

The basic starting point for any Gaussian integral is the identity

$$\int_{-\infty}^{\infty} dx e^{-\frac{1}{2}ax^2} = \sqrt{\frac{2\pi}{a}}, \tag{A.1}$$

for any $a \in \mathbb{R}^+$. In some situations the exponent of the integrand involved in a Gaussian integration is not purely quadratic but has both quadratic and linear terms, that is

$$\int_{-\infty}^{\infty} dx e^{-\frac{1}{2}ax^2+bx}, \tag{A.2}$$

for $b \in \mathbb{R}$. To obtain the value of this integral, that generalize the Eq. (A.1), we perform the change of variables $x = y + \frac{b}{a}$ which leads to the transformation

$$-\frac{1}{2}ax^2 + bx \Longrightarrow -\frac{1}{2}ay^2 + \frac{b^2}{2a}, \tag{A.3}$$

so that

$$\begin{aligned}\int_{-\infty}^{\infty} dx e^{-\frac{1}{2}ax^2+bx} =& \int_{-\infty}^{\infty} dy e^{-\frac{1}{2}ay^2+\frac{b^2}{2a}} \\ =& \sqrt{\frac{2\pi}{a}} e^{\frac{b^2}{2a}}.\end{aligned} \tag{A.4}$$

N-dimensional Gaussian integral

The multi-dimensional version of the integral (A.1) correspond to

$$\int d\mathbf{v} e^{-\frac{1}{2}\mathbf{v}^T\mathbf{A}\mathbf{v}} = \frac{(2\pi)^{\frac{N}{2}}}{(\det \mathbf{A})^{\frac{1}{2}}}, \tag{A.5}$$

where $\mathbf{A}$ is a defined positive symmetric[1] $N \times N$ matrix and $\mathbf{v}$ is an N-component real vector (a column vector) and $\mathbf{v}^T$ is the corresponding transposed (a row vector). To prove this, we must consider the fact

1 If the matrix $\mathbf{A}$ has an antisymmetric part, the contribution of this to $\mathbf{v}^T\mathbf{A}\mathbf{v}$ will vanish.

that $\mathbf{A}$ (by virtue of being symmetric) can be diagonalized by orthogonal transformation, $\mathbf{A} = \mathbf{O}^T\mathbf{D}\mathbf{O}$, where the matrix $\mathbf{O}$ is orthogonal ($\mathbf{O}\mathbf{O}^T = \mathbb{1}$) and $\mathbf{D}$ is a diagonal matrix which elements are all positive. Making the change of variables $\mathbf{w} = \mathbf{O}\mathbf{v}$ which has unit Jacobian, that is $\det \mathbf{O} = 1$, it possible to absorb the matrix $\mathbf{O}$ into the integration vector. As result

$$\int d\mathbf{v} e^{-\frac{1}{2}\mathbf{v}^T\mathbf{A}\mathbf{v}} = \int d\mathbf{w} e^{-\frac{1}{2}\mathbf{w}^T\mathbf{D}\mathbf{w}}. \tag{A.6}$$

The last integral is a product of N independent Gaussian integrals thanks to the diagonalizability of $\mathbf{D}$, each of them contributing with a factor $\sqrt{2\pi/d_i}$, where d_i with $(1, 2, ..., N)$ are the diagonal elements of the matrix $\mathbf{D}$. The result is

$$\begin{aligned}\prod_{i=1}^{N} \sqrt{2\pi/d_i} =& (2\pi)^{\frac{N}{2}} \prod_{i=1}^{N} \frac{1}{\sqrt{d_i}} \\ =& \frac{(2\pi)^{\frac{N}{2}}}{\sqrt{\det \mathbf{A}}},\end{aligned} \tag{A.7}$$

considering the fact that $\prod_{i=1}^{N} d_i = \det \mathbf{D} = \det \mathbf{A}$.
To obtain the equivalent of the Eq. (A.4) for multi-dimensional integration, in other words, the value of the integral

$$\int d\mathbf{v} e^{-\frac{1}{2}\mathbf{v}^T\mathbf{A}\mathbf{v}+\mathbf{b}^T\cdot\mathbf{v}} \tag{A.8}$$

where $\mathbf{b}$ is an arbitrary N-component vector, we must to proceed exactly as in the one-dimensional case, changing variables to $\mathbf{w}$ by shifting the integration vector $\mathbf{v}$ according[2]

$$\mathbf{v} = \mathbf{w} + \mathbf{A}^{-1}\mathbf{b}, \tag{A.9}$$

which does not transform the value of the integral but take out the linear term from the exponent in the integrand,

$$-\frac{1}{2}\mathbf{v}^T\mathbf{A}\mathbf{v} + \mathbf{b}^T\cdot\mathbf{v} \implies -\frac{1}{2}\mathbf{w}^T\mathbf{A}\mathbf{w} + \frac{1}{2}\mathbf{b}^T\mathbf{A}^{-1}\mathbf{b}\,, \tag{A.10}$$

resulting in a integral similar to the Eq. (A.5). We have

$$\begin{aligned}\int d\mathbf{v} e^{-\frac{1}{2}\mathbf{v}^T\mathbf{A}\mathbf{v}+\mathbf{b}^T\cdot\mathbf{v}} =& \int d\mathbf{w} e^{-\frac{1}{2}\mathbf{w}^T\mathbf{A}\mathbf{w}+\frac{1}{2}\mathbf{b}^T\mathbf{A}^{-1}\mathbf{b}} \\ =& \frac{(2\pi)^{\frac{N}{2}}}{\sqrt{\det \mathbf{A}}} e^{\frac{1}{2}\mathbf{b}^T\mathbf{A}^{-1}\mathbf{b}}.\end{aligned} \tag{A.11}$$

2 The existence of the matrix $\mathbf{A}^{-1}$ comes from the positivity assumed in the matrix $\mathbf{A}$.

This identity, besides being important because it allows to solve directly integrals of this type, also help to generate other useful expressions. Applying the differentiation operation $\partial^2_{ij}|_{\mathbf{b}=0}$ to the Eq. (A.11) we get[3]

$$\int d\mathbf{v}\, v_i v_j e^{-\frac{1}{2}\mathbf{v}^T\mathbf{A}\mathbf{v}} = \frac{(2\pi)^{\frac{N}{2}}}{\sqrt{\det \mathbf{A}}} A^{-1}_{ij}, \tag{A.12}$$

which, introducing the shorthand notation

$$\langle \cdots \rangle \equiv \frac{\sqrt{\det \mathbf{A}}}{(2\pi)^{\frac{N}{2}}} \int d\mathbf{v} e^{-\frac{1}{2}\mathbf{v}^T\mathbf{A}\mathbf{v}} (\cdots), \tag{A.13}$$

can be written in a compact way as

$$\langle v_i v_j \rangle = A^{-1}_{ij}, \tag{A.14}$$

suggesting in this way to interpret the Gaussian weight as a probability distribution.

How could it be expected, the process of differentiation over the Eq. (A.11) can be performed to any order. For example, differentiating four times, one obtains

$$\langle v_i v_j v_k v_l \rangle = A^{-1}_{ij} A^{-1}_{kl} + A^{-1}_{ik} A^{-1}_{jl} + A^{-1}_{il} A^{-1}_{jk}. \tag{A.15}$$

Notice that $\langle v_i v_j v_k v_l \rangle$ is given by the sum of all terms that can be formed multiplying in pairs the expression defining $\langle v_i v_j \rangle$ where are four vector under consideration. Taking this in account, the generalization for this kind of expression to arbitrary order, i.e. for the $2n$-fold derivative of Eq. (A.11) yields

$$\langle v_{i_1} v_{i_2} \cdots v_{i_{2n}} \rangle = \sum_{\substack{\text{pairs of} \\ (i_1,\ldots,i_{2n})}} A^{-1}_{i_{k_1} i_{k_2}} \cdots A^{-1}_{i_{k_{2n-1}} i_{k_{2n}}}. \tag{A.16}$$

This identity constitute the basis of Wick's theorem (for real bosonic fields), as can be seen in the section XXX, in chapter 1.

3 The symbol v_i is used to designate the i-th component of the vector $\mathbf{v}$ and A^{-1}_{ij} for the ij-element of the matrix $\mathbf{A}^{-1}$.

B THE LANDAU-GINZBURG MODEL

The origin of magnetism involves elements of quantum mechanics, therefore a microscopic description turns out to be complicated and dependent of the structure of the materials. If one were interested in to stablish the possible components that produce ferromagnetism this would be the way to follow. But, if the starting point is to assume that such a phenomenon exists and the focus is to describe its phenomenology then this is not necessary. Here, it is where the Landau-Ginzburg theory comes in, this theory is used to study the critical phenomena in ferromagnetic systems.

For instance, at the phase transition point of a system described by the Ising model, thermal fluctuations on large scales (at small momenta in the Fourier space) give arise to singular terms in the thermodynamic functions. The fine details of the probability distribution of the magnetization, the relevant variable to study magnetic phenomena, on a small scale are not important to derive general properties, as the value of the critical exponent. This is because the degrees of freedom which describe the transition between different phases correspond to collective excitations of spins with large wavelength.

As a starting point, the Landau-Ginzburg theory adopts a Hamiltonian $\mathcal{H}(\varphi_i)$ depending on a field variable φ_i defined in each one of the sites i of a lattice of spins, which corresponds to the average of the magnetization. The probability of a particular configuration of the field being proportional to $e^{-\mathcal{H}(\varphi_i)}$. So, the Landau-Ginzburg Hamiltonian that describes the comportment of a lattice of N spins near to the phase transition immersed in a d-dimensional space is written as

$$\mathcal{H}(\varphi_i) = a^d \sum_{i=1}^{N} \left[\frac{1}{2} (\nabla \varphi_i)^2 + \frac{1}{2} m_0^2(T) \varphi_i^2 + \frac{1}{4!} \lambda \varphi_i^4 \right] , \qquad \text{(B.1)}$$

where a is the lattice spacing, $m_0^2(T)$ is a parameter depending on the temperature T and λ is a positive constant. The symbol ∇ stands for the *discretized gradient* which accounts for interactions between nearest neighbours[1]. Despite having the Hamiltonian expressed in Eq. (B.1), the use in this form it is not very convenient due to the difficulties that may arise in relation to calculations. One prefers to adopt the continuous formulation where the position vector $\mathbf{x}_i$ varies

1 This gradient is defined as

$$\nabla \varphi(\mathbf{x}_i) = \frac{1}{a} \left[\varphi(\mathbf{x}_i + \mathbf{a}) - \varphi(\mathbf{x}_i) \right] .$$

Here, $\mathbf{x}_i$ is the position vector at site i, so that $\varphi_i = \varphi(\mathbf{x}_i)$, and $\mathbf{a}$ is a vector linking the site $\mathbf{x}_i$ to one of its nearest neighbours.

continuously over all the space occupied by the physical system, instead of being confined to the sites of some lattice. The procedure can be viewed as a *coarse-graining* of the original Ising model where the radius of the volume over which the magnetization must be averaged is large enough but much smaller than the correlation length[2], of the order of few lattice spacing. According to the above, the continuous version of the Landau-Ginzaburg Hamiltonian is given by

$$\mathcal{H}(\varphi) = \int d^d x \left[\frac{1}{2}\Big(\nabla\varphi(x)\Big)^2 + \frac{1}{2}m_0^2(T)\varphi^2(x) + \frac{1}{4!}\lambda\varphi^4(x)\right], \tag{B.2}$$

where ∇ now represents the usual gradient. At this stage, the various thermodynamic functions and its respective singular behaviour can be obtained from the corresponding partition function that in turn is got integrating over all configuration of the field $\varphi(x)$, that is

$$Z(h) = \int [d\varphi] e^{-\int d^d x \left[\frac{1}{2}(\nabla\varphi(x))^2 + \frac{1}{2}m_0^2(T)\varphi^2(x) + \frac{1}{4!}\lambda\varphi^4(x) - h(x)\varphi(x)\right]}. \tag{B.3}$$

In this equation $h(x)$ denotes a external magnetic field and $[d\varphi]$ is the integration measure. Applying the saddle-point or mean-field approximation we can write [48]

$$Z(h) \equiv e^{-F} \quad \texttt{and} \quad F \simeq \texttt{mim}_\varphi \left[\mathcal{H}(\varphi)\right], \tag{B.4}$$

The F is the free energy and $\texttt{mim}_\varphi \left[\mathcal{H}(\varphi)\right]$ represent the minimum of the Hamiltonian with respect to variations in the field. Supposing a uniform magnetic field h, the free energy density reads

$$\begin{aligned} f(h) &\equiv \frac{F}{V} \\ &= \frac{1}{2}m_0^2(T)\varphi^2 + \frac{1}{4!}\lambda\varphi^4 - h\varphi, \end{aligned} \tag{B.5}$$

The behaviour of $F(h)$ or equivalently of $f(h)$ depends on the sign of $m_0^2(T)$ as follows:

- For $m_0^2(T) > 0$ the quartic term can be ignored and the minimum occurs in $\varphi = h/m_0^2(T)$. When $h \to 0$ the magnetization vanishes.

- For $m_0^2(T) < 0$ the quartic term is required to ensure stability, i.e. to ensure that the magnetisation remains finite. There are two different minima, which occur at $\varphi = \pm\sqrt{-6m_0^2(T)/\lambda}$ when $h = 0$, being degenerated in this case. We have spontaneous breaking of rotational symmetry in spin space, in other words, a spontaneous magnetization. Switching on an infinitesimal field h such degeneracy of the ground state is broken.

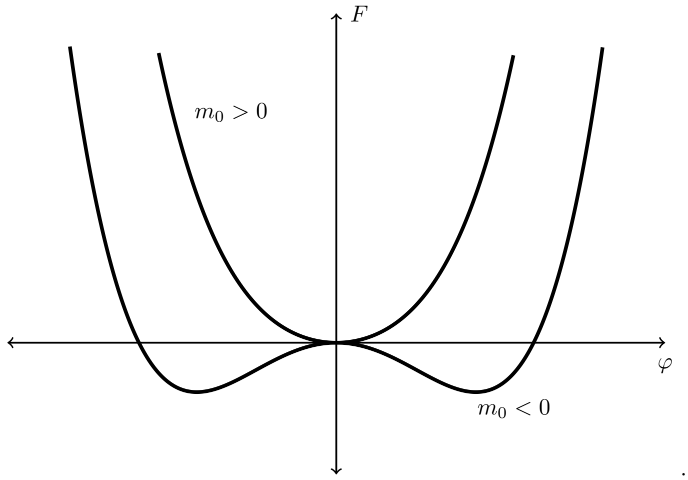

Figure 5: Schematic diagram of the mean-field Landau-Ginzburg free energy.

Concluding briefly, the Landau-Ginzburg Hamiltonian presents paramagnetic behaviour for $m_0^2(T) > 0$ and ferromagnetic behaviour for $m_0^2(T) < 0$, as it is showed in Fig. (). Without loss of generality, in this approximation we can identify the parameter $m_0^2(T)$ with the reduced temperature $m_0^2(T) = (T - T_c)/T_c$, where the critical temperature T_c takes place at $m_0^2(T_c) = 0$.

2 See Refs. [77] for details.

C

THE COMPOSITE OPERATOR

It is well known that at non-zero temperatures the perturbative expansion in powers of the coupling constant breaks down [155]. Therefore resummation schemes are necessary in order to obtain reliable results. On of these resummation schemes is called the CJT formalism [156, 157]. This formalism resums one-particle irreducible diagrams to all orders. The stationary conditions for the effective action are the Dyson-Schwinger equations, for the one and two-point correlation functions of the field theory model. The equations for the two-point correlation functions in the Hartree-Fock approximation give us self-consistent conditions or gap equations.

Let us define the generating functional for the Schwinger functions with the usual source $h(x)$, but with an additional contribution $K(x,y)$ which couples to $\frac{1}{2}\left[\varphi(x)\varphi(y)\right]$.

$$Z(h,K) = \int [d\varphi] \exp\Big[-S + \int d^d x h(x)\varphi(x) + \frac{1}{2}\int d^d x d^d y \varphi(x) K(x,y)\varphi(y)\Big], \tag{C.1}$$

where S is the action of the model. We can define the generating functional of connected Schwinger functions $W(h,K)$ defining

$$W(h,K) = \ln Z(h,K). \tag{C.2}$$

The normalized vacuum expectation value of the field $\varphi_0(x)$ and the connected two-point Schwinger function $G_c(x,y)$ are given by

$$\frac{\delta W(h,K)}{\delta h(x)} = \varphi_0(x) \tag{C.3}$$

and

$$\frac{\delta W(h,K)}{\delta K(x,y)} = \frac{1}{2}\Big[G_c(x,y) + \varphi_0(x)\varphi_0(y)\Big]. \tag{C.4}$$

Making use of a Legendre transformation, we obtain the effective action

$$\Gamma(\varphi_0, G_c) = W(h,K) - \varphi_0 h - \frac{1}{2}\varphi_0 K \varphi_0 - \frac{1}{2} G_c K, \tag{C.5}$$

where we are using that

$$\varphi_0 h = \int d^d x\, \varphi_0(x) h(x), \tag{C.6}$$

$$\varphi_0 K \varphi_0 = \int d^d x d^d y \varphi_0(x) K(x,y) \varphi_0(y). \tag{C.7}$$

and

$$\frac{1}{2} G_c K = \frac{1}{2} \int d^d x \, d^d y \, G_c(x,y) K(y,x). \tag{C.8}$$

Functional derivatives of the effective action $\Gamma(\varphi_0, G_c)$ give us

$$\frac{\delta \Gamma(\varphi_0, G_c)}{\delta \varphi_0(x)} = -h(x) - \int d^d y K(x,y) \varphi_0(y), \tag{C.9}$$

$$\frac{\delta \Gamma(\varphi_0, G_c)}{\delta \, G_c(x,y)} = -\frac{1}{2} K(x,y). \tag{C.10}$$

For vanishing sources, the stationary conditions which determine the normalized expectation value of the field $\varphi_0(x)$ and the two-point Schwinger function are

$$\left. \frac{\delta \Gamma(\varphi_0, G_c)}{\delta \varphi_0(x)} \right|_{\substack{\varphi_0 = \varphi \\ G_c = G_0}} = 0 \tag{C.11}$$

and

$$\left. \frac{\delta \Gamma(\varphi_0, G_c)}{\delta G_c(x,y)} \right|_{\substack{\varphi_0 = \varphi \\ G_c = G_0}} = 0. \tag{C.12}$$

The last equation corresponds to a Dyson-Schwinger equation for the full two-point Schwinger function. The effective action $\Gamma(\varphi_0, G_c)$ can be written as

$$\Gamma(\varphi_0, G_c) = I(\varphi_0) - \frac{1}{2} \mathrm{Tr}(\ln \, G_c^{-1}) - \frac{1}{2} \mathrm{Tr}(D^{-1} \, G_c - 1) + \Gamma_2(\varphi_0, G_c), \tag{C.13}$$

where D^{-1} is the inverse of the tree-level two-point Schwinger function and $\Gamma_2(\varphi_0, G_c)$ is the sum of all two-point irreducible diagrams where all lines represent free propagators. Let us assume that $\varphi_0(x) = \varphi_0$, i.e., a constant field. For such homogeneous system we can define the effective potential. Stationary condition gives us a Dyson-Schwinger equation.

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