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ESTADOS COERENTES APLICADOS À LOCALIZAÇÃO QUÂNTICA NO CÍRCULO E À PROBABILIDADE DE ERRO QUÂNTICO

Diego Noguera

Tese de Doutorado apresentada ao Programa de Pós-Graduação em Física do Centro Brasileiro de pesquisas Físicas-CBPF, como parte dos requisitos necessários à obtenção do título de Doutor em Ciências (Física).

Orientador: Evaldo Mendonça Fleury Curado

Coorientador: Jean-Pierre Gazeau

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Resumo

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Esta tese consiste na aplicação de estados coerentes em dois problemas: a quantização do movimento de uma partícula no círculo e no estudo do erro quântico em uma codificação de mensagems feita por estados coerentes não-lineares. No primeiro caso, usando estados coerentes do grupo Euclidiano E(2), a quantização integral covariante é aplicada ao movimento de uma partícula no círculo. O processo de quantização é implementado nos observáveis clássicos básicos, em particular na função ângulo e no momentum angular. Os análogos semi-clássicos chamados de "lower symbols" são calculados para os observáveis quantizados. A localizazão quântica é estudada usando propriedades do operador ângulo obtido, seu espectro, "lower symbol" e comutador com o momentum angular. No segundo caso, dois tipos de deformações da distribuição binomial são utilizadas para contruir estados coerentes não-lineares. Considerando-se a tranferência de informação com um alfabeto de dois estados coerentes não-lineares produzidos por um laser, a probabilidade de erro quântico (ou limite de Helstrom) é estudada. Tambem é analizada a possibilidade de otimizar o erro quântico em relação a estados coerentes lineares.

Palavras-chave: Estados Coerentes, Quantização Integral, Limite de Helstrom, Distribues Binomiais Deformadas.

Abstract

COHERENT STATES APPLIED TO QUANTUM LOCALIZATION ON THE CIRCLE AND QUANTUM ERROR PROBABILITY

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Orientador: Evaldo Mendonça Fleury Curado

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This thesis consists in the application of coherent states in two problems: the quantization of the motion of a particle on the circle and the study of the quantum error in the codification of messages made by non-linear coherent states. In the first case, covariant integral quantization using coherent states for semi-direct product groups is implemented for the motion of a particle on the circle. We carry out the corresponding quantizations of the basic classical observables, particularly the angular momentum, and the 2π-periodic discontinuous angle function. We compute their corresponding lower symbols. The quantum localization on the circle is examined through the properties of the angle operator, lower symbol, and commutator with the quantum angular momentum. Two types of deformation of the binomial distribution are used to construct nonlinear coherent states. In the second case, considering the transference of information with an alphabet of two nonlinear coherent states generated by a laser, the quantum error probability (Helstrom bound) is studied. Is also analized the possibility of optimization of the Helstrom bound in relation to linear coherent states.

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Chapter 1

Introduction

1.1 Coherent states: some definitions and generalizations.

What is now called *coherent states* (CS) were first studied by Schrödinger [1], Kennard [2] and Darwin [3]. These authors were interested in nonspreading wavepackets of the harmonic oscillator that can restore the classical behavior of the position operator. On the beginning of the sixties, the same quantum states were used by Klauder [4, 5] in his formulation of quantum mechanics, and they were reintroduced by Glauber [6, 7] and Sudarshan [8] to describe coherent light beams produced by lasers. In quantum optics, these "linear" CS (also called standard or canonical) are defined as the superposition of photon number states $|n\rangle$ given by

$$|\alpha\rangle := e^{-\frac{|\alpha|^2}{2}} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle \,, \quad \text{where } \alpha \in \mathbb{C} \,. \tag{1.1}$$

These states satisfy the following set of properties:

- The operator valued map $\alpha \mapsto D(\alpha) = e^{\alpha a^\dagger - \bar{\alpha} a}$ corresponds to a unitary irreducible representation of the Weyl-Heisenberg group up to a phase factor. The set of linear CS is the orbit of the Fock vacuum under the action of $D(\alpha)$

$$|\alpha\rangle = D(\alpha) |0\rangle \ . \tag{1.2}$$

- The CS are eigenvectors of the annihilation operator

$$a\,|\alpha\rangle = \alpha\,|\alpha\rangle \ . \tag{1.3}$$

- The CS satisfy the resolution of the identity

$$\int_{\mathbb{C}} \frac{\mathrm{d}^2\alpha}{\pi}\,|\alpha\rangle\,\langle\alpha| = I\,. \tag{1.4}$$

- For an operator A, the standard deviation is given by $\Delta A = \sqrt{\langle A^2\rangle - \langle A\rangle^2}$. Considering the position operator $Q = \frac{1}{\sqrt{2}}(a^\dagger + a)$ and momentum operator $P = \frac{\mathrm{i}}{\sqrt{2}}(a^\dagger - a)$ the CS saturate the Heisenberg inequality

$$\langle\alpha|\,\Delta Q\,|\alpha\rangle\,\langle\alpha|\,\Delta P\,|\alpha\rangle = \frac{\hbar}{2}\,. \tag{1.5}$$

The properties listed above can be used to make a generalization of the notion of CS. In general, we can find in the literature the following ones:

- One can generalize the expression (1.1) by changing the coefficients in the expansion

$$|\alpha\rangle = \sum_{n=0}^{\infty} c_n(\alpha)\,|n\rangle \ . \tag{1.6}$$

- The property given in the equation (1.2) is the basis for the group-theoretical generalization of CS developed by Gilmore [9] and Perelomov [10] (both developed independently). Let us consider the Lie group G and a homogeneous space X, viewed as the left coset manifold $X \sim G/H$, where the closed subgroup H is the stabilizer of some point of X[1]. Given the vector space V, let us also consider the representation $U : G \to \mathrm{GL}(V)$ and the section $\sigma : X \to G$. The CS associated with G are defined as the orbit of $\eta \in V$ under the action of U, i.e.

$$\eta_x := U(\sigma(x))\eta\,, \quad x \in X\,. \tag{1.7}$$

[1] Given the group action $X \ni x \mapsto g \cdot x \in X$ for $g \in G$, the stabilizer of $x \in X$ is the set $H = \{g \in G \,|\, g \cdot x = x\}$. This is a subgroup of G.

- The property given in the equation (1.3) is the basis for the generalization of Barut and Girardello [11], they constructed CS as eigenvectors of one generator of the Lie algebra $\mathfrak{su}(1,1)$. Basically, we try to find continuous family of eigenstates $|\eta_\alpha\rangle$ such that

$$A\,|\eta_\alpha\rangle = \alpha\,|\eta_\alpha\rangle \ , \tag{1.8}$$

  where the operator A is a generator of some Lie algebra.

- From the expression (1.5), a new set of CS can be defined as the set of states which minimize the uncertainty relation for the self-adjoint operators A and B.

In the next two subsections we will discuss two applications of CS that we will develop in this thesis.

1.2 Application of CS generalizations on quantum localization on the circle and quantum error probability

1.2.1 Quantum localization on the circle.

When one aims to establish the quantum version of the simple pendulum, one faces the difficulty of properly defining a localization operator on the circle, whereas such an object exists unambiguously for the quantum model of the motion on the line. Indeed, supposing that a 2π periodic wave functions $\psi(\alpha)$ exists on the circle, we cannot introduce an angle operator $\widehat{\alpha}$ as the multiplication operator $(\widehat{\alpha}\psi)(\alpha) = \alpha\psi(\alpha)$ without breaking the periodicity, except if the factor α stands for the 2π-periodic discontinuous angle function, i.e.,

$$(\widehat{\alpha}\psi)(\alpha) := \left(\alpha - 2\pi \left\lfloor \frac{\alpha}{2\pi} \right\rfloor\right) \psi(\alpha)\,, \tag{1.9}$$

as given for instance in [12] (see also [13]), and where $\lfloor\cdot\rfloor$ stands for the floor function. On a more mathematical level, if we require $\widehat{\alpha}$ to be a self-adjoint multiplication operator with spectrum supported by the period interval $[0, 2\pi)$, on which are defined these 2π periodic wave functions, it is well known that the canonical commutation rule $[\widehat{\alpha}, \widehat{p}_\alpha] = \mathrm{i}\hbar I$ cannot hold with a self-adjoint quantum angular momentum $\widehat{p}_\alpha = -\mathrm{i}\hbar \frac{\partial}{\partial \alpha}$. From definition (1.9), the right-hand side of the commutation rule reads instead as a kind of Dirac comb,

$$[\widehat{\alpha}, \widehat{p}_\alpha] = \mathrm{i}\hbar I \left[1 - 2\pi \sum_n \delta(\alpha - 2n\pi) \right] . \tag{1.10}$$

As a result, it is necessary to revisit the quantum localization on the circle and its related Heisenberg inequality $\Delta\widehat{\alpha}\, \Delta\widehat{p}_\alpha \geq$ *lower bound.* Most of the approaches and subsequent discussions rest upon the replacement of a hypothetical angle operator with the quantum version of a smooth periodic function of the classical angle, at the cost of the loss of satisfying localization properties.

Let us present a brief survey of the extensive literature on the problem of defining the angle operator conjugate to the quantum angular momentum, or its parent phase operator conjugate to the number operator. This problem goes back to Dirac [14], and since then most of the works addressed the question of the validity of commutation relations between these operators (see the celebrated review [15] and also [16] for a clear mathematical analysis). In the wake of Dirac's proposal, early works were fixed on the goal of attaining canonical commutation relations which reproduce the classical Poisson brackets, in analogy to position and momentum. It was soon realized that this was not possible, due to angular momentum operator domain issues [17]. The idea of using smooth periodic functions instead of the angle variable has been pursued in many works where a formal operator algebra is used to derive uncertainty relations [18, 19]. This approach was then made rigorous in [20, 21] with a self-adjoint phase operator having canonical commutation relations with the number operator, and in [22] where general properties of phase operators are considered. A rigorous treatment based on the canonical factorization theorem of the phase operator in the context of quantum electrodynamics is given in [23]. In [24] a no-go theorem is proved about the nonexistence of a phase operator along the lines of the previous works for systems with finite degrees of freedom. The use of non-hermitian (or rather, non-self-adjoint) operators such as the unitary $\exp \mathrm{i}\hat{\varphi}$ instead of the discontinuous angle operator can provide well-defined commutation relations with the angular momentum operator [12]. We cannot end this survey without mentioning the important contribution of Berezin, where a general quantization scheme for a phase space which is a complex Kähler manifold is developed and applied to the case of the cylinder using Weyl quantization, see [25].

One of our aims in the present work is to build acceptable angle operators from the classical angle function through a consistent and manageable quantisation procedure. We recall that the standard ($\sim$ canonical) quantisation is based on the replacement of the classical conjugate pair $(q,p) \in \mathbb{R}^2$, with $\{q,p\} = 1$, by its quantum counterpart (Q,P) made of two essentially self-adjoint operators having continuous spectrum $\mathbb{R}$ and such that $[Q,P] = i\hbar I$. As a result, the quantisation of a classical observable is the (not well-defined) map $f(q,p) \mapsto \mathrm{Sym} f(Q,P)$, where the symbol Sym stands for symmetrisation, which maps real functions to symmetric operators. Due to the pragmatic stance of the procedure, canonical quantisation is commonly accepted in view of its numerous experimental validations since the emergence of quantum physics. Now, when one wants to implement the method in dealing with geometries other than simple Euclidean spaces, particularly when one is concerned with impenetrable barriers, or when one wants to quantise singular functions, one may be faced with serious mathematical problems. This is precisely the case we are considering in this work, namely the discontinuous angle (or phase function $\arctan p/q$ in the above case), for which canonical quantisation is clearly unsuited.

In the present work we revisit the problem of the quantum angle through coherent state (CS) quantisation, which is a particular (and better manageable) method belonging to (covariant) integral quantisation [26, 27, 28]. Various families of coherent states have already been used for this purpose, as the standard or the so-called circle coherent states or even more general versions like the ones in [29, 30]. CS quantisation has also been applied in the finite-dimensional Hilbertian framework in [31], where infinite-dimensional limits are taken of mean-values of physical quantities in order to obtain the usual commutation relations between phase and number operators. The essential ingredient of CS quantisation or the more general integral quantisation is the resolution of the identity provided by a (positive) operator-valued measure. Here, our approach is group theoretical, based on the unitary irreducible representations of the (special) Euclidean group $\mathrm{E}(2) = \mathbb{R}^2 \rtimes \mathrm{SO}(2)$ [32, 33], and it is strongly influenced by the seminal paper by De Bièvre [34] and chapter 9 of the book [27]. Related group theoretical approaches are found in [35, 36, 37, 38]. The results obtained in this work were published in [39, 40].

1.2.2 Quantum error probability with nonlinear CS.

In quantum information processing the information carriers are quantum states, while the communication channels are quantum operations. For binary comunication, the sender uses an alphabet $\mathcal{A} = \{\rho_0, \rho_1\}$ composed by two well-defined quantum states. The receiver performs a measurement on the channel to determine which state was transmitted. Lets consider an alphabet formed by linear CS. The linear CS are non-orthogonal states. A consequence of the overlap between those states is the existence of a nonzero probability that the receiver will misinterpret the transmitted quantum state. The problem of distinguishing between non-orthogonal states can be addressed by obtimizing over all Positive Operator-Valued Measures (POVM) [2]. This leads to the quantum error probability or Helstrom bound [42], which is the smallest physically allowable error probability.

In [43], it was shown that it is possible to approach experimentally the Helstrom bound

[2]Given a set Ω and a σ-algebra Σ of subsets of Ω, a measure is a function $\mu : \Sigma \to [0, \infty]$ (which fulfil some conditions, see [41]). Measures for wich $\mu(\Omega) = 1$ are called probability measures. Let us consider the Hilbert space $\mathcal{H}$, and the space $\mathcal{L}_+(\mathcal{H})$ of positive bounded operators in $\mathcal{H}$. A POVM is a map $\mu : \Sigma \to \mathcal{L}_+(\mathcal{H})$ such that $\mu(\Omega) = I_{\mathcal{H}}$, where $I_{\mathcal{H}}$ is the identity operator [41]. In this work we deal with the most simple case: a finite set of self-adjoint positive semidefinite operators $\{M_i\}$ on a Hilbert space $\mathcal{H}$ such that

$$\sum_{i=1}^{n} M_i = I_{\mathcal{H}} \,.$$

using communication processes based on photodetection. A photocounter consists of a light detector connected to an electronic counter. The detector produces short electric pulses in response to the light beam and the counter registers the number of pulses that are emitted within a certain time interval. The average count rate is determined by the intensity of the light beam. Due to the discrete nature of the photons, a beam of light would not consist of a stream of photons with regular time intervals between them. There must be statistical fluctuations on short time-scales. Considering the probability $\mathcal{P}(n)$ of registering n counts, the fluctuations of that distribution about its mean value $\langle n\rangle$ are usually quantified in terms of the variance $\mathrm{Var}(n)$. The variance is defined by

$$\mathrm{Var}(n) := \sum_{n=0}^{\infty}(n - \langle n\rangle)^2\mathcal{P}(n)\,. \tag{1.11}$$

The standard deviation Δn is defined as: $\Delta n := \sqrt{\mathrm{Var}(n)}$. One of the properties of the Poisson distribution is that $\mathrm{Var}(n) = \langle n\rangle$. This provides a way to classify the photon distribution of a light beam as following:

- Sub-poissonian distribution: $\mathrm{Var}(n) < \langle n\rangle$
- Poissonian distribution: $\mathrm{Var}(n) = \langle n\rangle$
- Super-poissonian distribution: $\mathrm{Var}(n) > \langle n\rangle$

Lasers constitute a convenient way to realize CS, since a perfectly coherent light beam with a constant intensity has a Poissonian photon distribution. However, only an ideal laser will have a perfectly Poissonian distribution, because in real lasers perturbations may appear. The study of real lasers leads to almost-Poissonian (or non-Poissonian ones) photon distributions [44]. An interesting example of states with such properties are the generalized or nonlinear CS. In [45, 46] the nonlinear CS were defined as the eigenstates $|\alpha, f\rangle$ of the product of some function of the number operator N and the boson anihilation operator a: $f(N)a|\alpha, f\rangle = \alpha|\alpha, f\rangle$. From a physical standpoint, those states may appear as stationary states of a trapped ion, or in a nonlinear process like the frequency blue shift in high intensity photon beams. Many other examples of *nonlinear* generalizations of optical CS exist in the literature [47, 48, 49, 50, 51, 52]. In this work we will define the nonlinear CS in terms of their decomposition over the Fock basis. Therefore, the nonlinear CS fall under the generalizations based on various forms granted to the coefficients $c_n(\alpha)$ in the expression (1.6).

In the case of perfect detection, the photo-counter is ideally counting all photons. But in practice, available photo-counters are not ideally counting all photons, and their performances are limited by a efficiency parameter $\eta \in [0, 1]$, namely only a fraction η of the incoming photons leads to a *count*. In the case of imperfect photodetection, the binomial distribution allows to compute the probability to detect n-photons when the laser beam is in a state ρ. In [53], some preliminary theoretical and numerical explorations were presented concerning the properties of the Helstrom bound in binary communication involving nonlinear CS. There was proposed a deformation of the binomial distribution in order to compute the probability to detect n-photons for imperfect photodetection. This deformation of the binomial distribution was obtained in [54] and [55] using generating functions under some statistical constraints. Later on, it was called *asymmetric* deformation of the binomial distribution. In [56], a *symmetric* deformation of the binomial distribution was constructed in a similar fashion. Those distributions were used afterward in [57] to show some examples where the Boltzmann-Gibbs entropy is not extensive.

In this work we revisit the study of the optimization of quantum information with nonlinear CS, using the asymmetric and symmetric deformations of the binomial distribution (constructed from generating functions).

We study two specific examples of nonlinear CS. In the **first type** the nonlinear CS are defined in terms of a sequence of non-negative numbers χ, such that χ is generated by a function $\mathcal{N}$. To construct deformations of the binomial distribution with a complete probabilistic interpretation, a set of constraints must be imposed on $\mathcal{N}$. The

second type corresponds to the so-called Susskind-Glogower CS. The Susskind-Glogower operator $V := \frac{1}{\sqrt{N+1}}a$ indicates that the construction of nonlinear CS derives from the algebraic expression $f(N)a|\alpha, f\rangle = \alpha|\alpha, f\rangle$ (where $f(N) = \frac{1}{\sqrt{N+1}}$). Instead of that, in the work [58], the Susskind-Glogower CS are defined as $|\alpha\rangle_{\mathrm{SG}} = D_{\mathrm{SG}}|0\rangle$, where D_{SG} is the deformed displacement operator (which will be specified later).

1.3 Organization of this work.

Chapter 2 : This chapter is devoted to the study of quantum localization on the circle. In Section 2.1, we describe the CS associated with General Semidirect Product Groups. In Section 2.2 the corresponding covariant CS quantisation, denoted classical function $f \mapsto$ quantum operator A_f, is implemented. In Section 2.3 a family of probability distributions is constructed from these CS. They provide lower symbols $\check{f}(p, q)$ associated with the operator A_f. Section 2.4 is devoted to the study of the angle operator, related localization properties, and possible physical applications. Section 2.5 is devoted to the analytic and numerical study of the commutation relation between the quantum angle and the quantum angular momentum.

Chapter 3 : This chapter is devoted to the study of the quantum error probability with nonlinear CS. In Section 3.1 we study the nonlinear CS. In section 3.2 we present a short review of quantum error probability. In Section 3.3 we study the nonlinear CS generated by deformations of the binomial distribution. In Section 3.4 we study the Helstrom bound for nonlinear CS. In Section 3.5 we analyze questions related to the optimization of the Helstrom bound for nonlinear CS in comparison with linear CS.

Chapter 4 : in this chapter some conclusions are presented.

Chapter 2

Quantum localization on the circle

In Section 2.1 we study a generalization of the expression (1.7) using a group theoretical approach. Subsections 2.1.2 to 2.1.3 consist in a review of the material found in [27] and [34]. In Subsection 2.1.4 we apply the above formalism to one of the simplest cases, namely the Euclidean group E(2) which is the semidirect product $E(2) = \mathbb{R}^2 \rtimes \mathrm{SO}(2)$ and we introduce coherent states for E(2).

In Section 2.2 the corresponding covariant CS quantisation is implemented. In this case, the G-coset $X = G/H = \left(\mathbb{R}^2 \rtimes \mathrm{SO}(2)\right)/\mathbb{R}$ is represented by the cylinder $X = \mathbb{R} \times \mathbb{S}^1 = \{(p,q)\,,\, p \in \mathbb{R}\,,\, q \in [0, 2\pi) \bmod 2\pi\}$. The configuration manifold is the unit circle on which the motion of the particle takes place, and the velocity is parametrized by p. CS quantisation linearly maps functions or distributions $f(p,q)$ to operators A_f in the Hilbert space $\mathcal{H}$ carrying the group representation U of G. The covariance property of the quantisation map $f \mapsto A_f$ is made explicit. When f is real, i.e. when it is viewed as a classical observable, we expect that A_f be self-adjoint, or at least symmetric. We study the cases where the function $f(p,q) = u(q)$ does not depend on p and leads to a multiplication operator, the elementary example $u(q) = e^{inq}$, the quantum angular momentum issued from $f(p,q) = p$, the kinetic energy $f(p,q) = p^2$, as well as products of the type $p\,u(q)$ or $p^2 u(q)$, in order to cover the majority of the interesting Hamiltonians in quantum mechanics.

In Section 2.3 a family of probability distributions is constructed from the CS. They provide lower symbols $\check{f}(p,q)$ associated to the operator A_f. Explicit formulas are given for $\check{u}(q)$ and $\check{p}$.

Section 2.4 is devoted to the study of the angle operator, and related localization properties, resulting from the quantisation of the 2π-periodic discontinuous angle function, particularly its spectrum as a bounded self-adjoint multiplication operator. The study is illustrated analytically and numerically with the use of a particular family of smooth fiducial vectors η.

Section 2.5 is devoted to the analytic and numerical study of the commutation relation between the quantum angle and the quantum angular momentum, and the resulting uncertainty relation or Heisenberg inequality.

2.1 CS of General Semidirect Product Groups

2.1.1 Semidirect product groups

Let us consider an n-dimensional vector space V, a subgroup S of $GL(V)$ and the group $G = V \rtimes S$ with:

- the action $v \mapsto sv$ of S on V, for $v \in V$ and $s \in S$,
- the semidirect product law of composition $(x_1, s_1)(x_2, s_2) = (x_1 + s_1 x_2, s_1 s_2)$ for $x_1, x_2 \in V$ and $s_1, s_2 \in S$,
- the action $V^* \ni k \mapsto sk$ of S on the dual $V^* \sim V$, defined by $\langle sk; x\rangle = \langle k; s^{-1}x\rangle$ (dual pairing between V^* and V),
- the *adjoint action* of G on its Lie algebra $\mathfrak{g}$: $\mathrm{Ad}_g(X) = gXg^{-1}$ for $g \in G$ and $X \in \mathfrak{g}$,
- the *coadjoint action* of G on $\mathfrak{g}^*$: $\langle \mathrm{Ad}^{\#}_g(X^*); X\rangle_{\mathfrak{g}^*,\mathfrak{g}} = \langle X^*; \mathrm{Ad}_{g^{-1}}(X)\rangle_{\mathfrak{g}^*,\mathfrak{g}}$ for X^* (dual pairing between $\mathfrak{g}^*$ and $\mathfrak{g}$).

We now present some useful isomorphisms (summarized in the equation 2.3). Given the orbit of $k_0 \in V^*$ under the action of S

$$\mathcal{O}^* = \{k = sk_0 \in V^* \,|\, s \in S\}\,, \tag{2.1}$$

the *cotangent bundle* $T^*\mathcal{O}^* := \bigcup_{k\in\mathcal{O}^*} T^*_k\mathcal{O}^*$ admits a symplectic structure. Given the Lie algebras $\mathfrak{v}$ of V and $\mathfrak{s}$ of S respectively, the (coadjoint) orbit $\mathcal{O}^*_{(k_0,0)} = \{\mathrm{Ad}^{\#}_g(k_0,0) \in \mathfrak{g}^* \,|\, g \in G\}$ of $(k_0,0) \in \mathfrak{g}^*$ (for $k_0 \in \mathfrak{v}^*$ and $0 \in \mathfrak{s}^*$) is isomorphic to $T^*\mathcal{O}^*$ under the coadjoint action [27] . The stabilizer $H_0 = N_0 \rtimes S_0$ of $(k_0,0) \in \mathfrak{g}^*$ under the coadjoint action is the semi-direct product between the *annihilator*

$$N_0 = \left\{x \in V : \langle p; x\rangle = 0, \forall p \in T^*_{k_0}\mathcal{O}^*\right\}\,, \tag{2.2}$$

and the stabilizer $S_0 = \{s \in S | sk_0 = k_0\}$ of $k_0 \in V^*$ under the action of S. The left coset space $X = G/H_0$ is isomorphic to $T^*\mathcal{O}^*$. Considering the space $V_0 = T^*_{k_0}\mathcal{O}^*$, the space $T^*\mathcal{O}^*$ is isomorphic to $V_0 \times \mathcal{O}^*$ as a Borel space. We can summarize the isomorphisms given above in the following way [1]

$$\mathcal{O}_{(k_0,0)} \simeq T^*\mathcal{O}^* \simeq X = G/H_0 \simeq V_0 \times \mathcal{O}^*\,. \tag{2.3}$$

2.1.2 Induced representations for semi-direct product groups

Let us consider a one-dimensional unitary representation of V given by the character $\chi(v) = \exp(-\mathrm{i}\langle k_0; v\rangle)$ (for $k_0 \in V^*$ and $v \in V$), and a unitary irreducible representation $s \mapsto L(s)$ of S_0 (carried by the Hilbert space $\mathcal{K}$). Then one defines a unitary irreducible representation of $V \rtimes S_0$ as

$$(\chi \otimes L)\,(v,s) = e^{-\mathrm{i}\langle k_0;v\rangle} L(s) \quad \text{carried by } \mathcal{K}\,. \tag{2.4}$$

Given the relation $G/(V \rtimes S_0) \simeq \mathcal{O}^*$, one *induces* a representation of G from the representation $\chi \otimes L$ of $V \rtimes S_0$. Let us consider the bundle $S \xrightarrow{\pi_S} \mathcal{O}^*$ with the projection

[1] A detailed proof of this relation can be found in [27], Chapter 9, Section 9.2.2

$S \ni s \mapsto \pi_S(s) \in \mathcal{O}^*$, and the smooth section $\Lambda : \mathcal{O}^* \to S$ such that

$$\Lambda(k_0) = e \text{ (identity element of } S) \tag{2.5a}$$

$$\Lambda(k)k_0 = k,\ k \in \mathcal{O}^* . \tag{2.5b}$$

In this way any element $s \in S$ can be written as

$$s = \Lambda(k)s_0 \quad \text{for } k \in \mathcal{O}^*,\ s_0 \in S_0 . \tag{2.6}$$

Then one defines the action of S on $\mathcal{O}^*$ as $s'\pi_S(s) = \pi_S(s's)$ for $s', s \in S$. Considering this action and the property $\pi_S[\Lambda(sk)] = sk$ one gets

$$\pi_S[\Lambda(sk)] = \pi_S[s\Lambda(k)] . \tag{2.7}$$

Let us consider the bundle $G \xrightarrow{\pi_G} \mathcal{O}^*$ with the projection $\pi_G(x, s) = \pi_S(s)$. A smooth section $\lambda : \mathcal{O}^* \to G$ is defined by

$$\lambda(k) = (0, \Lambda(k)) . \tag{2.8}$$

According to the definition (2.7) and the equation (2.8), one finds the relation $\pi_G[(0, \Lambda(sk))] = \pi_G[(v, s\Lambda(k))]$. In other words $(0, \Lambda(sk))$ and $(v, s\Lambda(k))$ belong to the same fiber (equivalence class). Therefore,

$$(0, \Lambda(sk))h((v, s), k) = (v, s\Lambda(k)) . \tag{2.9}$$

The element $h((v, s), k) \in G$ defines the cocycles[2] $h : G \times \mathcal{O}^* \to V \rtimes S_0$ and $h_0 : S \times \mathcal{O}^* \to S_0$ by

$$h((v, s), k) = (\Lambda(sk)^{-1}v, h_0(s, k)) , \tag{2.10a}$$

$$h_0(s, k) = \Lambda(sk)^{-1}s\Lambda(k). \tag{2.10b}$$

[2]For $g, g_1, g_2 \in G$ and $k \in \mathcal{O}^*$ with $h'(g, k) = [h(g^{-1}, k)]^{-1} \in V \rtimes S_0$, the cocycle conditions are:

$$\begin{cases} h'(g_1g_2, k) &= h'(g_1, k)h'(g_2, g_1^{-1}k), \\ h'(e, k) &= e. \end{cases}$$

Using the representation $\chi \otimes L$ of $V \rtimes S_0$, one represents $h((v,s)^{-1},k) \in V \rtimes S_0$ as

$$(\chi \otimes L)(h((v,s)^{-1},k)) = e^{-\mathrm{i}\langle k_0;v\rangle} L\left(h_0(s^{-1},k)\right) . \tag{2.11}$$

Considering the space $\widetilde{\mathcal{H}} = \mathcal{K} \otimes L^2(\mathcal{O}^*, d\nu)$ of all square-integrable functions $\phi : \mathcal{O}^* \to \mathcal{K}$ in the norm $\|\phi\|^2_{\widetilde{\mathcal{H}}} = \int_{\mathcal{O}^*} \|\phi(k)\|^2_{\mathcal{K}} d\nu(k)$, one defines the representation $(v,s) \mapsto {}^{\chi L}U(v,s)$ of G (carried by $\widetilde{\mathcal{H}}$) as

$$\left({}^{\chi L}U(v,s)\phi\right)(k) = e^{\mathrm{i}\langle k_0;v\rangle} L\left(h_0(s^{-1},k)\right)^{-1} \phi(s^{-1}k)\,. \tag{2.12}$$

The expression (2.12) is a representation of G, which is *induced* by the representation $\chi \otimes L$ of $V \rtimes S_0$. The representation ${}^{\chi L}U$ of G is irreducible.

2.1.3 Coherent states for semi-direct product groups

From the isomorphisms (2.3) one constructs a section $V_0 \times \mathcal{O}^* \ni (\boldsymbol{p},\boldsymbol{q}) \mapsto \sigma(\boldsymbol{p},\boldsymbol{q}) \in G$, where $(\boldsymbol{p},\boldsymbol{q})$ are canonically conjugate pairs for the symplectic structure of the manifold $V_0 \times \mathcal{O}^*$. Given the invariant symplectic measure $\mathrm{d}\mu(\boldsymbol{p},\boldsymbol{q})$ for $V_0 \times \mathcal{O}^*$, the action of the induced representation (2.12) of G on a vector $\eta \in \widetilde{\mathcal{H}}$ gives a family of vectors $\eta^\sigma_{\boldsymbol{p},\boldsymbol{q}} \equiv \eta_{\boldsymbol{p},\boldsymbol{q}} \in \widetilde{\mathcal{H}}$ parametrized by $(\boldsymbol{p},\boldsymbol{q})$ (where $\boldsymbol{q} \in V_0$ and $\boldsymbol{p} \in \mathcal{O}^*$)

$$\eta_{\boldsymbol{p},\boldsymbol{q}}(k) = \left({}^{\chi L}U(\sigma(\boldsymbol{p},\boldsymbol{q}))\eta\right)(k) \quad \Leftrightarrow \quad |\,\eta_{\boldsymbol{p},\boldsymbol{q}}\,\rangle = {}^{\chi L}U(\sigma(\boldsymbol{p},\boldsymbol{q}))|\,\eta\,\rangle\,. \tag{2.13}$$

Since in the present paper the representation L is actually trivial, we dismiss $\mathcal{K}$ from now on, so that $\widetilde{\mathcal{H}} = \mathcal{H} = L^2(\mathcal{O}^*, d\nu)$. Let us consider the formal integral

$$\int_{V_0\times\mathcal{O}^*} \mathrm{d}\mu(\boldsymbol{p},\boldsymbol{q})\langle\,\phi\,|\,\eta_{\boldsymbol{p},\boldsymbol{q}}\,\rangle_{\mathcal{H}}\langle\,\eta_{\boldsymbol{p},\boldsymbol{q}}\,|\,\psi\,\rangle_{\mathcal{H}}\,,\ \text{where } \phi,\psi : \mathcal{O}^* \to \mathbb{C}\,. \tag{2.14}$$

If we prove that it is equal to $c_\eta\langle\,\phi\,|\,\psi\,\rangle$ for some constant $0 < c_\eta < \infty$, we obtain that the resolution of the identity

$$\frac{1}{c_\eta}\int_{V_0\times\mathcal{O}^*} \mathrm{d}\mu(\boldsymbol{p},\boldsymbol{q})|\,\eta_{\boldsymbol{p},\boldsymbol{q}}\,\rangle\langle\,\eta_{\boldsymbol{p},\boldsymbol{q}}\,| = I \quad \text{where } 0 < c_\eta < \infty \tag{2.15}$$

holds on $\mathcal{H}$. In the case we are considering in this paper, we will see that (2.15) holds by imposing restrictions on $\operatorname{supp}\eta$. When (2.15) is valid, the states (2.13) are our (covariant) *coherent states*, which generalize the Gilmore-Perelomov construction [9, 59].

2.1.4 Coherent States for $\mathbf{E}(2)$

The (special) Euclidean group is the semi-direct product of the translations in the plane with the rotations,

$$\mathrm{E}(2) = \mathbb{R}^2 \rtimes \mathrm{SO}(2) = \{(\boldsymbol{r}, \theta)\,,\, \boldsymbol{r} \in \mathbb{R}^2\,,\, \theta \in [0, 2\pi)\}\,, \tag{2.16}$$

equipped with the composition rule and the inverse[3]

$$(\boldsymbol{r}, \theta)(\boldsymbol{r}', \theta') = (\boldsymbol{r} + \mathcal{R}(\theta)\boldsymbol{r}', \theta + \theta')\,, \quad (\boldsymbol{r}, \theta)^{-1} = (-\mathcal{R}(-\theta)\boldsymbol{r}, -\theta)\,, \tag{2.17}$$

where $\mathcal{R}(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ rotates vectors in the plane by the angle θ. We denote by $L^2(\mathbb{S}^1, \mathrm{d}\alpha)$ the Hilbert space of 2π-periodic complex-valued functions $\psi(\alpha)$ which are square-integrable on a period interval $[\alpha_0, \alpha_0 + 2\pi]$, $\alpha_0 \in \mathbb{R}$,

$$\int_{\alpha_0}^{\alpha_0+2\pi} \mathrm{d}\alpha\, |\psi(\alpha)|^2 \equiv \int_{\mathbb{S}^1} \mathrm{d}\alpha\, |\psi(\alpha)|^2\,, \tag{2.18}$$

and equipped with the scalar product

$$\langle\phi|\psi\rangle = \int_{\mathbb{S}^1} \mathrm{d}\alpha\, \overline{\psi(\alpha)}\, \phi(\alpha)\,. \tag{2.19}$$

Given a real number $a \neq 0$, the action of the unitary irreducible representations of E(2) [32, 33] on $L^2(\mathbb{S}^1, \mathrm{d}\alpha)$ is realised as

$$\begin{aligned} L^2(\mathbb{S}^1, \mathrm{d}\alpha) \ni \psi(\alpha) \mapsto \left(U_a(\boldsymbol{r}, \theta)\psi\right)(\alpha) &= e^{\mathrm{i}a(r_1 \cos\alpha + r_2 \sin\alpha)}\psi(\alpha - \theta) \\ &\equiv e^{ia[\mathcal{R}(-\alpha)\mathbf{r}]_1}\psi(\alpha - \theta)\,. \end{aligned} \tag{2.20}$$

Besides this set of non-equivalent UIR's, there exists a degenerate one, corresponding to $a = 0$, which we ignore. From now on, we pick the value $a = 1$ and write simply $U_1 = U$.

The cotangent bundle $T^*\mathbb{S}^1$ is viewed as the classical phase space for a particle moving on a circle. It can be identified with a co-adjoint orbit of E(2), and, as a homogenous space of the latter, with the left coset of E(2)

$$T^*\mathbb{S}^1 \simeq (\mathbb{R}^2 \rtimes \mathrm{SO}(2))/H \simeq \mathbb{R} \times \mathbb{S}^1\,, \tag{2.21}$$

[3] Here and throughout the text, sums of angles are always to be understood module 2π, i.e., $\theta + \theta' \simeq (\theta + \theta') \bmod 2\pi$.

where $H \simeq \mathbb{R}$ is the isotropy subgroup of one point of the co-adjoint orbit. All technical details about the general construction of representations of semi-direct products of the type $G = V \rtimes S$, where V is an n-dimensional vector space and S is a subgroup of $\mathrm{GL}(V)$, where presented in in the previous section, and we recall that more developed material is found in [27] and [34]. We just retain from that framework the following choice for the subgroup H defining the group factorisation (2.21):

$$H \equiv H_{\hat{\boldsymbol{c}}} = \left\{(\boldsymbol{x}, 0) \in \mathrm{E}(2) \,|\, \hat{\boldsymbol{c}} \cdot \boldsymbol{x} = 0,\, \hat{\boldsymbol{c}} \in \mathbb{R}^2,\, \|\hat{\boldsymbol{c}}\| = 1, \text{ fixed}\right\}\,, \tag{2.22}$$

By adopting the usual phase space notations, $T^*\mathbb{S}^1$ carries canonical coordinates $(p, q) \in \mathbb{R} \times \mathbb{S}^1$ and the symplectic invariant measure $\mathrm{d}p\,\mathrm{d}q \equiv \mathrm{d}p \wedge \mathrm{d}q$.

In accordance with (2.21), phase space coordinates (p, q) are mapped to $\mathrm{E}(2)$ through a general section σ as

$$\mathbb{R} \times \mathbb{S}^1 \ni (p, q) \mapsto \sigma(p, q) = (\boldsymbol{f}(p, q), q) \in \mathrm{E}(2)\,. \tag{2.23}$$

where $\boldsymbol{f}(p, q)$ is a function to be determined. Precisely, we have the following result.

Theorem 2.1.1. *Given the unit vector $\hat{\boldsymbol{c}} \in \mathbb{R}^2$ and the corresponding subgroup $H_{\hat{\boldsymbol{c}}}$ defined by (2.22), there exists a family of affine sections $\sigma : \mathbb{R} \times \mathbb{S}^1 \to E(2)$ defined as*

$$\sigma(p, q) = (\mathcal{R}(q)(\boldsymbol{\kappa} p + \boldsymbol{\lambda}), q)\,, \tag{2.24}$$

where $\boldsymbol{\kappa}, \boldsymbol{\lambda} \in \mathbb{R}^2$ are constant vectors, and $\hat{\boldsymbol{c}} \cdot \boldsymbol{\kappa} \neq 0$.

The action of $E(2)$ on its left coset determined by these sections, through $(\boldsymbol{r}, \theta)\sigma(p, q) = \sigma(p', q')(\boldsymbol{x}, 0)$, $\hat{\boldsymbol{c}} \cdot \boldsymbol{x} = 0$, is given by

$$\begin{aligned} p' &= p + \frac{1}{\hat{\boldsymbol{c}} \cdot \boldsymbol{\kappa}}\, \mathcal{R}(q + \theta)\hat{\boldsymbol{c}} \cdot \boldsymbol{r}\,, \\ q' &= q + \theta\,. \end{aligned} \tag{2.25}$$

This action is canonical, $\mathrm{d}p' \wedge \mathrm{d}q' = \mathrm{d}p \wedge \mathrm{d}q$.

Proof. Let us consider the generic section $\sigma : \mathbb{R} \times \mathbb{S}^1 \to \mathrm{E}(2)$ given by (2.23). The factorization of $(\boldsymbol{r}, \theta) \in G$ according to the left coset $\mathrm{E}(2)/H_{\hat{\boldsymbol{c}}}$ is given by

$$(\boldsymbol{r}, \theta) = (\boldsymbol{r}', \theta)(\boldsymbol{x}, 0)\,, \quad \boldsymbol{r}' = \boldsymbol{r} - \mathcal{R}(\theta)\boldsymbol{x}\,, \quad \hat{\boldsymbol{c}} \cdot \boldsymbol{x} = 0\,. \tag{2.26}$$

With this expression at hand, one writes the action of E(2) on its left coset as

$$(\boldsymbol{r}, \theta)\sigma(p, q) = \sigma(p', q')(\boldsymbol{x}, 0)\,, \quad \hat{\boldsymbol{c}} \cdot \boldsymbol{x} = 0\,. \tag{2.27}$$

Taking into account the general form (2.23), this identity reads

$$(\boldsymbol{r} + \mathcal{R}(\theta)\boldsymbol{f}(p, q), \theta + q) = (\boldsymbol{f}(p', q') + \mathcal{R}(q')\boldsymbol{x}, q')\,. \tag{2.28}$$

Therefore, we arrive at the conditions

$$\boldsymbol{r} + \mathcal{R}(\theta)\boldsymbol{f}(p, q) = \boldsymbol{f}(p', q') + \mathcal{R}(q')\boldsymbol{x}\,, \tag{2.29a}$$

$$q' = \theta + q\,. \tag{2.29b}$$

These conditions determine the change of variables $(p, q) \to (p', q')$ for the angular coordinate q and its conjugate momentum p. The conditions (2.29) provide an explicit form for the function $q'(q, \theta)$, but not for the function $p'(p, q, \theta, \boldsymbol{r})$. From (2.29) the vector $\boldsymbol{x}$ is written as

$$\boldsymbol{x} = \mathcal{R}(-q - \theta)\boldsymbol{r} + \mathcal{R}(-q)\boldsymbol{f}(p, q) - \mathcal{R}(-q - \theta)\boldsymbol{f}(p'(p, q, \theta, \boldsymbol{r}), q'(q, \theta))\,. \tag{2.30}$$

Since $(\boldsymbol{x}, 0) \in H_{\hat{\boldsymbol{c}}}$, we have from (2.22) $\hat{\boldsymbol{c}} \cdot \boldsymbol{x} = 0$. For the particular case $\boldsymbol{r} = \boldsymbol{0}$, equation (2.30) becomes

$$\hat{\boldsymbol{c}} \cdot [\mathcal{R}(-q)\boldsymbol{f}(p, q) - \mathcal{R}(-q - \theta)\boldsymbol{f}\,(p'(p, q, \theta, \boldsymbol{0}), q + \theta)] = 0\,, \quad \text{for all} \quad q\,, p\,, \theta\,. \tag{2.31}$$

Choosing $q = 0$ leads to

$$\hat{\boldsymbol{c}} \cdot [\boldsymbol{f}(p, 0) - \mathcal{R}(-\theta)\boldsymbol{f}\,(p'(p, 0, \theta, \boldsymbol{0}), \theta)] = 0\,. \tag{2.32}$$

The vector $\hat{\boldsymbol{c}}$ is fixed, and when the value of p is also fixed, the right-hand side of the equation (2.32) should be independent of θ. Therefore $p'(p, 0, \theta, \mathbf{0})$ must be independent of θ, i.e., $p' = p'(p, 0, \mathbf{0})$. In order to eliminate the dependence on θ from (2.32), we write $\boldsymbol{f}\,(p'(p, 0, \mathbf{0}), \theta)$ as

$$\boldsymbol{f}\,(p'(p, 0, \mathbf{0}), \theta) = \mathcal{R}(\theta)\boldsymbol{g}(p'(p, 0, \mathbf{0}))\,, \tag{2.33}$$

where $\boldsymbol{g}$ is a function to be determined. Since $\mathrm{d}p \wedge \mathrm{d}q$ should be left invariant under the change of variables $(p, q) \mapsto (p', q')$, i.e. $\mathrm{d}\,p'\,\mathrm{d}q' = |J\,|\mathrm{d}p\,\mathrm{d}q$ with $|J| = 1$, the only possible choice is $\frac{\partial}{\partial p}p'(p, 0, \mathbf{0}) = 1$. Therefore $p'(p, 0, \mathbf{0})$ must be

$$p'(p, 0, \mathbf{0}) = p + \text{constant}\,. \tag{2.34}$$

Considering (2.33) and (2.34) we arrive at

$$\boldsymbol{f}(p + \text{constant}, \theta) = \mathcal{R}(\theta)\boldsymbol{g}(p + \text{constant})\,. \tag{2.35}$$

The simplest generalisation of (2.35) to the case $q \neq 0$ is

$$\boldsymbol{f}(p + \varphi(q), q + \theta) = \mathcal{R}(q + \theta)\boldsymbol{g}(p + \varphi(q))\,. \tag{2.36}$$

With this choice, the vector $\boldsymbol{f}(p, q)$ assumes the form

$$\boldsymbol{f}(p, q) = \mathcal{R}(q)\boldsymbol{g}(p)\,. \tag{2.37}$$

Now $\boldsymbol{g}(p)$ has the property $\boldsymbol{g}(p) \to \boldsymbol{g}(p + \varphi(q))$ when $p \to p'$. Hence, the simplest choice is $\boldsymbol{g}(p) = \boldsymbol{\kappa}p + \boldsymbol{\lambda}$ where $\boldsymbol{\kappa}, \boldsymbol{\lambda} \in \mathbb{R}^2$ are constant vectors

$$\boldsymbol{f}(p, q) = \mathcal{R}(q)(\boldsymbol{\kappa}p + \boldsymbol{\lambda})\,. \tag{2.38}$$

It is then straightforward to derive the general transform (2.25) and to check the invariance of $\mathrm{d}p \wedge \mathrm{d}q$.

The section $\sigma(p, q)$ given by (2.24) may not be the most general type of Borel section allowed in this problem, but is compatible with the conditions (2.29). $\square$

Definition 2.1.1. *Using the section (2.24), the representation (2.20) with* $a = 1$, *and a choice of vector* $\eta \in L^2(\mathbb{S}^1, \mathrm{d}\alpha)$, *we define the following family of states:*

$$|\eta_{p,q}\rangle = U(\sigma(p,q))|\eta\rangle \,. \tag{2.39}$$

Explicitely,

$$\eta_{p,q}(\alpha) = e^{\mathrm{i}[\mathcal{R}(q-\alpha)(\boldsymbol{\kappa}p+\boldsymbol{\lambda})]_1}\eta(\alpha - q) = e^{\mathrm{i}[\kappa p \cos(q-\alpha+\gamma)+\lambda\cos(q-\alpha+\zeta)]}\eta(\alpha - q)\,, \tag{2.40}$$

with $\boldsymbol{\kappa} = \kappa \begin{pmatrix}\cos\gamma \\ \sin\gamma\end{pmatrix}$ *and* $\boldsymbol{\lambda} = \lambda\begin{pmatrix}\cos\zeta \\ \sin\zeta\end{pmatrix}$, $\kappa = \|\boldsymbol{\kappa}\|$, $\lambda = \|\boldsymbol{\lambda}\|$, $\gamma = \arg\boldsymbol{\kappa}$, $\zeta = \arg\boldsymbol{\lambda}$.

Note that if we wish to take into account physical dimensions in the above formalism, the parameter κ has to carry the dimension of inverse momentum. Hence, if p is an angular momentum, then $\kappa \propto 1/\hbar$.

We now examine the question whether the states (2.39) are *coherent* in the sense that they solve the identity. This results in conditions on the vector η, which in this context makes it a *fiducial vector*.

Theorem 2.1.2. *The vectors* $\eta_{p,q}$ *form a family of coherent states for* $E(2)$ *which resolves the identity on* $L^2(\mathbb{S}^1, \mathrm{d}\alpha)$,

$$I = \int_{\mathbb{R}\times\mathbb{S}^1} \frac{\mathrm{d}p\,\mathrm{d}q}{c_\eta}\,|\,\eta_{p,q}\,\rangle\langle\,\eta_{p,q}\,|\,, \tag{2.41}$$

if $\eta(\alpha)$ *is* <u>*admissible*</u> *in the sense that* supp $\eta \in (\gamma - \pi, \gamma) \bmod 2\pi$, *and*

$$0 < c_\eta := \frac{2\pi}{\kappa}\int_{\mathbb{S}^1} \frac{|\eta(q)|^2}{\sin(\gamma - q)}\,\mathrm{d}q < \infty\,. \tag{2.42}$$

Proof. Let ψ and ϕ be two functions in $L^2(\mathbb{S}^1, \mathrm{d}\alpha)$. In order to prove the theorem, we must find the conditions for which their scalar product $\langle\,\phi\,|\,\psi\,\rangle$ is equal to the integral

$$I(\psi,\phi) = \int_{\mathbb{R}\times\mathbb{S}^1}\frac{\mathrm{d}p\,\mathrm{d}q}{c_\eta}\int_{\mathbb{S}^1}\mathrm{d}\alpha\,\overline{\psi(\alpha)}\,\eta_{p,q}(\alpha)\int_{\mathbb{S}^1}\mathrm{d}\alpha'\,\overline{\eta_{p,q}(\alpha')}\,\phi(\alpha')\,. \tag{2.43}$$

After integrating with respect to the variable p by using $\int_{\mathbb{R}}\mathrm{d}pe^{-\mathrm{i}pk} = 2\pi\delta(k)$, the integral (2.43) becomes

$$\begin{aligned} I(\psi,\phi) = \frac{2\pi}{c_\eta}\int_{\mathbb{S}^1}\mathrm{d}q\int_{\mathbb{S}^1}\mathrm{d}\alpha'\,\overline{\eta(\alpha'-q)}\phi(\alpha')\int_{\mathbb{S}^1}\mathrm{d}\alpha\,\overline{\psi(\alpha)}\eta(\alpha-q)e^{2\mathrm{i}\lambda S_\zeta(\alpha,\alpha',q)} \\ \times\delta(2\kappa S_\gamma(\alpha,\alpha',q))\,. \end{aligned} \tag{2.44}$$

where the function $S_x(\alpha,\alpha',q)$ is defined as

$$S_x(\alpha,\alpha',q) = \sin\left(\frac{\alpha-\alpha'}{2}\right)\sin\left(q+x-\frac{\alpha+\alpha'}{2}\right), \quad \text{for} \quad x=\gamma,\zeta\,. \tag{2.45}$$

Now the Dirac delta has the expansion

$$\delta(2\kappa S_x(\alpha,\alpha',q)) = \sum_k \frac{\delta\left(\alpha-\alpha_k\right)}{2\kappa\left|\partial_\alpha S_x(\alpha,\alpha',q)\right|_{\alpha=\alpha_k}} + \sum_{k'} \frac{\delta\left(\alpha-\alpha_{k'}\right)}{2\kappa\left|\partial_\alpha S_x(\alpha,\alpha',q)\right|_{\alpha=\alpha_{k'}}}, \tag{2.46}$$

where α_k and $\alpha_{k'}$ are the roots of $S_x(\alpha,\alpha',q)$ obtained when $\dfrac{\alpha-\alpha'}{2} = k\pi$ or $q+x-\dfrac{\alpha+\alpha'}{2} = k'\pi$ for $k,k'\in\mathbb{Z}$. Hence, $\alpha_k = \alpha'+2k\pi$ and $\alpha_{k'} = -\alpha'+2q+2x-2k'\pi$. The Dirac delta is now written as

$$\delta(2\kappa S_x(\alpha,\alpha',q)) = \sum_k \frac{\delta\left(\alpha-\alpha_k\right)}{\kappa\left|\sin\left(q+x-\alpha'\right)\right|} + \sum_{k'} \frac{\delta\left(\alpha-\alpha_{k'}\right)}{\kappa\left|\sin\left(q+x-\alpha'\right)\right|}\,. \tag{2.47}$$

With the help of expression (2.47), using the 2π periodicity of all involved functions and the fact that one integrates over one period interval, the integral (2.44) becomes

$$\begin{aligned} I(\psi,\phi) &= \frac{2\pi}{\kappa c_\eta}\int_{\mathbb{S}^1}\mathrm{d}\alpha'\,\overline{\psi(\alpha')}\phi(\alpha')\int_{\mathbb{S}^1}\mathrm{d}q\,\frac{\overline{\eta(\alpha'-q)}\eta(\alpha'-q)}{\left|\sin\left(\gamma-(\alpha'-q)\right)\right|} \\ &+\frac{2\pi}{\kappa c_\eta}\int_{\mathbb{S}^1}\mathrm{d}q\int_{\mathbb{S}^1}\mathrm{d}\alpha'\,\frac{\overline{\eta(\alpha'-q)}\phi(\alpha')}{\left|\sin\left(q+\gamma-\alpha'\right)\right|}e^{2\mathrm{i}\lambda\sin(\gamma+q-\alpha')\sin(\zeta-\gamma)} \\ &\qquad\times\overline{\psi(-\alpha'+2q+2\gamma)}\eta\left(q-\alpha'+2\gamma\right)\,. \end{aligned} \tag{2.48}$$

Performing the change of variable $q\mapsto q' = \alpha'-q$ in both integrals, and choosing $(\gamma-\pi,\gamma+\pi)$ as the integration interval for the q' variable, one has

$$\begin{aligned} I(\psi,\phi) &= \frac{2\pi}{\kappa c_\eta}\int_{\mathbb{S}^1}\mathrm{d}\alpha'\,\overline{\psi(\alpha')}\phi(\alpha')\int_{\gamma-\pi}^{\gamma+\pi}\mathrm{d}q'\,\frac{|\eta(q')|^2}{\left|\sin\left(\gamma-q'\right)\right|} \\ &+\frac{2\pi}{\kappa c_\eta}\int_{\mathbb{S}^1}\mathrm{d}\alpha'\,\phi(\alpha')\int_{\gamma-\pi}^{\gamma+\pi}\mathrm{d}q'\,\frac{\overline{\eta(q')}}{\left|\sin\left(\gamma-q'\right)\right|}e^{2\mathrm{i}\lambda\sin(\gamma-q')\sin(\zeta-\gamma)} \\ &\qquad\times\overline{\psi(\alpha'-2q'+2\gamma)}\eta\left(2\gamma-q'\right)\,. \end{aligned} \tag{2.49}$$

In order to avoid the singularity appearing in the denominator of the integrand of the first integral in (2.49), we impose that $|\sin(\gamma-q')|\neq 0$ for $q'\in\operatorname{supp}\eta$. Hence, we choose $\operatorname{supp}\eta\subset(\gamma-\pi,\gamma)\bmod 2\pi$. The second integral vanishes, since $2\gamma-q'\notin\operatorname{supp}\eta$. Thus (2.49) reduces to

$$I(\psi,\phi) = \langle\psi|\phi\rangle\,\frac{1}{c_\eta}\frac{2\pi}{\kappa}\int_{\gamma-\pi}^{\gamma}\mathrm{d}q\,\frac{|\eta(q)|^2}{\sin\left(\gamma-q\right)}\,. \tag{2.50}$$

Imposing the condition

$$c_\eta = \frac{2\pi}{\kappa} \int_{\gamma-\pi}^{\gamma} \mathrm{d}q \, \frac{|\eta(q)|^2}{\sin(\gamma - q)} < \infty \,, \tag{2.51}$$

gives $I(\psi, \phi) = \langle \psi | \phi \rangle$. With this result the integral (2.43) takes the form

$$\langle \psi \,|\, \phi \rangle = \int_{\mathbb{R}\times\mathbb{S}^1} \frac{\mathrm{d}p \, \mathrm{d}q}{c_\eta} \langle \psi \,|\, \eta_{p,q} \rangle \langle \eta_{p,q} \,|\, \phi \rangle \,. \tag{2.52}$$

Hence the vectors $\eta_{p,q}$ form a family of *coherent states* for E(2) which resolves the identity on $L^2(\mathbb{S}^1, \mathrm{d}\alpha)$. □

Remark We note that the de Bièvre coherent states [34] are recovered with the choices $\gamma = \pi/2$, $\kappa = 1$, and $\boldsymbol{\lambda} = \mathbf{0}$. At this point, let us examine the constants appearing in our approach. The arbitrary unit vector $\hat{\boldsymbol{c}}$ together with vectors $\boldsymbol{\kappa}$ and $\boldsymbol{\lambda}$ determine the section $\sigma \equiv \sigma_{\hat{\boldsymbol{c}},\boldsymbol{\kappa},\boldsymbol{\lambda}}$. The particular case presented in [34] hints at the meaning of these parameters. Phase space coordinates (p, q) map to $\sigma(p, q)$. Now, given $\hat{\boldsymbol{c}}$ we have the equivalence mod $H_{\hat{\boldsymbol{c}}}$:

$$\sigma(p, q) = (\mathcal{R}(q)(\boldsymbol{\kappa}p + \boldsymbol{\lambda}), q) \equiv (\mathcal{R}(q)(\boldsymbol{\kappa}p + \boldsymbol{\lambda}), q)(\boldsymbol{b}, 0) = (\mathcal{R}(q)(\boldsymbol{\kappa}p + \boldsymbol{\lambda} + \boldsymbol{b}), q) \,,$$

for all $\boldsymbol{b}$ such that $\boldsymbol{b} \cdot \hat{\boldsymbol{c}} = 0$. Then we choose $\hat{\boldsymbol{c}} = \hat{\boldsymbol{\imath}} = \boldsymbol{\kappa}$ and $\boldsymbol{\lambda} \perp \hat{\boldsymbol{\imath}}$. We get

$$\sigma(p, q) \equiv ((p + b_y)\mathcal{R}(q)\boldsymbol{\imath}, q) = \left((p + b_y + \lambda_y) \begin{pmatrix} -\sin q \\ \cos q \end{pmatrix}, q \right) \,.$$

This is the situation met in the example given in [34], up to the presence of the arbitrary $b_y + \lambda_y$. This translational freedom expresses the arbitrariness of the origin for the momentum coordinate p. The vector $\hat{\boldsymbol{c}}$ is seen then to guarantee translational invariance in the momentum coordinate. Similarly, the arbitrariness of $\boldsymbol{\kappa}$ has to do with the choice of angular origin (role of γ) and scaling (role of κ as a circle radius). This arbitrariness in the choice of the origin in the cylindric phase space **can** also be considered under the covariance perspective described by Eq.(2.58) below.

For the sake of later convenience, we introduce the following families of integrals.

Definition 2.1.2. *Given a 2π-periodic function $\eta(\alpha) \in L^2(\mathbb{S}^1, \mathrm{d}\alpha)$ with $\operatorname{supp}\eta \in (\gamma - \pi, \gamma) \bmod 2\pi$, $\gamma \in [0, 2\pi)$, we define the integrals,*

$$c_\nu(\eta,\gamma) = \int_{\mathbb{S}^1} \mathrm{d}\alpha \frac{|\eta(\alpha)|^2}{(\sin(\gamma-\alpha))^\nu}\,, \tag{2.53}$$

where $\nu \in \mathbb{C}$ is such that convergence is assured.

With this definition, $c_0 = 1$ from the normalisation of η, and the constant c_η is given by $c_\eta = \frac{2\pi}{\kappa}\, c_1(\eta,\gamma)$.

Definition 2.1.3. *For η of class C^k, we define for $j \le k$ the set of functions*

$$f_{j;m}(q) = \frac{\eta(q)\, \partial_q^j\, \overline{\eta(q)}}{(\sin(\gamma-q))^m}\,, \quad \textit{for } j,m \in \mathbb{N}\,. \tag{2.54}$$

2.2 CS Quantisation of Classical Observables

2.2.1 Quantisation map and its covariance

According to the general scheme of covariant integral quantisation [27], with coherent states $|\eta_{p,q}\rangle$ built from a section σ and a given admissible fiducial vector η, the quantisation of a classical observable $f(p,q)$ is defined as the linear map

$$f \mapsto A_f^\sigma = \int_{\mathbb{R}\times\mathbb{S}^1} \frac{\mathrm{d}p\,\mathrm{d}q}{c_\eta} f(p,q)\, |\, \eta_{p,q}\,\rangle\langle\, \eta_{p,q}\,|\,, \tag{2.55}$$

where the constant c_η has been introduced in (2.51). The covariance of this map holds in the following sense. Consider the sections $\sigma_g : \mathrm{E}(2)/H_{\hat{\boldsymbol{c}}} \to \mathrm{E}(2)$, which are covariant translates of σ under $g = (\boldsymbol{r},\theta) \in \mathrm{E}(2)$:

$$\sigma_g(p,q) = g\sigma(g^{-1}(p,q)) = \sigma(p,q)h(g, g^{-1}(p,q))\,. \tag{2.56}$$

and where the cocycle $h(g,(p,q))$ belongs to $H_{\hat{\boldsymbol{c}}}$. Explicitly,

$$\begin{aligned} \sigma_g(p,q) &= \left(\mathcal{R}(q)(\boldsymbol{\kappa}p + \boldsymbol{\lambda}) + \boldsymbol{r} - \mathcal{R}(q)\boldsymbol{\kappa}\,\frac{\mathcal{R}(q)\hat{\boldsymbol{c}}\cdot\boldsymbol{r}}{\hat{\boldsymbol{c}}\cdot\boldsymbol{\kappa}}, q\right) \\ &= (\mathcal{R}(q)(\boldsymbol{\kappa}p+\boldsymbol{\lambda}), q)\left(\mathcal{R}(-q)\boldsymbol{r} - \boldsymbol{\kappa}\,\frac{\mathcal{R}(q)\hat{\boldsymbol{c}}\cdot\boldsymbol{r}}{\hat{\boldsymbol{c}}\cdot\boldsymbol{\kappa}}, 0\right)\,. \end{aligned} \tag{2.57}$$

Then,

$$
\begin{aligned}
U(g)A^{\sigma}_{f}U(g)^{\dagger} &= A^{\sigma_g}_{\mathcal{U}_l(g)f}\,, \\
A^{\sigma_g}_{f} &:= \int_{\mathbb{R}\times\mathbb{S}^1} \frac{\mathrm{d}p\,\mathrm{d}q}{c_\eta} f(p,q)\,|\,U(\sigma_g(p,q)\eta\rangle\langle U(\sigma_g(p,q)\eta\,|\,,
\end{aligned}
\tag{2.58}
$$

with $\mathcal{U}_l(g)f(p,q) = f\left(g^{-1}(p,q)\right)$. We note that the section itself is invariant under pure rotations $g = (\mathbf{0},\theta)$.

In the sequel we will drop the superscript σ and write simply $A^{\sigma}_{f} = A_f$. This operator acts on the Hilbert space $L^2(\mathbb{S}^1, \mathrm{d}\alpha)$ as the integral operator

$$
(A_f\psi)(\alpha) = \int_{\mathbb{S}^1} \mathrm{d}\alpha'\, \mathcal{A}_f(\alpha,\alpha')\,\psi(\alpha')\,, \tag{2.59}
$$

whose kernel $\mathcal{A}_f$ is given by

$$
\mathcal{A}_f(\alpha,\alpha') = \frac{1}{c_\eta}\int_{\mathbb{S}^1} \mathrm{d}q\,\eta(\alpha-q)\overline{\eta(\alpha'-q)}e^{2\mathrm{i}\lambda S_\varsigma(\alpha,\alpha',q)}\int_{-\infty}^{+\infty}\mathrm{d}p\,e^{\mathrm{i}2\kappa S_\gamma(\alpha,\alpha',q)p}\,f(p,q)\,. \tag{2.60}
$$

The expression (2.60) is quite involved. Therefore, in the following we examine particular cases.

2.2.2 Quantisation of a function of the coordinate q

Let us introduce the positive 2π-periodic function

$$
E_{\eta;\gamma}(\alpha) := \frac{2\pi}{\kappa c_\eta}\,\frac{|\eta(\alpha)|^2}{\sin(\gamma-\alpha)}\,,\ \mathrm{supp}\,E_{\eta;\gamma}\subset(\gamma-\pi,\gamma)\,, \tag{2.61}
$$

for $0\leq\gamma<\pi$. It is normalised in the sense that

$$
\int_{\gamma-\pi}^{\gamma+\pi}\mathrm{d}\alpha\,E_{\eta;\gamma}(\alpha) = \int_{\gamma-\pi}^{\gamma}\mathrm{d}\alpha\,E_{\eta;\gamma}(\alpha) = 1\,. \tag{2.62}
$$

Thus it can be considered a probability distribution on the interval $[\gamma-\pi,\gamma]$ (or $[-\pi,\pi]$), and the average value of a function $f(\alpha)$ on the same interval will be denoted by

$$
\langle f\rangle_{E_{\eta;\gamma}} := \int_{-\pi}^{+\pi}\mathrm{d}\alpha\,f(\alpha)E_{\eta;\gamma}(\alpha) = \int_{\gamma-\pi}^{\gamma}\mathrm{d}\alpha\,f(\alpha)E_{\eta;\gamma}(\alpha)\,. \tag{2.63}
$$

The application of (2.55) and (2.59) to the quantisation of functions which only depend on the angle is straightforward and leads to the following result.

Proposition 2.2.1. *For $f(p,q) = u(q)$ with $u(q+2\pi) = u(q)$, A_u is the multiplication operator*

$$(A_u\psi)(\alpha) = (E_{\eta;\gamma} * u)(\alpha)\,\psi(\alpha)\,, \tag{2.64}$$

where the periodic convolution product on the circle is defined by

$$(E_{\eta;\gamma} * u)(\alpha) = \int_{\alpha-\gamma}^{\alpha+\pi-\gamma} \mathrm{d}q\, E_{\eta;\gamma}(\alpha - q)\, u(q)\,. \tag{2.65}$$

Moreover, since the function $E_{\eta;\gamma}$ is a probability distribution on a period interval, a standard result of Analysis [60] on convolution allows us to state the following.

Proposition 2.2.2. *If the 2π-periodic function u is bounded on a period interval, then the 2π-periodic convolution $E_{\eta;\gamma} * u$ is bounded and continuous.*

The expression (2.65) is expected to regularise the original $u(q)$. This depends of course on the regularity of the fiducial vector η. For instance, with $u(q) = \delta_{q_0}(q)$, A_u is the multiplication operator $E_{\eta;\gamma}(\alpha - q_0)$. Finally note the translation covariance mod 2π issued from Eq.(2.58),

$$U(\mathbf{0},\theta)A_f U(\mathbf{0},\theta)^\dagger = A^{\sigma_g}_{f(\cdot-\theta)}\,, \tag{2.66}$$

which conveys to the quantum description the transition map from one chart to another one on the circle.

An elementary example: the Fourier exponential

The operator A_{e_n} associated with the Fourier exponential $\mathit{e}_n(\alpha) = e^{\mathrm{i}n\alpha}$, $n \in \mathbb{Z}$, is given by (2.64). The convolution $E_{\eta;\gamma} * \mathit{e}_n$ takes the form

$$(E_{\eta;\gamma} * \mathit{e}_n)(\alpha) = \int_{\alpha-\gamma}^{\alpha+\pi-\gamma} \mathrm{d}q\, E_{\eta;\gamma}(\alpha - q)\, e^{\mathrm{i}nq}\,. \tag{2.67}$$

The change of variables $q \to \alpha - q$ yields the multiplication operator

$$(E_{\eta;\gamma} * \mathit{e}_n)(\alpha) = \left(\int_{\gamma-\pi}^{\gamma} \mathrm{d}q\, E_{\eta;\gamma}(q)\, e^{-\mathrm{i}nq}\right) e^{\mathrm{i}n\alpha} = 2\pi c_n\left(E_{\eta;\gamma}\right) e^{\mathrm{i}n\alpha}\,, \tag{2.68}$$

where $c_n\left(E_{\eta;\gamma}\right)$ is the nth Fourier coefficient of $E_{\eta;\gamma}$. Thus the quantum versions of simple trigonometric functions, such as $\sin\alpha$ and $\cos\alpha$, are multiplication operators defined by these classical functions, as is the case with many other approaches [12], up to the presence of a multiplicative constant. For a given $n \in \mathbb{Z}$, and with an appropriate choice of η, this constant can be put equal to 1.

2.2.3 Quantisation of the coordinate p

For the momentum $f(p,q) = p$, the expression (2.59) becomes

$$\begin{aligned}\left(A_p\psi\right)(\alpha) = \frac{-\mathrm{i}\pi}{\kappa c_\eta}\int_{\mathbb{S}^1}\mathrm{d}q\,\eta(\alpha-q)\int_{\mathbb{S}^1}\mathrm{d}\alpha'\,\overline{\eta(\alpha'-q)}\times \\ \times\, e^{2\mathrm{i}\lambda S_\zeta(\alpha,\alpha',q)}\psi\left(\alpha'\right)\frac{\partial\delta\left(2\kappa S_\gamma\left(\alpha,\alpha',q\right)\right)}{\partial S_\gamma\left(\alpha,\alpha',q\right)}\,.\end{aligned} \tag{2.69}$$

Taking into account the support of η from Theorem (2.1.2), the above integral reduces to

$$\begin{aligned}\left(A_p\psi\right)(\alpha) = \frac{-\mathrm{i}\pi}{\kappa^2 c_\eta}\int_{\mathbb{S}^1}\mathrm{d}q\,\frac{\eta(\alpha-q)}{\sin\left(q+\gamma-\alpha\right)}\int_{\mathbb{S}^1}\mathrm{d}\alpha'\,\overline{\eta(\alpha'-q)}e^{2\mathrm{i}\lambda S_\zeta(\alpha,\alpha',q)} \\ \times\left(\frac{\partial S_\gamma\left(\alpha,\alpha',q\right)}{\partial\alpha'}\right)^{-1}\psi\left(\alpha'\right)\frac{\partial}{\partial\alpha'}\delta\left(2\kappa S_\gamma\left(\alpha,\alpha',q\right)\right)\,.\end{aligned} \tag{2.70}$$

Integrating (2.70) with respect to α' while taking into account $\operatorname{supp}\eta$, we then perform the change of variables $q' = \alpha - q$, in order to obtain

$$\begin{aligned}\left(A_p\psi\right)(\alpha) = -\mathrm{i}\frac{c_2(\eta,\gamma)}{\kappa c_1(\eta,\gamma)}\partial_\alpha\psi\left(\alpha\right) - \mathrm{i}\frac{1}{\kappa c_1(\eta,\gamma)}\left(\int_{\mathbb{S}^1}\mathrm{d}q\,\cos\left(\gamma-q\right)f_{0;3}(q)\right. \\ \left. + \int_{\mathbb{S}^1}\mathrm{d}q\,f_{1;2}(q) - \mathrm{i}\lambda\int_{\mathbb{S}^1}\mathrm{d}q\,\sin\left(\zeta-q\right)f_{0;2}(q)\right)\psi\left(\alpha\right)\,,\end{aligned} \tag{2.71}$$

where the functions $f_{j;m}(q)$ are defined in (2.54). The quantity

$$\int_{\mathbb{S}^1}\mathrm{d}q\,\cos\left(\gamma-q\right)f_{0;3}(q) + \int_{\mathbb{S}^1}\mathrm{d}q\,f_{1;2}(q)$$

is purely imaginary, as can be seen from the definition of the functions $f_{j;m}(q)$. Integrating by parts, it can be shown that it is equal to minus its conjugate, and so it vanishes in the case of real η. Therefore, for real η, the expression (2.71) takes the simplified form

$$\left(A_p\psi\right)(\alpha) = \left(-\mathrm{i}\frac{c_2(\eta,\gamma)}{\kappa c_1(\eta,\gamma)}\frac{\partial}{\partial\alpha} - \lambda a\right)\psi\left(\alpha\right)\,, \tag{2.72}$$

where the constant a is

$$a = \frac{1}{\kappa c_1(\eta,\gamma)} \int_{\mathbb{S}^1} \mathrm{d}q \, \sin(\zeta - q) f_{0;2}(q) \, . \tag{2.73}$$

We note that, given η, one can choose the parameter κ such that

$$\kappa = \frac{c_2(\eta,\gamma)}{c_1(\eta,\gamma)} \tag{2.74}$$

in order to get, up to the addition of an irrelevant constant, the familiar self-adjoint angular momentum operator $-\mathrm{i}\partial/\partial\alpha$, with spectrum $n \in \mathbb{Z}$ and Fourier exponentials $e^{\mathrm{i}n\alpha}$ as corresponding eigenfunctions.

It is also interesting to note the role played by the parameter λ. It introduces a kind of gauge freedom, and since it is a free parameter, it can be chosen to be 0.

2.2.4 Quantisation of separable functions $f(q,p) = u(q)p^n$

Many physically relevant Hamiltonians for one-dimensional systems, if not all of them, can be written in the form $H = u_2(q)p^2 + u_1(q)p + u_0(q)$. Thus, it is useful to find the corresponding operator A_H under the integral quantisation map (2.55).

We start by examining the quadratic function $f(q,p) = p^2$, which is the classical kinetic term up to a multiplicative constant. In this case, one obtains the more elaborate expression

$$\begin{aligned}(A_{p^2}\psi)(\alpha) = -\left[\int_{\mathbb{S}^1} \mathrm{d}q B_1(q)\right] \partial_\alpha^2 \psi(\alpha) + \mathrm{i}\lambda \left[\int_{\mathbb{S}^1} \mathrm{d}q B_2(q)\right] \partial_\alpha \psi(\alpha) \\ + \left[\int_{\mathbb{S}^1} \mathrm{d}q B_3(q) + \lambda^2 \int_{\mathbb{S}^1} \mathrm{d}q B_4(q)\right] \psi(\alpha) \, ,\end{aligned} \tag{2.75}$$

where the periodic functions $B_j(q)$ are expressed in terms of the $f_{j;m}(q)$'s in the Appendix A.

For the simple separable functions $pu(q)$ and $p^2u(q)$ one obtains

$$(A_{pu}\psi)(\alpha) = \left[-\mathrm{i}(u * B_5)(\alpha)\partial_\alpha - \frac{\mathrm{i}}{2}\partial_\alpha(u * B_5)(\alpha) - \lambda(u * B_6)(\alpha)\right] \psi(\alpha) \, , \tag{2.76}$$

and

$$\left(A_{p^2u(q)}\psi\right)(\alpha) = \left\{-\left(u * B_1\right)(\alpha)\partial_\alpha^2 + \left[-\partial_\alpha\left(u * B_1\right)(\alpha) + 2\mathrm{i}\lambda\left(u * B_2\right)(\alpha)\right]\partial_\alpha\right\}\psi(\alpha) \\ + \left[-\left(u * B_3\right)(\alpha) + \lambda^2\left(u * B_4\right)(\alpha) + \mathrm{i}\lambda\partial_\alpha\left(u * B_2\right)(\alpha)\right]\psi(\alpha)\,, \tag{2.77}$$

where B_5 and B_6 are given in Appendix A.

In the above formulae, the systematic appearance of multiplicative constants and of extra additive terms is a by-product of the quantisation method. They are functions of the various free parameters κ, γ, λ, ζ, and parameters of the fiducial vector η. All these parameters can be adjusted to comply with experiments or observations determining the constants appearing in physical quantities.

One can easily guess that the quantisation of the general polynomial

$$f(q,p) = \sum_{k=0}^{N} u_k(q)\,p^k \tag{2.78}$$

would yield an operator of the form

$$\sum_{k=0}^{N} a_k(\alpha)(-\mathrm{i}\partial_\alpha)^k\,. \tag{2.79}$$

2.3 Computation of Lower Symbols

The semi-classical phase space portrait provided by the covariant [25] or lower [61] symbol $\check{f}(q,p)$ of the operator A_f completes the quantization map $f \mapsto A_f$. The lower symbol $\check{f}(q,p)$ is defined as the CS expectation value of A_f,

$$\check{f}(p,q) = \langle\, \eta_{p,q}\,|A_f|\,\eta_{p,q}\,\rangle = \int_{\mathbb{R}\times\mathbb{S}^1} \frac{\mathrm{d}p'\mathrm{d}q'}{c_\eta} f\left(p',q'\right)\left|\langle\eta_{p',q'}|\eta_{p,q}\rangle\right|^2\,. \tag{2.80}$$

It is the local average value of the original $f(q',p')$ with respect to the probability distribution $(q',p') \mapsto |\langle q,p|q',p'\rangle|^2$, i.e., the modulus squared of the overlap between two CS, on the phase space equipped with the measure $\dfrac{\mathrm{d}p'\mathrm{d}q'}{c_\eta}$.

From (2.64), one immediately obtains the lower symbol for $f(p,q) = u(q)$ as

$$\check{u}(q) = \left[|\widetilde{\eta}|^2 * \left(E_{\eta;\gamma} * u\right)\right](q)\,, \tag{2.81}$$

where $\widetilde{\eta}(\alpha) = \eta(-\alpha)$. As for the multiplication operator (2.64) this convolution is expected to regularise the original $u(q)$.

Likewise, the lower symbol for the momentum $f(q,p) = p$ is given by the integral

$$\begin{aligned} \check{p} = \frac{-\mathrm{i}\pi}{\kappa c_\eta} \int_{\mathbb{S}^1} \mathrm{d}q' \int_{\mathbb{S}^1} \mathrm{d}\alpha \, \overline{\eta(\alpha - q')} \eta(\alpha - q) \int_{\mathbb{S}^1} \mathrm{d}\alpha' \, \eta(\alpha' - q') \overline{\eta(\alpha' - q)} \\ \times e^{\mathrm{i}2\kappa S_\gamma(\alpha',\alpha,q)p} e^{\mathrm{i}2\lambda S_\zeta(\alpha,\alpha',q')} e^{\mathrm{i}2\lambda S_\zeta(\alpha',\alpha,q)} \frac{\partial}{\partial S_\gamma(\alpha',\alpha,q')} \delta\left(2\kappa S_\gamma(\alpha,\alpha',q')\right) . \end{aligned} \tag{2.82}$$

Assuming that η is a real function, one arrives at

$$\check{p} = \frac{c_2(\eta,\gamma)}{c_1(\eta,\gamma)} c_{-1}(\eta,\gamma)\, p + \lambda \frac{c_2(\eta,\gamma)}{\kappa c_1(\eta,\gamma)} c_{-1}(\eta,\zeta) - \lambda a \, . \tag{2.83}$$

where the constant a is given by (2.73).

With a Dirac delta-like fiducial vector η on the circle, it is expected that within the framework of semi-classical analysis that $\check{f}$ approaches f as η becomes more localized. As a matter of fact, this a confirmed, in the case of a specific family of fiducial vectors, by a numerical study of the lower symbol for the angle operator in subsection 2.4.2.

2.4 The Angle Operator

We now study the quantisation of the 2π-periodic and discontinuous angle function $a(\alpha)$, defined by $a(\alpha) = \alpha$ for $\alpha \in [0, 2\pi)$.

2.4.1 The angle operator: general properties

Let us introduce the following continuous function of $\alpha \in \mathbb{R}$,

$$G_{\eta;\gamma}(\alpha) = \int_{-\pi}^{\alpha} E_{\eta;\gamma}(q)\, \mathrm{d}q \, , \tag{2.84}$$

where $E_{\eta;\gamma}(q)$ is now viewed as a periodic function of $q \in \mathbb{R}$. The function (2.84) satisfies $G_{\eta;\gamma}(\alpha + 2\pi) = G_{\eta;\gamma}(\alpha) + 1$ and $G_{\eta;\gamma}(\alpha) = 1$ if $\alpha \in [\gamma, \gamma + \pi]$. Now consider the function

$$F_{\eta;\gamma}(\alpha) = 2\pi(1 - G_{\eta;\gamma}(\alpha)) \, . \tag{2.85}$$

The convolution $E_{\eta;\gamma} * a$ can now be expressed in terms of $F_{\eta;\gamma}$ as

$$(E_{\eta,\gamma} * a)(\alpha) = \alpha + F_{\eta;\gamma}(\alpha) - \int_{\gamma-\pi}^{\gamma} q\, E_{\eta,\gamma}(q)\, \mathrm{d}q\,. \tag{2.86}$$

Note that this is a continuous periodic function of α, since $\alpha + F_{\eta;\gamma}(\alpha)$ is periodic. The last term on the right-hand side of (2.86) is the mean value of the angle with respect to the probability distribution $E_{\eta,\gamma}(q)$, as defined in (2.63).

2.4.2 Angle operator: analytic and numerical results

A specific section σ is now used (which implies a choice of $\boldsymbol{\kappa}, \boldsymbol{\lambda} \in \mathbb{R}^2$). To simplify, we put $\lambda = 0$, $\gamma = \pi/2$, so $\operatorname{supp}\eta \subset (-\pi/2, \pi/2) \bmod 2\pi$. In order to study the relation between the localization of η and the spectrum of A_a, we pick the following familiar smooth and compactly supported test functions for distributions, namely,

$$\omega_s(x) = \begin{cases} \exp\left(-\dfrac{s}{1-x^2}\right) & 0 \leq |x| < 1\,, \\ 0 & |x| \geq 1\,, \end{cases} \tag{2.87}$$

where the parameter $s > 0$ determines the rate of decrease of ω_s. We also note that $0 \leq \omega_s(x) \leq e^{-s}$. Now we choose as fiducial vectors the family of 2π-periodic smooth even functions which have support $[-\epsilon, \epsilon] \subset (-\pi/2, \pi/2)$ mod 2π, and which are parametrized by $s > 0$ and $0 < \epsilon < \pi/2$,

$$\eta(\alpha) \equiv \eta^{(s,\epsilon)}(\alpha) = \frac{1}{\sqrt{\epsilon e_{2s}}}\, \omega_s\left(\frac{\alpha}{\epsilon}\right) \quad \text{where} \quad e_s := \int_{-1}^{1} \mathrm{d}x\, \omega_s(x)\,. \tag{2.88}$$

Note that we can enlarge at wish the set of free parameters in (2.88), besides ϵ and s, for instance by multiplying our initial choice η by a polynomial in $\sin(\gamma_n - \alpha)$ with arbitrary degree.

As a function of $s \in (0, +\infty)$, e_s decreases monotonically from 2 to 0. Defining

$$e_s(\epsilon, \nu) := \int_{-1}^{1} .\mathrm{d}x\, \frac{\omega_s(x)}{(\cos \epsilon x)^\nu}\,, \quad 0 < \epsilon < \frac{\pi}{2}\,, \tag{2.89}$$

the integrals $c_\nu\left(\eta, \frac{\pi}{2}\right)$ defined by (2.53) assume the simple form

$$c_\nu\left(\eta^{(s,\epsilon)}, \frac{\pi}{2}\right) = \frac{e_{2s}(\epsilon, \nu)}{e_{2s}}\,. \tag{2.90}$$

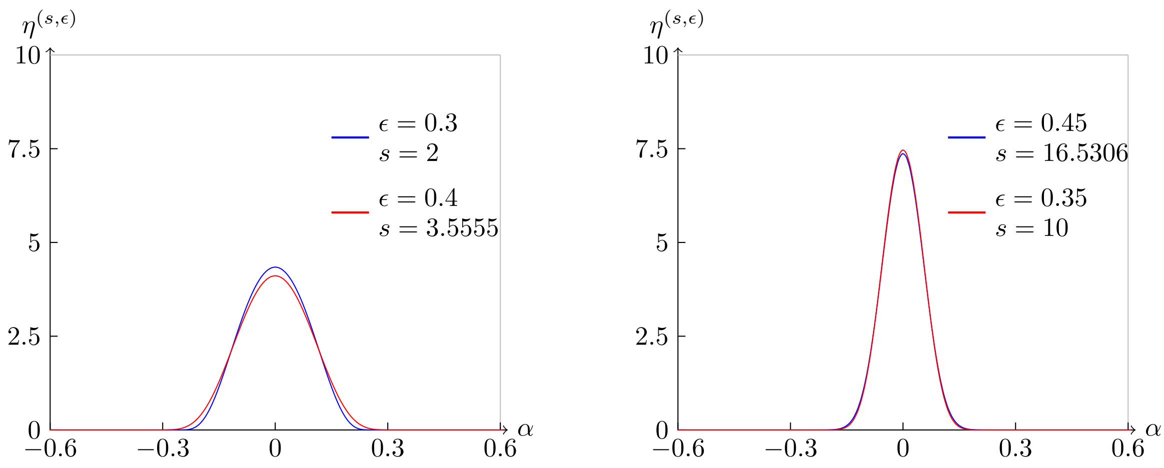
Graphs of the functions $\eta^{(s,\epsilon)}(\alpha)$ for a few values of the parameters ϵ and s are shown in Figures 2.1a and 2.1b. They give an idea of their localization properties. As a matter of fact, the family of the squares of these functions form a Dirac delta sequence with respect to each parameter,

$$\left(\eta^{(s,\epsilon)}\right)^2(\alpha) \to \delta(\alpha) \quad \text{as} \quad \epsilon \to 0 \quad \text{or} \quad \text{as} \quad s \to \infty. \tag{2.91}$$

With the above notations,

$$E_{\eta;\frac{\pi}{2}}(\alpha) = E_{\eta^{(s,\epsilon)};\frac{\pi}{2}}(\alpha) = \frac{1}{\epsilon e_{2s}(\epsilon,1)} \frac{\omega_{2s}\left(\frac{\alpha}{\epsilon}\right)}{\cos\alpha}\,. \tag{2.92}$$

In Figure 2.1 the parameters s and ϵ control the localization about $\alpha = 0 \bmod 2\pi$ of the



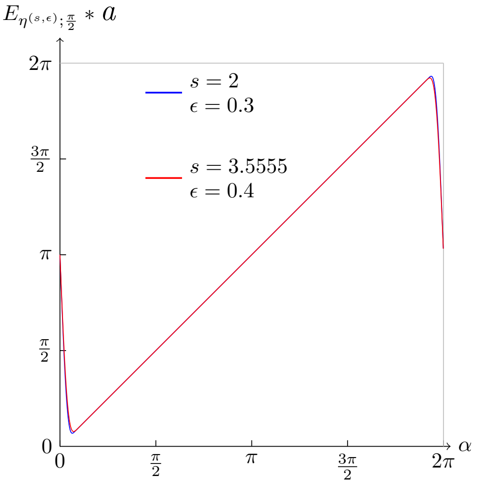
(a) $\tau = 22.22$ (b) $\tau = 81.63$

Figure 2.1: Plots of $\eta^{(s,\epsilon)}$ for various values of $\tau = \dfrac{s}{\epsilon^2}$.

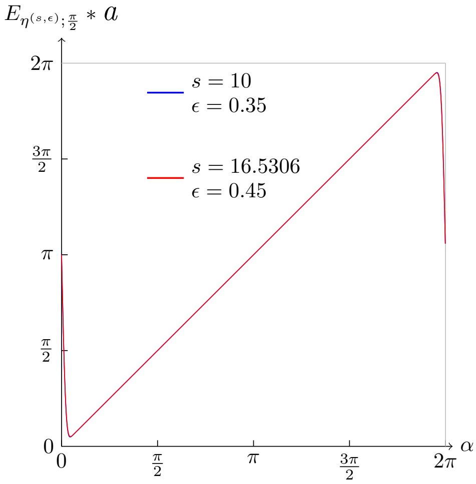
fiducial function $\eta^{(s,\epsilon)}(\alpha)$. For large s, the fiducial vector depends essentially on a single parameter, namely, the combination $\tau = s/\epsilon^2$, as is depicted in the plots.

In Figure 2.2b, we plot $\left(E_{\eta^{(s,\epsilon)};\frac{\pi}{2}} * \mathcal{a}\right)(\alpha)$ to note that it coincides with the angle function $\mathcal{a}$ inside $[\epsilon, 2\pi - \epsilon]$, while outside that interval the function $F_{\eta^{(s,\epsilon)};\frac{\pi}{2}}(\alpha)$ regularizes $\mathcal{a}$.

Since $E_{\eta^{(s,\epsilon)};\frac{\pi}{2}}$ is defined in terms of the Dirac delta sequence $\epsilon^{-1}\left[\eta^{(s,\epsilon)}\right]^2$ (at fixed s), we get $E_{\eta^{(s,\epsilon)};\frac{\pi}{2}} * \mathcal{a} \to \mathcal{a}$ as $\epsilon \to 0$. Likewise, in the limit $s \to \infty$, $E_{\eta^{(s,\epsilon)};\frac{\pi}{2}}$ behaves as a



(a) $\tau = 22.22$



(b) $\tau = 81.63$

Figure 2.2: Plots of $\left(E_{\eta^{(s,\epsilon)};\frac{\pi}{2}} * a\right)(\alpha)$ for various values of $\tau = \dfrac{s}{\epsilon^2}$.

delta sequence. We notice from the figure that the spectrum $\sigma\left(A_a\right)$ of the angle operator is continuous, as expected from the smoothness and non-stationarity of the convolution, and is, for a given ϵ and s, a closed interval strictly included in the interval $[0, 2\pi]$, i.e.,

$$\sigma\left(A_a\right) = [\pi - m(s,\epsilon), \pi + m(s,\epsilon)]\,, \quad 0 < m(s,\epsilon) < \pi\,, \tag{2.93}$$

with $m(s,\epsilon) \to \pi$ as $\epsilon \to 0$ or $s \to \infty$, i.e., the spectrum goes to $[0, 2\pi)$. For a fixed value of s, the real number $\pi - m(s,\epsilon)$ corresponds to the positive root $\alpha \in (0,\epsilon)$ of the equation

$$\omega_{2s}\left(\frac{\alpha}{\epsilon}\right) = \frac{\epsilon e_{2s}(\epsilon, 1)}{2\pi} \cos\alpha\,. \tag{2.94}$$

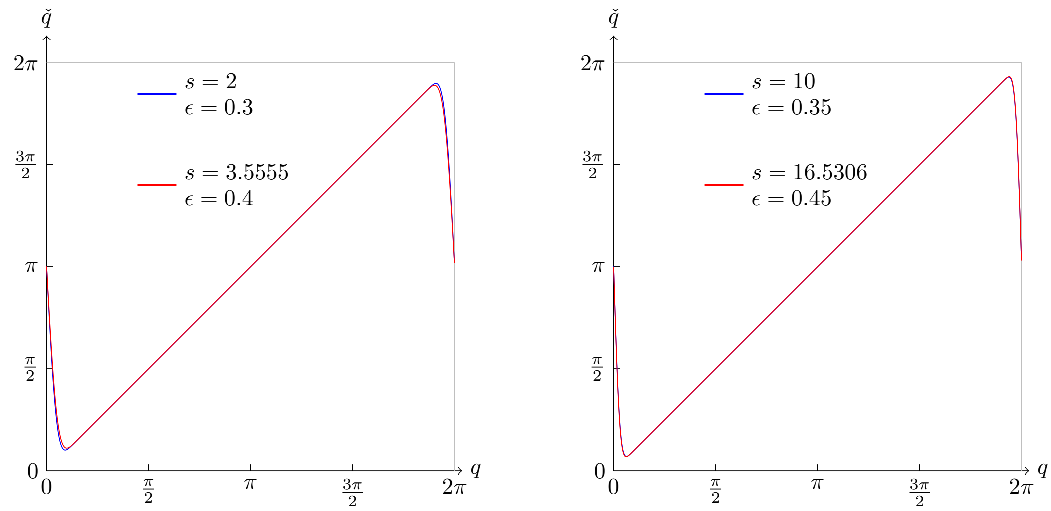
It is interesting to compare the function $\left(E_{\eta^{(s,\epsilon)};\frac{\pi}{2}} * a\right)(\alpha)$ with the lower symbol of A_a, the function $\check{q}$. Considering the function $\widetilde{\eta}^{(\epsilon,\delta)}(\alpha) = \eta^{(\epsilon,\delta)}(-\alpha)$, using the expression (2.81) the semiclassical portrait of A_a can be written as

$$\check{q}(q) = \left[\left(\widetilde{\eta}^{(\epsilon,\delta)}\right)^2 * \left(E_{\eta^{(\epsilon,\delta)};\frac{\pi}{2}} * a\right)\right](q). \tag{2.95}$$

Taking into account the support of $\eta^{(\epsilon,\delta)}$ one has explicitly

$$\check{q} = \begin{cases} \int_{q+2\pi-\delta}^{2\pi} \mathrm{d}\alpha \, \left(\eta^{(\epsilon,\delta)}(\alpha-q-2\pi)\right)^2 \left(E_{\eta^{(\epsilon,\delta)};\frac{\pi}{2}} * \mathsf{a}\right)(\alpha) & \\ \quad + \int_{0}^{q+\delta} \mathrm{d}\alpha \, \left(\eta^{(\epsilon,\delta)}(\alpha-q)\right)^2 \left(E_{\eta^{(\epsilon,\delta)};\frac{\pi}{2}} * \mathsf{a}\right)(\alpha) & 0 \leq q < \delta \\ \\ \int_{q-\delta}^{q+\delta} \mathrm{d}\alpha \, \left(\eta^{(\epsilon,\delta)}(\alpha-q)\right)^2 \left(E_{\eta^{(\epsilon,\delta)};\frac{\pi}{2}} * \mathsf{a}\right)(\alpha) & \delta \leq q < 2\pi-\delta \\ \\ \int_{0}^{q-2\pi+\delta} \mathrm{d}\alpha \, \left(\eta^{(\epsilon,\delta)}(\alpha-q+2\pi)\right)^2 \left(E_{\eta^{(\epsilon,\delta)};\frac{\pi}{2}} * \mathsf{a}\right)(\alpha) & \\ \quad + \int_{q-\delta}^{2\pi} \mathrm{d}\alpha \, \left(\eta^{(\epsilon,\delta)}(\alpha-q)\right)^2 \left(E_{\eta^{(\epsilon,\delta)};\frac{\pi}{2}} * \mathsf{a}\right)(\alpha) & 2\pi-\delta \leq q < 2\pi \end{cases} \tag{2.96}$$

The behaviour of $\check{q}$ is depicted in Figure 2.3 in order to be compared with $E_{\eta^{(\epsilon,\epsilon)};\frac{\pi}{2}} * \mathsf{a}$. One can see that the lower symbol $\check{q}(q)$ has the form of the angle function $\mathsf{a}(q)$, except at the border, where it regularizes the angle function.



(a) $\tau = 22.22$ (b) $\tau = 81.63$

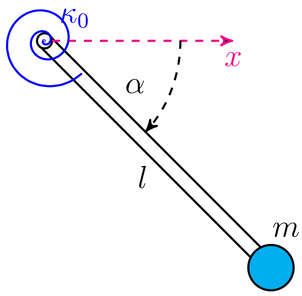
Figure 2.3: Plots of the lower symbol $\check{q}(q)$ of the angle operator A_{a} for various values of $\tau = \dfrac{s}{\epsilon^2}$.

2.4.3 Possible applications of the angle operator

We naturally expect that the spectrum of the angle operator is given by $[0, 2\pi)$. However, as we can see in the expression (2.93), the values near zero (or 2π) are forbidden unless $\tau \to \infty$, except, of course, if we proceed with a change of chart by using the E(2) covariance of our quantisation procedure. For this reason, a suitable application of the angle operator obtained here is the study of a restricted angular motion. The simple pendulum is a good example of that kind of system. The quantum version of that system is already well known since the potential $U(\theta) = \sin\theta$ can be easily quantized. A better example is given by the **torsion spring**.

In [62], the compressibility properties of LnFe(CN)6 were studied with the assumption that the bonds behave like torsion springs. In the most general case, the Hamiltonian of a lattice contains electromagnetic interaction terms between atoms, which leads to very complicated expressions. When we look at the response of a lattice under shear forces applying the ideas in [62], we can ask the question: can some atomic lattices be modeled as systems of coupled torsion springs?.

Lets consider a classical torsion pendulum with moment of inertia $I = mL^2$ and a torsion spring with elastic constant κ_0. In this system the movement is restricted to the interval $\alpha \in (-a, a)$ where $0 \leq a < \pi$.



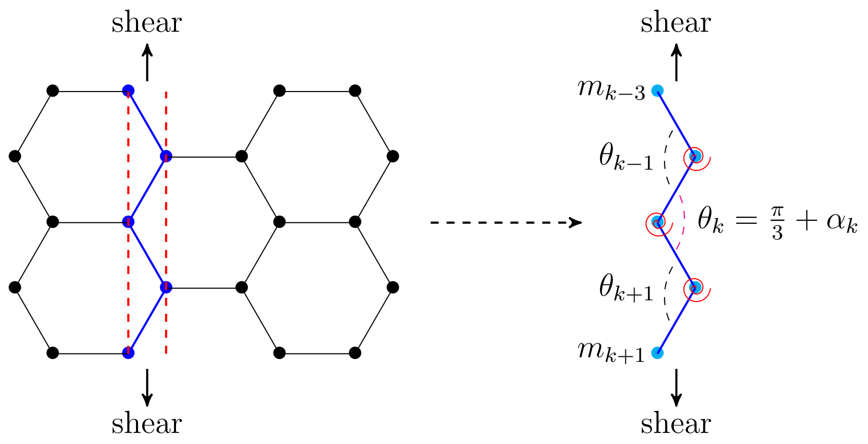
The force associated with the torsion spring is attractive and increases with α. Then the potential V_{Tor} for this system must be given for a convex function $v(\alpha)$ defined inside the interval $\alpha \in (-a, a)$, outside this interval V_{Tor} has to be infinite since the movement is restricted. With those considerations the potential is given by

$$V_{\text{Tor}}(\alpha) = \begin{cases} v(\alpha) & |\alpha| < a \\ \infty & \text{otherwise} \end{cases}, \quad \text{for} \quad [-\pi, \pi)\,. \tag{2.97}$$

With the angular momentum $p_\alpha = ml^2\dot{\alpha}$, the classical Hamiltonian takes the form

$$H(p_\alpha, \alpha) = \frac{1}{2I}p_\alpha^2 + V_{\text{Tor}}(\alpha)\,. \tag{2.98}$$

As an example, we can study the response of carbon nanotubes (which consist of sheets of graphene folded in a cylinder) under shear forces. Let us consider a toy model, where shear is applied to an hexagonal lattice of particles. The lattice can be decomposed into chains of torsion pendulums (the decomposition depends on the orientation of the force in relation to the lattice). This can be seen in the following figure



If we want to study a *quantum model* of the response under shear forces, this type of *effective model* can be a convenient simplification. But of course, various question should be answered first: Can we really use the potential (2.97) for small perturbations?. Since the Schrödinger equation for the potential (2.97) has no exact solution, can we use a potential with an exact solution, like the Poschl-Teller potential? and so on. Ultimately, only experimental evidence can really support those assumptions.

2.5 Angle-Angular Momentum: Commutation and Inequality

2.5.1 Commutation relation

For $\lambda = 0$ and $\psi(\alpha) \in L^2(\mathbb{S}^1, \mathrm{d}\alpha)$, using (2.72) we find the following (non-canonical) commutation rule between the angle operator and the momentum operator,

$$([A_p, A_{\mathfrak{a}}]\psi)(\alpha) = -\mathrm{i}\mathsf{c}\left(\frac{\mathrm{d}}{\mathrm{d}\alpha}(E_{\eta;\gamma} * \mathfrak{a})(\alpha)\right)\psi(\alpha)\,, \quad \mathsf{c} := \frac{c_2(\eta,\gamma)}{\kappa c_1(\eta,\gamma)}. \tag{2.99}$$

Considering Eq. (2.86), we arrive at the following result,

$$([A_p, A_{a}]\psi)(\alpha) = -\mathrm{i}\mathsf{c}\left(1 - 2\pi E_{\eta;\gamma}(\alpha)\right)\psi(\alpha)\,. \tag{2.100}$$

Notwithstanding the constant factor c, which can be made equal to 1, we have obtained a regularisation of the Dirac comb (1.10). The latter is recovered in the limit of the sequence of fiducial vectors. Indeed, since $\lim_{\epsilon\to 0}\frac{c_2(\eta^{(s,\epsilon)},\frac{\pi}{2})}{c_1(\eta^{(s,\epsilon)},\frac{\pi}{2})} = 1$ and $\lim_{\epsilon\to 0} E_{\eta;\gamma}(\alpha) = \delta(\alpha)$, with the choice $\kappa = 1$ the expression (2.100) gives, in the limit $\epsilon \to 0$, for $\alpha \in [0, 2\pi)$ mod 2π,

$$([A_p, A_{a}]\psi)(\alpha) = \left(-\mathrm{i} + \mathrm{i}2\pi\delta(\alpha)\right)\psi(\alpha)\,. \tag{2.101}$$

2.5.2 Heisenberg inequality

Let us now consider the Heisenberg inequality concerning the angle and angular momentum operators,

$$\Delta A_p\,\Delta A_{a} \geqslant \frac{1}{2}\left|\langle\,\phi\,|[A_p, A_{a}]|\,\phi\,\rangle\right|\,, \tag{2.102}$$

where $\phi \in L^2(\mathbb{S}^1, \mathrm{d}\alpha)$ and $(\Delta A)^2 = \langle\phi|A^2|\phi\rangle - \langle\phi|A|\phi\rangle^2$.

With coherent states

As discussed in the introduction, one of the main issues regarding the definition of an acceptable angle operator concerns the quantum angular dispersion versus the quantum angular momentum one. The Heisenberg inequality or uncertainty relation for the operators A_p and A_{a}, when computed with the coherent states $\eta_{p,q}$, is given by

$$\Delta A_p\,\Delta A_{a} \geqslant \frac{1}{2}\left|\langle\,\eta_{p,q}\,|[A_p, A_{a}]|\,\eta_{p,q}\,\rangle\right|\,. \tag{2.103}$$

Before calculating directly the product of dispersions on the left-hand side of (2.103), let us study in more detail the right-hand side of this inequality as a function of phase space variables and underlying constants. We recall that the factor $\mathsf{c} = \dfrac{c_2(\eta,\gamma)}{\kappa c_1(\eta,\gamma)}$ can be put equal to 1 following the remarks after (2.74). Thus, we only have to study the relative smallness of the mean value of the multiplication operator given by (2.100). With the choice (2.88), for $\widetilde{\eta}^{(s,\epsilon)}(\alpha) = \eta^{(s,\epsilon)}(-\alpha)$ the right-hand side of (2.103) reads

$$\frac{1}{2}\left|\langle\, \eta_{p,q}\,|[A_p, A_{a}]|\,\eta_{p,q}\,\rangle\right| = \frac{1}{2}\mathsf{c}\left|1 - 2\pi\,\left(\left(\widetilde{\eta}^{(s,\epsilon)}\right)^2 * E_{\eta^{(s,\epsilon)};\frac{\pi}{2}}\right)(q)\right|\,.$$

Let us now compute the left-hand side of (2.103) by using

$$\left[\Delta A_f(p,q)\right]^2 = \left\langle \eta^{(s,\epsilon)}_{p,q}\right| A_f^2 \left|\eta^{(s,\epsilon)}_{p,q}\right\rangle - \left(\left\langle \eta^{(s,\epsilon)}_{p,q}\right| A_f \left|\eta^{(s,\epsilon)}_{p,q}\right\rangle\right)^2\,. \tag{2.104}$$

With the particular case considered in Section 2.4.2, the value of $\left(\Delta A_{a}\right)^2$ is given by

$$\left[\Delta A_{a}(q)\right]^2 = \left[\left(\widetilde{\eta}^{(s,\epsilon)}\right)^2 * \left(E_{\eta^{(s,\epsilon)};\frac{\pi}{2}} * a\right)^2\right](q) - \left\{\left[\left(\widetilde{\eta}^{(s,\epsilon)}\right)^2 * \left(E_{\eta^{(s,\epsilon)};\frac{\pi}{2}} * a\right)\right](q)\right\}^2\,, \tag{2.105}$$

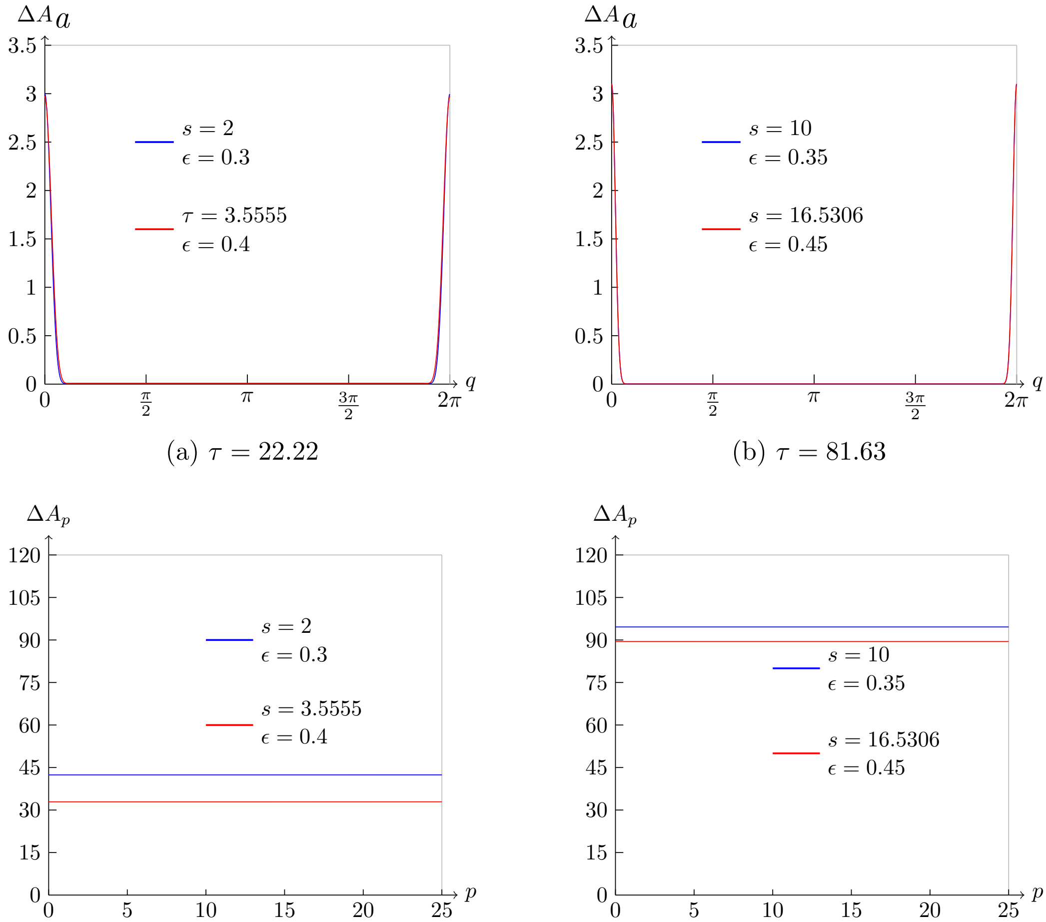
In Figures 2.4a and 2.4b, ΔA_{a} is approximately vanishing, except at the border, where the uncertainty increases sharply due to the poor behavior of the multiplication operator at $q = 0, 2\pi$. However, because of rotational invariance the coherent state is well-localized for all values of q, so the abrupt change in uncertainty at the border is an artifact of the choice of interval.

The value of $\left[\Delta A_p(p)\right]^2$ is given by

$$\begin{aligned}\left[\Delta A_p(p)\right]^2 = \mathsf{c}^2\frac{s}{\epsilon^2}\frac{1}{e_{2s}}\int_{-1}^{1}\mathrm{d}x\,\omega_{2s}(x)\left(\frac{2\,(1+3x^2)}{(1-x^2)^3} - s\frac{4x^2}{(1-x^2)^4}\right)\\ + \left(c_{-2}\left(\eta^{(s,\epsilon)},\frac{\pi}{2}\right) - c_{-1}\left(\eta^{(s,\epsilon)},\frac{\pi}{2}\right)^2\right)(\kappa\mathsf{c})^2\,p^2\,.\end{aligned} \tag{2.106}$$

As is seen in Figures 2.4c and 2.4d, for large values of τ, the dependence on p in the expression (2.106) can be neglected, therefore ΔA_p does not depend on p or q.

In Figures 2.5a and 2.5b we show that the state $|\,\eta^{(s,\epsilon)}_{p,q}\,\rangle$ saturates the uncertainty relation (2.103) for large values of $\tau = s/\epsilon^2$. In Figures 2.5c and 2.5d, we examine the behavior of the uncertainty relation with respect to τ for fixed q. We note that these plots are insensitive to changes of the value of q which do not lie near the border, that is, near $q = 0$ or $q = 2\pi$. The exponential decay of the left-hand side minus the right-hand side of



(a) $\tau = 22.22$ (b) $\tau = 81.63$

(c) $\tau = 22.22$ (d) $\tau = 81.63$

Figure 2.4: Plots of the dispersions ΔA_{a} and ΔA_p with respect to the coherent state $|\,\eta_{p,q}^{(s,\epsilon)}\,\rangle$ for various values of $\tau = \dfrac{s}{\epsilon^2}$.

(2.103) with respect to τ and the saturation of the uncertainty relation seen in first two plots follow from the Gaussian form of the coherent state $\eta^{(s,\epsilon)}$ for large values of s.

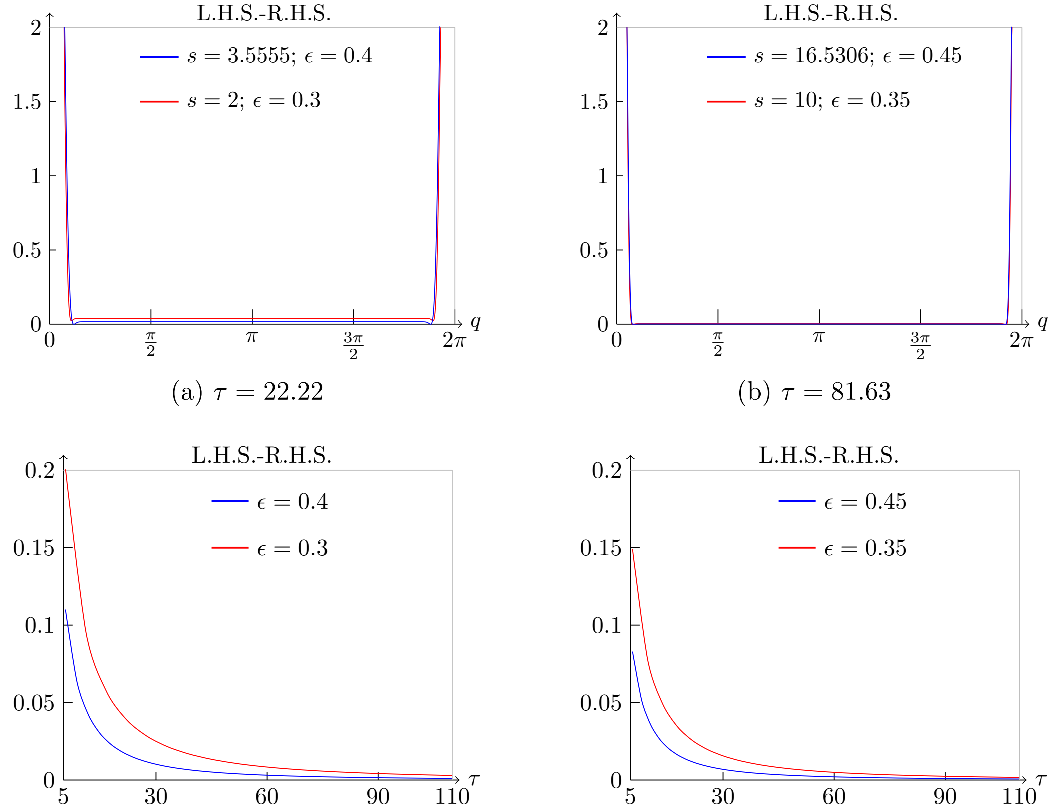
With Fourier exponentials as eigenfunctions of A_p

It is interesting to compare the above inequalities computed with coherent states with those calculated from the eigenstates $\varphi_m(\alpha) = \dfrac{e^{im\alpha}}{\sqrt{2\pi}}$ of A_p, $m \in \mathbb{Z}$,

$$\left(A_p^n \varphi_m\right)(\alpha) = \mathsf{c}^n m^n \varphi(\alpha)\,, \tag{2.107}$$

for which obviously $(\Delta A_p)^2 = 0$. The action of A_{a} on φ_m is

$$\left(A_{a}^n \varphi_m\right)(\alpha) = \left[\left(E_{\eta;\gamma} * a\right)(\alpha)\right]^n \varphi(\alpha)\,. \tag{2.108}$$



(a) $\tau = 22.22$

(b) $\tau = 81.63$

(c) $q = 1$

(d) $q = 1$

Figure 2.5: Plots of the difference L.H.S.-R.H.S. beetwen the left-hand side and right-hand side of the uncertainty relation with respect to the coherent state $|\eta_{p,q}^{(s,\epsilon)}\rangle$ for various values of $\tau = \frac{s}{\epsilon^2}$.

The expectation value $\langle\varphi_m| A^n_{\mathsf{a}} |\varphi_m\rangle$ is

$$\langle\varphi_m| A^n_{\mathsf{a}} |\varphi_m\rangle = \int_0^{2\pi} \mathrm{d}\alpha\, \overline{\varphi_m(\alpha)} \left(A^n_{\mathsf{a}}\varphi_m\right)(\alpha) = \frac{1}{2\pi}\int_0^{2\pi} \mathrm{d}\alpha\, \left[\left(E_{\eta;\gamma} * \mathsf{a}\right)(\alpha)\right]^n . \quad (2.109)$$

For $\epsilon = 0.3$, we calculate ΔA_{a} for different values of s:

s	$\langle A_{\mathsf{a}}\rangle$	$\left\langle A^2_{\mathsf{a}}\right\rangle$	ΔA_{a}
1	3.1404	12.8007	1.7143
2	3.1404	12.8662	1.7333
10	3.1403	13.0016	1.7720
100	3.1402	13.1040	1.8009

The above values are in agreement with the dispersion from (1.9), where $\Delta\hat{\theta} = \pi/\sqrt{3} = 1.8138$. Of course there is no contradiction with the inequality (2.102), since the average value of the commutator in the normalised Fourier exponentials is also vanishing.

2.6 Comparison with another "integral quantization"

It is not easy to compare the coherent states for the motion on the circle that were presented in the work[30] with the ones being presented here, since they are of a different nature (a detailed description can be found in the Appendix B). Those states are a generalisation of coherent states introduced in [63, 64, 65, 66, 67, 68, 69, 70]. The normalised states of Reference [30] are defined by the expansion

$$|p,q\rangle = \frac{1}{\sqrt{\mathcal{N}^{\sigma}(p)}}\sum_{n\in\mathbb{Z}}\sqrt{w_n^{\sigma}(p)}\;\mathrm{e}^{-\mathrm{i}nq}\,|e_n\rangle\,,\quad p\in\mathbb{R}\,,0\leq q<2\pi\,, \tag{2.110}$$

where the $|e_n\rangle$'s form an orthonormal basis of a Hilbert space $\mathcal{H}$, for instance the Fourier exponentials in $L^2(\mathbb{S}^1,\mathrm{d}\alpha)$, and $0<\mathcal{N}^{\sigma}(p)\overset{\text{def}}{=}\sum_{n\in\mathbb{Z}}w_n^{\sigma}(p)$ is the normalisation factor. The construction of states (2.110) rests upon a probability distribution $w^{\sigma}(p)$ on the range of the variable p. It is a non-negative, <u>even</u>, well-localized and normalized integrable function which is subject to certain conditions, an essential one being

$$w_n^{\sigma}(p) = w_0^{\sigma}(p-n)\,. \tag{2.111}$$

States (2.110) resolve the identity with respect to the measure $\mathcal{N}^{\sigma}(p)\,\frac{\mathrm{d}p\,\mathrm{d}q}{2\pi}$, and thus also allow a CS quantisation. Let us now compare states (2.110) with the CS introduced in the present work and given by (2.40). Their Fourier expansion reads

$$\eta_{p,q}(q) = \sum_{n\in\mathbb{Z}} c_n\left(\eta_{p,q}\right)e^{\mathrm{i}n\alpha}\,, \tag{2.112}$$

with coefficients given by

$$c_n\left(\eta_{p,q}\right) = \frac{1}{2\pi}\int_{\mathbb{S}^1}\mathrm{d}\alpha\,e^{-\mathrm{i}n\alpha}\eta_{p,q}(\alpha) = \frac{1}{2\pi}\int_{\mathbb{S}^1}\mathrm{d}\alpha\,e^{-\mathrm{i}n\alpha}e^{\mathrm{i}[\mathcal{R}(q-\alpha)(\kappa p+\lambda)]_1}\eta(\alpha-q)\,. \tag{2.113}$$

The change of variable $\alpha\to\alpha-q$ in (2.113) gives

$$c_n\left(\eta_{p,q}\right) = e^{-\mathrm{i}nq}\frac{1}{2\pi}\int_{\mathbb{S}^1}\mathrm{d}\alpha\,e^{-\mathrm{i}n\alpha+\mathrm{i}\kappa p\cos(\alpha-\gamma)+\mathrm{i}\lambda\cos(\alpha-\zeta)}\eta(\alpha)\,. \tag{2.114}$$

Comparing the Fourier coefficients (2.110) with the ones given in (2.114) yields the relation

$$\sqrt{w_n^{\sigma}(p)} = \frac{\sqrt{\mathcal{N}^{\sigma}(p)}}{2\pi}\int_{\mathbb{S}^1}\mathrm{d}\alpha\,e^{-\mathrm{i}n\alpha+\mathrm{i}\kappa p\cos(\alpha-\gamma)+\mathrm{i}\lambda\cos(\alpha-\zeta)}\eta(\alpha)\,. \tag{2.115}$$

Besides the positiveness condition imposed to the integral above, we immediately notice that (2.115) fails to fulfill the essential condition (2.111). Hence, it is not possible to make a direct connection between [30] and our present work.

Chapter 3

Quantum error probability with nonlinear CS

In Section 3.1 we study specific cases of the superposition of photon number states (1.6) called nonlinear CS. The Subsection 3.1.1 shows the relation between the Mandel parameter and the photon statistics for nonlinear CS.

A summary about quantum error probability is presented in Section 3.2.

Section 3.3 is devoted to the study of the non-linear CS generated by deformations of the binomial distribution. In Subsections 3.3.1 and 3.3.2, we review the main results in [55] and [56] about the asymmetric and symmetric deformations of the binomial distribution. The photon distribution for nonlinear CS associated with those deformations is analyzed using the Mandel parameter in Subsection 3.3.3.

In Section 3.4 we study the Helstrom bound for nonlinear CS. We examine the case of perfect detection in Subsection 3.4.1 and imperfect detection in Subsection 3.4.2. We summarize the asymmetric deformation for the photocounting distribution, and develop a symmetric deformation.

In Section 3.5 we analyze questions related to the optimization of the Helstrom bound for nonlinear CS in comparison with linear CS. We study an example of CS in generated by deformations of the binomial distribution in Subsection 3.5.3. In Subsection 3.5.4, we also give an example of another type of nonlinear CS, the Susskind-Glogower CS.

3.1 Nonlinear CS

The first generalization of CS that we will study, is a deformation of the expression (1.1). The nonlinear CS are defined as the family of states $|\alpha;\chi\rangle$ for a one-mode electromagnetic quantum field in the corresponding Fock space, with the following analytical expression

$$|\alpha;\chi\rangle := \sum_{n=0}^{\infty} \frac{1}{\sqrt{\mathcal{N}\left(|\alpha|^2\right)}} \frac{\alpha^n}{\sqrt{x_n!}} |n\rangle \ . \tag{3.1}$$

The state $|n\rangle$ is an eigenvector of the number operator $N = a^\dagger a$. The generalized *factorial* $x_n!$ is defined as

$$x_n! = x_1 x_2 \cdots x_n, \quad x_0! := 1\,, \tag{3.2}$$

where the values x_n for $n \in \mathbb{N}$ (with $x_0 := 0$), form a sequence of positive numbers,

$$\chi := \{x_0 = 0, x_1, x_2, \cdots, x_n, \cdots\}\,. \tag{3.3}$$

The states (3.1) are normalized, thus the function $\mathcal{N}$ is given by the expression

$$\mathcal{N}(t) = \sum_{n=0}^{\infty} \frac{t^n}{x_n!}\,, \quad t = |\alpha|^2\,. \tag{3.4}$$

We can construct a set of CS $|\alpha;\chi\rangle$ if there exists a measure $\mathrm{d}\lambda$ related to $x_n!$ through a moment condition

$$\frac{x_n!}{2\pi} = \int_0^L r^{2n} \mathrm{d}\lambda(r), \quad \frac{1}{2\pi} = \int_0^L \mathrm{d}\lambda(r), \quad \text{for } \alpha = r e^{\mathrm{i}\theta}\,, \tag{3.5}$$

with L being the radius of convergence of the series $\sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{x_n!}}$. If the condition (3.5) holds true, the set of states $|\alpha;\chi\rangle$ will satisfy the resolution of the identity

$$\int_0^{2\pi} \mathrm{d}\theta \int_0^L \mathrm{d}\lambda(r)\,\mathcal{N}(|\alpha|^2)\,|\alpha;\chi\rangle\,\langle\alpha;\chi| = \mathbb{I}\,. \tag{3.6}$$

Given the non-linear CS $|\alpha;\chi\rangle$, we can write the probability of detecting n photons as

$$\mathcal{P}_n\left(|\alpha;\chi\rangle\right) = |\,\langle n|\alpha;\chi\rangle\,|^2 = \frac{t^n}{\mathcal{N}(t)\,x_n!}\,. \tag{3.7}$$

3.1.1 Mandel parameter and photon statistics for nonlinear CS

The expected number of photons $\langle n \rangle$ for $|\alpha, \chi\rangle$ is given by

$$\langle n \rangle := \langle \alpha, \chi | a^\dagger a | \alpha, \chi \rangle = t \frac{d}{dt} \ln \mathcal{N}(t) \,. \tag{3.8}$$

The expected value $\langle n^2 \rangle$ for $|\alpha, \chi\rangle$ is given by

$$\langle n^2 \rangle := \langle \alpha, \chi | (a^\dagger a)^2 | \alpha, \chi \rangle = \frac{t}{\mathcal{N}(t)} \frac{d}{dt} t \frac{d}{dt} \mathcal{N}(t) \,. \tag{3.9}$$

The deviation of the distribution $|\langle n|\alpha, \chi\rangle|^2$ from the Poisson distribution can be measured with the *Mandel parameter* Q_M defined as

$$Q_M := \frac{\langle n^2 \rangle - \langle n \rangle^2 - \langle n \rangle}{\langle n \rangle} \,. \tag{3.10}$$

If $Q_M = 0$ (linear CS) the distribution $|\langle n|\alpha, \chi\rangle|^2$ is Poissonian. The distribution is sub-Poissonian if $Q_M < 0$, and super-Poissonian if $Q_M > 0$.

3.2 Quantum error probability

Let us consider a binary system where the sender uses an alphabet formed with the quantum states ρ_0 and ρ_1. The receiver tries to guess which state was transmitted by performing measurements. Those measurements are represented by the POVM

$$M_0 + M_1 = \mathbb{I} \,. \tag{3.11}$$

The receiver performs a measurement to decide which state was transmitted. The possible results are m_0 or m_1. If the result is m_k (for $k = 0, 1$) the receiver chooses the state ρ_k. The existence of an error implies that the probability of measure m_0 after the state ρ_1 was sent is non-zero (or measure m_1 after the state ρ_0 was sent). The respective conditional probabilities are

$$p(m_0|\rho_1) = \mathrm{Tr}[M_0 \rho_1] \,, \quad p(m_1|\rho_0) = \mathrm{Tr}[M_1 \rho_0] \,. \tag{3.12}$$

If we consider the probabilities ξ_0 and ξ_1 of sending the states ρ_0 and ρ_1 respectively, the total error probability is given by

$$p(M_0, M_1) = \xi_1\, p(m_0|\rho_1) + \xi_0\, p(m_1|\rho_0)\,, \tag{3.13}$$

which can be re-written as

$$p(M_0, M_1) = \xi_1 + \mathrm{Tr}[M_1\Gamma]\,, \quad \text{where} \quad \Gamma = \xi_0\rho_0 - \xi_1\rho_1\,. \tag{3.14}$$

Minimizing the error between all the possible POVMs (M_0, M_1) yields the definition of the *quantum error* (or *Helstrom bound*),

$$P_H := \min_{M_0,M_1} p(M_0, M_1) = \xi_1 + \min_{M_1} \mathrm{Tr}[M_1\Gamma]\,. \tag{3.15}$$

Let $\Gamma = \sum_n \lambda_n|\gamma_n\rangle\langle\gamma_n|$ be the spectral decomposition of Γ. One can write $\mathrm{Tr}[M_1\Gamma] = \sum_n \lambda_n\langle\gamma_n|M_1|\gamma_n\rangle$. Then the Helstrom bound can be expressed as $P_H = \xi_1 + \sum_{\lambda_n<0}\lambda_n$, which correspond to the case in which M_1 is the projector on all the eigenstates $|\gamma_n\rangle$ with negative λ_n. For pure states, where $\rho_0 = |\psi_0\rangle\langle\psi_0|$ and $\rho_1 = |\psi_1\rangle\langle\psi_1|$, the operator Γ has only one negative eigenvalue

$$\lambda_- = \frac{1}{2}\left(1 - \sqrt{1 - 4\xi_0\xi_1|\langle\psi_1|\psi_0\rangle|^2}\right) - \xi_1\,. \tag{3.16}$$

Therefore the Helstrom bound must be

$$P_H = \frac{1}{2}\left(1 - \sqrt{1 - 4\xi_0\xi_1|\langle\psi_1|\psi_0\rangle|^2}\right)\,. \tag{3.17}$$

3.3 Non-linear CS generated by deformations of the binomial distribution

The nonlinear CS are defined by a sequence $\chi = \{x_k\}$. As shown in (3.4), the sequence χ can be generated by the coefficients in the power series expansion of the function $\mathcal{N}$. In the works [55, 56], the function $\mathcal{N}$ is used to construct generating functions of deformations of the binomial distribution. In this section, we will study the nonlinear CS constructed with those sequences.

Let us consider an experiment with two possible outcomes: success with probability η and *loss* with probability $1-\eta$, where $\eta \in [0,1]$. The probability of k successes for a sequence of n independent experiments is given by the binomial distribution

$$p_k^{(n)}(\eta) = \frac{n!}{(n-k)!\,k!}\,\eta^k\,(1-\eta)^{n-k}\,. \tag{3.18}$$

We are interested in the deformations of the binomial law using an increasing sequence of nonnegative real numbers $\{x_n\}_{n\in\mathbb{N}}$, where $x_0 = 0$. Now, we will define some useful sets of functions:

Definition 3.3.1. *The set Σ is defined as the set of entire series $\mathcal{N}(t) = \sum_{n=0}^{\infty} a_n t^n$ possessing a non-vanishing radius of convergence and verifying $a_0 = 1$ and $\forall n \geq 1,\ a_n > 0$.*

Definition 3.3.2. *The set Σ_0 is defined as the set of entire series $f(t) = \sum_{n=0}^{\infty} a_n t^n$ possessing a non-vanishing radius of convergence and verifying $a_0 = 0,\ a_1 > 0$ and $\forall\, n \geq 2,\ a_n \geq 0$.*

Definition 3.3.3. *Given $F(t) = \sum_{k=0}^{\infty} a_k t^k \in \Sigma_0$, the set Σ_+ is defined as*

$$\Sigma_+ = \left\{ e^F \mid F \in \Sigma_0 \right\}\,. \tag{3.19}$$

The sequence $\chi = \{x_n\}$ which defines the nonlinear CS $|\alpha,\chi\rangle$ will be generated by the function $\mathcal{N} \in \Sigma_+$. The sequence χ will be given by the power series expansion (3.4). In order to construct a set of CS, the sequence χ must verify the moment condition (3.5).

3.3.1 Asymmetric deformation

The asymmetric deformation $\mathfrak{p}_k^{(n)}(\eta)$ of the binomial distribution is defined as

$$\mathfrak{p}_k^{(n)}(\eta) := \binom{x_n}{x_k}\eta^k p_{n-k}(\eta)\,, \tag{3.20}$$

where the generalized binomial coefficients $\binom{x_n}{x_k}$ are given by

$$\binom{x_n}{x_k} := \frac{x_n!}{x_{n-k}!x_k!}\,. \tag{3.21}$$

The polynomials $p_n(\eta)$ are constrained by the condition

$$\forall n, k \in \mathbb{N}, \quad \forall \eta \in [0,1], \quad \sum_{k=0}^{n} \mathfrak{p}_k^{(n)}(\eta) = 1, \quad \mathfrak{p}_k^{(n)}(\eta) \geq 0. \tag{3.22}$$

The distribution (3.20) is called asymmetric because is not invariant under the transformations $k \to n-k$ and $\eta \to 1-\eta$. The polynomials $p_n(\eta) = \mathfrak{p}_0^{(n)}$ have a probabilistic interpretation as long as they are nonnegative.

For $\mathcal{N}(t) \in \Sigma$, the generating function $G_{\mathcal{N},\eta} \in \Sigma_+$ of the polynomials $p_k(\eta)$ is given by

$$G_{\mathcal{N},\eta} := \frac{\mathcal{N}(t)}{\mathcal{N}(\eta t)} = \sum_{k=0}^{\infty} p_k(\eta)\frac{t^k}{x_k!}\,. \tag{3.23}$$

The polynomials p_n are issued from

$$p_n(\eta) = \frac{x_n!}{n!}\frac{d^n}{dt^n}\frac{\mathcal{N}(t)}{\mathcal{N}(\eta t)}\bigg|_{t=0}\,. \tag{3.24}$$

The decomposition of the function

$$\frac{1}{\mathcal{N}(t)} = \sum_{k=0}^{\infty}\frac{(-t)^k}{x_k!}I_k\,, \tag{3.25}$$

allows us to write the polynomials p_n in terms of the coefficients I_k as follows

$$p_n(\eta) = \sum_{k=0}^{n}\binom{x_n}{x_k}(-\eta)^k\, I_k\,. \tag{3.26}$$

The limit when $n \to \infty$ of the deformed binomial distribution with $\eta = t/x_n$ is the deformed Poisson distribution

$$\binom{x_n}{x_k}\eta^k p_{n-k}(\eta) \underset{n\to\infty}{\longrightarrow} \frac{t^k}{x_k!}\frac{1}{\mathcal{N}(t)}\,. \tag{3.27}$$

3.3.2 Symmetric deformation

The symmetric deformation $\mathfrak{p}_k^{(n)}(\eta)$ of the binomial distribution is defined as

$$\mathfrak{p}_k^{(n)}(\eta) = \binom{x_n}{x_k} q_k(\eta) q_{n-k}(1-\eta)\,, \tag{3.28}$$

where the $q_k(\eta)$ are polynomials of degree k. The polynomials $q_k(\eta)$ are constrained by:

$$\forall n, k \in \mathbb{N}, \quad \forall \eta \in [0,1], \quad \sum_{k=0}^{n} \mathfrak{p}_k^{(n)}(\eta) = 1, \quad \mathfrak{p}_k^{(n)}(\eta) \geq 0. \tag{3.29}$$

Since q_0 is a polynomial of degree 0, then $q_0(\eta) = \pm 1$. We choose here $q_0(\eta) = 1$. With this value for q_0

$$\forall n \in \mathbb{N}, \quad \forall \eta \in [0,1], \quad \mathfrak{p}_0^{(n)}(\eta) = q_n(1-\eta). \tag{3.30}$$

The distribution $\mathfrak{p}_k^{(n)}(\eta)$ can be interpreted as the probability of k successes in a sequence of n *correlated* experiments.

For $\mathcal{N}(t) \in \Sigma$, the generating function $G_{\mathcal{N},\eta} \in \Sigma_+$ of the polynomials q_n corresponds to

$$G_{\mathcal{N},\eta}(t) = (\mathcal{N}(t))^{\eta} = \sum_{n=0}^{\infty} q_n(\eta) \frac{t^n}{x_n!}, \quad \forall \eta \in [0,1]\,. \tag{3.31}$$

The polynomials q_n are issued from

$$q_n(\eta) = \frac{x_n!}{n!} \frac{d^n}{dt^n} (\mathcal{N}(t))^{\eta} \bigg|_{t=0}\,. \tag{3.32}$$

The polynomials q_n have the property $q_n(1) = 1$ and also fullfill the relation

$$\forall z_1, z_2 \in \mathbb{C}, \forall n \in \mathbb{N}, \sum_{k=0}^{n} \binom{x_n}{x_k} q_k(z_1) q_{n-k}(z_2) = q_n(z_1 + z_2)\,, \tag{3.33}$$

3.3.3 Mandel parameter and photon statistics

This section shows the contribution of this thesis to the analysis of the photon statistics for nonlinear CS generated by $\mathcal{N} \in \Sigma_+$, not discussed until now in the literature.

Let us examine the behaviour of the distribution $|\langle n|\alpha,\chi\rangle|^2$, where the sequence χ is generated by $\mathcal{N}(t)\in\Sigma_+$. Given $\mathcal{N}(t)=e^{F(t)}\in\Sigma_+$ like in 3.3.3 we have

$$\langle n^2\rangle = \frac{t}{\mathcal{N}(t)}\frac{d}{dt}t\frac{d}{dt}\mathcal{N}(t) = \sum_{k=1}^{\infty} k^2 a_k t^k + \left(\sum_{k=1}^{\infty} k a_k t^k\right)^2 , \tag{3.34}$$

and

$$\langle n\rangle^2 = \left(t\frac{d}{dt}\ln\mathcal{N}(t)\right)^2 = \left(\sum_{k=1}^{\infty} k a_k t^k\right)^2 , \tag{3.35}$$

therefore

$$\langle n^2\rangle - \langle n\rangle^2 - \langle n\rangle = \sum_{k=1}^{\infty}(k^2-k)a_k t^k . \tag{3.36}$$

Since $a_1 > 0$ and $\forall n \geq 2$ $a_n \geq 0$, the right-hand side of the equation (3.36) is positive. Thus, the Mandel parameter Q_M is positive. We can conclude that the nonlinear CS associated with a sequence χ generated by $\mathcal{N}\in\Sigma_+$ are super-Poissonian.

3.4 Helstrom bound for nonlinear CS

This section shows the contribution of this thesis to the Helstrom bound for nonlinear CS. In the work [53], the Helstrom bound for nonlinear CS using the binomial distribution and the asymmetric generalization of the binomial distribution was studied. Here, we expand those results adding the symmetric generalization of the binomial distribution.

The type of measurement implemented in the communication process will be photodetection. The quantum theory of photodetection was formulated in [71] using a theory of electromagnetic field measurement by means of photoionization. In the case of perfect detection, the photo-counter is ideally counting all photons. In practice however, available photo-counters are not ideally counting all photons, and their performances are limited by a efficiency parameter $\eta\in[0,1]$, namely only a fraction η of the incoming photons lead to a *count*. We will study the Helstrom bound for both cases.

3.4.1 Perfect detection

In this case, the efficiency is $\eta = 1$. Let us consider an alphabet $\mathcal{A} = \{|0\rangle, |\alpha\rangle\}$ of two linear CS generated by a laser beam. Using a combination of beam splitters and phase shifters, we can transform the alphabet $\mathcal{A}$, to a phase-shift keyed alphabet $\mathcal{A}' = \left\{\left|\frac{\alpha}{\sqrt{2}}\right\rangle, \left|\frac{e^{\mathrm{i}\varphi}\alpha}{\sqrt{2}}\right\rangle\right\}$. The mean value of the number operator is $\langle n\rangle = \langle\alpha|N|\alpha\rangle = |\alpha|^2$. The overlap $\langle\psi_1|\psi_0\rangle = \langle\alpha|0\rangle$ is $\langle\alpha|0\rangle = \exp(-\langle n\rangle/2)$. Therefore the Helstrom bound for perfect detection is given by

$$P_H = \frac{1}{2}\left(1 - \sqrt{1 - 4\xi_0\xi_1 e^{-\langle n\rangle}}\right) . \tag{3.37}$$

For the perfect detection with nonlinear CS, we consider the alphabet alphabet $\mathcal{A}_\chi = \{|0\rangle, |\alpha, \chi\rangle\}$. The overlap between $|\psi_0\rangle$ and $|\psi_1\rangle$ is

$$\langle\psi_1|\psi_0\rangle = \langle\alpha, \chi|0\rangle = \frac{1}{\sqrt{\mathcal{N}(t)}}. \tag{3.38}$$

Then, the Helstrom bound is given by

$$P_H = \frac{1}{2}\left(1 - \sqrt{1 - 4\xi_0\xi_1 \frac{1}{\mathcal{N}(t)}}\right) . \tag{3.39}$$

3.4.2 Imperfect detection

The probability $\mathcal{P}_n(\eta)$ to detect n-photons using a non-ideal photodetector ($\eta < 1$) for a single mode of frequency is given by the Poissonian distribution

$$\mathcal{P}_n(\eta) = \mathrm{Tr}\left[\varrho : \frac{(\eta a^\dagger a)^n}{n!} e^{-\eta a^\dagger a} :\right] , \tag{3.40}$$

where ϱ corresponds to the state of the laser beam and $:\cdots:$ to the normal ordering of field operators. The expression (3.40) yields to

$$\mathcal{P}_n(\eta) = \sum_{m=n}^{\infty} \binom{m}{n} \eta^n (1-\eta)^{m-n} \mathcal{P}_m(\eta = 1) , \tag{3.41}$$

where $\langle m|\,\varrho\,|m\rangle = \mathcal{P}_m(\eta = 1)$. Therefore, the photocounting distribution (3.41) depends on the binomial distribution $\binom{m}{n}\eta^n(1-\eta)^{m-n}$. We will study the distribution (3.41) when ϱ is given by linear and nonlinear CS. Afterward, the distribution (3.41) will be modified by a deformation on the binomial distribution.

Binomial distribution and linear CS

Lets consider the state of the light in the expression (3.41) as $\varrho = |\alpha\rangle\langle\alpha|$ where $|\alpha\rangle$ is a linear CS, then $\mathcal{P}_m(\eta = 1) = |\langle m|\alpha\rangle|^2 = \frac{e^{-t}t^m}{m!}$. To obtain $\mathcal{P}_n(\eta)$ we use the power series expansion of $e^{(1-\eta)t}$ which is given by

$$\sum_{j=0}^{\infty} \frac{(1-\eta)^j t^j}{j!} = e^{(1-\eta)t} . \tag{3.42}$$

Making $j = m - n$ and multiplying $\frac{\eta^n}{n!}$ at both sides yields to

$$\mathcal{P}_n(\eta) = |\langle n|\sqrt{\eta}\alpha\rangle|^2 . \tag{3.43}$$

Now, we will consider $|\psi_0\rangle = |0\rangle$ and $|\psi_1\rangle = \left|\sqrt{\eta}\alpha\right\rangle$. The overlap in (3.17) will be $|\langle\psi_1|\psi_0\rangle|^2 = |\langle\sqrt{\eta}\alpha|0\rangle|^2 = e^{-\eta\langle n\rangle}$. Therefore, the Helstrom bound is given by

$$P_H = \frac{1}{2}\left(1 - \sqrt{1 - 4\xi_0\xi_1 e^{-\eta\langle n\rangle}}\right) . \tag{3.44}$$

Binomial distribution and non-linear CS

Let us consider the state of the laser beam in (3.41) as $\varrho = |\alpha,\chi\rangle\langle\alpha,\chi|$ where $|\alpha,\chi\rangle$ is a nonlinear CS. The sequence $\chi = \{x_k\}$ is generated by $\mathcal{N} \in \Sigma_+$, then $\mathcal{P}_m(\eta = 1) = |\langle m|\alpha;\chi\rangle|^2 = \frac{t^m}{\mathcal{N}(t)\, x_m!}$. The expression (3.41) yields to

$$\mathcal{P}_n(t;\eta) = \sum_{m=n}^{\infty} \binom{m}{n} \eta^n (1-\eta)^{m-n} \frac{t^m}{\mathcal{N}(t) x_m!} . \tag{3.45}$$

The distribution above can be rewritten as

$$\mathcal{P}_n(t;\eta) = \frac{(\eta t)^n}{\mathcal{N}(\eta t) x_n!} \mathcal{C}_n(t) , \tag{3.46}$$

where the corrective factor $\mathcal{C}_n(t)$ is given by

$$\mathcal{C}_n(t) = \frac{\mathcal{N}(\eta t)}{\mathcal{N}(t)} \frac{x_n!}{n!} \sum_{k=0}^{\infty} \frac{(n+k)!}{x_{n+k}!} \frac{[t(1-\eta)]^k}{k!} . \tag{3.47}$$

The probability distribution (3.46) is associated with the set of normalized pure states

$$|\alpha;\eta\rangle = \sum_{n=0}^{\infty} \sqrt{\mathcal{P}_n(t;\eta)} e^{in\arg(\alpha)} |n\rangle , \tag{3.48}$$

where $|0;\eta\rangle = |0\rangle$, and $\arg(\alpha)$ is the argument of the complex number α. With the alphabet $\mathcal{A}_{\alpha,\eta} = \{|0;\eta\rangle, |\alpha;\eta\rangle\}$, the overlap in (3.17) reads as

$$|\langle\psi_1|\psi_0\rangle|^2 = |\langle\alpha;\eta|0;\eta\rangle|^2 = \frac{\mathcal{N}(t(1-\eta))}{\mathcal{N}(t)} . \tag{3.49}$$

Therefore, the Helstrom bound is given by

$$P_H^{\text{bin}} = \frac{1}{2}\left(1 - \sqrt{1 - 4\xi_0\xi_1\frac{\mathcal{N}(t(1-\eta))}{\mathcal{N}(t)}}\right) . \tag{3.50}$$

Asymmetric generalized binomial distributions and non-linear CS

This section shows the contribution of this thesis to the analysis of the symmetric case, not discussed until now in the literature.

Now we want to deform the binomial distribution inside (3.41) with the asymmetric deformation (3.20) in order to express $\mathcal{P}_n(\eta)$ as

$$\mathcal{P}_n(\eta) = |\langle n|\sqrt{\eta}\alpha;\chi\rangle|^2 . \tag{3.51}$$

To achieve this we use $\mathcal{N} \in \Sigma_+$ which generates the sequence $\chi = \{x_k\}$. For $\eta \in [0,1]$, with the help of (3.23) we can write

$$\sum_{j=0}^{\infty} p_j(\eta)\frac{t^j}{\mathcal{N}(t)\,x_j!} = \frac{1}{\mathcal{N}(\eta t)} . \tag{3.52}$$

Making $j = m - n$ and multiplying $\frac{\eta^n}{x_n!}$ at both sides yields to

$$\frac{(\eta t)^n}{\mathcal{N}(\eta t)\,x_n!} = \sum_{m=n}^{\infty} \frac{x_m!}{x_{m-n}!x_n!}\eta^n\, p_{m-n}(\eta)\frac{t^m}{\mathcal{N}(t)\,x_m!} . \tag{3.53}$$

Using the expression (3.7), the equation (3.53) can be re-written as

$$|\langle n|\sqrt{\eta}\alpha;\chi\rangle|^2 = \sum_{m=n}^{\infty} \binom{x_m}{x_n}\eta^n\, p_{m-n}(\eta)\,|\langle m|\alpha;\chi\rangle|^2 , \tag{3.54}$$

where $\mathcal{P}_m(\eta = 1) = |\langle m|\alpha;\chi\rangle|^2$. With the alphabet $\mathcal{A}_{\chi,\eta} = \left\{|0\rangle, |\alpha, \sqrt{\eta}\chi\rangle\right\}$, the overlap in (3.17) is given by $|\langle\psi_1|\psi_0\rangle|^2 = |\langle\sqrt{\eta}\alpha, \chi|0\rangle|^2 = \dfrac{1}{\mathcal{N}(\eta t)}$. Therefore, the Helstrom bound is given by

$$P_H^{\text{asymm}} = \frac{1}{2}\left(1 - \sqrt{1 - 4\xi_0\xi_1\frac{1}{\mathcal{N}(\eta t)}}\right) . \tag{3.55}$$

Symmetric generalized binomial distributions and non-linear CS

This section shows the contribution of this thesis to the analysis of the symmetric case, not discussed until now in the literature.

Let us deform the binomial distribution inside (3.41) with the symmetric deformation (3.28) in order to express $\mathcal{P}_n(\eta)$ as

$$\mathcal{P}_n(\eta) = |\langle n|\alpha;\Upsilon\rangle|^2 , \tag{3.56}$$

To achieve this we use $\mathcal{N} \in \Sigma_+$ which generates the sequence $\chi = \{x_k\}$. For $\xi \in [0,1]$, with the help of (3.31) we can write

$$\sum_{j=0}^{\infty} q_j(\xi)\frac{t^j}{x_j!} = \mathcal{N}(t)^{\xi} . \tag{3.57}$$

Making $\xi = 1-\eta$ and $j = m-n$ gives

$$\sum_{m=n}^{\infty} q_{m-n}(1-\eta)\frac{t^m t^{-n}}{x_{m-n}!} = \frac{\mathcal{N}(t)}{\mathcal{N}(t)^{\eta}} . \tag{3.58}$$

Multiplying $\frac{q_n(\eta)}{x_n!}$ at both sides of (3.58) yields

$$\frac{t^n}{\mathcal{N}(t)^{\eta}\frac{x_n!}{q_n(\eta)}} = \sum_{m=n}^{\infty}\frac{x_m!}{x_{m-n}!x_n!}q_n(\eta)q_{m-n}(1-\eta)\frac{t^m}{\mathcal{N}(t)\,x_m!} . \tag{3.59}$$

Since $\mathcal{N}(t)^{\eta} \in \Sigma$, there is a new sequence $\Upsilon = \{y_k\}$ where $y_n! = \frac{x_n!}{q_n(\eta)}$. Using the expression (3.7), the equation (3.59) becomes

$$|\langle n|\alpha;\Upsilon\rangle|^2 = \sum_{m=n}^{\infty}\binom{x_m!}{x_n!}q_n(\eta)q_{m-n}(1-\eta)\,|\langle m|\alpha;\chi\rangle|^2 . \tag{3.60}$$

where $\mathcal{P}_m(\eta=1) = |\langle m|\alpha;\chi\rangle|^2$. Let us now consider the alphabet $\mathcal{A}_\Upsilon = \{|0\rangle, |\alpha,\Upsilon\rangle\}$, the overlap in (3.17) will be $|\langle\psi_1|\psi_0\rangle|^2 = |\langle\alpha;\Upsilon|0\rangle|^2 = \dfrac{1}{\mathcal{N}(t)^{\eta}}$. Therefore, the Helstrom bound is given by

$$P_H^{\text{symm}} = \frac{1}{2}\left(1 - \sqrt{1 - 4\xi_0\xi_1\frac{1}{\mathcal{N}(t)^\eta}}\right). \tag{3.61}$$

Definition 3.4.1. *The function $\mathcal{N} \in \Sigma_+$ possesses logarithmic scale invariance if the sequence $\chi^s = \{x_n\}$ depends on the parameter s (that is $x_n \equiv x_n(s)$) in such a way that for $\mathcal{N}(t) \equiv \mathcal{N}(t,s)$ we have*

$$(\mathcal{N}(t,s))^\eta = \mathcal{N}(\eta t, \eta s). \tag{3.62}$$

If $\mathcal{N}$ has this property, the logarithmic scale invariance (3.62) allows to give the following expression for the polynomial q_n

$$q_n(\eta, s) = \eta^n \frac{x_n(s)!}{x_n(\eta s)!}. \tag{3.63}$$

With the relation (3.63), we can express $\mathcal{P}_n(\eta)$ in (3.59) as

$$|\langle n|\sqrt{\eta}\alpha, \chi^{\eta s}\rangle|^2 = \sum_{m=n}^{\infty} \binom{x_m(s)}{x_n(s)} q_n(\eta, s) q_{m-n}(1-\eta, s)\, |\langle m|\alpha; \chi^s\rangle|^2. \tag{3.64}$$

with $\mathcal{P}_m(\eta = 1) = |\langle m|\alpha; \chi^s\rangle|^2$ The alphabet $\mathcal{A}_\Upsilon$, becomes $\mathcal{A}_{\chi^{\eta s}} = \left\{|0\rangle, |\sqrt{\eta}\alpha, \chi^{\eta s}\rangle\right\}$. The overlap in (3.17) will be $|\langle\psi_1|\psi_0\rangle|^2 = |\langle\sqrt{\eta}\alpha, \chi^{\eta s}|0\rangle|^2 = \dfrac{1}{\mathcal{N}(\eta t, \eta s)}$. Therefore, the expression (3.61) becomes

$$P_H^{\text{symm}} = \frac{1}{2}\left(1 - \sqrt{1 - 4\xi_0\xi_1\frac{1}{\mathcal{N}(\eta t, \eta s)}}\right). \tag{3.65}$$

3.5 Optimization with nonlinear CS

This section shows the contribution of this thesis to the analysis of the optimization with nonlinear CS. In the work [53], the optimization using the binomial distribution and the asymmetric generalization of the binomial distribution was studied. Here, we expand those results by adding the symmetric generalization of the binomial distribution, making a complete analysis of the nonlinear CS generated by $\mathcal{N} \in \Sigma_+$, and studying an alphabet formed by Sussking-Glogower CS.

For *perfect detection*, the sender uses the alphabet $\mathcal{A}_\varphi = \{|\varphi_0\rangle, |\varphi_\alpha\rangle\}$ composed by two states from the family $\{|\varphi_\alpha\rangle\}$ parametrized by α, where $|\varphi_0\rangle = |0\rangle$. If we want to lower the Helstrom bound for $\mathcal{A}_\varphi$ in comparison with $\mathcal{A} = \{|0\rangle, |\alpha'\rangle\}$ so that

$$\frac{1}{2}\left(1 - \sqrt{1 - 4\xi_0\xi_1|\langle\varphi_\alpha|\varphi_0\rangle|^2}\right) < \frac{1}{2}\left(1 - \sqrt{1 - 4\xi_0\xi_1|\langle\alpha'|0\rangle|^2}\right). \tag{3.66}$$

The inequality above yields the condition

$$|\langle\varphi_\alpha|0\rangle|^2 < |\langle\alpha'|0\rangle|^2\,. \tag{3.67}$$

For *imperfect detection* the sender uses the alphabet $\mathcal{A}_{\varphi,\eta} = \{|\varphi_{0,\eta}\rangle, |\varphi_{\alpha,\eta}\rangle\}$ composed by two states from the family $\{|\varphi_{\alpha,\eta}\rangle\}$ parametrized by α and $\eta < 1$, where $|\varphi_{0,\eta}\rangle = |0\rangle$. The condition to lowering the Helstrom bound for $\mathcal{A}_{\varphi,\eta}$ in comparison with $\mathcal{A} = \{|0\rangle, |\alpha'\rangle\}$ amounts to

$$|\langle\varphi_{\alpha,\eta}|0\rangle|^2 < |\langle\sqrt{\eta}\alpha'|0\rangle|^2\,. \tag{3.68}$$

3.5.1 Perfect detection

Considering the alphabet $\mathcal{A}_\chi = \{|0\rangle, |\alpha,\chi\rangle\}$ formed by two nonlinear CS. The condition (3.67) becomes

$$|\langle\alpha,\chi|0\rangle|^2 < |\langle\alpha|0\rangle|^2\,. \tag{3.69}$$

The quantity $\langle\alpha'|N|\alpha'\rangle = |\alpha'|^2 = t'$ represents the expected number of photons when the laser beam is in a linear CS. The quantity $\langle\alpha,\chi|N|\alpha,\chi\rangle = t\frac{d}{dt}\ln\mathcal{N}(t)$ (for $|\alpha|^2 = t$) represents the expected number of photons when the laser beam is in a nonlinear CS. Clearly the parameter α generates different expected values for the linear and nonlinear CS, i.e. $\langle\alpha|N|\alpha\rangle \neq \langle\alpha,\chi|N|\alpha,\chi\rangle$. Therefore, we must choose different parameters in order to write $\langle n\rangle = \langle\alpha'|N|\alpha'\rangle = \langle\alpha,\chi|N|\alpha,\chi\rangle$, which allows to compare both sides of the inequality (3.69). This yields the condition

$$\mathcal{N}(t(\langle n\rangle)) > e^{\langle n\rangle}\,. \tag{3.70}$$

where $t(\langle n\rangle)$ is the inverse of $\langle n\rangle(t) = t\frac{d}{dt}\ln\mathcal{N}(t)$. Since $t(\langle n\rangle)$ is bijective (within the radius of convergence of $\mathcal{N}(t)$) we can write

$$\mathcal{N}(t) > e^{t\frac{d}{dt}\ln\mathcal{N}(t)}\,, \tag{3.71}$$

which yields the condition

$$\ln\mathcal{N}(t) > t\frac{d}{dt}\ln\mathcal{N}(t)\,. \tag{3.72}$$

Non-linear CS generated by $\mathcal{N} \in \Sigma_+$

We want to find $\mathcal{N} \in \Sigma_+$ such that (3.72) is satisfied. In the article [55] one can find the proof for the following statements:

- For all $\mathcal{N} \in \Sigma_+$, $\ln\mathcal{N} \in \Sigma_0$ and

$$\ln\mathcal{N}(t) = \sum_{k=1}^{\infty} \frac{(-p_k'(1))}{x_k!}\frac{t^k}{k}\,, \tag{3.73}$$

- $p_1'(1) = -1$ and $p_n'(1) \leq 0$.

Using (3.73) we write

$$t\frac{d}{dt}\ln\mathcal{N}(t) = \sum_{k=1}^{\infty} \frac{(-p_k'(1))}{x_k!}t^k\,. \tag{3.74}$$

Since $t = |\alpha|^2 > 0$, we conclude that $\ln\mathcal{N}(t) < t\frac{d}{dt}\ln\mathcal{N}(t)$. Therefore, the condition (3.72) cannot be satisfied with $\mathcal{N} \in \Sigma_+$.

3.5.2 Imperfect detection

Binomial distribution

Considering the alphabet $\mathcal{A}_{\alpha,\eta} = \{|0;\eta\rangle, |\alpha;\eta\rangle\}$. The family of normalized states $|\varphi_\alpha\rangle$ is given by the states $|\alpha;\eta\rangle$ defined in (3.48). The condition (3.68) becomes

$$|\langle\alpha;\eta|0;\eta\rangle|^2 < |\langle\sqrt{\eta}\alpha'|0\rangle|^2\,. \tag{3.75}$$

In subsection 3.5.1, we got the relations

$$\begin{aligned} &|\alpha'|^2 = t'\,,\\ &|\alpha|^2 = t\,,\\ &\langle n\rangle = t' = t\tfrac{d}{dt}\ln\mathcal{N}(t)\,. \end{aligned} \tag{3.76}$$

The expected value of N for the states $|\sqrt{\eta}\alpha'\rangle$ is given by

$$\langle\sqrt{\eta}\alpha'|N|\sqrt{\eta}\alpha'\rangle = |\sqrt{\eta}\alpha'|^2 = \eta\langle n\rangle\,. \tag{3.77}$$

Therefore, the expressions (3.49) and (3.75) yields to the condition

$$\frac{\mathcal{N}(t(\langle n\rangle))}{\mathcal{N}(t(\langle n\rangle)(1-\eta))} > e^{\eta\langle n\rangle}\,, \tag{3.78}$$

Since $t(\langle n\rangle)$ is bijective (within the radius of convergence of $\mathcal{N}(t)$) we can write

$$\frac{\mathcal{N}(t)}{\mathcal{N}(t(1-\eta))} > e^{\eta t\frac{d}{dt}\ln\mathcal{N}(t)}\,. \tag{3.79}$$

Taking the logarithm

$$\ln\mathcal{N}(t) - \ln\mathcal{N}(t(1-\eta)) > \eta t\frac{d}{dt}\ln\mathcal{N}(t)\,. \tag{3.80}$$

Non-linear CS generated by $\mathcal{N}\in\Sigma_+$: Using the properties (3.73) and (3.74) yields to the left hand-side of the inequality

$$\ln\mathcal{N}(t) - \ln\mathcal{N}(t(1-\eta)) = \sum_{k=1}^{\infty}\frac{(-p_k'(1))}{x_k!}[1-(1-\eta)^k]\frac{t^k}{k}\,. \tag{3.81}$$

The right hand-side of the inequality must be

$$\eta t\frac{d}{dt}\ln\mathcal{N}(t) = \sum_{k=1}^{\infty}\frac{(-p_k'(1))}{x_k!}(k\eta)\frac{t^k}{k}\,. \tag{3.82}$$

By induction we show that

$$\ln\mathcal{N}(t) - \ln\mathcal{N}(t(1-\eta)) < \eta t\frac{d}{dt}\ln\mathcal{N}(t)\,. \tag{3.83}$$

Therefore, the condition (3.75) cannot be satisfied with $\mathcal{N}\in\Sigma_+$.

Induction of (3.83)*: for* $k=2\to 1-(1-\eta)^2<2\eta$*, for* $k=3\to 1-(1-\eta)^3<3\eta$*. Let's consider it true for* k*, i.e.* $1-(1-\eta)^k<k\eta$*. Then, for* $k+1$

$$1-(1-\eta)^{k+1} = 1-(1-\eta)^k+\eta(1-\eta)^k \tag{3.84}$$

since $1-(1-\eta)^k<k\eta$ *and* $(1-\eta)^k<1$ *we have*

$$1-(1-\eta)^{k+1} < k\eta+\eta = (k+1)\eta\,. \quad \square \tag{3.85}$$

Asymmetric generalized binomial distribution

Considering the alphabet $\mathcal{A}_{\chi,\eta} = \big\{|0\rangle, |\alpha, \sqrt{\eta}\chi\rangle\big\}$. The family of normalized states $|\varphi_\alpha\rangle$ is given by the nonlinear CS $|\sqrt{\eta}\alpha, \chi\rangle$. The condition (3.68) becomes

$$|\langle\sqrt{\eta}\alpha, \chi|0\rangle|^2 < |\langle\sqrt{\eta}\alpha'|0\rangle|^2\,. \tag{3.86}$$

In subsection 3.5.1, we got the relations

$$\begin{aligned}&|\alpha'|^2 = t'\,,\\ &|\alpha|^2 = t\,,\\ &\langle n\rangle = t' = t\tfrac{d}{dt}\ln\mathcal{N}(t)\,.\end{aligned} \tag{3.87}$$

From equation (3.77) we know that $\langle\sqrt{\eta}\alpha'|N|\sqrt{\eta}\alpha'\rangle = \eta\langle n\rangle$. The overlap $|\langle\sqrt{\eta}\alpha, \chi|0\rangle|^2 = 1/\mathcal{N}(\eta t)$ and the inequality (3.86) yields the condition

$$\mathcal{N}(\eta t(\langle n\rangle)) > e^{\eta\langle n\rangle}\,. \tag{3.88}$$

where $\langle n\rangle$ is the expected number of photons for perfect detection.

Since $t(\langle n\rangle)$ is bijective (within the radius of convergence of $\mathcal{N}(t)$) we can write

$$\mathcal{N}(\eta t) > e^{\eta t\frac{d}{dt}\ln\mathcal{N}(t)}\,. \tag{3.89}$$

Therefore, we want to find $\mathcal{N} \in \Sigma_+$ such that

$$\ln\mathcal{N}(\eta t) > \eta t\frac{d}{dt}\ln\mathcal{N}(t)\,. \tag{3.90}$$

Non-linear CS generated by $\mathcal{N} \in \Sigma_+$: Since $\eta \in [0,1]$ and $t = |\alpha|^2 > 0$, using (3.73) we get

$$\ln\mathcal{N}(\eta t) = \sum_{k=1}^{\infty}\frac{(-p_k'(1))}{x_k!}\eta^k\frac{t^k}{k} < \eta\ln\mathcal{N}(t) \tag{3.91}$$

Using (3.73) again we write

$$\eta t\frac{d}{dt}\ln\mathcal{N}(t) = \eta\sum_{k=1}^{\infty}\frac{(-p_k'(1))}{x_k!}t^k\,. \tag{3.92}$$

From (3.73), (3.91) and (3.92) we conclude that $\ln\mathcal{N}(\eta t) < \eta t\frac{d}{dt}\ln\mathcal{N}(t)$. Therefore, the condition (3.90) cannot be satisfied with $\mathcal{N} \in \Sigma_+$.

Symmetric generalized binomial distributions and non-linear CS

Considering the alphabet $\mathcal{A}_{\Upsilon} = \{|0\rangle, |\alpha, \Upsilon\rangle\}$. The steps used in the previous case yields to the condition

$$\mathcal{N}(t)^{\eta} > e^{\eta t \frac{d}{dt} \ln \mathcal{N}(t)} . \tag{3.93}$$

The expression above leads to the inequality

$$\ln \mathcal{N}(t) > t \frac{d}{dt} \ln \mathcal{N}(t). \tag{3.94}$$

Non-linear CS generated by $\mathcal{N} \in \Sigma_+$: As in the previous case, the equation (3.73) and its derivative leads to the inequality $\ln \mathcal{N}(t) < t\frac{d}{dt} \ln \mathcal{N}(t)$, which is in contradiction with the condition (3.94). Thus, we must conclude that (3.94) cannot be satisfied with $\mathcal{N} \in \Sigma_+$.

3.5.3 Example 1: of nonlinear CS generated by $\mathcal{N} \in \Sigma_+$

Let us consider the q-exponential distribution

$$\mathcal{N}(t,s) = \left(1 - s^{-1}t\right)^{-s} = \sum_{n=0}^{\infty} \frac{(s)_n}{n! s^n} t^n \quad \text{for } s > 1 \,, \tag{3.95}$$

where $(s)_n$ is the Pochammer symbol defined as

$$(z)_k = \begin{cases} 1 & k = 0 \,, \\ z(z+1)(z+2)\cdots(z+k-1) & k > 0 \,. \end{cases} \tag{3.96}$$

The function (3.95) has logarithmic scale invariance, it generates the sequence $\chi^s = \{x_k(s)\}$ given by

$$x_k(s) = \frac{ks}{k+s-1} \,. \tag{3.97}$$

The asymmetric polynomials p_n issued from (3.24) are

$$p_n\left(\eta, s\right) = {}_2F_1\left(-n, -s; 1-s-n; \eta\right). \tag{3.98}$$

The symmetric polynomials q_n issued from (3.32) are

$$q_n\left(\eta, s\right) = \frac{(\eta s)_n}{(\eta)_n}. \tag{3.99}$$

The measure dλ defined as

$$\mathrm{d}\lambda\,(r) = \frac{(s-1)}{2\pi s}\left(1-\frac{r^2}{s}\right)^{s-2} r\mathrm{d}r\,, \tag{3.100}$$

satisfy the condition (3.5). Then, we can construct the set of nonlinear CS $|\alpha, \chi^s\rangle$, where $t = |\alpha|^2$. The Mandel Q_M parameter is given by

$$Q_M = \frac{s^{-1}t}{1-s^{-1}t}\,. \tag{3.101}$$

Perfect detection

The Helstrom bound for linear CS

$$P_H = \frac{1}{2}\left(1-\sqrt{1-4\xi_0\xi_1 e^{-t'}}\right)\,. \tag{3.102}$$

The Helstrom bound for nonlinear CS

$$P_H^{\text{asymm}} = \frac{1}{2}\left(1-\sqrt{1-4\xi_0\xi_1\left(1-s^{-1}t\right)^s}\right)\,. \tag{3.103}$$

The expected number of photons is $\langle n\rangle = t' = t\frac{d}{dt}\ln\mathcal{N}(t)$, then

$$t = \frac{s\langle n\rangle}{s+\langle n\rangle}\,. \tag{3.104}$$

Since $\lim_{s\to\infty} t = \langle n\rangle$, then $\lim_{s\to\infty}\left[1-s^{-1}t\right]^s = e^{-\langle n\rangle}$.

Imperfect detection

The Helstrom bound for the binomial distribution is given by

$$P_H^{\text{bin}} = \frac{1}{2}\left(1-\sqrt{1-4\xi_0\xi_1\frac{\left[1-s^{-1}t\right]^s}{\left[1-s^{-1}(1-\eta)t\right]^s}}\right)\,. \tag{3.105}$$

The Helstrom bound for the asymmetric deformation of the binomial distribution is given by

$$P_H^{\text{asymm}} = \frac{1}{2}\left(1-\sqrt{1-4\xi_0\xi_1\left[1-s^{-1}\eta t\right]^s}\right)\,. \tag{3.106}$$

The Helstrom bound for the symmetric deformation of the binomial distribution is given by

$$P_H^{\text{symm}} = \frac{1}{2}\left(1 - \sqrt{1 - 4\xi_0\xi_1\left[1-(\eta s)^{-1}\eta t\right]^{\eta s}}\right) . \tag{3.107}$$

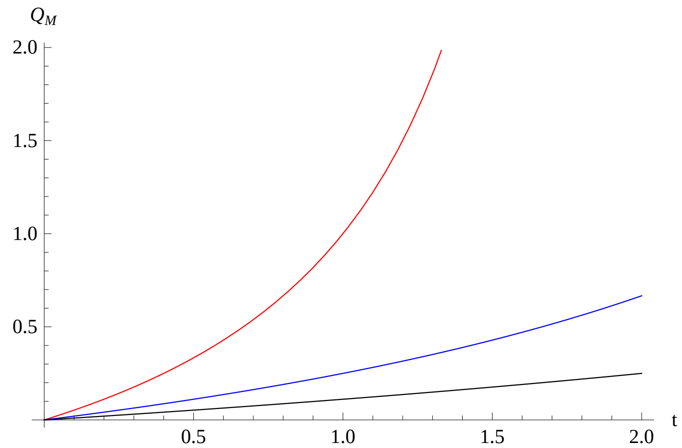


Figure 3.1: The Mandel parameter Q_M for nonlinear CS associated with the sequence (3.97) for $s = 2$ (red line), $s = 5$ (blue line) and $s = 10$ (black line).

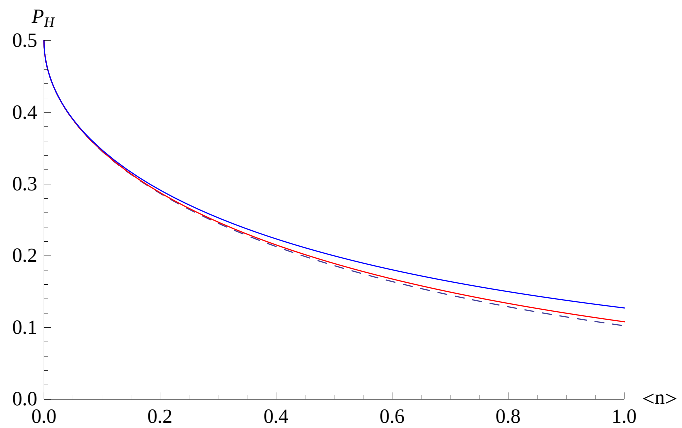


Figure 3.2: Helstrom bound P_H versus the expected number of photons $\langle n \rangle$ for **perfect detection** with $\xi_0 = \xi_1 = 1/2$. The dashed line corresponds to the Helstrom bound for linear CS. The Helstrom bound for nonlinear CS associated with the sequence (3.97) correspond to the blue line ($s = 2$) and the red line ($s = 10$).

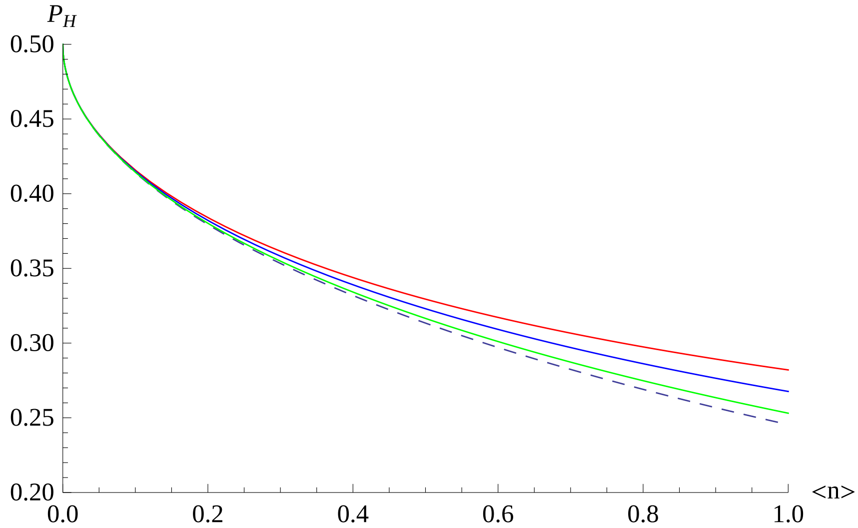


Figure 3.3: Helstrom bound versus the expected number of photons $\langle n\rangle$ for imperfect detection with $\eta = 0.3$, $s = 10$ and $\xi_0 = \xi_1 = 1/2$. The **dashed line** corresponds to the Helstrom bound for linear CS. The Helstrom bound for nonlinear CS associated with: the *asymmetric binomial* distribution correspond to the **red line**, the *symmetric binomial* distribution correspond to the **blue line**, the *binomial* distribution correspond to the **green line** .

Since $t = |\alpha|^2 = \frac{s\langle n\rangle}{s+\langle n\rangle}$, we have $\lim_{s\to\infty} t = \langle n\rangle$, then

$$\begin{aligned} \lim_{s\to\infty} \frac{\left[1 - s^{-1}t\right]^s}{\left[1 - s^{-1}(1-\eta)t\right]^s} &= e^{-\eta\langle n\rangle}\,, \\ \lim_{s\to\infty} \left[1 - s^{-1}\eta t\right]^s &= e^{-\eta\langle n\rangle}\,, \\ \lim_{s\to\infty} \left[1 - (\eta s)^{-1}\eta t\right]^{\eta s} &= e^{-\eta\langle n\rangle}\,, \end{aligned} \tag{3.108}$$

which explains the behaviour of the figure 3.3.

3.5.4 Example 2: Susskind-Glogower CS

Let us consider in this example a different type of CS which are not generated by a function $\mathcal{N} \in \Sigma_+$. The Susskind-Glogower operators V and $V^\dagger$ defined by

$$V := \frac{1}{\sqrt{N+1}} a = \sum_{n=0}^{\infty} |n\rangle\langle n+1|\,. \tag{3.109}$$

From (3.109) we get the properties $V|n\rangle = |n-1\rangle$ and $V^\dagger|n\rangle = |n+1\rangle$, where $[V, V^\dagger] = |0\rangle\langle 0|$. The Susskind-Glogower CS are defined as

$$|\alpha\rangle_{\mathrm{SG}} = e^{x(V^\dagger - V)}|0\rangle, \quad x = |\alpha|^2 \in \mathbb{R}\,. \tag{3.110}$$

These states can be expressed in terms of their decomposition over the Fock basis as,

$$|\alpha\rangle_{\mathrm{SG}} = \sum_{n=0}^{\infty}(n+1)\,\frac{J_{n+1}(2x)}{x}\,|n\rangle\,, \tag{3.111}$$

where $|\alpha(x=0)\rangle_{\mathrm{SG}} = |0\rangle$ and the Bessel function J_ν is given by

$$J_\nu(z) = \left(\frac{z}{2}\right)^{\nu}\sum_{k=0}^{\infty}\frac{(-1)^k\left(\frac{z}{2}\right)^{2k}}{k!\,\Gamma(\nu+k+1)}\,. \tag{3.112}$$

The expected number of photons $\langle n\rangle = {}_{\mathrm{SG}}\langle\alpha|N|\alpha\rangle_{\mathrm{SG}}$ is given by

$$\begin{aligned}\langle n\rangle &= (6x^2+1)J_0^2(2x) + (6x^2-1)J_1^2(2x) - 2xJ_0(2x)J_1(2x)\\ &\quad+\frac{2x^2}{3}\left[J_0(2x)J_2(2x) + J_1(2x)J_3(2x)\right] - 1\,.\end{aligned} \tag{3.113}$$

The mean value $\langle n^2\rangle = {}_{\mathrm{SG}}\langle\alpha|N^2|\alpha\rangle_{\mathrm{SG}}$ is given by

$$\langle n^2\rangle = 3x^2 - 2\langle n\rangle\,. \tag{3.114}$$

Therefore, the Mandel parameter is given by

$$Q_M = \frac{3x^2}{\langle n\rangle} - \langle n\rangle - 3\,. \tag{3.115}$$

Those results were presented in [72]. From this point, we use them to compute the Helstrom Bound for the Susskind-Glogower CS. If the family of normalized states $|\varphi_\alpha\rangle$ in our alphabet is given by the Susskind-Glogower CS $|\alpha\rangle_{\mathrm{SG}}$. The overlap $|\,{}_{\mathrm{SG}}\langle\alpha|0\rangle|^2$ will be

$$|\,{}_{\mathrm{SG}}\langle\alpha|0\rangle|^2 = \frac{J_1^2(2x(\langle n\rangle))}{x}\,. \tag{3.116}$$

Therefore, the Helstrom bound is given by

$$P_H = \frac{1}{2}\left(1 - \sqrt{1 - 4\xi_0\xi_1\frac{J_1^2(2x(\langle n\rangle))}{x}}\right)\,. \tag{3.117}$$

The expression (3.117) has quasi-periodic roots originated by the function $J_1(x)$. The Bessel function $J_1(x)$ oscillates “like” a sine function that decay proportionally to $1/\sqrt{x}$, although their roots are not generally periodic, except for large values of x. Thus, the Heltrom bound P_H will also oscillate, as we can see in Figure 3.5.

The condition (3.67) becomes

$$|\,_{\mathrm{SG}}\langle\alpha|0\rangle|^2 < |\langle\alpha|0\rangle|^2 \,. \tag{3.118}$$

Which can be expressed as

$$\frac{J_1^2(2x(\langle n\rangle))}{x} < e^{-\langle n\rangle}\,, \tag{3.119}$$

where $x(\langle n\rangle)$ is the inverse of (3.113).

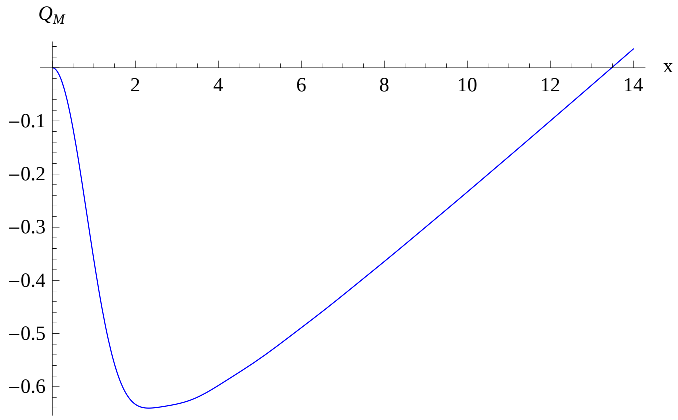


Figure 3.4: The Mandel parameter Q_M for Susskind-Glogower CS.

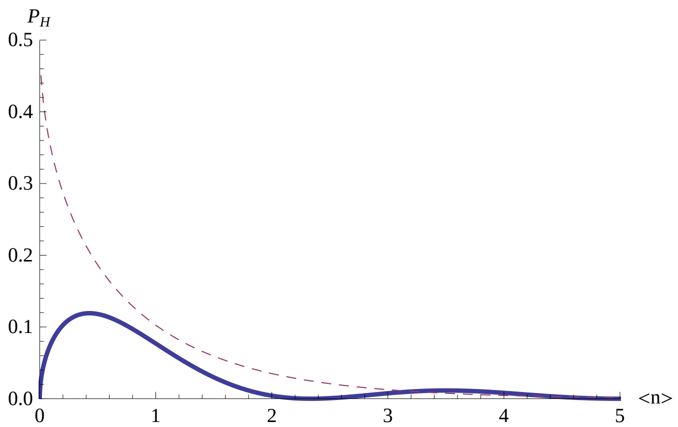


Figure 3.5: Helstrom bounds versus the expected number of phtons $\langle n\rangle$ for perfect detection. The dashed line corresponds to the Helstrom bound for linear CS. The Helstrom bounds for Susskind-Glogower CS correspond to the thick line.

The inequality (3.119) cannot hold for all values of $\langle n\rangle$. However, as we can see in Figure 3.5 for some values of $\langle n\rangle$, the Helstrom bound for Susskind-Glogower CS is lower than the Helstrom bound for linear CS.

In [73] a Hamiltonian that can be produced in ion-traps is presented. In the book [72], is shown that the Susskind-Glogower CS are eigenfunctions of a modified version the Hamiltonian used in [73]. This modified Hamiltonian is given by

$$H = \xi(V + V^\dagger)\,, \tag{3.120}$$

where ξ is a coupling coefficient. The physical realization of those states is then related to the eigenstates of trapped ions, although we wanted to use them to represent photons in a laser beam. Thus, a scheme to produce a beam of light in a Susskind-Glogower CS is still an open problem.

Chapter 4

Conclusion

In this thesis we have studied two problems through some generalizations of CS. The first one consists in the quantum localization on the circle, and the second one consists in the quantum error probability in binary communication with an alphabet of nonlinear CS.

Quantum localization on the circle. In this work we have presented a set of instructive outcomes of a quantisation based on the resolution of the identity provided by coherent states for the special Euclidean group E(2). The cylinder $\mathbb{R}\times\mathbb{S}^1$, which depicts the classical phase space of the motion of a particle on a circle, is indeed mathematically realized as the left coset $\mathrm{E}(2)/H$, where H is a stabilizer subgroup under a certain coadjoint action of E(2). The coherent states for E(2) are constructed from a unitary irreducible representation of the semi-direct product $\mathrm{E}(2) = \mathbb{R}^2 \rtimes \mathrm{SO}(2)$ restricted to a so-called affine section $\mathbb{R} \times \mathbb{S}^1 \ni (p,q) \mapsto \sigma(p,q) \in \mathrm{E}(2)$. For various functions on the cylindric phase space, the corresponding operators and lower symbols are determined . In the particular case of periodic functions $f(q)$ of the angular coordinate q, the operators A_f are multiplication operators whose spectra are given by periodic functions.

The angle function $a(\alpha)$, defined by $a(\alpha) = \alpha$ for $\alpha \in [0, 2\pi)$, is mapped to a self-adjoint multiplication angle operator A_a with continuous spectrum. For a particular family of coherent states, it is shown that the spectrum is $[\pi - m(s,\epsilon), \pi + m(s,\epsilon)]$, where $m(s,\epsilon) \to \pi$ as $\epsilon \to 0$ or $s \to \infty$. In other words, we are restricted to the motion on $[\pi - m(s,\epsilon), \pi + m(s,\epsilon)]$, the whole circle being recovered only in the limit of Dirac sequences built from fiducial vectors. Therefore systems like the classical pendulum or the torsion spring (where the angular motion is restricted) can be quantised without major issues. Is also shown that the lower symbol $\check{q}$ of $A_{\mathfrak{a}}$ can be made arbitrarily close to the values of the angle function $\mathfrak{a}(\alpha)$.

We found a (non-canonical) commutation rule between the angle operator and the momentum operator, as well as an expression for the uncertainty relation between them. The uncertainty relation with eigenstates of the momentum gives similar results to what one would expect working with (1.9).

In this work, we did not examine the question of the classical limit and related semi-classical approximations, like the link between Poisson brackets and commutators. It is in itself an appealing program. We prefer to postpone its careful study to a future work.

We finally observe that it is not possible to compare the coherent states for the motion on the circle that were presented in the work [30] with the ones being presented here, since they are of a different nature (a detailed description can be found in the Appendix B).

Quantum error probability with nonlinear CS.

We have studied the nonlinear CS, which are defined as superpositions of photon number states $|n\rangle$. The coefficients of that superposition are generated by the function $\mathcal{N}(t)$ defined in the equation (3.4). The power series expanssion of $\mathcal{N}(t)$ provides the sequence of positive numbers $\chi = \{x_0, x_1, \cdots\}$. If we want to construct a set of nonlinear CS, they must satisfy the moment condition (3.5). However, we did not provide a general solution for this problem. Therefore, the moment condition (3.5) must be verified for every example of nonlinear CS independently (as we do with the Example 1, in Subsection 3.5.3). We studied in particular, the nonlinear CS generated by the functions $\mathcal{N} \in \Sigma_+$ (the set Σ is presented in the Definition 3.3.2). With the function $\mathcal{N} \in \Sigma_+$ we can construct the asymmetric and symmetric deformations of the binomial distribution respectively. Using the Mandel parameter, we have shown that the photon distribution is super-Poissonian for those CS.

We studied the quantum error probability (or Helstrom bound) for binary communication using nonlinear CS. We considered the cases with perfect detection and imperfect detection respectively. For imperfect detection ($\eta < 1$), the photocounting distribution is associated with the binomial distribution (as we see in the expression (3.41)). In the work [53], an asymmetric deformation of the binomial distribution for the photocounting distribution was developed. In a similar way, we constructed here a symmetric deformation of the binomial distribution associated with the photocounting distribution. Using the photocounting distributions associated with the binomial distribution and their respective deformations, we compute the Helstrom bound with nonlinear CS for imperfect detection.

We have analyzed the optimization of the Helstrom bound for nonlinear CS in comparison with linear CS. This generates the inequality (3.69) for the optimization in the case of perfect detection. For nonlinear CS this inequality can be expressed in terms of the function $\mathcal{N}(t)$. For nonlinear CS generated by the functions $\mathcal{N} \in \Sigma_+$, we have shown that the optimization condition (3.69) cannot be fulfilled. Similar inequalities conditions were obtained in the case of imperfect detection. As in the previous case, for nonlinear CS generated by the functions $\mathcal{N} \in \Sigma_+$, we have shown that those inequalities cannot be fulfilled. Those results are illustrated in Subsection 3.5.3 where we have presented an example of CS generated by $\mathcal{N} \in \Sigma_+$: the q-exponential distribution. Given the parameter $s > 1$, as $s \to \infty$: the sequence χ^s generated by the q-exponential distribution approach $\mathbb{N}$ and $|\alpha, \chi^s\rangle \to |\alpha\rangle$. The states $|\alpha, \chi^s\rangle$, are super-Poissonian as expected. In [53], using nonlinear CS generated by the Delone sequences, the Helstrom bound for nonlinear CS was lowered in comparison with linear CS. The CS generated by those Delone sequences are shown to be sub-Poissonian. However, they cannot be associated with asymmetric or symmetric deformations of the binomial distribution with statistical interpretation (the expressions (3.20) and (3.28) cannot fulfill the condition $\mathfrak{p}_k^{(n)}(\eta) \geq 0$).

We also studied an example of another type of nonlinear CS in this work, the Susskind-

Glogower CS (Example 2, in Subsection 3.5.4). The Susskind-Glogower CS are particularly interesting for two reassons: they are sub-Poissonian (for a certain range of the parameter x (here we considered as $x = \mathrm{Re}\{\alpha\}$), as we can see in Figure 3.4), and they can be associated with a Hamiltonian that can be produced in ion-traps. Considering perfect detection, the inequality (3.119) was obtained as a condition for the optimization . As we can see in Figure 3.5, the Helstrom bound for Susskind-Glogower CS can be lowered in comparison with linear CS. In particular, the quantum error probability can be minimized for an expected value of photons $\langle n \rangle \approx 2$. The price we have to pay for this minimization is to fix the average value of incoming photons in the laser beam.

There are still some interesting topics left for a future work. We can search for other deformations of the binomial distribution in order to obtain CS with sub-Poissonian photon-statictics. The study of the extensivity for correlated systems described by asymetric deformations of the binomial distribution, since there appears to be a contraction of the phase space. A study of the Helstrom bound with Susskind-Glogower CS considering imperfect detection, as well as some physical realization of those CS in optical systems.

Bibliography

[1] Erwin Schrödinger. Der stetige übergang von der mikro-zur makromechanik. *Naturwissenschaften*, 14(28):664–666, 1926.

[2] Earle H Kennard. Zur quantenmechanik einfacher bewegungstypen. *Zeitschrift für Physik*, 44(4-5):326–352, 1927.

[3] Charles Galton Darwin. Free motion in the wave mechanics. *Proc. R. Soc. Lond. A*, 117(776):258–293, 1927.

[4] John R Klauder. Continuous-representation theory. i. postulates of continuous-representation theory. *Journal of Mathematical Physics*, 4(8):1055–1058, 1963.

[5] John R Klauder. Continuous-representation theory. ii. generalized relation between quantum and classical dynamics. *Journal of Mathematical Physics*, 4(8):1058–1073, 1963.

[6] Roy J Glauber. The quantum theory of optical coherence. *Physical Review*, 130(6):2529, 1963.

[7] Roy J Glauber. Rj glauber, phys. rev. 131, 2766 (1963). *Phys. Rev.*, 131:2766, 1963.

[8] ECG Sudarshan. Equivalence of semiclassical and quantum mechanical descriptions of statistical light beams. *Physical Review Letters*, 10(7):277, 1963.

[9] Robert Gilmore. Geometry of symmetrized states. *Annals of Physics*, 74(2):391–463, 1972.

[10] Askold Perelomov. Coherent states for arbitrary lie group. *Communications in Mathematical Physics*, 26(3):222–236, 1972.

[11] AO Barut and L Girardello. New coherent states associated with non-compact groups. *Communications in Mathematical Physics*, 21(1):41–55, 1971.

[12] Jean-Marc Lévy-Leblond. Who is afraid of nonhermitian operators? a quantum description of angle and phase. *Annals of Physics*, 101(1):319–341, 1976.

[13] K Kowalski, K Podlaski, and J Rembieliński. Quantum mechanics of a free particle on a plane with an extracted point. *Physical Review A*, 66(3):032118, 2002.

[14] Paul Adrien Maurice Dirac. The quantum theory of the emission and absorption of radiation. *Proc. R. Soc. Lond. A*, 114(767):243–265, 1927.

[15] P Carruthers and Michael Martin Nieto. Phase and angle variables in quantum mechanics. *Reviews of Modern Physics*, 40(2):411, 1968.

[16] Eric Galapon. Pauli's theorem and quantum canonical pairs: the consistency of a bounded, self–adjoint time operator canonically conjugate to a hamiltonian with non–empty point spectrum. In *Proceedings of the Royal Society of London A: Mathematical, Physical and Engineering Sciences*, volume 458, pages 451–472. The Royal Society, 2002.

[17] D Judge. On the uncertainty relation for lz and φ. *Physics Letters*, 5(3):189, 1963.

[18] Leonard Susskind and Jonathan Glogower. Quantum mechanical phase and time operator. *Physics Physique Fizika*, 1(1):49, 1964.

[19] EC Lerner, HW Huang, and GE Walters. Some mathematical properties of oscillator phase operators. *Journal of Mathematical Physics*, 11(5):1679–1684, 1970.

[20] John C Garrison and Jack Wong. Canonically conjugate pairs, uncertainty relations, and phase operators. *Journal of Mathematical Physics*, 11(8):2242–2249, 1970.

[21] A Galindo. Phase and number. *letters in mathematical physics*, 8(6):495–500, 1984.

[22] Evangelos K Ifantis. Abstract formulation of the quantum mechanical oscillator phase problem. *Journal of Mathematical Physics*, 12(6):1021–1026, 1971.

[23] Howard C Volkin. Phase operators and phase relations for photon states. *Journal of Mathematical Physics*, 14(12):1965–1976, 1973.

[24] F Rocca and M Sirugue. Phase operator and condensed systems. *Communications in Mathematical Physics*, 34(2):111–121, 1973.

[25] Felix A Berezin. Quantization. *Izvestiya: Mathematics*, 8(5):1109–1165, 1974.

[26] Hervé Bergeron and Jean Pierre Gazeau. Integral quantizations with two basic examples. *Annals of Physics*, 344:43–68, 2014.

[27] Syed Twareque Ali, Jean Pierre Antoine, and Jean Pierre Gazeau. *Coherent States, Wavelets, and Their Generalizations.* Springer Publishing Company, Incorporated, 2016.

[28] Jean Pierre Gazeau, Mario Cesar Baldiotti, and Rodrigo Fresneda. Three examples of covariant integral quantization. *PoS Proc. Sci*, 2014.

[29] Jean Pierre Gazeau and Franciszek Hugon Szafraniec. Three paths toward the quantum angle operator. *Annals of Physics*, 375:16–35, 2016.

[30] Isiaka Aremua, Jean Pierre Gazeau, and Mahouton Norbert Hounkonnou. Action-angle coherent states for quantum systems with cylindric phase space. *Journal of Physics A: Mathematical and Theoretical*, 45(33):335302, 2012.

[31] PL Garcia de Leon and Jean Pierre Gazeau. Coherent state quantization and phase operator. *Physics Letters A*, 361(4-5):301–304, 2007.

[32] Elmar Thoma. Die unitären darstellungen der universellen überlagerungsgruppe der bewegungsgruppe desr 2. *Mathematische Annalen*, 134(5):428–452, 1958.

[33] Naum IAkovlevich Vilenkin. *Special functions and the theory of group representations*, volume 22. American Mathematical Soc., 1978.

[34] S De Bièvre. Coherent states over symplectic homogeneous spaces. *Journal of mathematical physics*, 30(7):1401–1407, 1989.

[35] Charles P Boyer and Kurt Bernardo Wolf. Canonical transforms. iii. configuration and phase descriptions of quantum systems possessing an sl (2, r) dynamical algebra. *Journal of Mathematical Physics*, 16(7):1493–1502, 1975.

[36] CJ Isham. Relativity, groups and topology ii. *Proc. 1983 Les Houches Summer School ed B DeWitt and R Stora (Amsterdam: North-Holland) p*, 1059, 1984.

[37] Luis Miguel Nieto, Natig M Atakishiyev, Sergey M Chumakov, and Kurt Bernardo Wolf. Wigner distribution function for euclidean systems. *Journal of Physics A: Mathematical and General*, 31(16):3875, 1998.

[38] Hans A Kastrup. Quantization of the canonically conjugate pair angle and orbital angular momentum. *Physical Review A*, 73(5):052104, 2006.

[39] Diego Noguera. Quantum angle from e (2) coherent states quantization of motion on the circle. In *Physical and Mathematical Aspects of Symmetries*, pages 397–399. Springer, 2017.

[40] Rodrigo Fresneda, Jean Pierre Gazeau, and Diego Noguera. Quantum localisation on the circle. *Journal of Mathematical Physics*, 59(5):052105, 2018.

[41] Kedar S Ranade. Functional analysis and quantum mechanics: an introduction for physicists. *Fortschritte der Physik*, 63(9-10):644–658, 2015.

[42] Carl W Helstrom. Quantum detection and estimation theory (new york: Academic). 1976.

[43] Robert L Cook, Paul J Martin, and John M Geremia. Optical coherent state discrimination using a closed-loop quantum measurement. *Nature*, 446(7137):774–777, 2007.

[44] Jan Perina. *Quantum statistics of linear and nonlinear optical phenomena*. Springer Science & Business Media, 1991.

[45] RL de Matos Filho and W Vogel. Nonlinear coherent states. *Physical Review A*, 54(5):4560, 1996.

[46] VI Man'ko, G Marmo, ECG Sudarshan, and F Zaccaria. f-oscillators and nonlinear coherent states. *Physica Scripta*, 55(5):528, 1997.

[47] M Arik and Darryl D Coon. Hilbert spaces of analytic functions and generalized coherent states. *Journal of Mathematical Physics*, 17(4):524–527, 1976.

[48] VI Man'ko, G Marmo, S Solimeno, and F Zaccaria. Physical nonlinear aspects of classical and quantum q-oscillators. *International Journal of Modern Physics A*, 8(20):3577–3597, 1993.

[49] Jacob Katriel and Allan I Solomon. Nonideal lasers, nonclassical light, and deformed photon states. *Physical Review A*, 49(6):5149, 1994.

[50] S Sivakumar. Photon-added coherent states as nonlinear coherent states. *Journal of Physics A: Mathematical and General*, 32(18):3441, 1999.

[51] Xiao-Guang Wang and Hong-Chen Fu. Negative binomial states of the radiation field and their excitations are nonlinear coherent states. *Modern physics letters B*, 13(18):617–623, 1999.

[52] Z Kis, W Vogel, and L Davidovich. Nonlinear coherent states of trapped-atom motion. *Physical Review A*, 64(3):033401, 2001.

[53] EMF Curado, JP Gazeau, and Ligia MCS Rodrigues. Nonlinear coherent states for optimizing quantum information. *Physica Scripta*, 82(3):038108, 2010.

[54] EMF Curado, JP Gazeau, and Ligia MCS Rodrigues. On a generalization of the binomial distribution and its poisson-like limit. *Journal of Statistical Physics*, 146(2):264–280, 2012.

[55] H Bergeron, EMF Curado, JP Gazeau, and Ligia MCS Rodrigues. Generating functions for generalized binomial distributions. *Journal of Mathematical Physics*, 53(10):103304, 2012.

[56] H Bergeron, EMF Curado, JP Gazeau, and Ligia MCS Rodrigues. Symmetric generalized binomial distributions. *Journal of Mathematical Physics*, 54(12):123301, 2013.

[57] H Bergeron, EMF Curado, JP Gazeau, and Ligia MCS Rodrigues. Symmetric deformed binomial distributions: An analytical example where the boltzmann-gibbs entropy is not extensive. *Journal of Mathematical Physics*, 57(2):023301, 2016.

[58] R de J León-Montiel, H Moya-Cessa, and F Soto-Eguibar. Nonlinear coherent states for the susskind-glogower operators. *Revista Mexicana de Física*, 57(3):133–147, 2011.

[59] Askold Perelomov. *Generalized coherent states and their applications*. Springer Science & Business Media, 2012.

[60] René L Schilling. *Measures, integrals and martingales*. Cambridge University Press, 2006.

[61] Elliott H Lieb. The classical limit of quantum spin systems.

[62] Samuel G Duyker, Vanessa K Peterson, Gordon J Kearley, Andrew J Studer, and Cameron J Kepert. Extreme compressibility in lnfe (cn) 6 coordination framework materials via molecular gears and torsion springs. *Nature chemistry*, 8(3):270, 2016.

[63] Stephan De Bieèvre and José A Gonález. Semiclassical behaviour of coherent states on the circle. In *Quantization and Coherent States Methods*, pages 152–157. World Scientific, 1993.

[64] K Kowalski, J Rembielinski, and LC Papaloucas. Coherent states for a quantum particle on a circle. *Journal of Physics A: Mathematical and General*, 29(14):4149, 1996.

[65] José A González and Mariano A Del Olmo. Coherent states on the circle. *Journal of Physics A: Mathematical and General*, 31(44):8841, 1998.

[66] Brian C Hall and Jeffrey J Mitchell. Coherent states on spheres. *Journal of Mathematical Physics*, 43(3):1211–1236, 2002.

[67] K Kowalski and J Rembieliński. Exotic behaviour of a quantum particle on a circle. *Physics Letters A*, 293(3-4):109–115, 2002.

[68] K Kowalski and J Rembieliński. On the uncertainty relations and squeezed states for the quantum mechanics on a circle. *Journal of Physics A: Mathematical and General*, 35(6):1405, 2002.

[69] DA Trifonov. Comment on on the uncertainty relations and squeezed states for the quantum mechanics on a circle. *Journal of Physics A: Mathematical and General*, 36(8):2197, 2003.

[70] K Kowalski and J Rembielinski. Reply to the comment on"on the uncertainty relations and squeezed states for the quantum mechanics on a circle". *arXiv preprint quant-ph/0305069*, 2003.

[71] PL Kelley and WH Kleiner. Theory of electromagnetic field measurement and photoelectron counting. *Physical Review*, 136(2A):A316, 1964.

[72] Héctor Manuel Moya-Cessa and Francisco Soto-Eguibar. Introduction to quantum optics. *Differential equations*, 2011.

[73] S Wallentowitz, W Vogel, and PL Knight. High-order nonlinearities in the motion of a trapped atom. *Physical Review A*, 59(1):531, 1999.

Appendix A

$B_n(q)$ Functions

The following functions are referred to in subsection 2.2.4, and are given terms of the $f_{j;m}(q)$'s defined in (2.54),

$$B_1(q) = \frac{1}{\kappa^2 c_1(\eta,\gamma)} f_{0;3}(q)\,, \tag{A.1}$$

$$B_2(q) = \frac{2}{\kappa^2 c_1(\eta,\gamma)} f_{0;3}(q)\sin(\zeta - q)\,, \tag{A.2}$$

$$\begin{aligned} B_3(q) = \frac{1}{\kappa^2 c_1(\eta,\gamma)} & \left[f_{2;3}(q) + 3f_{1;4}(q)\cos(\gamma - q)\right. \\ & \left. +3f_{0;5}(q)(\cos(\gamma - q))^2 + f_{0;3}(q)\right]\,, \end{aligned} \tag{A.3}$$

$$B_4(q) = \frac{1}{\kappa^2 c_1(\eta,\gamma)} (\sin(\zeta - q))^2 f_{0;3}(q)\,, \tag{A.4}$$

$$B_5(q) = \frac{1}{\kappa c_1(\eta,\gamma)} f_{0;2}(q)\,, \tag{A.5}$$

$$B_6(q) = \frac{1}{\kappa c_1(\eta,\gamma)} \sin(\zeta - q)\, f_{0;2}(q)\,. \tag{A.6}$$

Appendix B

Other quantum angle for cylindric phase space

In this appendix, we give a summary of the work [30] where other coherent states for the motion on the circle and their associated integral quantisation quantisation were presented.

B.1 Other coherent states

We start with the cylindric phase space $\mathbb{R} \times [0, 2\pi] = \{(p, q), \mid p \in \mathbb{R}\,, 0 \leq q < 2\pi\}$, equipped with the measure $\frac{1}{2\pi}\,\mathrm{d}p\,\mathrm{d}q$. We introduce a probability distribution on the range of the variable p. It is a non-negative, even, well localized and normalized integrable function

$$\mathbb{R} \ni p \mapsto w^{\sigma}(p)\,, \quad w^{\sigma}(p) = w^{\sigma}(-p)\,, \quad \int_{-\infty}^{+\infty} \mathrm{d}p\, w^{\sigma}(p) = 1\,, \tag{B.1}$$

where $\sigma > 0$ is a width parameter. This function must obey the following conditions:

Conditions B.1.1. *(i)* $0 < \mathcal{N}^{\sigma}(p) \stackrel{\text{def}}{=} \sum_{n\in\mathbb{Z}} w_n^{\sigma}(p) < \infty$ *for all* $p \in \mathbb{R}$, *where*

$$w_n^{\sigma}(p) \stackrel{\text{def}}{=} w^{\sigma}(p-n)\,. \tag{B.2}$$

(ii) the Poisson summation formula is applicable to $\mathcal{N}^{\sigma}$*:*

$$\mathcal{N}^{\sigma}(p) = \sum_{n\in\mathbb{Z}} w_n^{\sigma}(p) = \sqrt{2\pi} \sum_{n\in\mathbb{Z}} \mathrm{e}^{-2\pi \mathrm{i} n p}\, \widehat{w^{\sigma}}(2\pi n)\,,$$

where $\widehat{w^\sigma}$ *is the Fourier transform of* w^σ*,*

(iii) its limit at $\sigma \to 0$*, in a distributional sense, is the Dirac distribution:*

$$w^\sigma(p) \underset{\sigma\to 0}{\to} \delta(p)\,,$$

(iv) the limit at $\sigma \to \infty$ *of its Fourier transform is proportional to the characteristic function of the singleton* $\{0\}$*:*

$$\widehat{w^\sigma}(k) \underset{\sigma\to\infty}{\to} \frac{1}{\sqrt{2\pi}}\,\delta_{k0}\,,$$

(v) considering the overlap matrix *of the two distributions* $p \mapsto w^\sigma_n(p)$*,* $p \mapsto w^\sigma_{n'}(p)$ *with matrix elements,*

$$w^\sigma_{n,n'} = \int_{-\infty}^{+\infty} \mathrm{d}p\, \sqrt{w^\sigma_n(p)\, w^\sigma_{n'}(p)} \leq 1\,,$$

we impose the two conditions

$$w^\sigma_{n,n'} \to 0 \quad \textit{as} \quad n - n' \to \infty \quad \textit{at fixed } \sigma\,, \tag{a}$$

$$\exists\, n_M \geq 1 \quad \textit{such that} \quad w^\sigma_{n,n'} \underset{\sigma\to\infty}{\to} 1 \quad \textit{provided } |n-n'| \leq n_M\,. \tag{b}$$

Properties (ii) and (iv) entail that $\mathcal{N}^\sigma(p) \underset{\sigma\to\infty}{\to} 1$. Also note the properties of the overlap matrix elements $w^\sigma_{n,n'}$ due to the properties of w^σ:

$$w^\sigma_{n,n'} = w^\sigma_{n',n} = w^\sigma_{0,n'-n} = w^\sigma_{-n,-n'}\,, \quad w^\sigma_{n,n} = 1 \quad \forall\, n,n' \in \mathbb{Z}\,.$$

The most immediate choice for $w^\sigma(p)$ is Gaussian, i.e. $w^\sigma(p) = \sqrt{\frac{1}{2\pi\sigma^2}}\,\mathrm{e}^{-\frac{1}{2\sigma^2}p^2}$ (for which the n_M in (b) is ∞) Let us now introduce the weighted Fourier exponentials:

$$\phi_n(p,\alpha) = \sqrt{w^\sigma_n(p)}\,\mathrm{e}^{\mathrm{i}n\alpha}\,, \quad n \in \mathbb{Z}\,.$$

These functions form the countable orthonormal system in $L^2(\mathbb{S}^1 \times \mathbb{R}, \mathrm{d}p\,\mathrm{d}q/2\pi)$ needed to construct coherent states in agreement with a general procedure explained, for instance, in

[27]. Let $\mathcal{H}$ be a Hilbert space with orthonormal basis $\{|e_n\rangle \,|\, n \in \mathbb{Z}\}$, e.g. $\mathcal{H} = L^2(\mathbb{S}^1, \mathrm{d}\alpha\}$ with $e_n(\alpha) = \frac{1}{\sqrt{2\pi}} e^{in\alpha}$. The correspondent family of coherent states on the circle reads as:

$$|p, q\rangle = \frac{1}{\sqrt{\mathcal{N}^\sigma(p)}} \sum_{n\in\mathbb{Z}} \sqrt{w_n^\sigma(p)}\ \mathrm{e}^{-\mathrm{i}nq}\, |e_n\rangle\,. \tag{B.3}$$

These states are normalized and resolve the unity. They overlap as:

$$\langle p, q|p', q'\rangle = \frac{1}{\sqrt{\mathcal{N}^\sigma(p)\,\mathcal{N}^\sigma(p)}} \sum_{n\in\mathbb{Z}} \sqrt{w_n^\sigma(p)\, w_n^\sigma(p')}\ \mathrm{e}^{-\mathrm{i}n(q-q')}\,.$$

The function $w^\sigma(p)$ gives rise to a double probabilistic interpretation [27]:

- For all p viewed as a shape parameter, there is the discrete distribution,

$$\mathbb{Z} \ni n \mapsto |\langle e_n|p, \alpha\rangle|^2 = \frac{w_n^\sigma(p)}{\mathcal{N}^\sigma(p)}\,. \tag{B.4}$$

  This probability, of genuine quantum nature, concerns experiments performed on the system described by the Hilbert space $\mathcal{H}$ within some experimental protocol, in order to measure the spectral values of a self-adjoint operator acting in $\mathcal{H}$ and having the discrete spectral resolution $\sum_n a_n|e_n\rangle\langle e_n|$. For $a_n = n$ this operator is the number or quantum angular momentum operator.

- For each n, there is the continuous distribution on the cylinder $\mathbb{R}\times\mathbb{S}^1$ (reps. on $\mathbb{R}$) equipped with its measure $\mathrm{d}p\,\mathrm{d}q/2\pi$ (resp. $\mathrm{d}p$),

$$(p, q) \mapsto |\phi_n(p, q)|^2 = w_n^\sigma(p) \quad (\text{resp.} \quad \mathbb{R} \ni p \mapsto w_n^\sigma(p))\,. \tag{B.5}$$

  This probability, of classical nature and uniform on the circle, determines the CS quantisation of functions of p.

B.2 CS quantisation

By virtue of the CS quantisation scheme, the quantum operator (acting on $\mathcal{H}$) associated with functions $f(p, q)$ on the cylinder is obtained through

$$A_f := \int_{\mathbb{R}\times[0,2\pi]} f(p, q)|p, q\rangle\langle p, q|\, \mathcal{N}^\sigma(p)\, \frac{\mathrm{d}p\,\mathrm{d}q}{2\pi} = \sum_{n,n'} (A_f)_{nn'}\ |e_n\rangle\langle e_{n'}|\,, \tag{B.6}$$

where

$$(A_f)_{nn'} = \int_{-\infty}^{+\infty} \mathrm{d}p \, \sqrt{w_n^\sigma(p)\, w_{n'}^\sigma(p)}\, \frac{1}{2\pi} \int_0^{2\pi} \mathrm{d}q \; \mathrm{e}^{-\mathrm{i}(n-n')q} \; f(p,q)\,. \tag{B.7}$$

The lower symbol of f is given by:

$$\check{f}(p,q) = \langle p,q|A_f|p,q\rangle = \int_{-\infty}^{+\infty} \mathrm{d}p' \int_0^{2\pi} \frac{\mathrm{d}q'}{2\pi} \mathcal{N}^\sigma(p')\, f(p',q')\, |\langle p,q|p',q'\rangle|^2\,. \tag{B.8}$$

If f is depends on p only, $f(p,q) \equiv v(p)$, then A_f is diagonal with matrix elements that are w^σ transforms of $v(p)$:

$$\left(A_{v(p)}\right)_{nn'} = \delta_{nn'} \int_{-\infty}^{+\infty} \mathrm{d}p\, w_n^\sigma(p)\, v(p) = \delta_{nn'} \langle v\rangle_{w_n^\sigma}\,,$$

where $\langle\cdot\rangle_{w_n^\sigma}$ designates the mean value w.r.t. the distribution $p \mapsto w_n^\sigma(p)$. For the most basic case, $v(p) = p$, our assumptions on w^σ give

$$A_p = \int_{\mathbb{S}^1\times\mathbb{R}} \frac{\mathrm{d}p\,\mathrm{d}\alpha}{2\pi} \mathcal{N}^\sigma(p)\, p\, |p,\alpha\rangle\langle p,\alpha| = \sum_{n\in\mathbb{Z}} n\, |e_n\rangle\langle e_n| = N\,. \tag{B.9}$$

This is nothing but the number or angular momentum operator (in unit $\hbar = 1$), which reads $A_p = -\mathrm{i}\partial/\partial\alpha$ in angular position representation, i.e. when $\mathcal{H}$ is chosen as $L^2(\mathbb{S}^1, \mathrm{d}\alpha)$.

Let us define the unitary representation $\theta \mapsto U_{\mathbb{S}^1}(\theta)$ of $\mathbb{S}^1$ on $\mathcal{H}$ as the diagonal operator $U_{\mathbb{S}^1}(\theta)|e_n\rangle = \mathrm{e}^{in\theta}\,|e_n\rangle$, i.e. $U_{\mathbb{S}^1}(\theta) = \mathrm{e}^{\mathrm{i}\theta N}$. We easily infer from the straightforward covariance property of the coherent states :

$$U_{\mathbb{S}^1}(\theta)|p,q\rangle = |p, q-\theta\rangle\,,$$

the rotational covariance of A_f itself,

$$U_{\mathbb{S}^1}(\theta) A_f U_{\mathbb{S}^1}(-\theta) = A_{T^{-1}(\theta)f}\,,$$

where $T^{-1}(\theta)f(\alpha) \stackrel{\mathsf{def}}{=} f(\alpha+\theta)$.

If f depends on q only, $f(p,q) = u(q)$, we have

$$A_{u(q)} = \int_{\mathbb{R}\times[0,2\pi]} \frac{\mathrm{d}p\,\mathrm{d}q}{2\pi} \mathcal{N}^\sigma(p) v(q)\, |p,q\rangle\langle p,q| \tag{B.10}$$

$$= \sum_{n,n'\in\mathbb{Z}} w_{n,n'}^\sigma\, c_{n-n'}(v) |e_n\rangle\langle e_{n'}|\,, \tag{B.11}$$

where $c_n(v)$ is the nth Fourier coefficient of v. In particular, we have the angle operator corresponding to the 2π-periodic angle function $\mathcal{a}(q)$ previously defined as the periodic extension of $\mathcal{a}(q) = q$ for $0 \leq q < 2\pi$

$$A_{\mathcal{a}} = \pi I + \mathrm{i} \sum_{n \neq n'} \frac{w^{\sigma}_{n,n'}}{n - n'} \, |e_n\rangle\langle e_{n'}| \, , \tag{B.12}$$

This operator is bounded self-adjoint. Its covariance property is

$$U_{\mathbb{S}^1}(\theta) A_{\mathcal{a}} U_{\mathbb{S}^1}(-\theta) = A_{\mathcal{a}} + (\theta \operatorname{mod}(2\pi)) I \, . \tag{B.13}$$

Note the operator corresponding to the elementary Fourier exponential,

$$A_{\mathrm{e}^{\pm \mathrm{i}q}} = w^{\sigma}_{1,0} \sum_{n} |e_{n\pm 1}\rangle\langle e_n| \, , \quad A^{\dagger}_{\mathrm{e}^{\pm \mathrm{i}q}} = A_{\mathrm{e}^{\mp \mathrm{i}q}} \, . \tag{B.14}$$

We remark that $A_{\mathrm{e}^{\pm \mathrm{i}q}} A^{\dagger}_{\mathrm{e}^{\pm \mathrm{i}q}} = A^{\dagger}_{\mathrm{e}^{\pm \mathrm{i}q}} A_{\mathrm{e}^{\pm \mathrm{i}q}} = (w^{\sigma}_{1,0})^2 1_d$. Therefore this operator fails to be unitary. It is "asymptotically" unitary at large σ since the factor $(w^{\sigma}_{1,0})^2$ can be made arbitrarily close to 1 at large σ as a consequence of Requirement (b). In the Fourier series realization of $\mathcal{H}$, for which the kets $|e_n\rangle$ are the Fourier exponentials $\mathrm{e}^{\mathrm{i}\, n\alpha}/\sqrt{2\pi}$, the operators $A_{\mathrm{e}^{\pm \mathrm{i}q}}$ are multiplication operators by $\mathrm{e}^{\pm \mathrm{i}\alpha}$ up to the factor $w^{\sigma}_{1,0}$. Finally, the commutator of angular momentum and angle operators is given by the expansion

$$[A_p, A_{\mathcal{a}}] = \mathrm{i} \sum_{n \neq n'} w^{\sigma}_{n,n'} \, |e_n\rangle\langle e_{n'}| \, . \tag{B.15}$$

One observes that the overlap matrix completely encodes this basic commutator. Because of the required properties of the distribution w^{σ} the departure of the r.h.s. of (B.15) from the canonical r.h.s. $-\mathrm{i}I$ can be bypassed by examining the behavior of the lower symbols at large σ. For an original function depending on q only we have the Fourier series

$$\check{f}(p_0, q_0) = \langle p_0, q_0 | A_f | p_0, q_0 \rangle = c_0(f) + \sum_{m \neq 0} d^{\sigma}_m(p_0) \, w^{\sigma}_{0,m} \, c_m(f) \, \mathrm{e}^{\mathrm{i} m q_0} \, , \tag{B.16}$$

with

$$d^{\sigma}_m(p) = \frac{1}{\mathcal{N}^{\sigma}(p)} \sum_{r=-\infty}^{+\infty} \sqrt{w^{\sigma}_r(p) w^{\sigma}_{m+r}(p)} \leq 1 \, , \tag{B.17}$$

the last inequality resulting from Condition (i) and Cauchy-Schwarz inequality. If we further impose the condition that $d^\sigma_m(p) \to 1$ uniformly as $\sigma \to +\infty$, then the lower symbol $\check{u}(p_0, q_0)$ tends to the Fourier series of the original function $u(q)$. A similar result is obtained for the lower symbol of the commutator (B.15):

$$\langle p_0, q_0 | [A_p, A_{\mathfrak{a}}] | p_0, q_0 \rangle = \mathrm{i} \sum_{m \neq 0} d^\sigma_m(p_0)\, w^\sigma_{0,m}\, \mathrm{e}^{\mathrm{i} m q_0} \,. \tag{B.18}$$

Therefore, with the condition that $d^\sigma_m(p) \to 1$ uniformly as $\sigma \to \infty$, we obtain at this limit the result similar to (**??**),

$$\langle p_0, q_0 | [A_p, A_{\mathfrak{a}}] | p_0, q_0 \rangle \underset{\sigma \to \infty}{\to} -\mathrm{i} + \mathrm{i} \sum_m \delta(q_0 - 2\pi m) \,. \tag{B.19}$$

So we asymptotically (almost) recover the classical canonical commutation rule except for the singularity at the origin mod 2π, a logical consequence of the discontinuities of the saw function $\mathfrak{a}(q)$ at these points.