NOTAS DE FÍSICA

SUPLEMENTO AO VOLUME VII

ON THE INTERACTION OF THE ELEMENTARY PARTICLES

(K-Meson-Deuteron Inelastic and Charge-Exchange Scattering)

by

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A thesis presented for the degree of Doctor of Philosophy of the University of London

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ABSTRACT

The formal expressions of the scattering operators for meson-deuteron processes are expanded in terms of two-particle scattering operators. The terms of this expansion which represent first and second order processes are explicitly evaluated. Formulae for inelastic ($K^+d \rightarrow K^+np$) and charge-exchange ($K^+d \rightarrow K^o pp$) processes are written in terms of the parameters describing the two-particle processes involved. It is assumed that the K meson-nucleon interactions in both I = 1 and I = 0 isotopic spin states are purely S-wave. Comparison of the calculated differential cross-sections with the experimental results seems to confirm this assumption.

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PREFACE

The work described in this thesis was carried out in the Department of Mathematics, Imperial College of Science and Technology, University of London, between January 1959 and June 1960, under the supervision of Professor A. Salam.

Except where stated in the text, the material contained in this thesis is original and has not previously been presented for a degree in this or any other university.

The author wishes to thank Professor Abdus Salam and Dr. P.T. Matthews for encouragement, interest and advice during the pursuit of this work and other research carried out during his stay at Imperial College. The author had many helpful discussions with Dr. S. Kamefuchi, Dr. I.R. Gatland, Dr. Y. Yamaguchi and Drs. G. and S. Goldhaber.

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CHAPTER I

INTRODUCTION

In the experimental study of the interactions of the strongly interacting elementary particles, protons are in a very special situation. The hydrogen atom is a neutral system consisting of only protons and of the much more weakly interacting electrons. Due to the small mass of the electron, the proton is nearly at rest in the system of the centre of mass of the atom. Thus, for the purpose of the study of the strong interactions, a target consisting of hydrogen atoms is almost the same as an ideal target consisting of free protons at rest. In a hydrogen bubble chamber the interaction of all the elementary particles with protons can be studied with accuracy.

On the other hand, mesons and hyperons are not stable, and are not known to form any stable system. We do not know how to make targets containing these particles. Interactions among them can at present be studied only by indirect means. They can be produced and beams formed with them, so that before they decay they can be made to interact with stable systems.

The neutron falls between these two extremes. It is not stable but, together with protons, neutrons form stable systems. With the exception of hydrogen, any kind of common matter can be used as a target containing neutrons, with which other particles can be made to interact.

But it is not simple to understand and interpret in terms of elementary interactions the experiments in which beams of particles are scattered by matter in general. Neutrons and protons are closely packed together to form most of the nuclei, so that the interaction of the incident particle with only one of the nucleons of the nucleus without the others strongly participating is almost impossible. During and after the interaction of the incident particle with one of the nucleons of the nucleus, this nucleon will interact strongly with the other nucleons. The incident particle itself will very likely interact with two or more nucleons at a time, or at least suffer multiple scattering, since the scattering centres (the nucleons) are so close to each other. On the other hand, the nucleons are not at rest inside the nucleus, and their motion should be known if properties of the elementary two-particle interactions are to be used or deduced. This knowledge is not available for most of the nuclei.

In studying the experiments in which beams of particles are scattered by heavy or medium nuclei, these are considered as acting through an average effective potential. It is the collective behaviour of the nucleons which is of primary importance rather than their individual properties.

The deuteron is a rather special system among the nuclei. It consists of a single neutron loosely bound to a single proton. The two nucleons in the deuteron are separated by a relatively large distance, so that the incident particle probably interacts strongly with only one nucleon at a time. If the interaction between the incident particle and one nucleon lasts only a relatively short time, the presence of the second nucleon will not affect much the state of motion of the first nucleon during this interval of time, and the characteristics of the two-particle interaction will be approximately obeyed. Also, the equations of the motion of the nucleons in the deuteron are quite well known. Thus the deuteron provides us with a set of circumstances that may enable us to approximately describe scattering events in terms of a series of two-particle interactions. These properties of the deuteron were first recognized by G.F. Chew (Ref. 1) who introduced what is called the Impulse Approximation to treat the problem of scattering on deuterons.

In the Impulse Approximation the scattering amplitude for a complex nucleus is represented as a superposition of scattering amplitudes for free nucleons which have the same momentum distribution as the actual bound nucleons. It is based on the following accumptions.

I. The range of the forces between the incident particles and the nucleons is shorter than the average distance between two nucleons.

II. The nucleus is rather "transparent" to the incident wave so that the amplitude falling on each nucleon is approximately the same as if the nucleon were alone.

III. Multiple scattering processes have small probability.

IV. The forces that bind the nucleons only have the effect of giving to each nucleon a certain distribution of momentum.

The conditions of applicability of the Impulse Approximation were qualitatively discussed by several authors (Refs. 1 - 3). G.F. Chew and M.L. Goldberger (Ref. 4) expanded the formal expression for the transition probability for elastic scattering of a particle by a complex nucleus in terms of two-particle scattering amplitudes and showed how the terms corresponding to the Impulse Approximation appear naturally in this case. They left open the question of the quantitative estimate of the corrections to the Impulse Approximation. S. Fernbach, T.A. Green and K.M. Watson (Ref. 5) applied the Impulse Approximation to pion-deuteron scattering, and related the amplitudes for the elementary pion-nucleon processes to the pion-deuteron elastic and inelastic cross-section. R.M. Rockmore (Ref. 6) treated the same pion-deuteron problem using a phase-shift expansion for the pion-nucleon amplitudes. Comparison of Rockmore's analysis with experimental results on pion-deuteron scattering (Ref. 7) verified the reliability of the use of the Impulse Approximation in this problem. Later the Impulse Approximation was applied to K meson-deuteron scattering by M. Gourdin and A. Martin (Ref. 8) and by the present author (Ref. 9). Then M. Gourdin and A. Martin (Ref. 10) improved the calculation of the inelastic and charge exchange K^+-deuteron scattering by taking into account the interaction of the two nucleons in the final state by analogy with the case of the photodisintegration of the deuteron.

We intend here to make a more complete quantitative analysis of the meson-deuteron inelastic scattering. We start by writing the formal expressions for the scattering amplitudes for inelastic scattering of a particle by a complex nucleus (Chapter II). This amplitude is then expanded in terms of two-particle scattering amplitudes (Chapter III), and physical meaning is given to the several terms of this expansion. For the case of the deuteron the terms representing double scattering of the incident meson and those representing a meson-nucleon scattering followed by a nucleon-nucleon collision are explicitly written in Chapter IV, where the contributions from the nucleon spin variables are discussed, and expressions for the cross-section are given. In Chapter V a complete evaluation of the terms of the meson scattering amplitude representing double scattering and nucleon-nucleon interaction in final state is made for the case of inelastic K meson-deuteron scattering. Differential and total cross-section for K^+-deuteron inelastic and charge exchange scattering are calculated and discussed in Chapter V. In Chapter VI an analysis is made of the experimental results on K^+-deuteron interactions available at present.

We must now emphasize the importance of this analysis of the K^+-deuteron experiments.

According to the present ideas, the π^+, π^0 and π^- mesons are the components of a triplet of isotopic spin I = 1 and protons and neutrons are the components of a doublet of isotopic spin $I = \frac{1}{2}$. Thus, for the pion-nucleon system the possible values of the total isotopic spin are then $I = {}^3\!/_2$ and $I = \frac{1}{2}$. Assuming charge independence (i.e. conservation of total isotopic spin) all the pion-nucleon interactions can be described by the two amplitudes corresponding to these two isotopic spin states.

Experiments on the scattering of beams of the two charged pions by protons are in principle sufficient to give complete information on these two amplitudes. Thus, if charge independence in the strong interactions is a valid concept, experiments on scattering of pions by neutrons (i.e. by deuterons) do not give any additional information (though they can serve as a direct check on charge independence).

However, the situation is not the same in the K meson-nucleon system. K^+ and K^0 mesons being the two components of an $I = \frac{1}{2}$ doublet, the K meson-nucleon system has two possible values of the total isotopic spin, namely $I = 1$ and $I = 0$. Assuming charge independence, two independent amplitudes are sufficient to describe all the K meson-nucleon processes. Now the K^+ proton interaction can only give information on the scattering amplitude for the isotopic spin $I = 1$ state so that, since experiments with K^0 mesons are not easy to perform, access to the $I = 0$ state must be made through K^+ neutron processes. We have to make experiments with beams of K^+ mesons incident on deuterons, apply the results of the analysis in terms of the two-particle processes, and extract the values of the required amplitude.

<u>The Structure of the Deuteron</u>

The possibility of success by this method is dependent on our having a fairly good knowledge of the structure of the neutron-proton bound state. In the article on The Two-Nucleon Problem written by L. Hulthèn and M. Sugawara for the Encyclopedia of Physics (Ref. 11) the structure of the deuteron is described and discussed. We detail here the properties which will be needed later.

The deuteron binding energy is 2.226 MeV. The spin is 1. The ground state is mainly an S-state, with a small admixture (3 or 4% in the probability) of D-state. The momentum spectra of the nucleons in the S-and D-states is of almost the same shape. Also, there is a correlation between the direction of the deuteron spin and the direction of the orbital angular momentum in the D-state. But unless there is a strong spin dependence of the meson-nucleon interaction, the presence of the D-state can be ignored for our purposes. The small value of the binding energy indicates that the deuteron is a rather diffuse, loosely bound structure. It is accepted that the $^3S_1$ ground state is well represented by the Hulthèn wave function

$$\psi_D(r) = N\left(e^{-\alpha r} - e^{-\beta r}\right)\Big/ r \tag{I.1}$$

where r is the internucleon distance, and

$$\begin{aligned} \alpha &= 45.7 \text{ MeV}/\hbar c \\ \beta &= 7\alpha \, . \end{aligned} \tag{I.2}$$

The normalization constant is

$$N = \left[\alpha\beta(\alpha+\beta)\Big/2\pi(\beta-\alpha)^2\right]^{\frac{1}{2}} \tag{I.3}$$

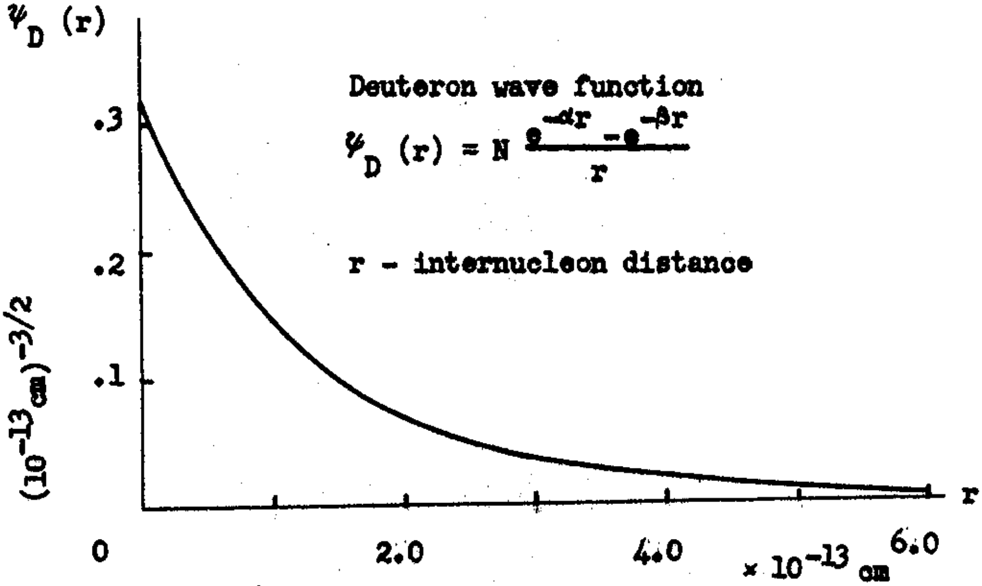
The momentum space representation of Eq. (I.1) is

$$\Phi_D(\ell) = N\sqrt{\frac{2}{\pi}}\left[\frac{1}{\alpha^2+\ell^2} - \frac{1}{\beta^2+\ell^2}\right] \tag{I.4}$$

where ℓ is the momentum common to both the nucleons in the rest system of the deuteron.

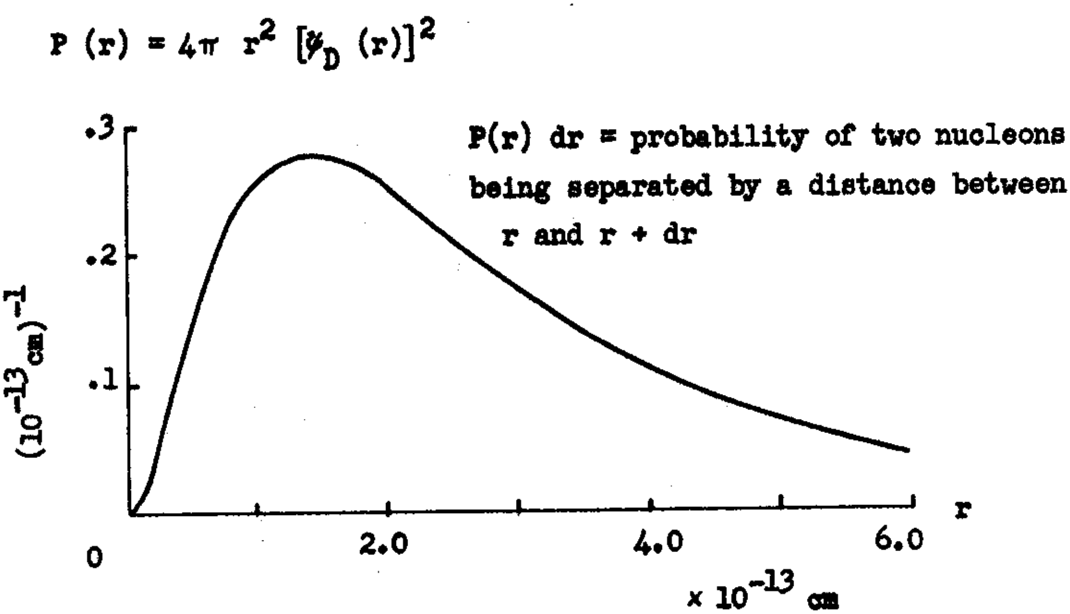
Figures I.1 and I.3 represent the wave functions of Eqs. (I.1) and (I.4), and Figs. I.2 and I.3 represent their corresponding probabilities.

* * *



ψD (r)
Deuteron wave function
ψD (r) = N (e^-αr - e^-βr)/r
r - internucleon distance
.3
.2
.1
(10^-13 cm)^-3/2
0
2.0
4.0
6.0
r
× 10^-13 cm

Fig. I.1



P (r) = 4π r² [ψD (r)]²
P(r) dr = probability of two nucleons being separated by a distance between r and r + dr
.3
.2
.1
(10^-13 cm)^-1
0
2.0
4.0
6.0
r
× 10^-13 cm

Fig. I.2

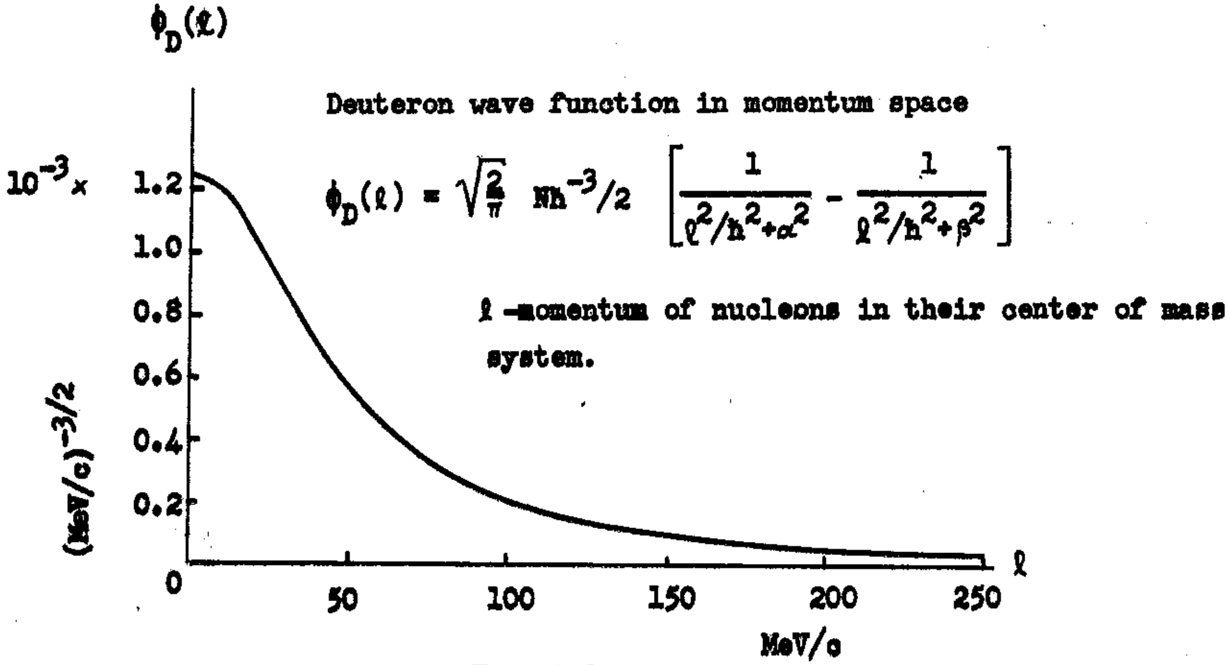

$\phi_D(\ell)$
Deuteron wave function in momentum space
$\phi_D(\ell) = \sqrt{\frac{2}{\pi}}\ N h^{-3}/2 \left[\frac{1}{\ell^2/h^2+\alpha^2} - \frac{1}{\ell^2/h^2+\beta^2}\right]$
ℓ -momentum of nucleons in their center of mass system.
$10^{-3}\times$
1.2
1.0
0.8
0.6
0.4
0.2
0
$(\mathrm{MeV/c})^{-3/2}$
50
100
150
200
250
ℓ
MeV/c

Fig. I.3

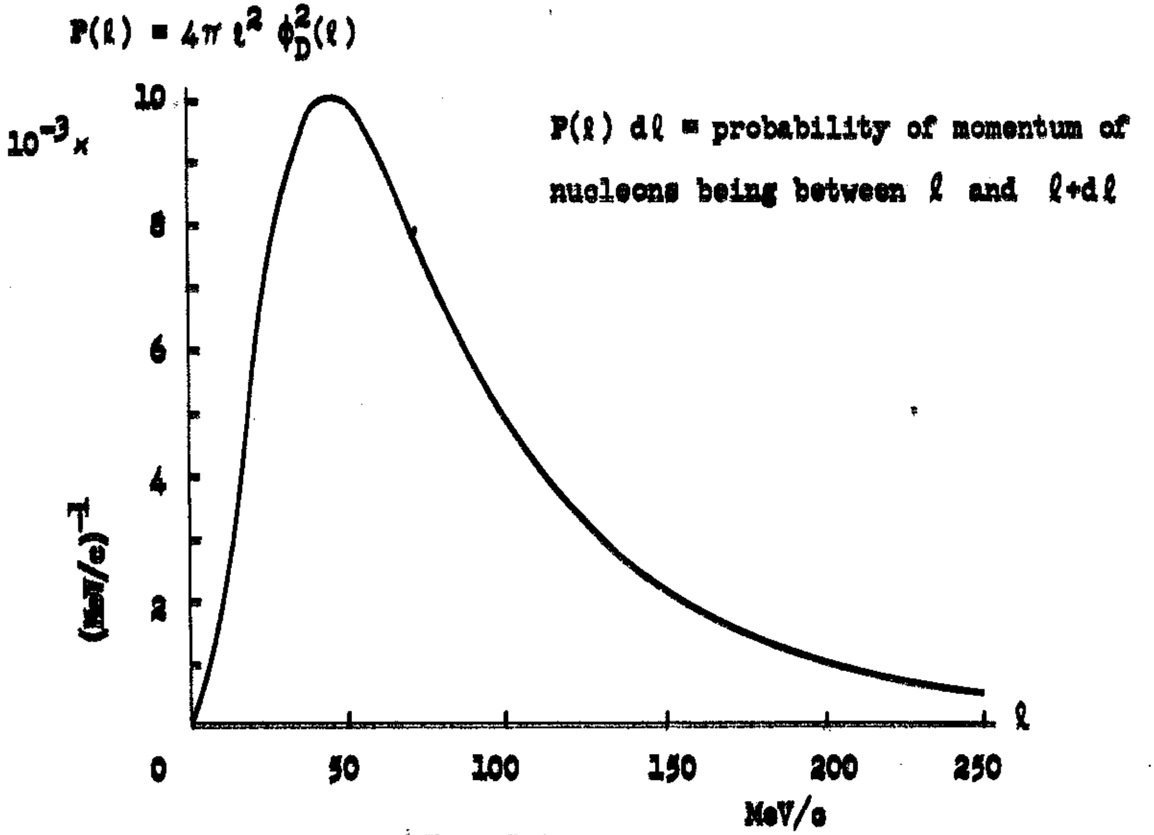

$P(\ell) = 4\pi \ell^2 \phi_D^2(\ell)$
$P(\ell)\, d\ell$ = probability of momentum of nucleons being between ℓ and $\ell + d\ell$
$10^{-3}\times$
10
8
6
4
2
0
$(\mathrm{MeV/c})^{-1}$
50
100
150
200
250
ℓ
MeV/c

Fig. I.4

CHAPTER II

FORMAL DESCRIPTION OF K-DEUTERON SCATTERING

1. Introduction

To study the scattering of K mesons by deuterons, we have to derive the expressions for the probability amplitudes for transitions from the initial state, which consists of a free meson incident on a deuteron, to a final state, which can be either a meson and a deuteron or a system of three free particles, i.e. one meson and two nucleons. (We call the first kind of processes elastic scattering, the second kind inelastic scattering.)

We do not consider processes in which the meson forms a bound state with a nucleon. The fact that the deuteron is the only possible bound state simplifies our problem, since then we have either elastic scattering (with the same deuteron in final state) or we have a completely unbound system of three particles in the final state.

We are not concerned with a relativistic field theoretic problem, that is, we are not going to consider processes of creation or absorption of particles. Our system will always consist of three particles, one meson and two nucleons. We will be working with K-meson incident energies such that no particles can be created or such that creation of particles (e.g. π mesons) has a very small probability.

We want to write the transition amplitudes for elastic and inelastic scattering of mesons by deuterons in terms of the quantities describing the interactions between pairs of particles of our system of three particles. That is, we intend to describe the meson-deuteron scattering in terms of meson-proton, meson-neutron, and nucleon-nucleon interactions.

Let us introduce a potential U describing the nucleon-nucleon interaction. It is responsible for the formation of the bound state of the neutron-proton system. The initial state, being a deuteron and a free meson, is an eigenstate ψ_a of the Hamiltonian K + U, where K is the operator for the total kinetic energy, that is, we have

$$(K + U)\psi_a = E_a\psi_a \ . \tag{II.1}$$

In the same way we introduce potentials V_p and V_n that are responsible for the interaction between the meson and the proton and neutron respectively. The total Hamiltonian is

$$H = K + U + V_p + V_n \ . \tag{II.2}$$

Strictly speaking, these potentials connect several reaction channels and must really have a matrix structure. For example, V_n must describe both the processes $K^+n \rightarrow K^+n$ and $K^+n \rightarrow K^op$. U must describe n-p and p-p interactions at least, since these two pairs of nucleons will occur in our problem. But this does not affect the formalism presented in this and in the next chapters.

An outgoing scattering state $\psi_a^{(+)}$, an exact solution of H with the asymptotic behaviour of a free meson and a free deuteron wave plus an outgoing wave of a meson and a deuteron, satisfies

$$\psi_a^{(+)} = \psi_a + \frac{1}{E_a - K - U + i\epsilon} (V_p + V_n)\psi_a^{(+)} . \qquad \text{(II.3)}$$

An outgoing scattering state $\Phi_b^{(+)}$ with asymptotic behaviour corresponding to plane plus outgoing waves of three free particles satisfies

$$\Phi_b^{(+)} = \Phi_b + \frac{1}{E_b - K + i\epsilon} (V_p + V_n + U)\Phi_b^{(+)} \qquad \text{(II.4)}$$

where Φ_b is a plane-wave state of three free particles, satisfying

$$K \Phi_b = E_b \Phi_b . \qquad \text{(II.5)}$$

We can find an explicit formal solution for Eq. (II.3) using the technique introduced by Gell-Mann and Goldberger (Ref. 12) in the following way. Using the general operator identity

$$\frac{1}{A} - \frac{1}{B} = \frac{1}{A}(B-A)\frac{1}{B} \qquad \text{(II.6)}$$

with $A = (E_a - K - U - V_p - V_n + i\epsilon)$ and $B = (E_a - K - U + i\epsilon)$ we get

$$\frac{1}{E_a - K - U + i\epsilon} = \frac{1}{E_a - K - U - V_p - V_n + i\epsilon} \left(1 - (V_p + V_n) \frac{1}{E_a - K - U + i\epsilon}\right) .$$

Substituting in Eq. (II.3) we get

$$\psi_a^{(+)} = \psi_a + \frac{1}{E_a - K - U - V_p - V_n + i\epsilon} (V_p + V_n) \left[1 - \frac{1}{E_a - K - U + i\epsilon} (V_p + V_n)\right] \psi_a^{(+)} .$$

Using Eq. (II.3) again for simplifying the last part of this expression, we finally obtain

$$\psi_a^{(+)} = \psi_a + \frac{1}{E_a - K - U - V_p - V_n + i\epsilon} (V_p + V_n)\psi_a \tag{II.7}$$

which does not contain $\psi_a^{(+)}$ in the right-hand side. In a precisely analogous way we can obtain

$$\Phi_b^{(+)} = \Phi_b + \frac{1}{E_b - K - V_p - V_n - U + i\epsilon} (V_p + V_n + U)\Phi_b \tag{II.8}$$

which is an explicit expression for $\Phi_b^{(+)}$. Of course, these solutions for $\psi_a^{(+)}$ and $\Phi_b^{(+)}$ are only formal, since we do not know how to find the inverse operators $(E-K-U-V_p-V_n+i\epsilon)^{-1}$ which are in the right-hand side of these expressions. To know how to calculate these inverse operators would correspond to knowing how to solve the Schroedinger equation

$$(K + U + V_p + V_n)\psi = E\psi \ .$$

Equations (II.7) and (II.8) can be regarded as giving the transformations from the sets of plane-wave states to the set of scattering states with specified asymptotic conditions. If we note that, due to Eqs.(II.1), (II.2) and (II.5), we have

$$(V_p + V_n)\psi_a = (H - E_a)\psi_a \tag{II.9}$$

$$(V_p + V_n + U)\Phi_b = (H - E_b)\Phi_a \tag{II.10}$$

we see that Eq. (II.7) and Eq. (II.8) can be written in a single equation

$$\Psi^{(+)} = \Psi + \frac{1}{E - H + i\epsilon} (H-E)\Psi = \Omega_+ \Psi \tag{II.11}$$

where Ψ is any of the states of the set of "plane-wave states" of the system of three particles. This set includes the states in which we have a deuteron and a meson as separate plane waves, and the states in which we have three separate plane waves. $\Psi^{(+)}$ represents the corresponding outgoing scattering states. As Ψ and $\Psi^{(+)}$ form two complete sets of states and there is a one to one correspondence between the elements of these sets, Eq. (II.10) defines completely the wave operator $\Omega^{(+)}$.

We can similarly define the ingoing scattering state

$$\psi_a^{(-)} = \psi_a + \frac{1}{E_a - K - U - V_p - V_n - i\epsilon} (V_p + V_n)\psi_a \tag{II.12}$$

for the case of proton and neutron being bound to form a deuteron, and

$$\Phi_b^{(-)} = \Phi_b + \frac{1}{E_b - K - U - V_p - V_n - i\epsilon} (V_p + V_n + U)\Phi_b \tag{II.13}$$

for the case of p and n being asymptotically free. Again the two relations can be combined in a single expression

$$\Psi^{(-)} = \Psi + \frac{1}{E - H - i\epsilon} (H - E)\Psi = \Omega_- \Psi \tag{II.14}$$

which defines the wave operator Ω_- connecting the complete set of plane-wave states Ψ with the complete set of ingoing scattering states $\Psi^{(-)}$.

2. The Expressions for the Collision Operators

We now proceed to write expressions for operators T^+ and T^-, which we call collision operators, related to these wave operators Ω_+ and Ω_- and which, when operating between states of equal energies, reduce to the collision operators defined by Lippman and Schwinger (Ref. 13), their square being then proportional to the transition probability between the two states.

The case of elastic scattering has been treated by Chew and Goldberger (Ref. 4). They define two operators

$$T_{el}^{+} = (V_p + V_n) + (V_p + V_n) \frac{1}{E_i - K - U - V_p - V_n + i\epsilon} (V_p + V_n) \qquad \text{(II.15)}$$

and

$$T_{el}^{-} = (V_p + V_n) + (V_p + V_n) \frac{1}{E_f - K - U - V_p - V_n + i\epsilon} (V_p + V_n) \qquad \text{(II.16)}$$

where E_i is the energy of the meson-deuteron state on the right of the operator T_{el}^+, when it is acting between two states, and E_f is the energy of the state on the left of T_{el}^-. In other words, the matrix elements of T_{el}^+ and T_{el}^- between states i and f of the meson-deuteron system are

$$(T_{el}^{+})_{fi} = \langle \psi_f | T_{el}^{+} | \psi_i \rangle = \langle \psi_f | (V_p + V_n) \psi_i^{(+)} \rangle \qquad \text{(II.17)}$$

$$(T_{el}^{-})_{fi} = \langle \psi_f | T_{el}^{-} | \psi_i \rangle = \langle \psi_f^{(-)} | (V_p + V_n) \psi_i \rangle \qquad \text{(II.18)}$$

where ψ_i and ψ_f satisfy Eq. (II.1), $\psi_i^{(+)}$ satisfies Eq. (II.3) and $\psi_f^{(-)}$ satisfies Eq. (II.11). For $E_f = E_i$, of course, we have

$(T^{+}_{el})_{fi} = (T^{-}_{el})_{fi}$, since the only difference between them is in the parameters E_f and E_i in the denominators, and they coincide with the usual T_{fi} matrix element of scattering theory. T_{fi} is related to the S-matrix element by

$$S_{fi} = \delta_{fi} - 2\pi i\delta(E_f - E_i)T_{fi} \ . \tag{II.19}$$

Let us now consider the case of inelastic scattering of mesons by deuterons. The expression for the transition amplitude from the bound to an unbound state was obtained by Gell-Mann and Goldberger (Ref. 12). The result they obtained is

$$(T_{inel})_{fi} = \langle \Phi_f | (U + V_p + V_n) | \psi_i^{(+)} \rangle \tag{II.20}$$

where $\psi_i^{(+)}$ obeys Eq. (II.3), Φ_f is a state of three free particles, and $E_f = E_i$.

Given Eqs.(II.20) and (II.7) we see that an extension of the definition of T_{inel} to include off-the-energy-shell matrix elements is obtained by defining the operator

$$T^{+}_{inel} = (U + V_p + V_n) + (U + V_p + V_n) \frac{1}{E_i - K - U + i\epsilon - V_p - V_n} (V_p + V_n). \tag{II.21}$$

To define the corresponding operator T^{-}_{inel} we shall need an expression for the transition amplitude, equivalent to Eq. (II.20), but written as a matrix element between states $\Phi_f^{(-)}$ and ψ_i instead of between Φ_f and $\psi_i^{(+)}$. We shall now obtain such an expression by using the same method as Gell-Mann and Goldberger (Ref. 12) used to obtain Eq. (II.20). The transition rate from the plane-wave state is

$$\dot{\omega}_{fi} = \frac{\partial}{\partial t} \left| \langle \Phi_f^- | e^{i(K+V_p+V_n+U)t} \; e^{-i(U+K)t} \; | \psi_i \rangle \right|^2$$

where the time dependent exponentials are responsible for the time variation of the state vectors in the Schroedinger representation. At t = 0 we get

$$\dot{\omega}_{fi} = i \langle \Phi_f^{(-)} | V_p + V_n | \psi_i \rangle \langle \Phi_f^{(-)} | \psi_i \rangle^* + \text{c.c.}$$

But using Eq. (II.13) we obtain

$$\langle \Phi_f^{(-)} | \psi_i \rangle = \langle \Phi_f + \frac{1}{E_f - K - U - V_p - V_n - i\epsilon} (V_p + V_n + U) \Phi_f | \psi_i \rangle =$$

$$= \langle \Phi_f + \left(\frac{1}{E_f - K - U - i\epsilon} + \right.$$

$$\left. + \frac{1}{E_f - K - U - i\epsilon} (V_p + V_n) \frac{1}{E_f - K - U - V_p - V_n - i\epsilon} \right) (V_p + V_n + U) \Phi_f | \psi_i \rangle =$$

$$= \langle \Phi_f + \frac{1}{E_f - K - U - i\epsilon} (V_p + V_n + U) \Phi_f +$$

$$+ \frac{1}{E_f - K - U - i\epsilon} (V_p + V_n)(\Phi_f^{(-)} - \Phi_f) | \psi_i \rangle =$$

$$= \langle \Phi_f + \frac{1}{E_f - K - U - i\epsilon} U \Phi_f + \frac{1}{E_f - K - U - i\epsilon} (V_p + V_n) \Phi_f^{(-)} | \psi_i \rangle =$$

$$= \langle \Phi_f + U \frac{1}{E_f - E_i - i\epsilon} \Phi_f + \frac{1}{E_f - E_i - i\epsilon} (V_p + V_n) \Phi_f^{(-)} | \psi_i \rangle =$$

$$= \langle \Phi_f | 1 - \frac{1}{E_i - K - i\epsilon} U | \psi_i \rangle + \frac{1}{E_f - E_i + i\epsilon} \langle \Phi_f^{(-)} | V_p + V_n | \psi_i \rangle =$$

$$= \frac{1}{E_f - E_i + i\epsilon} \langle \Phi_f^{(-)} | V_p + V_n | \psi_i \rangle .$$

In the last step we have used the fact that the deuteron wave function satisfies $\psi_i = (E_i - K)^{-1} U \psi_i$.

Substituting the above result into the expression for the transition rate we obtain

$$\dot{\omega}_{fi} = \left| \langle \Phi_f^{(-)} | V_p + V_n | \psi_i \rangle \right|^2 \left[\frac{i}{E_f - E_i + i\epsilon} - \frac{i}{E_f - E_i - i\epsilon} \right]$$

$$= 2\pi \delta (E_f - E_i) \left| \langle \Phi_f^{(-)} | V_p + V_n | \psi_i \rangle \right|^2 .$$

Comparing with the usual expression for the transition rate we see that

$$(T_{inel})_{fi} = \langle \Phi_f^{(-)} | V_p + V_n | \psi_i \rangle . \qquad \text{(II.22)}$$

An adequate definition of the collision operator T_{inel}^- is then

$$T_{inel}^- = (V_p + V_n + U) \frac{1}{E_f - K - U - V_p - V_n + i\epsilon} (V_p + V_n) + (V_p + V_n). \qquad \text{(II.23)}$$

Comparing the formulae (II.21) and (II.23), we see that T^{+}_{inel} and T^{-}_{inel} differ not only by the values of the energy E_i and E_f in the denominator, but also by an extra U that appears in T^{-}_{inel}. We can prove that this difference also disappears on the energy shell. In fact, since the integrals over closed surfaces of the flux of Φ_f and ψ_i vanish, and K is hermitian, we have

$$\langle \Phi_f | U | \psi_i \rangle = \langle \Phi_f | (K+U) - K | \psi_i \rangle = (E_i - E_f) \langle \Phi_f | \psi_i \rangle$$

and this is zero for $E_f = E_i$.

We now obtain general expressions for the operators T^{+} and T^{-} which apply to both elastic and inelastic scatterings, and so combine Eqs.(II.15) and (II.21) into one expression, and Eqs. (II.16) and (II.23) into another expression. As before, we designate by φ a "plane wave" state, which can either consist of a free meson and a free deuteron, or of three free particles, one meson and two nucleons. Scattering states $\varphi^{(+)}$ and $\varphi^{(-)}$, and operators Ω_+ and Ω_- are defined by Eqs. (II.11) and (II.14). By considering Eqs. (II.9) and (II.10) we can easily see that Eqs. (II.15) and (II.21) can both be given the form

$$(T^{+})_{fi} = (H - E_f) + (H - E_f) \frac{1}{E_i - H + i\epsilon} (H - E_i) = (H - E_f)\Omega_+(E_i) \quad \text{(II.24)}$$

and that Eqs.(II.16) and (II.23) can be written as

$$(T^{-})_{fi} = (H - E_i) + (H - E_f) \frac{1}{E_f - H + i\epsilon} (H - E_i) = \Omega_-^{*}(E_f)(H - E_i). \quad \text{(II.25)}$$

* * *

CHAPTER III

EXPANSIONS OF THE COLLISION OPERATORS FOR K-d PROCESSES IN TERMS OF TWO-PARTICLE OPERATORS

1. The Two-Particle Collision Operators

The purpose of our analysis is to obtain (approximate) expressions relating quantities which can be obtained directly or indirectly from experiments. In Chapter II we have written the formulae for the collision operators for elastic and inelastic scattering of mesons by deuterons in terms of the potentials V_p, V_n, U between pairs of particles. But we cannot expect to extract from the results of a limited number of experiments on scattering of mesons by nucleons very useful information about the potentials V_p and V_n. In fact, almost nothing can be said to be known about them at the moment. What can be directly obtained from real or ideal scattering experiments is information on quantities like matrix elements of scattering operators (and trivially related quantities, such as our collision operators). Thus it is in terms of these two-particle collision operators that the K-d scattering is to be analysed.

As the neutron and proton form a bound state and are, in general, a much more extensively studied system than the K-nucleon system, we have some knowledge of the potential U causing their interaction. In the aspects of our problem, in fact, which involve the bound state of the neutron-proton system, we shall make use of this knowledge of U (especially by explicitly using the deuteron wave function). In the analysis of the inelastic K-d scattering, when two unbound nucleons come out from the reaction, the collision operator for the system of two nucleons will also appear as an important quantity.

We now proceed to define these quantities (the two-particle collision operators) in terms of which we intend to describe the K-d scattering.

For the meson-proton system we define the collision operators

$$t_p^+ = V_p + V_p \frac{1}{E_i - K - V_p + i\epsilon} V_p \qquad \text{(III.1)}$$

$$t_p^- = V_p + V_p \frac{1}{E_f - K - V_p + i\epsilon} V_p \qquad \text{(III.2)}$$

where E_i is the energy of the plane-wave state on the right and E_f that of the plane-wave state on the left when these operators are acting between two states. These are extensions for regions off-the-energy shell (Ref. 4) of the usual operators of the scattering theory. They are, of course, obtained by directly extending the expressions

$$(t_p)_{fi} = \langle \Phi_f, V_p \Phi_i^{(+)} \rangle = \langle \Phi_f^{(-)}, V_p \Phi_i \rangle$$

where

$$\Phi_i^{(+)} = \Phi_i + \frac{1}{E_i - K + i\epsilon} V_p \Phi_i^{(+)} = \Phi_i + \frac{1}{E_i - K - V_p + i\epsilon} V_p \Phi_i \qquad (III.3)$$

$$\Phi_f^{(-)} = \Phi_f + \frac{1}{E_f - K - i\epsilon} V_p \Phi_i^{(-)} = \Phi_f + \frac{1}{E_f - K - V_p - i\epsilon} V_p \Phi_f \qquad (III.4)$$

which were defined by Lippman and Schwinger (Ref. 13) only for $E_f = E_i$. We need these extended definitions because we will be concerned with off-the-energy-shell matrix elements.

For the meson-neutron interaction we define perfectly analogous expressions merely by changing the index p to n. Also we define nucleon-nucleon collision operators

$$t_u^+ = U + U \frac{1}{E_i - K - U + i\epsilon} U \qquad (III.5)$$

$$t_u^- = U + U \frac{1}{E_f - K - U + i\epsilon} U \ . \qquad (III.6)$$

The definitions we have given for the t^+ operators (we drop for a while the indices n,p,U) assume that they are to act between states of which the state at the right is an eigenstate of the kinetic energy operator K, with energy E_i. This definition of the operator is not complete, in the sense that it does not tell how the operator acts on an arbitrary state, i.e. on a superposition $\sum_\ell c_\ell \Phi_\ell$ of free particle states Φ_ℓ (with $K \Phi_\ell = E_\ell \Phi_\ell$). t^+ must have the following property

$$t^+ \sum_\ell c_\ell \Phi_\ell \rangle = \sum_\ell c_\ell \left(V + V \frac{1}{E_\ell - K - V + i\epsilon} V \right) \Phi_\ell \rangle \qquad (III.7)$$

We can find an explicit expression for such an operator, namely:

$$t^{+} = \sum_{j} \left(V + V \frac{1}{E_j - K - V + i\epsilon} V \right) \Phi_j \rangle\langle \Phi_j \, . \qquad \text{(III.8)}$$

Here we need to include explicitly the bra and ket symbols that we kept implicit before.

Similarly the complete definition of the t^{-} operator is

$$t^{-} = \sum_{j} \Phi_j \rangle\langle \Phi_j \left(V + V \frac{1}{E_j - K - V + i\epsilon} V \right) . \qquad \text{(III.9)}$$

For brevity, we shall suppress the bra, ket and summation symbols when writing the expressions for the $t^{\pm}$ operators. In other words, we shall write expressions like Eqs. (III.1) and (III.2), but keep in mind the complete expressions (III.8) and (III.9) and actually use them whenever we are operating with t^{+} or t^{-} on packets of free waves.

2. Analysis of the Elastic K-d Scattering

We have now to tackle the problem of expressing the collision operators T of the K-d system, which, in Chapter II, were written in terms of the above-defined operators referring to two-particle scattering. We have to deal separately with elastic and inelastic scattering for two reasons. The first is that the explicit expressions for $T^{\pm}_{el}$ and $T^{\pm}_{inel}$ in terms of the two-particle potentials are not the same [see formulae (II.15), (II.16), (II.21) and (II.23)]. The second is that the arrangement and interpretation of the terms which we shall have in the two cases are different. This is due to the fact that in the case of inelastic scattering the two nucleons in the final state are free and we do have the collision operator t_u acting on this state, while in the elastic scattering we are concerned with the bound neutron-proton system in the final state and not with a scattering state of these two particles.

Chew and Goldberger (Ref. 4) have already considered the problem of the elastic scattering of a particle by a number of interacting centres of force. They obtained the expansion which corresponds in our problem to expressing the collision operator T_{el} for elastic scattering of mesons by deuterons in terms of $t_p^{\pm}$, $t_n^{\pm}$ and U. We now use their method to derive another expansion, which differs from theirs in the arrangement and interpretation of some terms.

We are interested in the collision operator for elastic K-d scattering on the energy shell. So we take $E_f = E_i$, $T^+_{el} = T^-_{el} = T_{el}$. This only gives simplicity of notation, and has no effects at all on the results.

The initial and final state consist of a deuteron and a free meson, and can be represented by a superposition of plane-wave states of three particles:

$$|\psi_i > = \sum_{\ell} c_\ell |\Phi_\ell > \qquad \text{(III.10)}$$

where c_ℓ is determined by our knowledge of the structure of the deuteron - it is in fact the deuteron wave function in momentum space. We call E_ℓ the energy of the free particles in the component Φ_ℓ of this wave packet, i.e.

$$K|\Phi_\ell> = E_\ell|\Phi_\ell> .$$

By applying the operator identity

$$\frac{1}{A} = \frac{1}{B} + \frac{1}{A}(B-A)\frac{1}{B} \qquad \text{(III.11)}$$

to the expression (III.8) defining t_p^+ after multiplication on the left by $\frac{1}{E_\ell + i\epsilon - K}$, we obtain

$$\frac{1}{E_\ell - K - V_p + i\epsilon} V_p|\Phi_\ell> = \frac{1}{E_\ell + i\epsilon - K} t_p^+ |\Phi_\ell> . \qquad \text{(III.12)}$$

By multiplying Eq. (III.9) on the right by $\frac{1}{E_{\ell'} + i\epsilon - K}$, we obtain analogously

$$<\Phi_{\ell'}|V_p \frac{1}{E_{\ell'} - K - V_p + i\epsilon} = <\Phi_{\ell'}|t_p^- \frac{1}{E_{\ell'} + i\epsilon - K} . \qquad \text{(III.13)}$$

Equations (III.12) and (III.13) are very useful relations. We can write analogous expressions for V_n and U, of course.

By repeatedly using Eqs. (III.11), (III.12), (III.13), etc., we have

$$T_{el}|\psi_i> = \left[(V_p + V_n) + (V_p + V_n)\frac{1}{E-K-U-V_p-V_n+i\epsilon}(V_p + V_n)\right]|\psi_i> =$$

$$= \sum_\ell c_\ell\left\{V_p + V_n + V_p\left[\frac{1}{E_\ell - K - V_p + i\epsilon} + \frac{1}{E-K-U-V_p-V_n+i\epsilon}(E_\ell - E + U + V_n)\right. \times\right.$$

$$\times \frac{1}{E_\ell - K - V_p + i\epsilon}\Big] V_p + V_p \frac{1}{E - K - U - V_p - V_n + i\epsilon} V_n +$$

$$+ V_n \frac{1}{E - K - U - V_p - V_n + i\epsilon} V_p + V_n \Big[\frac{1}{E_\ell - K - V_n + i\epsilon} +$$

$$+ \frac{1}{E - K - U - V_p - V_n + i\epsilon} (E_\ell - E + U + V_p) \frac{1}{E_\ell - K - V_n + i\epsilon}\Big] V_n \Big\} |\Phi_\ell>$$

$$= \sum_\ell c_\ell \Big\{ t_p^+ + t_n^+ + V_p \frac{1}{E - K - U - V_p - V_n + i\epsilon} (E_\ell - E + U + V_n) \frac{1}{E_\ell - K + i\epsilon} t_p^+ +$$

$$+ V_n \frac{1}{E - K - U - V_p - V_n + i\epsilon} (E_\ell - E + U + V_p) \frac{1}{E_\ell - K + i\epsilon} t_n^+ +$$

$$+ V_p \Big[\frac{1}{E_\ell - K - V_n + i\epsilon} + \frac{1}{E - K - U - V_p - V_n + i\epsilon} (U + V_p + E_\ell - E) \frac{1}{E_\ell - K - V_n + i\epsilon} \Big] V_n +$$

$$+ V_n \Big[\frac{1}{E_\ell - K - V_p + i\epsilon} + \frac{1}{E - K - U - V_p - V_n + i\epsilon} (U + V_n + E_\ell - E) \frac{1}{E_\ell - K - V_p + i\epsilon} \Big] V_p \Big\} |\Phi_\ell>$$

$$= \sum_\ell c_\ell \Big\{ t_p^+ + t_n^+ + \Big[V_p \frac{1}{E - K - U - V_p - V_n + i\epsilon} (E_\ell - E + U + V_n) +$$

$$+ V_n + V_n \frac{1}{E - K - U - V_p - V_n + i\epsilon} (U + V_n + E_\ell - E) \Big] \frac{1}{E_\ell - K + i\epsilon} t_p^+ +$$

$$+ \Big[V_n \frac{1}{E - K - U - V_p - V_n + i\epsilon} (E_\ell - E + U + V_p) + V_p +$$

$$+ V_p \frac{1}{E - K - U - V_p - V_n + i\epsilon} (U + V_p + E_\ell - E)\Big] \frac{1}{E_\ell - K + i\epsilon} t_n^+ \Big\} |\Phi_\ell>$$

$$= \Big\{ t_p^+ + t_n^+ + \Big[V_n + T_{el} \frac{1}{E - K - U + i\epsilon} (E_\ell - E + U + V_n) \Big] \frac{1}{E_\ell - K + i\epsilon} t_p^+ +$$

$$+ \Big[V_p + T_{el} \frac{1}{E - K - U + i\epsilon} (E_\ell - E + U + V_p) \Big] \frac{1}{E_\ell - K + i\epsilon} t_n^+ \Big\} |\psi_i> .$$

In this step we used the relation

$$(V_p + V_n) \frac{1}{E - K - U - V_p - V_n + i\epsilon} = T_{el} \frac{1}{E - K - U + i\epsilon} \qquad \text{(III.14)}$$

which was obtained by the same method as (III.13): multiplying T_{el} by $(E - K - U + i\epsilon)^{-1}$ and using identity (III.11).

The purpose of this series of transformations has been that of eliminating the "unobservable" potentials and introducing the "observable" collision operators. To transform the terms V_n and V_p that are still left inside the brackets, we must remember that the above expression is to be multiplied on the left by the final state $< \psi_f | = \sum_{\ell'} c^*_{\ell'} < \Phi_{\ell'} |$. We obtain

$$<\psi_f|T_{el}|\psi_i> = <\psi_f| \Big\{ t_p^+ + t_n^+ + T_{el} \frac{1}{E-K-U+i\epsilon} (E_\ell - E + U) \frac{1}{E_\ell - K + i\epsilon} t_p^+ +$$

$$+ T_{el} \frac{1}{E - K - U + i\epsilon} (E_\ell - E + U) \frac{1}{E_\ell - K + i\epsilon} t_n^+ + \qquad \text{(III.15)}$$

$$+ t_n^- \frac{1}{E_\ell - K + i\epsilon} t_p^+ + t_p^- \frac{1}{E_\ell - K + i\epsilon} t_n^+ \Big\} |\psi_i>$$

+ terms of higher orders.

By "terms of higher orders" we mean terms which when expressed in the form of products of collision operators and propagators will consist of a product of two or more propagators and three or more collision operators.

The meaning of these terms can be understood in the following manner. We have an initial state $|\psi_i\rangle = \sum_\ell c_\ell |\Phi_\ell\rangle$ which consists of a packet (deuteron bound state) of free waves of proton and neutron and a free incident meson. Similarly for the final state. The term

$$\langle\psi_f| t_p^+ |\psi_i\rangle = \sum_{\ell,\ell'} c_\ell c_{\ell'}^* \ \langle\Phi_{\ell'}| t_p^+ |\Phi_\ell\rangle \tag{III.16}$$

represents collisions of the incident meson with a proton of momentum labelled by ℓ, the meson taking the momentum specified in the final state of the system, and the proton passing to a free state of momentum labelled by ℓ'. The sums over ℓ and ℓ' correspond to using all the components of the initial and final wave-packets. The collision operator t_p^+ contains a δ-function of momentum variables as a factor so that the total momentum of mesons and protons is conserved in this collision. This means that for each value of ℓ only one value of ℓ' contributes to the above sum. This value of ℓ' which is fixed by the momentum conservation is specified by the initial and final meson momenta. Energy, of course, is not necessarily conserved in these two-particle collisions. Similarly, $\langle\psi_f| t_n^+ |\psi_i\rangle = \sum_{\ell,\ell'} c_{\ell'}^* c_\ell \langle\Phi_{\ell'}| t_n^+ |\Phi_\ell\rangle$ represents single scattering of the incident meson by the neutron alone.

Terms like

$$<\psi_f | t_n^- \frac{1}{E_\ell - K + i\epsilon} t_p^+ | \psi_i> = \sum_{\ell,\ell'} c_{\ell'}^* c_\ell <\Phi_{\ell'} | t_n^- \frac{1}{E_\ell - K + i\epsilon} t_p^+ | \Phi_\ell> \quad \text{(III.17)}$$

represent double scattering processes. Here the incident meson collides with a proton of a certain momentum labelled ℓ; the system of three free particles of energy E_ℓ then "propagates" (factor $1/E_\ell - K + i\epsilon$) until there is scattering of the meson by the neutron, leading the system to the component ℓ' of the packet of waves of the final state. There is conservation of momentum, but not necessarily of energy, in each of these two collisions, due to the fact that each operator t contains a δ-function of momentum variables as a factor. Again we sum over all components of the initial and the final wave packets. Analogously, some of the terms of higher order in the expansion of T_{el} will represent multiple scattering processes.

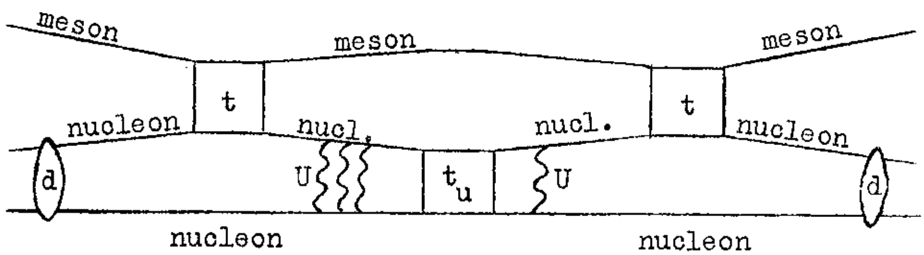
The term

$$T_{el} \frac{1}{E - K - U + i\epsilon} (E_\ell - E + U) \frac{1}{E_\ell - K + i\epsilon} t_p^+ \quad \text{(III.18)}$$

represents a complication to these double and single scattering processes. It brings out the fact that the two nucleons are not free, and strictly speaking cannot be considered as such during the interval of time during which collision processes occur. The simplest terms contributing to the above-mentioned term are

$$\sum_{\ell,\ell'} c_\ell c_{\ell'}^* <\Phi_{\ell'} | (t_p^- + t_n^-) \frac{1}{E_{\ell'} - K + i\epsilon} (E_\ell - E + U) \frac{1}{E_\ell - K + i\epsilon} t_p^+ | \Phi_\ell>$$

which are obtained by iterations using $\langle\psi_f|T_{el} = \langle\psi_f|t_p^- + t_n^- + \ldots$ and Eq. (III.11) to transform the propagator $(E - K - U + i\epsilon)^{-1}$. This expression is represented by the figure below, which shows the nucleons interacting between two meson-nucleon collisions.



Terms like Eq. (III.18) correspond to three-body effects which cannot be reduced to combinations of two-body processes: it is not possible to express Eq. (III.18) in terms of t_p, t_n, t_u only, eliminating U.

Much has already been said in the literature (Refs. 2-4) on the conditions under which the double scattering and these additional potential effects are small compared to the single scattering processes. In our case of K-deuteron scattering, these conditions are well satisfied for incident mesons of medium or high energies. The meson being fast and its interaction with the nucleon being of short range, we expect that during the short time in which the meson-nucleon interaction takes place, the nucleon-nucleon binding has very small effects. The short range of the meson-nucleon interaction as compared to an average internucleonic distance in the deuteron, causes double scattering processes to be much less important than the single scattering ones.

An estimate of the value of these double scattering contributions can be given in the following way (Ref. 2). Let us take p and n as two sources of scattering, separated by a distance $R = 4.3 \times 10^{-13}$ cm which is the average inter-nucleonic distance in the deuteron. The amplitude of the wave scattered by one nucleon, calculated at the position of the other, is $f/R = \sqrt{\sigma/4\pi}\,/R$ where f is the scattering amplitude and σ is the total cross-section. Phase factors have been neglected. For K^+-proton scattering, we have $\sigma \sim 16$ mb (Ref. 14), and then $f/R = 1/12$, i.e. the wave scattered by the proton, when hitting the neutron, is twelve times weaker than the wave incident directly on the neutron. This indicates that double scattering gives a much smaller contribution to the K^+-deuterons processes than single scattering. For K^--deuteron scattering, double scattering processes become more important due to the fact that the total K^--proton cross-section is several times higher than the K^+-proton cross-section. Taking $\sigma \sim 60$ mb (Ref. 15) for example, the ratio of the contributions to the matrix element coming from double and single scattering could be of the order $f/R = 1/6$, which is too high to justify neglect of double scattering terms in the K^- case, if a reliable comparison with experimental data is to be made.

It would be interesting to calculate the contributions of double scattering terms and of the simplest potential corrections to the matrix element of T_{el}. We have not done so in this case of elastic meson-deuteron scattering, though we have in the case of inelastic scattering, which will be treated in the next section and in following chapters.

The approximation whereby one assumes that

$$\langle \psi_f | T_{el} | \psi_i \rangle = \langle \psi_f | t_p^+ + t_n^+ | \psi_i \rangle \tag{III.19}$$

is usually called the Impulse Approximation. As we have just shown, we expect it to give results correct to within a few percent (perhaps 10%) for K^+-deuteron scattering cross-section, but it may not be so good in the case of K^--deuteron scattering.

3. Analysis of the Inelastic K-d Scattering

The inelastic scattering of mesons by deuterons is governed by the matrix elements of the collision operator T_{inel} ($T_{inel} = T^+_{inel} = T^-_{inel}$ for on-the-energy-shell matrix elements) between an initial state consisting of a deuteron and a free incident meson, and a final state consisting of a free meson and two free nucleons. The expression for T_{inel} in terms of the potentials V_p, V_n, U, has been given in Chapter II. Our task in this section is to expand this expression in such a way that the two-particle collision operators t_p, t_u, t_n appear in the most important terms instead of the potentials V_p, V_n, U.

We again represent the deuteron by a superposition $|\psi_i\rangle = \sum_\ell c_\ell |\Phi_\ell\rangle$ of free waves, with $K|\Phi_\ell\rangle = E_\ell |\Phi_\ell\rangle$. The final state satisfies $K|\Phi_f\rangle = E|\Phi_f\rangle$ and the initial state obeys $(K + U)|\psi_i\rangle = E|\psi_i\rangle$.

By using the operator identity (III.11) and relations like Eqs. (III.12) and (III.13) we obtain

$$T_{inel}|\psi_0> = \left[(U+V_p+V_n) + (U+V_p+V_n) \frac{1}{E-K-U-V_p-V_n+i\epsilon} (V_p+V_n) \right] |\psi_i> =$$

$$= \sum_{\ell} c_\ell \Big\{ (U+V_p+V_n) + (U+V_p+V_n) \Big[\frac{1}{E_\ell+i\epsilon-K-V_p} V_p + \frac{1}{E+i\epsilon-K-U-V_p-V_n} \times$$

$$\times (E_\ell-E+U+V_n) \frac{1}{E_\ell+i\epsilon-K-V_p} V_p + \frac{1}{E_\ell+i\epsilon-K-V_n} V_n +$$

$$+ \frac{1}{E+i\epsilon-K-U-V_p-V_n} (E_\ell-E+U+V_p) \frac{1}{E_\ell+i\epsilon-K-V_n} V_n \Big] \Big\} |\Phi_\ell>$$

$$= \sum_{\ell} c_\ell \Big\{ U + \left(V_p + V_p \frac{1}{E_\ell+i\epsilon-K-V_p} V_p \right) + \left(V_n + V_n \frac{1}{E_\ell+i\epsilon-K-V_n} V_n \right) +$$

$$+ (U+V_p+V_n) \frac{1}{E+i\epsilon-K-U-V_p-V_n} (E_\ell-E) \Big[\frac{1}{E_\ell+i\epsilon-K-V_p} V_p +$$

$$+ \frac{1}{E_\ell+i\epsilon-K-V_n} V_n \Big] + (U+V_n) \frac{1}{E_\ell+i\epsilon-K-V_p} V_p +$$

$$+ (U+V_p+V_n) \frac{1}{E+i\epsilon-K-U-V_p-V_n} (U+V_n) \frac{1}{E_\ell+i\epsilon-K-V_p} V_p +$$

$$+ (U+V_p) \frac{1}{E_\ell+i\epsilon-K-V_n} V_n + (U+V_p+V_n) \frac{1}{E+i\epsilon-K-U-V_p-V_n} (U+V_p) \times$$

$$\times \frac{1}{E_\ell + i\epsilon - K - V_n} V_n \Big\} |\Phi_\ell> =$$

$$= \sum_\ell c_\ell \Big\{ U + t_p^+ + t_n^+ + (U+V_p+V_n) \frac{1}{E+i\epsilon-K-U-V_p-V_n} (E_\ell - E) \frac{1}{E_\ell + i\epsilon - K} (t_p^+ + t_n^+) +$$

$$+ \Big[(U+V_n) + (U+V_p+V_n) \frac{1}{E+i\epsilon-K-U-V_p-V_n} (U+V_n) \Big] \frac{1}{E_\ell + i\epsilon - K} t_p^+ +$$

$$+ \Big[(U+V_p) + (U+V_p+V_n) \frac{1}{E+i\epsilon-K-U-V_p-V_n} (U+V_p) \Big] \frac{1}{E_\ell + i\epsilon - K} t_n^+ \Big\} |\Phi_\ell> .$$

Let us evaluate the contribution from one of these terms in the square brackets. In order to do this, we apply the operators inside the brackets to the final state $< \Phi_f |$ on the left:

$$<\Phi_f| \Big[(U+V_n) + (U+V_p+V_n) \frac{1}{E+i\epsilon-K-U-V_p-V_n} (U+V_n) \Big] \frac{1}{E_\ell + i\epsilon - K} t_p^+ | \psi_i >$$

$$= <\Phi_f| \Big\{ U+V_n+V_p \frac{1}{E+i\epsilon-K-U-V_p-V_n} (U+V_n) + U \frac{1}{E+i\epsilon-K-U-V_p-V_n} V_n +$$

$$+ V_n \frac{1}{E+i\epsilon-K-U-V_p-V_n} U + U \Big[\frac{1}{E+i\epsilon-K-U} + \frac{1}{E+i\epsilon-K-U} (V_p+V_n) \frac{1}{E+i\epsilon-K-U-V_p-V_n} \Big]$$

$$\times U+V_n \Big[\frac{1}{E+i\epsilon-K-V_n} + \frac{1}{E+i\epsilon-K-V_n} (U+V_p) \frac{1}{E+i\epsilon-K-U-V_p-V_n} \Big] V_n \Big\} \times \frac{1}{E_\ell + i\epsilon - K} t_p^+ | \psi_i >$$

$$= <\Phi_f| \Big\{ t_u^- + t_n^- + t_u^- \frac{1}{E + i\epsilon - K} (V_p + V_n) \frac{1}{E + i\epsilon - K - U - V_p - V_n} U +$$

$$+ t_n^- \frac{1}{E+i\epsilon-K} (U+V_p) \frac{1}{E+i\epsilon-K-U-V_p-V_n} V_n + V_p \frac{1}{E+i\epsilon-K-U-V_p-V_n} (U+V_n) +$$

$$+ U \frac{1}{E+i\epsilon-K-U-V_p-V_n} V_n + V_n \frac{1}{E+i\epsilon-K-U-V_p-V_n} U \Bigg\} \frac{1}{E_\ell+i\epsilon-K} t_p^+ |\psi_i\rangle .$$

We thus obtain products of the type $t' G_0 t'' G_0 t''' \ldots$ [where $G_0 = (E-K+i\epsilon)^{-1} =$ = free particle propagator] of higher and higher orders, which correspond to multiple scattering effects, together with terms which represent the corrections to them from "essentially three-body effects".

Separating contributions from processes up to the second order we have

$$\langle \Phi_f | T_{inel} | \psi_i \rangle = \sum_\ell c_\ell \langle \Phi_f | t_p^+ + t_n^+ + t_u^- \frac{1}{E_\ell+i\epsilon-K} t_p^+ + t_n^- \frac{1}{E_\ell+i\epsilon-K} t_p^+ +$$

$$+ t_u^- \frac{1}{E_\ell+i\epsilon-K} t_n^+ + t_p^- \frac{1}{E_\ell+i\epsilon-K} t_n^+ + \text{remainder} \, |\Phi_\ell\rangle \qquad \text{(III.20)}$$

where

$$\text{remainder} = (U+V_p+V_n) \frac{1}{E+i\epsilon-K-U-V_p-V_n} (E_\ell - E) \frac{1}{E_\ell+i\epsilon-K} (t_p^+ + t_n^+)$$

$$+ \text{terms of higher order} =$$

$$= T_{inel} \frac{1}{E+i\epsilon-K-U} (E_\ell - E) \frac{1}{E_\ell+i\epsilon-K} (t_p^+ + t_n^+) \qquad \text{(III.21)}$$

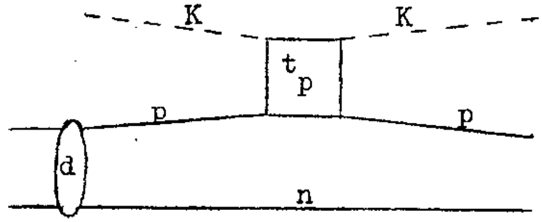
$$+ \text{terms of higher order.}$$

It is easy to understand the meaning of the terms in Eq. (III.20). Those with t_p^+ and t_n^+ alone correspond to single scattering by the proton and by the neutron, respectively. Only one of the nucleons is hit, the other being left alone. The δ-function in momentum variables which is contained as a factor in the t-operators imposes conservation of momentum on the two-particle interaction and on the whole process (this will be seen in more detail in Chapter IV). However, energy cannot be conserved in the two-particle collision, i.e. $\langle \Phi_f | t_p^+ | \Phi_\ell \rangle$ is necessarily an off-the-energy-shell matrix element. This is so for the following reason. Since we impose conservation of energy in the whole process, $E_f = E_i$. But E_i = incident meson energy + + deuteron mass, and E_ℓ = incident meson energy + proton mass + + neutron mass + proton kinetic energy + neutron kinetic energy, so that $E_\ell > E_i$, which implies $E_\ell > E_f$ for any ℓ.

In addition to the double scattering terms $t_p^- \frac{1}{E_\ell + i\epsilon - K} t_n^+$ and $t_n^- \frac{1}{E_\ell + i\epsilon - K} t_p^+$ similar to those which we had in the case of elastic scattering, we now have two other second order terms: $t_u^- \frac{1}{E_\ell + i\epsilon - K} t_p^+$ and $t_u^- \frac{1}{E_\ell + i\epsilon - K} t_n^+$. They represent the collision of the meson with one of the nucleons followed by a scattering of the two nucleons, and are the simplest form of "potential corrections" to our multiple scattering model.

These terms are diagramatically represented by the figures below. Nucleons are represented by heavy lines, mesons by dotted lines. The fact that the particle may be "virtual" is indicated by a wavy line over the line that represents it.

$$\langle \Phi_f | T_{inel} | \psi_i \rangle =$$



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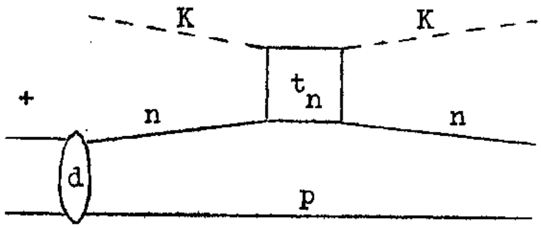

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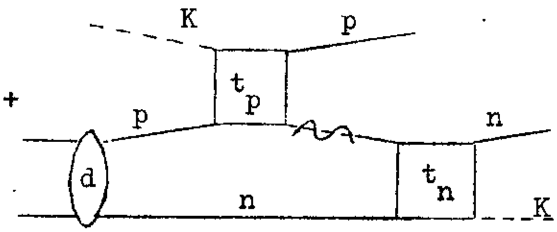

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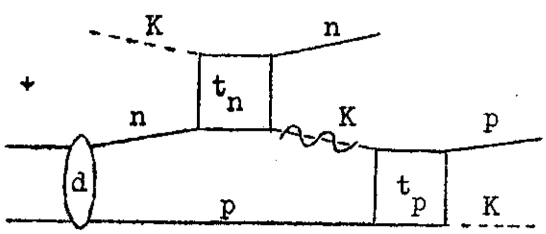

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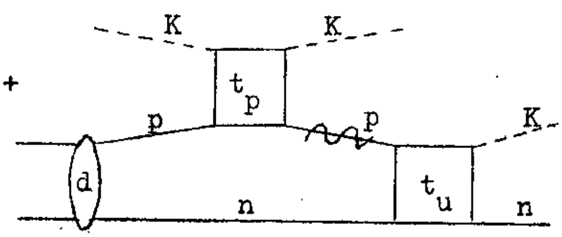

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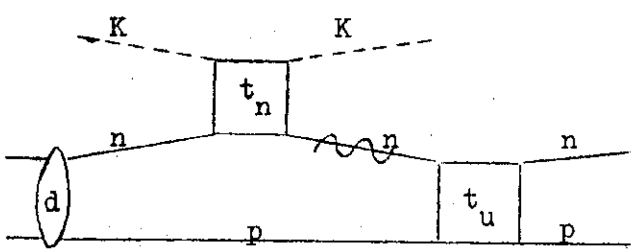
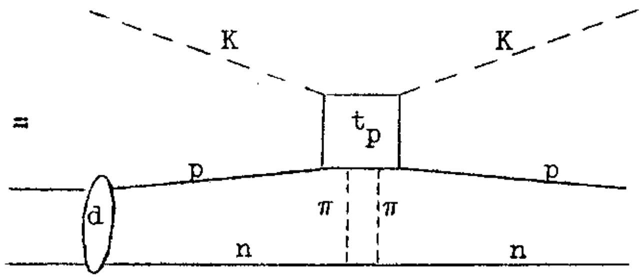


+ remainder

As in the case of elastic scattering discussed in Section III.2, the "remainder" represents multiple scattering processes of high order, and essentially three-body processes which cannot be expressed in terms of two-particle operators.

The residual terms correspond to modifications of the processes which the above diagrams describe. For example, pions may be exchanged between the two nucleons while interactions with the meson are taking place. In the language of diagrams, this modification appears as

Remainder =



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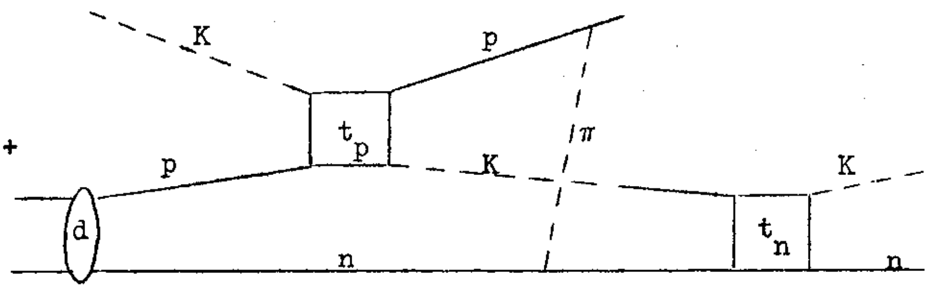

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+ ....

We can obtain a different expansion for T_{inel} in the following way. We first apply T_{inel} to $< \Phi_f|$ from the right, obtaining

$$<\Phi_f|T_{inel} = <\Phi_f|t_p^- + t_p^- \frac{1}{E+i\epsilon-K}\left[V_n + (U+V_n)\frac{1}{E+i\epsilon-K-U-V_p-V_n}(V_p+V_n)\right]$$

$$+ t_n^- + t_n^- \frac{1}{E+i\epsilon-K}\left[V_p + (U+V_p)\frac{1}{E+i\epsilon-K-U-V_p-V_n}(V_p+V_n)\right]$$

$$+ t_u^- \frac{1}{E+i\epsilon-K}\left[(V_p+V_n) + (V_p+V_n)\frac{1}{E+i\epsilon-K-U-V_p-V_n}(V_p+V_n)\right] =$$

$$= <\Phi_f|t_p^- + t_n^- + t_p^- \frac{1}{E+i\epsilon-K}\left[V_n + (U+V_n)\frac{1}{E+i\epsilon-K-U-V_p-V_n}(V_p+V_n)\right] +$$

$$+ t_n^- \frac{1}{E+i\epsilon-K}\left[V_p+(U+V_p)\frac{1}{E+i\epsilon-K-U-V_p-V_n}(V_p+V_n)\right] + t_u^- \frac{1}{E+i\epsilon-K}T_{el} .$$

By applying this expression from the left to $|\psi_i> = \sum_\ell c_\ell |\Phi_\ell>$ we obtain, collecting terms up to second order,

$$<\Phi_f|T_{inel}|\psi_i> = \sum_\ell c_\ell \; <\Phi_f|t_p^- + t_n^- + t_p^- \frac{1}{E+i\epsilon-K} t_n^+ +$$

$$+ t_n^- \frac{1}{E+i\epsilon-K} t_p^+ + t_u^- \frac{1}{E+i\epsilon-K} t_n^+ + t_u^- \frac{1}{E+i\epsilon-K} t_p^+$$

$$+ \text{remainder} \; |\Phi_\ell> \qquad \text{(III.22)}$$

where

$$\langle \Phi_f | \text{remainder} | \Phi_\ell \rangle =$$

$$= \sum_\ell c_\ell \langle \Phi_f | t_u^- \frac{1}{E+i\epsilon-K} T_{el} \frac{1}{E-K-U+i\epsilon} (E_\ell - E + U) \frac{1}{E_\ell - K + i\epsilon} (t_p^+ + t_n^+) | \Phi_\ell \rangle$$

+ terms of same and higher orders. (III.23)

Comparing the two expansions (III.20) and (III.22) we see that they differ, firstly in the single scattering terms by the fact that in one case [(III.20)] we have t_p^+ and t_n^+ and in the other case [III.22)] we have t_p^- and t_n^-. As we are concerned with matrix elements of these operators off-the-energy-shell, this is a real difference. Secondly, in Eq. (III.20) we have E_ℓ as energy parameter in the propagators that appear in the second order terms, while in Eq. (III.22) we have E.

The "remainder" for Eq. (III.22) is of one "order" higher than that for Eq. (III.20). This could suggest that Eq. (III.22) is a better expansion than Eq. (III.20). However, we have not been able to evaluate the contributions coming from these residues. Also we do not know anything about the behaviour of off-the-energy shell matrix elements of the collision operators. Thus we could not justify preference for one or other of the two expansions. We can expect that their difference is smaller than the error involved in neglecting the residual terms of the expansions.

We can try to write Eqs. (III.20) or (III.22) in terms of two-particle scattering states. Let us consider Eq. (III.22). <u>If we assume</u> $t_p^+ = t_p^- = t_p$ and $t_n^- = t_n^+ = t_n$ we can group the "single scattering" and "potential correction" terms into the form

$$\left\langle\Phi_f\right| \left(1 + t_u^- \frac{1}{E + i\epsilon - K}\right)\left(t_p + t_n\right) = \left\langle\Phi_{f_u}^{(-)}\right|\left(t_p + t_n\right) \qquad \text{(III.24)}$$

where

$$\Phi_{f_u}^{(-)} = \left(1 + \frac{1}{E - K - i\epsilon}\, t_u^{-*}\right)\Phi_f = \left(1 + \frac{1}{E - K - i\epsilon - U}\, U\right)\Phi_f \qquad \text{(III.25)}$$

represents a free meson plane wave and an ingoing wave scattering state of the two-nucleon system. It is a solution of the Schrödinger equation with Hamiltonian $K + U$ with specified asymptotic behaviour. Equation (III.22) would then become

$$\begin{aligned} \langle \Phi_f | T_{inel} | \psi_i \rangle = {} & \langle \Phi_{f_u}^{(-)} | t_p + t_n | \psi_i \rangle + \\ & + \langle \Phi_f | t_p^- \frac{1}{E+i\epsilon-K}\, t_n^+ + t_n^- \frac{1}{E+i\epsilon-K}\, t_p^+ | \psi_i \rangle + \qquad \text{(III.26)} \\ & + \text{remainder.} \end{aligned}$$

The first term in the right-hand side of Eq. (III.26), with its ingoing wave-scattering state on the left-hand side of the matrix element, resembles the usual form of the Final State Interaction Theory (Ref. 16). It has been adopted in calculations by Gourdin and Martin (Ref. 10) and by Karplus and Rodberg (Ref. 17) as an improvement to the pure Impulse Approximation (which takes into account single scattering terms only). This term does not include effects of double scattering, which are described by the second term in Eq. (III.26).

The double scattering terms can also be expressed in terms of the meson-nucleon scattering states. Since Φ_f is a three-free-particle state with energy E,

$$\Phi_{f_p}^{(-)} = \left(1 + \frac{1}{E - K - i\epsilon}\, t_p^{-*}\right)\Phi_f = \left(1 + \frac{1}{E - K - i\epsilon - V_p}\, V_p\right)\Phi_f \qquad \text{(III.27)}$$

is a solution of the Schrödinger equation with Hamiltonian $K + V_p$ representing a free neutron and an ingoing wave scattering state of the meson-proton system. We have an analogous expression for $\Phi_{f_n}^{(-)}$, the meson-neutron scattering state. Equation (III.26) can then be written

$$\begin{aligned} \langle\Phi_f|T_{inel}|\psi_i\rangle = {} & \langle\Phi_{f_u}^{(-)}|t_p + t_n|\psi_i\rangle + \langle\Phi_{f_p}^{(-)}|t_n|\psi_i\rangle + \\ & + \langle\Phi_{f_n}^{(-)}|t_p|\psi_i\rangle - \langle\Phi_f|t_p + t_n|\psi_i\rangle + \\ & + \text{remainder} \end{aligned} \qquad \text{(III.28)}$$

where we have assumed that matrix elements of scattering operators t^+ and t^- are equivalent. If we do not make this assumption we must write

$$\begin{aligned} \langle\Phi_f|T_{inel}|\psi_i\rangle = {} & \langle\Phi_{f_u}^{(-)}|t_p^+ + t_n^+|\psi_i\rangle + \langle\Phi_{f_p}^{(-)}|t_n^+|\psi_i\rangle + \\ & + \langle\Phi_{f_n}^{(-)}|t_p^+|\psi_i\rangle - \langle\Phi_f|2t_p^+ + 2t_n^+ - t_p^- - t_n^-|\psi_i\rangle + \\ & + \text{remainder.} \end{aligned}$$

As far as the general structure of our three-body system is concerned, we have no reasons to believe that double scattering processes are less important than the "potential correction" effects. The competition between the two kinds of processes will depend, among other things, on the value of the matrix elements of the collision operators t_p, t_n, t_u, i.e. on peculiarities of the particular system studied. If we keep all the second order terms we shall have what we believe to be a good approximation to the meson-deuteron inelastic scattering. This will be discussed in more detail in Chapter V.

* * *

CHAPTER IV

DYNAMICAL VARIABLES AND SPIN SUMS

1. Dynamical Variables and Representation of States

In Chapter III we expressed formally the collision operators for elastic and inelastic meson-deuteron scattering in terms of the several two-particle collision operators. We now introduce explicitly the dynamical variables describing the system, and show how the main terms of the expansions we have obtained depend on these variables and on the quantities describing two-particle processes more directly.

Let us define the following symbols:

$\vec{q}$ - meson momentum variable in lab. system;

$\vec{q}_o$ - value of initial meson momentum in lab. system;

$\vec{q}_f$ - value of final meson momentum in lab. system;

E_q - meson total energy variable in lab. system;

E_{q_o} - value of meson total energy in initial state in lab.;

E_{q_f} - value of final meson total energy in lab.;

$\vec{p}$ - proton lab. momentum variable;

$\vec{p}_f$ - value of final proton lab. momentum, when it is free (inelastic scattering);

$\vec{n}$ - neutron lab. momentum variable;

$\vec{n}_f$ - value of neutron lab. momentum in final state, in case of plane-waves final state (inelastic scattering);

$\vec{K}$ - total momentum of centre of mass of deuteron;

$\vec{K}_0$ - value of momentum of centre of mass of deuteron in initial state;

$\vec{K}_f$ - value of momentum of centre of mass of deuteron in final state;

$\vec{\ell}$ - proton-neutron relative momentum variable (momentum of proton relative to the centre of mass of the neutron-proton system);

$\vec{\ell}_f$ - value of proton-neutron relative momentum in final state, in case of free particles in this state (inelastic scattering);

We have the following relations:

$$\vec{K} = \vec{p} + \vec{n} \; ; \qquad \vec{\ell} = \frac{E_n\vec{p} - E_p\vec{n}}{E_n + E_p} \; . \tag{IV.1}$$

For non-relativistic particles, this becomes

$$\vec{\ell} = \frac{M_n\vec{p} - M_p\vec{n}}{M_n + M_p} \; .$$

For $M_n = M_p$ this is $\vec{\ell} = \frac{1}{2}(\vec{p} - \vec{n})$.

The inverse relations are

$$\vec{p} = \vec{\ell} + \frac{E_p}{E_n + E_p}\vec{K} \quad \text{and} \quad \vec{n} = -\vec{\ell} + \frac{E_n}{E_n + E_p}\vec{K} \; . \tag{IV.2}$$

Our system of three particles can be described either by the set of variables $\vec{q}$, $\vec{p}$, $\vec{n}$ or by the set $\vec{q}$, $\vec{K}$, $\vec{\ell}$.

The meson-proton relative momentum (i.e. the momentum of the meson in the centre-of-mass system of meson and proton) is:

$$\vec{k}_p = \frac{E_p\vec{q} - E_q\vec{p}}{E_p + E_q} \qquad \text{(IV.3)}$$

and the relative meson-neutron momentum is

$$\vec{k}_n = \frac{E_n\vec{q} - E_q\vec{n}}{E_n + E_q} \; . \qquad \text{(IV.4)}$$

In our problem only non-relativistic motion of nucleons will occur; under these conditions we may define the canonically conjugate coordinates to the set $\vec{q}$, $\vec{K}$, $\vec{\ell}$. We call $\vec{R}$ the position vector of the centre of mass of the neutron-proton system, $\vec{r}$ the relative position vector from the neutron to the proton, and $\vec{\rho}$ the position vector of the meson.

The initial state consists of a free incident meson of momentum $\vec{q}_o$ and a deuteron with a total momentum $\vec{K}_o$. Let us indicate this state by $|\psi_i\rangle = |\psi_i(\vec{q}_o,\vec{K}_o)\rangle$. Its representative in momentum space is

$$\langle\vec{\ell},\vec{q},\vec{K}|\psi_i(\vec{q}_o,\vec{K}_o)\rangle = \delta(\vec{q}-\vec{q}_o)\delta(\vec{K}-\vec{K}_o)\psi_D(\ell) \qquad \text{(IV.5)}$$

where $\psi_D(\ell)$ is the deuteron wave function in momentum space. We keep the spin variables implicit in this and the next section to avoid unnecessary complications. In Eq. (IV.5) we used the δ-function normalization for the plane-waves, i.e. the state is normalized in such a way that

$$\langle\psi_i(\vec{q}_1,\vec{K}_1)|\psi_i(\vec{q}_o,\vec{K}_o)\rangle = \delta(\vec{K}_1-\vec{K}_o)\delta(\vec{q}_1-\vec{q}_o) \; .$$

The representative of this initial state in configuration space is

$$\langle\vec{r},\vec{\rho},\vec{R}|\psi_i(\vec{q}_0,\vec{K}_0)\rangle = \frac{1}{(2\pi)^{3/2}} e^{i\vec{q}_0\cdot\vec{\rho}} \frac{1}{(2\pi)^{3/2}} e^{i\vec{K}_0\cdot\vec{R}} \psi_D(r)$$

where $\psi_D(r)$ is the deuteron wave function in configuration space.

For the final state, if the two nucleons are bound, we have, analogously to the initial state

$$\langle\vec{\ell},\vec{q},\vec{K}|\psi_f(\vec{q}_f,\vec{K}_f)\rangle = \delta(\vec{q}-\vec{q}_f)\delta(\vec{K}-\vec{K}_f)\psi_f(\ell) \tag{IV.6}$$

where $\psi_F(\ell)$ is a function, with normalization

$$\int \psi_F^*(\ell)\psi_F(\ell)d_3\vec{\ell} = 1$$

describing the nucleon-nucleon bound state.

If the final state consists of three free particles, its representative in momentum space is

$$\langle\vec{q},\vec{K},\vec{\ell}|\Phi_F(\vec{q}_f,\vec{K}_f,\vec{\ell}_f)\rangle = \delta(\vec{q}-\vec{q}_f)\delta(\vec{K}-\vec{K}_f)\delta(\vec{\ell}-\vec{\ell}_f)$$

or (IV.7)

$$\langle\vec{q},\vec{p},\vec{n}|\Phi_F(\vec{q}_f,\vec{n}_f,\vec{p}_f)\rangle = \delta(\vec{q}-\vec{q}_f)\delta(\vec{n}-\vec{n}_f)\delta(\vec{p}-\vec{p}_f)$$

while its configuration space representative is

$$\langle\vec{r},\vec{\rho},\vec{R}|\Phi_F(\vec{q}_f,\vec{K}_f,\vec{\ell}_f)\rangle = \frac{1}{(2\pi)^{3/2}} e^{i\vec{g}_f\cdot\vec{\rho}} \frac{1}{(2\pi)^{3/2}} e^{i\vec{K}_f\cdot\vec{R}} \frac{1}{(2\pi)^{3/2}} e^{i\vec{\ell}_f\cdot\vec{r}} .$$

This three-free-particle state is normalized so that

$$\langle\Phi_F(\vec{\ell}_f',\vec{q}_f',\vec{K}_f')|\Phi_F(\vec{q}_f,\vec{K}_f,\vec{\ell}_f)\rangle = \delta(\vec{\ell}_f'-\vec{\ell}_f)\delta(\vec{q}_f'-\vec{q}_f)\delta(\vec{K}_f'-\vec{K}_f) .$$

We are not interested in the case of a bound meson-nucleon state which could, in principle, exist but in fact does not occur in our actual problem.

2. The Structure of the t-operators

As t_p does not act on the neutron, we have

$$\langle \vec{p}\,',\vec{n}\,',\vec{q}\,'|t_p|\vec{p},\vec{n},\vec{q}\rangle = \langle \vec{n}\,'|\vec{n}\rangle\langle \vec{p}\,',\vec{q}\,'|t_p|\vec{p},\vec{q}\rangle =$$

$$= \delta(\vec{n}\,'-\vec{n})\langle \vec{p}\,',\vec{q}\,'|t_p|\vec{p},\vec{q}\rangle \ .$$

$\langle \vec{p}\,',\vec{q}\,'|t_p|\vec{p},\vec{q}\rangle$ contains as a factor a δ-function responsible for conservation of momentum in the collision of the two particles. We define a new operator r_p by

$$\langle \vec{p}\,',\vec{q}\,'|t_p|\vec{p},\vec{q}\rangle = -\delta(\vec{p}\,'+\vec{q}\,'-\vec{p}-\vec{q})\langle \vec{p}\,',\vec{q}\,'|r_p|\vec{p},\vec{q}\rangle \ . \quad \text{(IV.8)}$$

$\langle \vec{p}\,',\vec{q}\,'|r_p|\vec{p},\vec{q}\rangle$ depends, in fact, only on the relative meson-proton momentum, so that

$$\langle \vec{p}\,',\vec{q}\,'|t_p|\vec{p},\vec{q}\rangle = -\delta(\vec{p}\,'+\vec{q}\,'-\vec{p}-\vec{q})\langle \vec{k}_p\,'|r_p|\vec{k}_p\rangle =$$

$$= -\delta(\vec{p}\,'+\vec{q}\,'-\vec{p}-\vec{q})\left\langle \frac{E_{p'}\vec{q}\,'-E_{q'}\vec{p}\,'}{E_{p'}+E_{q'}}\right| r_p \left|\frac{E_p\vec{q}-E_q\vec{p}}{E_p+E_q}\right\rangle . \quad \text{(IV.9)}$$

We can introduce in the second member the set of variables $\vec{\ell}$, $\vec{K}$, $\vec{q}$ in the place of $\vec{q}$, $\vec{p}$, $\vec{n}$. Let us note that $\vec{p}\,'+\vec{q}\,'=\vec{p}+\vec{q}$

corresponds to

$$\vec{\ell}' = \frac{E_{n'}\vec{p}' - E_{p'}\vec{n}'}{E_{n'} + E_{p'}} = \frac{E_{n'}(\vec{p} + \vec{q} - \vec{q}\,') - E_{p'}\vec{n}}{E_{n'} + E_{p'}} =$$

$$= \vec{\ell} + \frac{E_{n'}}{E_{n'} + E_{p'}}(\vec{q} - \vec{q}\,') + \vec{K}\left(\frac{E_p}{E_n + E_p} - \frac{E_{p'}}{E_{n'} + E_{p'}}\right)$$

so that

$$\langle \vec{p}\,', \vec{n}', \vec{q}\,' | t_p | \vec{p}, \vec{n}, \vec{q} \rangle =$$

$$= - \delta(\vec{K}' + \vec{q}\,' - \vec{K} - \vec{q})\, \delta\left[\vec{\ell}' - \vec{\ell} - \frac{E_{n'}}{E_{n'} + E_{p'}}(\vec{q} - \vec{q}\,') - \vec{K}\left(\frac{E_p}{E_n + E_p} - \frac{E_{p'}}{E_{n'} + E_{p'}}\right)\right]$$

$$\times \langle \vec{k}'_p | r_p | \vec{k}_p \rangle \qquad \text{(IV.10}$$

where

$$\vec{k}'_p = \frac{1}{E_{p'} + E_{q'}}\left[E_{p'}\vec{q}\,' - E_{q'}\vec{\ell}' - E_{q'}\frac{E_{p'}}{E_{n'} + E_{p'}}\vec{K}'\right]$$

$$\text{(IV.11)}$$

and

$$\vec{k}_p = \frac{1}{E_p + E_q}\left[E_p\vec{q} - E_q\vec{\ell} - E_q\frac{E_p}{E_n + E_p}\vec{K}\right] .$$

In the same way we can find the explicit dependence of $\langle \vec{p}\,', \vec{q}\,', \vec{n}' | t_n | \vec{p}, \vec{q}, \vec{n} \rangle$ on the momentum variables. As $\vec{q}\,' + \vec{n}' = \vec{q} + \vec{n}$ implies that

$$\vec{\ell}' = \vec{\ell} - \frac{E_{p'}}{E_{n'} + E_{p'}}(\vec{q}_o - \vec{q}_f) + \vec{K}\left(\frac{E_p}{E_n + E_p} - \frac{E_{p'}}{E_{n'} + E_{p'}}\right)$$

we obtain

$$\langle \vec{p}', \vec{n}', \vec{q}' | t_n | \vec{p}, \vec{n}, \vec{q} \rangle =$$

$$= - \delta(\vec{K}' + \vec{q}' - \vec{K} - \vec{q}) \, \delta \left[\vec{\ell}' - \vec{\ell} + \frac{E_{p'}}{E_{n'} + E_{p'}} (\vec{q}_0 - \vec{q}_f) - \vec{K} \left(\frac{E_p}{E_n + E_p} - \frac{E_{p'}}{E_{n'} + E_{p'}} \right) \right]$$

$$\times \langle \vec{k}_n' | {}^r p | \vec{k}_n \rangle \qquad \text{(IV.12)}$$

where

$$\vec{k}_n' = \frac{1}{E_{n'} + E_{q'}} \left[E_{n'} \vec{q}' + E_{q'} \vec{\ell}' - E_{q'} \frac{E_{n'}}{E_{n'} + E_{p'}} \vec{K}' \right]$$

and (IV.13)

$$\vec{k}_n = \frac{1}{E_n + E_q} \left[E_n \vec{q} + E_q \vec{\ell} - E_q \frac{E_n}{E_n + E_p} \vec{K} \right] .$$

Only t_u remains to be discussed. As it involves only the nucleon-nucleon interaction, the meson is left alone, and analogously to the previous cases we have

$$\langle \vec{p}', \vec{q}', \vec{n}' | t_u | \vec{p}, \vec{q}, \vec{n} \rangle = - \delta(\vec{q}' - \vec{q}) \delta(\vec{p}' + \vec{n}' - \vec{p} - \vec{n}) \langle \vec{\ell}' | r_u | \vec{\ell} \rangle$$

$$= - \delta(\vec{q}' + \vec{K}' - \vec{q} - \vec{k}) \delta(\vec{K}' - \vec{K}) \langle \vec{\ell}' | r_u | \vec{\ell} \rangle . \qquad \text{(IV.14)}$$

For off-the-energy shell matrix elements, the two kinds of collision operators, t^+ and t^- are different. We must then define two corresponding operators r^+ and r^-:

$$\langle \vec{p}_1', \vec{p}_2' | t^{\pm} | \vec{p}_1, \vec{p}_2 \rangle = - \delta(\vec{p}_1' + \vec{p}_2' - \vec{p}_1 - \vec{p}_2) \langle \vec{p}_1', \vec{p}_2' | r^{\pm} | \vec{p}_1, \vec{p}_2 \rangle .$$

The differential cross-section is given as usual by

$$\frac{d\sigma}{d\Omega} = \int \sum \frac{(2\pi)^4}{v} \left| < f | r | i > \right|^2 \delta(E_f - E_i)\, p'^2 dp'$$

where v is relative velocity of the two particles, p′ is the momentum of one of the outgoing particles, and Σ means sum over final and average over initial spin states.

For a non-relativistic collision, an on-the-energy shell matrix element $< \vec{k}' | r | \vec{k} >$ is related to the scattering amplitude $f(\Theta)$ by

$$f(\Theta) = (2\pi)^2 \mu < \vec{k}' | r | \vec{k} >$$

where Θ is the angle between $\vec{k}'$ and $\vec{k}$, and μ is the reduced mass of the system of the two colliding particles.

3. The Single-Scattering Terms in the Expansion of T_{el} and T_{inel}

For scattering by the proton we have

$$< \psi_f | t_p | \psi_i > = \int d_3\vec{q}' d_3\vec{K}' d_3\vec{\ell}' < \psi_f | \vec{\ell}' \vec{q}' \vec{K}' > < \vec{\ell}' \vec{q}' \vec{K}' | t_p | \vec{\ell} \vec{q} \vec{K} > < \vec{\ell} \vec{q} \vec{K} | \psi_i > d_3\vec{q} d_3\vec{K} d_3\vec{\ell} \,. \qquad \text{(IV.15)}$$

By using Eqs.(IV.5), (IV.6) and (IV.10) and choosing $\vec{K}_0 = 0$ (deuteron at rest in the laboratory system) we get

$$<\psi_f|t_p|\psi_i> = -\,\delta(\vec{K}_f + \vec{q}_f - \vec{q}_o)\int \psi_F^*(\ell')\,\delta\left(\vec{\ell}' - \vec{\ell} - \frac{E_{n'}}{E_{n'} + E_{p'}}\,(\vec{q}_o - \vec{q}_f)\right)$$

$$<\vec{k}_p'|r_p|\vec{k}_p> \times \psi_D(\ell)\,d_3\vec{\ell}'\,d_3\vec{\ell} \qquad \text{(IV.16)}$$

where $\vec{k}_p'$, $\vec{k}_p$ are given by Eq. (IV.11) with $\vec{q}' \to \vec{q}_f$, $\vec{q} \to \vec{q}_o$, $\vec{K} = 0$, $\vec{K}' = \vec{K}_f$. For elastic scattering we have

$$<\psi_f(\vec{q}_f,\vec{K}_f)|t_p|\psi_i(\vec{q}_o,\vec{K}_o = 0)> = -\,\delta(\vec{K}_f + \vec{q}_f - \vec{q}_o)\int \psi_D^*\left(\vec{\ell} + \tfrac{1}{2}(\vec{q}_o - \vec{q}_f)\right)$$

$$\left\langle \frac{M\vec{q}_f - E_{q_f}\vec{\ell} - E_{q_f}(\vec{q}_o - \vec{q}_f)}{M + E_{q_f}} \,\middle|\, r_p \,\middle|\, \frac{M\vec{q}_o - E_{q_o}\vec{\ell}}{M + E_{q_o}} \right\rangle \times \psi_D(\ell)\,d_3\vec{\ell}\,. \qquad \text{(IV.17)}$$

For inelastic scattering, with non-relativistic nucleons in the final state, we obtain

$$<\Phi_f(\vec{q}_f,\vec{n}_f,\vec{p}_f)|t_p|\psi_i(\vec{q}_o,\vec{K}_o = 0)> =$$

$$= -\,\delta(\vec{K}_f + \vec{q}_f - \vec{q}_o)\,\psi_D(n_f)\left\langle \frac{M\vec{q}_f - E_{q_f}\vec{p}_f}{M + E_{q_f}} \,\middle|\, r_p \,\middle|\, \frac{M\vec{q}_o + E_{q_o}\vec{n}_f}{M + E_{q_o}} \right\rangle . \qquad \text{(IV.18)}$$

For single scattering by the neutron we have

$$<\psi_f(\vec{q}_f,\vec{K}_f)|t_n|\psi_i(\vec{q}_o,\vec{K}_o = 0)> = -\,\delta(\vec{K}_f + \vec{q}_f - \vec{q}_o) \times$$

$$\times \int \psi_F^*(\ell')\,\delta\left(\vec{\ell}' - \vec{\ell} + \frac{E_{p'}}{E_{n'} + E_{p'}}\,(\vec{q}_o - \vec{q}_f)\right) \times <\vec{k}_n'|r_n|\vec{k}_n>\,\psi_D(\ell)\,d_3\vec{\ell}'\,d_3\vec{\ell}$$

$$\text{(IV.19)}$$

with $\vec{k}'_n, \vec{k}_n$ given by Eq. (IV.13) with $\vec{q} = \vec{q}_0$, $\vec{K} = \vec{K}_0 = 0$, $\vec{K}' = \vec{K}_f$, $\vec{q}' = \vec{q}_f$. For elastic scattering

$$\left\langle \psi_f(\vec{q}_f,\vec{K}_f)\right| t_n \left|\psi_i(\vec{q}_0,\vec{K}_0 = 0)\right\rangle = - \delta(\vec{K}_f + \vec{q}_f - \vec{q}_0) \times \int \psi_D^*\left(\vec{\ell} - \tfrac{1}{2}(\vec{q}_0 - \vec{q}_f)\right) \times$$
$$\left\langle \frac{M\vec{q}_f + E_{q_f}\vec{\ell} - E_{q_f}(\vec{q}_0 - \vec{q}_f)}{M + E_{q_f}} \left| r_n \right| \frac{M\vec{q}_0 + E_{q_0}\vec{\ell}}{M + E_{q_0}} \right\rangle \psi_D(\ell) d_3\vec{\ell} \; . \qquad \text{(IV.20)}$$

For inelastic scattering we obtain

$$\left\langle \Phi_f(\vec{q}_f,\vec{n}_f,\vec{p}_f)\right| t_n \left|\psi_i(\vec{q}_0,\vec{K}_0 = 0)\right\rangle =$$
$$= - \delta(\vec{K}_f + \vec{q}_f - \vec{q}_0)\psi_D(p_f) \left\langle \frac{M\vec{q}_f - E_{q_f}\vec{n}_f}{M + E_{q_f}} \left| r_n \right| \frac{M\vec{q}_0 + E_{q_0}\vec{p}_f}{M + E_{q_0}} \right\rangle \; . \qquad \text{(IV.21)}$$

4. The Double-Scattering Terms

Using Eqs. (IV.6), (IV.10) and (IV.12) we get

$$\left\langle \psi_f \right| t_n \frac{1}{E - K + i\epsilon} t_p \left| \psi_i \right\rangle =$$
$$= \int \left\langle \psi_f(\vec{q}_f,\vec{K}_f) \middle| \vec{\ell}'\vec{q}'\vec{K}' \right\rangle \left\langle \vec{\ell}'\vec{q}'\vec{K}' \right| t_n \frac{1}{E - K + i\epsilon} t_p \left| \vec{\ell}\,\vec{q}\,\vec{K} \right\rangle \times$$
$$\times \left\langle \vec{\ell}\vec{q}\,\vec{K} \middle| \psi_i(\vec{q}_0,\vec{K}_0 = 0)\right\rangle d_3\vec{\ell}'\, d_3\vec{q}'\, d_3\vec{K}'\, d_3\vec{\ell}\, d_3\vec{q}\, d_3\vec{K} \; .$$

Introducing complete sets of free particle states between t_n, t_p and the propagator we may write this as

$$\int \psi_F^*(\ell') \left\langle \vec{n}', \vec{q}_f | r_n | \vec{n}_m, \vec{q}_m \right\rangle \delta(\vec{p}' - \vec{p}_m)\delta(\vec{n}' + \vec{q}_f - \vec{n}_m - \vec{q}_m) \times$$

$$\times \frac{1}{E - E_m + i\epsilon} \left\langle \vec{q}_m \vec{p}_m | r_p | \vec{q}_0 \vec{p} \right\rangle \delta(\vec{n}_m - \vec{n})\delta(\vec{p}_m + \vec{q}_m - \vec{p} - \vec{q}_0) \times$$

$$\times \psi_D(\ell)\, d_3\vec{\ell}'\, d_3\vec{\ell}\, d_3\vec{p}_m\, d_3\vec{n}_m\, d_3\vec{q}_m \ .$$

For elastic scattering we obtain

$$\left\langle \psi_f(\vec{K}_f \vec{q}_f) \right| t_n \frac{1}{E - K + i\epsilon} t_p \left| \psi_i(\vec{q}_0, \vec{K}_0 = 0) \right\rangle = \delta(\vec{K}_f + \vec{q}_f - \vec{q}_0) \times$$

$$\times \int \psi_D^*(\ell') \left\langle \vec{n}', \vec{q}_f | r_n | \vec{n}, \vec{q}_m \right\rangle \frac{1}{E - E_m + i\epsilon} \left\langle \vec{q}_m \vec{p}' | r_p | \vec{q}_0 \vec{p} \right\rangle$$

$$\times \psi_D(\ell)\, d_3\vec{\ell}'\, d_3\vec{\ell} \qquad \text{(IV.22)}$$

where $\vec{q}_m = \vec{n}' + \vec{q}_f - \vec{n} = \vec{q}_0 + \vec{p} - \vec{p}'$, $\vec{p} = \vec{\ell} = -\vec{n}$, $\vec{p}' = \vec{\ell}' + \frac{1}{2}\vec{K}_f$,
$\vec{n}' = -\vec{\ell}' + \frac{1}{2}\vec{K}_f$.

E_m is the total energy of the state where momenta of the particles are $\vec{q}_m$, $\vec{p}'$, $\vec{n}$. We have

$$E_m = M_n + \frac{\ell^2}{2M_n} + M_p + \frac{\vec{p}'^2}{2M_p} + \sqrt{m^2 + q_m^2}$$

for a relativistic meson (mass m) in the intermediate state. For inelastic scattering we put $\psi_F(\ell') = \delta(\vec{\ell}' - \vec{\ell}_f)$ and get

$$\left\langle \Phi_f(\vec{q}_f \vec{p}_f \vec{n}_f) \right| t_n \frac{1}{E - K + i\epsilon} t_p \left| \psi_i(\vec{q}_0, \vec{K}_0 = 0) \right\rangle = \delta(\vec{K}_f + \vec{q}_f - \vec{q}_0) \times$$

$$\times \int d_3\vec{\ell} \left\langle \vec{n}_f \vec{q}_f | r_n | \vec{n}\ \vec{q}_m \right\rangle \frac{1}{E - E_m + i\epsilon} \left\langle \vec{q}_m \vec{p}_f | r_p | \vec{q}_0, \vec{p} \right\rangle \psi_D(\ell) \qquad \text{(IV.23)}$$

where

$$\vec{p} = -\vec{n} = \vec{\ell}$$

$$\vec{q}_m = \vec{q}_0 + \vec{\ell} - \vec{p}_f = \vec{q}_f + \vec{\ell} + \vec{n}_f$$

$$E_m = M_n + \frac{\ell^2}{2M_n} + M_p + \frac{\vec{p}_f^{\,2}}{2M_p} + \sqrt{m^2 + q_m^2} \,.$$

More explicitly,

$$\left\langle \Phi_f(\vec{q}_f \vec{p}_f \vec{n}_f) \right| t_n \frac{1}{E - K + i\epsilon} t_p \left| \psi_i(\vec{q}_0, \vec{K}_0 = 0) \right\rangle = \delta(\vec{K}_f + \vec{q}_f - \vec{q}_0) \times$$

$$\times \int d_3\vec{\ell} \left\langle \frac{M\vec{q}_f - E_{q_f}\vec{n}_f}{M + E_{q_f}} \right| r_n \left| \frac{M(\vec{q}_0 + \vec{\ell} - \vec{p}_f) - E_{q_m}\vec{\ell}}{M + E_{q_m}} \right\rangle \frac{1}{E - E_m + i\epsilon} \times$$

$$\times \left\langle \frac{M\vec{q}_m - E_{q_m}\vec{p}_f}{M + E_{q_m}} \right| r_p \left| \frac{M\vec{q}_0 - E_{q_0}\vec{\ell}}{M + E_{q_0}} \right\rangle \psi_D(\ell) \qquad \text{(IV.24)}$$

When the first collision is on the neutron, we obtain in an entirely analgous way, for elastic scattering

$$<\psi_f(\vec{q}_f,\vec{K}_f)\left|t_p \frac{1}{E-K+i\epsilon} t_n\right| \psi_i(\vec{q}_0,\vec{K}_0=0)> = \delta(\vec{K}_f+\vec{q}_f-\vec{q}_0)\times$$

$$\times\int \psi_D^*(\vec{\ell}')\times <\vec{p}',\vec{q}_f|r_p|\vec{p},\vec{q}_m> \frac{1}{E-E_m+i\epsilon} <\vec{q}_m\vec{n}'|r_n|\vec{q}_0,\vec{n}>$$

$$\psi_D(\ell)\, d_3\vec{\ell}'\, d_3\vec{\ell} \qquad \text{(IV.25)}$$

where $\vec{p}=\vec{\ell}=-\vec{n}$, $\vec{q}_m=\vec{q}_0+\vec{n}-\vec{n}'=\vec{p}'+\vec{q}_f-\vec{p}$ and E_m is the energy of the state in which momenta of the particles are $\vec{q}_m$, $\vec{n}'$, $\vec{p}$, i.e.

$$E_m = M_p + \frac{\ell^2}{2M_p} + M_n + \frac{\vec{n}'^2}{2M_n} + \sqrt{m^2+\vec{q}_m^{\,2}}\ .$$

For inelastic scattering

$$<\Phi_f(\vec{p}_f\vec{n}_f\vec{q}_f)\left|t_p \frac{1}{E-K+i\epsilon} t_n\right| \psi_i(\vec{q}_0,\vec{K}_0=0)> = \delta(\vec{K}_f+\vec{q}_f-\vec{q}_0)\times$$

$$\times\int d_3\vec{\ell} <\vec{p}_f\vec{q}_f|r_p|\vec{p}\ \vec{q}_m> \frac{1}{E-E_m+i\epsilon} <\vec{q}_m\vec{n}_f|r_n|\vec{q}_0,\vec{n}> \psi_D(\ell) \qquad \text{(IV.26)}$$

where $\vec{q}_m=\vec{q}_0+\vec{n}-\vec{n}_f=\vec{q}_0-\vec{\ell}-\vec{n}_f=\vec{q}_f+\vec{p}_f-\vec{\ell}$

and $$E_m = M_p + \frac{\ell^2}{2M_p} + M_n + \frac{\vec{n}_f^{\,2}}{2M_n} + \sqrt{m^2+\vec{q}_m^{\,2}}\ .$$

5. The Potential Correction Terms for Inelastic Scattering

We now consider the terms of the type $t_u \frac{1}{E-K+i\epsilon} t_p$ which occur in our expansion of T_{inel}. By introducing sets of free particle states in the appropriate places we obtain

$$\langle \Phi_f(\vec{q}_f\vec{p}_f\vec{n}_f) \Big| t_u \frac{1}{E-K+i\epsilon} t_p \Big| \psi_i(\vec{q}_0,\vec{K}_0 = 0) \rangle =$$

$$= \int \langle \Phi_f(\vec{q}_f\vec{p}_f\vec{n}_f) | \vec{\ell}'\vec{q}'\vec{K}' \rangle \langle \vec{\ell}'\vec{q}'\vec{K}' | t_u | \vec{\ell}_m\vec{q}_m\vec{K}_m \rangle \frac{1}{E-E_m+i\epsilon} \times$$

$$\times \langle \vec{\ell}_m\vec{q}_m\vec{K}_m | t_p | \vec{\ell}\,\vec{q}\,\vec{K} \rangle \langle \vec{\ell}\,\vec{q}\,\vec{K} | \psi_i \rangle \, d_3\vec{\ell}' \, d_3\vec{q}' \, d_3\vec{K}' \, d_3\vec{\ell} \, d_3\vec{q} \, d_3\vec{K} \, d_3\vec{\ell}_m \times$$

$$\times \, d_3\vec{q}_m \, d_3\vec{K}_m \, .$$

Using the representatives of the final and initial states and the properties of t_u and t_p as given by (IV.10) and (IV.14), we obtain

$$\langle \Phi_f(\vec{p}_f\vec{n}_f\vec{q}_f) \Big| t_u \frac{1}{E-K+i\epsilon} t_p \Big| \psi_i(\vec{q}_0,\vec{K}_0 = 0) \rangle = \delta(\vec{K}_f + \vec{q}_f - \vec{q}_0) \times$$

$$\times \int d_3\vec{\ell} \, \langle \vec{p}_f\vec{n}_f | r_u | \vec{p}_m\vec{n} \rangle \frac{1}{E-E_m+i\epsilon} \langle \vec{q}_f\vec{p}_m | r_p | \vec{q}_0\vec{p} \rangle \qquad \text{(IV.27)}$$

where $\vec{p}_m = \vec{p} + \vec{q}_0 - \vec{q}_f = \vec{\ell} + \vec{q}_0 - \vec{q}_f$ and E_m is the energy of the intermediate state

$$E_m = E_{q_f} + M_n + \frac{\ell^2}{2M_n} + M_p + \frac{(\vec{\ell} + \vec{q}_0 - \vec{q}_f)^2}{2M_p} \, .$$

Analogously we get

$$\langle \Phi_f(\vec{p}_f\vec{q}_f\vec{n}_f) \Big| t_u \frac{1}{E - K + i\epsilon} t_n \Big| \psi_i(\vec{q}_0, \vec{K}_0 = 0) \rangle = \delta(\vec{K}_f + \vec{q}_f - \vec{q}_0) \times$$

$$\times \int d_3\vec{\ell} \langle \vec{p}_f\vec{n}_f | t_u | \vec{p}\ \vec{n}_m \rangle \frac{1}{E - E_m + i\epsilon} \langle \vec{q}_f\vec{n}_m | t_n | \vec{q}_0\vec{n} \rangle \psi_D(\ell) \qquad \text{(IV.28)}$$

where $\qquad \vec{n}_m = \vec{q}_0 + \vec{n} - \vec{q}_f = \vec{q}_0 - \vec{q}_f - \vec{\ell}$

and $\qquad E_m = E_{q_f} + M_p + \frac{\ell^2}{2M_p} + M_n + \frac{(\vec{q}_0 - \vec{q}_f - \vec{\ell})^2}{2M_n}$.

6. Expressions for the Cross-Section

We see that all the terms contributing to the expansions of T_{el} and T_{inel} have $\delta(\vec{K}_f + \vec{q}_f - \vec{q}_0 - \vec{K}_0)$ as a factor, as they should. We define the operators R_{el} and R_{inel} by

$$(T_{el})_{fi} = - \delta(\vec{K}_f + \vec{q}_f - \vec{q}_0 - \vec{K}_0)(R_{el})_{fi} \qquad \text{(IV.29)}$$

$$(T_{inel})_{fi} = - \delta(\vec{K}_f + \vec{q}_f - \vec{q}_0 - \vec{K}_0)(R_{inel})_{fi} \ . \qquad \text{(IV.30)}$$

The elastic scattering differential cross-section will then be given by

$$d\sigma_{el} = \int \sum{}' \frac{(2\pi)^4}{v} \delta(E_f - E_i) \Big| (R_{el})_{fi} \Big|^2 q_f^2 dq_f \, d\Omega_{q_f} \qquad \text{(IV.31)}$$

where Σ' represente the appropriate sum and average over the final and initial spin states. $d\Omega_{q_f}$ is the element of solid angle about the direction of $\vec{q}_f$. v is the velocity of the incident meson. If we want the differential cross-section we do not integrate over $d\Omega_{q_f}$.

The inelastic scattering cross-section is given by

$$d\sigma_{inel} = \int \sum{}' \frac{(2\pi)^4}{v} \delta(E_f - E_i) \left| (R_{inel})_{fi} \right|^2 d_3\vec{q}_f \, d_3\vec{p}_f . \quad (IV.32)$$

7. Spin Variables and Spin Sums

The most general meson-proton interaction can be described by an operator

$$r_p = a_p + \vec{\sigma}_p \cdot \vec{b}_p \quad (IV.33)$$

where $\vec{\sigma}_p$ is the proton spin matrix and a_p and $\vec{b}_p$ do not depend on spin variables. The vector b_p depends on the relative momenta of the two colliding particles in the initial and final states. Analogously, for the meson-neutron interaction we have

$$r_n = a_n + \vec{\sigma}_n \cdot \vec{b}_n . \quad (IV.34)$$

The general fermion-fermion interaction has a much more complicated dependence on spin variables (Ref. 18). In our problem we shall be concerned with relative energies of the nucleon-nucleon system which are not very high, and we can assume they interact in S-and P-waves only. Let us first assume the nucleon-nucleon interaction is purely S-wave.

A. Nucleons interacting in S-waves only

Under this condition the most general form for r_u is

$$r_u = a_u + b_u(\vec{\sigma}_p \cdot \vec{\sigma}_n)$$

where a_u and b_u do not depend on spin variables. We prefer to write this as

$$r_u = c_s\left(\frac{1 - \vec{\sigma}_p \cdot \vec{\sigma}_n}{4}\right) + c_t\left(\frac{3 + \vec{\sigma}_p \cdot \vec{\sigma}_n}{4}\right) = c_s P_s + c_t P_t \qquad \text{(IV.35)}$$

where $P_s = \frac{1 - \vec{\sigma}_p \cdot \vec{\sigma}_n}{4}$ and $P_t = \frac{3 + \vec{\sigma}_p \cdot \vec{\sigma}_n}{4}$ are the projection operators for the singlet and triplet states of the nucleon-nucleon system respectively. c_s and c_t do not depend on spin variables.

In the terms of the expansions of T_{el} and T_{inel}, there appear products $r_u r_p$, $r_u r_n$, $r_p r_n$ besides the single scattering terms r_p and r_n. So, as far as spin variables are concerned, the most general form of matrix element which we have to consider (if we include up to second order processes and if the nucleon-nucleon interaction in the final state is purely S-wave), is the matrix element of

$$R_1 = A + \vec{B}\cdot\vec{\sigma}_p + \vec{C}\cdot\vec{\sigma}_n + (\vec{D}\cdot\vec{\sigma}_p)(\vec{E}\cdot\vec{\sigma}_n) + F\,P_s + G\,P_t +$$

$$+ P_s(\vec{H}\cdot\vec{\sigma}_p) + P_s(\vec{I}\cdot\vec{\sigma}_n) + P_t(\vec{J}\cdot\vec{\sigma}_p) + P_t(\vec{K}\cdot\vec{\sigma}_n) \qquad \text{(IV.36)}$$

calculated between the initial and final state of our system of one meson and two nucleons. The first three terms refer to single scattering of the meson by a nucleon, the fourth one refers to the double scattering processes, and the others to meson-nucleon collisions followed by nucleon-nucleon interactions.

The initial state is a triplet (spin of deuteron is one). Our problem is to square the matrix element $< f|R_1|i >$ and then sum over some or all of the possible polarizations in the final state, and to take the average over the three directions of polarizations of the deuteron in the initial state. We have two independent spin variables $\vec{\sigma}_p$ and $\vec{\sigma}_n$ belonging to independent spaces. Their correlation in initial or final states can be taken into account by means of the appropriate projection operators; we then sum over the two states of polarization of each of the two nucleons. We get the following results.

i) Initial State is a Triplet State, Final State is any one.

$$\frac{1}{3}\sum_{f,i} \langle i|R_1^+|f\rangle\langle f|R_1 P_t|i\rangle = \frac{1}{3}\,\mathrm{Tr}_p\,\mathrm{Tr}_n\,(R_1^+ R_1 P_t) =$$

$$= \frac{1}{3}\Big\{3(A^+ + G^+)(A+G) + (A^+ + G^+)(\vec{D}\cdot\vec{E}) + (A+G)(\vec{D}^+\cdot\vec{E}^+) +$$

$$+ 3(\vec{D}^+\cdot\vec{D})(\vec{E}^+\cdot\vec{E}) + 2(\vec{J}^+ + \vec{K}^+ + \vec{C}^+ + \vec{B}^+)\cdot(\vec{J}+\vec{K}+\vec{C}+\vec{B}) +$$

$$+ (\vec{H}^+ - \vec{I}^+ - \vec{C}^+ + \vec{B}^+)\cdot(\vec{H}-\vec{I}-\vec{C}+\vec{B}) - (\vec{D}^+\cdot\vec{E}^+)(\vec{D}\cdot\vec{E}) +$$

$$+ (\vec{D}\cdot\vec{E}^+)(\vec{D}^+\cdot\vec{E}) + i\Big[(\vec{B}^+ - \vec{C}^+ - \vec{I}^+ + \vec{H}^+)\cdot(\vec{D}\wedge\vec{E}) -$$

$$- (\vec{B}-\vec{C}-\vec{I}+\vec{H})\cdot(\vec{D}^+\wedge\vec{E}^+)\Big]\Big\}. \qquad \text{(IV.37)}$$

ii) Initial State is Triplet, Final State is Triplet.

$$\frac{1}{3}\sum_{f,i} \langle i|R_1^+ P_t|f\rangle\langle f|R_1 P_t|i\rangle = \frac{1}{3}\,\mathrm{Tr}_p\,\mathrm{Tr}_n\,(R_1^+ P_t R_1 P_t) =$$

$$= \frac{1}{3}\Big\{ 3(A^+ + G^+)(A+G) + (A^+ + G^+)(\vec{D}\cdot\vec{E}) + (A+G)(\vec{D}^+\cdot\vec{E}^+) +$$

$$+ 2(\vec{B}^+ + \vec{J}^+ + \vec{C}^+ + \vec{K}^+)\cdot(\vec{B}+\vec{J}+\vec{C}+\vec{K}) + 2(\vec{D}^+\cdot\vec{D})(\vec{E}^+\cdot\vec{E}) -$$

$$- (\vec{D}^+\cdot\vec{E}^+)(\vec{D}\cdot\vec{E}) + 2(\vec{D}^+\cdot\vec{E})(\vec{E}^+\cdot\vec{D}) \Big\} \qquad \text{(IV.38)}$$

iii) Initial State is Triplet, Final State is Singlet.

$$\frac{1}{3}\sum_{f,i} \langle i|R_1^+ P_s|f\rangle\langle f|R_1 P_t|i\rangle = \frac{1}{3}\,\mathrm{Tr}_p\,\mathrm{Tr}_n\,(R_1^+ P_s R_1 P_t) =$$

$$= \frac{1}{3}\Big\{ (\vec{H}^+ - \vec{I}^+ - \vec{C}^+ + \vec{B}^+)\cdot(\vec{H}-\vec{I}-\vec{C}+\vec{B}) - (\vec{D}^+\cdot\vec{E})(\vec{E}^+\cdot\vec{D}) +$$

$$+ (\vec{D}^+\cdot\vec{D})(\vec{E}^+\cdot\vec{E}) + i\Big[(\vec{B}^+ + \vec{H}^+ - \vec{C}^+ - \vec{I}^+)\cdot(\vec{D}\wedge\vec{E}) -$$

$$- (\vec{B}+\vec{H}-\vec{C}-\vec{I})\cdot(\vec{D}^+\wedge\vec{E}^+)\Big]\Big\}. \qquad \text{(IV.39)}$$

In deriving these expressions the following formulae were used:

$$\mathrm{Tr}_p\ \mathrm{Tr}_n\left[(\vec{a}\cdot\vec{\sigma}_p)(\vec{b}\cdot\vec{\sigma}_p)1_n\right] = 4(\vec{a}\cdot\vec{b})$$

$$\mathrm{Tr}_p\ \mathrm{Tr}_n\left[(\vec{a}\cdot\vec{\sigma}_p)(\vec{b}\cdot\vec{\sigma}_n)(\vec{\sigma}_p\cdot\vec{\sigma}_n)\right] = 4(\vec{a}\cdot\vec{b})$$

$$(\vec{\sigma}_p\cdot\vec{\sigma}_n)(\vec{a}\cdot\vec{\sigma}_p)(\vec{\sigma}_p\cdot\vec{\sigma}_n) = -(\vec{a}\cdot\vec{\sigma}_p)\ 1_n + 2(\vec{a}\cdot\vec{\sigma}_n)1_p$$

$$\mathrm{Tr}_p\ \mathrm{Tr}_n\left[(\vec{a}\cdot\vec{\sigma}_p)(\vec{\sigma}_p\cdot\vec{\sigma}_n)(\vec{b}\cdot\vec{\sigma}_p)(\vec{\sigma}_p\cdot\vec{\sigma}_n)\right] = -4\ \vec{a}\cdot\vec{b}$$

$$\mathrm{Tr}_p\ \mathrm{Tr}_n\left[(\vec{a}\cdot\vec{\sigma}_p)(\vec{\sigma}_p\cdot\vec{\sigma}_n)(\vec{b}\cdot\vec{\sigma}_n)(\vec{\sigma}_p\cdot\vec{\sigma}_n)\right] = 8\ \vec{a}\cdot\vec{b}$$

$$\mathrm{Tr}_p\ \mathrm{Tr}_n\left[(\vec{a}\cdot\vec{\sigma}_p)(\vec{\sigma}_p\cdot\vec{\sigma}_n)(\vec{b}\cdot\vec{\sigma}_p)(\vec{c}\cdot\vec{\sigma}_n)(\vec{\sigma}_p\cdot\vec{\sigma}_n)\right] = 0 \qquad \text{(IV.40)}$$

$$\mathrm{Tr}_p\ \mathrm{Tr}_n\left[(\vec{a}\cdot\vec{\sigma}_p)(\vec{b}\cdot\vec{\sigma}_p)(\vec{c}\cdot\vec{\sigma}_p)\ 1_n\right] = 4i\ \vec{a}\cdot(\vec{b}\wedge\vec{c})$$

$$\mathrm{Tr}_p\ \mathrm{Tr}_n\left[(\vec{\sigma}_p\cdot\vec{\sigma}_n)(\vec{a}\cdot\vec{\sigma}_p)(\vec{b}\cdot\vec{\sigma}_p)(\vec{c}\cdot\vec{\sigma}_n)(\vec{d}\cdot\vec{\sigma}_n)\right] = 4\left[(\vec{d}\cdot\vec{a})(\vec{b}\cdot\vec{c}) - (\vec{d}\cdot\vec{b})(\vec{a}\cdot\vec{c})\right]$$

$$\mathrm{Tr}_p\ \mathrm{Tr}_n\left[(\vec{a}\cdot\vec{\sigma}_p)(\vec{b}\cdot\vec{\sigma}_p)(\vec{c}\cdot\vec{\sigma}_p)(\vec{d}\cdot\vec{\sigma}_p)1_n\right] = 4\left[(\vec{a}\cdot\vec{b})(\vec{c}\cdot\vec{d}) - (\vec{b}\cdot\vec{d})(\vec{a}\cdot\vec{c}) + (\vec{a}\cdot\vec{d})(\vec{b}\cdot\vec{c})\right]$$

$$\mathrm{Tr}_p\ \mathrm{Tr}_n\left[(\vec{a}\cdot\vec{\sigma}_p)(\vec{b}\cdot\vec{\sigma}_n)(\vec{\sigma}_p\cdot\vec{\sigma}_n)(\vec{c}\cdot\vec{\sigma}_p)(\vec{d}\cdot\vec{\sigma}_n)(\vec{\sigma}_p\cdot\vec{\sigma}_n)\right] = 4\left[2(\vec{a}\cdot\vec{d})(\vec{c}\cdot\vec{b}) + 2(\vec{a}\cdot\vec{b})(\vec{c}\cdot\vec{d}) - (\vec{b}\cdot\vec{d})(\vec{a}\cdot\vec{c})\right].$$

Here 1_n and 1_p are the unit matrices in the neutron and proton spin spaces respectively, and of course we get valid formulae analogous to the above by interchanging p and n indices. In the expression (IV.36) for R these unit matrices have not been explicitly written.

B. The P-waves in the nucleon-nucleon interaction

Now let us consider the contributions of the P-waves to the nucleon-nucleon interaction. If the total spin of the two nucleons is $S = 0$, only one value of the total angular momentum is possible, namely $J = L = 1$. There is no spin dependence of the nucleon-nucleon interaction in this particular state, and its contribution can be absorbed in Eq. (IV.36) by suitably modifying the parameters $F, \vec{H}, \vec{I}$ of the S-wave case.

If $S = 1$, three values of J are possible, namely 0,1,2 with three corresponding independent scattering amplitudes. The spin dependence of the scattering in these states can be obtained in the following way. We expand $P_1(\cos\Theta)\chi_1^m = \cos\Theta\, \chi_1^m$, where Θ is the scattering angle in the centre-of-mass system, and χ_1^m is the spin-function with $S = 1$ and spin component m along direction of quantization (which we choose to be the direction of the final relative momentum), in terms of the simultaneous eigenfunctions of J^2, L^2, S^2, J_z. This expansion is easily obtained with the help of a table of Clebsch-Gordan coefficients. It is

$$Y_{1,0}(\Theta)\chi_1^m = \left(\frac{3}{4\pi}\right)^{\frac{1}{2}} P_1(\cos\Theta)\chi_1^m = -\sqrt{\frac{1}{3}}\,\delta_{m,0}\,\mathcal{Y}^{0}_{0,1,1} + \left(\sqrt{\frac{1}{2}}\,\delta_{m,-1}\,\mathcal{Y}^{-1}_{1,1,1}\right.$$

$$\left. - \sqrt{\frac{1}{2}}\,\delta_{m,1}\,\mathcal{Y}^{1}_{1,1,1}\right) + \left(\sqrt{\frac{1}{2}}\,\delta_{m,1}\,\mathcal{Y}^{1}_{2,1,1} + \sqrt{\frac{2}{3}}\,\delta_{m,0}\,\mathcal{Y}^{0}_{2,1,1} + \sqrt{\frac{1}{2}}\,\delta_{m,-1}\,\mathcal{Y}^{-1}_{2,1,1}\right) \tag{IV.41}$$

where $\mathcal{Y}^{M}_{J,L,S}$ indicates the normalized eigenfunctions of J^2, L^2, S^2, J_z with eigenvalues $J(J+1)$, $L(L+1)$, $S(S+1)$, M. We can then separate the J = 2,1,0 parts and write each of them again in terms of angle and spin functions by using

$$\mathcal{Y}^{0}_{0,1,1} = \sqrt{\tfrac{1}{3}}\, Y_{1,1}(\Theta,\varphi)\chi_1^{-1} + \sqrt{\tfrac{1}{3}}\, Y_{1,-1}(\Theta,\varphi)\chi_1^{1} - \sqrt{\tfrac{1}{3}}\, Y_{1,0}(\Theta,\varphi)\chi_1^{0}$$

$$\mathcal{Y}^{1}_{1,1,1} = \sqrt{\tfrac{1}{2}}\, Y_{1,1}(\Theta,\varphi)\chi_1^{0} - \sqrt{\tfrac{1}{2}}\, Y_{1,0}(\Theta,\varphi)\chi_1^{1}$$

$$\mathcal{Y}^{0}_{1,1,1} = \sqrt{\tfrac{1}{2}}\, Y_{1,1}(\Theta,\varphi)\chi_1^{-1} - \sqrt{\tfrac{1}{2}}\, Y_{1,-1}(\Theta,\varphi)\chi_1^{1}$$

$$\mathcal{Y}^{-1}_{1,1,1} = \sqrt{\tfrac{1}{2}}\, Y_{1,0}(\Theta,\varphi)\chi_1^{-1} - \sqrt{\tfrac{1}{2}}\, Y_{1,-1}(\Theta,\varphi)\chi_1^{0} \qquad \text{(IV.42)}$$

$$\mathcal{Y}^{1}_{2,1,1} = \sqrt{\tfrac{1}{2}}\, Y_{1,1}(\Theta,\varphi)\chi_1^{0} + \sqrt{\tfrac{1}{2}}\, Y_{1,0}(\Theta,\varphi)\chi_1^{1}$$

$$\mathcal{Y}^{0}_{2,1,1} = \sqrt{\tfrac{1}{6}}\, Y_{1,1}(\Theta,\varphi)\chi_1^{-1} + \sqrt{\tfrac{1}{6}}\, Y_{1,-1}(\Theta,\varphi)\chi_1^{1} + \sqrt{\tfrac{2}{3}}\, Y_{1,0}(\Theta,\varphi)\chi_1^{0}$$

$$\mathcal{Y}^{-1}_{2,1,1} = \sqrt{\tfrac{1}{2}}\, Y_{1,-1}(\Theta,\varphi)\chi_1^{0} + \sqrt{\tfrac{1}{2}}\, Y_{1,0}\chi_1^{-1}$$

which were obtained by again consulting a table of Clebsch-Gordan coefficients. These formulae are general for addition of angular momentum vectors $\vec{L}$ and $\vec{S}$ with magnitudes L = S = 1. We can now specify the spin $\vec{S}$ as being the result of the sum of the intrinsic spins of two spin-$\frac{1}{2}$ particles: $\vec{S} = \frac{1}{2}(\vec{\sigma}_p + \vec{\sigma}_n)$. Introducing the corresponding spin functions χ_1^m, using convenient projection operators to eliminate the Kronecker δ-symbols, utilising properties of the Pauli spin matrices, etc., we obtain (Ref. 19) that the $J = 0$ part of $\cos\Theta\ \chi_1^m$ can be written

$$\frac{1}{3}\ \frac{1}{2}(\vec{\sigma}_p - \vec{\sigma}_n) \cdot \frac{\vec{k}_i}{k_i}\ \frac{1}{2}(\vec{\sigma}_p - \vec{\sigma}_n) \cdot \frac{\vec{k}_f}{k_f}\ \chi_1^m \tag{IV.43}$$

the $J = 1$ part is

$$\frac{1}{2}\ \frac{1}{2}(\vec{\sigma}_p + \vec{\sigma}_n) \cdot \frac{\vec{k}_i}{k_i}\ \frac{1}{2}(\vec{\sigma}_p + \vec{\sigma}_n) \cdot \frac{\vec{k}_f}{k_f}\ \chi_1^m \tag{IV.44}$$

and the $J = 2$ part is $\cos\Theta\ \chi_1^m$ minus these two parts. Here $\vec{k}_f$ is a vector along the z-axis and $\vec{k}_i$ is a vector which forms an angle Θ with $\vec{k}_f$. If Θ is a scattering angle, $\vec{k}_i$ and $\vec{k}_f$ will be the relative momenta in the centre-of-mass system before and after the collision.

There will be different scattering parameters in these three states with different values of J. In our problem the nucleon-nucleon interaction occurs in the final state, after a meson-nucleon collision. We shall then have to calculate the matrix element of

$$P_t \left\{ (a_0 - a_2)\frac{1}{3}\ \frac{1}{2}(\vec{\sigma}_p - \vec{\sigma}_n) \cdot \frac{\vec{k}_i}{k_i}\ \frac{1}{2}(\vec{\sigma}_p - \vec{\sigma}_n) \cdot \frac{\vec{k}_f}{k_f} + a_2 \cos\Theta + \right.$$
$$\left. + (a_1 - a_2)\frac{1}{2}\ \frac{1}{2}(\vec{\sigma}_p + \vec{\sigma}_n) \cdot \frac{\vec{k}_i}{k_i}\ \frac{1}{2}(\vec{\sigma}_p + \vec{\sigma}_n) \cdot \frac{\vec{k}_f}{k_f} \right\} (Q + \vec{S} \cdot \vec{\sigma}_p + \vec{T} \cdot \vec{\sigma}_n) .$$

The term $a_2\cos\Theta(Q+\vec{S}\cdot\vec{\sigma}_p+\vec{T}\cdot\vec{\sigma}_n)$ can be considered as absorbed in the equivalent (with respect to spin variables) terms of (IV.36). We can then write in this more compact form, which maintains all the necessary spin dependence:

$$R_2 = P_t\Big[(\vec{L}\cdot\vec{\sigma}_p)(\vec{N}\cdot\vec{\sigma}_p) + (\vec{L}\cdot\vec{\sigma}_n)(\vec{N}\cdot\vec{\sigma}_n) + M(\vec{L}\cdot\vec{\sigma}_p)(\vec{N}\cdot\vec{\sigma}_n) + M(\vec{L}\cdot\vec{\sigma}_n)(\vec{N}\cdot\vec{\sigma}_p)\Big](Q+\vec{S}\cdot\vec{\sigma}_p+\vec{T}\cdot\vec{\sigma}_n) \; . \qquad (IV.45)$$

We have to square the matrix element of $R = R_1 + R_2$ and sum over the possible polarization of the two nucleons. R_2 will not contribute to final singlet states, and the transition probability to singlet states will be given by Eq. (IV.39). For final triplet states we have

$$\frac{1}{3}\sum_{f,i} \langle i|R^+P_t|f\rangle\langle f|R\,P_t|i\rangle = \frac{1}{3}\,\mathrm{Tr}_p\,\mathrm{Tr}_n(R_1^+\,P_t\,R_1\,P_t) + \frac{1}{3}\,\mathrm{Tr}_p\,\mathrm{Tr}_n\Big[R_1^+\,P_t\,R_2\,P_t + R_2^+\,P_t\,R_1\,P_t\Big] + \frac{1}{3}\,\mathrm{Tr}_p\,\mathrm{Tr}_n\Big[R_2^+\,P_t\,R_2\,P_t\Big] . \qquad (IV.46)$$

The first term of the right-hand side is given by Eq. (IV.38). For the second term we obtain

$$\frac{1}{3}\,\mathrm{Tr}_p\,\mathrm{Tr}_n\Big[R_1^+\,P_t\,R_2\,P_t + R_2^+\,P_t\,R_1\,P_t\Big] = \Big\{\frac{2}{3}(A^+ + G^+)\Big[(3+M)(\vec{L}\cdot\vec{N})Q + 2i\,(\vec{T}+\vec{S})\cdot(\vec{L}\wedge\vec{N})\Big] + \frac{2}{3}(\vec{B}^+ + \vec{J}^+ + \vec{C}^+ + \vec{K}^+)\cdot\Big[(\vec{L}\wedge\vec{N})2iQ + (\vec{S}+\vec{T})2(\vec{L}\cdot\vec{N}) +$$

$$+ \vec{L}\,\vec{N}\cdot(\vec{S}+\vec{T})(M+1) + \vec{N}\,\vec{L}\cdot(\vec{S}+\vec{T})(M-1) \Big] +$$

$$+ \frac{2}{3} Q \Big[2M(\vec{D}^+\cdot\vec{L})(\vec{E}^+\cdot\vec{N}) + 2M(\vec{D}^+\cdot\vec{N})(\vec{E}^+\cdot\vec{L}) + (-M+1)(\vec{D}^+\cdot\vec{E}^+)(\vec{L}\cdot\vec{N}) \Big]$$

$$+ \frac{2}{3} i \,(\vec{S}+\vec{T})\cdot \Big[\vec{D}^+\,\vec{E}^+\cdot(\vec{L}\wedge\vec{N}) + \vec{E}^+\,\vec{D}^+\cdot(\vec{L}\wedge\vec{N}) + M(\vec{E}^+\wedge\vec{N})(\vec{D}^+\cdot\vec{L}) +$$

$$+ M(\vec{D}^+\wedge\vec{N})(\vec{E}^+\cdot\vec{L}) + M(\vec{E}^+\wedge\vec{L})(\vec{D}^+\cdot\vec{N}) + M(\vec{D}^+\wedge\vec{L})(\vec{E}^+\cdot\vec{N}) \Big] \Big\} +$$

$$+ \text{complex conjugate} . \tag{IV.47}$$

For the third term in Eq. (IV.46) we obtain

$$\frac{1}{3} \mathrm{Tr}_p \, \mathrm{Tr}_n \Big[R_2^+ \, P_t \, R_2 \, P_t \Big] = \frac{4}{3} Q^+ Q \Big\{ (\vec{N}^+\cdot\vec{L}^+)(\vec{N}\cdot\vec{L})(3+M+M^+-M^+M) +$$

$$+ 2(\vec{N}^+\cdot\vec{N})(\vec{L}^+\cdot\vec{L})(1+M^+M) + 2(\vec{N}^+\cdot\vec{L})(\vec{L}^+\cdot\vec{N})(-1+M^+M) \Big\} +$$

$$+ \frac{4i}{3} \Big[Q^+(\vec{S}+\vec{T}) + Q(\vec{S}^+ + \vec{T}^+) \Big] \cdot \Big\{ (\vec{N}^+\wedge\vec{L}^+)2(\vec{L}\cdot\vec{N}) + (\vec{L}\wedge\vec{N})2(\vec{L}^+\cdot\vec{N}^+) +$$

$$+ (\vec{L}\wedge\vec{N}) \quad (\vec{N}^+\wedge\vec{L}^+) + \vec{L}\, M(\vec{N}^+\wedge\vec{L}^+)\cdot\vec{N} - \vec{L}\, M^+(\vec{N}\wedge\vec{L})\cdot\vec{N}^+$$

$$+ \vec{N}\, M(\vec{N}^+\wedge\vec{L}^+)\cdot\vec{L} - \vec{N}^+\, M^+(\vec{N}\wedge\vec{L})\cdot\vec{L}^+ + (\vec{L}^+\wedge\vec{L})M^+M(\vec{N}^+\cdot\vec{N})$$

$$+ (\vec{N}^+\wedge\vec{N})M^+M(\vec{L}^+\cdot\vec{L}) + (\vec{L}^+\wedge\vec{N})M^+M(\vec{N}^+\cdot\vec{L}) - (\vec{L}\wedge\vec{N}^+)M^+M(\vec{N}\cdot\vec{L}^+) \Big\}$$

$$\mp \frac{8}{3} (\vec{S}^+ + \vec{T}^+)\cdot(\vec{S}+\vec{T}) \Big\{ (\vec{L}\cdot\vec{N})(\vec{L}^+\cdot\vec{N}^+)(1-M^+M) +$$

$$+ (\vec{N}^+\cdot\vec{L})(\vec{L}^+\cdot\vec{N})(-1+M^+M) + (\vec{N}^+\cdot\vec{N})(\vec{L}^+\cdot\vec{L})(1+M^+M) \Big\}$$

$$+ \frac{4}{3}\left[(\vec{S}^{+} + \vec{T}^{+})\wedge(\vec{S} + \vec{T})\right] \cdot \left\{(\vec{L}\wedge\vec{N})(\vec{L}^{+} \cdot \vec{N}^{+})(1 + M^{+}) + (\vec{N}^{+}\wedge \vec{L}^{+})(\vec{N} \cdot \vec{L})(1 + M)\right\} +$$

$$+ \frac{4}{3}\left[\vec{N}^{+} \cdot (\vec{S} + \vec{T})\right]\left\{(\vec{L}^{+} \cdot \vec{N})\left[\vec{L} \cdot (\vec{S}^{+} + \vec{T}^{+})\right](1 + M - M^{+} - M^{+}M) +\right.$$

$$\left. + (\vec{L}^{+} \cdot \vec{L})\left[\vec{N} \cdot (\vec{S}^{+} + \vec{T}^{+})\right](-1 + M + M^{+} - M^{+}M)\right\} + \qquad \text{(IV.48)}$$

$$+ \frac{4}{3}\left[L^{+} \cdot (\vec{S} + \vec{T})\right]\left\{(\vec{N}^{+} \cdot \vec{L})\left[\vec{N} \cdot (\vec{S}^{+} + \vec{T}^{+})\right](1 + M^{+} - M - M^{+}M) +\right.$$

$$\left. + (\vec{N}^{+} \cdot \vec{N})\left[\vec{L} \cdot (\vec{S}^{+} + \vec{T}^{+})\right](-1 - M - M^{+} - M^{+}M)\right\} +$$

$$+ \frac{4}{3}(M + M^{+}M)(\vec{N}^{+} \cdot \vec{L}^{+})\left\{\left[\vec{N} \cdot (\vec{S}^{+} + \vec{T}^{+})\right]\left[\vec{L} \cdot (\vec{S} + \vec{T})\right] + \left[\vec{L} \cdot (\vec{S}^{+} + \vec{T}^{+})\right]\left[\vec{N} \cdot (\vec{S} + \vec{T})\right]\right\} +$$

$$+ \frac{4}{3}(M^{+} + M^{+}M)(\vec{N} \cdot \vec{L})\left\{\left[\vec{N}^{+} \cdot (\vec{S} + \vec{T})\right]\left[\vec{L}^{+} \cdot (\vec{S}^{+} + \vec{T}^{+})\right] + \left[\vec{L}^{+} \cdot (\vec{S} + \vec{T})\right]\left[\vec{N}^{+} \cdot (\vec{S}^{+} + \vec{T}^{+})\right]\right\}$$

In the evaluation of the traces the following formulae have been used:

$$\mathrm{Tr}_p\, \mathrm{Tr}_n \left\{(\vec{\sigma}_p \cdot \vec{\sigma}_n)(\vec{A} \cdot \vec{\sigma}_p)(\vec{\sigma}_p \cdot \vec{\sigma}_n) f(\vec{\sigma}_p, \vec{\sigma}_n)\right\} = \mathrm{Tr}_p\, \mathrm{Tr}_n \left\{-(\vec{A} \cdot \vec{\sigma}_p) f(\vec{\sigma}_p, \vec{\sigma}_n) + 2(\vec{A} \cdot \vec{\sigma}_n) f(\vec{\sigma}_p, \vec{\sigma}_n)\right\}$$

$$(\vec{\sigma}_p \cdot \vec{\sigma}_n)(\vec{A} \cdot \vec{\sigma}_p)(\vec{B} \cdot \vec{\sigma}_p)(\vec{\sigma}_p \cdot \vec{\sigma}_n) = 2(\vec{A} \cdot \vec{B})(\vec{\sigma}_p \cdot \vec{\sigma}_n) + 2(\vec{A} \cdot \vec{\sigma}_n)(\vec{B} \cdot \vec{\sigma}_p) - (\vec{A} \cdot \vec{\sigma}_p)(\vec{B} \cdot \vec{\sigma}_n) - 2(\vec{A} \cdot \vec{B})$$

$$(\vec{\sigma}_p\cdot\vec{\sigma}_n)(\vec{A}\cdot\vec{\sigma}_p)(\vec{B}\cdot\vec{\sigma}_p)(\vec{\sigma}_p\cdot\vec{\sigma}_n) = (\vec{A}\cdot\vec{\sigma}_n)(\vec{B}\cdot\vec{\sigma}_p)(\vec{\sigma}_p\cdot\vec{\sigma}_n) +$$

$$+ (\vec{\sigma}_p\cdot\vec{\sigma}_n)(\vec{A}\cdot\vec{\sigma}_p)(\vec{B}\cdot\vec{\sigma}_n) - (\vec{A}\cdot\vec{\sigma}_n)(\vec{B}\cdot\vec{\sigma}_n) - (\vec{A}\cdot\vec{\sigma}_n)(\vec{B}\cdot\vec{\sigma}_p)$$

$$+ i\,\vec{\sigma}_p\cdot(\vec{A}\wedge\vec{B}) + i\,\vec{\sigma}_n\cdot(\vec{A}\wedge\vec{B}) - (\vec{A}\cdot\vec{\sigma}_p)(\vec{B}\cdot\vec{\sigma}_n) + 2(\vec{A}\cdot\vec{B})$$

$$\mathrm{Tr}_p\,\mathrm{Tr}_n\left\{(\vec{\sigma}_p\cdot\vec{\sigma}_n)(\vec{A}\cdot\vec{\sigma}_p)(\vec{B}\cdot\vec{\sigma}_n)(\vec{\sigma}_p\cdot\vec{\sigma}_n)\,f(\vec{\sigma}_p)\right\} =$$

$$= -2(\vec{A}\cdot\vec{B})\cdot\mathrm{Tr}_n\,\mathbb{1}_n \times \mathrm{Tr}_p\left[f(\vec{\sigma}_p)\right] \qquad \text{(IV.49)}$$

* * *

CHAPTER V

INELASTIC MESON-DEUTERON SCATTERING

1. Introduction

We must now proceed to the explicit evaluation of the terms contributing to the collision operator T. We shall be concerned with inelastic scattering only. The terms corresponding to single scattering of the incident meson by the proton and by the neutron are already given explicitly by Eqs. (IV.18) and (IV.21) respectively. To obtain the contributions coming from double scattering and potential correction terms we have to evaluate the integrals in Eqs. (IV.24), (IV.26), (IV.27) and (IV.28).

We first notice in the integrands the presence of the matrix elements of r_p, r_n, r_u with arguments which depend on the variable of integration. In Eq. (IV.24) the dependence of the arguments on $\vec{\ell}$ is explicitly exhibited. The values of ℓ that contribute to the integral are those available in the deuteron wave function, i.e. those which make $\ell^2 \psi_D(\ell)$ large. These values of ℓ lie between zero and about 150 MeV/c. As $\vec{\ell}$ varies in modulus and direction within this range of values, the relative momentum of the two colliding particles and the scattering angle vary. If q_0 is not small, the relative momentum $\vec{k}_p$ of the meson-nucleon system will vary within a not very wide solid angle and the scattering angle will correspondingly not have a large fluctuation. We try to illustrate this situation in the figure below. Most of the

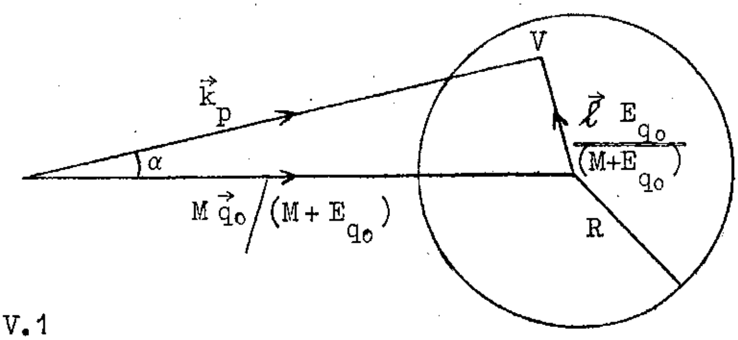


Fig. V.1

contributions to the integral come from values of $\vec{k}_p$ such that the point V is inside the sphere of radius $R = \frac{E_{q_0}}{M+E_{q_0}} \ell_{max}$. For incident K mesons of momentum $q_0 = 200$ MeV/c ($E_{q_0} = 540$ MeV), and $\ell_{max} = 150$ MeV/c, the value of the angle α_{max} is 20 degrees, and the modulus of k_p varies from 180 to 80 MeV/c. If $q_0 = 500$ MeV/c ($E_{q_0} \doteq 700$ MeV), $\alpha_{max} = 13$ degrees and the modulus k_p varies from 140 to 350 MeV/c.

By using the relation

$$\frac{1}{E - E_m \pm i\epsilon} = P\frac{1}{E - E_m} \mp i\pi\,\delta(E - E_m) \qquad (V.1)$$

where P means principal value, we can separate the integrals representing the second order processes into two parts, one taking into account the contributions coming from values of E_m on the energy shell $E = E_m$, and the other involving values of E_m which are different from $E = E_f$. In all integrals [Eqs. (IV.24), (IV.26), (IV.27) and (IV.28)] $E_m = E$ is the energy shell for the second interaction in the double scattering processes represented by them. We shall show that due to the particular loosely bound structure of the deuteron, the on-the-energy shell part is of the same order of magnitude as the contributions coming from off-the-energy shell. Now we must notice that to on-the-energy shell matrix elements of the collision operator for the second scattering correspond off-the-energy shell matrix elements of the collision operator representing the first scattering. This is because in the calculation of transition probabilities and cross-sections we are concerned only with matrix elements of T_{inel} for which the total energies in final and initial states are equal, i.e. for which $E = E_f = E_i = M_0 + E_{q_0}$. Now, for any value of relative momentum ℓ in the deuteron wave function we have that the sum of the kinetic energies of the three particles is $E_\ell = 2M + \frac{\ell^2}{M} + E_{q_0} > M_0 + E_{q_0}$, so that it can never be $E_\ell = E_f$. The larger the value of ℓ, the farther from the shell $E_\ell = E_f = E$ is the matrix element for the first interaction. This is so if we use the expansion (III.22) for T_{inel}, as we did all through Chapter IV. If, instead, we adopt the expansion (III.20) the strong contribution will come from values of $\vec{\ell}$ such that $E_m = E_\ell$, which is the energy shell for the first interaction in the second order processes.

So, strictly speaking, a knowledge of the behaviour of the off-the-energy shell matrix elements of the collision operators of the meson-nucleon system is essential in our problem. This knowledge is not available at present, however (if it were, the two-particle interaction potential would virtually be known, and perhaps our whole analysis would be completely unnecessary). In fact, even the magnitudes of the on-the-energy shell matrix elements for the K meson-nucleon systems are scarcely known at present. We shall then have to assume some sort of behaviour of these matrix elements off-the-energy shell, perhaps that they have a constant value, or that there is a decrease in value as the distance to the energy shell increases. As the deuteron wave function contains momenta up to a value which is not very large ($\ell_{max} \sim 150$ MeV/c), only matrix elements which are not very far from the energy shall will have important contributions to the processes we have, and probably it will not be very bad to assume a constant value.

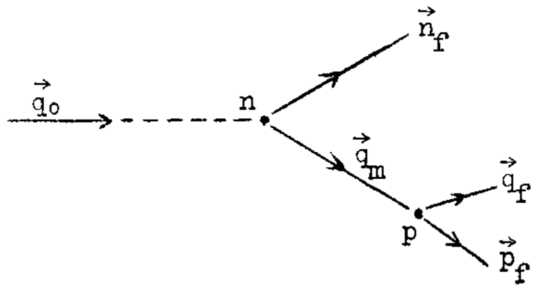
For the $K^+p \rightarrow K^+p$ interaction the experimental results (Ref. 14) seem to indicate an almost constant and isotropic differential cross-section over a wide range of energies. This implies that the modulus $|\langle |r_p| \rangle|$ depends little on the relative momentum and on the scattering angle over a wide range of values of relative momentum, though nothing can be said about the phase of the complex quantity $\langle |r_p| \rangle$.

For the K^+n interaction, there is not so much available data, and that which exist are not so reliable as the K^+p data. At high energies (above 500 MeV, up to 1,500 MeV), data (Ref. 14) seem to indicate that the K^+n cross-section does not vary much with the energy. At lower energies there are practically no available data, except that coming from the measurement of the charge exchange cross-section, with the assumption of charge independence. These data are not detailed or reliable enough to suggest any kind of variation of $\langle r_n \rangle$ with energy or with angle.

We thus see that at least we are not contradicting any available experimental evidence by assuming that $\langle |r_p| \rangle$ and $\langle |r_n| \rangle$ can be extracted from inside the integral signs as being approximately constant over the most important values of the variables.

2. The Double-Scattering Terms

Let us consider the process in which the meson is scattered by a neutron and then rescattered by the proton:



The matrix element is given by [Eq. (IV.26)]

$$\delta(\vec{K}_f+\vec{q}_f-\vec{q}_0)\ J_{pn} = \delta(\vec{K}_f+\vec{q}_f-\vec{q}_0)\int d_3\vec{\ell}\ \langle \vec{q}_f\vec{p}_f|r_p|\vec{q}_m\vec{\ell}\rangle \frac{1}{E-E_m+i\epsilon}\ \times$$

$$\times \langle \vec{q}_m\vec{n}_f|r_n|\vec{q}_0,-\vec{\ell}\rangle\ \psi_D(\ell) \qquad (V.2)$$

where if the intermediate meson is non-relativistic we have

$$E_m = 2M + \frac{n_f^2}{2M} + \frac{\ell^2}{2M} + m + \frac{(\vec{q}_0-\vec{n}_f-\vec{\ell})^2}{2m}\ . \qquad (V.3)$$

Let us call

$$\vec{K} = \frac{\mu}{m}\,(\vec{q}_0 - \vec{n}_f)$$

$$C = \mu\left(E_{q_f} + \frac{p_f^2}{2M} - m - \frac{K^2}{2m}\right) = \mu\left(\frac{p_f^2}{2M} - \frac{p_f^2}{2m} - \frac{\vec{q}_f\cdot\vec{p}_f}{m}\right) \qquad (V.4)$$

$$\mu = \frac{Mm}{M+m}\ .$$

The integral (V.2) is singular in a spherical surface

$$(\vec{\ell} - \vec{K})^2 = R^2 = K^2 + 2C = \frac{\mu^2}{m^2}\left(\vec{q}_f - \frac{m}{M}\vec{p}_f\right)^2 = k_f^2 \qquad \text{(V.5)}$$

where $\vec{k}_f$ is the momentum of the meson in the final state, relative to the centre of mass of the meson-proton system.

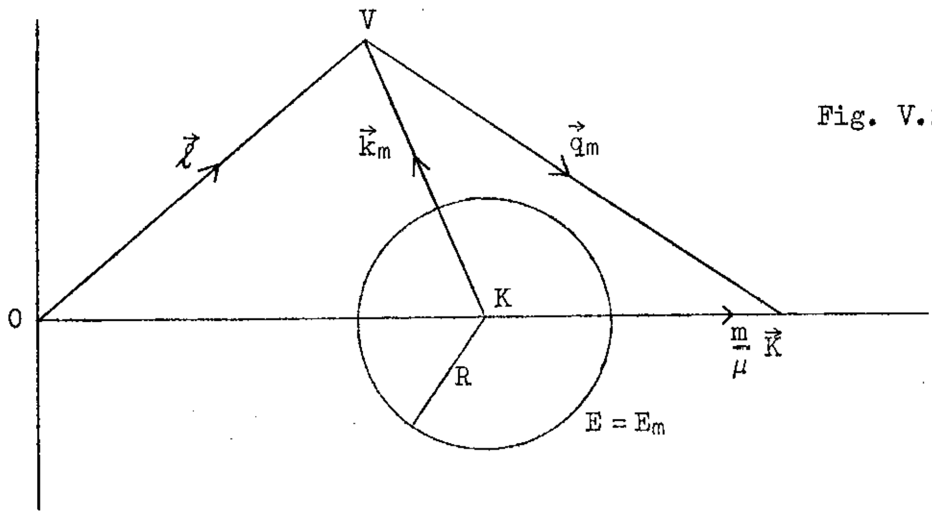


Fig. V.2

$\vec{k}_m$ in the figure above is the momentum of the meson in the intermediate state relative to the centre of mass of the meson-proton system. We can write

$$E - E_m = \frac{k_f^2}{2\mu} - \frac{k_m^2}{2\mu} \, . \qquad \text{(V.6)}$$

Using Eq. (V.1) we can separate Eq. (V.2) into two parts. For the on-the-energy shell part we obtain, assuming that $\langle |r_p| \rangle$ and $\langle |r_n| \rangle$ are constants equal to a_p and a_n respectively,

$$i\, a_p a_n\, J^{(1)}_{pn} = \int d_3\vec{\ell} < \vec{q}_f \vec{p}_f | r_p | \vec{q}_m \vec{\ell} > (-i\pi)\, \delta\, (E - E_m) < \vec{q}_m \vec{n}_f | r_n | \vec{q}_o, -\vec{\ell} > \psi_D(\ell) =$$

$$= a_p a_n \left(\frac{\mu}{K}\, 2\pi\, \sqrt{2\pi}\, N \right) \times \frac{i}{2} \left[\ln \frac{\alpha^2 + (K-R)^2}{\beta^2 + (K-R)^2} - \ln \frac{\alpha^2 + (K+R)^2}{\beta^2 + (K+R)^2} \right] \tag{V.7}$$

where $N = \sqrt{\frac{2\alpha\beta\,(\alpha+\beta)}{4\pi\,(\beta-\alpha)^2}}$ is the normalization constant in the deuteron (Hulthén) wave function.

Let us notice that in the integration over the sphere $E = E_m$ the relative meson-to-proton momentum $\vec{k}_m$ varies only in direction, its modulus being constant. So, extracting $<|r_p|>$ from the integrand means only to assume that it is independent of the scattering angle (that is, that it has an S-wave-like behaviour). On the other hand, the energy shell $E_\ell = E_m$ for the scattering by the neutron is a sphere with centre at the point $-K\,\frac{M+m}{M-m}$ (at the left of the point 0 in the figure above) and radius $\frac{1}{(M-m)}\,|M\,\vec{q}_o - m\,\vec{n}_f|$. This sphere does not cross the surface $E = E_m$, and so only off-the-energy shell matrix elements of r_n are involved in the integration in Eq. (V.7). We cannot tell if approximating these matrix elements by a constant a_n is a good approximation.

Let us consider now the principal part of the integral in Eq. (V.2). We obtain, by considering the matrix elements as constant

$$a_p a_n\, J^{(2)}_{pn} = P \int d_3\vec{\ell} < \vec{q}_f \vec{p}_f | r_p | \vec{q}_m \vec{\ell} > \frac{1}{E - E_m} < \vec{q}_m, \vec{n}_f | r_n | \vec{q}_o, -\vec{\ell} > \psi_D(\ell) =$$

$$= a_p a_n \left(\frac{\mu}{K}\, 2\pi\, \sqrt{2\pi}\, N \right) \times \left\{ \tan^{-1} \left(\frac{\beta K}{C + \frac{1}{2}\beta^2} \right) - \tan^{-1} \left(\frac{\alpha K}{C + \frac{1}{2}\alpha^2} \right) \right\}. \tag{V.8}$$

It is not easy to compare directly and in a generally valid way the values of Eqs. (V.7) and (v.8), because two of the three quantities C, K, R have a certain freedom of variation with respect to each other, which is only restricted by the total energy conservation imposed by the $\delta(E_f - E_i)$ that appears in the expression for the cross-sections (IV.32). This is due to the fact that, as we have three particles in the final state, for each scattering angle there will be a spectrum of values of momentum and energy of the particles. We have evaluated numerically Eqs. (V.7) and (V.8) for several values of the momenta of the particles in the final state, trying to cover all the spectra of possible values. We obtained that the two parts, $J^{(1)}_{pn}$ and $J^{(2)}_{pn}$, are in general of the same order of magnitude, one or other predominating in the different regions in the spectrum.

It is instructive to discuss the way in which the integral (V.8) is formed. It is particularly interesting to see whether or not important contributions to this integral come from values near the energy shell.

Using the variable $\vec{k}_m = \vec{\ell} - \vec{K}$ we can write

$$J^{(2)}_{pn} = \left(\frac{\pi}{K}\sqrt{\frac{2}{\pi}}\,N\right)\int_{k_m=0}^{\infty} \frac{k_m dk_m}{(k_m+R)(k_m-R)}\left[\ln\frac{\beta^2 + (k_m-K)^2}{\alpha^2 + (k_m-K)^2} - \ln\frac{\beta^2 + (k_m+K)^2}{\alpha^2 + (k_m+K)^2}\right]. \quad (V.9)$$

We have a pole at $k_m = R$. The factor $k_m/(k_m+R)$ is regular in the range of integration and varies slowly from $k_m = 0$ to $k_m = \infty$. (If R is small this variation is not so slow, and occurs near the origin, where the total quantity inside the brackets in the expression above is small.) The function $\ln\left\{[\alpha^2 + (k_m + K)^2]/[\beta^2 + (k_m + K)^2]\right\}$ has a bell-like shape, centered at $k_m = -K$. The other logarithm function has the same shape and is centered at $k_m = K$. Forgetting for the moment the factor $k_m/(k_m + R)$ we have the functions indicated in the figure below.

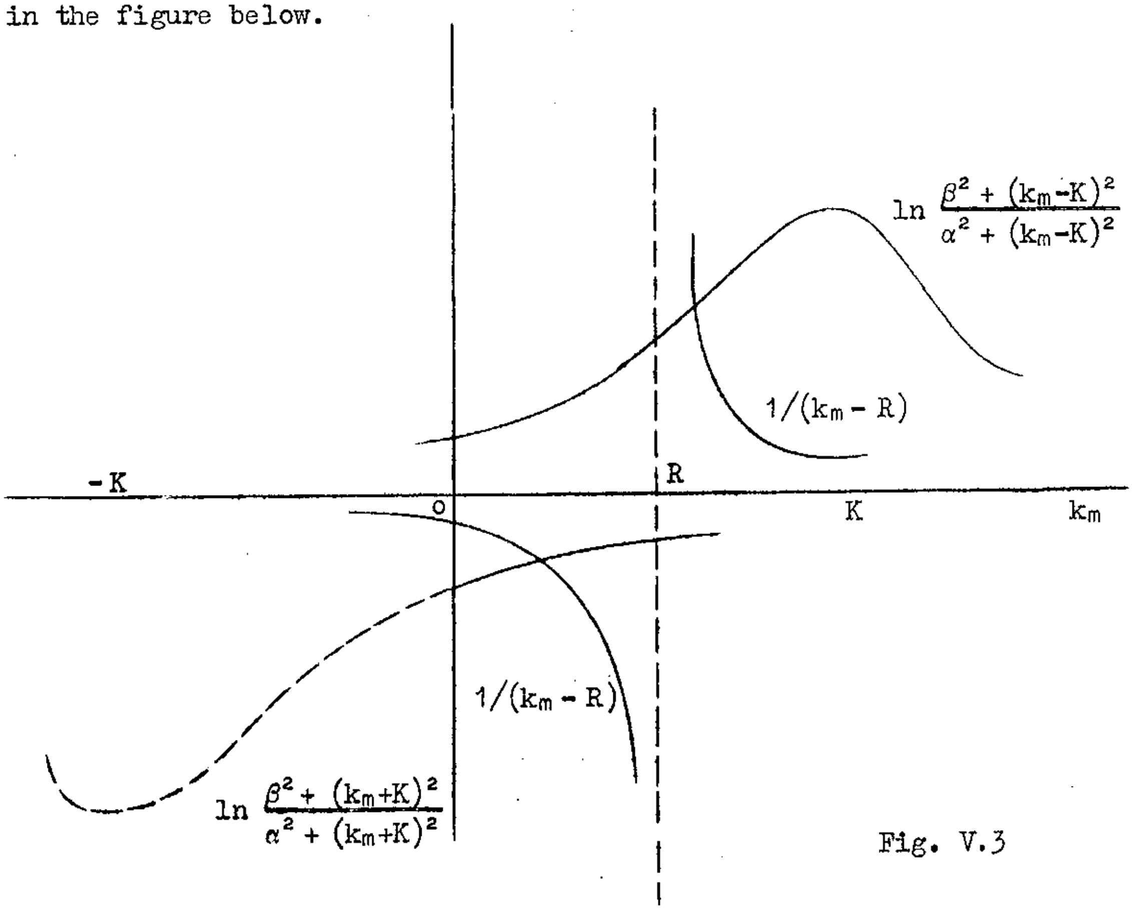


Fig. V.3

Let us first consider that $R(= k_f)$ is not very small. (By a small value of R we mean a value for which a considerable part of the left-hand side branch of the pole function is outside the range of integration.) In a more or less narrow strip around $k_m = R$, the contributions coming from each of the two sides of the pole $k_m = R$ are not very different and tend to cancel each other. If K is far from R, the product of the bell-shaped and the pole functions will be small, and so will be the integral $J_{pn}^{(2)}$. If $K \to R$ there tends to be cancellation because of the symmetry in the bell-shaped function (and antisymmetry in the pole-function). For a certain range of values between K and R the integral assumes values that are not small. Most of the contribution to $J_{pn}^{(2)}$ in these cases comes from the region of integration between R and K, that is, from a region which is not far from the energy shell. If instead of having a deuteron wave function (which gave rise to the two subtracting bell-shaped logarithmic functions) we had a constant function in momentum space (which would correspond to the two scatterers n and p being at short distance from each other, i.e. to the deuteron having a small "radius"), there would be a stronger cancellation of the contributions coming from the two sides in the neighbourhood of the energy shell, and contributions to the integral would come from all the range of values of k_m from 2R to infinity, i.e. from values of the variable that are very far from the energy shell.

Now let us consider the case in which R is small (the meson and proton in the final state having a small relative momentum). The energy shell is then near the origin, and a large part of the negative branch of the pole function is outside the range of integration. In the region between 0 and R the two logarithmic functions cancel each other partially. The function $k_m/(k_m + R)$, which increases from 0 at $k_m = 0$ to $\frac{1}{2}$ at $k_m = R$, helps in reducing the value of the contribution coming from the region on the left-hand side of the energy shell. So unless $K \to 0$, the integral $J_{pn}^{(2)}$ will have a large value in this case, contributions coming mainly from the region near the energy shell, extending from $k_m = R$ to some value of k_m a little greater than K.

So, we shall conclude that by assuming $\langle |r_p| \rangle$ to be constant in the integration is possibly not a bad approximation, since the most important contributions come from values not very far from the energy shell. By introducing a "cut-off" factor that decreases as the distance to the energy shell increases we could perhaps obtain a better result.

For the double scattering process in which the meson first hits the proton and then is scattered by the neutron, we obtain results of the same form as Eqs. (V.7) and (V.8) with the roles of proton and neutron interchanged in the definitions (V.9).

Comparison with First Order Processes

We must now compare the magnitudes of the contributions to the transition amplitude for meson-deuteron inelastic scattering, of the single and double scattering processes. For a final state in which the momenta of the particles are $\vec{q}_f$, $\vec{p}_f$, $\vec{n}_f$, we have that the scattering amplitude for single scattering by the proton is proportional to $-a_p\psi_D(n_f)$ and if the scattering is by the neutron it is proportional to $-a_n\psi_D(p_f)$ (cf. Chapter IV, Section 3). The values of these amplitudes vary along the spectra of possible values of $\vec{p}_f, \vec{n}_f$, but to have a value characteristic of the important part of the spectra, we may take the value of the deuteron wave function at the origin, which is of about $\sqrt{\frac{2}{\pi}}\, N \frac{1}{\alpha^2}$. The amplitudes for double scattering processes are given by Eqs. (V.7), (V.8) and the corresponding expression for the case in which the proton is the first scatterer. The expression that multiplies $a_p a_n \left(\frac{\mu}{K}\, 2\pi \sqrt{2\pi}\, N\right)$ in Eq. (V.7) has an interval of variation which is inside the interval

$\left(0,\ \ln\frac{\beta}{\alpha} = 2\right)$. The corresponding expression in Eq. (V.8) varies in an interval which is enclosed by the interval $(-\pi, +\pi)$. (In fact, due to energy conservation the allowed range of values is smaller than indicated by these intervals.) Let us then take $a_p a_n \mu \left(\frac{1}{K_{np}} + \frac{1}{K_{pn}}\right) 2\pi\sqrt{2\pi}\ N \times 2$ as a typical value of the double scattering terms. Here $\vec{K}_{np} = \frac{\mu}{m}(\vec{q}_0 - \vec{p}_f)$ and $\vec{K}_{pn} = \frac{\mu}{m}(\vec{q}_0 - \vec{n}_f)$. We have to compare this with $(a_p + a_n)\sqrt{\frac{2}{\pi}}\ N\frac{1}{\alpha^2}$ which gives the order of magnitude of the single scattering terms. Let us take a_p and a_n to be of the same value. We obtain

$$\frac{\text{2nd order}}{\text{1st order}} \approx \alpha\left(\frac{1}{K_{pn}} + \frac{1}{K_{np}}\right)\frac{1}{3}\sqrt{\sigma_T(\text{barns})} \qquad (V.10)$$

where $\sigma_T(\text{barns})$ is the total cross-section for meson-nucleon scattering, measured in barns. The parameter α of the Hulthen wave function is $\alpha = 45$ MeV/c. For K^+ meson-nucleon scattering we have $\sigma_T \sim 0.016$ barns, so that we have

$$\frac{\text{2nd order}}{\text{1st order}} \sim \alpha\left(\frac{1}{K_{pn}} + \frac{1}{K_{np}}\right)\frac{1}{20}\,.$$

We thus see that for values of $\vec{n}_f$, $\vec{p}_f$ such that both $\vec{K}_{pn}$ and $\vec{K}_{np}$ have moduli not small compared to $\alpha(= 45$ MeV/c$)$, the contributions coming from double scattering processes are small compared to those coming from single scattering terms. For small values of K_p or K_n, which <u>are</u> allowed by energy conservation, the double scattering terms may become important. (We must note that Eq. (V.10) is not valid for $K_p, K_n \to 0$, since then we have that both $\tan^{-1}$ and logarithm functions in Eqs. (V.7) and (V.8) also tend to zero, keeping the matrix elements finite.) This will happen in parts of the energy and angular spectra, and will affect strongly certain differential cross-sections. We can expect that total cross-sections will not be seriously affected by these double scattering processes in this case of K-deuteron inelastic scattering. If the meson-nucleon cross-section were, say, ten times bigger, we would have a much stronger double scattering effect. This partially explains why the pure impulse approximation (single scattering terms only) has given bad results when applied to K^--deuteron scattering.

3. The "Potential-Correction" Terms

Let us now consider the second order process in which the meson collides with the proton, which recoils and is then scattered by the neutron. The integrand in

$$I_{up} \quad \int d_3\vec{\ell} \, \langle \vec{n}_f \vec{p}_f | r_u | \vec{p}_m, -\vec{\ell} \rangle \frac{1}{E - E_m + i\epsilon} \langle \vec{p}_m, \vec{q}_f | r_p | \vec{\ell}, \vec{q}_0 \rangle \, \psi_D(\ell) \qquad (V.11)$$

where

$$E - E_m = \frac{\ell_f^2}{M} - \frac{1}{M}\left[\vec{\ell} + \tfrac{1}{2}(\vec{q}_0 - \vec{q}_f)\right]^2 \qquad (V.12)$$

is singular on the surface of the sphere of radius $\vec{\ell}_f$ and centre at the point $-\frac{1}{2}(\vec{q}_0 - \vec{q}_f)$. This sphere is the energy shell of $\langle \vec{n}_f \vec{p}_f | r_u | \vec{p}_m, -\vec{\ell} \rangle$. The energy shell for $\langle \vec{p}_m \vec{q}_f | r_p | \vec{\ell}, \vec{q}_0 \rangle$ is a plane orthogonal to the vector $(\vec{q}_0 - \vec{q}_f)$. This plane does not cross the sphere if we impose energy conservation to the whole process, $E = E_i = E_{q_0} + M_D$.

The integral in Eq. (V.11) is similar to that in Eq. (V.2). So here we could make the same considerations about the behaviour of the integrand near and far from the energy shell as we did with the double scattering terms.

Let us call

$$\vec{\Delta} = \vec{q}_f - \vec{q}_0 \ . \tag{V.13}$$

By considering $\langle |r_p| \rangle$ and $\langle |r_u| \rangle$ as constants respectively equal to a_p and a_u, we obtain for the on-the-energy shell part of Eq. (V.11)

$$i\ a_u a_p\ I_{up}^{(1)} = a_p a_u \left[\frac{M}{\Delta}\ (2\pi)^{3/2} N\right] \frac{i}{2} \left[\ln \frac{\alpha^2 + (\frac{1}{2}\Delta - \ell_f)^2}{\beta^2 + (\frac{1}{2}\Delta - \ell_f)^2} - \ln \frac{\alpha^2 + (\frac{1}{2}\Delta + \ell_f)^2}{\beta^2 + (\frac{1}{2}\Delta + \ell_f)^2}\right] \tag{V.14}$$

and for the principal part

$$a_u a_p\ I_{up}^{(2)} = a_p a_u \left[\frac{M}{\Delta}\ (2\pi)^{3/2} N\right] \left\{\tan^{-1}\left(\frac{\beta\Delta}{\ell_f^2 - \frac{1}{4}\Delta^2 + \beta^2}\right) - \tan^{-1}\left(\frac{\alpha\Delta}{\ell_f^2 - \frac{1}{4}\Delta^2 + \alpha^2}\right)\right\} . \tag{V.15}$$

With $\langle |r_u| \rangle = a_u = \text{const.}$,

$$I_{up} = a_u a_p \left(i\ I_{up}^{(1)} + I_{up}^{(2)}\right) . \tag{V.16}$$

If the first collision is with the neutron we have

$$E - E_m = \frac{\ell_f^2}{M} - \frac{1}{2}\left[\vec{\ell} + \frac{1}{2}\vec{\Delta}\right]^2 \tag{V.17}$$

instead of Eq. (V.12), but the evaluation of the matrix element gives exactly the same expressions as Eqs. (V.14) and (V.15), the only change being that a_p is substituted by a_n.

Now a few considerations impose. First, the nucleon-nucleon interaction has a range (judged by the value of its cross-section) much longer than that of the meson-nucleon interaction. This means that the matrix elements of the collision operator for nucleon-nucleon scattering must decrease more rapidly as the distance to the energy shell increases. The approximation of assuming a constant value for the matrix elements of the two-particle processes in the principal-part integrals might not be so good in the case of nucleon-nucleon interaction as it was assumed to be in the case of meson-nucleon interactions. The introduction of some sort of cut-off might be necessary.

Secondly, for meson incident momenta over a certain value, the recoil energies of the nucleons will be such that the nucleon-nucleon interaction will occur rather strongly in S, P and higher waves. These higher waves may be important in the nucleon-nucleon interaction in the final state. Particularly the P-wave may be important in the p-p interaction (which would occur after a charge exchange $K^+n \rightarrow K^0p$) in the triplet state, where S- and D-waves are excluded due to the Pauli principle.

It thus seems important to take into account the finite range of the nuclear forces and to include P_waves in our treatment of the nucleon-nucleon interaction. Both these tasks can be more easily accomplished if we write the expression for the matrix element (V.11) in configuration space. We have

$$\langle f|t_p|i\rangle = \int \langle f|\vec{\ell}',\vec{K}',\vec{q}'\rangle \langle \vec{\ell}',\vec{K}',\vec{q}'|t_p|\vec{\ell},\vec{K},\vec{q}\rangle \langle \vec{\ell}\,\vec{K}\,\vec{q}|i\rangle \; \times$$

$$\times\; d_3\vec{\ell}'\; d_3\vec{K}'\; d_3\vec{q}'\; d_3\vec{\ell}\; d_3\vec{K}\; d_3\vec{q} =$$

$$= -\,\delta(\vec{P}_f-\vec{P}_i)\int \langle f|\vec{\ell}'\rangle\, \delta\!\left(\vec{\ell}'-\vec{\ell}+\frac{\vec{q}_f-\vec{q}_o}{2}\right) \langle \vec{k}'|r_p|\vec{k}\rangle \langle \ell|i\rangle\, d_3\vec{\ell}'\; d_3\vec{\ell}\,. \tag{V.18}$$

Assuming that $\langle \vec{k}'|r_p|\vec{k}\rangle$ is a constant a_p, and introducing the Fourier transforms of the quantities in the integrand, we obtain

$$\langle f|t_p|i\rangle = -\,\delta(\vec{P}_f-\vec{P}_i)a_p \int \langle f|r\rangle\, e^{-i\frac{\vec{\Delta}}{2}\cdot\vec{r}} \langle r|i\rangle\, d_3\vec{r}\,. \tag{V.19}$$

Thus, I_{up} can be written as

$$I_{up} = -\,a_p \int \langle f|r\rangle\, e^{-i\frac{\vec{\Delta}}{2}\cdot\vec{r}}\, \psi_D(r) d_3\vec{r} \tag{V.20}$$

where $\psi_D(r)$ is the deuteron wave function in configuration space and

$$\langle r|f\rangle = \langle r|\,\frac{1}{E-K-i\epsilon}\,t_u|\Phi_f\rangle = -\int \frac{e^{+i\vec{\ell}'\cdot\vec{r}}}{(2\pi)^{3/2}}\,\frac{1}{E-E_{\ell'}-i\epsilon} \langle \vec{\ell}'|r_u|\vec{\ell}_f\rangle\, d_3\vec{\ell}' \tag{V.21}$$

$\langle r|f\rangle$ is the configuration space representation of the scattered waves in the nucleon-nucleon interaction. It is a solution of the Schroedinger equation with a certain asymptotic behaviour: it represents the ingoing-wave scattering state ψ_u^- minus the incident plane wave part, r is the relative proton-to-neutron coordinate.

First let us consider the case of S-waves in the nucleon-nucleon interaction. We substitute $\langle \vec{\ell}' | r_u | \vec{\ell}_f \rangle$ by a_u (a constant) in Eq. (V.21) and obtain

$$\langle r | f_S \rangle = - a_u \frac{M}{2} \ell_f \frac{4\pi^2}{(2\pi)^{3/2}} \left[i \; j_0(\ell_f r) + n_0(\ell_f r) \right] . \quad (V.22)$$

The first part, $j_0(\ell_f r)$, comes from the on-the-energy shell part of the integral. The part in $n_0(\ell_f r)$ comes from the principal part. Now this is a valid solution of the Schroedinger equation only for values of r that are outside the range of the nuclear forces. For $r \to 0$, $n_0(\ell_f r)$ tends to infinity. To avoid this, we have to introduce a cut-off in $\langle \vec{\ell}' | r_u | \vec{\ell}_f \rangle$: it must tend to zero as $|\ell' - \ell_f|$ increases. This would affect the part $n_0(\ell_f r)$, transforming it in a function which converges as $r \to 0$, leaving the part $j_0(\ell_f r)$ as it is. The best way to introduce this effect is directly in the result (V.22): we can either cut-off $n_0(\ell_f r)$ for distances r smaller than the range of nuclear forces, or introduce a convenient convergence factor, for example $(1 - e^{-Zr})$ (Ref. 10) where Z is a parameter related to the range of nuclear forces. We shall then write

$$\langle r | f_S \rangle = - a_u \frac{M}{2} \ell_f \frac{4\pi^2}{(2\pi)^{3/2}} \left[i \; j_0(\ell_f r) + n_0(\ell_f r)(1 - e^{-Zr}) \right] . \quad (V.23)$$

Now let us consider the case of P-waves in the nucleon-nucleon interaction. Let us substitute $\langle \vec{\ell}' | r_u | \vec{\ell}_f \rangle$ by $b_u \cos \Theta'$, where Θ' is the scattering angle in the centre-of-mass system of the two nucleons, i.e. the angle between $\vec{\ell}_f$ and $\vec{\ell}'$. We then have

$$\langle \vec{\ell}' | r_u | \vec{\ell} \rangle = a_u + b_u \cos (\vec{\ell}', \vec{\ell}) \; . \qquad \text{(V.24)}$$

We obtain

$$\langle r | f_P \rangle = - b_u \frac{M}{2} \ell_f \frac{4\pi^2}{(2\pi)^{3/2}} (+i) P_1 (\cos \Theta) \Big\{ i \, j_1 (\ell_f r) +$$

$$+ \frac{2(1-e^{-Zr})}{\pi} \Big[j_1 (\ell_f r) \, Ci (\ell_f r) + n_1 (\ell_f r) \, Si (\ell_f r) \Big] \Big\} . \qquad \text{(V.25)}$$

As in Eq. (V.23) we have introduced the convergence factor $(1 - e^{-Zr})$ to cancel the divergence resulting from the principal part of the integral. This divergence was due to the fact that the amplitude b_u was kept constant.

When substituting $\langle r | f \rangle = \langle r | f_S \rangle + \langle r | f_P \rangle$ taken from Eqs. (V.23) and (V.25) into Eqs. (V.20) we have to note that $\langle f | r \rangle = \langle r | f \rangle^*$.

The angular part of the integral (V.20) can be easily evaluated by using the expansion

$$e^{-i \frac{\vec{\Delta}}{2} \cdot \vec{r}} = \sum_{\ell} (-i)^{\ell} (2\ell + 1) \, P_\ell \left[\cos \left(\frac{\vec{\Delta}}{2}, \vec{r} \right) \right] j_\ell \left(\frac{\Delta}{2} r \right) \qquad \text{(V.26)}$$

and the addition theorem for Legendre polynomials, which gives

$$\int P_\ell \left[\cos \left(\frac{\vec{\Delta}}{2}, r \right) \right] P_{\ell'} \left[\cos \, (\vec{\ell}_f, \vec{r}) \right] d\Omega_{\vec{r}} = \frac{4\pi}{2\ell + 1} \, \delta_{\ell \ell'} P_\ell \left[\cos \left(\frac{\vec{\Delta}}{2}, \ell_f \right) \right] \; .$$

(V.27)

Thus, of the expansion (V.26) only the term $j_0\left(\frac{\Delta}{2}r\right)$ will survive multiplying $\langle f_S|r\rangle$ and only $j_1\left(\frac{\Delta}{2}r\right)$ will be multiplying $\langle f_p|r\rangle$ after the angular integration is effectuated.

We may observe that due to the $r^2\psi_D(r)$ appearing in the integrand, Eq. (V.20) is finite even without the introduction of the factors $(1-e^{-Zr})$. However finite, the values of the parts of the integral coming from the divergent functions would be too large. We try to exemplify this by the graphs below in which we plot the integrand in Eq. (V.20) (after the angular integrations) for the case (V.25) of P-wave interaction. Curve C_1 represents the part of the integrand coming from the part i $j_1(\ell_f r)$ (the on-the-energy shell part) in Eq. (V.25). Curve C_2 represents the contribution of the part $\frac{2}{\pi}\left[j_1(\ell_f r)\mathrm{Ci}(\ell_f r)+n_1(\ell_f r)\mathrm{Si}(\ell_f r)\right]$ to the integrand in Eq. (V.20), without cut-off factor being introduced. Curve C_3 is such that $C_3 = C_2(1-e^{-Z_3 r})$, where $Z_3 = 10\times 10^{12}\ \mathrm{cm}^{-1}$ = 200 MeV. In C_4 we have another value given to the cut-off parameter, $C_4 = C_2(1-e^{-Z_4 r})$, with $Z_4 = 5\times 10^{12}\ \mathrm{cm}^{-1} = 100$ MeV. We used the values $\ell_f = \Delta/2 = 100$ MeV/c, which are compatible with energy conservation if the incident meson momentum is $q_0 = 200$ MeV/c. The values of Z are chosen so that $(1-e^{-Zr})$ has a strong effect on the wave function $\langle r|f_p\rangle$ only inside the range of the nuclear forces. With Z = 200 MeV the value of $1-e^{-Zr}$ for $r = 2\times 10^{-13}$ cm (the range of the nuclear forces in the triplet state) is 0.865. For Z = 100 MeV, $(1-e^{-Zr}) = 0.63$ at the same value of r, and this is certainly a too strong reduction in the value of the wave function. We must choose Z = 200 MeV or larger, and we see from the curves that its effect in the integral (V.20) can be expected to be very small. In the case of meson-nucleon forces the corresponding effect would be

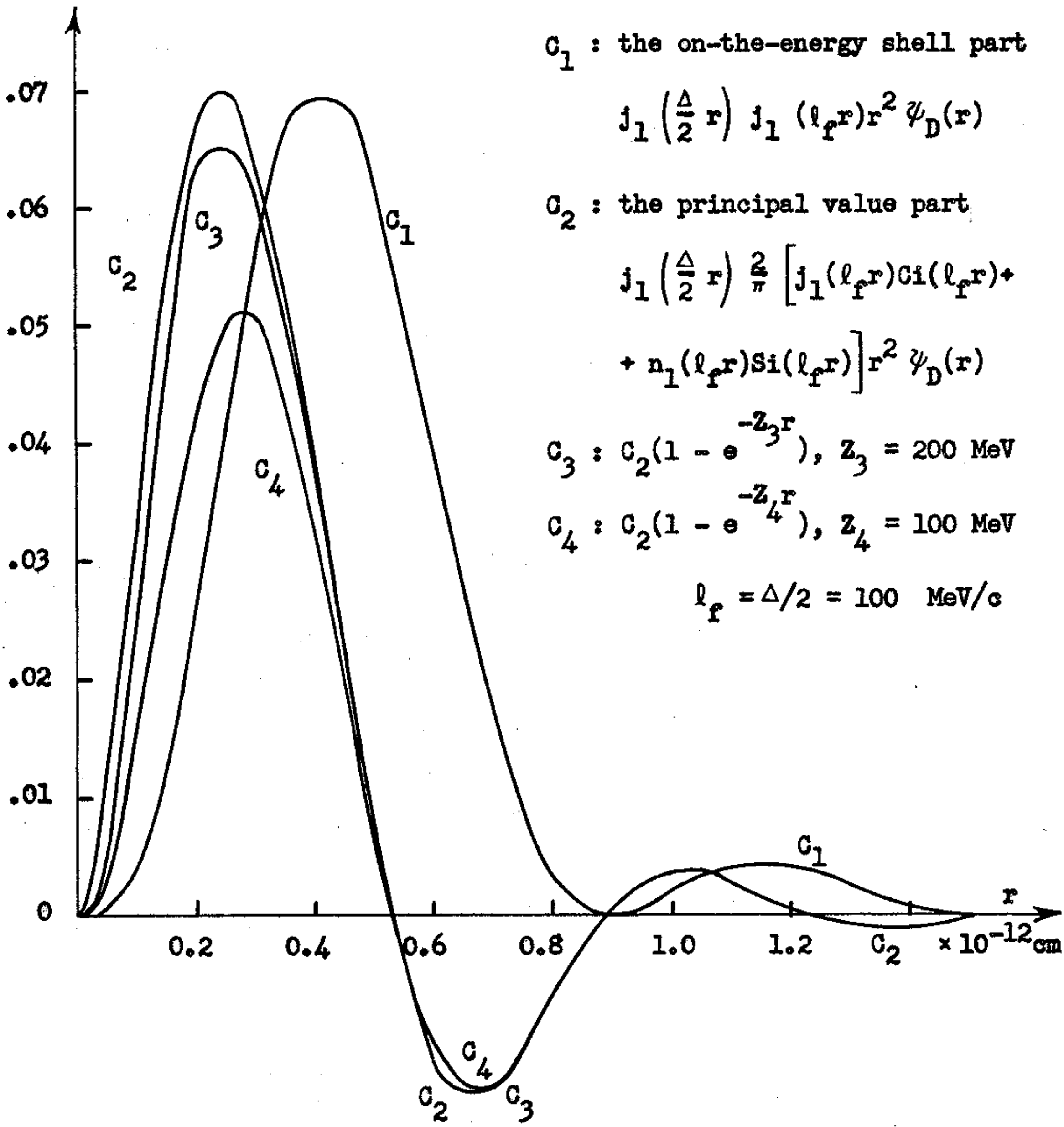


Fig. V.4

P-WAVE PARTS OF THE INTEGRAND IN (V.20) WITH DIFFERENT VALUES OF THE CUT-OFF PARAMETER.

much weaker (larger values of the parameter Z), as it was assumed when we treated the double-scattering terms.

For the S-wave part we have the same sort of behaviour as for the P-wave part quoted above.

If the first collision is on the neutron, then we have instead of Eq. (V.20)

$$I_{un} = - a_n \int < f|r > e^{+\frac{i}{2}\vec{\Delta}\cdot\vec{r}} \psi_D(r)\, d_3\vec{r} \qquad (V.28)$$

so that the S-wave part will give the same result as in the proton case, but the P-wave part will have the opposite sign. We then have

$$I_{up} + I_{un} =$$

$$M\, \ell_f (2\pi)^{3/2} a_u (a_p + a_n) \int_0^\infty \left\{ - i\, j_0(\ell_f r) + n_0(\ell_f r)(1 - e^{-Zr}) \right\} j_0\left(\frac{\Delta}{2} r\right) \psi_D(r) r^2 dr \, +$$

$$+ M\, \ell_f (2\pi)^{3/2} b_u \cos(\vec{\ell}_f, \vec{\Delta})(a_n - a_p) \int_0^\infty \left\{ -i\, j_1(\ell_f r) + \frac{2}{\pi}(1 - e^{-Zr}) \left[j_1(\ell_f r) \mathrm{Ci}(\ell_f r) \, + \right.\right.$$

$$\left.\left. + \, n_1(\ell_f r) \mathrm{Si}(\ell_f r) \right] \right\} j_1\left(\frac{\Delta}{2} r\right) \psi_D(r) r^2 dr \; . \qquad (V.29)$$

All these integrals, except those involving $\mathrm{Si}(\ell_f r)$ and $\mathrm{Ci}(\ell_f r)$, can be analytically evaluated. The fact that a complete analytic evaluation is not possible is not a problem, since numerical computations with these expressions can be made easily. Also, since

Ci(x) and Si(x) are functions with a simple behaviour, and since, due to the presence of $\psi_D(r) j_1\left(\frac{\Delta}{2} r\right) r^2$ in the integrand, all the important contributions to the integral come from a very limited range of values of r, approximate expressions for Ci(x) and Si(x) in terms of other functions can be used. For example, for values of x in the interval from zero to three, we have

$$\mathrm{Si}(x) \simeq 1.85 \sin\left(\frac{x}{1.85}\right)$$

$$\mathrm{Ci}(x) \simeq -0.453 + \ln x + 1.03 \cos(0.7\,x)$$

with an accuracy better than 2%. We may note that for values of ℓ_f in the range from zero to 100 MeV/c, this is the only region that has to be considered.

A natural modification in the function

$$\frac{2}{\pi}\left[j_1(\ell_f r)\mathrm{Ci}(\ell_f r) + n_1(\ell_f r)\mathrm{Si}(\ell_f r)\right]$$

which may, in fact, constitute an improvement, is the following. For large values of r this function becomes $n_1(\ell_f r)$. This was to be expected since $n_1(\ell_f r)$ is a solution of the Schroedinger equation for angular momentum equal to one in regions where the potential is zero, and $\frac{2}{\pi}[j_1(\ell_f r)\mathrm{Ci}(\ell_f r) + n_1(\ell_f r)\mathrm{Si}(\ell_f r)]$ was obtained as a solution of the Schroedinger equation with angular momentum equal to one for some potential which must decrease with r. [Of course, there is the other solution of the Schroedinger equation, the "on-the-energy shell" part $j_1(\ell_f r)$.]

Now, we have some knowledge about the nuclear potential, and we may use it. We know that the nuclear forces are restricted to a region of radius R, and then from r = R to infinity the function $n_1(\ell_f r)$ is a true solution of the Schroedinger equation. For r < R we may then adopt the function

$$(1 - e^{-Zr})\frac{2}{\pi}\left[j_1(\ell_f r)Ci(\ell_f r) + n_1(\ell_f r)Si(\ell_f r)\right]$$

or then $(1 - e^{-Zr})^2 n_1(\ell_f r)$ (Ref. 10) which is also regular. By evaluating these two expressions for Z = 50 MeV and several values of ℓ_f, we find that there is no important difference between them [at least as far as their effects in the integrand in Eq. (V.29) are concerned]. The form $(1 - e^{-Zr})^2 n_1(\ell_f r)$ has the advantage of making the integral simpler for analytical evaluation.

We shall then write

$$I_{up} + I_{un} =$$

$$M\,\ell_f(2\pi)^{3/2}a_u(a_p+a_n)\int_0^\infty\left[-i\,j_0(\ell_f r) + n_0(\ell_f r)(1-e^{-Zr})\right]j_0\left(\frac{\Delta}{2}r\right)\psi_D(r)r^2dr$$

$$+ M\,\ell_f(2\pi)^{3/2}b_u(a_n-a_p)\cos(\vec{\ell}_f,\vec{\Delta})\int_0^\infty\left[-i\,j_1(\ell_f r) +\right.$$

$$\left. + n_1(\ell_f r)(1 - e^{-Zr})^2\right]j_1\left(\frac{\Delta}{2}r\right)\psi_D(r)r^2dr \qquad \textbf{(V.30)}$$

instead of Eq. (V.29).

The integral involving $j_0(\ell_f r) j_0\left(\frac{\Delta}{2} r\right)$ gives the expression (V.14), the only change being that we now have $a_p + a_n$ instead of only a_p. The integral with $n_0(\ell_f r) j_0\left(\frac{\Delta}{2} r\right)$ gives Eq. (V.15) minus the same Eq. (V.15) where we substitute $\alpha \to \alpha + Z$, $\beta \to \beta + Z$. This is due to the cut-off factor that has been introduced.

For the part of Eq. (V.30) corresponding to on-the-energy-shell P-waves we obtain

$$(a_n - a_p) b_u \, i \, L_u^{(1)} \cos(\vec{\ell}_f, \vec{\Delta}) =$$

$$= b_u (a_n - a_p) \cos(\vec{\ell}_f, \vec{\Delta}) \left(\frac{M}{\Delta} 2\pi \sqrt{2\pi} \, N\right) i \left\{ \frac{\beta^2 + \ell_f^2 + \frac{\Delta^2}{4}}{2 \; {}_f\Delta} \ln \frac{\beta^2 + \left(\ell_f + \frac{\Delta}{2}\right)^2}{\beta^2 + \left(\ell_f - \frac{\Delta}{2}\right)^2} - \right.$$

$$\left. - \frac{\alpha^2 + \ell_f^2 + \frac{\Delta^2}{4}}{2\ell_f \Delta} \ln \frac{\alpha^2 + \left(\ell_f + \frac{\Delta}{2}\right)^2}{\alpha^2 + \left(\ell_f - \frac{\Delta}{2}\right)^2} \right\} \qquad \text{(V.31)}$$

and for the integral involving $n_1(\ell_f r) j_1\left(\frac{\Delta}{2} r\right)$ we obtain

$$(a_n - a_p) b_u \, L_u^{(2)} \cos(\vec{\ell}_f, \vec{\Delta}) =$$

$$= b_u (a_n - a_p) \cos(\vec{\ell}_f, \vec{\Delta}) \left(\frac{M}{\Delta} 2\pi \sqrt{2\pi} \, N\right) \left\{ \frac{\beta^2 + \ell_f + \left(\frac{\Delta}{2}\right)^2}{\Delta \ell_f} \tan^{-1} \frac{\beta \Delta}{\beta^2 + \ell_f^2 - \frac{\Delta^2}{4}} - \right.$$

$$\left. - \frac{\alpha^2 + \ell_f^2 + \left(\frac{\Delta}{2}\right)^2}{\Delta \ell_f} \tan^{-1} \frac{\alpha \Delta}{\alpha^2 + \ell_f^2 - \frac{\Delta^2}{4}} \right\} \qquad \text{(V.32)}$$

minus twice this same expression with $\alpha \to \alpha + Z$, $\beta \to \beta + Z$ plus this same expression with the substitution $\alpha \to \alpha + 2Z$, $\beta \to \beta + 2Z$. (This is due to the cut-off factor $(1 - e^{-Zr})^2$.)

Comparison with First Order Processes

For $\Delta \to 0$, all these expressions contributing to $I_{up} + I_{un}$ tend to zero, in spite of the presence of Δ in the denominator.

For $\ell_f \to 0$ the S-wave parts (V.14) and (V.15), and also (V.31), remain finite, but (V.32) increases like $1/\ell_f$, and the matrix element diverges if we consider b_u as constant. This can be modified by noticing that the P-wave scattering amplitude b_u must tend rapidly to zero with the relative momentum of the two colliding particles ($b_u \sim \ell_f^3$ for low energies). Thus, all the matrix elements are always finite.

To compare the magnitudes of these nucleon-nucleon interaction effects with the first order terms, we can do as we did in the discussion of the double scattering terms - notice that the ln and $\tan^{-1}$ functions that appear can only vary inside a limited interval, and then take typical values for them, as well as for the first order terms. We obtain

$$\frac{\text{2nd Order (nucleon-nucleon interaction)}}{\text{1st Order}} \simeq \frac{\alpha}{\Delta}\sqrt{\sigma^T_{NN}(\text{barns})} \qquad (V.33)$$

where σ^T_{NN}(barns) is the nucleon-nucleon total cross-section measured in barns. For low-energy n-p scattering in the triplet state we have $\sigma^T_{NN} \approx 2.4$ barns, and the relation above ...

indicates that the effect of the nucleon-nucleon interaction in the final state may be very strong. For example, for incident mesons of momentum q_0 = 200 MeV/c, a momentum transfer of 150 or 200 MeV/c is "typical", and the ratio between 2nd and 1st order matrix elements is then of about $\frac{1}{3}$ or $\frac{1}{2}$. We can expect that at least some differential cross-sections will be strongly affected by the nucleon-nucleon interactions in the final state.

4. Evaluation of Cross-Sections for Inelastic Scattering

We have obtained explicit expressions for all terms, corresponding to first and second order processes, that contribute to the transition amplitude for meson-deuteron inelastic scattering. We have now to square this transition amplitude and make the necessary sums over spin variables.

Let us assume that there is no spin-dependence in the meson-nucleon interactions. As the deuteron has spin 1, only triplet final states will occur.

We must note that in our system the relative momentum of the two nucleons before they interact is a variable of integration. Using the decomposition of the nucleon-nucleon triplet P-wave in terms of eigenstates of J as given by Eqs. (IV.43) and (IV.44) we have that the matrix element of the transition operator is of the form

$$(R_{inel})_{fi} = \langle f \left| A + P_t \int d_3\vec{\ell}_i \Big[(b_0 - b_2)\frac{1}{3}\frac{1}{2}\,(\vec{\sigma}_p - \vec{\sigma}_n)\cdot\frac{\vec{\ell}_i}{\ell_i}\,\frac{1}{2}\,(\vec{\sigma}_p - \vec{\sigma}_n)\cdot\frac{\vec{\ell}_f}{\ell_f} + \right.$$

$$\left. + (b_1 - b_2)\frac{1}{2}\,\frac{1}{2}\,(\vec{\sigma}_p + \vec{\sigma}_n)\cdot\frac{\vec{\ell}_i}{\ell_i}\,\frac{1}{2}\,(\vec{\sigma}_p + \vec{\sigma}_n)\cdot\frac{\vec{\ell}_f}{\ell_f} + b_2 \cos(\vec{\ell}_f, \vec{\ell}_i)\Big] Q(\vec{\ell}_i) \right| i\rangle \tag{V.34}$$

where b_0, b_1, b_2 are parameters that describe the scattering in the $J = 0,1,2$ states respectively. Rearranging the terms we can put this in a form convenient for comparison with Eqs. (IV.36) and (IV.45), so as to use the formulae (IV.38), (IV.47) and (IV.48). We obtain

$$\sum{}' \left|(R_{inel})_{fi}\right|^2 = \left|A + \int b_2\, Q(\vec{\ell}_i)\cos(\vec{\ell}_f, \vec{\ell}_i)\, d_3\vec{\ell}_i\right|^2 +$$

$$+ \Big\{ \frac{2}{3}\Big(\frac{1}{6}b_0 + \frac{1}{2}b_1 - \frac{2}{3}b_2\Big)\Big[A^* + \int b_2^* Q^*(\vec{\ell}_i')\cos(\vec{\ell}_f, \vec{\ell}_i')\, d_3\vec{\ell}_i'\Big] \int Q(\vec{\ell}_i)\cos(\vec{\ell}_f, \vec{\ell}_i)\, d_3\vec{\ell}_i$$

$$+ \text{complex conjugate} \Big\} + \tag{V.35}$$

$$+ \frac{1}{12}(b_1^* - b_2^*)(b_1 - b_2)\int Q^*(\vec{\ell}_i')\cos(\vec{\ell}_f, \vec{\ell}_i')\, d_3\vec{\ell}_i' \int Q(\vec{\ell}_i)\cos(\vec{\ell}_f, \vec{\ell}_i)\, d_3\vec{\ell}_i +$$

$$+ \Big[\frac{1}{12}(b_1^* - b_2^*)(b_1 - b_2) + \frac{1}{27}(b_0^* - b_2^*)(b_0 - b_2)\Big] \int Q^*(\vec{\ell}_i')\, d_3\vec{\ell}_i' \int Q(\vec{\ell}_i)\cos(\vec{\ell}_i', \vec{\ell}_i)\, d_3\vec{\ell}_i .$$

We can write

$$\cos(\vec{\ell}_i', \vec{\ell}_i) = \cos(\vec{\ell}_i', \vec{\ell}_f)\cos(\vec{\ell}_i, \vec{\ell}_f) + $$

$$+ \sin(\vec{\ell}_i', \vec{\ell}_f)\sin(\vec{\ell}_i, \vec{\ell}_f)\cos(\varphi_i' - \varphi_i)$$

the last part giving zero in the integration over φ_i, since $Q(\vec{\ell}_i)$ depends only on the angle $(\vec{\ell}_f, \vec{\ell}_i)$.

Grouping conveniently the terms, we obtain

$$\sum{}' \left|(R_{inel})_{fi}\right|^2 = A^*A + \frac{1}{9}(b_0 + 3b_1 + 5b_2)\, A^*B \; +$$

$$+ \frac{1}{9}(b_0^* + 3b_1^* + 5b_2^*)\, AB^* + \frac{1}{3}\left[\frac{1}{2}(b_1^* + b_2^*)(b_1 + b_2) + \frac{1}{9}(b_0^* + 2b_2^*)(b_0 + 2b_2)\right] B^*B \; . \tag{V.36}$$

A includes the single and double scattering effects, and the potential correction terms involving S-waves in the nucleon-nucleon interaction. B includes the processes with nucleon-nucleon interaction in P-waves. It is interesting that the parameters b_0, b_1, b_2 appear in the last term of Eq. (V.36) in a combination different from the combination with statistical weights $b_0 + 3b_1 + 5b_2$, which is the only one that occurs in the scattering of unpolarized nucleons. This is, of course, due to the correlation of the spins of the two nucleons in the deuteron.

For the problem of K^+ deuteron inelastic scattering, we have the possibility of a double exchange process in which the K^+ hits the neutron giving K^0 and proton, and then the K^0 hits the proton producing K^+ and neutron. We then have

$$A = a_p \psi_D(\vec{n}_f) + a_n \psi_D(\vec{p}_f) - a_p a_n \left[i\, J_{pn}^{(1)} + J_{pn}^{(2)} \right] - \left[a_p a_n + a_{ex}^2 \right]\left[i\, J_{np}^{(1)} + J_{np}^{(2)} \right] - \left[a_p + a_n \right] a_u \left[i\, I_{up}^{(1)} + I_{up}^{(2)} \right] \tag{V.37}$$

where $I_{up}^{(1)}$, $I_{up}^{(2)}$, $J_{pn}^{(1)}$, $J_{pn}^{(2)}$ are given by Eqs. (V.14), (V.15), (V.7), (V.8). $J_{np}^{(1)}$ and $J_{np}^{(2)}$ are obtained from $J_{pn}^{(1)}$ and $J_{pn}^{(2)}$ by exchanging neutron and proton variables. We must not forget to modify $I_{up}^{(2)}$ by subtracting from Eq. (V.14) the same expression with $\alpha \to \alpha + Z$, $\beta \to \beta + Z$. B will be given by

$$B = (i\, L_u^{(1)} + L_u^{(2)}) \cos (\vec{\ell}_f, \vec{\Delta})(a_p - a_n) \qquad (V.38)$$

with $L_u^{(1)}$ and $L_u^{(2)}$ given by Eqs. (V.31) and (V.32).

b_0, b_1, b_2 are related to the P-wave phase-shifts for the neutron-proton interaction in the triplet state by

$$b_J = \frac{2 \times 3}{M(2\pi)^2 \ell_f} \sin \delta_1^J \, e^{i\delta_1^J} . \qquad (V.39)$$

a_u is related to the S-wave phase-shift in the triplet state by

$$a_u = \frac{2}{M(2\pi)^2 \ell_f} \sin \delta_0^1 \, e^{i\delta_0^1} . \qquad (V.40)$$

The cross-section is given by

$$d\sigma = \frac{(2\pi)^4}{v} \int \sum{}' \left| (R_{inel})_{fi} \right|^2 \delta(E_f - E_i)\, d_3\vec{q}_f \, d_3\vec{\ell}_f \qquad (V.41)$$

with $$E_f - E_i = E_{q_f} + \frac{\ell_f^2}{M} + \frac{(\vec{q}_0 - \vec{q}_f)^2}{4M} + 2M - M_D - E_{q_0} . \qquad (V.42)$$

The integral over ℓ_f can be made at once, since $E_f - E_i$ does not depend on the direction of $\vec{\ell}_f$ but only on its modulus. The integral over q_f will be limited to the interval from $q_f = 0$ to a $q_{f\,max}$ which is the root of

$$E_{q_{f\,max}} + \frac{q_{f\,max}^2}{4M} - \frac{q_o}{2M} q_{f\,max} \cos\Theta = E_{q_o} + M_D - 2M - \frac{q_o^2}{4M} \qquad (V.43)$$

where Θ is the angle between $\vec{q}_f$ and $\vec{q}_o$.

We can express $\psi_D(n_f)$ and $\psi_D(p_f)$ as functions of $\vec{\ell}_f$ and $\vec{\Delta}$ by writing

$$\psi_D(p_f) = \frac{1}{(2\pi)^{3/2}} \int e^{-i\vec{p}_f\cdot\vec{r}} \psi_D(r) d_3\vec{r} = \frac{1}{(2\pi)^{3/2}} \int e^{-i\vec{\ell}_f\cdot\vec{r}} e^{\frac{i}{2}\vec{\Delta}\cdot\vec{r}} \psi_D(r) d_3\vec{r} =$$

$$= \frac{4\pi}{(2\pi)^{3/2}} \sum_\ell (2\ell+1) P_\ell\left[\cos(\vec{\ell}_f,\Delta)\right] \int j_\ell(\ell_f r)\, j_\ell\left(\frac{\Delta}{2} r\right) \psi_D(r) r^2 dr =$$

$$= \frac{4\pi}{(2\pi)^{3/2}} \sum (2\ell+1) P_\ell\left[\cos(\vec{\ell}_f,\vec{\Delta})\right] \Gamma_\ell(\ell_f,\Delta) \qquad (V.44)$$

and

$$\psi_D(n_f) = \frac{4\pi}{(2\pi)^{3/2}} \sum_\ell (2\ell+1)(-1)^\ell P_\ell\left[\cos(\vec{\ell}_f,\vec{\Delta})\right] \Gamma_\ell(\ell_f,\Delta) \qquad (V.45)$$

where we have called

$$\Gamma_\ell(\ell_f,\Delta) = \int_0^\infty j_\ell(\ell_f r) j_\ell\left(\frac{\Delta}{2} r\right) \psi_D(r) r^2 dr =$$

$$= \frac{N}{\ell_f \Delta} Q_\ell\left(\frac{\alpha^2 + \ell_f^2 + \frac{1}{4}\Delta^2}{\ell_f \Delta}\right) - \frac{N}{\ell_f \Delta} Q_\ell\left(\frac{\beta^2 + \ell_f^2 + \frac{1}{4}\Delta^2}{\ell_f \Delta}\right) \qquad (V.46)$$

where the Q_ℓ's are the Legendre polynomials of the second kind.

Let us neglect double scattering processes in the K^+-deuteron inelastic scattering. We have seen that their effects are small due to the smallness of the K^+-nucleon scattering parameters. Then both quantities A and B that appear in Eq. (V.36) can be expressed in terms of Δ, ℓ_f and the angle between $\vec{\ell}_f$ and $\vec{\Delta}$. The integration over all directions of $\vec{\ell}_f$ can then be made without need of the approximation, which was made in the previous papers dealing with the analysis of the meson-deuteron scattering, that the energy in the final state E_f, given by Eq. (V.42), does not depend on ℓ_f. We obtain

$$d\sigma = \frac{(2\pi)^4}{v} \frac{M\ell_f}{2} d_3\vec{q}_f \left\{ 8 \sum_{\ell=0}^{\infty} (2\ell+1) \left[a_p^2 + a_n^2 + (-1)^\ell (a_p^* a_n + a_n^* a_p) \right] \Gamma_\ell^2(\ell_f,\Delta) - \right.$$

$$- \frac{(4\pi)^2}{(2\pi)^{3/2}} \Gamma_0(\ell_f,\Delta)(a_p + a_n)(a_p^* + a_n^*) a_u^* \left[- i\, I_{up}^{(1)} + I_{up}^{(2)} \right] -$$

$$- \frac{(4\pi)^2}{(2\pi)^{3/2}} \Gamma_0(\ell_f,\Delta)(a_p^* + a_n^*)(a_p + a_n) a_u \left[i\, I_{up}^{(1)} + I_{up}^{(2)} \right] +$$

$$+ 4\pi \left|a_p + a_n\right|^2 a_u^2 \left[I_{up}^{(1)^2} + I_{up}^{(2)^2} \right] - \left[\frac{1}{9}(b_0 + 3b_1 + 5b_2)(i L_u^{(1)} + L_u^{(2)}) + \right.$$

$$\left. + \frac{1}{9} (b_0^* + 3b_1^* + 5b_2^*)(-i \; L_u^{(1)} + L_u^{(2)}) \right] \frac{(4\pi)^2}{(2\pi)^{3/2}} \Gamma_1 (\ell_f, \Delta) \left|a_n - a_p\right|^2 +$$

$$+ \frac{1}{3} \left[\frac{1}{2} (b_1^* + b_2^*)(b_1 + b_2) + \frac{1}{9}(b_0^* + 2b_2^*)(b_0 + 2b_2) \right] \left|a_n - a_p\right|^2 \times$$

$$\left. \times \left[L_u^{(1)^2} + L_u^{(2)^2} \right] \frac{4\pi}{3} \right\} \qquad \text{(V.47)}$$

The integrals Γ_ℓ in Eq. (V.46) decrease rapidly to zero as ℓ increases above a certain value. This can be seen in the following way. The function $Nr(e^{-\alpha r} - e^{-\beta r}) = r^2\psi_D(r)$ has the value zero at $r = 0$, increases to a maximum at $(\alpha r) \approx 1$, and then decreases rapidly: at $(\alpha r) = 5$ its value is $1/10$ of the value at the maximum. The functions $j_\ell(ar)$ with $\ell > 1$ start from zero at $r = 0$, and remain very small up to $ar \sim \ell$, where a bump starts. So, if ℓ is so large that the bump in the function j_ℓ starts, say, after $\alpha r = 5$, where the $r^2\psi_D(r)$ is very small, Γ_ℓ will be very small. So we expect that only few terms in the sum $\Sigma(2\ell + 1)\Gamma_\ell^2$ that appears in the expression for the cross-section will be important. For the largest values of ℓ_f and $\Delta/2$ the integrals Γ_ℓ will have the largest values, because then the bumps in $j_\ell(\ell_f r)$ and $j_\ell\left(\frac{\Delta}{2} r\right)$ will occur for smaller values of r, and there will be stronger intersections with $r^2\psi_D(r)$. We can adopt the following criterion to decide where to stop the sum $\sum_{\ell}(2\ell+1)\Gamma_{\ell}^{2}(\ell_f,\Delta/2)$. For a given incident momentum q_0 we choose the highest values of ℓ_f and $\Delta/2$ that are compatible with energy conservation [$E_f = E_i$ in Eq. (V.42)]. For example, for q_0 = 200 MeV/c we may take ℓ_f = 150 MeV/c, $\Delta/2$ = 120 MeV/c. For q_0 = 100 MeV/c we may take $\ell_f = \Delta/2$ = 70 MeV/c. Substituting these values in the explicit expressions for the Legendre polynomials Q_ℓ we can find the value of ℓ for which $(2\ell+1)\Gamma_{\ell}^{2}$ can be neglected. For q_0 = 200 MeV/c, for example, we find that cutting the series at $\ell = 4$ causes an error which is smaller than 3% at the extreme values ℓ_f = 150 MeV/c and $\Delta/2$ = 120 MeV/c. For other values of ℓ_f and Δ the error is much smaller. For higher incident energies we have to take more terms in the sum if we want to keep the error negligible.

With $\beta = 7\alpha$ and for not very high incident energies, the second Q_ℓ in Eq. (V.46) for $\ell \geq 1$ is very small and can be neglected.

If the scattered mesons momenta are not experimentally measured, it is interesting to integrate Eq. (V.47) over the spectrum of values of q_f for each scattering angle Θ. This integration can be made with the help of an electronic computer. The dependence of the neutron-proton scattering parameters a_u, b_0, b_1, b_2 on the relative momentum ℓ_f can easily be taken into account in the integration. For the S-wave phase-shift δ_0^1 we take the shape independent approximation

$$\cot \delta_0^1 = -\frac{1}{a_t \ell_f} + \frac{1}{2} r_{ot} \ell_f \qquad \text{(V.48)}$$

with $a_t = 5.38 \times 10^{-13}$ cm, $r_{0t} = 1.70 \times 10^{-13}$ cm. The P-wave phase-shifts for nucleon-nucleon scattering are small and not very well determined at low energies. The terms of Eq. (V.47) involving P-waves in the n-p interaction which could give non-negligible contributions are then those containing $L_u^{(2)}$. These depend only on the "average" phase-shift in P-wave $\bar{\delta}_p = (\delta_1^0 + 3\delta_1^1 + 5\delta_1^2)$ which seems to have a value of about 1° at $E_{lab} = 20$ MeV ($\ell_f \approx 100$ MeV/c). We adopted a linear dependence of $\bar{\delta}_p$ on ℓ_f

$$\bar{\delta}_p = C\,\ell_f$$

with C chosen so as to satisfy the above mentioned value of $\bar{\delta}_p = 1°$ at $\ell_f = 100$ MeV/c. Then the numerical computations showed that the effect of the P-wave interaction of the two nucleons is negligible in K^+d inelastic scattering for an incident meson momentum of $q_0 = 200$ MeV/c.

We now present the results of numerical computations made for $q_0 = 200$ MeV/c. In terms of the scattering amplitudes in the isotopic spin states I = 1 and I = 0 we have

$$a_p = a_1$$
$$a_n = \tfrac{1}{2}(a_1 + a_0) \; . \qquad \text{(V.49)}$$

Here we are neglecting the Coulomb interaction between the K^+ meson and the proton. This has very little importance in the inelastic scattering for $q_0 \geq 200$ MeV/c. It can only deform a little the shape of the differential inelastic cross-section at small angles.

We call s_1 and s_0 the scattering lengths for I = 1 and I = 0 respectively. The relation between the parameters a_1 and a_0 and the scattering lengths are

$$s_1 = (2\pi)^2 \mu a_1 \qquad (V.50)$$

and the analogous expression for s_0. $\mu = 323.6$ MeV is the reduced mass of the K^+-nucleon system. The experimentally measured value of the K^+p total cross-section at 200 MeV/c is 14 mb, and the differential cross-section shows a constructive interference between Coulomb and nuclear interaction (Ref. 14), so that we have

$$s_1 = -0.3338 \times 10^{-13} \text{ cm} . \qquad (V.51)$$

The value of s_0 is to be determined from the analysis of the experiments made with deuterium bubble chambers. As we shall see in the next section, the measured value of the cross-section for the charge exchange scattering implies that $|1 - s_0/s_1|$ is less than one. A value of 0.7 for this quantity will give $\sigma_{exch} \approx 0.95$ mb, which is close to the measured value [with poor statistics (Ref. 20)]. The inelastic cross-section has not yet been determined experimentally. As the phase-angle between the two quantities s_0 and s_1 is unknown, we cannot determine uniquely the value of s_0. The relation $|1 - s_0/s_1| = 0.7$ implies that $|s_0/s_1|$ is somewhere between 0.3 and 1.7 and that the phase-angle is between 0 and 45°. More detailed information can only come through measurement of the inelastic cross-section.

Figures V.5 and V.6 represent the momentum spectra of the K^+ mesons emergent from the process $K^+d \rightarrow K^+np$ at two different angles (lab. angles $\Theta = 45°$ and $\Theta = 90°$). For each angle we traced two curves, one corresponding to $s_0 = 0$, the other to $s_0 = -0.1 \times 10^{-13}$ cm, to show how the choice of the parameter s_0 can affect the momentum spectrum of the outgoing mesons. For $s_0 = s_1$ the spectrum is the most peaked, with the minimum of tail. Unless the statistics become very good, it appears that a determination of the meson momentum spectrum is not a convenient way of fixing the value of s_0.

The differential cross-section for inelastic scattering can be written in the form

$$\frac{d\sigma^{inel}}{d\Omega} = \left(\frac{0.10136\ E_{q_0}}{q_0}\right)\Big[|3s_1 + s_0|^2 F(q_0,\Theta) + + |s_1 - s_0|^2 G(q_0,\Theta)\Big] . \tag{V.52}$$

The functions $F(q_0,\Theta)$ and $G(q_0,\Theta)$ were evaluated numerically. In Fig. V.7, curve C_1 represents the "inelastic + elastic" cross-section for $s_0 = 0$ in the closure approximation (Refs. 5, 21). This is obtained by neglecting the final state interaction. Curve C_2 represents the inelastic differential cross-section for the same value of the parameter s_0. Comparison of C_2 with C_1 shows that if the closure approximation is valid, the elastic scattering must predominate at this energy (meson incident momentum 200 MeV/c).

Curves in Fig. V.8 show the inelastic differential cross-section for two different values of s_0.

The total inelastic cross-section at 200 MeV/c is given by

$$\sigma^{inel}\ (q_0\ = 200\ \mathrm{MeV/c})\ = 0.50|3s_1 + s_0|^2 + 0.89|s_1 - s_0|^2. \quad (V.53)$$

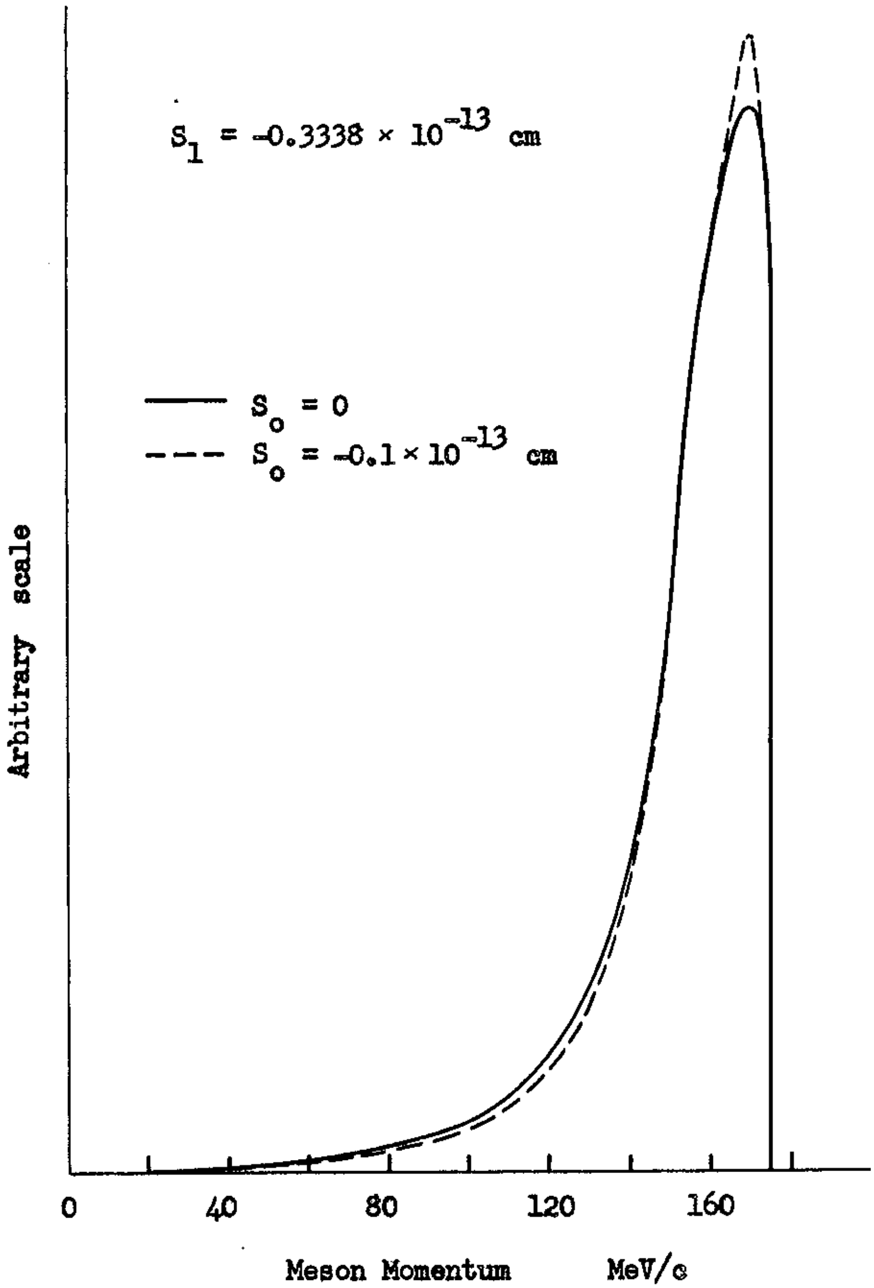


Fig. V.5

MOMENTUM SPECTRUM OF MESONS SCATTERED AT 45° IN $K^{+} d \longrightarrow K^{+}$ np INCIDENT MESON MOMENTUM q_o = 200 MeV/c.

S_1, S_o SCATTERING LENGTHS IN ISOSPIN STATES T = 1 AND T = 0.

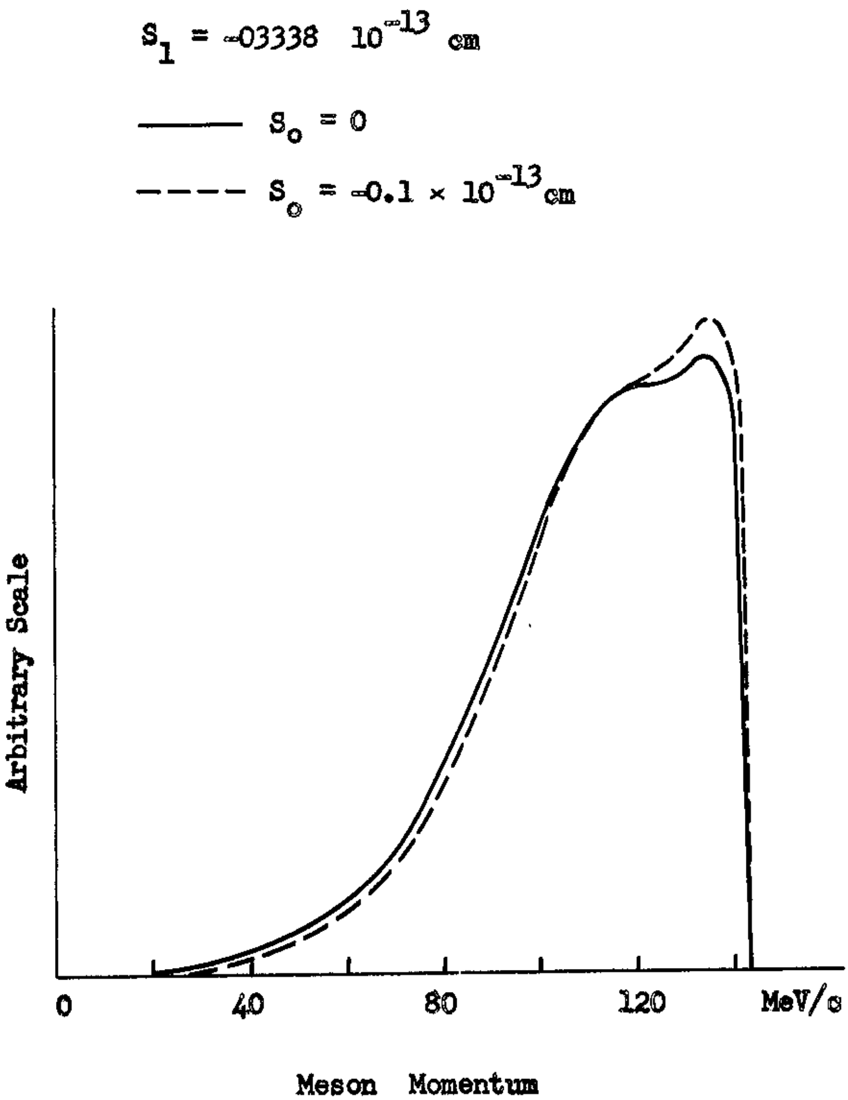


Fig. V.6

MOMENTUM SPECTRUM OF MESONS SCATTERED AT 90° IN $K^+d \rightarrow K^+np$

INCIDENT MESON MOMENTUM q_o = 200 MeV/c.

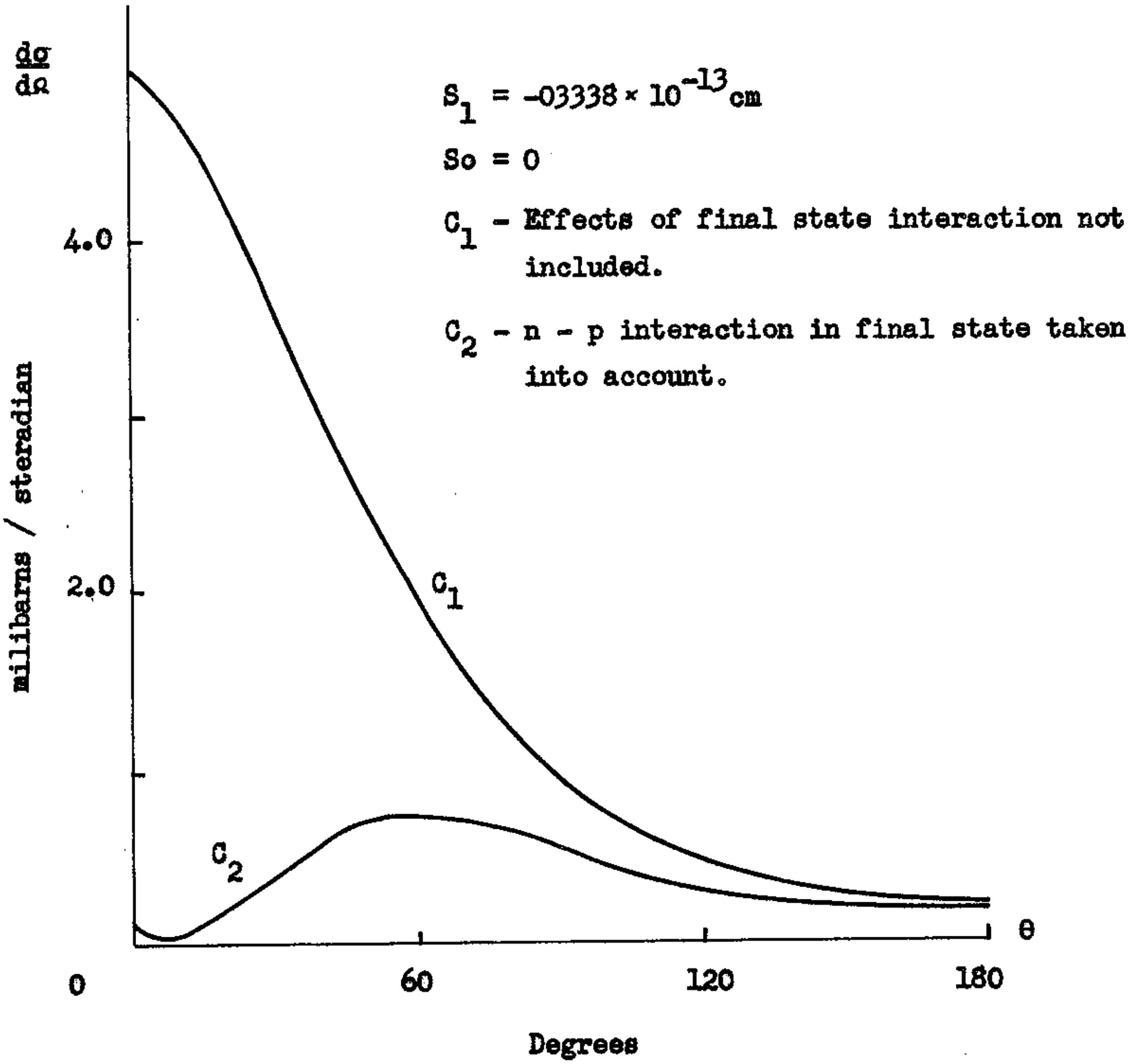


Fig. V.7

DIFFERENTIAL CROSS SECTION FOR INELASTIC SCATTERING $K^+d \rightarrow K^+np$. INCIDENT MESON MOMENTUM q_o = 200 MeV/c. θ - SCATTERING ANGLE OF MESON IN LABORATORY SYSTEM. C_1 REPRESENTS MORE APPROXIMATELY THE SUM OF ELASTIC AND INELASTIC CROSS SECTIONS (CLOSURE APPROXIMATION) THAN THE PURELY INELASTIC CROSS SECTION.

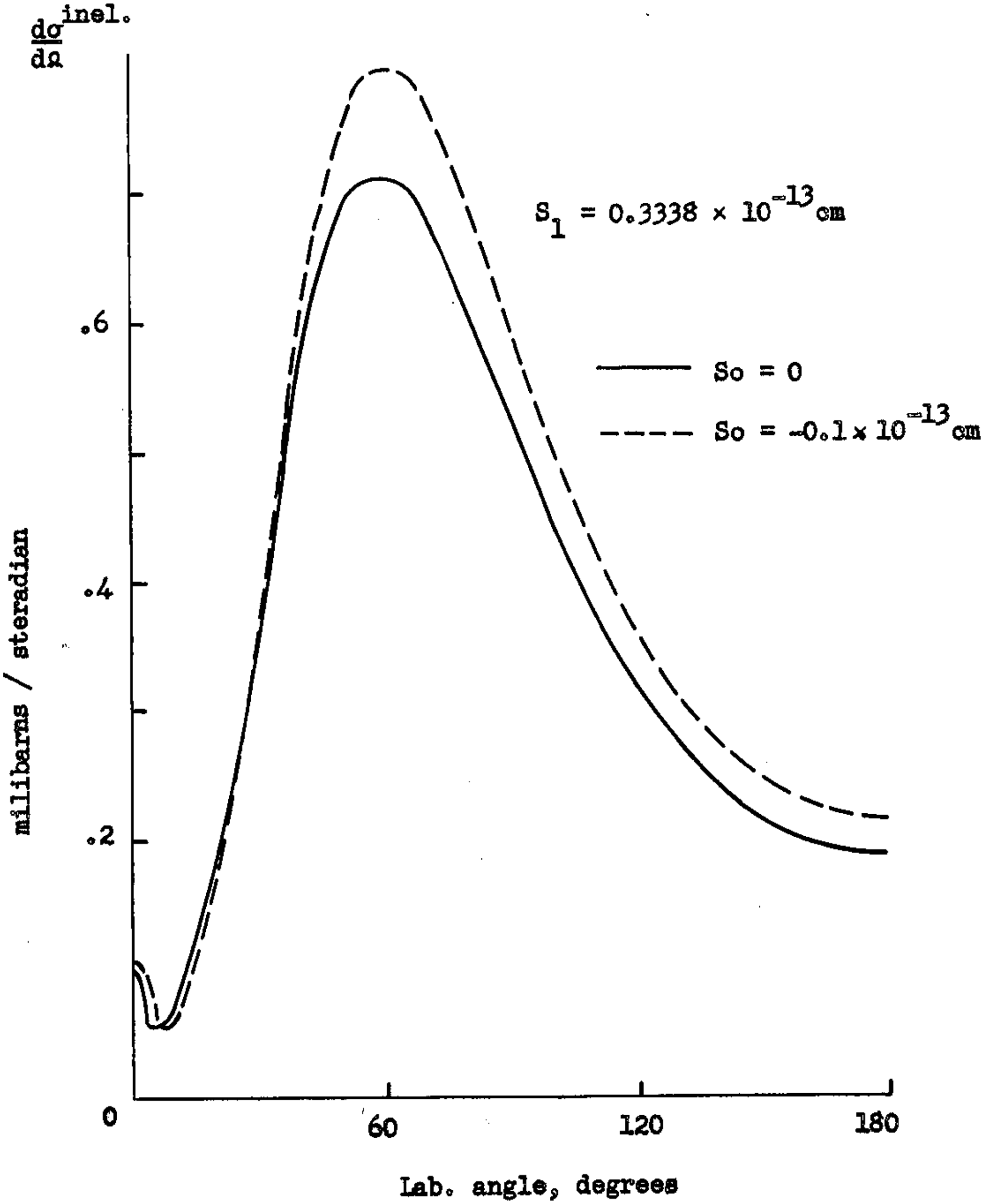


Fig. V.8

INELASTIC SCATTERING $K^+ \; d \longrightarrow K^+ \; np$ DIFFERENTIAL CROSS SECTION.

INCIDENT MESON MOMENTUM $q_o = 200$ MeV/c.

5. Charge-Exchange Scattering

We now consider the process $K^+d \to K^o p\,p$. We may have first a collision, with charge exchange, of the K^+ with the neutron, followed by the interaction of the two protons in the final state or a rescattering of the (K^o) meson. Or we may have first a collision of the incident K^+ meson on the proton followed by a scattering, with charge exchange, of the K^+ by the neutron. Let us call a_{ex} the value of the matrix element of the collision operator for charge-exchange $K^+n \to K^o p$ scattering, and a_{op} the value of the matrix element of the collision operator for the scattering $K^o p \to K^o p$. Besides the fact that these different amplitudes have to be considered, we have in this case of charge-exchange scattering two other important differences with respect to the inelastic $(K^+d \to K^+n\,p)$ scattering. One is that in the present case we have two identical particles (protons) in the final state, and the Pauli principle requires the final state wave function to be antisymmetric with respect to the coordinates of the two protons. The other difference lies in the presence of the Coulomb interaction between the two protons. This must not be neglected a priori since the two nucleons in the final state will not always have a high energy of relative motion. Another fact that must not be forgotten is the mass difference of the K^+ and K^o mesons.

The Coulomb interaction in the final state

The terms in the expansion of the collision operator for charge exchange meson-deuteron scattering which correspond to a single scattering (with charge exchange) of the meson by the neutron and to this scattering followed by a proton-proton interaction give (forgetting the antisymmetrization for the moment)

$$\left\langle f \middle| t_n^{exch} \middle| i \right\rangle + \left\langle f \middle| t_u^{pp} \frac{1}{E-K+i\epsilon} t_n^{exch} \middle| i \right\rangle =$$

$$= -\delta(\vec{P}_f - \vec{P}_i)\, a_{ex} \int \langle \psi^{(-)} | r \rangle\, e^{+\frac{i}{2}\vec{\Delta}\cdot\vec{r}}\, \psi_D(r)\, d_3\vec{r} \qquad (V.51)$$

where the amplitude for charge exchange K^+n scattering has been taken as constant over the range of integration. $\langle \psi^{(-)} | r \rangle$ is the configuration space representation of the scattering state (with proper asymptotic behaviour) of the proton-proton system with nuclear and Coulomb interactions. Outside the range of the nuclear forces we have

$$\langle \psi^{(-)} | r \rangle = \frac{e^{-\frac{1}{2}n\pi}}{(2\pi)^{3/2}} \sum_{\ell=0}^{\infty} \frac{\Gamma(\ell+1+in)}{(2\ell)!} (2i\ell_f r)^{\ell} e^{i\ell_f r} (-1)^{\ell} P_\ell[\cos(\vec{\ell}_f,\vec{r})]$$

$$\times\left\{F(\ell+1+in, 2\ell+2, -2i\ell_f r) + (e^{2i\delta_\ell}-1) W_1(\ell+1+in, 2\ell+2, -2i\ell_f r)\right\} \qquad (V.52)$$

where

$$n = \frac{Me^2}{2\ell_f} = \frac{M}{274\,\ell_f} \qquad (V.53)$$

and δ_ℓ are the nuclear phase-shifts.

F is the well-known confluent hypergeometric function. W_1 is related to the Whittaker function $W_{\lambda\mu}(z)$ by

$$W_1(\mu+\tfrac{1}{2}-\lambda,\ 2\mu+1,\ z) = \frac{\Gamma(2\mu+1)}{\Gamma(\mu+\frac{1}{2}+\lambda)}\,(-1)^{\mu+\frac{1}{2}+\lambda}\,z^{-(\mu+\frac{1}{2})}\,W_{\lambda\mu}(z)\ . \qquad (V.54)$$

F is regular at r = 0, but $r^{\ell} W_1 | \ell+1+in,\ 2\ell+2,\ -2i\ell_f r)$ diverges like $^1/r^{\ell+1}$ when $r \to 0$. Thus to extend Eq. (V.52) to the region inside the range of the nuclear interaction we must introduce a cut-off factor of some kind in the part containing W_1 to eliminate this divergence.

Our problem now is to evaluate the integral in Eq. (V.51) using the expression in Eq. (V.52) for $< \psi^{(-)}|r>$. Let us first consider the pure Coulomb part. It is

$$C = a_{ex}\int \frac{e^{-\frac{1}{2}n\pi}}{(2\pi)^{3/2}}\sum_{\ell=0}^{\infty}\frac{\Gamma(\ell+1+in)}{(2\ell)!}\,(2i\ell_f r)^{\ell}\,e^{i\ell_f r}\,(-1)^{\ell}P_{\ell}[\cos(\vec{\ell}_f,\vec{r})]\ \times$$

$$\times\ F(\ell+1+in,\ 2+\ell 2,\ -2i\ell_f r)\ e^{+i\frac{\vec{\Delta}}{2}\cdot\vec{r}}\ \psi_D(r)\ d_3\vec{r} =$$

$$= a_{ex}\frac{e^{-\frac{1}{2}n\pi}}{(2\pi)^{3/2}}\sum_{\ell=0}^{\infty}\frac{\Gamma(\ell+1+in)}{(2\ell)!}P_{\ell}[\cos(\vec{\ell}_f,\vec{\Delta})](4\pi)(-i)^{\ell}\ \times \qquad (V.55)$$

$$\times\int_{r=0}^{\infty}(2i\ell_f r)^{\ell}\ e^{i\ell_f r}\ j_{\ell}\left(\frac{\Delta}{2}\right)\psi_D(r)F(\ell+1+in,2\ell+2,-2i\ell_f r)\ r^2 dr$$

the integral over angles having been effectuated in the usual way. Now we substitute F by its integral representation (Ref. 22)

$$F(\ell+1+in, 2\ell+2, -2i\ell_f r) = \\ = \frac{\Gamma(2\ell+2)}{\Gamma(\ell+1+in)\Gamma(\ell+1-in)} \int_{u=0}^{1} e^{-2i\ell_f ru}\, u^{\ell+in} (1-u)^{\ell-in}\, du \qquad (V.56)$$

and integrate over r first. We obtain

$$C = -a_{ex} \frac{e^{-\frac{1}{2}n\pi}}{(2\pi)^{3/2}} 4\pi N \sum_{\ell} \frac{2^{\ell}}{(2\ell)!} (\ell_f \Delta)^{\ell} P_{\ell}[\cos(\vec{\ell}_f, \vec{\Delta})] \frac{\Gamma(2\ell+2)\Gamma(\ell+1)}{\Gamma(\ell+1-in)} \times$$

$$\times \int_0^1 du\, u^{\ell+in} (1-u)^{\ell-in} \left\{ \frac{1}{\left\{\frac{\Delta^2}{4} + \left[\alpha + i\ell_f(2u-1)\right]^2\right\}^{\ell+1}} - \right.$$

$$\left. - \frac{1}{\left\{\frac{\Delta^2}{4} + \left[\beta + i\ell_f(2u-1)\right]^2\right\}^{\ell+1}} \right\}. \qquad (V.57)$$

Substituting $u/(1-u) = t$ the integral in the expression above becomes

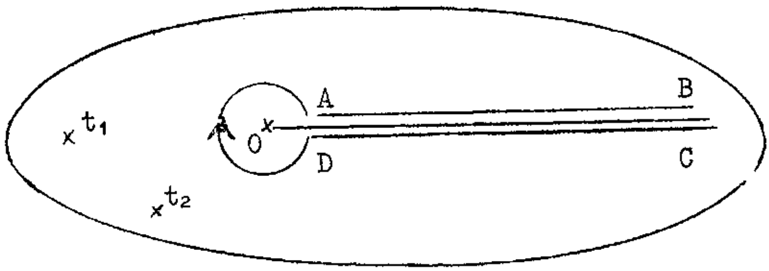
$$\int_{t=0}^{\infty} dt \left\{ \frac{1}{\left[\frac{\Delta^2}{4} + (\alpha + i\ell_f)^2\right]^{\ell+1}} \quad \frac{t^{\ell+in}}{(t-t_1)^{\ell+1}(t-t_2)^{\ell+1}} \right\} - \int_{t=0}^{\infty} dt \left\{\alpha \to \beta\right\}$$

with

$$t_1(\alpha) = \frac{-\alpha^2 - \left(\frac{\Delta}{2} + \ell_f\right)^2}{\frac{\Delta^2}{4} + (\alpha + i\ell_f)^2}$$

$$t_2(\alpha) = \frac{-\alpha^2 - \left(\frac{\Delta}{2} - \ell_f\right)^2}{\frac{\Delta^2}{4} + (\alpha + i\ell_f)^2} \qquad (V.58)$$

and analogous expressions for $t_1(\beta)$, $t_2(\beta)$. The poles at $t = t_1$, $t = t_2$ are not in the path of integration. Let us go to the complex plane. We have a branch point at $t = 0$. We now make a cut from 0 to infinity along the real axis, and integrate over the contour indicated below.



The integral over the curve at infinity and over the infinitely small circle around $t = 0$ both vanish. The integral over CD is $-(e^{2\pi i})^{in} = -e^{-2\pi n}$ times the integral over AB, which corresponds to the one we want to evaluate. We then obtain, by the Theorem of the Residues, that

$$\int_{t=0}^{\infty} \frac{t^{\ell+in}}{(t-t_1)^{\ell+1}(t-t_2)^{\ell+2}}\,dt = \frac{2\pi i}{1-e^{-2\pi n}}\frac{1}{\ell!}\left\{\left(\frac{d^\ell}{dt^\ell}\frac{t^{\ell+in}}{(t-t_1)^{\ell+1}}\right)_{t=t_2} + \left(\frac{d^\ell}{dt^\ell}\frac{t^{\ell+in}}{(t-t_2)^{\ell+1}}\right)_{t=t_1}\right\}. \qquad (V.59)$$

We thus finally have for C

$$C = a_{ex} \frac{e^{-\frac{1}{2}n\pi}}{(2\pi)^{3/2}} 4\pi N \frac{2\pi i}{(1-e^{-2\pi n})} \sum_{\ell} (2\ell_f\Delta)^{\ell} \frac{2\ell+1}{\Gamma(\ell+1-in)} P_{\ell}\left[\cos\,(\vec{\ell}_f,\vec{\Delta})\right] \times$$

$$\left[\left\{\frac{1}{\left[\frac{\Delta^2}{4} + (\alpha + i\ell_f)^2\right]^{\ell+1}} \left(\frac{d^{\ell}}{dt^{\ell}} \frac{t^{\ell+in}}{(t-t_1)^{\ell+1}}\right)_{t=t_2(\alpha)} + \right.\right.$$

$$\left. + \frac{1}{\left[\frac{\Delta^2}{4} + (\alpha + i\ell_f)^2\right]^{\ell+1}} \left(\frac{d^{\ell}}{dt^{\ell}} \frac{t^{\ell+in}}{(t-t_2)^{\ell+1}}\right)_{t=t_1(\alpha)}\right\} -$$

$$\left. - \left\{\alpha \to \beta\right\}\right] = \sum_{\ell} C_{\ell} \; . \qquad (V.60)$$

The evaluation of the terms in this series is straightforward.

Now for the term that depends on the nuclear phase-shifts. The proton-proton scattering phase-shifts in P-waves are very small. In the case of the triplet state of the p-p system, the antisymmetrization introduced by the Pauli principle will require the final state wave function to be negligible for very small relative momenta of the two protons. Thus $n = \frac{1}{274} \frac{M}{\ell_f}$ will not have very high values, being in general small compared to unity. We can then put $n = 0$ in the argument of the function W_1 in Eq. (V.52) and thus reduce the second term of this expression to the usual combination of spherical Bessel functions that represent the pure nuclear scattered wave. A purely nuclear term like this has already been treated in Section V.4, where $I_{up}^{(1)}$, $I_{up}^{(2)}$, $L_u^{(1)}$ and $L_u^{(2)}$ were obtained. This approximation corresponds to writing the scattering amplitude for p-p scattering as a sum of the scattering amplitude for a pure Coulomb interaction with the amplitude for a purely nuclear interaction, namely

$$t_u = t_u^C + t_u^N \quad . \tag{V.61}$$

The single scattering term in the left-hand side of Eq. (V.51) gives

$$\langle f|t_n^{ex}|i\rangle = -\,\delta(\vec{P}_f-\vec{P}_i)\; a_{ex}\int \frac{e^{-i\vec{\ell}_f\cdot\vec{r}}}{(2\pi)^{3/2}}\, e^{+\frac{i}{2}\vec{\Delta}\cdot\vec{r}}\,\psi_D(r)d_3\vec{r} \;=$$

$$= -\,\delta(\vec{P}_f-\vec{P}_i)\; a_{ex}\frac{4\pi}{(2\pi)^{3/2}}\sum_\ell (2\ell+1)P_\ell\left[\cos\,(\vec{\ell}_f,\vec{\Delta})\right]\Gamma_\ell(\ell_f,\Delta)$$

and we have, without having yet antisymmetrized the final state

$$\langle f|t_u\frac{1}{E-K+i\epsilon}\,t_n^{ex}|i\rangle = -\,\delta(\vec{P}_f-\vec{P}_i)\sum_\ell\Big\{\, C_\ell\; -$$

$$- \,a_{ex}\frac{4\pi}{(2\pi)^{3/2}}\,(2\ell+1)\;P_\ell\left[\cos\;(\vec{\ell}_f,\vec{\Delta})\right]\Gamma_\ell\,(\ell_f,\Delta)\Big\}\; +$$

$$+ \langle f|t_u^N\;\frac{1}{E-K+i\epsilon}\;t_n^{ex}|i\rangle \;. \tag{V.62}$$

Now, the wave scattered by the Coulomb field, which gives rise to all the part of the above expression which is embraced by the $\sum_\ell$ symbol, will consist of only a few angular momentum waves. This is due to the fact that n is not very small (Coulomb scattering at not very low energies is described by only a few spherical waves). Thus, the series above will converge rapidly, only a few terms being necessary for our purposes. In the case of a triplet spin-state of the two protons only the term with $\ell = 1$ has to be considered ($\ell = 0$ and $\ell = 2$ are excluded by antisymmetrization and $\ell = 3$ already gives very small contribution).

Charge-Exchange Scattering Cross-Section

Let us consider the case in which there is no spin dependence in the scattering amplitude a_{ex} for the charge-exchange process $K^+n \rightarrow K^0p$, as well as in the amplitudes a_p and a_{op}. Then the two protons will be in the triplet state, as they are in the deuteron. Antisymmetrization of the wave function will eliminate all even parity states. Squaring the properly antisymmetrized amplitude, and summing over spin variables, we obtain for $|(R_{inel})_{fi}|^2$ an expression of the form (V.36), with

$$A = +\sqrt{2} \sum_{\ell \text{ odd}} C_\ell - \frac{1}{\sqrt{2}} a_{ex} a_{op} \left[i\, J^{(1)}_{p_2 p_1} - i\, J^{(1)}_{p_1 p_2} + J^{(2)}_{p_2 p_1} - J^{(2)}_{p_1 p_2} \right] -$$

$$- \frac{1}{\sqrt{2}} a_{ex} a_p \left[i\, J^{(1)}_{p_2 p_1} - i\, J^{(1)}_{p_1 p_2} + J^{(2)}_{p_2 p_1} - J^{(2)}_{p_1 p_2} \right]$$

$$B = -\sqrt{2}\; a_{ex} \cos(\vec{\ell}_s, \vec{\Delta}) \left[i\, L^{(1)}_u + L^{(2)}_u \right] \qquad \text{(V.63)}$$

where p_1, p_2 indicate the two protons in the final state and $\vec{p}_{1f}, \vec{p}_{2f}$ are their momenta. $\vec{\ell}_f$ is the vector $\frac{1}{2}(\vec{p}_{1f} - \vec{p}_{2f})$.

The C_ℓ's include both the single scattering terms and the terms which involve the Coulomb interaction in the final state. We have seen that the contributions coming from the single scattering processes can be represented in the form of a series in ℓ, the contributions from the terms of the series decreasing rapidly as ℓ increases. Also, very few values of ℓ (and possible only one) will give non-negligible contributions to the Coulomb interaction. If only P-waves are important we have for the differential cross-section, after integration over $d_3\vec{\ell}_f$, and neglecting double scattering processes,

$$d\sigma = \frac{(2\pi)^4}{v} \frac{M\ell_f}{2} a_{ex}^2 \, d_3 q_f \left\{ K + \sum_{\ell=3,5,7\ldots} 16(2\ell+1)\, \Gamma_\ell^2(\ell_f,\Delta) \right. -$$

$$- \frac{2}{9}(b_0 + 3b_1 + 5b_2)(\, i\, L_u^{(1)} + L_u^{(2)}) \frac{(4\pi)^2}{(2\pi)^{3/2}} \Gamma_1(\ell_f,\Delta) -$$

$$- \frac{2}{9}(b_0^* + 3b_1^* + 5b_2^*)(-i\, L_u^{(1)} + L_u^{(2)}) \frac{(4\pi)^2}{(2\pi)^{3/2}} \Gamma_1(\ell_f,\Delta) +$$

$$\left. - \frac{2}{3}\left[\frac{1}{2}|b_1 + b_2|^2 - \frac{1}{9}|b_0 + 2b_2|^2\right] \frac{4\pi}{3}\left[L_u^{(1)^2} + L_u^{(2)^2}\right] \right\} \qquad (V.64)$$

where

$$K = \frac{16 \times 6\pi\ N^2}{(1-e^{-2\pi n})n(1+n^2)\Delta^2 \ell_f^2} \left\{ e^{n\varphi(\alpha)} \left[n \cos\left(\frac{n}{2} \log \frac{\alpha^2 + \left(\ell_f - \frac{\Delta}{2}\right)^2}{\alpha^2 + \left(\ell_f + \frac{\Delta}{2}\right)^2} \right) + \right. \right.$$

$$+ \frac{\alpha^2 + \frac{1}{4}\Delta^2 + \ell_f^2}{\ell_f \Delta} \sin\left(\frac{n}{2} \log \frac{\alpha^2 + \left(\ell_f - \frac{\Delta}{2}\right)^2}{\alpha^2 + \left(\ell_f + \frac{\Delta}{2}\right)^2} \right) \Bigg] -$$

$$- e^{n\varphi(\beta)} \left[n \cos\left(\frac{n}{2} \log \frac{\beta^2 + \left(\ell_f - \frac{\Delta}{2}\right)^2}{\beta^2 + \left(\ell_f + \frac{\Delta}{2}\right)^2} \right) + \right.$$

$$\left. \left. + \frac{\beta^2 + \frac{1}{4}\Delta^2 + \ell_f^2}{\ell_f \Delta} \sin\left(\frac{n}{2} \log \frac{\beta^2 + \left(\ell_f - \frac{\Delta}{2}\right)^2}{\beta^2 + \left(\ell_f + \frac{\Delta}{2}\right)^2} \right) \right] \right\}^2 . \qquad \text{(V.65)}$$

K accounts for the Coulomb interaction between the two protons. If $n \to 0$, $K \to 16 \times 3\ a_{ex}^2\ \Gamma_1^2(\ell_f, \Delta)$ as it should.

In Eq. (V.65)

$$\varphi(\alpha) = \text{arc tan} \frac{2\ \alpha\ \ell_f}{\frac{1}{4}\Delta^2 + \alpha^2 - \ell_f^2} \qquad \text{(V.66)}$$

with an analogous expression for $\varphi(\beta)$.

Fig. V.9 shows the charge-exchange differential cross-section for an incident meson momentum $q_0 = 200$ MeV/c, and scattering lengths $s_1 = -0.3338 \times 10^{-13}$ cm and $s_0 = 0$. In curve

C_1 the Coulomb interaction of the two protons was taken into account, and in curve C_2 it has been neglected. We see that the shape of the angular distribution is not much altered by the Coulomb effect at this energy, but that the value of the cross-section is uniformly increased at all angles. The increase in the calculated total cross-section, obtained by including the Coulomb interaction, is 20% for 200 MeV/c incident meson momentum; 13% for 350 MeV/c; 7.5% for 530 MeV/c; and 6% for 642 MeV/c.

The general form of the differential charge-exchange scattering cross-section in the Lab. system is

$$\frac{d\sigma^{exch.}}{d\Omega} = \left(\frac{0.10136\ E_{q_0}}{q_0}\right) \left|s_1 - s_0\right|^2 H(q_0,\Theta)\ . \qquad (V.67)$$

The function $H(q_0,\Theta)$ has to be calculated numerically. Fig. V.10 shows the shape of the differential charge-exchange cross-section for $q_0 =$ 200 MeV/c. Due to the smallness of the number of observed charge-exchange events at this energy (Ref. 20), The angular distribution is not yet known experimentally. With $s_1 = - 0.3338 \times 10^{-13}$ cm, the total charge-exchange cross-section for 200 MeV/c is given by

$$\sigma^{exch.}(q_0 = 200\ \mathrm{MeV/c}) = 1.924\left|1 - \frac{s_0}{s_1}\right|^2 .$$

The reported experimental value is $0.9^{+0.7}_{-0.3}$ mb at 230 MeV/c. This implies that the value of $\left|1 - \frac{s_0}{s_1}\right|$ is of about 0.7 at this energy.

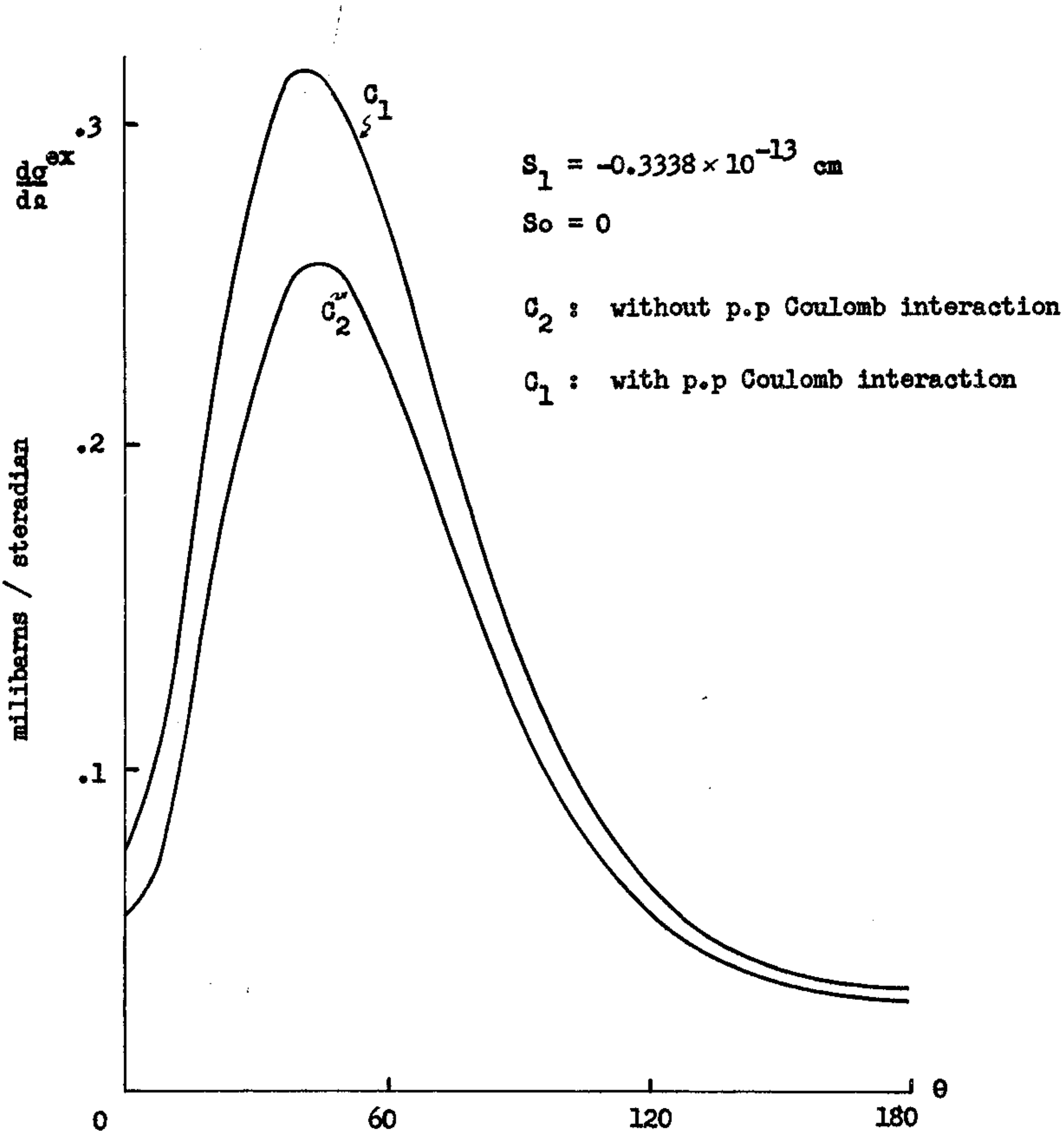


Fig. V.9

THE EFFECT OF THE COULOMB INTERACTION OF THE TWO PROTONS IN $K^+ d \longrightarrow K^o pp$ FOR AN INCIDENT MESON MOMENTUM OF q_o = 200 MeV/c

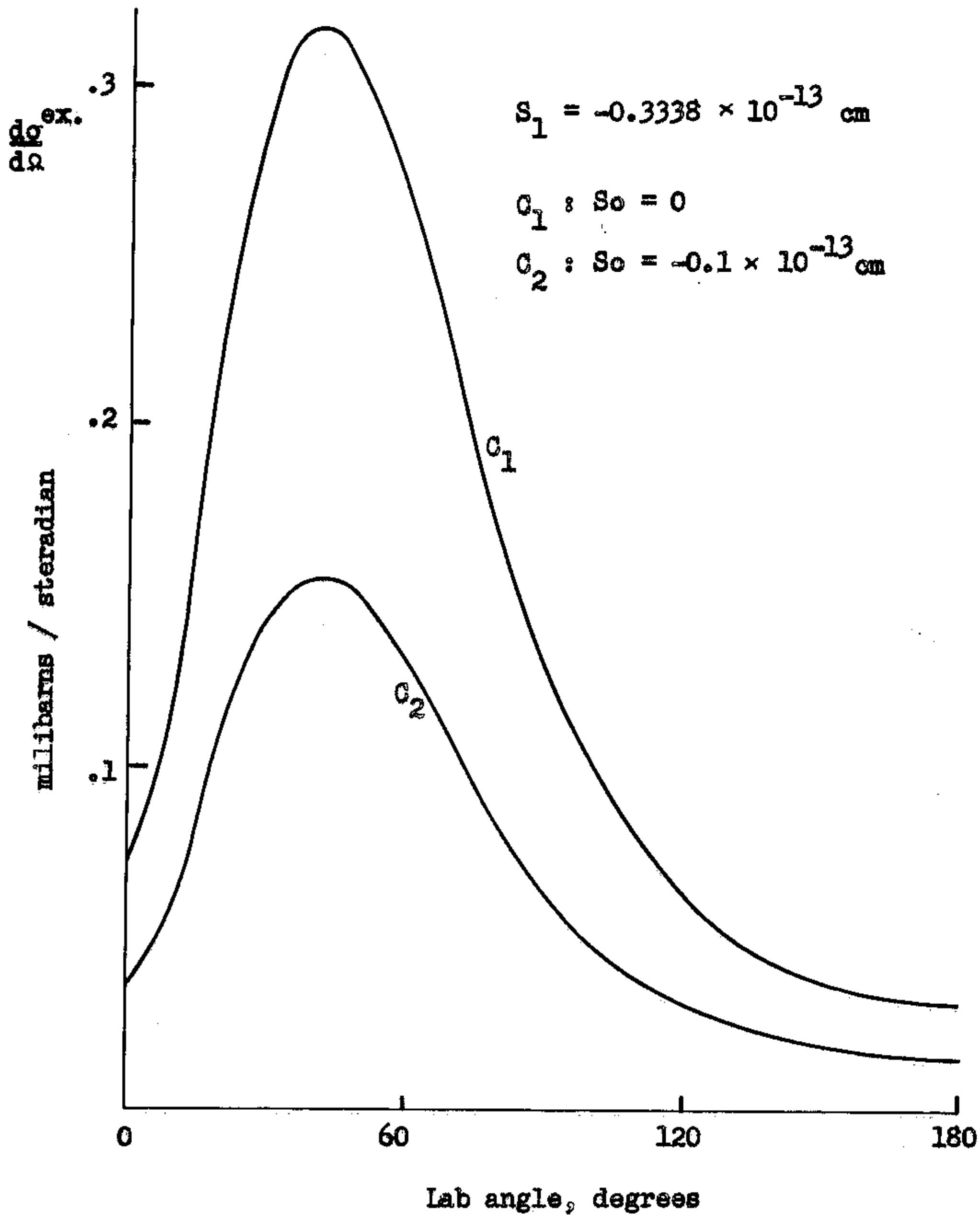


Fig. V.10

DIFFERENTIAL CROSS SECTION FOR CHARGE EXCHANGE SCATTERING $K^+d \rightarrow K^0pp$ FOR AN INCIDENT MESON MOMENTUM q_0 = 200 MeV/c.

CHAPTER VI

COMPARISON WITH EXPERIMENTAL DATA

Data on the differential and total cross-sections for the charge-exchange ($K^+d \rightarrow K^0pp$) scattering for K^+-meson beams of several energies have been obtained with deuterium bubble chambers (Ref. 20).

The results for the differential cross-sections are shown in the curves at the end of this chapter. The measured values of the total exchange cross-section are reproduced below.

Lab. mom.	Lab. en.	σ_{ex}
230 MeV/c	52 MeV	$0.9 {+0.7 \atop -0.3}$
350 MeV/c	111 MeV	3.0 ± 0.3
530 MeV/c	230 MeV	5.8 ± 0.6
642 MeV/c	315 MeV	6.7 ± 0.6

A K^+ beam of higher momentum (810 MeV/c) is also available for the same experimental group, but data with deuterium at this energy have not yet been obtained. The scattering of this beam by free protons in a hydrogen bubble chamber showed that the $K^+p \rightarrow K^+p$ differential cross-section is not inconsistent with a pure S-wave interaction, the total cross-section being of 16 mb.

The cross-sections for inelastic scattering $K^+d \to K^0 np$ have not yet been obtained.

In the previous chapters we adopted a model in which it is assumed that the meson-deuteron processes can be approximately described by a series of two-particle processes. Assuming that the K^+n and K^+p interactions are purely in S-waves, and neglecting multiple scattering (but not neglecting the nucleon-nucleon interaction in the final state), in Chapter V we obtained that the inelastic and charge-exchange differential cross-sections are given by

$$\frac{d\sigma^{inel}}{d\Omega} = \left(\frac{0.10136\ E_{q_0}}{q_0}\right) |s_1|^2 \left\{F(q_0,\Theta)\left|3+\frac{s_0}{s_1}\right|^2 + G(q_0,\Theta)\left|1-\frac{s_0}{s_1}\right|^2\right\} \quad \text{(VI.1)}$$

$$\frac{d\sigma^{exch}}{d\Omega} = \left(\frac{0.10136\ E_{q_0}}{q_0}\right) |s_1|^2\ H(q_0,\Theta)\left|1-\frac{s_0}{s_1}\right|^2. \quad \text{(VI.2)}$$

The functions F, G and H were calculated for five different values of the incident meson momentum q_0. Their values are tabulated at the end of this chapter.

s_0 and s_1 are the scattering lengths in the I = 0 and I = 1 isotopic spin states. The value of s_1 is to be taken directly from the $K^+p \to K^+p$ experiments. We can adopt the value $s_1 = -0.357 \times 10^{-13}$ which corresponds to a total cross-section of 16 mb.

The interesting feature of the experimental values of the charge-exchange cross-section is the rather fast increase of the total cross-section with the incident energy. This feature led some authors to believe in the existence of strong P-waves in the $K^+ \to n$ interaction.

The value of the total charge-exchange cross-section fixes the value of $|1 - s_0/s_1|^2$. By integrating Eq. (VI.2) over angles we obtain

$$\begin{aligned} \sigma^{ex} &= 3.4\ |1 - s_0/s_1|^2 \text{ mb at } 350 \text{ MeV/c} \\ \sigma^{ex} &= 4.86 |1 - s_0/s_1|^2 \text{ mb at } 530 \text{ MeV/c} \\ \sigma^{ex} &= 5.5\ |1 - s_0/s_1|^2 \text{ mb at } 642 \text{ MeV/c} \\ \sigma^{ex} &= 6.1\ |1 - s_0/s_1|^2 \text{ mb at } 810 \text{ MeV/c} . \end{aligned} \qquad \text{(VI.3)}$$

Comparing these formulae with the experimental results we find that the quantity $|1 - s_0/s_1|$ has the values of 0.94 at 350 MeV/c and 1.1 at 530 and 642 MeV/c. Thus we see that we do not need more than a more or less constant value of the S-wave amplitude in the I = 0 state to explain the rise in the total exchange cross-section with the energy. This rise is already contained in the functions H, and can be attributed to a rapid increase of the phase-space factors with the incident energy, due to the fact that there are three free particles in the final state to share the available energy.

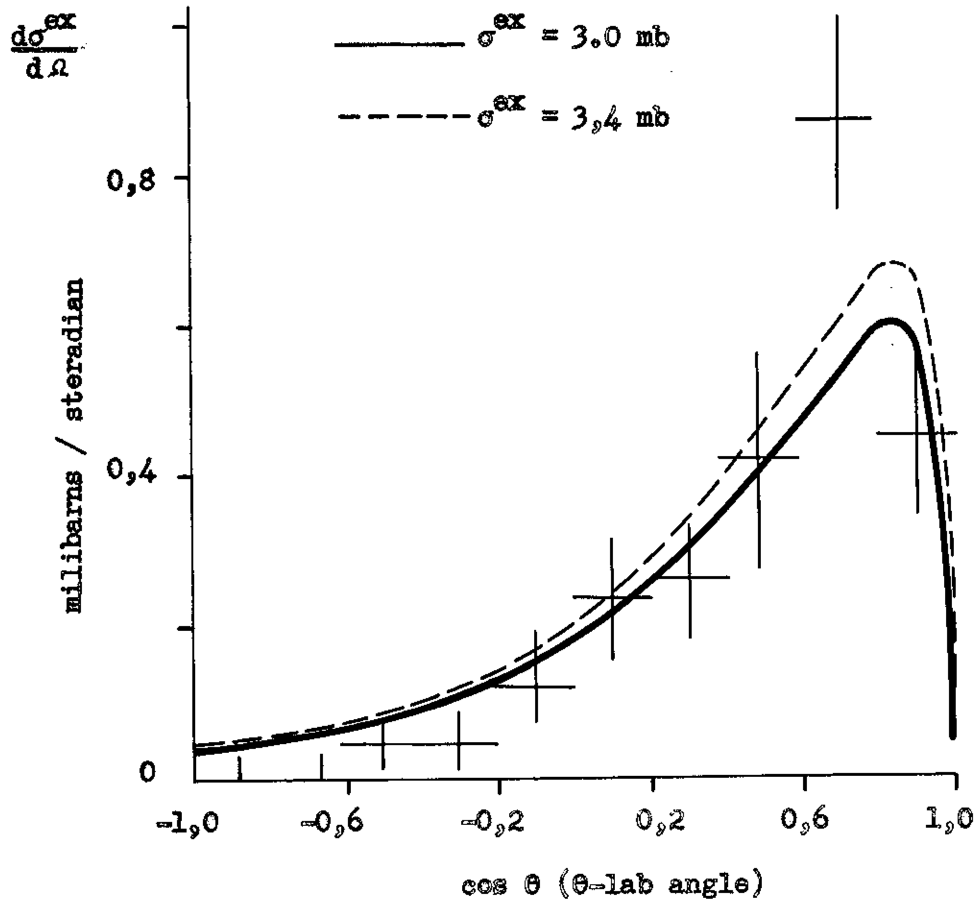
The total exchange cross-section at 810 MeV/c has not yet been measured. If we assume that the value of $|1 - s_0/s_1|$ will remain more or less constant at about 1.1, we can make the prediction that the value of the total exchange cross-section at this energy is of about 7.5 millibarns.

If the adopted model is good, the angular distribution must be well described by Eq. (VI.2). There is no possibility of playing with parameters to adapt the shape of the angular distribution in the charge-exchange scattering, except by multiplying it by the proper constant $|1 - s_0/s_1|^2$ so as to obtain the required value of the total cross-section. For q_0 = 642 MeV/c the theoretical curves fit the experimental points well. For q_0 = 530 MeV/c the agreement is still fairly good, but the curves for 530 and 350 MeV/c together show that there may be a systematic deviation of the experimental points from the theoretical curves. This may be due to essential three-body effects, which we would expect to be stronger at the lower energies. Also it may be due to a P-wave contribution in the K^+n interaction, but we would expect this to be also strong in the higher energy (642 MeV/c) where the agreement is so good with S-waves only.

Besides an improvement in the statistics in the measurements at the lower energies, it seems desirable to have the experimental determination of the differential exchange cross-section at 810 MeV/c. We expect the independent particle model to be valid at this energy, as it seems to be at 642 MeV/c.

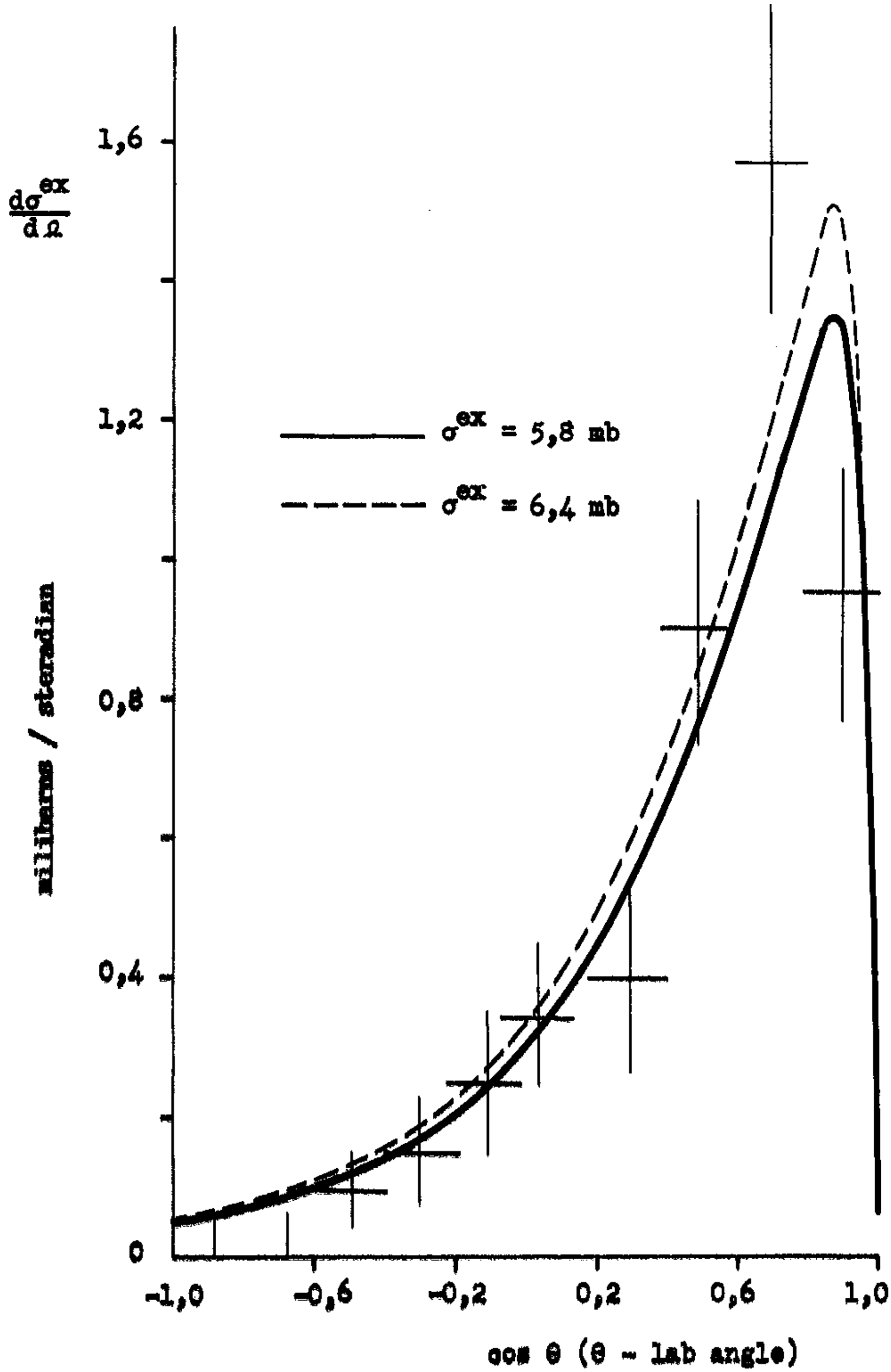
The phase angle between the two quantities s_0 and s_1 is unknown, so that the measurement of the charge-exchange cross-section alone cannot give very definite information on s_0. The value $|1 - s_0/s_1|^2 = 1.1$, for example, can be obtained with any value of $|s_0/s_1|$ from 0.1 to 2.1, the phase angle ranging accordingly from π to 0. Since the inelastic cross-section [Eq. (VI.1)] involves another linear combination, namely $|3s_1 + s_0|$, its determination is important. We suggest that effort be put into the determination of the inelastic scattering total cross-section at 530 MeV/c and higher energies.

* * *

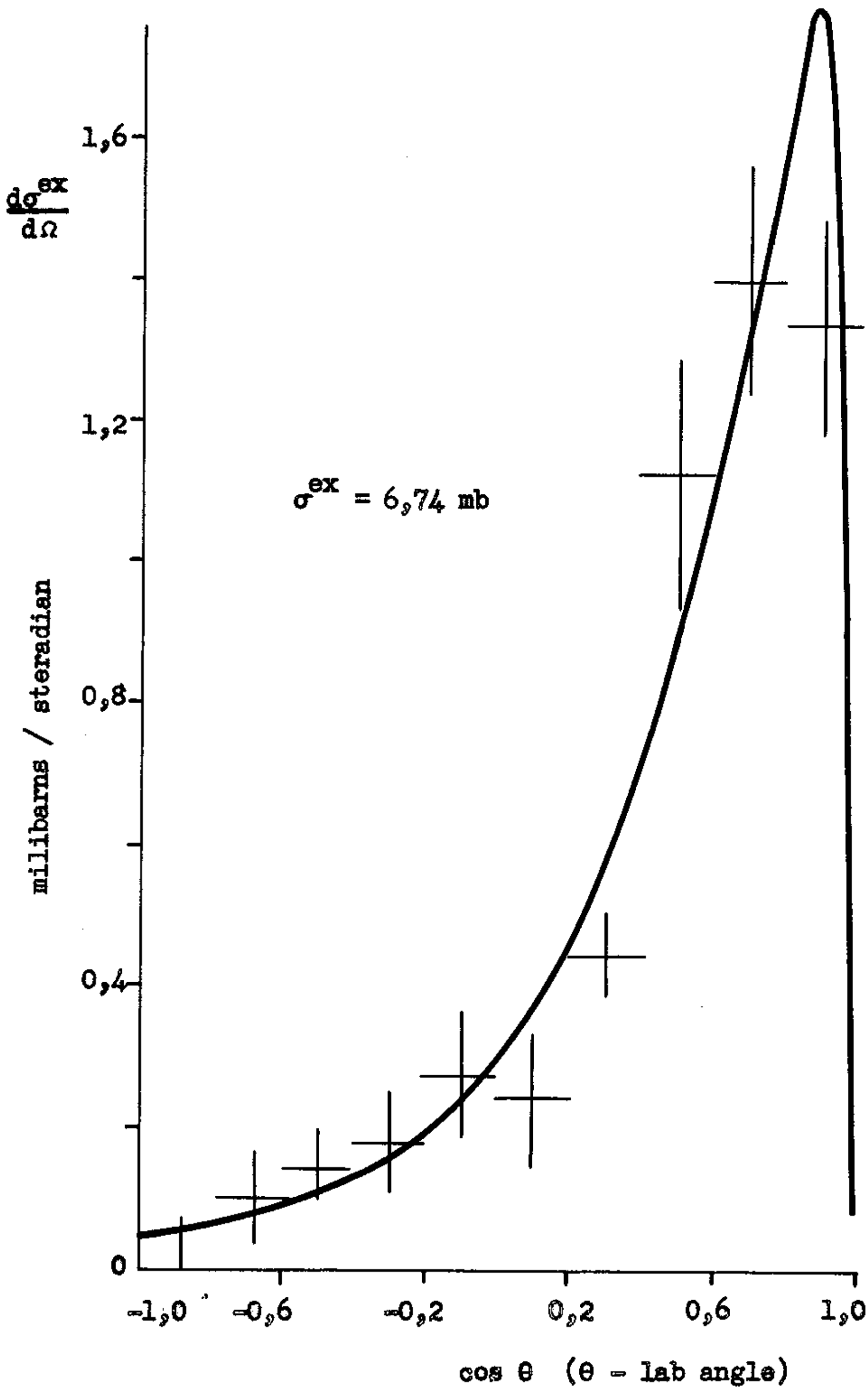


DIFFERENTIAL CHARGE EXCHANGE CROSS SECTION MESON INCIDENT MOMENTUM

q_o = 350 MeV/c.



DIFFERENTIAL CHARGE EXCHANGE CROSS SECTION q_o = 530 MeV/c.



DIFFERENTIAL CHARGE EXCHANGE CROSS SECTION

q_o = 642 MeV/c.

TABLE I

q_o = 200 MeV/c

Θ	F	G	H
0°	0.0302	0.0609	0.1846
10	0.0142	0.1185	0.2928
20	0.0460	0.2400	0.5332
25	0.0669	0.2901	0.6527
30	0.1007	0.3319	0.7474
35	0.1393	0.3607	0.8138
40	0.1682	0.3764	0.8494
45	0.1945	0.3806	0.8561
50	0.2136	0.3744	0.8386
55	0.2254	0.3595	0.8014
60	0.2299	0.3392	0.7501
65	0.2286	0.3153	0.6905
70	0.2228	0.2891	0.6293
80	0.2016	0.2360	0.4992
90	0.1748	0.1880	0.3881
100	0.1485	0.1484	0.2993
110	0.1254	0.1176	0.2312
120	0.1064	0.0946	0.1837
130	0.0917	0.0780	0.1492
140	0.0806	0.0662	0.1253
150	0.0728	0.0581	0.1090
160	0.0675	0.0528	0.0985
170	0.0644	0.0499	0.0928
180	0.0635	0.0489	0.0909

TABLE II

$q_0 = 350$ MeV/c

Θ	cos Θ	F	G	H
0°	1.000	0.0120	0.0922	0.2970
10	0.985	0.0630	0.4792	1.3728
20	0.940	0.3865	0.9840	2.5669
25	0.906	0.6021	1.1837	2.9484
30	0.866	0.7760	1.2765	3.1042
35	0.819	0.9129	1.3123	3.0990
40	0.766	1.0054	1.3007	3.0147
45	0.707	1.0429	1.2638	2.8709
50	0.643	1.0433	1.1921	2.6984
55	0.574	1.0153	1.1085	2.4618
60	0.500	0.9648	1.0147	2.2268
65	0.423	0.9021	0.9181	1.9945
70	0.342	0.8273	0.8209	1.7668
80	0.174	0.6756	0.6415	1.3577
90	0.000	0.5359	0.4906	1.0232
100	- 0.174	0.4204	0.3730	0.7680
110	- 0.342	0.3312	0.2861	0.5829
120	- 0.500	0.2650	0.2237	0.4520
130	- 0.643	0.2172	0.1799	0.3608
140	- 0.766	0.1837	0.1497	0.2988
150	- 0.866	0.1607	0.1294	0.2573
160	- 0.940	0.1458	0.1164	0.2309
170	- 0.985	0.1375	0.1093	0.2164
180	- 1.000	0.1349	0.1070	0.2118
		$\int F d\Omega = 6.7867$	$\int G d\Omega = 7.4226$	$\int H d\Omega = 16.4648$

TABLE III

$q_o = 530$ MeV/c

Θ	cos Θ	F	G	H
0°	1.000	0.0543	0.1473	0.3883
10	0.985	0.3535	1.4641	4.1953
20	0.940	1.6555	2.4560	5.9668
25	0.906	2.1122	2.8988	6.6138
30	0.866	2.3745	2.8833	6.5613
35	0.819	2.4670	2.7894	6.2438
40	0.766	2.4993	2.6692	5.8639
45	0.707	2.3843	2.4536	5.3436
50	0.643	2.2307	2.2276	4.8075
55	0.574	2.0394	1.9927	4.2660
60	0.500	1.8328	1.7576	3.7358
65	0.423	1.6267	1.5355	3.2442
70	0.342	1.4233	1.3272	2.7883
80	0.174	1.0658	0.9727	2.0231
90	0.000	0.7818	0.7007	1.4445
100	- 0.174	0.5725	0.5044	1.0320
110	- 0.342	0.4243	0.3688	0.7500
120	- 0.500	0.3223	0.2767	0.5599
130	- 0.643	0.2532	0.2151	0.4316
140	- 0.766	0.2069	0.1744	0.3506
150	- 0.866	0.1765	0.1478	0.2966
160	- 0.940	0.1573	0.1312	0.2629
170	- 0.985	0.1469	0.1222	0.2431
180	- 1.000	0.1437	0.1193	0.2374
		$\int F d\Omega = 12.7350$	$\int G d\Omega = 13.03185$	$\int H d\Omega = 28.2825$

TABLE IV

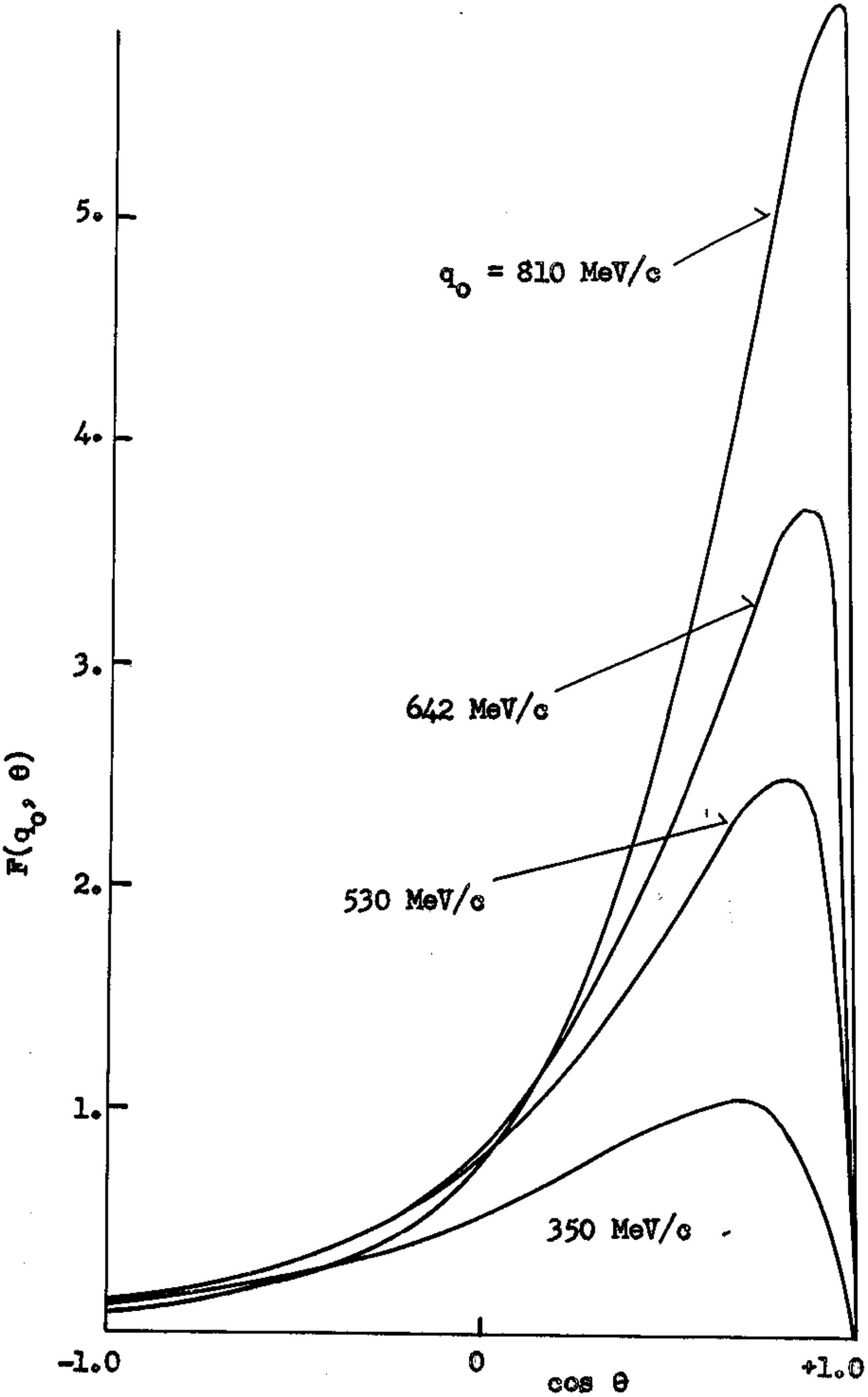
$q_0 = 642$ MeV/c

Θ	cos Θ	F	G	H
0°	1.000	0.0628	0.1468	0.4329
10	0.985	0.5102	1.9884	5.7962
20	0.940	2.6890	3.6942	8.2996
25	0.906	3.4232	4.1463	8.9932
30	0.855	3.6845	4.0771	8.7282
35	0.819	3.6758	3.7929	8.1353
40	0.766	3.5698	3.5783	7.5224
45	0.707	3.2593	3.2065	6.7065
50	0.643	2.9495	2.8436	5.9175
55	0.574	2.6228	2.4850	5.1480
60	0.500	2.2887	2.1415	4.4195
65	0.423	1.9765	1.8289	3.7607
70	0.342	1.6855	1.5450	3.1661
80	0.174	1.2019	1.0828	2.2050
90	0.000	0.8409	0.7468	1.5120
100	- 0.174	0.5888	0.5165	1.0412
110	- 0.342	0.4202	0.3644	0.7291
120	- 0.500	0.3090	0.2654	0.5290
130	- 0.643	0.2365	0.2016	0.4027
140	- 0.766	0.1895	0.1605	0.3180
150	- 0.866	0.1592	0.1342	0.2655
160	- 0.940	0.1404	0.1180	0.2333
170	- 0.985	0.1305	0.1093	0.2159
180	- 1.000	0.1272	0.1066	0.2105
		$\int F d\Omega = 16.1020$	$\int G d\Omega = 16.0863$	$\int H d\Omega = 39.2936$

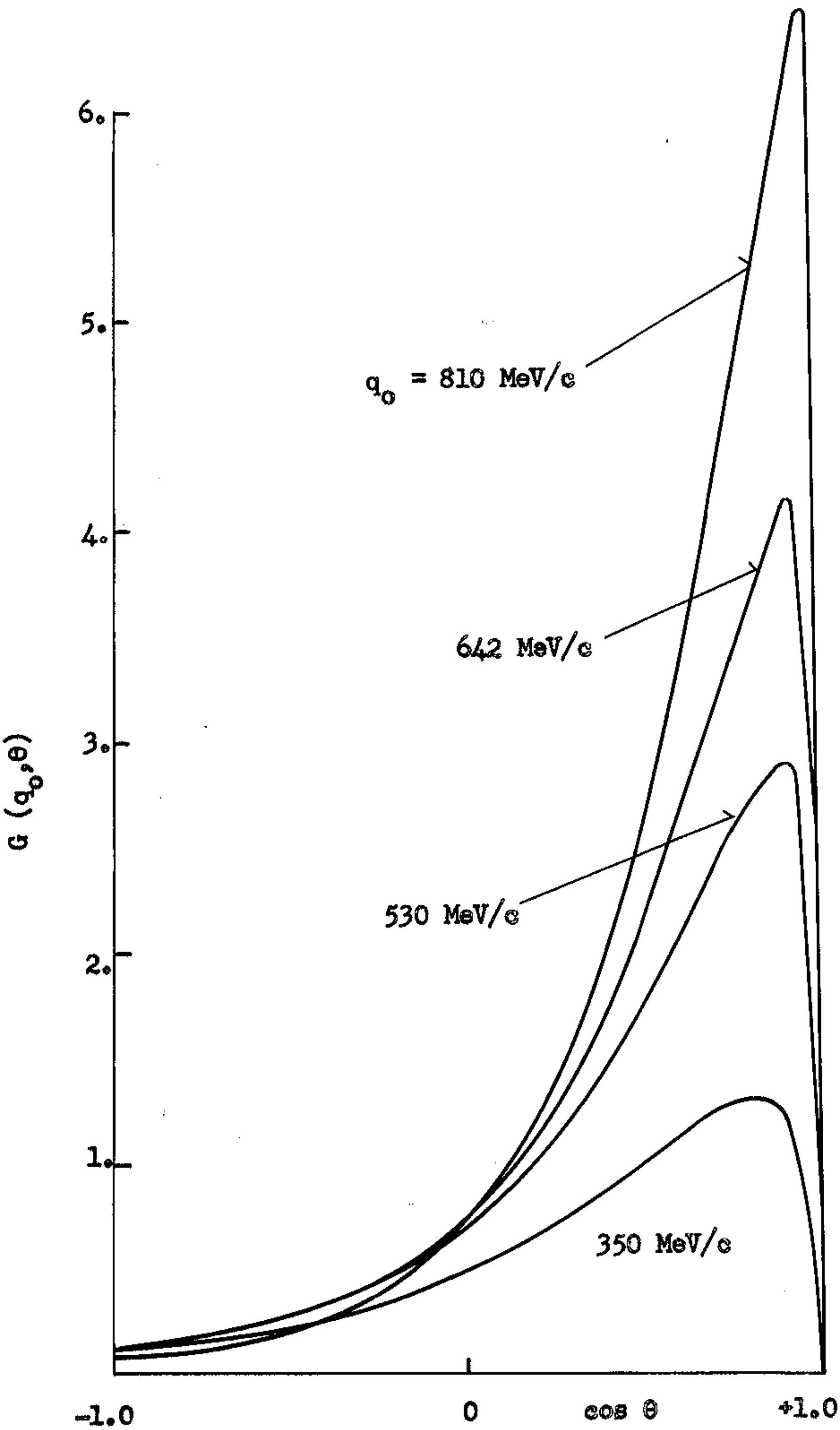
TABLE V

$q_0 = 810$ MeV/c

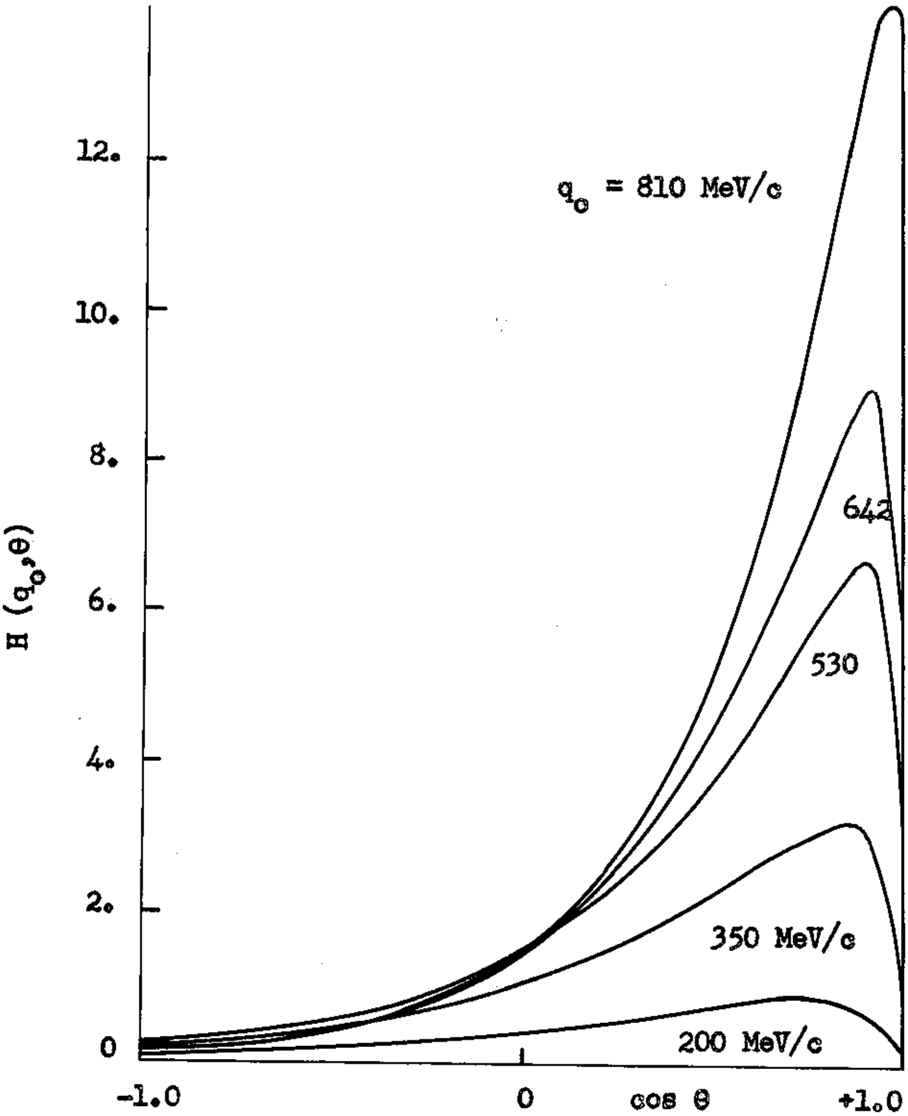
Θ	cos Θ	F	G	H
0°	1.000	0.1080	0.2600	0.5820
10	0.985	2.1441	5.8271	12.5470
20	0.940	5.3424	6.4364	14.0307
25	0.906	5.9642	6.4423	13.7279
30	0.866	5.8452	5.9734	12.8380
35	0.819	5.6159	5.5490	11.6164
40	0.766	5.0646	4.9065	10.1588
45	0.707	4.5344	4.2966	8.8975
50	0.643	3.9419	3.6813	7.5927
55	0.574	3.3637	3.1064	6.3853
60	0.500	2.8244	2.5827	5.2920
65	0.423	2.3450	2.1264	4.3446
70	0.342	1.9233	1.7297	3.5243
80	0.174	1.2649	1.1220	2.2742
90	0.000	0.8171	0.7160	1.4446
100	- 0.174	0.5309	0.4603	0.9216
110	- 0.342	0.3555	0.3054	0.6081
120	- 0.500	0.2476	0.2111	0.4192
130	- 0.643	0.1817	0.1540	0.3049
140	- 0.766	0.1412	0.1190	0.2352
150	- 0.866	0.1160	0.0974	0.1923
160	- 0.940	0.1008	0.0845	0.1666
170	- 0.985	0.0930	0.0776	0.1530
180	- 1.000	0.0905	0.0755	0.1488
		$\int F d\Omega = 20.9790$	$\int G d\Omega = 20.8461$	$\int H d\Omega = 43.3570$



THE FUNCTION $F(q_o, \theta)$



THE FUNCTION G (q_o, θ)



THE FUNCTION $H(q_o,\theta)$

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