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TWO-COMPONENT SPINORS IN THE HAMILTONIAN FORMULATION OF GENERAL RELATIVITY

by

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ABSTRACT OF DISSERTATION

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ABSTRACT

It is known that the Riemann properties of the four-dimensional space can be obtained from a given field of four 2 × 2 spin matrices.[1] With six 2 × 2 Hermitian matrices a canonical formalism is set up, following the methods originated by Dirac.[2]

It is shown that a complete set of dynamical variables can be constructed which are related to each other by a total number of ten constraints at each point of the three-dimensional hypersurface where the state of the system is defined. The Hamiltonian of the theory is a linear combination of the constraints.

The meaning of these constraints is discussed in terms of the generators of the invariance group of the theory.

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I. INTRODUCTION

The two-component spinor theory has recently gained importance in the treatment of purely classical problems, such as gravitational radiation, asymptotic conditions on the curvature tensor, and the initial-value problem for null hypersurfaces. In this article we shall develop the Dirac Hamiltonian formulation of General Relativity in terms of two-component spinors and spin matrices. The principal motivation for this work arises from two different reasons.

First, any quantum field theory of gravitation eventually will involve the interaction of the gravitational with other fields. It is well known that fermion fields can be included in our description only when we use tetrads (or, equivalent, spin matrices) for the description of the gravitational field. Since every known field is capable of interacting with gravitation, the spinor formulation of Dirac's Hamiltonian theory seems to be a useful step toward a future quantum theory of General Relativity.

The other motivation for this treatment arises from the possibility that a spinor formulation may be more suitable for the discussion of the problem of fluctuations in the q-number metric field than the conventional tensor formulation. In a spinor formulation the signature of the metric field is fixed; the metric field is given entirely in terms of four 2 × 2 Hermitian spin matrices. If we allow fluctuations in the q-number Hermitian matrices, these fluctuations will not change the signature of the metric, either. Hence the range of integration of c-number field variables, as called for in Feynman's method of integrating over histories, for instance, is signifi cantly different in the spinor and in the conventional formulations of general relativity. This difference may be expected to lead to inequivalent quantum theories of gravitation, which eventually will deserve separate exploration.

Our formulation is related to Dirac's tetrad method; [3] but we are not restricting the orientation of the tetrad axes which are intrinsically associated with the spinor field at each point.

Since the techniques employed in this article involve two-component spinors and its properties, we shall first review briefly the basic properties of the standard spinor theory of van der Waerden, in Section II. No attempt has been made to render this review complete as this material can be found in the literature. The notation used will be the following: Greek letters designate tensor indices from 0 to 3. Small Latin letters designate tensor indices running from 1 to 3, whereas capital Latin letters refer to spinor indices; the latter are of two different types and will be denoted by dotted and undotted capital Latin letters, both running from 1 to 2. Small Greek letters inside a parenthesis refer to local (or tetrad) indices; they are related to spin degrees of freedom as explained in Section II. They range from 0 to 3; no system of triads is used in this paper.

Because of its simplicity the index-free notation (or matrix notation) for spinors will be used as far as possible. However, for the definition of the Poisson bracket between spin dynamical variables we shall need explicitly the index notation.

The metric tensor $g_{\mu\nu}$(and $g^{\mu\nu}$) has the signature -2; it can be brought to the diagonal from (1, -1, -1, -1) at each individual world point; we shall denote this diagonal form by $\overset{\circ}{g}_{\mu\nu}$. (In special relativity this can be done globally). The remaining sections of this paper can be outlined as follows: Section III is a brief account of Dirac's Hamiltonian formulation of general relativity in the standard tensor form; all relationships required for subsequent use have been furnished in this section. The fourth section treats the problem of the determination of the dynamical variables in the framework of the spinor theory of general relativity. In Section V we discuss the emergence of the new six spin constraints, as well as the meaning of these constraints in terms of infinitesimal canonical transformations, and the associated generators. Finally, Section VI deals with an application of the formalism, namely the determination of the classical commutator for infinitesimally separated times, up to the first order.

We use units such that $\frac{16\pi k}{C^4} = 1$, $C = 1$, where k is the Newtonian gravitational constant.

II. FOUR-DIMENSIONAL FORMULATION OF GENERAL RELATIVITY IN TERMS OF TWO-COMPONENT SPINORS

At each point in the four-dimensional space-time we can define a two-dimensional linear vector space. The elements of this space are the ordered complex functions of the coordinates,[4]

$$u(x) = \begin{pmatrix} u^1(x) \\ u^2(x) \end{pmatrix}$$

We shall call this linear vector space in short spin-space and its elements spinor fields. Just as in tensor calculus these spinor fields can be described in terms of different base systems by forming linear combinations of their components. The coefficients of this transformation matrix are arbitrary complex functions of the coordinates, but they are restricted by the requirement that the determinant of the matrix be equal to 1. With respect to this transformation group the spinor fields can be divided into four types,

$$u'^A = M^A{}_K\, u^K, \qquad u^K = M^{-1K}{}_A\, u'^A, \tag{1.1}$$

$$u'^{\dot A} = M^{\dot A}{}_{\mathring{K}}\, u^{\mathring{K}}, \qquad u^{\mathring{K}} = M^{-1\mathring{K}}{}_{\dot A}\, u'^{\dot A}, \tag{1.2}$$

$$u_A = u'_K\, M^K{}_A, \qquad u'_K = u_A\, M^{-1A}{}_K, \tag{2.1}$$

$$u_A = u'_K\, M^K{}_A, \qquad u'_K = u_A\, M^{-1A}{}_K, \tag{2.2}$$

where, $M^A{}_k$ is the coordinate-independent transformation matrix

$$M^A{}_K = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}, \qquad M^{\mathring{A}}{}_{\mathring{K}} = \begin{pmatrix} \bar\alpha & \bar\beta \\ \bar\gamma & \bar\delta \end{pmatrix}, \tag{3.1}$$

which satisfies the condition

$$\alpha\delta - \beta\gamma = 1 . \tag{3.2}$$

$\bar\alpha$ means the complex conjugate of α; and a dot on all indices is the same as taking the complex conjugate. The relation (3.2) is the unimodular condition. The relations (1), (2) can be written in matrix notation as:

$$u' = M u , \qquad u = M^{-1} u' ,$$
$$\bar{u}' = \bar{M} \bar{u} , \qquad \bar{u} = \bar{M}^{-1} \bar{u}' ,$$
$$v = v' M , \qquad v' = v M^{-1} ,$$
$$\bar{v} = \bar{v}' \bar{M} , \qquad \bar{v}' = \bar{v} \bar{M}^{-1} ,$$

where, $v = (u_A)$ can be interpreted as a row vector. Spinor indices are raised and lowered by means of the spin matrices ϵ^{AB}, ϵ_{AB} and their complex conjugate,

$$\epsilon^{AB} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \epsilon^{\dot{A}\dot{B}} , \tag{4}$$

$$\epsilon^{AB} \epsilon_{BC} = - \delta^A_C , \tag{5}$$

$$u^A = {}^{AB} u_B , \tag{6.1}$$

$$u_A = u^B \epsilon_{BA} = - \epsilon_{AB} u^B . \tag{6.2}$$

From Eqs. (1), (2), and (6) it follows that,

$$\epsilon'^{AL} = M^A{}_k M^L{}_B \epsilon^{KB} ,$$

or,

$$\epsilon' = M \epsilon M^T . \tag{7}$$

Using Eq. (4) together with the unimodular condition (3.2) we can prove that the ϵ matrices have the same expression in all base systems

$$\epsilon'^{AB} = \epsilon^{AB} .$$

Since $\epsilon_{AB} = \epsilon^{AB}$, the same result holds for the components ϵ_{AB}.

Higher-rank spinors transform in the same way as the dydadic product of spin-vectors, for instance $x^{AB}{}_{\dot{C}}$ transforms like $u^A w^B v_{\dot{C}}$ under the action of the unimodular transformation group.

A hermitian second-rank spinor is defined by,

$$w_{\dot{A}C} = w_{C\dot{A}} ,$$

$$w = (w_{\dot{A}C}), \; w^T = (w_{\dot{C}A}), \; \bar{w} = (w_{A\dot{C}}) ,$$

$$w^\dagger = (w_{C\dot{A}}) = \bar{w}^T = w .$$

Hermicity is preserved under unimodular spin transformations. Any Hermitian spinor of rank two can be described by a set of four real numbers (or functions). Associated with each point of the four-dimensional coordinate space, we shall define a set of four linearly independent Hermitian spin matrices $\sigma_\mu{}^{K\dot{M}}$. Under unimodular transformations they transform as,

$$\sigma'^{K\dot{M}} = M^K{}_S \; M^{\dot{M}}{}_{\dot{P}} \; \sigma_\mu{}^{S\dot{P}} , \qquad (8)$$

or,

$$\sigma'_\mu = M(x) \; \sigma_\mu(x) \; M^\dagger(x) . \qquad (9)$$

With respect to the index μ they transform as a four-vector under all coordinate transformations,

$$\sigma'_\rho(x') = \frac{\partial x^\mu}{\partial x'^\rho} \sigma_\mu(x) ,$$

$$x'^\alpha = x'^\alpha(x) .$$

The spin matrices $\sigma_\mu^{\ K\dot{M}}$ allow us to associate with any tensor $T^{\mu\nu\ldots}$ a spinor $T^{K\dot{M},R\dot{P}\ldots}$, such that to each tensor index corresponds a pair of spinor indices, one dotted, the other undotted,

$$T^{K\dot{M},R\dot{P}\ldots} = \sigma_\mu^{\ K\dot{M}}\, \sigma_\nu^{\ R\dot{P}} \ldots T^{\mu\nu} \ldots ,$$

and

$$T^{\mu\nu\ldots} = \left(\tfrac{1}{2}\right)^n \sigma^\mu_{K\dot{M}}\, \sigma^\nu_{R\dot{P}} \ldots T^{K\dot{M},R\dot{P}\ldots} , \; *$$

From the condition of Hermicity of the σ_μ, it follows that:

$$T^{K\dot{M},R\dot{P}} = T^{\dot{M}K,\dot{P}R} .$$

This relation has the effect of maintaining the same total number of components in both sides of the above equation.

From these relations we see that the reality of tensors is translated into Hermicity of the correspondent spinors.

We define the set of matrices τ_μ according to,

$$\begin{aligned} \tau_\mu &= \epsilon\, \bar{\sigma}_\mu\, \epsilon , \\ \tau_\mu^\dagger &= \tau_\mu . \end{aligned} \qquad (10)$$

Under the local transformation group they transform as

$$\tau'_\mu(x) = \overset{\dagger}{M}{}^{-1}(x)\, \tau_\mu(x)\, M^{-1}(x) , \qquad (11)$$

$$x^\mu = \text{constant}.$$

Eq. (11) is a consequence of the definition (10) along with Eqs. (7) and (9).

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* The contravariant σ matrices will be defined later on. The number n denotes the rank of the tensor, for instance, for a vector $n = 1$.

From (9) and (11) we conclude that the matrix product of a σ and a τ matrix has the following transformation law at the point x^μ,

$$\sigma'_\mu \tau'_\nu = M \sigma_\mu \tau_\nu M^{-1} . \qquad (12)$$

Now, we introduce the set of Hermitian spin matrices $\overset{o}{\sigma}{}^\mu$, which are the Pauli spin matrices and the two-by-two identity matrix,

$$\overset{o}{\sigma}{}^{1,K\dot{M}} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \overset{o}{\sigma}{}^{2,K\dot{M}} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \overset{o}{\sigma}{}^{3,K\dot{M}} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

$$\overset{o}{\sigma}{}^{0,K\dot{M}} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} . \qquad (13)$$

The matrices $\overset{o}{\sigma}{}^\mu$ allow us to write the elements of σ_μ as,

$$\sigma_\mu = h_{\mu(\alpha)} \overset{o}{\sigma}{}^{\alpha} . \qquad (14)$$

From the Hermiticity of σ_μ and $\overset{o}{\sigma}{}^\alpha$ it follows that $h_{\mu(\alpha)}$ are a set of 16 real functions; the role of these $h_{\mu(\alpha)}$ will be clarified in what will follow. Similarly we can write,

$$\tau_\mu = h_{\mu(\alpha)} \overset{o}{\tau}{}^{\alpha} , \qquad (15)$$

$$\overset{o}{\tau}{}^{\alpha} = \epsilon \, \bar{\overset{o}{\sigma}}{}^{\alpha} \epsilon . \qquad (16)$$

By using the $\overset{o}{\sigma}{}^\alpha$ as given by (13), we can write the components of the $\overset{o}{\tau}{}^\alpha$ matrices from (16),

$$\overset{o}{\tau}{}^{i} = \overset{o}{\sigma}{}^{i}, \quad \overset{o}{\tau}{}^{0} = - \overset{o}{\sigma}{}^{0} . \qquad (17)$$

A strightforward calculation shows that,

$$\overset{o}{\sigma}{}^{\alpha} \overset{o}{\tau}{}^{\beta} + \overset{o}{\sigma}{}^{\beta} \overset{o}{\tau}{}^{\alpha} = -2 \overset{o}{g}{}^{\alpha\beta} \cdot 1 . ^{*} \qquad (18)$$

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* By taking Hermitian conjugate on both sides of (18) we conclude that an equation identical to (18) also holds with the $\overset{o}{\tau}$ matrices written at the left. A similar result holds for Eq. (20).

Where $\overset{\circ}{g}{}^{\alpha\beta}$ is defined in the Introduction, and 1 is the two-by-two identity matrix.

If we form the symmetrized product of a σ matrix and a τ matrix, and use Eqs. (14), (15), and (18), we obtain:

$$\sigma_\mu \tau_\nu + \sigma_\nu \tau_\mu = -2\, h_{\mu(\alpha)}\, h_{\nu(\beta)}\, \overset{\circ}{g}{}^{\alpha\beta} \cdot 1 \;. \tag{19}$$

This symmetrized $\sigma\tau$ product has all the properties of a metric tensor and it will be interpreted in that way. [1]

$$\sigma_\mu \tau_\nu + \sigma_\nu \tau_\mu = -2\, g_{\mu\nu} \cdot 1 \;, \tag{20}$$

$$g_{\mu\nu} = h_{\mu(\alpha)}\, h_{\nu(\beta)}\, \overset{\circ}{g}{}^{\alpha\beta} \;. \tag{21}$$

Therefore, the $h_{\mu(\alpha)}$ introduced in Eq. (14) are just the set of four tetrad axes at the point x^μ.

From Eqs. (12) and (20) it follows that the value of the metric field is the same for all possible choices of spin frame at any point x^μ.

The spin transformation (9) can be generated by giving to the tetrad axes a suitable four-rotation (or Lorentz transformation), as it is clear from the relation (14).

The important result to be noted for subsequent exploitation is that a given field of four Hermitian spin matrices σ_μ defines uniquely the metric field by means of (20).

The local (or tetrad) indices are raised and lowered by means of the $\overset{\circ}{g}{}^{\mu\rho}$ and $\overset{\circ}{g}_{\mu\rho}$; as an example, given the covariant local components $A_{(\mu)}$ we can form the contravariant components $A^{(\mu)}$ according to,

$$A^{(\mu)} = \overset{\circ}{g}{}^{\mu\rho} A_{(\rho)} ,$$

and reciprocally,

$$A_{(\mu)} = \overset{\circ}{g}_{\mu\rho} A^{(\rho)} .$$

In this paper tetrads will be used only as an intermediate aid in some of the calculations; the fundamental variables taking over the role of the metric tensor will be the spinor fields σ_μ. However, in order to clarify some of the relations involving tetrads, some further results of the tetrad calculus will be given briefly. From the relation,

$$h_{\mu(\lambda)} h^{\mu(\rho)} = \delta^{(\rho)}_{(\lambda)} , \qquad (22)$$

it follows that $\overset{\circ}{\sigma}{}^\mu$ can be interpreted as the tetrad components of σ_μ; this allows us to raise or lower the indices in these "local matrices" similarly as we did for an arbitrary tetrad vector,

$$\overset{\circ}{\sigma}{}^\alpha = h^{\mu(\alpha)} \sigma_\mu{}^* ,$$

$$\overset{\circ}{\sigma}_\alpha = \overset{\circ}{g}_{\alpha\rho} \overset{\circ}{\sigma}{}^\rho .$$

The contravariant tetrad vector (contravariant with respect to the index μ) $h^{\mu(\alpha)}$ has not yet been defined; the definition of this vector is equivalent to the definition of the components of the contravariant metric tensor, since:

$$g^{\mu\nu} = h^{\mu(\alpha)} h^{\nu(\beta)} \overset{\circ}{g}_{\alpha\beta} = h^{\mu(0)} h^{\nu(0)} - (h^{\mu(1)} h^{\nu(1)} + h^{\mu(2)} h^{\nu(2)} + h^{\mu(3)} h^{\nu(3)}) . \qquad (23)$$

If we form the relations, [1]

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* The tetrad components of any given tensor are defined by the relation:

$$A^{(\alpha)(\beta)\ldots} = h^{\mu(\alpha)} h^{\nu(\beta)} \ldots A_{\mu\nu\ldots} .$$

$$\Delta = \varepsilon^{\mu\alpha\beta\lambda} h_{\mu(0)} h_{\alpha(1)} h_{\beta(2)} h_{\lambda(3)} , \tag{24}$$

$$h^{\mu(0)} = \Delta^{-1} \varepsilon^{\mu\alpha\beta\lambda} h_{\alpha(1)} h_{\beta(2)} h_{\lambda(3)} , \tag{25.1}$$

$$h^{\mu(1)} = -\Delta^{-1} \varepsilon^{\mu\alpha\beta\lambda} h_{\alpha(0)} h_{\beta(2)} h_{\lambda(3)} , \tag{25.2}$$

$$h^{\mu(2)} = \Delta^{-1} \varepsilon^{\mu\alpha\beta\lambda} h_{\alpha(0)} h_{\beta(1)} h_{\lambda(3)} , \tag{25.3}$$

$$h^{\mu(3)} = -\Delta^{-1} \varepsilon^{\mu\alpha\beta\lambda} h_{\alpha(0)} h_{\beta(1)} h_{\lambda(3)} , \tag{25.4}$$

where $\varepsilon^{\mu\alpha\beta\lambda}$ is the permutation symbol, defined to be +1 for the order 0123. We can prove that the $h^{\mu(\alpha)}$ of (25) satisfy the relation (22). Equivalently, from Eqs. (21), (23), (24), and (25) we can show that the above $g^{\mu\nu}$ satisfies,

$$g_{\mu\lambda}\, g^{\lambda\rho} = \delta_{\mu}^{\rho} .$$

Therefore, the relations (23) and (25) define the contravariant metric tensor and the contravariant tetrad vector $h^{\mu(\alpha)}$. We note that Eq. (24) gives the determinant of the matrix $(h_{\mu(\alpha)})$, where the first index (the coordinate index) designates the rows and the tetrad index the columns.

$$\Delta = |h_{\mu(\alpha)}| .$$

Now, we consider the following expression:

$$D = \varepsilon^{\alpha\beta\gamma\delta} \sigma_{\alpha} \tau_{\beta} \sigma_{\gamma} \tau_{\delta} . \tag{26}$$

Using the Eq. (14) and the definition of determinant, we can write the above equation as,

$$D = \varepsilon^{(\mu)(\lambda)(\varepsilon)(\nu)} \tilde{\Delta}\, \overset{\circ}{\sigma}_{\mu} \overset{\circ}{\tau}_{\lambda} \overset{\circ}{\sigma}_{\varepsilon} \overset{\circ}{\tau}_{\nu} , \quad \tilde{\Delta} = |h_{\mu}^{(\alpha)}| ;$$

a direct calculation shows that,

$$\varepsilon^{(\mu)(\lambda)(\varepsilon)(\nu)} \overset{\circ}{\sigma}_{\mu} \overset{\circ}{\tau}_{\lambda} \overset{\circ}{\sigma}_{\varepsilon} \overset{\circ}{\tau}_{\nu} = 4!\ (-i) \cdot 1 \ , \ ^{*}$$

then,

$$D = 4!(-i)\tilde{\Delta} \cdot 1 \ .$$

From (21) we have: **

$$\tilde{\Delta} = \sqrt{-{}^{4}g} \ , \quad {}^{4}g = |g_{\mu\nu}| \ .$$

Which gives,

$$D = 4!\ (-i)\sqrt{-{}^{4}g} \cdot 1 \ . \qquad (27)$$

We define the quantity

$$V^{\mu} = \varepsilon^{\mu\nu\rho\sigma}\, \sigma_{\nu}\, \tau_{\rho}\, \sigma_{\sigma} = \varepsilon^{\mu\nu\rho\sigma}\, h_{\nu(\alpha)}\, h_{\rho(\beta)}\, h_{\sigma(\lambda)}\, \overset{\circ}{\sigma}{}^{\alpha} \overset{\circ}{\tau}{}^{\beta} \overset{\circ}{\sigma}{}^{\lambda} \ , \qquad (28)$$

as a consequence of the definition of determinant, we have:

$$\varepsilon^{\mu\nu\rho\sigma}\, h_{\nu}^{(\alpha)}\, h_{\rho}^{(\beta)}\, h_{\sigma}^{(\lambda)} = h^{\mu}_{(\tau)}\ \varepsilon^{(\tau)(\alpha)(\beta)(\lambda)}\, \tilde{\Delta} \ ,$$

using this relation in the Eq. (28), we obtain:

$$V^{\mu} = h^{\mu}_{(\tau)}\, \varepsilon^{(\tau)(\alpha)(\beta)(\lambda)}\ \tilde{\Delta}\, \overset{\circ}{\sigma}_{\alpha} \overset{\circ}{\tau}_{\beta} \overset{\circ}{\sigma}_{\lambda} \ ,$$

since,

$$\varepsilon^{(\tau)(\alpha)(\beta)(\lambda)} \overset{\circ}{\sigma}_{\alpha} \overset{\circ}{\tau}_{\beta}\ \overset{\circ}{\sigma}_{\lambda} = -3!\ i\ \overset{\circ}{\sigma}{}^{\tau} ,$$

we can write,

$$\sigma^{\mu} = h^{\mu}\ \ \overset{\circ}{\sigma}{}^{\tau} = i(3!\tilde{\Delta})^{-1}\, V^{\mu} = \overset{\dagger}{\sigma}{}^{\mu} \ . \qquad (29)$$

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* $\varepsilon^{(0)(1)(2)(3)} = +1$.

** $g_{\mu\nu} = h_{\mu}^{(\alpha)}\ h_{\nu}^{(\beta)}\ \overset{\circ}{g}_{\alpha\beta}$.

This relation defines the contravariant components of the σ matrices. For the τ matrices we have,

$$\tau^{\mu} = -i\,(3!\,\tilde{\Delta})^{-1}\, R^{\mu} = \overset{+}{\tau}{}^{\mu}\,, \qquad (30)$$

$$R^{\mu} = \epsilon\,\bar{V}^{\mu}\epsilon = \varepsilon^{\mu\nu\rho\sigma}\,\tau_{\nu}\,\sigma_{\rho}\,\tau_{\sigma}\,. \qquad (31)$$

Due to the fact that the spin transformation matrix M is an arbitrary function of the coordinates, it follows that the derivatives of a spinor do not transform as a spinor field under spin transformations; the quantities which transform as a spinor field are,

$$u^{K}_{;\rho} = u^{K}_{,\rho} + \Gamma^{K}_{\rho\;V}\, u^{V}\,,$$

that is,

$$u'^{K}_{;\rho} = M^{K}_{\;\;L}\, u^{L}_{\;;\rho}\,,$$

this means that the Γ_{ρ} transforms as

$$\Gamma'^{K}_{\rho\;L} = \left(M^{K}_{\;\;V}\,\Gamma^{V}_{\rho\;P} - M^{K}_{,\rho\;P}\right) M^{-1P}_{\;\;\;\;L}$$

under spin transformations. With respect to coordinate transformations the derivatives of a spinor field transform as a vector, the same being valid for the Γ_{ρ}:

$$u'_{;\rho} = \frac{\partial x^{\alpha}}{\partial x'^{\rho}}\, u_{;\alpha}\,,$$

$$x'^{\rho} = x'^{\rho}(x^{\alpha})\,.$$

The $u_{;\rho}$ defines the covariant derivative of the spinor u, and the $\Gamma^{K}_{\rho\;L}$ are the spin-affine connection.

If we require, [1]

$$(\sigma^{\mu}_{\dot{K}R})_{;\rho} = (\sigma^{\mu}_{\dot{K}R})_{,\rho} + \left\{{\mu \atop \alpha\rho}\right\} \sigma^{\alpha}_{\dot{K}R} - \sigma^{\mu}_{\dot{K}L} \Gamma_{\rho}{}^{L}{}_{R} - \sigma^{\mu}_{\dot{L}R} \bar{\Gamma}_{\rho}{}^{\dot{L}}{}_{\dot{K}} = 0 ,$$

$$\epsilon^{KR}_{;\beta} = \Gamma_{\beta}{}^{K}{}_{S}\, \epsilon^{SR} + \Gamma_{\beta}{}^{R}{}_{S}\, \epsilon^{KS} = 0 .$$

we find the following expression for Γ_{ρ},

$$\Gamma_{\rho}{}^{K}{}_{L} = -\frac{1}{4} \sigma_{\mu L\dot{R}} \left\{ \sigma^{\mu\dot{R}K}_{,\rho} + \left\{{\mu \atop \alpha\rho}\right\} \sigma^{\alpha\dot{R}K} \right\}.$$

In the above expressions, $\left\{{\mu \atop \alpha\rho}\right\}$ are the Christoffel symbols, the metric being defined by Eq. (20).

The spin curvature is defined by the relation,

$$u^{A}_{;\rho\sigma} - u^{A}_{;\sigma\rho} = P^{A}_{\rho\sigma\ B}\, u^{B} ,$$

$$P^{A}_{\rho\sigma B} = \Gamma^{A}_{\rho,\sigma\ B} - \Gamma^{\ A}_{\sigma,\rho\ B} - \Gamma_{\rho}^{A}{}_{C}\, \Gamma^{C}_{\sigma\ B} + \Gamma_{\sigma}^{A}{}_{C}\, \Gamma_{\rho}^{C}{}_{B} .$$

These equations in matrix notation read as,

$$u_{;\rho\sigma} - u_{;\sigma\rho} = P_{\rho\sigma}\, u ,$$

$$P_{\rho\sigma} = \Gamma_{\rho,\sigma} - \Gamma_{\sigma,\rho} - \Gamma_{\rho} \Gamma_{\sigma} + \Gamma_{\sigma} \Gamma_{\rho} .$$

We shall use this notation in the next relations; an equation relating the components of the Riemann curvature tensor with the spin curvature, is obtained from the relations,

$$\sigma_{\mu;\alpha\beta} - \sigma_{\mu;\beta\alpha} = \sigma^{\lambda} R_{\lambda\mu\alpha\beta} - \sigma_{\mu} P_{\alpha\beta} - P^{+}_{\alpha\beta} \sigma_{\mu} ,$$

$$\tau_{\mu;\alpha\beta} - \tau_{\mu;\beta\alpha} = \tau^{\lambda} R_{\lambda\mu\alpha\beta} + P_{\alpha\beta} \tau_{\mu} + \tau_{\mu} P^{+}_{\alpha\beta} .$$

Since we have taken the covariant derivatives of the σ, τ matrices equal to zero, we have:

$$\sigma^{\lambda} R_{\lambda\mu\alpha\beta} - \sigma_{\mu} P_{\alpha\beta} - P^{+}_{\alpha\beta} \sigma_{\mu} = 0 ,$$
$$\tau^{\lambda} R_{\lambda\mu\alpha\beta} + P_{\alpha\beta} \tau_{\mu} + \tau_{\mu} P^{+}_{\alpha\beta} = 0 .$$

These relations allow us to express the two curvatures in terms of each other,

$$R_{\lambda\mu\alpha\beta} = \frac{1}{4} \mathrm{Tr}\left[P^{+}_{\lambda\mu}(\sigma_{\alpha}\tau_{\beta} - \sigma_{\beta}\tau_{\alpha}) + (\tau_{\beta}\sigma_{\alpha} - \tau_{\alpha}\sigma_{\beta})P_{\lambda\mu}\right] ,$$
$$P_{\alpha\beta} = \frac{1}{4} \tau^{\lambda}\sigma^{\mu} R_{\alpha\beta\lambda\mu} .$$

Due to the fact that $R_{\alpha\beta\lambda\mu}$ is antisymmetric in (λ, μ), we obtain:

$$\mathrm{Tr}\; P_{\alpha\beta} = 0 .$$

We note that Γ_{ρ} is not a Hermitian spin matrix (which implies that $P_{\rho\sigma}$ is not Hermitian) a conclusion easily acceptable considering that both indices of Γ_{ρ}, and $P_{\rho\sigma}$, are of the same type (undotted indices). From the above expressions we can derive the components of the Ricci tensor and the scalar curvature.[1]

III. DIRAC CANONICAL FORMULATION OF THE GRAVITATIONAL FIELD

The physical state of a given system in the Hamiltonian theory is described in terms of a complete * system of dynamical variables and their conjugate momentum densities at a given instant of time. In a theory that from its natural starting point is fully covariant,

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* By "complete" we mean a set of canonical variables, such as g_{mn}, p^{mn}, which describe uniquely the state of the system. For instance, all Weyl scalars depend only on the g_{mn}, p^{mn}.[5]

this amounts to the necessity of giving the initial data on a three-dimensional hypersurface of the four-dimensional Riemann space.

Dirac [2] has shown how we can obtain a formulation of general relativity in terms of Cauchy data on a given three-dimensional hypersurface.

The continuation in time out of the hypersurface, which should be obtained by integration of the canonical field equations, is arbitrary insofar as the Hamiltonian itself contains arbitrary coefficients. Some of the components of the metric, which are not to be found among the Cauchy data, are present as coefficients in the Hamiltonian; their free choice reflects the arbitrariness in the choice of coordinates outside the hypersurface.

The set of twelve canonical field variables chosen as the Cauchy data are not all independent of each other; they are related by a set of four constraints (relations involving the dynamical variables but not their time derivatives). The values of the canonical field variables at any given time must therefore be chosen consistent with these constraints (the "representative point" should lie on the constraint hypersurface of the phase space of the theory). Because Dirac's constraints are "first class" (their Poisson brackets vanish on the constraint hypersurface), and because the Hamiltonian is a linear combination of the constraints, it follows that the contraints have vanishing Poisson brackets with the Hamiltonian. Therefore, if the set of canonical variables chosen at one time satisfies these constraints, the variables will continue to satisfy the constraints at all later times. Let us denote by C_a the set of contraints (the range of index "a" is not important for the present discussion). The definition of first-class is that

$$\left[C_a, C'_b\right] = f^{c}_{[ab]}\ C_c$$

on the constraints hypersurface, $C_c = 0$. The Hamiltonian is given by the relation

$$H = \int \alpha^{a}\ C_a\ d_3\, x$$

Then:

$$\frac{dC_b}{dt} = \left[C_b,\ H\right] = \int d_3x'\ \alpha^{a}(x')f^{c}_{[ba]}(x,x')\ C_c(x)\ ,$$

which vanishes on the constraint hypersurface. Hence the trajectories in accordance with Hamilton's equations of motion do not lead off the constraint hypersurface.

After these initial comments, we shall review the Dirac's Hamiltonian formulation of the gravitational field.

First of all we shall introduce the concept of D-invariance, which we shall refer to frequently in what follows.

We shall call D-invariant any function (or functional) defined on a given three-dimensional space-like hypersurface of the four-dimensional Riemann space whose infinitesimal transformation law under infinitesimal coordinate transformations does not involve the partial derivatives of the δx^{α} normal to the hypersurface, i.e. the $\delta x^{\alpha}{}_{,\mu}\ell^{\mu}$, where ℓ^{μ} is the unit normal to the hypersurface. *

* Geometrically, they are quantities which are independent of the continuation of the coordinate system immediately outside the points of the hypersurface.

Higher-order normal derivatives are also excluded from the transformation law of D-invariants. The unit normal to the hypersurface is defined by

$$\ell^{\mu} = \frac{g^{o\mu}}{\sqrt{g^{oo}}} , \quad \ell_{\mu} = g_{\mu\alpha}\, \ell^{\alpha} = \frac{\delta^{o}_{\mu}}{\sqrt{g^{oo}}} , \tag{32}$$

$$\ell_{\mu}\ell^{\mu} \equiv 1 , \tag{33}$$

Over an infinitesimal volume containing a given point x^{μ} the ℓ^{μ} take the Galilean value,

$$\ell^{\mu} = \delta^{\mu}_{o} , \qquad \ell_{\mu} = \delta^{o}_{\mu} .$$

If we consider the δx^{μ} as an infinitesimal displacement, we can rewrite the normal derivatives to the hyperfusrface as usual time derivatives.

The covariant spatial components of any four-tensor form a D-invariant, since:

$$\bar{\delta}\, T_{ij} = - \xi^{\beta}_{,j}\, T_{i\beta} - \xi^{\alpha}_{,i}\, T_{\alpha j} - T_{ij,\alpha}\, \xi^{\alpha} ,$$

$$\bar{\delta}\, T_{ij}(x) = T'_{ij}(x) - T_{ij}(x) ,$$

where

$$x'^{\mu} = x^{\mu} + \xi^{\mu}(x) ,$$

the same obviously holds for tensors of higher rank, $T_{\alpha\beta\ldots}$. However, the contravariant spatial components of any four-tensor do not form a D-invariant; for instance,

$$\bar{\delta}\, T^{ij} = \xi^{j}_{,\beta}\, T^{i\beta} + \xi^{i}_{,\alpha}\, T^{\alpha j} - T^{ij}_{,\alpha}\, \xi^{\alpha} ,$$

contains $\xi^{i}_{,o}$.

If we define the quantities,

$$Q^{\rho}{}_{\sigma} = \delta^{\rho}_{\sigma} - \ell^{\rho}\,\ell_{\sigma}\,,$$
$$Q^{\rho}{}_{\sigma}\, Q^{\sigma}{}_{\lambda} = Q^{\rho}{}_{\lambda}\,,$$

which serve as projectors, we can construct from the components $T^{\alpha\beta\cdots\lambda}$ of an arbitrary four-tensor the D-invariant,

$$\tilde{T}^{\rho\mu\cdots\nu} = Q^{\rho}{}_{\alpha}\; Q^{\mu}{}_{\beta}\cdots Q^{\nu}{}_{\lambda}\; T^{\alpha\beta\cdots\lambda}\,. \tag{34}$$

Since the components Q^{o}_{α} vanish, we can write this relation simply as,

$$\tilde{T}^{ij\cdots k} = Q^{i}{}_{\alpha}\; Q^{j}{}_{\beta}\cdots Q^{k}{}_{\nu}\; T^{\alpha\beta\cdots\nu}\,, \tag{35.1}$$
$$\tilde{T}^{o\;\cdots\nu} = 0\,.$$

The explicit expressions for Q^{i}_{α} are:

$$Q^{i}{}_{j} = \delta^{i}_{j}\,, \tag{36.1}$$

$$Q^{i}{}_{o} = (g^{oo})^{-1} g^{oi}\,. \tag{36.2}$$

For the special case of a contravariant four-vector, Eq. (34) turns into:

$$\tilde{T}^{\rho} = Q^{\rho}{}_{\alpha}\; T^{\alpha}\,,$$

and from (35) and (36),

$$\tilde{T}^{o} = 0\,,$$
$$\tilde{T}^{i} = T^{i} - (g^{oo})^{-1}\, g^{oi}\, T^{o}\,,$$

therefore, given the contravariant components T^{α} we can form the D-invariant $\tilde{T}^{i}$ by means of the above relation. We note that $\tilde{T}^{i}$ depends on T^{o} as well as on T^{i}.

The spatial components of the covariant metric four-tensor, g_{ik} form a D-invariant. From the contravariant components we can form the following D-invariant:

$$\tilde{g}^{\rho\mu} = Q^{\rho}{}_{\alpha}\; Q^{\mu}{}_{\nu}\; g^{\alpha\nu} ,$$

$$\tilde{g}^{o\mu} = 0 ,$$

$$\tilde{g}^{ik} = g^{ik} - \frac{g^{oi}\; g^{ok}}{g^{oo}} \equiv e^{ik} . \qquad (37)$$

The notation e^{ik} was first introduced by Dirac. We shall use it in all that follows. e^{ik} is the reciprocal of g_{ik},

$$e^{ik}\; g_{k\ell} = \delta^{i}_{\ell} \qquad (38)$$

With the components $T^{\mu\nu\ldots\lambda}$, we can also form the quantity

$$T_L = \ell_\mu \ell_\nu \;\ldots \ell_\lambda \;\; T^{\mu\nu\ldots\lambda} , \qquad (39)$$

which is a D-invariant.

In particular, from (33) we have,

$$g_L = \ell_\mu \ell_\nu \;\; g^{\mu\nu} = 1 . ^{*}$$

It is a consequence of the previous results that we can construct the following D-invariants with a given scalar f(x),

$$f_{,i} ,$$

$$f_{,\alpha}\, \ell^{\alpha} = f_{,i}\, \frac{g^{oi}}{\sqrt{g^{oo}}} + f_{,o} \sqrt{g^{oo}} .$$

In what follows we shall have the opportunity to introduce other D-invariants.

We shall now proceed to introduce Dirac's theory in some

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* We can also form the four quantities, $v_\alpha = g_{\alpha\beta}\, \ell^{\beta}$. v_o is not a D-invariant, as it equals $(g^{oo})^{-1/2}$; the remaining v_i vanish (they are equal to ℓ_i). Thus, from $g_{\alpha\beta}$ we can form only six D-invariant nontrivial components, the g_{rs}.

detail. Given Einstein's Lagrangian density,

$$L = \sqrt{-{}^4g}\; g^{\mu\nu}\left(\begin{Bmatrix}\sigma\\ \mu\rho\end{Bmatrix}\begin{Bmatrix}\rho\\ \nu\sigma\end{Bmatrix} - \begin{Bmatrix}\rho\\ \mu\nu\end{Bmatrix}\begin{Bmatrix}\sigma\\ \rho\sigma\end{Bmatrix}\right),$$

we can construct the canonical momentum densities conjugate to $g_{\mu\nu}$ by the usual relations,

$$p^{\mu\nu} \equiv \frac{\partial L}{\partial \dot{g}_{\mu\nu}} . \tag{40}$$

Because of the covariance of the theory, it is found that there exist four relations among the twenty canonical variables $g_{\mu\nu}$, $p^{\mu\nu}$, the so-called primary constraints,

$$p^{o\mu} + \phi^{\mu}(g_{\alpha\beta}, g_{\alpha\beta,i}) = 0 . \tag{41}$$

As a consequence of (41) we cannot solve (40) for all the "velocities" $\dot{g}_{\mu\nu}$ in terms of $p^{\mu\nu}$; namely $\dot{g}_{o\mu}$ cannot be written as function of the momenta. As a consequence the Hamiltonian densi ty,

$$H = p^{\mu\nu}\,\dot{g}_{\mu\nu} - L ,$$

cannot be written as a function of only the field variables $g_{\mu\nu}$, $p^{\mu\nu}$.

Dirac has solved this difficulty by adding to L a suitable divergence, which has no effect on the field equations.

$$L_D = L + \mathcal{D} .$$

The addition of this divergence does change the definition of the $p^{\mu\nu}$. With the new Lagrangian density L_D we find that

$$p_D^{o\mu} = \frac{\partial L_D}{\partial \dot{g}_{o\mu}} = 0 ,$$

instead of the constraint equations (41). In this scheme the Hamiltonian is

$$\mathcal{H}_D = \int d_3x \left(p_D^{ij} \, \dot{g}_{ij} - L_D \right),$$

$$p_D^{ij} = \frac{\partial L_D}{\partial \dot{g}_{ij}} .$$

The p_D^{ij} have the form

$$p_D^{ij} = \sqrt{-g} \left(e^{i\ell} e^{jm} - e^{ij} \, e^{\ell m} \right) v_{\ell m} ,$$

$$g = |g_{ik}| ,$$

where, v_{ij} are the so-called Dirac "invariant velocities",

$$v_{ij} = \frac{1}{\sqrt{g^{oo}}} \begin{Bmatrix} o \\ ij \end{Bmatrix} .$$

These v_{ij} are D-invariant; hence the momenta p_D^{ij} are also D-invariant. Accordingly, the twelve canonical field variables of Dirac's theory, the g_{ij} and the p_D^{ij} are all D-invariants.

Dirac's Hamiltonian still contains the $g_{o\mu}$ (which is not a D-invariant) as coefficients, the explicit expression for H_D being,

$$\mathcal{H}_D = \int d_3x \; H_D = \int d_3x \left\{ (g^{oo})^{-\frac{1}{2}} \, H_L + g_{or} H^r \right\}, \tag{42}$$

with H_L and H^r independent of $g_{o\mu}$,

$$H^r = e^{rs} \left[p^{uv} g_{uv,s} - 2(p^{uv} g_{us})_{,v} \right] = -2p^{ru}{}_{|u} , \tag{43}$$

$$H_L = K^{-1} \left[p^{rs} \; p_{rs} - \tfrac{1}{2} \, p_r{}^r \; p_s{}^s \right] + K \, e^{rs} \; S_{rs} , \tag{44}$$

where,

$$p_{rs} = g_{rv} \; g_{sn} \; p^{vn} ,$$

$$K^2 = -g ,$$

$$S_{rs} = \Gamma^{\ell}_{rs,\ell} - \Gamma^{\ell}_{r\ell,s} + \Gamma^{\ell}_{rs}\,\Gamma^{v}_{\ell v} - \Gamma^{v}_{r\ell}\Gamma^{\ell}_{sv} \,, \tag{45}$$

$$\Gamma^{u}_{kv} = \frac{1}{2}\, e^{us}\left(g_{sk,v} + g_{sv,k} - g_{kv,s}\right). \tag{46}$$

By a vertical bar we indicate the covariant derivative with respect to the three-dimensional metric g_{ik} and its reciprocal e^{ik}. All relations from here on will be written in this three-dimensional geometry on the space-like hypersurface of the four-space. For instance, the "covariant" derivative of a given T^{ij} is, three-dimensionally,

$$T^{ij}_{|r} = T^{ij}_{,r} + \Gamma^{i}_{\ell r}\, T^{\ell j} + \Gamma^{j}_{\ell r}\, T^{i\ell} \,.$$

If we require that the primary constraints, $p^{o\mu} = 0$, are maintained in the course of time, then if $p^{o\mu} = 0$ at $t = t_o$ and if $t = t_o + \lambda$ (with λ a first-order infinitesimal) we have, to the first order in λ,

$$p^{o\mu}(\vec{x},t) = p^{o\mu}(\vec{x},t_o) + \lambda\left[p^{o\mu}(\vec{x},t_o),\, \mathcal{H}_D\right] \,, \tag{47.1}$$

hence

$$\left[p^{o\mu}(\vec{x},t_o),\, \mathcal{H}_D\right] = 0 \,. \tag{47.2}$$

The Poisson bracket is defined here in terms of all the canonical variables $g_{\mu\nu}$, $p^{\mu\nu}$,

$$[A,\, B] = \int d_3x \left(\frac{\delta A}{\delta g^{(x)}_{\mu\nu}}\,\frac{\delta B}{\delta p^{\mu\nu}_{(x)}} - \frac{\delta A}{\delta p^{\mu\nu}_{(x)}}\,\frac{\delta B}{\delta g^{(x)}_{\mu\nu}}\right), \tag{48}$$

for two given functionals of the canonical variables. Using Eq. (48), we have for (47.2),

$$\left[p^{ko}(\vec{x},t_o),\, \mathcal{H}_D\right] = H^{k}(\vec{x},t_o) + \frac{1}{2}\,\frac{g^{ko}}{\sqrt{g^{oo}}}\, H_L(\vec{x},t_o) = 0 \,,$$

$$\left[p^{oo}(\vec{x},t_o), \mathcal{H}_D\right] = \frac{1}{2}\sqrt{g^{oo}}\; H_L(\vec{x},t_o) = 0 .$$

Then,

$$H_L = 0 , \tag{49}$$

$$H^k = 0 , \tag{50}$$

which are the secondary constraints (see an analogous discussion for the electromagnetic field in the Appendix).

These secondary constraints coincide with four of the gravitational field equations, the equations

$$G_{\mu o} = R_{\mu o} - \frac{1}{2}\, g_{\mu o}\, R = 0 ,$$

which correspond to the equation $\nabla\cdot\vec{E} = 0$ in the case of the electromagnetic field.

These secondary constraints are first-class, the expression for their Poisson brackets being, [6]

$$[H_\alpha, H^0_\beta] = H_\lambda \left(\delta^r_\beta\, \delta^\lambda_\alpha - \delta^r_\alpha \delta^\lambda_\beta\right) \; \delta_{,r}(\vec{x}-\vec{x}') ,$$

$$H_\alpha = \ell_\alpha H_L + Q^r{}_\alpha \;\; g_{rk}\; H^k .$$

Hence, in accordance with the argument presented at the beginning of this section, the set of constraints H_L, H^r is complete, and no additional constraints are obtained by further differentiation with respect to x^o.

The three secondary constraints

$$H^k = -2\, p^{ku}{}_{|u} = 0$$

resemble the constraint for free electromagnetic fields, $\nabla\cdot\vec{p} = 0$ (see Appendix), except that the divergences are now "three-dimensionally covariant".

If we give g_{ij}, p^{ij} at some initial time, it is possible in principle to get the value of these variables at a later time by means of the Hamiltonian equations of motion; however, even if we give $g_{o\mu}$ at the same initial time, it will remain arbitrary at any later time. This arbitrariness in $g_{o\mu}$ in turn affects the determination of g_{ij}, p^{ij} at the later time, since $g_{o\mu}$ is present in the equations of motion. Hence if we merely give g_{ij}, p^{ij} at the initial time, we cannot predict uniquely their values at a later time. This property is characteristic of any generally covariant field theory, and in particular of general relativity.

Equivalently, we can say that the continuation of the coordinate system outside the three-dimensional hypersurface x^o = constant, where the Cauchy data was given, remains arbitrary.

The dependence of $g_{o\mu}$ on the choice of coordinates corresponds to the behaviour of the scalar potential $\emptyset$ with respect to the arbitrary choice of gauge (if we want to draw an analogy between the two theories), as is explained in the Appendix.

To finish this section we analyze the total number of degrees of freedom involved in the set g_{ij}, p^{ij}. These data are complete but redundant, in the sense that they do not represent a minimal set describing a field of spin 2 and zero rest mass. As we have four constraints relating these twelve variables, only 8 are independent. Besides we also have to choose our coordinate system; this choice could be made by making four functions of g_{ij}, p^{ij} equal the coordinates x^μ,

$$x^\mu = f^\mu(g_{ij}, p^{ij}) .$$

The arbitrary variables $g_{o\mu}$ should then be fixed by the condition

$$\partial_t f^\mu = \delta_o^\mu .$$

Once the coordinates have been fixed, we are left with but four independent field variables, a number appropriate to a massless field of spin 2. This kinematical consideration obviously holds both in the full nonlinear theory and in the weak field approximation.

IV. THE DYNAMICAL VARIABLES IN THE SPINOR FORMULATION

In this section we shall exploit the results of Section II as applied to the Hamiltonian formulation of the gravitational field. In Section II we have shown that the knowledge of the set of linearly independent Hermitian spin matrices σ_μ is sufficient for the determination of all properties of the Riemann space; presently we shall look for a set of dynamical spin variables from which we can derive all the properties of the Dirac theory outlined in Section III.

In order to follow the same line of approach as was used in Section III, we shall first introduce the full set of D-invariants which we can form from the spin matrices σ_μ. In passing we note that the definition of D-invariance is related only to transformation properties with respect to (infinitesimal) coordinate transformations but not to spin transformations (or equivalently to tetrad rotations), which are transformations at a fixed point of the four-space. Hence the search for D-invariants associated with σ_μ reduces to the search for D-invariants associated with a four-vector V_μ of the coordinate space.

Thus, from σ_μ we can form the D-invariants σ_i and σ_L,

$$\sigma_L = \ell_\mu \sigma^\mu = \sigma_L^{\dagger} . \ * \tag{51}$$

The variable σ_o is not a D-invariant, just as $g_{o\mu}$ is not a D-invariant. The role played by these two variables is about the same in each formalism, even the total number of components of each of them is the same (4 real numbers).

We can also define the D-invariants,

$$\tau_i , \ \tau_L = \tau_\mu \ \ell^\mu ,$$

which are algebraically dependent on the set (51), as from the normalization condition (33) it follows that

$$\tau_L = - \sigma_L^{-1} .$$

Using Eq. (29) and,

$$\sigma_L = \frac{\sigma^o}{\sqrt{g^{oo}}} = \frac{\sigma^o}{\sqrt{|\sigma^o|}} , \tag{52}$$

we obtain,

$$\sigma_L = \frac{i}{3!} \ \frac{\varepsilon^{\varrho\nu\rho\sigma} \sigma_\nu \tau_\rho \ \sigma_\sigma}{\sqrt{-|g_{rs}|}} = \frac{i}{3!} \ \frac{\varepsilon^{ijk} \ \sigma_i \tau_j \sigma_k}{\sqrt{-|g_{rs}|}} . \tag{53}$$

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* It is simple to verify this explicitly: consider the infinitesimal coordinate transformation $x'^\mu = x^\mu + \xi^\mu(x)$, under which the components of ℓ^μ (Eq. (32)) transform as,

$$\ell'^\mu = \ell^\mu - (g^{oo})^{-1/2} \ \ell^\mu \ell^\beta \xi^o_{,\beta} + \ell^\beta \xi^\mu_{,\beta} + (g^{oo})^{-1/2} \ \xi^o_{,\beta} \ g^{\mu\beta} ,$$

for a given four-vector,

$$V'_\mu \ (x') = V_\mu(x) - \xi^\alpha_{,\mu} \ V_\alpha(x) \ ;$$

the contracted product $V'_\mu \ \ \ell'^\mu$ is then equal to $V_\mu \ \ell^\mu$, plus terms independent of the time derivative of ξ^α.

(we have used that, $\sqrt{-{}^4g} = \sqrt{\dfrac{-|g_{rs}|}{g^{oo}}}$ and $g^{oo} = |\sigma^o|$). The expression for the determinant of g_{rs} is,

$$|g_{rs}| = \frac{1}{3!}\, \varepsilon^{mnv}\, \varepsilon^{ijk}\, g_{im}\, g_{jn}\, g_{kv} =$$

$$= -\left(\frac{1}{2}\right)^3 \frac{1}{3!}\, \varepsilon^{mnv}\, \varepsilon^{ijk}\, \tau_{iK\dot{M}}\, \tau_{jS\dot{P}}\, \tau_{KV\dot{R}}\, \sigma_m{}^{\dot{M}K}\, \sigma_n{}^{\dot{P}S}\, \sigma_v{}^{\dot{R}V}. \quad (54)$$

To obtain this relation first note that Eq. (10) for $g_{\mu\nu}$ may be written as

$$\sigma_\mu{}^{K\dot{M}}\, \sigma_{\nu \dot{M}R} + \sigma_\nu{}^{K\dot{M}}\, \sigma_{\mu \dot{M}R} = 2\, g_{\mu\nu} \delta^K_R\,, * \quad (55)$$

then,

$$g_{\mu\nu} = \frac{1}{4}\left(\sigma_\mu^{K\dot{M}}\, \sigma_{\nu\, \dot{M}K} + \sigma_\nu{}^{K\dot{M}}\, \sigma_{\mu\, \dot{M}K}\right), \quad (56)$$

or,

$$g_{\mu\nu} = -\frac{1}{4}\, \mathrm{Tr}(\sigma_\mu \tau_\nu + \sigma_\nu \tau_\mu)\,. \quad (57)$$

This can be further simplified if we recall that the σ, τ matrices are Hermitian, which allow us to write,

$$\mathrm{Tr}(\sigma_\mu\ \tau_\nu) = \mathrm{Tr}(\sigma_\nu\ \tau_\mu)\,. \quad (58)$$

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* From Eq. (10) we have,

$$\tau_{\mu L\dot{K}} = \epsilon_{LR}\, \bar{\sigma}_\mu{}^{R\dot{P}}\, \epsilon_{\dot{P}\dot{K}} = -\bar{\sigma}^{R\dot{P}}\, \epsilon_{RL}\, \epsilon_{\dot{P}\dot{K}} = -\bar{\sigma}_{\mu L\dot{K}} = -\overset{T}{\sigma}_{\mu L\dot{K}}\,.$$

Using this we rewrite Eq. (20) in the form of (55) above. It should be noted that in this equation we have the symmetrized product of the matrix,

$$\sigma_\mu{}^{K\dot{M}} = \begin{pmatrix} \sigma^{1\dot{1}} & \sigma^{1\dot{2}} \\ \sigma^{2\dot{1}} & \sigma^{2\dot{2}} \end{pmatrix},$$

with the transpose of σ_ν with all indices lowered:

$$-\left[\sigma_{(\mu}\ \tau_{\nu)}\right]^K{}_R = \left[\sigma_{(\mu}\ \overset{T}{\sigma}_{\nu)}\right]^K{}_R = 2\, g_{\mu\nu}\, \delta^K_R\,.$$

Then, Eq. (57) takes the form:

$$g_{\mu\nu} = -\frac{1}{2}\,\mathrm{Tr}(\sigma_\mu\,\tau_\nu)\;. \tag{59}$$

A similar result holds, both for (58) and (59), with the Υ matrices written at the left. We go from one set to the other simply by taking Hermitian conjugates.

Eq. (59), taken for the spatial indices, proves that the relation (54) is correct. This relation is obviously D-invariant, as it should be.

From Eq. (52) we see that the σ_L matrix has a determinant equal to 1, this property translated in terms of the relation (53) means that the square of the denominator is equal to the determinant of the matrix standing in the numerator; actually we can indeed show that this is true; first of all we can show that the spin matrix,

$$M = \varepsilon^{ijk}\,\sigma_i\,\Upsilon_j\,\sigma_k = -M^\dagger\,,$$

is generally not a multiple of the identity matrix. For proof we use tetrads and write:

$$M = \varepsilon^{ijk}\,h_i^{(\rho)}\,h_j^{(\mu)}\,h_k^{(\nu)}\,\mathring{\sigma}_\rho\,\mathring{\tau}_\mu\,\mathring{\sigma}_\nu\;.$$

Taking the sum over the index ρ we get:

$$M = \varepsilon^{ijk}\,h_i^{(o)}h_j^{(\mu)}h_k^{(\nu)}\,\mathring{\tau}_\mu\mathring{\sigma}_\nu + \varepsilon^{ijk}\,h_i^{(r)}\,h_j^{(\mu)}\,h_k^{(\nu)}\,\mathring{\sigma}_r\,\mathring{\tau}_\mu\,\mathring{\sigma}_\nu\;.$$

For an easier understanding of the properties of this matrix we shall perform explicitly all sums indicated, the result being.

$$M = \varepsilon^{ijk}\,h_i^{(o)}\,h_j^{(r)}\,h_k^{(s)}\,\mathring{\tau}_r\,\mathring{\sigma}_s + \varepsilon^{ijk}\,h_i^{(r)}\,h_j^{(s)}\,h_k^{(\ell)}\,\mathring{\sigma}_r\,\mathring{\tau}_s\,\mathring{\sigma}_\ell\;.$$

Whenever r, s is equal to 1,3 or 2,3 in the sum over r,s in the first term, we shall have nondiagonal matrices $\mathring{\sigma}_2$, $\mathring{\sigma}_1$ contributing to M. The second term in the above expression is proportional to the identity matrix, since ε^{ijk} is completely antisymmetric. The trace of M comes entirely from this latter term:

$$\mathrm{Tr}\; M = 12\, i\, \varepsilon^{ijk}\, h_i^{(1)}\, h_j^{(2)}\, h_k^{(3)}.$$

The matrix M is clearly not proportional to the identity matrix unless the three tetrad components $h_i^{(o)}$ all vanish. By contrast in four dimensions we had Eqs. (26) and (27).

If we want to take this further, we may define the two-dimensional products

$$N^k = \varepsilon^{kij}\, \sigma_i\, \tau_j\ ,$$

which are also not proportional to the identity matrix,

$$N^k = 2\, \varepsilon^{kij}\, h_i^{(o)}\, h_j^{(r)}\, \overset{\circ}{\sigma}_r + \varepsilon^{kij}\, h_i^{(r)}\, h_j^{(s)}\, \overset{\circ}{\sigma}_r\, \overset{\circ}{\tau}_s\ ,$$

since nondiagonal matrices occur when we take r = 1,2 in the first term, or r,s equal to 1,3 or 2,3 in the second term. Since all diagonal matrices are here proportional to $\overset{\circ}{\sigma}_3$, we conclude that the matrices N^k have vanishing traces unlike M. This is so because in M we have terms proportional to the identity matrix, but no such terms are present in N^k.

We can take the determinant of M and show by a straightforward calculation that this is the same as taking the determinant of g_{rs} according to Eq. (54),

$$|g_{rs}| = \frac{1}{3!}\, \varepsilon^{ijk}\, \varepsilon^{\ell mn}\, g_{i\ell}\, g_{jm}\, g_{kn}\ .$$

Where we use tetrads,

$$g_{i\ell} = h_i^{(\mu)}\, h_\ell^{(\nu)}\, \overset{\circ}{g}_{\mu\nu}\ ,$$

then,

$$|g_{rs}| = \frac{1}{3!}\,\varepsilon^{ijk}\,\varepsilon^{\ell mn}\left\{h_i^{(\mu)}\,h_\ell^{(\nu)}\,h_j^{(\rho)}\,h_m^{(\tau)}\,h_k^{(\alpha)}\,h_n^{(\beta)}\,\mathring{g}_{\mu\nu}\mathring{g}_{\rho\tau}\mathring{g}_{\alpha\beta}\right\} ;$$

the final result being,

$$\left(\frac{i}{3!}\right)^2 \left|\varepsilon^{ijk}\,\sigma_i\,\tau_j\,\sigma_k\right| = -\,|g_{rs}| .$$

Care should be taken with the notation. The determinant on the left hand side refers to spinor indices of the matrix, whereas at the right side it refers to the three-dimensional vector indices (see Eq. (54)). The determinant of the left hand side is calculated by using the previous expansion of M in terms of the tetrad four-vectors. The above relation is naturally equivalent to (54). Actually, the relation (54) is simpler for practical calculations than the above relation.

To summarize, σ_L of Eq. (53) has a determinant equal to 1, and it is D-invariant. From the set of matrices σ_μ we can form only the three independent D-invariants σ_i, the remaining D-invariant σ_L being algebraically dependent on the σ_i because of Eqs. (53) and (54). Finally the property that $|\sigma_L| = 1$ corresponds to $g_L = 1$ of Dirac's theory.

The next subject to be discussed is the construction of D-invariants with the contravariant spin matrices. From Eq. (29) we have

$$\sigma^\mu = \frac{i\sqrt{g^{oo}}\,\varepsilon^{\mu\nu\rho\sigma}\,\sigma_\nu\,\tau_\rho\,\sigma_\sigma}{3!\,\sqrt{-|g_{rs}|}} . \qquad (60.1)$$

This can also be written as,

$$\sigma^\mu = \frac{4\,\varepsilon^{\mu\nu\rho\sigma}\,\sigma_\nu\,\tau_\rho\,\sigma_\sigma}{\varepsilon^{\alpha\beta\gamma\delta}\sigma_\alpha\,\tau_\beta\,\sigma_\gamma\,\tau_\delta} , \qquad (60.2)$$

where we have made use of the relations (26) and (27). The contra variant σ^{μ} are not D-invariant, in accordance with our previous discussion, as we cannot form D-invariants using only the contravariant components of a four-vector.

We have shown previously that some combinations can be defined among the contravariants components and the Q^{μ}_{α} which are D-invariant. Let us see how they look presently. According to the definition given by the Eq. (34),

$$\tilde{\sigma}^{\mu} = Q^{\mu}{}_{\nu}\ \sigma^{\nu} , \tag{61.1}$$

$$Q^{\mu}{}_{\nu} = \delta^{\mu}_{\nu} - \ell^{\mu}\ell_{\nu},\ Q^{o}{}_{\nu} = 0 , \tag{61.2}$$

or,

$$\tilde{\sigma}^{o} = 0 , \tag{62.1}$$

$$\tilde{\sigma}^{i} = \sigma^{i} - \ell^{i}\ \sigma_{L} , \tag{62.2}$$

since in Eq. (61.1) the σ^{ν} are given by (60), in the relation (62.2) the σ^{i} are given by (60) taken for $\mu = i$; the σ_{L} are given by (53).

The explicit from for $\tilde{\sigma}^{i}$ is then,

$$\tilde{\sigma}^{i} = \frac{4\ \varepsilon^{i\mu\nu\rho}\ \sigma_{\mu}\ \tau_{\nu}\ \rho_{\rho}}{\varepsilon^{\alpha\beta\gamma\delta}\sigma_{\alpha}\tau_{\beta}\sigma_{\gamma}\tau_{\delta}} - \frac{i\ \ell^{i}\ \varepsilon^{\ell mn}\ \sigma_{\ell}\ \tau_{m}\ \sigma_{n}}{3!\ \sqrt{-|g_{rs}|}} . \tag{63}$$

Similarly we can define expressions like,

$$\Lambda^{\mu\nu} = Q^{\mu}{}_{\alpha}\ \ Q^{\nu}{}_{\beta}\ \ \sigma^{\alpha}\tau^{\beta} = \tilde{\sigma}^{\mu}\ \tilde{\tau}^{\nu} , \tag{64}$$

up to any desired order in the σ^{μ}; the quantities σ^{μ}, $\Lambda^{\mu\nu}$ are the D-invariants that we can form with the contravariant σ^{μ}. These relations are more complicated that the relation giving the D-invariant associated to the σ_μ.

The components of the above $\Lambda^{\mu\nu}$ are,

$$\Lambda^{o\nu} = 0 \,,$$

$$\Lambda^{ij} = \sigma^i \gamma^j - \ell^j \sigma^i \gamma_L - \ell^i \sigma_L \gamma^j - \ell^i \ell^j \,, \tag{65}$$

we used that,

$$\tilde{\gamma}^\nu = Q^\nu{}_\alpha \gamma^\alpha = \gamma^\nu - \ell^\nu \gamma_L \,,$$
$$\tilde{\gamma}^o = 0 \,, \tag{66}$$

and, by Eq. (30)

$$\gamma^\nu = \frac{-4\, \varepsilon^{\nu\mu\lambda\rho} \gamma_\mu \sigma_\lambda \gamma_\rho}{\varepsilon^{\alpha\beta\gamma\delta} \sigma_\alpha \gamma_\beta \sigma_\gamma \gamma_\delta} \tag{67}$$

The relation (65) symmetrized over the indices ij takes the form,

$$\Lambda^{(ij)} \equiv \Lambda^{ij} + \Lambda^{ji} \,,$$

$$\Lambda^{(ij)} = -2(g^{ij} + \ell^i \ell^j) - \ell^i(\sigma_L \gamma^j + \sigma^j \gamma_L) - \ell^j(\sigma^i \gamma_L + \sigma_L \gamma^i) \,,$$

since

$$\sigma_L \gamma^j + \sigma^j \gamma_L = -2\ell^j \cdot 1 \,, \tag{68}$$

$$\gamma^j \sigma_L + \gamma_L \sigma^j = -2\ell^j \cdot 1 \,, \tag{69}$$

(use the Eq. (20) written for contravariant components, and the definition of the ℓ^μ).

We obtain,

$$\Lambda^{(ij)} = \tilde{\sigma}^{(i}\tilde{\gamma}^{j)} = -2(g^{ij} - \ell^i \ell^j) \cdot 1 \,. \tag{70}$$

Using Eq. (37) we rewrite (70) as,

$$\tilde{\sigma}^{(i}\tilde{\gamma}^{j)} = -2e^{ij} \cdot 1 \,, \tag{71}$$

in this equation both sides are manifestly D-invariant. In the next sections we will use this relation in order to write the e^{ij} in terms of spin matrices in such expressions as the secondary constraints.

In order to write the $\tilde{\sigma}^i$ entirely in terms of our set of spin matrices σ_μ we still need to give the expression of the normal ℓ^μ in function of these matrices.

Now, from the Eq. (68) or (69) we get,

$$\ell^i = -\frac{1}{2}\,\mathrm{Tr}(\sigma_L\,\gamma^i) = -\frac{1}{2}\,\mathrm{Tr}(\tau_L\,\sigma^i)\,, \qquad (72)$$

(we used that the σ_L(or γ_L) is Hermitian, a property which follows from the fact that the ℓ^μ are real).

Using (60), (72), and (53) we write (62.2) as:

$$\tilde{\sigma}^i = \frac{i\,\varepsilon^{i\nu\rho\sigma}\,\sigma_\nu\,\tau_\rho\,\sigma_\sigma}{3!\,\sqrt{-|g_{\mu\nu}|}} - \frac{i\,\ell^i\,\varepsilon^{rsk}\,\sigma_r\,\tau_s\,\sigma_k}{3!\,\sqrt{-|g_{sr}|}}\,, \qquad (73)$$

$$\ell^i = -\frac{\varepsilon^{i\alpha\beta\gamma}\varepsilon^{\ell mn}\,\mathrm{Tr}\left(\sigma_\ell\tau_m\,\sigma_n\tau_\alpha\,\sigma_\beta\tau_\gamma\right)}{2(3!)^2\sqrt{-|g_{rs}|}\,\sqrt{-|g_{\mu\nu}|}}\,, \qquad (74)$$

with $|g_{rs}|$ and $|g_{\mu\nu}|$ given by (54) and (27) respectively. Eq. (73) is equivalent to Eq. (63). Similarly,

$$\tilde{\tau}^i = \frac{-i\,\varepsilon^{i\nu\rho\sigma}\,\tau_\nu\sigma_\rho\,\tau_\sigma}{3!\,\sqrt{-|g_{\mu\nu}|}} + \frac{i\,\ell^i\,\varepsilon^{rsk}\,\tau_r\,\sigma_s\,\tau_k}{3!\,\sqrt{-|g_{rs}|}}\,, \qquad (75)$$

with ℓ^i given by (74). From Eqs. (73) and (75) we can write explicitly the expression of e^{ij} in term of the spin matrices; note that form (71),

$$e^{ij} = -\frac{1}{4}\,\mathrm{Tr}\left[\tilde{\sigma}^{(i}\,\tilde{\tau}^{j)}\right]. \qquad (76)$$

The discussion concerning the construction of D-invariants from the spin matrices ends here. The remaining of this section will be used for the discussion of the problem of determination of the canonical variables for the gravitational field in the spinor theory.

We make the substitution of $g_{\mu\nu}$ in L_D in terms of spin matrices by using the relation (56). With this substitution L_D becomes a function of σ_μ, $\sigma_{\mu,i}$, and of the "spin velocities" $\dot{\sigma}_\mu$. Therefore, we can calculate the conjugated momenta associated to σ_μ,

$$\pi^{\mu}_{K\dot{M}} = \frac{\partial L_D}{\partial \dot{\sigma}^{K\dot{M}}_{\mu}} , \qquad (77.1)$$

we can write this as follows:

$$\pi^{\mu}_{K\dot{M}} = \frac{\partial L_D}{\partial \dot{g}_{\rho\sigma}} \frac{\partial \dot{g}_{\rho\sigma}}{\partial \dot{\sigma}^{K\dot{M}}_{\mu}} = p^{\rho\sigma} \frac{\partial \dot{g}_{\rho\sigma}}{\partial \dot{\sigma}^{K\dot{M}}_{\mu}} , \qquad (77.2)$$

a straightforward calculation using (55) or (56) gives,

$$\frac{\partial \dot{g}_{\rho\sigma}}{\partial \dot{\sigma}^{K\dot{M}}_{\mu}} = \frac{1}{2} \left(\delta^{\mu}_{\rho} \, \sigma_{\sigma\dot{M}K} + \delta^{\mu}_{\sigma} \, \sigma_{\rho\dot{M}K} \right) , \qquad (78)$$

substituting (78) into (77) and using the symmetry of $p^{\rho\sigma}$ we obtain,

$$\pi^{\mu} = \frac{\partial L_D}{\partial \dot{\sigma}_\mu} = - p^{\alpha\mu} \, \tau_\alpha , \qquad (79)$$

for the conjugated momenta to σ_μ. The primary constraint $p^{o\mu} = 0$, implies that:

$$\pi^{o} = 0 ,$$

which is also a primary constraint. Therefore the equation (79) reduces to,

$$\pi^{i} = - p^{ij} \, \tau_j , \qquad (80)$$

$$\pi^{i+} = \pi^{i} ,$$

these π^i are D-invariant since p^{ij} and γ_j are both D-invariant. We have twelve variables σ_i and twelve variables π^i; later on, it will be shown that we have six primary constraints relating these twenty four canonical variables at each space-time point. Presently we note that this number of variables can be decomposed into real components if we use tetrads; that is, if we use Eqs. (14) and (15),

$$\sigma_i = h_i^{(\alpha)} \overset{\circ}{\sigma}_\alpha \ , \tag{81.1}$$

$$\pi^i = -p^{ij} h_j^{(\alpha)} \overset{\circ}{\gamma}_\alpha = -\chi^{i(\alpha)} \overset{\circ}{\gamma}_\alpha, \tag{81.2}$$

the $h_i^{(\alpha)}$, $\chi^{i(\alpha)}$ represents twenty-four real functions of the coordinates.

We can also define the variable,

$$\lambda^i = \frac{\partial L_D}{\partial \dot{\gamma}_i} = - p^{ij} \sigma_j = \epsilon \bar{\pi}^i \epsilon, \tag{82}$$

which is algebraically dependent on the set given by π^i.

The knowledge of σ_i is sufficient for the determination of the covariant spatial components of the metric, which is Dirac's theory represent half of the dynamical variables; now we ask whether knowledge of the π^i is sufficient for the determination of the Dirac's p^{ij}. In other words, we ask if the set σ_i, π^i is sufficient for the determination of the g_{ij}, p^{ij} of the theory outlined in Section III. In order to answer this question we need to solve Eq. (80) for the p_{ij} in terms of the spin variables. In order to do that, we multiply both sides of Eq. (80) by σ_k,

$$\pi^i \sigma_k = - p^{ij} \gamma_j \sigma_k \ ,$$

this can also be written as,

$$\pi^i \sigma_k = -\frac{1}{2} p^{ij} \left(\tau_j \sigma_k + \tau_k \sigma_j\right) - \frac{1}{2} p^{ij} \left(\tau_j \sigma_k - \tau_k \sigma_j\right) ,$$

using Eq. (20) and the definition given by (82), we obtain,

$$\frac{1}{2} \pi^i \sigma_k + \frac{1}{2} \tau_k \epsilon \bar{\pi}^i \epsilon = p^{ij} g_{jk} \cdot 1 ,$$

taking trace on both sides, and using:

$$\mathrm{Tr}\left(\tau_k \epsilon \bar{\pi}^i \epsilon\right) = \mathrm{Tr}\left(\pi^i \sigma_k\right) ,$$

we get the result,

$$p^{ij} g_{jk} = \frac{1}{2} \mathrm{Tr}\left(\pi^i \sigma_k\right) . \tag{83}$$

In order to solve this equation explicitly for the p^{ij} we multiply both sides by the $e^{k\ell}$ defined by the Eq. (37), the final result being

$$p^{i\ell} = \frac{1}{2} e^{k\ell} \mathrm{Tr}\left(\pi^i \sigma_k\right) , \tag{84}$$

$$e^{k\ell} \mathrm{Tr}\left(\pi^i \sigma_k\right) - e^{ki} \mathrm{Tr}\left(\pi^\ell \sigma_k\right) = 0 ; \tag{85}$$

the relation (85) has been imposed in order that the $p^{i\ell}$ of (84) be symmetric. In these relations $e^{\ell k}$ is now written in terms of the spin matrices (see Eq. (76)), hence $p^{i\ell}$ is given entirely as a function of the spin matrices. However, we should note that in this equation not only the σ_i, π^i are present in the right-hand side, but we also have the $e^{\ell k}$. These latter, in spite of being D-invariant, are complicated functions of the spin matrices σ_i. (See (76), (75), and (73)). The fact that $p^{i\ell}$ is such a complicated function of the spin variables is not to be considered a draw-back of the present method; from the present point of view, the π^i, not the $p^{i\ell}$, are basic variables. Therefore, the knowledge of the variables σ_i, π^i is sufficient for the determina tion of the g_{ij}, p^{ij} by means of the relations (20), (84). If we add to this scheme the six primary spin constraints (which will be studied later on), and also the six conditions for the determination of a spin frame (or orientation of the tetrad axes) we will have twelve independent components in the set σ_i, π^i which completes the correspondence with the g_{ij}, p^{ij}. This discussion allows us to interpret the set of D-invariants variables σ_i, π^i as the dynamical variables for the gravitational field in the spinor theory.

In the next section we shall discuss the meaning of the relation (85).

V. THE HAMILTONIAN AND THE PRIMARY SPIN CONSTRAINTS

In this section we shall introduce the expressions for the primary spin constraints; besides, we shall also write the Hamiltonian of Dirac's theory as a function of the spin variables, and introduce the commutator algebra of the spinor theory, these considerations will allow us to interpret the spin constraints as the generators of the spin transformation group.

If we consider the σ_i, π^i as a set of Hermitian spin matrices, they should contain a total number of twenty-four real numbers in their matrix elements. However, inspection of the relation (80) shows that the π^i contain actually only six variables; that is to say, given σ_i, the τ_i are determined from (10), and therefore, the π^i contain additionally only the p^{ij} which represent but six real functions.

Therefore, six components of π^i are eliminated by the definition (80); the reason is as follows: the Dirac Lagrangian density is invariant under unimodular spin transformations, a property which follows from the fact that all spin matrices appear in L_D only through the spin-invariant combinations g_{ij}, $\dot{g}_{ij}$ and $g_{ij,r}$,* this invariance property implies that some of the components of π^i will vanish; a similar situation prevails in the case of the Maxwell Lagrangian density, where the gauge invariance of L implies that the momentum density conjugate to the scalar potential vanishes, since we cannot form any gauge-invariant quantity out of the time derivatives of the scalar potential. We can also as well draw an analogy with the invariance of the Einstein-Dirac Lagrangian density L_D with respect to arbitrary coordinate transformations, which implies that four components of $p^{\mu\nu}$ vanish, the $p^{o\mu}$. Presently we should expect that six real components of π^i vanish because of the spin-invariance of L_D; and this is just what happens. The number six is related to the fact that the spin transformations contains six real parameters (or descriptors) on account of the unimodular condition, and we know that we have as many

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* **Under the transformation, $\sigma_\mu = M\,\sigma_\mu\,M^\dagger$, $\tau'_\mu = \overset{\dagger}{M}{}^{-1}\,\tau_\mu\,M^{-1}$; the g_{ij} given by (20) are invariant: $1\cdot g_{ij} = 1\cdot g'_{ij}$; if we take the derivatives of σ_μ, τ_ν they will transform as:**

$$\sigma'_{\mu,\nu} = M_{,\nu}\,\sigma_\mu\,M^\dagger + M\,\sigma_{\mu,\nu}\,M^\dagger + M\,\sigma_\mu\,M^\dagger_{,\nu}\;;\quad \tau'_{\mu,\nu} = \overset{\dagger}{M}{}^{-1}_{,\nu}\,\tau_\mu\,M^{-1} + \overset{\dagger}{M}{}^{-1}\,\tau_{\mu,\nu}\,M^{-1} + \overset{\dagger}{M}{}^{-1}\,\tau_\mu\,M^{-1}_{,\nu}\,,$$

then, the quantities, $1\cdot g_{\mu\nu,\alpha} = -1/2\left[\sigma_{(\mu,\alpha}\tau_{\nu)} + \sigma_{(\mu}\tau_{\nu),\alpha}\right]$, are also invariant under the spin transformations, $1\cdot g'_{\mu\nu,\alpha} = 1\cdot g_{\mu\nu,\alpha}$.

primary constraints as we have parameters in the associated transformation. Indeed, we shall prove later on that the primary spin constraints are the generators of the unimodular spin transformation group.

We can present a decomposition of π^i which shows explicitly its twelve real components, by writing,

$$\pi^i = - p^{ik} \tau_k + \alpha^{[ik]} \tau_k + \beta^i \tau_L . \qquad (86)$$

The Hermiticity of π^i and of the τ_k and τ_L assures that the variables p^{ik}, $\alpha^{[ik]}$ and β^i are all real. The relation (86) can be solved for the coefficients $\alpha^{[ik]}$ and β^i; a similar solution for p^{ik} was done in Eqs. (84) and (85). In Eq. (86) $\alpha^{[ik]}$ means the antisymmetric part of α^{ik}. The inverse relations are

$$\beta^i = - \frac{1}{2} \mathrm{Tr}(\sigma_L \pi^i) , \qquad (87)$$

$$\alpha^{[ik]} = \frac{1}{4} \mathrm{Tr}(\sigma^i \pi^k - \sigma^k \pi^i) , \qquad (88)$$

where,

$$\sigma^i = e^{ik} \sigma_k , \qquad (89)$$

$$\mathrm{Tr}(\sigma^i \tau_k) = e^{im} \mathrm{Tr}(\sigma_m \tau_k) = - 2\delta^i{}_k . \qquad (90)$$

In order to obtain the Eq. (87) we have used the relation

$$\mathrm{Tr}(\sigma_L \tau_k) = 0 . \qquad (91)$$

The proof of (91) is obtained as follows. First we use the fundamental "anticommutation relation" taken for mixed indices,

$$\sigma^\lambda \tau_\nu + \sigma_\nu \tau^\lambda = - 2 \delta^\lambda_\nu \cdot 1 ,$$

where we take $\lambda = 0$ and $\nu = i$; from these relations we get

$$\mathrm{Tr}(\sigma^{0}\,\gamma_{i}) = -\,\mathrm{Tr}(\sigma_{i}\,\gamma^{0}) \;; \tag{92}$$

from the Hermiticity of σ and γ we get,

$$\mathrm{Tr}(\sigma^{0}\,\gamma_{i}) = \mathrm{Tr}(\sigma_{i}\,\gamma^{0}) \;. \tag{93}$$

Comparison of (92) and (93) shows that

$$\mathrm{Tr}(\sigma^{0}\,\gamma_{i}) = 0 \;. \tag{94}$$

As σ_L is just σ^0 "normalized", according to Eq. (52), the relation (91) is confirmed.

From Eqs. (86), (87), (88) along with (80) we can see that the six primary spin constraints must be equivalent to the requirements

$$\beta^{i} = 0 \;, \tag{95}$$

$$\alpha^{[ik]} = 0 \;. \tag{96}$$

We can present these constraints entirely in terms of our chosen set of canonical variables, the σ_i, π^i. Before doing so we note that the relation (96) is the same as the relation (85) derived in the last section. The primary spin constraints have then the effect of eliminating six of the initial twelve components of π^i.

We define the set of six Hermitian spin matrices,

$$M_{ij} = \pi^{\ell}(\sigma_i\gamma_j - \sigma_j\gamma_i)\,\sigma_\ell + \sigma_\ell(\gamma_j\sigma_i - \gamma_i\sigma_j)\,\pi^{\ell} \;, \tag{97}$$

$$N_i = \pi^{\ell}\,\sigma_L\,\gamma_i\,\sigma_\ell + \sigma_\ell\,\gamma_i\,\sigma_L\,\pi^{\ell} \;. \tag{98}$$

A straightforward calculation shows that

$$\mathrm{Tr}\,N_i = 4\,g_{i\ell}\,\beta^{\ell} \;, \tag{99}$$

$$\mathrm{Tr}\,M_{ij} = 4\,g_{im}\,g_{jn}\,\alpha^{[mn]} \tag{100}$$

In these relations β^{ℓ} and $\alpha^{[mn]}$ are given by Eqs. (87) and (88). The proof of the relation (100) follows straight from (88) and (97); but the proof for (99) is a little more elaborated. We note that Eq. (98) can also be written as

$$N_i = \frac{1}{2}\pi^{\ell}\sigma_L\tau_{(i}\sigma_{\ell)} + \frac{1}{2}\sigma_{(\ell}\tau_{i)}\sigma_L\pi^{\ell} + \frac{1}{2}\pi^{\ell}\sigma_L\tau_{[i}\sigma_{\ell]} + \frac{1}{2}\sigma_{[\ell}\tau_{i]}\sigma_L\pi^{\ell},$$

where,

$$\tau_{(i}\sigma_{j)} = \tau_i\sigma_j + \tau_j\sigma_i \;;\; \tau_{[i}\sigma_{j]} = \tau_i\sigma_j - \tau_j\sigma_i .$$

Hence

$$N_i = -\pi^{\ell}\sigma_L g_{i\ell} - \sigma_L\pi^{\ell}g_{\ell i} + \frac{1}{2}\pi^{\ell}\sigma_L\tau_{[i}\sigma_{\ell]} + \frac{1}{2}\sigma_{[\ell}\tau_{i]}\sigma_L\pi^{\ell} .$$

Taking the trace on both sides we obtain:

$$\mathrm{Tr}\,N_i = -2g_{i\ell}\,\mathrm{Tr}(\sigma_L\pi^{\ell}) + \frac{1}{2}\mathrm{Tr}\left[\pi^{\ell}(\sigma_L\tau_{[i}\sigma_{\ell]} + \sigma_{[\ell}\tau_{i]}\sigma_L)\right]. \qquad (101)$$

Now we can show that the spin matrix,

$$X_{i\ell} = \sigma_L(\tau_i\sigma_\ell - \tau_\ell\sigma_i) + (\sigma_\ell\tau_i - \sigma_i\tau_\ell)\sigma_L , \qquad (102)$$

vanishes * by virtue of the "fundamental anti-commutation rela-

* It is interesting to note that the matrix introduced by the Eq. (102) is of the form,

$$X_{i\ell} = Y_{i\ell} + Y_{i\ell}^{\dagger} ;$$

since the Eqs. (103) can be written as, $\sigma_L\tau_i + \sigma_i\tau_L = 0$, and $\tau_L\sigma_i + \tau_i\sigma_L = 0$, it follows that Y_{il} is anti-Hermitian, and therefore $X_{il} = 0$.

tions",

$$\sigma^{o}\gamma_{i} + \sigma_{i}\gamma^{o} = 0 , \tag{103.1}$$

$$\gamma^{o}\sigma_{i} + \gamma_{i}\sigma^{o} = 0 . \tag{103.2}$$

Then,

$$\mathrm{Tr}\ N_{i} = - 2\ g_{i}\ \ \mathrm{Tr}\left(\sigma_{L}\ \pi^{\ell}\right) ;$$

which completes the proof of the relation (99). Therefore, the primary spin constraint can also be presented in the form,

$$A_{ij} = \mathrm{Tr}\ M_{ij} = 0 , \tag{104}$$

$$B_{i} = \mathrm{Tr}\ N_{i} = 0 , \tag{105}$$

which is more adequate for the subsequent discussion relating to the generators of the unimodular spin transformation group.

We can rewrite Eq. (86) as,

$$\tilde{\pi}^{i} = \pi^{i} + \frac{1}{4}\ e^{ir}\ e^{ks}\ A_{rs}\ \gamma_{k} + \frac{1}{4}\ e^{ij}\ B_{j}\ \gamma_{L} , \tag{106}$$

with,

$$\pi^{i} = - p^{ik}\ \gamma_{k} .$$

In the Eq. (106) we have used the notation $\tilde{\pi}^{i}$ for the momenta π^{i} off the spin constraint hypersurface of the phase of the theory; as we said before, the momenta which follow from the Dirac's Lagrangian density already refer to points on the spin constraint hypersurface, as L_D is invariant with respect to the unimodular spin transformations.

Incidentally, a more simple proof of Eq. (99) can be obtained from the relations,

$$\sigma_{L}\ \gamma_{i} + \sigma_{i}\ \gamma_{L} = 0 ; \qquad \gamma_{L}\ \sigma_{i} + \gamma_{i}\ \sigma_{L} = 0 .$$

Indeed, using (98):

$$\mathrm{Tr}\ N_i = \mathrm{Tr}\left\{\pi^{\ell}\ K_{i\ell}\right\};\ K_{i\ell} = \sigma_L\ \gamma_i\ \sigma_\ell + \sigma_\ell\ \gamma_i\ \sigma_L\ ,$$

then,

$$K_{i\ell} = \sigma_i\ \gamma_\ell\ \sigma_L + \sigma_\ell\ \gamma_i\ \sigma_L = -2\ g_{i\ell}\ \sigma_L\ .$$

Substituting this result in the previous equation for the trace of N_i we get Eq. (99).

For future reference we now write down some expressions involving the σ_i and γ_i matrices, which have a vanishing trace,

$$\mathrm{Tr}\ \left(\gamma_i\ \sigma_j - \gamma_j\ \sigma_i\right) = 0\ . \qquad (107)$$

We need to understand that the relation (107) and also (91), in spite of being relations involving the twelve components of the σ_i, actually do not represent any limitation on these components; that is, we still have twelve independent real variables in σ_i. The proof of this statement is made more simple if we use tetrads, and represent the twelve real variables in σ_i by the $h_i^{(\alpha)}$,

$$\sigma_i = h_i^{(\alpha)}\ \mathring{\sigma}_\alpha\ .$$

The Eq. (107) takes the form,

$$h_i^{(\mu)}\ h_j^{(\nu)}\ \mathrm{Tr}\ \left(\mathring{\gamma}_\mu\,\mathring{\sigma}_\nu - \mathring{\gamma}_\nu\,\mathring{\sigma}_\mu\right) = 0\ ,$$

and we see that this equation is always satisfied, independently of any particular form for the tetrad four-vectors. For Eq. (91) we have,

$$\begin{aligned}\sigma_L &= \ell_\mu\ \sigma^\mu\ ,\\ &= \ell_\mu\ h^\mu_{(\alpha)}\ \mathring{\sigma}^\alpha\ .\end{aligned}$$

Using the previous definition of tetrad components we see that we can rewrite this relation as,

$$\sigma_L = \ell_{(\alpha)} \,\overset{\circ}{\sigma}{}^{\alpha} .$$

The explicit value for $\ell^{(\alpha)}$ is obtained from Eq. (32),

$$\ell_{(\alpha)} = h^{o}_{(\alpha)} \left(h^{o}_{(\lambda)}\, h^{o(\lambda)} \right)^{-\frac{1}{2}} .$$

Hence,

$$\mathrm{Tr}(\sigma_L \,\tau_k) = -2\, h_k^{(\rho)}\, h^{o(\alpha)}\, \overset{\circ}{g}_{\alpha\rho} \left(h^{o}_{(\lambda)}\, h^{o(\lambda)} \right)^{-\frac{1}{2}} = 0 ,$$

which again places no limitation on the components of the tetrads.

From the relations (97), (98), (104), (105), and (106) we see that not only the constraints are D-invariant, but also the expression for $\tilde{\pi}^i$ is D-invariant.

Now we proceed to work out the second subject of this section, namely the determination of the spinor form of the Hamiltonian theory treated in the Section III. In order to do this we need first to derive some relationship between Dirac's canonical variables and our variables. Since all these relations are easy to obtain, they will be presented without an explicit proof. For π^i defined on the spin constraint hypersurface of the phase space,

$$p_{ij}\, p^{ij} = -\tfrac{1}{2}\, \mathrm{Tr}(\lambda^i\, \pi_i) , \qquad (108)$$

$$p_{ij} = g_{i\ell}\, g_{jm}\, p^{\ell m}; \quad \pi_i = g_{ij}\, \pi^j , \qquad (109)$$

$$p^{ij}\, g_{ij} = \tfrac{1}{2}\, \mathrm{Tr}(\pi^i\, \sigma_i) , \qquad (110)$$

$$p^{ij}\, g_{ij,r} = \mathrm{Tr}(\pi^i\, \sigma_{i,r}) , \qquad (111)$$

$$g_{\ell i,r} = -\tfrac{1}{2}\, \mathrm{Tr}\left(\tau_i\, \sigma_{\ell,r} + \tau_\ell\, \sigma_{i,r} \right) , \qquad (112)$$

$$p^{ki}\, g_{ir} = \tfrac{1}{2}\, \mathrm{Tr}\left(\pi^k\, \sigma_r \right) , \qquad (113)$$

For π^i off the spin constraint hypersurface, that is for $\tilde{\pi}^i$ given by Eq. (106), we have:

$$\tilde{\lambda}^i = - p^{ik} \sigma_k + \alpha^{[ik]} \sigma_k + \beta^i \sigma_L , \qquad (114.1)$$

$$= \lambda^i + \frac{1}{4} e^{ir} e^{ks} A_{rs} \sigma_k + \frac{1}{4} e^{ij} B_j \sigma_L , \qquad (114.2)$$

which gives

$$Tr(\tilde{\pi}_i \tilde{\lambda}^i) = -2 p_{uv} p^{uv} - 2\left(\beta_i \beta^i + \alpha_{[ij]} \alpha^{[ij]}\right), \qquad (115)$$

where, as before,

$$\tilde{\pi}_i = g_{ij} \tilde{\pi}^j ; \quad \beta_i = g_{ij} \beta^j ; \quad \alpha_{[ij]} = g_{im} g_{in} \alpha^{[mn]} .$$

Using Eq. (114.2) we rewrite (115) as

$$Tr(\tilde{\pi}_i \tilde{\lambda}^i) = - 2 p_{uv} p^{uv} - \frac{1}{8} \left(e^{ik} B_i B_k + e^{ir} e^{js} A_{ij} A_{rs}\right) . \qquad (116)$$

If the spin constraints are satisfied, the relation (116) goes over into Eq. (108) derived previously.

We can also show that

$$Tr(\tilde{\pi}^i \sigma_i) = Tr(\pi^i \sigma_i) , \qquad (117)$$

which is a consequence of Eqs. (91) and (107). The equivalent of the relation (111) is here

$$Tr(\tilde{\pi}^i \sigma_{i,r}) = Tr(\pi^i \sigma_{i,r}) + \frac{1}{4} e^{ir} e^{ks} A_{rs} Tr(\gamma_k \sigma_{i,r}) + \frac{1}{4} e^{ir} B_r Tr(\gamma_L \sigma_{i,r}),$$

or,

$$Tr(\tilde{\pi}^i \sigma_{i,r}) = p^{ij} g_{ij,r} + \frac{1}{8} e^{im} e^{ks} A_{ms} Tr(\gamma_k \sigma_{i,r} - \gamma_i \sigma_{k,r})$$

$$+ \frac{1}{4} e^{im} B_m Tr(\gamma_L \sigma_{i,r}) . \qquad (118)$$

From Eq. (91) we can write

$$Tr(\gamma_L \sigma_{i,r}) = - Tr(\gamma_{L,r} \sigma_i) .$$

Finally, the equivalent to the relation (113) is here

$$Tr(\tilde{\pi}^k \sigma_r) = 2 p^{ki} g_{ir} - \frac{1}{2} e^{km} A_{mr} . \qquad (119)$$

We have seen that the primary spin constraints have the effect of eliminating six out of the twelve components of π^i. Nevertheless, we still have twelve real components in the σ_i, so that the theory is asymmetric in this respect; a somewhat similar behaviour was obtained for the case of the electromagnetic field, where the secondary constraint associated with the gauge invariance of the Lagrangian density has the effect of eliminating the three components $\vec{p}_L$ out of the $\vec{p}$ (see the Appendix), and we are left with three momenta $\vec{p}_T$ and the "coordinate" variables $\vec{A}$; in order to get rid of the remaining three unphysical (nongauge-invariant) variables $\vec{A}_L$, a gauge frame is used explicitly (a gauge condition is taken), so that we finally obtain a symmetric theory with the momenta $\vec{p}_T$ and "coordinates" $\vec{A}_T$. For the gravitational field in the framework of Dirac's theory we saw that the invariance of the Lagrangian density has the effect of eliminating four components $p^{0\mu}$ of the initial ten components $p^{\mu\nu}$; we are left with six momenta p^{ij} and ten "coordinate" variables $g_{\mu\nu}$. In order to get rid of four of these variables which are not D-invariant, the $g_{0\mu}$ (which are unphysical in the sense that their values depend on the choice of the coordinates outside the three-dimensional hypersurface on which they are defined), a coordinate system is chosen, and this choice in effect reduces the number of field variables to a symmetric theory with six p^{ij} and six g_{ij}.

By a similar procedure we should expect that we can eliminate six variables in the σ_i by taking a certain spin frame, or at least by considering a certain restricted spin transformation group. This is equivalent to a choice for the local orientation of the tetrad axes, and we can see what should be the restriction on the matrix associated to the tetrad rotations, or equivalently on the matrix M of unimodular spin transformations.

Since we have six components in g_{ij}, we should expect that we need only six components in σ_i, if these two versions are to be equivalent; so that since the beginning six of the σ_i are redundant. Nevertheless, the situation presently is somewhat different from the two cases discussed before, in the sense that all the twelve components of σ_i are D-invariant, and hence are physical variables.

After these comments, we return to our original objective, to rewrite the Hamiltonian of Section III so as to turn into a function of the spin variables.

For π^i belonging to the spin constraint hypersurface, that is, for the situation where Eqs. (108) through (113) are valid, we have:

$$H = \mathrm{Tr}\left[\pi^i f_{ir} - \pi^i_{\ ,i}\ \sigma_r\right] , \tag{120}$$

with

$$f_{ir} = \sigma_{i,r} - \sigma_{r,i} . \tag{121}$$

$H_r = g_{rs} H^s$ is the "transverse" Hamiltonian constraint given by Eq. (43). In order to obtain the relation (120) we have made use of the relations (108) through (113).

From Eqs. (108) and (110) we have:

$$p^{rs}p_{rs} - \frac{1}{2} p_r{}^r \, p_s{}^s = - \frac{1}{2} \mathrm{Tr}(\lambda^i \, \pi_i) - \frac{1}{8} \mathrm{Tr}(\pi^i \, \sigma_i) \, \mathrm{Tr}(\pi^j \, \sigma_j). \quad (122)$$

Substituting (122) into the expression (44) we obtain

$$H_L = - \frac{K^{-1}}{2} \left\{ \mathrm{Tr}(\lambda^i \, \pi_i) + \frac{1}{4} \mathrm{Tr}(\pi^i \, \sigma_i) \, \mathrm{Tr}(\pi^j \, \sigma_j) \right\} + Ke^{rs} \, S_{rs} \, , \quad (123)$$

the expression for K being given by Eq. (54),

$$- K^2 = |g_{rs}| \, .$$

In order to write H_L entirely as a function of the spin variables we also require the form of S_{rs} in terms of the spin variables.

The expression for the three-dimensional Christoffel symbol of Eq. (46) in terms of spin matrices is,

$$\Gamma_{k,ij} = - \frac{1}{4} \mathrm{Tr} \left(\tau_i \, f_{kj} + \tau_k \, S_{ij} + \tau_j \, f_{ki} \right), \quad (124)$$

where f_{ij} is given by (121), and s_{ij} is short for,

$$s_{ij} = \sigma_{i,j} + \sigma_{j,i} \, . \quad (125)$$

In Eq. (124) $\Gamma_{k,ij}$ is, as usual,

$$\Gamma_{k,ij} = g_{k\ell} \, \Gamma^{\ell}_{ij} \, .$$

Since the spin matrices ϵ are independent of the coordinates, we can define the quantities

$$\mathring{m}_{kj} = \epsilon \, \bar{f}_{kj} \, \epsilon = \tau_{k,j} - \tau_{j,k} \, , \quad (126)$$

$$n_{kj} = \epsilon \, \bar{s}_{kj} \, \epsilon = \tau_{k,j} + \tau_{j,k} \, . \quad (127)$$

Then we have the following expression for the derivatives of $\Gamma_{k,ij}$:

$$\partial_\ell \, \Gamma_{k,ij} = - \frac{1}{4} \mathrm{Tr} \Big[\tau_i \, f_{kj,\ell} + \tau_k \, s_{ij,\ell} + \tau_j \, f_{ki,\ell} + \sigma_{i,\ell} \, m_{kj} + \sigma_{k,\ell} \, n_{kj} + \sigma_{j,\ell} \, m_{ki} \Big]. \quad (128)$$

The expression for the three-dimensional Ricci tensor is then given by Eq. (45), which reads:

$$S_{rs} = \left(e^{\ell u}\,\Gamma_{u,rs}\right)_{,\ell} - \left(e^{\ell u}\,\Gamma_{u,r\ell}\right)_{,s} + e^{\ell u}\,e^{vn}\,\Gamma_{u,rs}\,\Gamma_{n,\ell v} - e^{vn}\,e^{\ell u} - \Gamma_{n,r\ell}\,\Gamma_{u,sv}\ .$$

The $\Gamma_{k,ij}$ are given by (124), their derivatives by (128), and the e^{ik} are given by Eq. (76). The above relation for S_{rs}, though far from compact, is sufficient for practical calculations called for in the Hamiltonian theory, such as the calculation of the Poisson bracket of S_{rs} with some functional of the momenta π^{i}.

Now, by using the relation

$$g_{or}\,e^{rs} = -\,(g^{oo})^{-1}\,g^{os}\ ,$$

we can rewrite the Dirac Hamiltonian density (given by Eq. (42)) as:

$$H = \ell_o(H_L - \ell^s\,H_s)\ . \qquad (129)$$

The expression for ℓ^i in terms of spin matrices was obtained in Section IV, Eq. (74). For the ℓ_o we have from (52)

$$\ell_o = \frac{1}{\sqrt{g^{oo}}} = |\sigma^o|^{-\frac{1}{2}}. \qquad (130)$$

These ℓ^i, ℓ_o are not D-invariant, they represent the arbitrary coefficients of the constraints in the Hamiltonian; the comparison of this situation with the corresponding case for the electromagnetic field shows that these coefficients correspond to the role of the scalar potential $\emptyset$, which is not gauge invariant and which appears in the electromagnetic Hamiltonian density as the coefficient of the secondary constraint.

Similar computations can be performed for points outside the spin constraint hypersurface; from Eq. (43) and from Eqs. (118) and (119) we get

$$\widetilde{H}_\ell = g_{\ell r}\ \widetilde{H}^r = \mathrm{Tr}\left(\widetilde{\pi}^i\ f_{i\ell} - \widetilde{\pi}^i_{,i}\ \sigma_\ell\right) + C_\ell^{mn}\ A_{mn} + D_\ell^m\ B_m - \frac{1}{2}\ e^{vm}\ A_{m\ell,v}\ , \tag{131}$$

where

$$C_\ell^{mn} = -\frac{1}{8}\ e^{im}\ e^{kn}\ \mathrm{Tr}(\tau_k\ \sigma_{i,\ell} - \tau_i\sigma_{k,\ell}) - \frac{1}{2}\ e^{vm}_{,v}\ \delta^n_\ell\ , \tag{132}$$

$$D_\ell^m = -\frac{1}{4}\ e^{im}\ \mathrm{Tr}(\tau_L\ \sigma_{i,\ell})\ . \tag{133}$$

By $\widetilde{H}_\ell$ we have indicated the value of H_ℓ (given by Eq. (43)) when we substitute the $p^{i\ell}$ and $g_{i\ell}$ (and their derivatives) as functions of the $\widetilde{\pi}^i$, σ_i by means of the relations (118) and (119). If we take $\widetilde{\pi}^i = \pi^i$, then, $A_{mn} = 0$, $B_m = 0$ * and we get again the relation (124). We should note that the A_{mn}, B_m of (138) are given by the expressions (104), (105) along with (97) and (98), but we must take $\widetilde{\pi}^i$ inside the expressions giving the traces. The same obviously holds for Eq. (106) defining $\widetilde{\pi}^i$. In this relation the A_{rs} and B_j are the previous traces taken for $\widetilde{\pi}^i$.

Using the relation (44), along with the relations (116) and (117), we get,

$$\widetilde{H} = -\frac{K^{-1}}{2}\left\{\mathrm{Tr}(\widetilde{\lambda}^i\ \widetilde{\pi}_i) + \frac{1}{4}\ \mathrm{Tr}(\widetilde{\pi}^i\ \sigma_i)\ \mathrm{Tr}(\widetilde{\pi}^j\ \sigma_j)\right\} +$$

$$+ K\ e^{rs}\ S_{rs} + E^{ij}\ A_{ij} + F^i\ B_i\ , \tag{134}$$

where,

$$E^{ij} = -\frac{K^{-1}}{16}\ e^{ir}\ e^{js}\ A_{rs}\ , \tag{135}$$

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* If we take $\widetilde{\pi}^i = \pi^i$ in Eq. (106) we get

$$\frac{1}{4}\, e^{ir}\, e^{ks}\ A_{rs}\,\tau_k + \frac{1}{4}\ e^{ij}\ B_j\ \tau_L = 0\ .$$

Multiplying this relation by σ_L and by σ_m, and taking traces in both cases, we find directly that $A_{rs} = 0$, and $B_j = 0$.

$$F^i = -\frac{K^{-1}}{16} e^{ik} B_k \qquad (136)$$

when $\tilde{\pi}^i = \pi^i$, we see that the $\tilde{H}_L$ of (134) goes over the previous expression given by (123).

Using Eqs. (129), (131), and (134) we can introduce the quantity $\tilde{H}$ given by

$$\tilde{H} = \ell_o \left(\tilde{H}_L - \ell^s \tilde{H}_s \right) . \qquad (137)$$

This quantity is equivalent to a term identical to the previous H, where we have $\tilde{\pi}^i$ in the place of π^i, plus a given combination of the primary spin constraints (this combination contains terms linear and quadratic in those constraints, as we have seen); if we do not use the symbol $\sim$ in the π^i, we can write (137) as,

$$\tilde{H} = H + J^{ij} A_{ij} + K^i B_i - \frac{1}{2} \ell_o \ell^s e^{vm} A_{ms,v} , \qquad (138)$$

with

$$J^{ij} = \ell_o E^{ij} + \ell_o \ell^s C_s^{ij} , \qquad (139)$$

$$K^i = \ell_o F^i - \ell_o \ell^s D_s^i , \qquad (140)$$

in this expression the momenta π^i are taken outside the spin constraint hypersurface (they are the $\tilde{\pi}^i$).

Since all terms standing in the right hand side of Eq. (138) vanish weakly (in Dirac's terminology), we see that the $\tilde{H}$ has the same degree of arbitrariness as the Hamiltonian given by the relation (129). Therefore, we can use $\tilde{H}$ as the Hamiltonian, as a functional of the $\tilde{\pi}^i$, σ_i. The correspondence with Dirac's theory of Section III is obtained when we go over the spin constraint hypersurface, and take adequate conditions for the fixation of the spin frame.

The next subject is the construction of the classical commutator algebra for the spinor Hamiltonian theory; in order to do this, we need first to introduce the concept of functional derivative.

Let us consider the functional of some given set of functions $\eta_a(x)$, $\pi^a(x)$ which we will take as a canonical set of functions later on; this functional is denoted by the symbol $A\{\eta, \pi\}$; we shall restrict ourselves to functionals of the form,

$$A = \int \mathcal{A}(\eta_a(x), \pi^a(x))\, d_3x \qquad (141)$$

The functional derivative of the fundamental A with respect to any one of these functions, is defined by:

$$\frac{\delta A\{\eta,\pi\}}{\delta \eta_i(y)} = \lim_{k \to 0} \lim_{\lambda \to 0} \frac{1}{\lambda} \int d_3x\, \mathcal{A}\Big(\eta_a(x), \pi^a(x); \eta_i + \lambda \delta_k(x-y)\Big) - \int d_3x\, \mathcal{A}\Big(\eta_a(x), \pi^a(x)\Big), \qquad (142)$$

that is, we kept all $\eta_a(x)$, $\pi^a(x)$ fixed, with exception of $\eta_i(x)$, which goes from $\eta_i(x)$ to $\eta_i(x)+\lambda\delta_k$; here, δ_k is any well-behaved function which in the limit $k \to 0$ tends to the three-dimensional Dirac delta function.

If we take

$$\mathcal{A}(\eta_a(x), \pi^a(x)) = f^a(x)\, \eta_a(x) + \omega_a(x)\, \pi^a(x) ,$$

we obtain,

$$A\{\eta,\pi\} = \int d_3x \left[f^a(x)\, \eta_a(x) + \omega_a(x)\, \pi^a(x)\right]. \qquad (143)$$

Hence

$$\frac{\delta A\{\eta,\pi\}}{\delta \eta_i(y)} = \lim_{k \to 0} \lim_{\lambda \to 0} \frac{1}{\lambda} \int d_3x \Big\{ f^a(x)\, \eta_a(x) + \omega_a(x)\, \pi^a(x) +$$

$$+ f^i(x)\left[\eta_i(x) + \lambda \delta_k(x-y)\right]\Big\} - \int d_3x \left\{ f^a(x)\, \eta_a(x) + f^i(x)\eta_i(x) + \omega_a(x)\pi^a(x)\right\}. \qquad (144)$$

In Eq. (142) as well as in (144) the index "i" is one of the elements of the set "a"; that is "i" in (144) is not a summation index (dummy). The calculation indicated in (144) gives

$$\frac{\delta A\{\eta,\pi\}}{\delta\eta_i(y)} = \lim_{k\to 0}\int d_3x\ f^i(x)\ \delta_k(x-y) = f^i(y)\ . \tag{145}$$

Similarly,

$$\frac{\delta A\{\eta,\pi\}}{\delta\pi^i(y)} = \omega_i(y)\ . \tag{146}$$

We can obtain these same relations by taking the variations of A due to variations in the functions η, π (which is essentially what we did before),

$$\delta A\ \{\eta,\pi\} = \int\left[\frac{\delta A}{\delta\eta_a(x)}\ \delta\eta_a(x) + \frac{\delta A}{\delta\pi^a(x)}\ \delta\pi^a(x)\right]d_3x\ . \tag{147}$$

This last form of definition for $\frac{\delta A}{\delta\eta_a(x)}$ allows us to see directly that

$$\delta\eta_a(x) = \int d_3x'\ \delta(\vec{x}-\vec{x}')\ \delta\eta_a(x') = \int\frac{\delta\eta_a(x)}{\delta\eta_b(x')}\ \delta\eta_b(x')\ d_3x'\ ,$$

hence,

$$\frac{\delta\eta_a(x)}{\delta\eta_b(x')} = \delta_a^b\ \delta(\vec{x}-\vec{x}')\ . \tag{148}$$

This last result will be used subsequently. Clearly, a result similar to (148) also holds if we replace η by π.

With this definition of fundamental derivative, we can now define the Poisson bracket between two given functionals of the canonical variables, which, for the time being, we shall again denote by η_a, π^a. Our definition of Poisson bracket is:

$$\left[A\{\eta,\pi\},\ B\{\eta,\pi\}\right] = \int d_3x \left(\frac{\delta A}{\delta \eta_a(x)} \frac{\delta B}{\pi^a(x)} - \frac{\delta A}{\delta \pi^a(x)} \frac{\delta B}{\delta \eta_a(x)} \right) . \qquad (149)$$

From this definition we obtain directly the expressions for the Poisson brackets between the canonical variables themselves,

$$\left[\eta_a(x),\ \eta_b(y)\right]_{x^o=y^o} = 0 \ , \qquad (150)$$

$$\left[\pi^a(x),\ \pi^b(y)\right]_{x^o=y^o} = 0 \ , \qquad (151)$$

$$\left[\eta_a(x),\ \pi^b(y)\right]_{x^o=y^o} = \delta_a^b\ \delta(\vec{x}-\vec{y}) \ . \qquad (152)$$

The relations (150) through (152) are the fundamental Poisson brackets. Knowledge of these brackets allows us to derive the Poisson bracket between any two functionals of the complete set of canonical variables η_a, π^a.

The Hamiltonian equations of motion in the notation of functional derivatives read as follows:

$$\dot{\eta}_a(x) = \left[\eta_a(x),\ H\{\eta,\pi\}\right] = \frac{\delta H}{\delta \pi^a(x)} \ , \qquad (153)$$

$$\dot{\pi}^a(x) = \left[\pi^a(x),\ H\{\eta,\pi\}\right] = -\frac{\delta H}{\delta \eta_a(x)} \ , \qquad (154)$$

$$H = \int d_3x\ \mathcal{H}\left(\eta_a(x),\ \pi^a(x)\right) \ . \qquad (155)$$

Now, we shall apply the definition (149) to our situation, where the set σ_i, π^i are the canonical variables. The relation (149) takes the form

$$[A,B] = \int d_3x \left(\frac{\delta A}{\delta\sigma_i{}^{K\dot{M}}(x)} \frac{\delta B}{\delta\pi^i_{K\dot{M}}(x)} - \frac{\delta A}{\delta\pi^i_{K\dot{M}}(x)} \frac{\delta B}{\delta\sigma_i^{K\dot{M}}(x)} \right) . \tag{156}$$

The fundamental Poisson brackets which follow from this relation are,

$$\left[\sigma_j^{R\dot{S}}(x),\ \sigma_k^{P\dot{V}}(y)\right]_{x^o=y^o} = 0 , \tag{157}$$

$$\left[\pi^j_{R\dot{S}}(x),\ \pi^k_{P\dot{V}}(y)\right]_{x^o=y^o} = 0 , \tag{158}$$

$$\left[\sigma_j{}^{R\dot{S}}(x),\ \pi^k_{P\dot{V}}(y)\right]_{x^o=y^o} = \delta^k_j\ \delta^R_P\ \delta^{\dot{S}}_{\dot{V}}\ \delta(\vec{x}-\vec{y}) . \tag{159}$$

These expressions are equivalent to Eqs. (150) through (152). In order to obtain the relation (159) we have used the relation

$$\int d_3z \frac{\delta\sigma_i^{R\dot{S}}(x)}{\delta\sigma^{K\dot{M}}(z)} \frac{\delta\pi^k_{V\dot{P}}(y)}{\delta\pi^\ell_{K\dot{M}}(z)} = \int d_3z\, \delta(\vec{x}-\vec{z})\,\delta(\vec{y}-\vec{z})\ \delta^\ell_i\ \delta^k_\ell\ \delta^R_K\ \delta^{\dot{S}}_{\dot{M}}\ \delta^K_V\ \delta^{\dot{M}}_{\dot{P}} ,$$

$$= \delta(\vec{x}-\vec{y})\ \delta^k_i\ \delta^R_V\ \delta^{\dot{S}}_{\dot{P}} ,$$

which is an application of Eq. (148). Some care must be taken in calculations of this type. We should note that

$$\frac{\delta\sigma_i^{R\dot{S}}(x)}{\delta\sigma_j{}^{K\dot{M}}(z)} = \delta^j_i\ \delta^R_K\ \delta^{\dot{S}}_{\dot{M}}\ \delta(\vec{x}-\vec{z}) ,$$

but that

$$\frac{\delta\sigma_i^{R\dot{S}}(x)}{\delta\sigma_j^{K\dot{M}}(z)} = \delta^j_i \left(\delta^R_K\ \delta^{\dot{S}}_{\dot{M}} + \delta^R_{\dot{M}}\ \delta^{\dot{S}}_K \right)\ \delta(\vec{x}-\vec{z}) ,$$

would be incorrect, since from this last relation we should obtain that

$$\frac{\delta\sigma_i^{1\dot{2}}(x)}{\delta\sigma_j^{1\dot{2}}(z)} = \frac{\delta\sigma_i^{1\dot{2}}(x)}{\delta\sigma_j^{\dot{1}2}(z)}$$

which implies that $\sigma_j^{1\dot{2}} = \sigma_j^{\dot{1}2}$; this is contrary to the fact that the anti-diagonal matrix elements of an Hermitian matrix are complex. Indeed, if we write,

$$\sigma_\ell^{K\dot{M}} = \begin{pmatrix} \sigma_\ell^{1\dot{1}} & \sigma_\ell^{1\dot{2}} \\ \sigma_\ell^{2\dot{1}} & \sigma_\ell^{2\dot{2}} \end{pmatrix} = \begin{pmatrix} \alpha_\ell & \gamma_\ell \\ \bar{\gamma}_\ell & \beta_\ell \end{pmatrix}$$

the above equality would reduce to $\gamma_\ell = \bar{\gamma}_\ell$, whereas actually only α_ℓ and β_ℓ are real. Another way of confirming the arrangement of indices on the right-hand side of (159) is to consider the product $\sigma_\mu^{K\dot{M}}\ \sigma^\mu_{R\dot{S}}$, which has all spinor indices free, just as in (159). This product is equal to the product $\overset{\circ}{\sigma}_\mu{}^{K\dot{M}}\ \overset{\circ}{\sigma}{}^\mu_{R\dot{S}}$, as can be shown easily by decomposition in the tetrad axes; but $\overset{\circ}{\sigma}{}_\mu^{K\dot{M}}\ \overset{\circ}{\sigma}{}^\mu_{R\dot{S}}$ is proportional to $\delta^K_R\ \delta^{\dot{M}}_{\dot{S}}$, [4] as in (159).

From Eq. (159) we obtain,

$$\left[\sigma_j{}^{R\dot{S}}(x),\ \pi^{kL\dot{N}}_{(y)}\right]_{x^o=y^o} = \delta^k_j\ \epsilon^{\dot{N}\dot{S}}\ \epsilon^{LR}\ \delta(\vec{x}-\vec{y})\ , \qquad (160)$$

where we have multiplied both sides of (159) by the constant matrices $\epsilon^{\dot{N}\dot{V}}\ \epsilon^{LP}$.

Now, we have the necessary mathematical foundation for discussing the meaning of the previous spin constraints; it is known from the theory of infinitesimal canonical transformations that the infinitesimal variation in a given function (or functional) of the dynamical variables, due to a transformation generated by some other given functional of those variables is obtained by taking the Poisson bracket between these two functionals. Therefore, we look for a functional of the σ_i, π^i to be denoted by $G\left\{\sigma,\pi\right\}$ which will generate the previously considered spin transformations,

$$\delta\sigma_k^{K\dot{M}}(x) = \left[\sigma_k^{K\dot{M}}(x),\, G\left\{\sigma,\pi\right\}\right] . \tag{161}$$

From the "commutation" relation (159) it follows that the functional G of (161) must contain a term of the form,

$$G\left\{\sigma,\pi\right\} = \int d_3y\ \pi^k_{K\dot{M}}(y)\ \delta\sigma_k^{K\dot{M}}(y) . \tag{162}$$

The infinitesimal spin transformation is given by the equation (9) if we take

$$M = 1 + V . \tag{163}$$

Since the matrix M has a determinant equal to one, the infinitesimal matrix V has vanishing trace,

$$\mathrm{Tr}\ V = 0 . \tag{164}$$

A matrix satisfying this property is,

$$V = \varepsilon^{ijk}\ v_k(\sigma_i\tau_j - \sigma_j\tau_i) + \omega^i\ \sigma_L\ \tau_i , \tag{165}$$

where v_m and w^k are a set of six real infinitesimal parameters, all of these parameters being arbitrary functions of the coordinates (the matrix V describes an infinitesimal local transformation). The above matrix V is not Hermitian, a result that we know to be true since M is not Hermitian. The Hermitian conjugate of V is,

$$V^\dagger = \varepsilon^{ijk}\ v_k(\tau_j\sigma_i - \tau_i\sigma_j) + \omega^i\ \tau_i\ \sigma_L . \tag{166}$$

Before going on with the construction of the generator G, it is interesting to note that the relations (91) and (107), which guarantee that the trace of V vanishes, are covariant relations, with respect both to arbitrary coordinate transformations and to unimodular spin transformations. If we consider the last case, we have under a spin transformation

$$\sigma_L' = \ell_\mu \, \sigma'^\mu = \ell_\mu \, M \sigma^\mu M^\dagger \, ,$$

$$\gamma_i' = M^{\dagger -1} \gamma_i \, M^{-1} \; ; \quad \sigma_i' = M \sigma_i \, M^\dagger \, ,$$

then

$$\mathrm{Tr}(\sigma_L' \, \gamma_i') = \mathrm{Tr}(\sigma_L \, \gamma_i) \, ,$$

$$\mathrm{Tr}(\gamma_j' \, \sigma_i' - \gamma_i' \, \sigma_j') = \mathrm{Tr}(\gamma_j \, \sigma_i - \gamma_i \, \sigma_j) \, .$$

Under coordinate transformation,

$$x'^\mu = x^\mu + \xi^\mu(x) \, ,$$

with infinitesimal functions $\xi^\mu(x)$, the quantities σ_L, σ_i, and γ_i transform as

$$\sigma_L' = \sigma_L + \frac{1}{\sqrt{g^{oo}}} \left(g^{\mu\beta} - \frac{g^{\mu o} \, g^{\beta o}}{g^{oo}} \right) \sigma_\mu \quad \xi^o{}_{,\beta} \, ,$$

$$\begin{pmatrix} \sigma_i' \\ \gamma_i \end{pmatrix} = \begin{pmatrix} \sigma_i \\ \gamma_i \end{pmatrix} - \xi^\alpha_{,i} \begin{pmatrix} \sigma_\alpha \\ \gamma_\alpha \end{pmatrix},$$

from which we can check directly the D-invariance of these quantities. Then:

$$\mathrm{Tr}(\sigma_L' \gamma_i') = \mathrm{Tr}(\sigma_L \gamma_j) - \xi^j_{,i} \, \mathrm{Tr}(\sigma_L \gamma_j) \, ,$$

$$\mathrm{Tr}(\gamma_j' \sigma_i' - \gamma_i' \sigma_j') = \mathrm{Tr}(\gamma_j \sigma_i - \gamma_i \sigma_j) - \xi^k_{,i} \, \mathrm{Tr}(\gamma_j \sigma_k - \gamma_k \sigma_j) +$$

$$+ \xi^k_{,j} \, \mathrm{Tr}(\gamma_i \sigma_k - \gamma_k \sigma_i) \, .$$

Hence Eqs. (91) and (107) are verified for any choice of coordinate system.

Turning back to the problem of constructing the generator G, we note that under the action of the matrix given by (163) the components of σ_i transform as

$$\sigma_i' = \sigma_i + V\,\sigma_i + \sigma_i\,V^\dagger$$

or

$$\delta\sigma_i^{K\dot{M}} = \left(V^K{}_S\,\delta^{\dot{M}}_{\dot{P}} + V^{\dot{M}}{}_{\dot{P}}\,\delta^K_S\right)\sigma_i^{S\dot{P}}$$

Using Eq. (165) we obtain

$$\delta\sigma_\ell^{K\dot{M}} = \varepsilon^{ijk}\,v_k\left[\delta^{\dot{M}}_{\dot{P}}\left(\sigma_i^{K\dot{N}}\,\Upsilon_{J\dot{N}S} - \sigma_j^{K\dot{N}}\,\Upsilon_{i\dot{N}S}\right) + \right.$$

$$\left. + \delta^K_S\left(\sigma_i^{\dot{M}N}\,\Upsilon_{jN\dot{P}} - \sigma_j^{\dot{M}N}\,\Upsilon_{iN\dot{P}}\right)\right]\sigma_\ell^{S\dot{P}}$$

$$+\,\omega^i\left[\delta^{\dot{M}}_P\left(\sigma_L^{K\dot{N}}\,\Upsilon_{i\dot{N}S} + \delta^K_S\,\sigma_L^{\dot{M}N}\,\Upsilon_{iN\dot{P}}\right)\right]\sigma_\ell^{S\dot{P}}\;. \qquad (167)$$

Substituting (167) into (162) we get

$$G = \int d_3x\,\varepsilon^{ijk}\,v_k\,A_{ij} + \int d_3x\,\omega^i\,B_i\;. \qquad (168)^*$$

Therefore, the six primary spin constraints given by the relations (104) and (105) are the integrands of the functional G which generates the unimodular spin transformations. Similar results

* Note that the Eq. (162) can be written as,

$$G = \int d_3x\,\mathrm{Tr}(\pi^k\,\delta\,\sigma_k)\;.$$

Using (167) we get

$$\mathrm{Tr}(\pi^k\,\delta\,\sigma_k) = \mathrm{Tr}\,(\pi^k\,V\,\sigma_k + \sigma_k\,V^\dagger\,\pi^k)\;,$$

where we have made a cyclic permutation in the last factor. From this relation we get Eq. (168).

hold for the secondary Hamiltonian constraints with respect to coordinate transformations.[7]

The ten generator densities H_L, H_r, A_{ij}, and B_i are all first class among themselves. If we consider the Hamiltonian given by $\widetilde{H}$ (Eq. (138)), we can define an observable as a functional of the $\tilde{\pi}^i$, σ_i which has a null Poisson bracket with $\widetilde{H}$ on the grounds that $\widetilde{H}$ is a linear combination of all the ten generators of the invariance group of the theory.

Given some functional of the dynamical variables, say F, it is possible to construct an observable F^* by adding to F the necessary coordinate conditions,[5] as well as by choosing a spin frame; these conditions have the effect of fixing the arbitrary coefficients of the integrand of $\widetilde{\mathcal{H}}$; the F^* is obtained by adding to F a linear combination of all the constraints (if we impose the ten conditions fixing coordinates and spin frame, we shall have in all a set of twenty constraints which have nonvanishing Poisson brackets among themselves), so that the F^* commutes with all of them.

VI. THE POISSON BRACKET FOR POINTS ON DIFFERENT HYPERSURFACES, TO FIRST ORDER.

In this section we intend to give an application of the above method; namely the calculation of the classical commutator for infinitesimal separated points, both in space and in time; the relations which will be obtained in this section can be considered as a generalization of the same-time classical commutators derived previously (Eqs. (157) through (159)). Since the method employed in the derivation of these different-time commutators is essentially a power series expansion in the time separation, an approximation procedure is in order, which is also unavoidable because of the growing complexity in the calculations. We will perform the computation of these generalized Poisson brackets up to the first order in the time separation; higher-order terms can be included by a straightforward but otherwise rather lengthy calculation.

As our prototype we shall deal with the Poisson bracket given by Eq. (159). Let us introduce a function

$$f^{K\mathring{M},\ell}_{\dot{R}S,i}(x,x') = \left[\sigma_i{}^{K\dot{M}}(x'),\ \pi^{\ell}_{\dot{R}S}(x)\right]_{x'^{o} = x^{o} + \Upsilon} . \qquad (169)$$

Eqs. (157) and (158) may be generalized by the same technique. Υ is a first-order infinitesimal. For future reference we shall denote the Poisson brackets of (157) and (158) by the respective symbols

$$j^{jk}_{\dot{R}S,P\dot{V}} \quad \text{and} \quad m^{R\dot{S},P\dot{V}}_{jk} ,$$

$$j^{jk}_{R\dot{S},P\dot{V}}(x,x') = \left[\pi^{j}_{R\dot{S}}(x'),\ \pi^{k}_{P\dot{V}}(x)\right]_{x'^{o} = x^{o} + \Upsilon} , \qquad (170)$$

$$m^{R\dot{S},P\dot{V}}_{jk}(x,x') = \left[\sigma_j{}^{R\dot{S}}(x'),\ \sigma_k{}^{P\dot{V}}(x)\right]_{x'^{o} = x^{o} + \Upsilon} . \qquad (171)$$

Up to the first order in Υ, we have:

$$\sigma_i{}^{K\dot{M}}(\vec{x},\ x^{o} + \Upsilon) = \sigma_i{}^{K\dot{M}}(\vec{x},\ x^{o}) + \Upsilon\left(\frac{\partial \sigma_i{}^{K\dot{M}}}{\partial x^{o}}\right)_{\Upsilon=0} .$$

At the point $\vec{x}$ of the hypersurface x^o = constant the time derivative of σ_i equals its Poisson bracket with the Hamiltonian. Then,

$$\left[\sigma_i{}^{K\dot{M}}(\vec{x}', x'^o), \pi^{\ell}_{\dot{R}S}(\vec{x}, x^o)\right] = \left[\sigma_i{}^{K\dot{M}}(\vec{x}', x^o), \pi^{\ell}_{\dot{R}S}(\vec{x}, x^o)\right] +$$
$$+ \tau\left[\left[\sigma_i{}^{K\dot{M}}(\vec{x}', x^o), \mathcal{H}\right], \pi^{\ell}_{\dot{R}S}(\vec{x}, x^o)\right] .$$

In the expressions which will follow we shall take the expression given by Eq. (129) as the Hamiltonian, that is, we shall restrict ourselves to points on the spin constraint hypersurface (equivalently, we shall consider π^i as given by the relation (80) which we have seen is a solution of the six constraint relations). This restriction will simplify the results of the present calculation; besides we achieve correspondence with Dirac's theory when we adopt the spin constraints of the theory. Hence we are not losing any important result by taking this simplification.

The expressions for the quantities f, j, and m have the form

$$f^{K\dot{M},\ell}_{\dot{R}S,i} = \delta^{\ell}_{i}\delta^{K}_{S}\delta^{\dot{M}}_{\dot{R}}\delta(\vec{x},\vec{x}') + \tau\left[\left[\sigma_i{}^{K\dot{M}}(\vec{x}', x^o), \mathcal{H}\right], \pi^{\ell}_{\dot{R}S}(\vec{x}, x^o)\right], \qquad (172)$$

$$j^{jk}_{\dot{R}\dot{S},P\dot{V}} = \tau\left[\left[\pi^{j}_{\dot{R}\dot{S}}(\vec{x}', x^o), \mathcal{H}\right], \pi^{k}_{P\dot{V}}(\vec{x}, x^o)\right], \qquad (173)$$

$$m^{\dot{R}\dot{S},P\dot{V}}_{jk} = \tau\left[\left[\sigma_j{}^{\dot{R}\dot{S}}(\vec{x}', x^o), \mathcal{H}\right], \sigma_k{}^{P\dot{V}}(\vec{x}, x^o)\right] . \qquad (174)$$

We shall give first a simple example of the type of computation to be performed, consider a scalar real meson field in the framework of the Lorentz covariant field theory. The expressions for the Lagrangian, Hamiltonian, and the quantities f, j, and m are,

$$\mathcal{L} = \int d_3x \; L(x) \; ,$$

$$L(x) = \frac{1}{2} \overset{\circ}{g}{}^{\mu\nu} \phi_{,\mu} \phi_{,\nu} - \frac{1}{2} m^2 \phi^2 \; ,$$

$$\pi = \frac{\partial L}{\partial \dot{\phi}} = \dot{\phi} \; ,$$

$$\mathcal{H} = \int d_3x \; H(x) \; ,$$

$$H(x) = \frac{1}{2} \pi^2 + \frac{1}{2} m^2 \phi^2 + \frac{1}{2} (\nabla\phi)^2 \; .$$

The fundamental Poisson brackets are

$$\left[\phi(\vec{x}', x^0), \pi(\vec{x}, x^0)\right] = \delta(\vec{x} - \vec{x}') \; , \qquad (175.1)$$

$$\left[\phi(\vec{x}', x^0), \phi(\vec{x}', x^0)\right] = 0 \; , \qquad (175.2)$$

$$\left[\pi(\vec{x}', x^0), \pi(\vec{x}, x^0)\right] = 0 \; . \qquad (175.3)$$

The variables f, j, and m turn out to be, to the first order in Υ,

$$f(x, x') = \delta(\vec{x}, \vec{x}') \; , \qquad (176.1)$$

$$j(x, x') = \Upsilon \left\{ m^2 \; \delta(x, x') + \vec{\nabla}^2 \; \delta(\vec{x}, \vec{x}') \right\} \; , \qquad (176.2)$$

$$m(x, x') = - \Upsilon \, \delta(\vec{x}, \vec{x}') \; . \qquad (176.3)$$

Thus the different-time classical commutator between two field variables ϕ, namely, the quantity $m(x, x')$, has a first-order term different from zero; this result, well known in field theory, is the same as the one that occurs when we pass from the three-dimensional delta function standing on the right side of Eq. (175.1) to the so-called invariant function $\Delta(x, x')$; to obtain Δ completely we should have to determine all the terms of the power series expansion.

The remaining considerations of this section will be related to the computation of the quantity f of Eq. (172).

On the right-hand side of this equation all operations involve only the usual Poisson brackets (or same-time classical commutators). We have

$$\left[\sigma_i^{K\dot{M}}(\vec{x}, x^0), \mathcal{H}\right] = \int d_3y\ \ell_0(\vec{y}, x^0)\left[\sigma_i^{K\dot{M}}(\vec{x}, x^0),\ H_L(\vec{y}, x^0)\right]$$
$$- \int d_3y\ \ell_0(\vec{y}, x^0)\ \ell^s(\vec{y}, x^0)\left[\sigma_i^{K\dot{M}}(\vec{x}, x^0),\ H_s(\vec{y}, x^0)\right]$$

Using (159), (123) and (160) we obtain:

$$\left[\sigma_i^{K\dot{M}}(\vec{x}, x^0),\ H_L(\vec{y}, x^0)\right] = - K^{-1}(\vec{y}, x^0)\left\{\pi^{j\ \dot{M}K}(\vec{y}, x^0)\sigma_{j\ V\dot{T}}(\vec{y}, x^0)\ .\right.$$
$$\left.\cdot\ \sigma_i^{V\dot{T}}(\vec{y}, x^0) + \tfrac{1}{2}\ \sigma_j^{S\dot{R}}(\vec{y}, x^0)\ \sigma_i^{\dot{M}K}(\vec{y}, x^0)\ \pi^{j}_{\ \dot{S}R}(\vec{y}, x^0)\right\}\ \delta(\vec{x}, \vec{y})\ . \quad (177)$$

By (159) and (120- we have:

$$\left[\sigma_i^{K\dot{M}}(\vec{x}, x^0),\ H_s(\vec{y}, x^0)\right] = \left\{\sigma^{K\dot{M}}_{i,s}(\vec{y}, x^0)\ -\right.$$
$$\left. - \sigma^{K\dot{M}}_{s,i}(y, x^0)\right\}\ \delta(\vec{x}, \vec{y}) - \sigma_s^{K\dot{M}}(\vec{y}, x^0)\ \delta_{,i}(\vec{x}, \vec{y}). \quad (178)$$

Hence the commutator between σ_i and the Hamiltonian becomes:

$$\left[\sigma_i(\vec{x}, x^0), \mathcal{H}\right] = - K^{-1}(\vec{x}, x^0)\ \ell_0(\vec{x}, x^0)\left\{\pi^{j\ \dot{M}K}(\vec{x}, x^0)\ .\right.$$
$$\left.\sigma_{j\ \dot{V}T}(\vec{x}, x^0)\ \sigma_i^{\dot{V}T}(\vec{x}, x^0) + \tfrac{1}{2}\ \sigma_j^{\dot{S}R}(\vec{x}, x^0)\ \sigma_i^{\dot{M}K}(\vec{x}, x^0)\ \pi^{j}_{\ \dot{S}R}(\vec{x}, x^0)\right\}$$
$$- \ell_0(\vec{x}, x^0)\ \ell^s(\vec{x}, x^0)\left\{\sigma^{K\dot{M}}_{i,s}(\vec{x}, x^0) - \sigma^{K\dot{M}}_{s,i}(\vec{x}, x^0)\right\}$$
$$+ \int d_3y\ \ell_0(\vec{y}, x^0)\ \ell^s(\vec{y}, x^0)\ \sigma_s^{K\dot{M}}(\vec{y}, x^0)\ \delta_{,i}(\vec{x}, \vec{y})\ . \quad (179)$$

Taking the Poisson bracket of this expression with the momenta π^ℓ we obtain:

$$\left[\left[\left[\sigma_i^{K\dot{M}}(\vec{x'}, x^o), \mathcal{H}\right]\right], \pi^{\ell}_{\dot{R}S}(\vec{x}, x^o)\right] = K^{-1}\ell_o \, \pi^{\ell, \dot{M}K} \, \sigma_{i\dot{R}S} \, \delta(\vec{x}, \vec{x'}) -$$

$$- K^{-1} \, \ell_o \, \pi^{j \, \dot{M}K} \, \sigma_{j \, \dot{R}S} \, \delta^{\ell}_{i} \, \delta(\vec{x}, \vec{x'}) - \tfrac{1}{2} K^{-1} \, \ell_o \, \pi^{\ell}_{\dot{R}S} \, \sigma_i^{\dot{M}K} \, \delta(\vec{x}, \vec{x'}) -$$

$$- \tfrac{1}{2} K^{-1} \, \ell_o \, \pi^{j}_{\dot{V}T} \, \sigma_j^{\dot{V}T} \delta^{\ell}_{i} \, \delta^{\dot{M}}_{\dot{R}} \, \delta^{K}_{S} \, \delta(\vec{x}, \vec{x'}) - 2 \, \ell_o \, \ell^s \delta^{K}_{S} \delta^{\dot{M}}_{\dot{R}} \, \delta_{,s}(\vec{x}, \vec{x'})$$

$$+ 4 \, \ell_o \, \ell^{\ell} \, \delta^{K}_{S} \, \delta^{\dot{M}}_{\dot{R}} \, \delta_{,i}(\vec{x}, \vec{x'}) + B^{\ell, K\dot{M}}_{i, \dot{R}S}(\vec{x}, \vec{x'}; x^o) \; . \qquad (180)$$

The factor $B^{l}_{i}, {}^{K\dot{M}}_{\dot{R}S}$ in Eq. (180) refers to the additional Poisson bracket which depends on the arbitrary variable σ_o. Since in Eq. (180) some of these variables are also present as coefficients (the ℓ_o for instance), all terms in this equation are equally arbitrary. Therefore, the above decomposition is only a matter of convenience and has no further meaning.

We have:

$$B^{\ell, K\dot{M}}_{i, \dot{R}S} = - \pi^{j, \dot{M}K} \, \sigma_{j \, \dot{V}T} \, \sigma_i^{\dot{V}T} \left[K^{-1'} \, \ell_o', \pi^{\ell}_{\dot{R}S}\right] -$$

$$- \tfrac{1}{2} \, \sigma_j^{\dot{V}T} \, \sigma_i^{\dot{M}K} \, \pi^{j}_{\dot{V}T} \left[K^{-1'} \ell_o', \pi^{\ell}_{\dot{R}S}\right] -$$

$$- \left\{\sigma^{K\dot{M}}_{i,s} - \sigma^{K\dot{M}}_{s,i}\right\} \left[\ell_o' \, \ell'^s, \pi^{\ell}_{\dot{R}S}\right] +$$

$$+ \int d_3y \, \sigma_s^{K\dot{M}} \, \delta_{,i}(\vec{y}, \vec{x'}) \left[\ell_o(\vec{y}, x^o) \, \ell^s(\vec{y}, x^o), \pi^{\ell}_{\dot{R}S}(\vec{x}, x^o)\right] . \qquad (181)$$

In this expression ℓ' means the value of ℓ taken at $x = x'$. All the time coordinates x_o have the same value.

Now using Eq. (54) we get:

$$\left[K^{-1}(\vec{x}', x^{o}), \pi^{\ell}_{\dot{R}S}(\vec{x}, x^{o})\right] = \frac{1}{4!} (-|g_{rs}|)^{-3/2} \varepsilon^{mnq} \varepsilon^{prv} \delta(\vec{x}, \vec{x'}) \cdot$$

$$\cdot\Big\{\delta^{\ell}_{q} \sigma_{pU\dot{X}} \sigma_{rN\dot{L}} \sigma_{vS\dot{R}} \sigma_{m}{}^{U\dot{X}} \sigma_{n}{}^{\dot{L}N} + \delta^{\ell}_{n} \sigma_{pU\dot{X}} \sigma_{rS\dot{R}} \sigma_{vH\dot{F}} \sigma_{m}{}^{U\dot{X}} \sigma_{q}{}^{H\dot{F}} +$$

$$+ \delta^{\ell}_{m} \sigma_{pS\dot{R}} \sigma_{rN\dot{L}} \sigma_{vH\dot{F}} \sigma_{n}{}^{\dot{L}N} \sigma_{q}{}^{H\dot{F}} + \delta^{\ell}_{v} \sigma_{pU\dot{X}} \sigma_{rN\dot{L}} \sigma_{m}{}^{U\dot{X}} \sigma_{n}{}^{N\dot{L}} \sigma_{qS\dot{R}} +$$

$$+ \delta^{\ell}_{r} \sigma_{pU\dot{X}} \sigma_{vH\dot{F}} \sigma_{m}{}^{U\dot{X}} \sigma_{n\dot{R}S} \sigma_{q}{}^{H\dot{F}} + \delta^{\ell}_{p} \sigma_{rN\dot{L}} \sigma_{vH\dot{F}} \sigma_{m\dot{R}S} \sigma_{n}{}^{N\dot{L}} \sigma_{q}{}^{H\dot{F}}\Big\} . \tag{182}$$

From the relation

$$\lambda^{\mu} = -\frac{1}{2} \mathrm{Tr}(\tau^{\mu} \sigma_{L})$$

we have

$$\ell_{o} = \frac{1}{2} \sigma_{oK\dot{V}} \sigma_{L}{}^{K\dot{V}} ,$$

hence

$$\left[\ell'_{o}, \pi^{\ell}_{\mathring{R}S}\right] = \frac{1}{2} \sigma'_{oK\mathring{V}} \left[\sigma'_{L}{}^{K\dot{V}}, \pi^{\ell}_{\dot{R}S}\right] . \tag{183}$$

Using Eq. (53), we have

$$\left[\sigma_{L}{}^{K\dot{M}}(\vec{x}', x^{o}), \pi^{\ell}_{\mathring{R}S}(\vec{x}, x^{o})\right] = -\frac{i}{3!} \left(-|g_{rs}|\right)^{-\frac{1}{2}} \varepsilon^{ijk} \Big\{\delta^{\ell}_{k}\delta^{\dot{M}}_{\dot{R}} \sigma_{i}{}^{K\dot{P}} \sigma_{j\dot{P}S}$$

$$+ \delta^{\ell}_{j} \sigma_{i}{}^{K\dot{P}} \sigma_{k}{}^{V\dot{M}} \epsilon_{\dot{R}\dot{P}} \epsilon_{SV} + \delta^{\ell}_{i} \sigma_{j\dot{R}V} \sigma_{k}{}^{V\dot{M}} \delta^{K}_{s}\Big\} \delta(\vec{x}, \vec{x'}) -$$

$$- \frac{i}{3!\,4!} (-|g_{rs}|)^{-3/2} \varepsilon^{ijk} \varepsilon^{mnq} \varepsilon^{prv} \sigma_{i}{}^{K\dot{P}} \sigma_{j\mathring{P}V} \sigma_{k}{}^{V\dot{M}} \Big\{\delta^{\ell}_{q} \sigma_{pU\dot{X}} \cdot$$

$$\cdot \sigma_{rN\dot{L}} \sigma_{vS\dot{R}} \sigma_{m}{}^{U\dot{X}} \sigma_{n}{}^{N\dot{L}} + \delta^{\ell}_{n} \sigma_{pU\dot{X}} \sigma_{rS\dot{R}} \sigma_{vH\dot{F}} \sigma_{m}{}^{U\dot{X}} \sigma_{q}{}^{H\dot{F}} +$$

$$+ \delta^{\ell}_{m} \sigma_{pS\dot{R}} \sigma_{rN\dot{L}} \sigma_{vH\dot{F}} \sigma_{n}{}^{N\dot{L}} \sigma_{q}{}^{H\dot{F}} + \delta^{\ell}_{v} \sigma_{pU\dot{X}} \sigma_{rN\dot{L}} \sigma_{m}{}^{U\dot{X}} \sigma_{n}{}^{N\dot{L}} \sigma_{q\dot{R}S} +$$

$$+ \delta^{\ell}_{r} \sigma_{pU\dot{X}} \sigma_{vH\dot{F}} \sigma_{m}{}^{U\dot{X}} \sigma_{n\dot{R}S} \sigma_{q}{}^{H\dot{F}} + p\, \sigma_{rN\dot{L}} \sigma_{vH\dot{F}} \sigma_{m\dot{R}S} \sigma_{n}{}^{N\dot{L}} \sigma_{q}{}^{H\dot{F}}\Big\} \delta(\vec{x}, \vec{x'}). \tag{184}$$

From the relations

$$-\ell_o \ell^s = g_{or}\ e^{rs}\ , \qquad (185.1)$$

$$g_{or} = \frac{1}{2}\ \sigma_{rK\dot{V}}\ \sigma_o^{\ K\dot{V}}\ , \qquad (185.2)$$

$$\left[e^{i\ell}(x),\ \psi\right] = -\ e^{ij}(x)\left[g_{jk}(x),\ \psi\right]e^{k\ell}(x)\ , \qquad (185.3)$$

$$g_{jk} = \frac{1}{2}\ \sigma_{jK\dot{V}}\ \sigma_k^{\ K\dot{V}}\ , \qquad (185.4)$$

(where (185.3) is a consequence of the relation (38) along with the properties of the Poisson bracket) we can calculate

$$\left[\ell_o\ \ell^s,\ \pi'^{\ell}_{\dot{R}S}\right]\ . \qquad (186)$$

Hence all Poisson brackets present in the Eq. (181) can be written down by the use of Eqs. (182), (183), (184) and (185). The integral present in the last term of (181) disappears at the end of the calculations, since the Poisson bracket of Eq. (186) is proportional to the Dirac delta function $\delta(\vec{x},\ \vec{x}')$.

Thus we have completed the calculation up to first order of the different-time classical commutator given by the relation (169) (or equivalently by (172)). The Poisson brackets of the relations (173) and (174) can be obtained similarly.

VII. APPENDIX

THE HAMILTONIAN FORMULATION OF THE ELECTROMAGNETIC FIELD IN THE FRAMEWORK OF LORENTZ-COVARIANT FIELD THEORY

Since this formulation is well known, we shall keep our exposition brief. Its purpose is to clarify some of the aspects of the corresponding theory for the gravitational field. The similarity between these two theories arises from the fact that both are invariant with respect to well-defined function groups; the Lagrangian of the Electromagnetic field is invariant with respect to the group of gauge transformations, whereas the Lagrangian of Einstein's gravitational field theory is invariant with respect to arbitrary transformations of the four coordinates. The difference between these two groups is, for our present purposes primarily a difference in the number of arbitrary functions involved in the corresponding transformations.

All results relating to the appearance of the constraints are formally similar in the two theories, and this is just what we propose to show.

The Lagrangian density for the Maxwell field is *

$$L = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - j_\mu A^\mu = L_f + L_i .$$

Indices are raised and lowered by $\overset{0}{g}{}^{\mu\nu}$ and $\overset{0}{g}_{\mu\nu}$, since we have a Lorentz-covariant theory.

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* Here we use the Heaviside system of units.

$$F^{\mu\nu} = \frac{\partial A^{\mu}}{\partial x_{\nu}} - \frac{\partial A^{\nu}}{\partial x_{\mu}} ,$$

$$F_{\mu\nu} = \overset{\circ}{g}_{\mu\lambda} \, \overset{\circ}{g}_{\nu\rho} \, F^{\lambda\rho} ,$$

$$L_f = -\frac{1}{4} F_{\mu\nu} \, F^{\mu\nu} .$$

The free Lagrangian L_f is invariant under the gauge transformation of the potentials A^{μ},

$$A'^{\mu} = A^{\mu} + \frac{\partial \Lambda}{\partial x_{\mu}} ,$$

Λ being an arbitrary function of the four coordinates. The interaction term L_i depends on the type of fields which form the four-vector current density j_{μ}. If we consider a Dirac field, they are:

$$j_{\mu} = \bar{\psi} \gamma_{\mu} \psi , \; \bar{\psi} = \psi^{\dagger} \gamma_{0} .$$

Only the total Lagrangian density,

$$L_T = L_f + L_i + L_e ,$$

(where L_e is the Lagrangian for the external field interacting with the Maxwell field) is gauge-invariant (beside L_f, which is clearly gauge-invariant). This invariance obviously refers to the simultaneous gauge transformation of A^{μ} and of the given external field. As an example, if we take the Dirac field as the external field, we can define the simultaneous transformations.

$$\psi' = \psi \, e^{-i\Lambda} ,$$

$$\bar{\psi}' = \bar{\psi} \, e^{i\Lambda} ,$$

$$A'^{\mu} = A^{\mu} + \frac{\partial \Lambda}{\partial x_{\mu}} ,$$

under which L_T is invariant.

We define,

$$\frac{\partial}{\partial x^\mu} = \left(\partial_t, \nabla\right), \quad \frac{\partial}{\partial x_\mu} = \left(\partial_t, -\nabla\right),$$

so that,

$$\vec{A}' = (A'^i) = \vec{A} - \nabla\Lambda; \; \phi' = A'_0 = \phi + \dot{\Lambda}.$$

The Maxwell equations obtained from the variation of L with respect to A^μ are,

$$\frac{\partial F^{\mu\nu}}{\partial x^\nu} = j^\mu,$$

which are equivalent to four separated D'Alembert equations plus the Lorentz condition on the potentials,

$$\Box A^\mu = j^\mu; \quad \frac{\partial A^\nu}{\partial x^\nu} = 0.$$

The term L_e will not contribute to the expression of the canonical momenta for the Maxwell field:

$$L = \frac{1}{2}(\dot{\vec{A}} + \nabla\phi)^2 - \frac{1}{2}(\nabla \times \vec{A})^2 - \rho\phi + \vec{j}\cdot\vec{A},$$

$$\vec{p} = \frac{\partial L}{\partial \dot{\vec{A}}} = \dot{\vec{A}} + \nabla\phi = -\vec{E},$$

$$\pi = \frac{\partial L}{\partial \dot{\phi}} = 0.$$

The relation $\pi = 0$ is the primary constraint. Its appearance is related to the gauge invariance of the theory; indeed, we note that with $\dot{\vec{A}}$ we can form the gauge-invariant expression $\dot{\vec{A}} + \nabla\phi$, but with ϕ we can form only the expression $\dot{\phi} + \nabla\cdot\vec{A}$ which is the left-hand side of the Lorentz condition, and which transforms as follows:

$$\dot{\phi}' + \nabla\cdot\vec{A}' = \dot{\phi} + \nabla\cdot\vec{A} + \Box\Lambda.$$

This expression, then, is gauge invariant if we restrict the transformation function Λ to be a solution of D'Alembert's equation,

$$\Box\Lambda = 0 .$$

but this condition also characterizes the gauge transformations that preserve the Lorentz condition itself,

$$\dot{\phi} + \nabla\cdot\vec{A} = A^{\mu}_{,\mu} = 0 .$$

Hence we cannot form any nonvanishing gauge-invariant expression with $\dot{\phi}$. Hence, L_f must be independent of $\dot{\phi}$ (gauge invariance of L_f holds).

The equivalent situation in the case of the gravitational field is given by the primary constraints $p^{0\mu} = 0$, which are related to the invariance of Einstein's Lagrangian density under arbitrary transformation of the four coordinates.

The Hamiltonian for the Maxwell field is

$$\mathcal{H} = \int H \, d_3x ,$$

$$H = \vec{p}\cdot\dot{\vec{A}} - L = \frac{1}{2}\vec{p}^{\,2} + \frac{1}{2}(\nabla\times\vec{A})^2 - \vec{j}\cdot\vec{A} + \phi\,(\nabla\cdot\vec{p} + \rho) .$$

We can verify that the constraint $\pi = 0$ is not the only constraint of the theory. Indeed, let us examine the time derivative of π,

$$\dot{\pi}(x) = [\pi(x), \mathcal{H}] = \int d_3y \, [\pi(x), H(y)] ,$$

which must vanish if the primary constraint it to be maintained in the course of time. * Nevertheless, this Poisson bracket does not vanish automatically; rather, we find:

$$\dot{\pi} = \nabla \cdot \vec{p} + \rho \ ,$$

thus, if we require that the primary constraint is maintained in the course of time, we have

$$\nabla \cdot \vec{p} + \rho = 0 \ ,$$

which is an additional constraint in the theory; this constraint is called a secondary constraint.

From a different point of view, we can say that some of the field equations written in terms of the canonical variables will represent constraints. In the electromagnetic theory there is only one equation with this behaviour,

$$- \nabla \cdot \vec{E} + \rho = 0 \ .$$

(since $p = - \vec{E}$).

There are no additional constraints in the theory; the secondary constraint is automatically maintained at all times. For proof we write out explicitly the canonical Poisson brackets:

$$\left[p^i(x), A_j(y)\right]_{x^o = y^o} = \delta^i_j \ \delta(\vec{x} - \vec{y}) \ ,$$

$$\left[p^i(x), p^j(y)\right]_{x^o = y^o} = 0 \ ,$$

---

* Take $t' = t + \lambda$. A power series expansion to the first order in λ gives:

$$\pi(t') = \pi(t) + \lambda \dot{\pi}(t) = \pi(t) + \lambda \left[\pi(t), \mathcal{H}\right] \ .$$

Then, if $\mathring{\pi}(t) = 0$, we have $\pi(t') = \pi(t)$.

$$\left[A_i(x), A_j(y)\right]_{x^0 = y^0} = 0 .$$

Then, taking the time derivative of the secondary constraint, we obtain:

$$S \equiv \nabla\cdot\vec{p} + \rho ,$$

$$\frac{dS}{dt} = \nabla\cdot\frac{d\vec{p}}{dt} + \frac{\partial\rho}{\partial t} ,$$

for the first term we get, by using the above fundamental Poisson brackets:

$$\nabla\cdot\frac{d\vec{p}}{dt} = \nabla\cdot[\vec{p}, \mathcal{H}] = \nabla\cdot\vec{j} . *$$

Thus, the secondary constraint will be maintained at all times if the law of conservation of the four-vector of current j^{μ} holds,

$$\dot{S} = \nabla\cdot\vec{j} + \frac{\partial\rho}{\partial t} = 0 .$$

The Hamiltonian has a term which represents the product of the secondary constraint by the scalar potential ϕ. That this potential is not gauge-invariant implies that the Hamiltonian is not identically invariant with respect to gauge transformations. If we take a free electromagnetic field (which is more appropriate for comparisons with the free gravitational field considered in this paper), the Hamiltonian,

$$H = \frac{1}{2}\vec{p}^{\,2} + \frac{1}{2}(\nabla\times\vec{A})^2 + \phi\,\nabla\cdot\vec{p} ,$$

is not invariant with respect to gauge transformations; however, it reduces to the usual gauge-invariant energy density on the constraint hypersurface.

We also note that the values of ϕ and $\dot{\phi}$ at a given time t_o

* We denote by $\vec{j}$ the components j^i, so that the term $-\vec{j}\cdot\vec{A}$ is equal to $-j^iA^i$, or equivalently to $+j^iA_i$. We have used this later form in the previous calculation.

do not determine ϕ for all later times,

$$\phi(t) = \phi(t_o) + \lambda\dot{\phi}(t_o) \ ,$$

with $t = t_o + \lambda$, where λ is infinitesimal of the first order. This is because ϕ is arbitrary up to a function $\dot{\Lambda}$; hence, it remains undetermined at the time t.

All these considerations possess their analogs in the theory of the gravitational field, as was made clear in Section III.

The secondary constraint can be solved easily if we perform the decomposition

$$\vec{E} = \vec{E}_T + \vec{E}_L \ ,$$
$$\nabla \cdot \vec{E}_T = 0 \ ,$$
$$\nabla \times \vec{E}_L = 0 \ ,$$

and the same for $\vec{A}$; since the constraint equation now reads as,

$$\nabla \cdot \vec{E}_L = 0 \ ,$$

we conclude that $\vec{E}_L = 0$,* and the Hamiltonian takes the form,

$$H = \frac{1}{2} \vec{E}_T^2 + \frac{1}{2} (\nabla \times \vec{A})^2 \ ,$$

the canonical pair of variables, after the constraint has been solved for, are $-\vec{E}_T$ and $\vec{A}_T$. The remaining part of $\vec{A}$, $\vec{A}_L$, is still arbitrary; indeed, it is easy to see that $\vec{A}_T$ is gauge-invariant, but $\vec{A}_L$ it is not. Under a gauge transformation the transverse component remains unchanged, whereas the longitudinal part $\vec{A}_L$ changes like $\vec{A}$,

---

* **If $\rho \neq 0$, the $\vec{p}_L = -\vec{E}_L$ is given entirely as a function of ρ; symbolically we may write, $\vec{p}_L = -\nabla\left(\frac{1}{\nabla^2}\right)\rho$ where ∇^{-2} is the Coulomb integral operator.**

$$\vec{A}'_T = \vec{A}_T ,$$

$$\vec{A}'_L = \vec{A}_L - \nabla\Lambda .$$

Therefore, we need to choose a condition on Λ in order to fix the value of $\vec{A}_L$; a choice being, for instance, $\vec{A}'_L = 0$, which is obtained if we impose on Λ the condition,

$$\vec{A}_L - \nabla\Lambda = 0 .$$

To obtain the independent canonical variables $\vec{P}_T$, $\vec{A}_T$ we not only solved the constraint equation, but also chose a gauge condition so as to fix the remaining unphysical variable $\vec{A}_L$. Solution of the constraint equation also fixes the variable $\vec{P}_L$, which is physical but redundant.

A comparison with the gravitational field shows that here the $\vec{A}_L$ play a similar role as the ℓ^{μ} (or $g_{o\mu}$).

* * *

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