Aspects of Global and Conformal SuperSymmetry in One Dimension

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To My Family:

Abdul-Rahman *and* ***Toobaa***,

Abdul-Hamid *and* ***Chonoor***; ***Ghomri*** *and* ***Gharib***; ***Najmaddin*** *and* ***Banaaz***,

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Abstract

After a short introduction to the global supersymmetry in one dimension, presenting the basic definitions and tools, we construct the non-minimal linear representations of the $\mathcal{N}=4$ Extended Supersymmetry in one-dimension. They act on 8 bosonic and 8 fermionic fields. Inequivalent representations are specified by the mass-dimension of the fields, the connectivity of the associated graphs, commuting group and node choice group. Due to the important role that graphical representation plays for this classification, we investigate the correspondence between graphical representation, and algebraic representations. The distinction between the notion of equivalence for pure supermultiplets and the notion of equivalence for their associated graphs (Adinkras) is discussed.

"Pure" homogeneous linear supermultiplets (minimal and non-minimal) of the $\mathcal{N}=4$-Extended one-dimensional Supersymmetry Algebra are classified. "Pure" means that they admit at least one graphical presentation (the corresponding graph/graphs are known as "Adinkras").

Discrete properties such as "chirality" and "coloring" can discriminate different supermultiplets. The tools used in our classification include, among others, the notion of *field content*, *connectivity symbol*, *commuting group*, *node choice group* and so on. The oxidation to minimal $\mathcal{N}=5$ linear representations are given.

We further prove the existence of "entangled" linear supermultiplets which do not admit a graphical presentation, by constructing an explicit example of an entangled $\mathcal{N}=4$ supermultiplet with field content $(3,8,5)$. It interpolates between two inequivalent pure $\mathcal{N}=4$ supermultiplets with the same field content. The one-dimensional $\mathcal{N}=4$ sigma-

model with a three-dimensional target based on the entangled supermultiplet is presented.

Two types of $\mathcal{N}=4$ σ-models based on non-minimal representations are obtained: the resulting off-shell actions are either manifestly invariant or depend on a constrained prepotential. The connectivity properties of the graphs play a decisive role in discriminating inequivalent actions. These results find application in partial breaking of supersymmetric theories.

Based on minimal representation of the global $\mathcal{N}$-Extended one-dimensional Supersymmetry algebra we construct D-module representations of superconformal algebras in one dimension. We found that at critical values of the scaling dimension λ, these representations induce D-module representations of finite superconformal algebras (the latters being identified in terms of the global supermultiplet and its critical scaling dimension).

For $\mathcal{N}=4,8$ and global supermultiplets $(k,\mathcal{N},\mathcal{N}-k)$, the exceptional superalgebras $D(2,1;\alpha)$ are recovered for $\mathcal{N}=4$, with a relation between α and the scaling dimension given by $\alpha=(2-k)\lambda$. For $\mathcal{N}=8$ and $k\neq 4$ all four $\mathcal{N}=8$ finite superconformal algebras are recovered, at the critical values $\lambda_k=\frac{1}{k-4}$, with the following identifications: $D(4,1)$ for $k=0,8$, $F(4)$ for $k=1,7$, $A(3,1)$ for $k=2,6$ and $D(2,2)$ for $k=3,5$.

The $\mathcal{N}=7$ global supermultiplet $(1,7,7,1)$ induces, at $\lambda=-\frac{1}{4}$, a D-module representation of the exceptional superalgebra $G(3)$.

D-module representations are applicable to the construction of superconformal mechanics in a Lagrangian setting. The isomorphism of the $D(2,1;\alpha)$ algebras under an S_3 group action on α, coupled with the relation between α and the scaling dimension λ, induces non-trivial constraints on the admissible models of $\mathcal{N}=4$ superconformal mechanics. The existence of new superconformal models is pointed out. E.g., coupled $(1,4,3)$ and $(3,4,1)$ supermultiplets generate an $\mathcal{N}=4$ superconformal mechanics if λ is related to the golden ratio.

The relation between classical versus quantum D-module representations is presented.

Resumo

Após uma pequena introdução sobre a supersimetria global unidimensional, apresentando as definições e ferramentas básicas, construímos as representações não-mínimais lineares da Supersimetria Estendida $\mathcal{N} = 4$ unidimensional. Eles atuam em campos com 8 bósons e 8 férmions. Representações não equivalentes são especificadas pela dimensão de massa dos campos, a conectividade dos gráficos associados, o grupo comutante e o "node choice group". Devido ao papel importante que desempenha a representação gráfica para esta classificação, investigamos a correspondência entre as representações gráficas e as representações algébricas.

Discutimos a diferença entre a noção de equivalência para supermultipletos puros e a noção de equivalência para seus gráficos associados (Adinkras) .

Classificamos os Supermultipletos lineares homogêneos puros (minimais e não-minimais) da álgebra da supersimetria estendida unidimensional $\mathcal{N} = 4$. O termo Puro significa que eles admitem pelo menos uma representação gráfica (o gráfico correspondente / gráficos são conhecidos como "Adinkras").

Propriedades discretas tais como, "quiralidade" e coloração podem discriminar diferentes supermultipletos.

As ferramentas utilizadas na nossa classificação incluem, entre outros, a noção de *conteúdo do campo*, *símbolo de conectividade*, *grupo comutante*, *node choice group* e outros mais. Mostramos a oxidação minimal para representações lineares $\mathcal{N} = 5$.

Provamos ainda a existência de supermultipletos lineares emaranhados que não admitem uma representação gráfica, através da construção de um exemplo explícito de um

supermultipleto emaranhado $\mathcal{N} = 4$ com componentes do campo $(3, 8, 5)$. Isto interpola entre dois sumermultipletos puros $\mathcal{N} = 4$ não equivalentes com as mesmas componentes do campo. Apresentamos o modelo sigma unidimensional $\mathcal{N} = 4$ com um alvo tridimensional baseado no supermultipleto emaranhado.

São obtidos dois tipos de modelos-σ $\mathcal{N} = 4$ nas representações não-minimais: as ações off-shell resultantes são manifestamente invariante ou dependem de um prepotencial vinculado. As propriedades da conectividade dos gráficos desempenham um papel decisivo em discriminar ações não equivalentes. Estes resultados encontram aplicações em quebra parcial das teorias supersimétricas.

Baseado na representação minimal da álgebra $\mathcal{N}$-estendida supersimétrica unidimensional, construimos representações D-módulos de álgebras superconforme em uma dimensão. Descobrimos que em valores críticos da dimensão de scala λ, essas representações induzem representações D-módulos de álgebras superconforme finitas.

Para $\mathcal{N} = 4, 8$ e supermultipletos globais $(k, \mathcal{N}, \mathcal{N} - k)$, as superálgebras excepcionais $D(2, 1; \alpha)$ são recuperadas por $\mathcal{N} = 4$, com uma relação entre α e a dimensão de escala dada por $\alpha = (2 - k)\lambda$. Para $\mathcal{N} = 8$ e $k \neq 4$ todas as quatro álgebras superconforme finitas $\mathcal{N} = 8$ são recuperadas, para valores críticos $\lambda_k = \frac{1}{k-4}$, com as seguintes identificações: $D(4, 1)$ for $k = 0, 8$, $F(4)$ for $k = 1, 7$, $A(3, 1)$ for $k = 2, 6$ e $D(2, 2)$ for $k = 3, 5$.

O supermultipleto global $\mathcal{N} = 7$ $(1, 7, 7, 1)$ induz, para $\lambda = -\frac{1}{4}$, uma representação D-módulo da superálgebra excepcional $G(3)$.

As representações D-módulos são aplicadas na construção da mecânica superconforme no cenário Lagrangiano. O isomorfismo das álgebras $D(2, 1; \alpha)$ para o grupo S_3 agindo em α, aclopados a relação entre α e a dimensão de escala λ, induz vínculos não triviais no modelos admissível da mecânica superconforme $\mathcal{N} = 4$. A existência de um novo modelo superconforme é apontado para fora. Por exemplo, os supermultipletos $(1, 4, 3)$ e $(3, 4, 1)$ geram uma mecânica superconforme $\mathcal{N} = 4$ si λ está relacionada a razão de ouro .

Apresentamos a relação entre as representações D-módulo clássicas e quânticas.

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Chapter 1

Introduction

Supersymmetry is the supreme symmetry: from the viewpoint of a unification theory, it unifies space-time symmetries with internal symmetries, accommodate bosons and fermions within a multiplet and (for supergravity) gravity with matter. Under quite general assumptions it is the largest possible symmetry of the S-matrix. In the sixties it was believed that it was possible to find a symmetry with particles with different spins on the same multiplet. In the year 1967, Sidney Coleman and Jeffrey Mandula published a paper on "All Possible Symmetries of the S Matrix" containing their famous "Coleman-Mandula no-go theorem" stating that "space-time and internal symmetries cannot be combined in any but a trivial way" and the only conserved quantities apart from the generators of the Poincaré group must be Lorentz scalars [1]. However, each no-go theorem can be applied if all its hypotheses are satisfied. The Coleman-Mandula theorem starts from the following assumptions:

1) the S-matrix is based on local, relativistic quantum field theory in four-dimensional spacetime;

2) there is only a finite number of different particles associated with one-particle states of a given mass;

3) there is an energy gap between the vacuum and the one particle states.

Loosening up some hypotheses there are options to bypass the theorem. In the year 1975, Haag, Lopuszanski and Sohnius investigated “All possible generators of supersymmetries of the S-matrix”, mentioning that supersymmetry may be considered a possible “loophole” of the Coleman-Mandula theorem because it contains additional generators (supercharges) that are not scalars but rather spinors [2]. This loophole is possible because supersymmetry is a Lie Superalgebra, not an ordinary Lie algebra. The corresponding theorem for supersymmetric theories with a mass gap is the Haag-Lopuszanski-Sohnius theorem. By relaxing the mass gap assumption, the Lie algebra of symmetries of the S-matrix could be a tensor product of the conformal algebra with an internal Lie algebra.

From the point of view of theoretical physics, supersymmetry is a crucial ingredient in superstring theory. As a symmetry of the two-dimensional “world sheet” of strings it was proposed in the beginning of the 70s by Pierre Ramond, John H. Schwarz and Andre Neveu [3]. Almost at the same time J.L. Gervais and B. Sakita wrote down the first supersymmetric action which was the two-dimensional superstring action (1971) [4]. Golfand and Likhtman extended Poincaré algebra to the Super Poincaré algebra and built the first supersymmetrtic field theory in four-dimension (1971) [5].

Volkov and Akulov found a nonlinear realization of the same supersymmetric algebra that they used to write a geometrical Lagrangian (1972) [6]. Nonlinear realizations play an important role in theories with spontaneously broken symmetries.

In 1973, Wess and Zumino published three works on the building of supersymmetric gauge theories in four dimensions “A Lagrangian Model Invariant Under Supergauge Transformations”, “Supergauge Transformations in Four-Dimensions ” and “Supergauge Invariant Extension of Quantum Electrodynamics ” [7]. After these works, in 1974 Salam and Strathdee, Ferrara and colleagues found the realization of supersymmetry generators on a superspace of coordinates and introduced superfields over it to describe a supersymmetry multiplet. It lead to a rapid development in finding extensions of ordinary field theories and in studying their properties.

Later on at the beginning of the 80s the Minimal supersymmetric standard model was completed. There are many good reasons to believe that the Standard Model (SM) is not the ultimate theory of nature, since it is unable to answer many fundamental questions, among of them the hierarchy problem, why and how the electroweak scale and the Planck scale are so hierarchically ($M_w/M_p \sim 10^{32}$) separated? Renormalization effects will mix these two mass scales, ruining the separation. Even if we use the fine-tuning at any order, the problem remains and one has to perform an infinite number of distinct fine tuning at each order of perturbation theory. Among other attempts such as conformal theories, extra dimensions and braneworld models, one fascinating solution to the hierarchy problem is to include supersymmetry, local and global. This has motivated the Minimal Supersymmetric extension of the Standard Model (MSSM) as the underlying theory at scales of the order of TeV. Since in supersymmetric theories the higher order interactions do not renormalize the mass scale, this means that one fine-tuning at the beginning is enough and we don't have to fine-tune these parameters at each order of perturbation.

Unbroken supersymmetry leads to degeneracy between the spectra of the fermions and bosons in the unified theory. Since this is not observed in nature, we need to have the supersymmetry broken. There are several ways of breaking supersymmetry, including soft breaking and spontaneously breaking of supersymmetry.

Another feature of supersymmetric field theories is that they are less ultraviolet divergent than the corresponding non-supersymmetric theory, due to the miraculous cancellations of the divergences of the fermionic loops with those of the bosonic loops. Models with more than one supersymmetry show even more ultraviolet convergence. For example it has been shown that $N = 4$ super Yang-Mills theory and certain versions of $N = 2$ super Yang-Mills theories are finite theories to all orders, which indicates the power of supersymmetry [8, 9]. In some sense, these theories answer Dirac's old objection to quantum field theory, that "renormalization theory is contrived and artificial".

Supersymmetry had become a natural framework for some unified theories after the discovery of supersymmetric extensions of a theory of gravity, the $N = 1$ supergravity theory, that was proposed in 1976 by Freedman, van Nieuwenhuizen and Ferrara. Going from global supersymmetric theory to local supersymmetries, automatically gives raise to gravitational interaction and leads to the possibility of unifying gravitational interactions with other known interactions[10].

Unlike most of the theories in physics, supersymmetry does not build on a well-understood mathematics, rather it has created its own, rich and truly new mathematics, one of the rare moments, when mathematicians overlooked a beautiful and useful structure and came to appreciate it only at the demand of physicists. Supersymmetry created by the physicists, became important for mathematicians due to its richness and the surprising connections to well established and developed concepts. For mathematicians supersymmetry provides a virtual playground of structures which beg for a rigorous foundation and complete classification. However, from a mathematical standpoint, physical supersymmetry has yet to be fully and properly formulated. This is especially so regarding the classification of off-shell representations of supersymmetry. Unfortunately, the term supersymmetry has come to mean slightly different things to physicists and mathematicians. This has caused some unfortunate mis-communication, which has partially hindered the historic synergy between these respective fields. In mathematics, the term "supersymmetry" is used to describe algebraic structures which possess a Z_2-grading and obey standard sign conventions related to that grading. In physics, the term "supersymmetry" is much more specific, referring to structures which are equivalent with respect to the super Poincaré group, the super Poincaré algebra, or their many variants.

At the present time there is no direct experimental evidence that supersymmetry is a fundamental symmetry of nature. LHC has put stringent constraints on simple model constructions and a large parameter region of the TeV scale for supersymmetric models has been already excluded [12]. On 8 November 2012 the LHCb team reported on an experiment seen as a "golden" test of supersymmetry theories in physics, by measuring the very rare decay of the B_s meson into two muons ($B_s^0 \longrightarrow \mu^+ + \mu^-$). The results, which match those predicted by the non-supersymmetrical Standard Model rather than the predictions by supersymmetric models, show that the decays are less common than some forms of supersymmetry predict, though could still match the predictions of other versions of supersymmetry theory. The results as initially drafted are stated to be short of a proof but at a relatively high 3.5σ level of significance.[14]

Of course, these results are not enough to convince theoretical physicists to abandon the whole elegant idea of supersymmetry. Indeed, all consistent unification of gravity with other three fundamental interactions require supersymmetry, while in principle the breaking scale could be even close to the Planck scale. For string theory to be consistent, supersymmetry appears to be required at some level (although it may be broken strongly). Even if supersymmetry, supergravity and superstrings are not the ultimate theories, their study will increase our understanding of classical and quantum field theory and they may be an important step in the understanding of some yet unknown, correct theory of nature. Summarizing, supersymmetric theories give us a theoretical laboratory to study field theories with radically different properties. They mix isospin and space-time symmetries giving hope of putting all subatomic particles in the same irreducible representation. In some cases we can cancel enough divergences to construct theories that are finite to all orders of perturbation theory. The local version of supersymmetric theories necessarily contain gravity. The ideas of supersymmetry have stimulated new approaches to other branches of physics like atomic, molecular, nuclear, statistical and condensed matter physics, a well as nonrelativistic quantum mechanics.

In the context of understanding the breaking down of supersymmetry in field theory, the subject of supersymmetric quantum mechanics has been introduced for the first time. In 1981, Edward Witten used a one-dimensional supersymmetric model, the supersymmetric quantum mechanics, as a simple model where dynamical supersymmetry breaking really does occur [15]. This model had led to new lines of research and applications in various areas of physics.

The SQM (supersymmetric quantum mechanics) has become important on its own right. It was realized that SQM gives insight into the factorization method of Infeld and Hull, which was the first attempt to categorize the analytically solvable potential problems. A whole technology has been evolved based on SQM to understand solvable potential problems and even discover new solvable potential problems. For potentials which are not exactly solvable a powerful new supersymmetric approximation method has been developed [16]. Nowadays SQM still represents a very active domain of research with a wide range of applications in different fields of theoretical and mathematical physics, such as non-linear equations, statistical physics, inverse scattering methods, and exactly solvable models. Also SQM plays an important role in different lines of research, as spontaneously breaking of SUSY [17, 18], the description of the motion of test particle near horizon of black holes [19, 20, 21, 22, 23], magnetic monopoles [24] , AdS^2 / CFT_1 correspondence [25, 26], superconformal quantum mechanics [27, 28, 29, 30, 31, 32] and investigating the light-cone dynamics of supersymmetric theories [33].

Since 70's, although many supersymmetric theories have been known and investigated, there is still no concrete classification of supermultiplets, even in the simple one dimensional case, which has been subject of several investigations during last few decades. Great attention has been given to one-dimensional N-extended supersymmetric models due to the fact that by dimensional reduction one can relate a supersymmetric model living in higher dimensions to another supersymmetric model living in one dimension [34]. Many properties (including the representation theory) of supersymmetric theories in higher dimensions are encoded in some way in the representations of the corresponding one-dimensional $\mathcal{N}$-extended theory obtained by dimensional reduction, where powerful mathematical technology and theoretical framework have been developed. The first formalism that has been used to attack this program is the superspace and superfield formalism. Indeed it is quite convenient to construct manifest invariants. Although, for small $\mathcal{N}$ ($\mathcal{N} < 8$) superspace formalism works well, however, for large values of $\mathcal{N}$, the associated superfields are getting too reducible. They require introducing more constraints to extract the irreducible representations to the point of becoming soon impractical [35].

Another alternative formalism for constructing, understanding and classifying one-dimensional supersymmetrical theories was introduced by A. Pashnev and F. Toppan [38]. Based on this approach the classification of the irreducible off-shell representations of the one-dimensional $\mathcal{N}$-extended supersymmetry has been given, based on the connection with well known mathematics such as Clifford algebras, division algebras and etc. [37, 38].

The question of constructing and classifying $\mathcal{N}$-extended SQM theories has not been fully completed yet. Our focus in this thesis is specifically to the classification of off-shell representations of arbitrary $\mathcal{N}$-extended one-dimensional superalgebras, including superconformal algebras.

The work has been structured as follows. In the continuations of this chapter we briefly review the technical framework that has been developed during the last 15 years by S.J Gates and collaborators and by F. Toppan and collaborators.

In chapter **2** we construct and classify the non-minimal linear representations of the $\mathcal{N}=4$ Extended Supersymmetry in one-dimension. "Pure" homogeneous linear supermultiplets (minimal and non-minimal) of the $\mathcal{N}=4$-Extended one-dimensional Supersymmetry Algebra are classified.

We further prove the existence of "entangled" linear supermultiplets which do not admit a graphical presentation, by constructing an explicit example of an entangled $\mathcal{N}=4$ supermultiplet with field content $(3,8,5)$. The distinction between the notion of equivalence for pure supermultiplets and the notion of equivalence for their associated graphs (Adinkras) is discussed.

In chapter **3** σ-models based on non-minimal representations are constructed: the resulting off-shell actions are either manifestly invariant or depend on a constrained prepotential. The connectivity properties of the graphs play a decisive role in discriminating inequivalent actions. The one-dimensional $\mathcal{N}=4$ sigma-model with a three-dimensional target based on the entangled supermultiplet is presented.

Based on representation of global supersymmetry in one dimension, we construct the D-module representations of superconformal algebra in chapter **4**. We found that this construction is only possible for some critical values of the scaling dimension λ (the lowest engineering dimension of fields entering the representation).

In chapter **5** these D-module representations are applied to the construction of superconformal mechanics in a Lagrangian setting. Non-trivial constraints on the admissible models of $\mathcal{N} = 4$ superconformal mechanics are discussed and we spend some words on superconformal quantum mechanics.

1.1 Review of the one dimensional Supersymmetry

Here we briefly review the theory, definitions and properties characterizing the linear representations of the one-dimensional $\mathcal{N}$-Extended Superalgebra.

Higher-dimensional $\mathcal{N} = 1$supersymmetric gauge theories are known to lead to gauge theories with extended supersymmetries in four dimensions. Well known examples are $\mathcal{N} = 1$ supersymmetric gauge theories in ten and six dimensions leading to $\mathcal{N} = 4$ and $\mathcal{N} = 2$ supersymmetric gauge theories in four dimensions under trivial reduction [52, 53]. Some supersymmetric quantum mechanics with large extended number of supersymmetries ($\mathcal{N}$) are obtained as "*shadows*" of higher-dimensional supersymmetric theories.

By dimensional reduction, we mean an equivalence relation between a quantum field theory and another quantum field theory (*shadow* theory) defined on the submanifold of the target manifold of the first theory. A specific dimensional reduction that we use to get one dimensional supersymmetric theories is the most radical reduction, where all space-dimensions are frozen and only one coordinate dependence is left, which we choose to be the time coordinate. This process reduces quantum field theory to quantum mechanics and will lead to a one-dimensional theory that maintains all of the supersymmetry apparent in the higher dimension. In this case, the representation theory of the higher dimensional off-shell supertheories are contained in the one-dimensional theories, where we can employ more known and powerful mathematical tools (based on the available classification of Clifford algebras) which are not available in higher dimensions.

The reverse process of dimensional reduction is "*oxidation*". By oxidation, we mean to construct a higher dimensional theory from information encoded in the theories living in lower dimension. It is an easier process to construct a lower dimensional theory from a given higher dimensional theory. On the other hand, it is more challenging to construct a higher dimensional supersymmetric model out of lower dimensional model(s), or to determine whether a lower dimensional model actually is a shadow.

Although all higher dimensional theories have shadows, not all lower dimensional supersymmetric theories may be interpreted as shadows. This statement is specifically true when we consider one dimensional supersymmetric models, where some supersymmetric one-dimensional theories seem to exist only in one dimension. They are still important on their own right as SQM models.

Deeper understanding of the oxidation process is highly demanded. Particularly because, together with complete classification of one-dimensional supersymmetry, it provides a systematical tool to construct and investigate all possible supersymmetric theories in an arbitrary dimension. This has motivated a branch of researches, but it still remains an open problem [54, 55].

We address the question of *off-shell* realizations of supersymmetric theories here. This question was forwarded by J. Gates as follow [56]:

"Why is it that in most theories involving supersymmetry, we (are) not able to describe them in a way that is independent of their dynamics?"

The question concerns the way that supersymmetry generators act on the fields. In off-shell representations they act on fields, disregarding the specific dynamics of fields, while on-shell representations are dynamically dependent and restrict the action to fields which satisfy the equations of motion.

Although on-shell representations are more complicated, they are more common in physics, and only few supersymmetric theories are well understood in their off-shell formalism.

It is possible to obtain several on-shell representations, from one off-shell representation by restrict it using different Lagrangians. The reverse, finding the off-shell representations for all supersymmetric theories proved to be challenging [57, 58].

One way to attack this problem (proposed by Gates *et al*) is studying off-shell supersymmetric representations in terms of their dimensional reductions to one dimension, noting that off-shell representations of one-dimensional supersymmetry can be constructed based on Clifford algebra [56].

1.1.1 N-extended supersymmetric quantum mechanic algebra and Clifford algebra

$\mathcal{N}$-extended supersymmetric algebra in one dimension, the algebra of super quantum mechanics, includes $\mathcal{N}$ fermionic generators Q_I ($I = 1, ..., \mathcal{N}$.) together with one bosonic operator H which we choose to be the hamiltonian operator, generating the temporal evolution of the system, satisfying the following (anti)commutators:

$$\begin{aligned} \{Q_I, Q_J\} &= \delta_{IJ} H \\ [Q_I, H] &= 0, \end{aligned} \tag{1.1}$$

where $I, J = 1, ..., \mathcal{N}$, and H is represented by the time-derivative $H \equiv \frac{\partial}{\partial_t}$.

(Herein, we focus on supersymmetry with no central charge.)

The irreducible linear representations are given by supermultiplets, which contain finite number of graded fields (bosonic and fermionic fields with different gradings). These fields are depending on a single coordinate t (the time). The generators Q_Is are represented as matrices, acting on the supermultiplet, whose entries are polynomials on the time-derivative (in the case of linear representations, they are either c-numbers or time-derivatives). In one dimension, the concept of " spin" is no longer present. From the dimension reduction, we know that fermionic fields are Grassmannian (anti-commuting fields) and according to the dimensional analysis, fields with different spins appear with different "mass-dimension" in the higher dimensional theories, as well the dimensionally reduced theory (see [59] and tables therein).

We use the "mass-dimension" of the fields as a grading. Since the hamiltonian H is proportional to the time-derivative operator $\partial \equiv \frac{d}{dt}$, it has a dimension 1, therefore the dimension 1/2 is associated to the supercharges Q_I's.

A grading, the ***Engineering dimension*** d, can be assigned to any field entering a linear representation. Conventionally one can associate bosonic (fermionic) fields with integer (respectively, half-integer) engineering dimension.

Field content*:*

Each finite linear representation is characterized by its "*Field content*" [38, 37], i.e. the set of integers $(n_1, n_2, \ldots, n_l)$ specifying the number n_i of fields of engineering dimension d_i ($d_i = d_1 + \frac{i-1}{2}$, with d_1 an arbitrary constant as the lowest engineering dimension of supermultiplet) entering the representation. Either $n_1, n_3, \ldots$ correspond to the bosonic fields (therefore $n_2, n_4, \ldots$ specify the fermionic fields) or vice versa.
In both cases the equality $n_1 + n_3 + \ldots = n_2 + n_4 + \ldots = n$ is guaranteed. Physically, the n_l fields of highest dimension are the auxiliary fields which transform as time-derivative of the field under any supersymmetry generator. The maximal value l (corresponding to the maximal dimensionality d_l) is defined to be the *length* of the representation. A representation which has length $l = 2$ called a "*root*" representation. The root supermultiplet is specified by the $\mathcal{N}$ supersymmetry operators $\widehat{Q}_I$ $(I = 1, \ldots, \mathcal{N})$, expressed in matrix form as

$$\widehat{Q}_J = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & \gamma_J \\ -\gamma_J \cdot H & 0 \end{pmatrix}, \quad \widehat{Q}_{\mathcal{N}} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & \mathbf{1}_n \\ \mathbf{1}_n \cdot H & 0 \end{pmatrix}, \tag{1.2}$$

where the γ_J matrices $(J = 1, \ldots, \mathcal{N} - 1)$ satisfy the Euclidean Clifford algebra

$$\{\gamma_I, \gamma_J\} = -2\delta_{IJ}\mathbf{1}_n. \tag{1.3}$$

Higher-length pure supermultiplets are obtained by applying a "*dressing transformation*" [38] to the length-2 root supermultiplet. For example the length-3 supermultiplets are specified by the $\mathcal{N}$ operators Q_i, given by the dressing transformation

$$Q_I = D\widehat{Q}_I D^{-1}, \tag{1.4}$$

where $\widehat{Q}_I$ is a root supercharge and D is a diagonal dressing matrix such that

$$D = \begin{pmatrix} \widetilde{D} & 0 \\ 0 & \mathbf{1}_n \end{pmatrix}, \tag{1.5}$$

with $\widetilde{D}$ an $n \times n$ diagonal matrix whose diagonal entries are either 1 or the derivative operator ∂_t. We treat the inverse of ∂_t entering D^{-1} matrix as an algebraic object. A dressing matrix is an acceptable (regular) dressing transformation if and only if at the left hand side of 1.4 the inverse of ∂_t does not appear explicitly.

In this framework, the classification of irreducible representations of the one dimensional supersymmetric algebra is reduced to the classification of root supermultiplets(equivalent to the classification of Clifford algebra) and classifying all regular dressing transformations.

According to Clifford algebra representation, the total number n of bosonic fields entering an irreducible representation (which equals the total number of fermionic fields) is expressed, for any given value of $\mathcal{N}$, by the following relation

$$n = 2^{4p} G(m) \tag{1.6}$$

where $\mathcal{N} = 8p + m$ and $G(m)$ is Radon-Hurwitz function with following values:

m	1	2	3	4	5	6	7	8
$G(m)$	1	2	4	4	8	8	8	8

(1.7)

We present the minimum number of bosonic (and equally fermionic) fields, required for an irreducable representation up to $\mathcal{N} = 32$, in the following table:

$\mathcal{N}=1$	1	$\mathcal{N}=9$	16	$\mathcal{N}=17$	256	$\mathcal{N}=25$	4096
$\mathcal{N}=2$	2	$\mathcal{N}=10$	32	$\mathcal{N}=18$	512	$\mathcal{N}=26$	8192
$\mathcal{N}=3$	4	$\mathcal{N}=11$	64	$\mathcal{N}=19$	1024	$\mathcal{N}=27$	16384
$\mathcal{N}=4$	4	$\mathcal{N}=12$	64	$\mathcal{N}=20$	1024	$\mathcal{N}=28$	16384
$\mathcal{N}=5$	8	$\mathcal{N}=13$	128	$\mathcal{N}=21$	2048	$\mathcal{N}=29$	32768
$\mathcal{N}=6$	8	$\mathcal{N}=14$	128	$\mathcal{N}=22$	2048	$\mathcal{N}=30$	32768
$\mathcal{N}=7$	8	$\mathcal{N}=15$	128	$\mathcal{N}=23$	2048	$\mathcal{N}=31$	32768
$\mathcal{N}=8$	8	$\mathcal{N}=16$	128	$\mathcal{N}=24$	2048	$\mathcal{N}=32$	32768

(1.8)

More details on these subjects could be found in [38, 60, 61]. Due to the importance of length-3 multiplets for our work, we emphasize another result, stating that for any value of $\mathcal{N}$, all length-3 multiplets of the type $(n-k, n, k)$ are an irrep of the $\mathcal{N}$-extended supersymmetry.

1.1.2 Graphical representations

Association with graphs:

There are examples, in theoretical physics, which the diagrammatic technics has been used to help our imagination, or employed as powerful tools to calculate more sophisticated processes. A famous example is Feynman diagrams. A powerful diagrammatic technique which usefully encodes many aspects of supersymmetry multiplets has been introduced by J. Gates and M. Faux [62, 36]. According to them, each supermultiplet has a corresponding distinctive symbolic form which has been called *adinkra symbol*. Meanwhile another graphical representation was introduced by F. Toppan and his colleagues with slightly different conventions [38, 61]. Throughout this paper we introduce and use the last one. It turns out that the correspondence between supermultiplets and supergraphs is not one-to-one, as it will be shown in the next chapter. We call those supermultiplets that admit graphical representation "*pure supermultiplets*".

"Supergraphs" (Adinkras) are $\mathcal{N}$-colored oriented graphs with vertices and edges, which in a pictorial way encode all details of the supersymmetry transformations on the component fields within a supermultiplet. The correspondence between supermultiplets and adinkras is not a one-to-one correspondence. Still this graphical technics plays an important role and they provide a systematic classification tool for representations of supersymmetry. The association between pure linear supersymmetry transformations and $\mathcal{N}$-colored oriented graphs [62] goes as follows. The fields (bosonic and fermionic) entering a representation are expressed as vertices. They can be accommodated into an $X - Y$ plane. The Y coordinate can be chosen to correspond to the engineering dimension d of the fields. Conventionally, the lowest dimensional fields can be associated to vertices lying on the X axis. The higher dimensional fields have positive, integer or half-integer values of Y. A colored edge links two vertices which are connected by a supersymmetry transformation. Each one of the $\mathcal{N}$ Q_i supersymmetry generators is associated to a given color. The edges are oriented. The orientation reflects the sign (positive or negative) of the corresponding supersymmetry transformation connecting the two vertices. Instead of using arrows, alternatively, solid or dashed lines can be associated, respectively, to positive or negative signs. No colored line is drawn for supersymmetry transformations connecting a field with the time-derivative of a lower dimensional field. This is in particular true for the auxiliary fields (the fields of highest dimension in the representation) which are necessarily mapped, under supersymmetry transformations, in the time-derivative of lower-dimensional fields.

The *pure* irreducible supersymmetry transformations can be presented (the identification is not unique) through an oriented $\mathcal{N}$-colored graph with $2n$ vertices. The graph is such that precisely $\mathcal{N}$ edges, one for each color, are linked to any given vertex which represents either a 0-engineering dimension or a $\frac{1}{2}$-engineering dimension field.

For sake of clarity, we present a few selected graphs associated to non-minimal supermultiplets Figures **1.1 - 1.1**.

An unoriented "color-blind" graph can be associated to the initial graph by disregard-

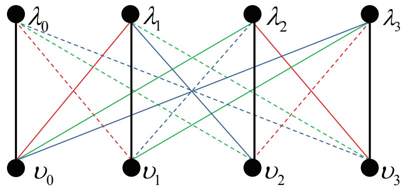

Figure 1.1: $(4, 4)$, Minimal $\mathcal{N} = 4$ (Root supermultiplet).

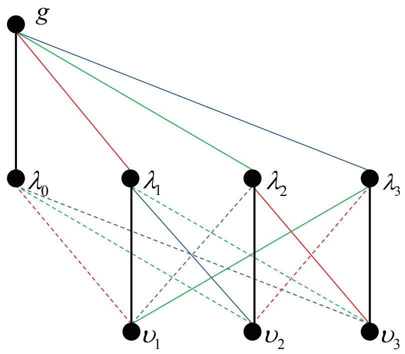

Figure 1.2: $(3, 4, 1)$, Minimal $\mathcal{N} = 4$, C.S. 4_1.

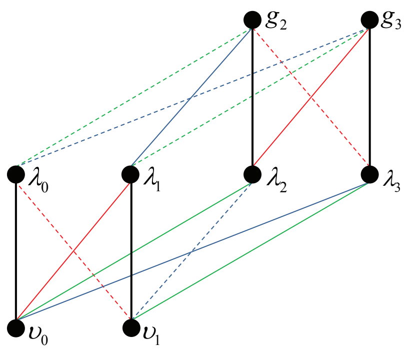

Figure 1.3: $(2, 4, 2)$, Minimal $\mathcal{N} = 4$, C.S. 4_2.

ing the orientation of the edges and their colors (all edges are painted in black). Some topological characteristics of colored blind graphs has been used to classify supergraphs. Among them, due to the importance of length-3 supermultiplets for physics, *Connectivity symbol* has been introduced:

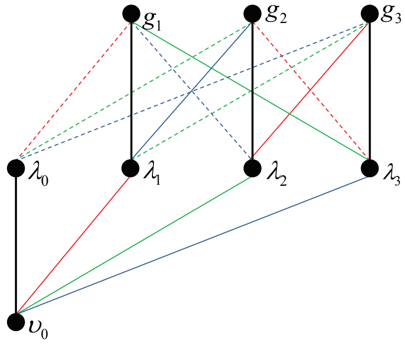

Figure 1.4: $(1,4,3)$, Minimal $\mathcal{N}=4$, C.S. 4_3.

Connectivity symbol:

A characterization of length $l=3$ color-blind, unoriented graphs can be expressed through the connectivity symbol ψ_g [63], defined as follows

$$\psi_g \quad = \quad (m_1)_{s_1} + (m_2)_{s_2} + \ldots + (m_Z)_{s_Z}. \tag{1.9}$$

The ψ_g symbol encodes the information on the partition of the n $\frac{1}{2}$-engineering dimension fields (vertices) into the sets of m_z vertices ($z = 1, \ldots, Z$) with s_z edges connecting them to the $n-k$ 1-engineering dimension auxiliary fields. We have

$$m_1 + m_2 + \ldots + m_Z \quad = \quad n, \tag{1.10}$$

while $s_z \neq s_{z'}$ for $z \neq z'$.

Node choice group:

Obviously a color blind graph, does not contain all information. "*Node choice group*" is a topological characteristic. It encodes color information. Given a graph associated to an $\mathcal{N}$-Extended pure supermultiplet, its node choice group [64] is the set of $\mathcal{N}$-character strings (of 0's and 1's), closed under the term-by-term $\mathbf{Z}_2$ addition ($0+0 = 1+1 = 0$, $0+1 = 1+0 = 1$). An $\mathcal{N}$-character string of r 1's (associated to the supersymmetry generators $Q_{i_1}, \ldots, Q_{i_r}$) and $\mathcal{N}-r$ 0's (associated to the remaining supersymmetry generators) belongs to the node choice group if and only if for any vertex of the graph (denoted as V_{in}), the path $Q_{i_1}, \ldots Q_{i_r}$ produces a final vertex V_{fin} with the same engineering dimension as V_{in}. Obviously r must necessarily be an even number.

A node choice group will be presented either by its set of generators (they will be denoted as "$< \cdot, \cdot, \ldots >$"), or by its total set of elements (denoted as "$\{\cdot, \cdot, \ldots\}$"). We have, for instance, $< 1100, 0011 > \equiv \{0000, 1100, 0011, 1111\}$.

Chirality of the $\mathcal{N} = 4$ minimal supermultiplets:

The $\mathcal{N} = 4$ root supermultiplet has a chirality associated with the overall sign ($\eta = \pm 1$) of the totally antisymmetric tensor ϵ_{ijk} [61]. Its supersymmetry transformations are given by

$$\begin{aligned} Q_i(\upsilon_0, \upsilon_j; \lambda_0, \lambda_j) &= (\lambda_i, -\delta_{ij}\lambda_0 - \eta\epsilon_{ijk}\lambda_k; -\dot{\upsilon}_i, \delta_{ij}\dot{\upsilon}_0 + \eta\epsilon_{ijk}\dot{\upsilon}_k), \\ Q_4(\upsilon_0, \upsilon_j; \lambda_0, \lambda_j) &= (\lambda_0, \lambda_j; \dot{\upsilon}_0, \dot{\upsilon}_j), \end{aligned} \tag{1.11}$$

for $i, j, k = 1, 2, 3$.

The notion of chirality for the root supermultiplet is extended and applied to the chirality of its dressed supermultiplets.

One can define *overall chirality* for a collection of ($r = 1, 2, \ldots, n$) independent minimal $\mathcal{N} = 4$ supermultiplets with chirality η_r. The modulus $\Delta = |\sum_r \eta_r|$ is their *overall chirality*. In the next chapter we discuss the importance of this notion.

For the sake of completeness we review some other definitions of the properties characterizing the homogeneous linear representations which applied in our work:

Dual supermultiplet:

A dual supermultiplet is obtained by mirror-reversing, upside-down, the graph associated to the original supermultiplet.

Commuting group:

For a given supermultiplet, its *commuting group* [65] is the maximal group of linear transformations of the component fields which commute with all supersymmetry transformations. For a root supermultiplet, its commuting group is read from the Schur's character (real, complex of quaternionic) of the associated Clifford algebra, see [65]. For dressed supermultiplets, the commuting group generators must commute with the dressing operator. The commuting group of the non-minimal supermultiplets has been discussed in the text.

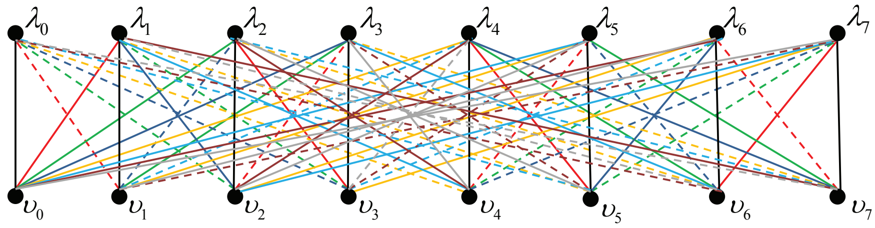

Figure 1.5: $(8, 8)$, $\mathcal{N} = 8$ (Root supermultiplet).

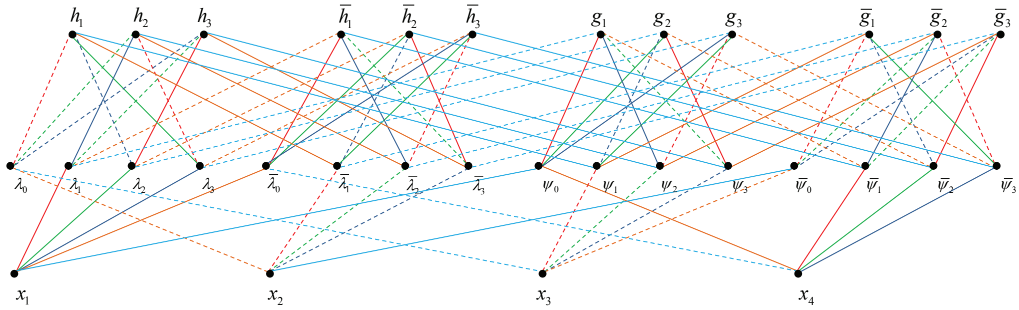

Figure 1.6: $(4, 16, 12)$, Connected $\mathcal{N} = 5$, C.S. $12_4 + 4_3$.

Chapter 2

Non-minimal, Pure and Entangled Supermultiplets

This chapter is an edited version of first parts of [66]and [67] written in collaboration with M. Gonzales, K. Iga, and F. Toppan.

2.1 Introduction

The $1D$ $\mathcal{N}$-Extended Superalgebra, with $\mathcal{N}$ odd generators Q_I ($I = 1, 2, \ldots, \mathcal{N}$) and a single even generator H satisfying the (anti)-commutation relations

$$\begin{aligned} \{Q_I, Q_J\} &= \delta_{IJ} H, \\ [H, Q_I] &= 0, \end{aligned} \tag{2.1}$$

is the superalgebra underlying the Supersymmetric Quantum Mechanics [15]. In recent years the structure of its linear and non-linear representations has been unveiled by a series of works (upon which the present investigation is based) [38]–[51], that we will briefly comment.

In this chapter, at first we present a systematical investigation of the inequivalent non-minimal linear supermultiplets carrying a representation of the one-dimensional $\mathcal{N} = 4$-Extended Superalgebra as an example. The construction of associated $\mathcal{N} = 4$-invariant

σ model will be presented in the next chapter.

In the second part of this chapter we present a class of entangled supermultiplets. We illustrate the notion by an example from $\mathcal{N} = 4$. The associated $\mathcal{N} = 4$-invariant σ model of this example will be presented in the next chapter.

The linear representations under considerations (supermultiplets) contain a finite, equal number of bosonic and fermionic fields depending on a single coordinate (the time). The operators Q_I and H act as graded differential operators. The linear representations are characterized by a series of properties and tools. They include, among others, the notions of *engineering dimension* (or, equally used terminology *mass-dimension*), *length of a supermultiplet*, *mirror symmetry duality*, *connectivity symbol*, *node choice group*, *commuting group*, possible *chirality and/or coloring of the supermultiplets*). They are revised partially in the Introduction and further discussed in the text, when needed.

The *minimal* linear representations (also called *irreducible supermultiplets*) are given by the minimal number n_{min} of bosonic (fermionic) fields for a given value of $\mathcal{N}$. The value n_{min} is given [38] by the formula

$$\begin{aligned} \mathcal{N} &= 8l + m, \\ n_{min} &= 2^{4l} G(m), \end{aligned} \tag{2.2}$$

where $l = 0, 1, 2, \ldots$ and $m = 1, 2, 3, 4, 5, 6, 7, 8$.

$G(m)$ appearing in (2.2) is the Radon-Hurwitz function

m	1	2	3	4	5	6	7	8
$G(m)$	1	2	4	4	8	8	8	8

(2.3)

Non-minimal linear representations, which are basically those representations whose their field contents contain a larger number of fields than the minimal one, have been discussed in [37, 64, 51, 68, 65]. An important subclass of non-minimal representations is given by the reducible but indecomposable supermultiplets (see [37]). For this subclass the associated graph is connected (there is a path connecting any two given vertices).

The maximal *finite* number n_{max} of bosonic (fermionic) fields entering a non-minimal representation, that still gives a connected associated graphical representation, is given by [64, 51]

$$n_{max} = 2^{\mathcal{N}-1}. \tag{2.4}$$

(equal to the number of vertices of a hyper $\mathcal{N}$-cube)

For $\mathcal{N} = 4$ we have that $n_{min} = 4$ and $n_{max} = 8$. As a consequence, there are only two subclasses of non-minimal $\mathcal{N} = 4$ representations. Besides the irreducible but indecomposable subclass, we have the subclass of fully reducible representations given by the direct sum of two minimal $\mathcal{N} = 4$ representations (the associated graph is disconnected and given by two separate minimal $\mathcal{N} = 4$ graphs).

While most of the properties that have been used to distinguish between different linear realizations of the algebra with the same field contents involve the graphical association and topological features of them, an important key issue concerns the distinction between the equivalence class of pure supermultiplets and the equivalence class of their associated graphs.

Equivalent graphs are related by two types of moves:

i) *local moves*, based on the permutation of vertices with the same engineering dimension and

ii) *global moves*, based on the permutation of the colored edges with or without a sign flipping.

Global moves are responsible for properties such as the global chirality or the global color of a graph expressed by a combination of disconnected subgraphs. Indeed, if a graph is a disjoint union of disconnected subgraphs, local moves can only affect some subgraphs, while leaving unaffected the remaining ones. On the other hand, the global moves produce a global effect.

The local moves are a special type (corresponding to permutations of the vertices) of linear transformations of fields with the same engineering dimension. Among the linear transformations of fields with the same engineering dimension another special type of transformations deserves the name (in reference to the famous story and for reasons that will soon be clear) of *gordian transformations*. Acting on a pure supermultiplet a gordian transformation maps its associated graph into another graph which cannot be recovered from the previous one under *i* and *ii* moves. Even if the two graphs belong to inequivalent classes under *i* and *ii* moves they describe, nevertheless, the same pure supermultiplet. We discus this relation precisely, presenting an explicit example of a pure supermultiplet (the non-minimal $\mathcal{N} = 4$ supermultiplet with $(4, 8, 4)$ field content and 8_2 connectivity symbol) which can be presented either as a fully connected graph or as a disconnected graph (the disjoint union of two disconnected subgraphs).

We further prove the existence of "entangled" linear supermultiplets which do not admit a graphical presentation, by constructing an explicit example of an entangled $\mathcal{N} = 4$ supermultiplet with field content $(3, 8, 5)$. It interpolates between two inequivalent pure $\mathcal{N} = 4$ supermultiplets with the same field content. The scheme of this section is the following.

In section **2** we briefly review the classification of the pure supermultiplets for $\mathcal{N} = 3$. We also discuss the notion of "coloring" for the $\mathcal{N} = 3$ supermultiplet with field content $(2, 4, 2)$. In section **3** we present the classification of the pure (minimal and non-minimal) homogeneous linear supermultiplets for $\mathcal{N} = 4$. They will be discriminated by their *field content*, *connectivity symbol*, *node choice group*, *commuting group*. In certain cases the notions of *chirality* and *coloring* apply.

We also present the so-called "oxidation diagrams" connecting the non-minimal $\mathcal{N} = 4$ representations with the minimal $\mathcal{N} = 5$ ones.[1] We employ the techniques introduced in [69]. The presented results answer the following question: which minimal $\mathcal{N} = 5$ su-

[1]We postpone to the Conclusions the discussion about the meaning of the term *oxidation* and of the physical importance of the so-called *oxidation program*, see [56, 70, 65].

permultiplets can be obtained by adding an extra supersymmetry operator to a given non-minimal $\mathcal{N} = 4$ supermultiplet in such a way to guarantee an overall $\mathcal{N} = 5$ (2.1) superalgebra. We recall that, due to (2.2), the minimal $\mathcal{N} = 5$ representations contain the same number of fields (8 bosons and 8 fermions) as the non-minimal $\mathcal{N} = 4$ representations.

In Section **4** an explicit example of an $\mathcal{N} = 4$ supermultiplet which cannot be realized as a pure supermultiplet is constructed ("entangled supermultiplet"). It corresponds to an interpolation, with a given angle, of two pure $\mathcal{N} = 4$ supermultiplets with field content $(3, 8, 5)$.

In the Discussion we conclude the results and make further comments.

For sake of clarity, we present a few selected graphs associated to non-minimal supermultiplets at the end of this Chapter (Figures **2.5-2.8**).

2.2 The $\mathcal{N} = 3$ supermultiplets

We present at first the list of the $\mathcal{N} = 3$ minimal supermultiplets. This is a necessary preliminary step for producing minimal and non-minimal $\mathcal{N} = 4$ pure supermultiplets. Indeed, these ones are obtained from the $\mathcal{N} = 3$ supermultiplets by adding a compatible fourth supertransformation (in the case of a non-minimal $\mathcal{N} = 4$ supermultiplet two separate $\mathcal{N} = 3$ supermultiplets are employed).

All minimal $\mathcal{N} = 3$ supermultiplets are pure supermultiplets associated with a graphical presentation. They contain 4 bosonic and 4 fermionic component fields. Their list specifying their properties (field content $F.C.$, connectivity symbol $C.S.$, commuting group $C.G.$, node choice group $N.C.G.$ and coloring $col.$, see the Introduction) is the following

$F.C.$	$C.S.$	$C.G.$	$N.C.G.$	*col.*
$(4,4)$	4_0	$SU(2)$	$\{evens\}$	1
$(1,4,3)$	1_3+3_2	$\mathbf{1}$	$< 000 >$	1
$(2,4,2)$	2_2+2_1	$U(1)$	$< 110 >$	3
$(3,4,1)$	3_1+1_0	$\mathbf{1}$	$< 000 >$	1
$(1,3,3,1)$			$< 000 >$	1

(2.5)

(where "$\{evens\}$" denotes the set containing the words with even number of 1's).

The above supermultiplets come out in two variants (bosonic or fermionic) according to the grading (even or odd) of the component fields with lowest engineering dimension (in application to supersymmetric models the fermionic version of the $(4,4)$ root supermultiplet is often denoted as "$(0,4,4)$").

The above supermultiplets are non-chiral.

The special case $(2,4,2)$ supermultiplet appears in 3 different *colorings* , related to its presentation in terms of the node choice group, its node choice group admits three different presentations related by global moves:
$NCG_1 =< 110 >\equiv \{110,000\}$, $NCG_2 =< 101 >\equiv \{101,000\}$, $NCG_3 =< 011 >\equiv \{011,000\}$. Let $j\neq 1$ be a third root of unity ($j^3=1$). We can associate to NCG_i the root $j_i=j^i$, respectively. For a collection of several independent ($r=1,2,\ldots,n$) $\mathcal{N}=3$ $(2,4,2)$ supermultiplets the modulus $C=|\sum_r j_r|$ is their overall color. Like its chiral counterpart, the modulus C is invariant under both local and global moves.

For graphs with $\mathcal{N}>3$ the notion of coloring is extended to inequivalent (under local moves only) presentations of its node choice group.The three colorings are related by global moves (permutation of the supertransformations), so that, under global moves, the three colorings belong to the same class of equivalence. On the other hand, the modulus $C=|\sum_{r=1}^{n} j_r|$ specifies the different classes of equivalence, under local and global moves, of a collection of n independent $(2,4,2)$ $\mathcal{N}=3$ supermultiplets, each one characterized by

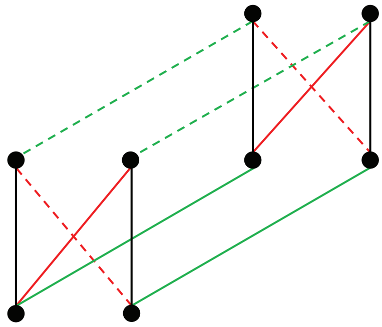

Figure 2.1: $N.C.G. < 110 >$

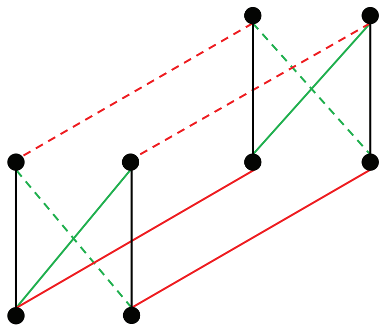

Figure 2.2: $N.C.G. < 101 >$

its coloring j_r. In this construction the three colorings are put in correspondence with the three third roots of unity ($j_r^3 = 1$, for $r = 1, 2, 3$). The graph characterizing the collection of the n independent supermultiplets is the disjoint union of the n graphs associated to the independent minimal pure supermultiplets.

For clarity we present the graphs (Adinkras) associated to the three colorings of the $(2, 4, 2)$ $\mathcal{N} = 3$ supermultiplet. The three supertransformations are painted in black, red and green. The graphs are presented in Figures **2.1**, **2.2** and **2.3**.

For graphs with $\mathcal{N} > 3$ the notion of coloring is extended to inequivalent (under local moves only) presentations of its node choice group.

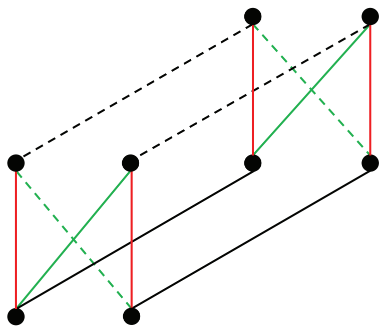

Figure 2.3: $N.C.G. < 011 >$

2.3 The classification of pure $\mathcal{N} = 4$ supermultiplets

2.3.1 Minimal $\mathcal{N} = 4$ supermultiplets

The minimal $\mathcal{N} = 4$ supermultiplets contain 4 bosonic and 4 fermionic fields. They are all pure supermultiplets associated with a graphical presentation. Their complete list, together with their properties, is given by the table below (one should note that, among the $\mathcal{N} = 3$ supermultiplets, the length-4 $(1, 3, 3, 1)$ is the only one which cannot be extended to $\mathcal{N} = 4$ by adding a compatible fourth supertransformation)

$F.C.$	$C.S.$	$C.G.$	$N.C.G.$	$col.$
$(4, 4)$	4_0	$SU(2)$	$\{evens\}$	1
$(1, 4, 3)$	4_3	$\mathbf{1}$	$< 0000 >$	1
$(2, 4, 2)$	4_2	$U(1)$	$< 1100, 0011 >$	3
$(3, 4, 1)$	4_1	$\mathbf{1}$	$< 0000 >$	1

(2.6)

The above supermultiplets come out in four variants. Similarly to the $\mathcal{N} = 3$ supermultiplets they are either bosonic or fermionic. Unlike the $\mathcal{N} = 3$ supermultiplets they are either chiral or antichiral ($\eta = \pm 1$, see Introduction). The chirality is flipped by global moves so that, for a single supermultiplet, there is only one class of equivalence under global moves.

For a collection of several ($r = 1, 2, \dots, n$) independent minimal $\mathcal{N} = 4$ supermultiplets with chirality η_r, the modulus $\Delta = |\sum_r \eta_r|$ is their *overall chirality*. The modulus is left invariant under local and global moves. For $\mathcal{N} = 3$ supermultiplets (disregarding Q_4) the chirality is not defined because one can flip the sign of η with local moves only.

2.3.2 Non-minimal, pure, disconnected supermultiplets

The non-minimal, pure, $\mathcal{N} = 4$ supermultiplets consist of 8 bosonic and 8 fermionic fields (that is, twice the number of the component fields of the minimal $\mathcal{N} = 4$ supermultiplets). Two classes of graphs are associated to the non-minimal, pure supermultiplets: the disconnected graphs, obtained by the disjoint union of 2 graphs associated with the minimal supermultiplets, and the connected graphs (presenting a path of colored edges connecting each vertex to any other vertex). The complete list of disconnected graphs for non-minimal, pure supermultiplets of length $l = 2, 3$ is presented in the table below. The data reported in the table are the field content ($F.C.$), the connectivity symbol ($C.S.$), the commuting group ($C.G.$), the node choice group ($N.C.G.$), together with the decomposition into direct sum of minimal $\mathcal{N} = 4$ supermultiplets and the label used to name the graphs ("FR" stands for "Fully Reducible"). We have

$F.C.$	$label$	$decomposition$	$C.S.$	$C.G.$	$N.C.G.$
$(8,8)$	FR	$(4,4)\oplus(4,4)$	8_0	$SU(2)\otimes SU(2)\otimes\mathbb{R}$	$\{evens\}$
$(1,8,7)$	FR	$(1,4,3)\oplus(0,4,4)$	4_4+4_3	$SU(2)\otimes\mathbf{1}_2\otimes\mathbb{R}$	$<1111>$
$(2,8,6)$	a	$(2,4,2)\oplus(0,4,4)$	4_4+4_2	$SU(2)\otimes U(1)\otimes\mathbb{R}$	$<1100,0011>$
	b	$(1,4,3)\oplus(1,4,3)$	8_3	$\mathbf{1}_2\otimes\mathbf{1}_2\otimes\mathbb{R}$	$<1111>$
$(3,8,5)$	a	$(3,4,1)\oplus(0,4,4)$	4_4+4_1	$SU(2)\otimes\mathbf{1}_2\otimes\mathbb{R}$	$<1111>$
	b	$(2,4,2)\oplus(1,4,3)$	4_3+4_2	$U(1)\otimes\mathbf{1}_2\otimes\mathbb{R}$	$<1111>$
$(4,8,4)$	a	$(4,4,0)\oplus(0,4,4)$	4_4+4_0	$SU(2)\otimes SU(2)\otimes\mathbb{R}$	$\{evens\}$
	b	$(3,4,1)\oplus(1,4,3)$	4_3+4_1	$\mathbf{1}_2\otimes\mathbf{1}_2\otimes\mathbb{R}$	$<1111>$
	c	$(2,4,2)\oplus(2,4,2)$	8_2	$U(1)\otimes U(1)\otimes\mathbb{R}$	$<1100,0011>$
	d	$(2,4,2)\oplus(2,4,2)$	8_2	$U(1)\otimes U(1)\otimes\mathbb{R}$	$<1111>$
$(5,8,3)$	a	$(4,4,0)\oplus(1,4,3)$	4_4+4_3	$SU(2)\otimes\mathbf{1}_2\otimes\mathbb{R}$	$<1111>$
	b	$(3,4,1)\oplus(2,4,2)$	4_2+4_1	$U(1)\otimes\mathbf{1}_2\otimes\mathbb{R}$	$<1111>$
$(6,8,2)$	a	$(4,4,0)\oplus(2,4,2)$	4_2+4_0	$SU(2)\otimes U(1)\otimes\mathbb{R}$	$<1100,0011>$
	b	$(3,4,1)\oplus(3,4,1)$	8_1	$\mathbf{1}_2\otimes\mathbf{1}_2\otimes\mathbb{R}$	$<1111>$
$(7,8,1)$	FR	$(4,4,0)\oplus(3,4,1)$	4_1+4_0	$SU(2)\otimes\mathbf{1}_2\otimes\mathbb{R}$	$<1111>$

(2.7)

The above supermultiplets come out in four variants: either bosonic or fermionic and either chiral ($\Delta = 2$) or non-chiral ($\Delta = 0$). For $\Delta = 2$ ($\Delta = 0$) the supermultiplet is decomposed into 2 minimal supermultiplets of same (opposite) chirality. The suffix Δ is used to discriminate the two inequivalent cases (therefore, the non-chiral supermultiplet $(3,8,5)_b$ will be denoted as "$(3,8,5)_{b,\Delta=0}$").

The supermultiplets $(4,8,4)_c$ and $(4,8,4)_d$ possess a different node choice group. This is a consequence of the decomposition into two minimal $(2,4,2)$ supermultiplets with same coloring (in the $(4,8,4)_c$ case) or different coloring (in the $(4,8,4)_d$ case).

In terms of node choice group presentations (colorings), the supermultiplets in (2.7) admitting inequivalent (under local moves) presentations are

$$(2,8,6)_a, \quad (4,8,4)_c, \quad (6,8,2)_a,$$

coming out in 3 colorings belonging to the same equivalence class under global moves.

The commuting group of the supermultiplets in (2.7) is the tensor products of three groups: the two independent commuting groups acting on the left (right) minimal supermultiplets of the decomposition and $\mathbb{R}$, acting as $+\mathbb{I}$ on the component fields of the left minimal supermultiplet and as $-\mathbb{I}$ on the component fields of the right minimal supermultiplet.

By suitably adjusting the relative engineering dimension of the two minimal supermultiplets, non-minimal disconnected supermultiplets with length $l > 3$ can be constructed. We just mention two examples: the length-4 supermultiplet with field content

$$(4,4,4,4) \quad = \quad (4,4,0,0) \oplus (0,0,4,4), \tag{2.8}$$

which comes out in four variants (either bosonic or fermionic and either chiral or non-chiral) and the interesting case of the supermultiplet with field content

$$(2,6,6,2) \quad = \quad (2,4,2,0) \oplus (0,2,4,2), \tag{2.9}$$

which comes out in 8 variants, namely either bosonic or fermionic, either chiral or non-chiral, either with $C = 2$ (same coloring of the $(2,4,2)$ minimal supermultiplets) or $C = 1$ (different coloring of the $(2,4,2)$ minimal supermultiplets).

2.3.3 Non-minimal pure supermultiplets with a connected graph

A graph is connected if any given vertex is connected to any other vertex by a path of colored edges representing the supersymmetry transformations of the component fields. The list, together with their properties, of the non-minimal $\mathcal{N} = 4$ pure supermultiplets of length $l = 2, 3$ represented by a connected graph is given in the table below. All these supermultiplets are non-chiral and can be *oxidized* (in the language of [71]) to $\mathcal{N}=8$ (i.e., four extra supersymmetry transformations can be consistently introduced so that $\mathcal{N}=8$ is the maximal number of supersymmetries acting on the given component fields). We have

F.C.	*label:*	*C.S.*	*C.G.*	*N.C.G.*
$(8,8)$	*conn.*	8_0	$SU(2)\otimes SU(2)\otimes \mathbb{R}$	$\{evens\}$
$(1,8,7)$	*conn.*	4_4+4_3	$\mathbf{1}$	$<0000>$
$(2,8,6)$	A	$2_4+4_3+2_2$	$U(1)$	$<1100>$
	B	8_3	$\mathbb{R}$	$<1111>$
$(3,8,5)$	A	$1_4+3_3+3_2+1_1$	$\mathbf{1}$	$<0000>$
	B	4_3+4_2	$\mathbf{1}$	$<0000>$
$(4,8,4)$	A	$1_4+6_2+1_0$	$\mathbf{1}$	$<0000>$
	B	4_3+4_1	$SU(2)$	$<0110,0101>$
	C	$2_3+4_2+2_1$	$\mathbf{1}$	$<0000>$
	D	8_2	$U(1)\otimes U(1)\otimes\mathbb{R}$	$<1100,0011>$
$(5,8,3)$	A	$1_3+3_2+3_1+1_0$	$\mathbf{1}$	$<0000>$
	B	4_2+4_1	$\mathbf{1}$	$<0000>$
$(6,8,2)$	A	$2_2+4_1+2_0$	$U(1)$	$<1100>$
	B	8_1	$\mathbb{R}$	$<1111>$
$(7,8,1)$	*conn.*	4_1+4_0	$\mathbf{1}$	$<0000>$

(2.10)

The above supermultiplets are either bosonic or fermionic (it is worth recalling that the bosonic or fermionic character of a supermultiplet corresponds to the grading, even or odd, of its component fields with lowest engineering dimension).

Inequivalent presentations of the node choice group under local moves (inequivalent colorings) are encountered in the following cases:

$(2,8,6)_A$ and $(6,8,2)_A$ (6 colorings), $(4,8,4)_B$ (12 colorings) and $(4,8,4)_D$ (3 colorings).

We present here an explicit construction of the non-minimal, reducible but indecomposable $\mathcal{N}=4$ supermultiplets of length $l=2,3$.

We start with the $\mathcal{N}=8$ $(8,8)$ root supermultiplet expressed through (1.2), with the 7 matrices γ_j given by

$$\gamma_1=\tau_1\otimes\mathbf{1}_2\otimes\tau_A,\quad \gamma_2=\tau_2\otimes\mathbf{1}_2\otimes\tau_A,\quad \gamma_3=\tau_A\otimes\tau_1\otimes\mathbf{1}_2,\quad \gamma_4=\tau_A\otimes\tau_2\otimes\mathbf{1}_2,$$
$$\gamma_5=\mathbf{1}_2\otimes\tau_A\otimes\tau_1,\quad \gamma_6=\mathbf{1}_2\otimes\tau_A\otimes\tau_2,\quad \gamma_7=\tau_A\otimes\tau_A\otimes\tau_A,$$

where τ_1, τ_2, τ_A, $\mathbf{1}_2$ are 2×2 matrices given by (e_{mn} is the 2×2 matrix with entry 1 at the m^{th} row, n^{th} column and 0 otherwise)

$$\tau_1=e_{12}+e_{21},\quad \tau_2=e_{11}-e_{22},\quad \tau_A=e_{12}-e_{21},\quad \mathbf{1}_2=e_{11}+e_{22}.$$

We can select, e.g., the 4 operators producing the non-minimal $\mathcal{N}=4$ supermultiplet of length $l=2$ (with connected graph) to be given by $\widehat{Q}_2$, $\widehat{Q}_5$, $\widehat{Q}_6$, $\widehat{Q}_7$. For this choice of root operators, the dressing transformations (1.4) producing the inequivalent length $l=3$ non-minimal, reducible but indecomposable $\mathcal{N}=4$ supermultiplets are obtained by applying the following diagonal dressing matrices $\widetilde{D}$, see (1.5):

field content:	*label:*	*dressing matrix* $\widetilde{D}$*:*
$(1,8,7)$		$diag(1,\partial,\partial,\partial,\partial,\partial,\partial,\partial)$
$(2,8,6)$	A	$diag(1,1,\partial,\partial,\partial,\partial,\partial,\partial)$
	B	$diag(1,\partial,\partial,\partial,\partial,\partial,\partial,1)$
$(3,8,5)$	A	$diag(1,1,1,\partial,\partial,\partial,\partial,\partial)$
	B	$diag(1,1,\partial,\partial,\partial,\partial,\partial,1)$
$(4,8,4)$	A	$diag(\partial,1,1,1,1,\partial,\partial,\partial)$
	B	$diag(1,1,1,1,\partial,\partial,\partial,\partial)$
	C	$diag(1,1,1,\partial,1,\partial,\partial,\partial)$
	D	$diag(1,1,\partial,\partial,\partial,\partial,1,1)$
$(5,8,3)$	A	$diag(1,1,1,1,1,\partial,\partial,\partial)$
	B	$diag(\partial,1,1,1,1,1,\partial,\partial)$
$(6,8,2)$	A	$diag(1,1,1,1,1,1,\partial,\partial)$
	B	$diag(\partial,1,1,1,1,1,1,\partial)$
$(7,8,1)$		$diag(1,1,1,1,1,1,1,\partial)$

(2.11)

The Schur's lemma states that the irreducible representations of the Clifford algebras are of three types (real, almost complex or quaternionic), according to the most general matrix commuting with all Clifford generators (see [72]). Since minimal root supermultiplets (see [38]) are uniquely determined by their associated Euclidean Clifford algebra, they inherit the corresponding Schur's property. The determination of the Schur's property of higher-length supermultiplets requires the compatibility with the dressing (this implies that the most general commuting matrix of the root supermultiplet is restricted by the further requirement of commuting with the dressing matrix D discussed in Introduction). The analysis for the minimal supermultiplets has been presented in [65]. We extend here the investigation of [65] to the case of the non-minimal $\mathcal{N}=4$ supermultiplets.

We are looking for the most general real-valued antisymmetric matrix Σ of the form

$$\Sigma = \left(\begin{array}{cc} \Sigma^{up} & 0 \\ 0 & \Sigma^{down} \end{array} \right), \tag{2.12}$$

where Σ^{up} (Σ^{down}) is an 8×8 matrix acting on the bosonic (respectively, fermionic) fields, commuting with the 4 non-minimal supersymmetry operators $\widehat{Q}_I$, i.e.

$$[\widehat{Q}_I, \Sigma] \;=\; 0, \quad for \quad I = 1,2,3,4. \tag{2.13}$$

For the non-minimal, reducible but indecomposable $(8,8)$ root supermultiplet, the most general matrix Σ can be written as $\Sigma = \sum_{i=1}^{i=6} \lambda_i \Sigma_i$, where the matrices $\Sigma_i = {\Sigma_i}^{up} \oplus {\Sigma_i}^{down}$ are the generators of the $su(2)\oplus su(2)$ Lie algebra.

Without loss of generality we can work with our previous convention. We have, explicitly,

$$\begin{array}{lll} {\Sigma_1}^{up} = \tau_2\otimes\tau_2\otimes\tau_A, & {\Sigma_2}^{up} = \mathbf{1}_2\otimes\tau_A\otimes\mathbf{1}_2, & {\Sigma_3}^{up} = \tau_2\otimes\tau_1\otimes\tau_A, \\ {\Sigma_4}^{up} = \tau_A\otimes\tau_1\otimes\mathbf{1}_2, & {\Sigma_5}^{up} = \tau_1\otimes\mathbf{1}_2\otimes\tau_A, & {\Sigma_6}^{up} = \tau_A\otimes\tau_2\otimes\mathbf{1}_2, \end{array} \tag{2.14}$$

while ${\Sigma_i}^{down} = {\Sigma_i}^{up}$ for $i = 1,2,3$ and ${\Sigma_i}^{down} = -{\Sigma_i}^{up}$ for $i = 4,5,6$.

The unitary invariant groups, commuting with the 4 supersymmetry operators of the length-3, reducible but indecomposable, non-minimal supermultiplets are given by the table

supermultiplet:	*commuting group:*
$(2,8,6)_A$	$U(1)$
$(2,8,6)_B$	$\mathbf{1}$
$(4,8,4)_A$	$\mathbf{1}$
$(4,8,4)_B$	$SU(2)$
$(4,8,4)_C$	$\mathbf{1}$
$(4,8,4)_D$	$U(1)\otimes U(1)$
$(6,8,2)_A$	$U(1)$
$(6,8,2)_B$	$\mathbf{1}$

(2.15)

In the remaining cases, for field content $(k, 8, 8-k)$ with k odd, the most general unitary group is just the identity group $\mathbf{1}$.

For higher length ($l = 4, 5$), the list of $\mathcal{N} = 4$ non-minimal supermultiplets with a connected graph is given by the table

field content:	$\mathcal{N}_{max}$:
$(1, 4, 6, 4, 1)$	4
$(1, 4, 7, 4) \leftrightarrow (4, 7, 4, 1)$	4
$(1, 5, 7, 3) \leftrightarrow (3, 7, 5, 1)$	5
$(1, 6, 7, 2) \leftrightarrow (2, 7, 6, 1)$	5
$(2, 6, 6, 2)$	6
$(1, 7, 7, 1)$	7

(2.16)

which reports their field content and the maximal number $\mathcal{N}_{max}$ of their oxidized supersymmetry. The supermultiplets connected by arrows are dual under mirror symmetry, while the remaining ones are self-dual. The above supermultiplets are non-chiral and appear in two variants (bosonic or fermionic).

The tables (2.10) and (2.16) present the complete list of $\mathcal{N} = 4$ non-minimal supermultiplets associated to a connected graph.

Some remarks should be made. The notion of *connectivity symbol* allows to discriminate inequivalent supermultiplets possessing the same field content, commuting group and node choice group. Indeed, if we compare the $(4, 8, 4)_A$ with the $(4, 8, 4)_C$ supermultiplet we notice that their only difference lies in their respective connectivity symbol.

2.3.4 Gordian transformation

Four cases in table (2.10), involving the supermultiplets $(8, 8)_{conn}$, $(2, 8, 6)_B$, $(4, 8, 4)_D$ and $(6, 8, 2)_B$, are particularly intriguing. For each such supermultiplet a related non-chiral pure supermultiplet entering table (2.7) (that is, admitting a disconnected graph presentation) and possessing the same field content, connectivity symbol, commuting group and node choice group, can be found. We have the equivalence

$$\begin{array}{|ccc|} \hline Connected: & & Disconnected: \\ \hline (8,8)_{conn} & \Leftrightarrow & (8,8)_{FR,\Delta=0} \\ \hline (2,8,6)_B & \Leftrightarrow & (2,8,6)_{b,\Delta=0} \\ \hline (4,8,4)_D & \Leftrightarrow & (4,8,4)_{c,\Delta=0} \\ \hline (6,8,2)_B & \Leftrightarrow & (6,8,2)_{b,\Delta=0} \\ \hline \end{array} \tag{2.17}$$

It is possible to prove, under general considerations, that identical properties (field content, connectivity symbol, commuting group and node choice group) shared by inequivalent graphs imply that they represent the same supermultiplet (as far as supersymmetry transformations are concerned). A supermultiplet can be associated with inequivalent graph presentations (the group of equivalence for graphs is based on the local and global moves discussed in the Introduction). The gordian transformations presented in the Introduction induce an equivalence relation for supermultiplets. They cannot be regarded, however, to be an equivalence relations for the graphs.

The four equivalence relations expressed in (2.17)admit explicit gordian transformations which allow "cutting" each one of the connected graphs on the left into the union of two separate disconnected graphs on the right. it is important to distinguish between the equivalence class of pure supermultiplets and the equivalence class of their associated graphs.

We present here, explicitly, the gordian transformation involving the supermultiplets in (2.17) with field content $(4,8,4)$ (the three remaining cases in (2.17) admit similar gordian transformations). Let $(\upsilon_0,\upsilon_1,\bar{\upsilon}_0,\bar{\upsilon}_1;\lambda_0,\lambda_1,\lambda_2,\lambda_3,\bar{\lambda}_0,\bar{\lambda}_1,\bar{\lambda}_2,\bar{\lambda}_3;g_2,g_3,\bar{g}_2,\bar{g}_3)$ be the component fields associated to the connected $(4,8,4)_D$ graph and the component fields $(u_0,u_1,\bar{u}_0,\bar{u}_1;\psi_0,\psi_1,\psi_2,\psi_3,\bar{\psi}_0,\bar{\psi}_1,\bar{\psi}_2,\bar{\psi}_3;f_2,f_3,\bar{f}_2,\bar{f}_3)$ associated to the disconnected nonchiral graph $(4,8,4)_{c,\Delta=0}$. The four supertransformations (painted in blue, red, green and yellow) are directly read from Figure **2.4**. The fields of lowest engineering dimension are related through the gordian transformation

$$u_0 = \upsilon_0 - \bar{\upsilon}_0, \quad u_1 = \upsilon_1 - \bar{\upsilon}_1, \quad \bar{u}_0 = \upsilon_0 + \bar{\upsilon}_0, \quad \bar{u}_1 = \upsilon_1 + \bar{\upsilon}_1 \tag{2.18}$$

(the gordian transformations relating the fields with higher engineering dimension are directly read from (2.18)).

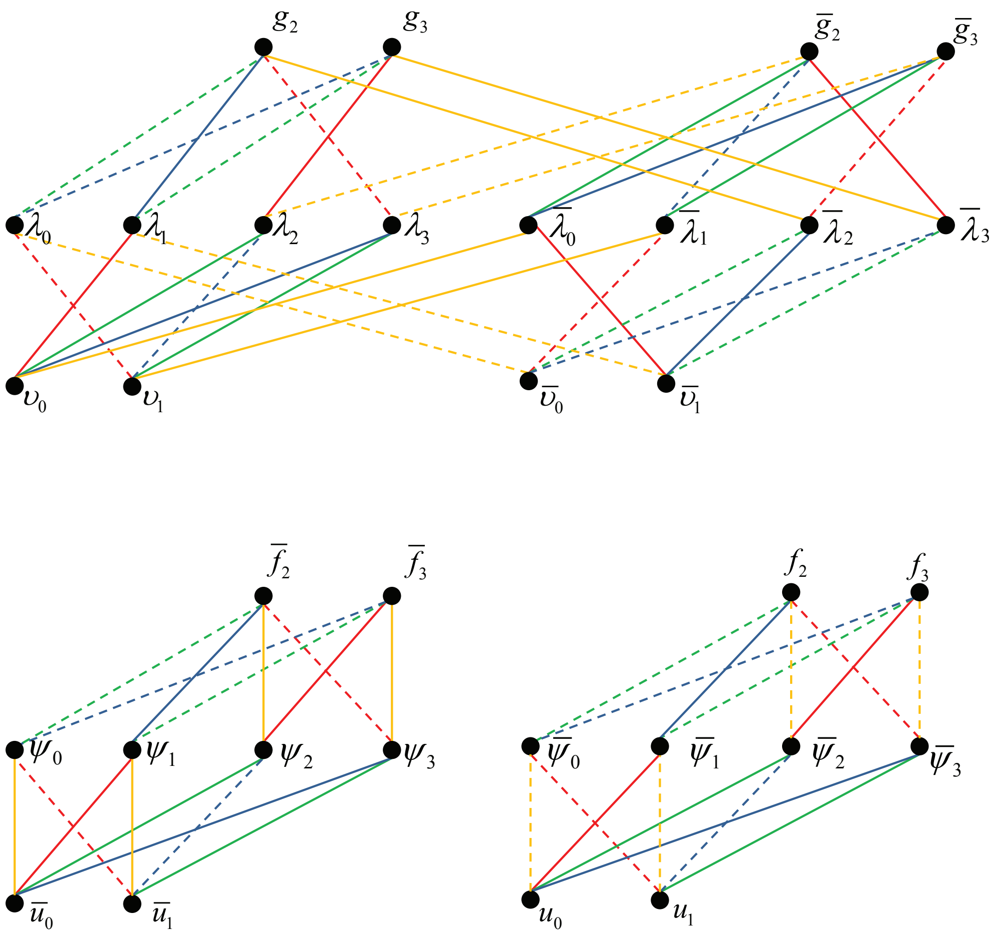

Figure 2.4: The $(4,8,4)_D$ connected graph (above) and the $(4,8,4)_{c,\Delta=0}$ disconnected graph (below), related by the gordian transformation $u_0 = \upsilon_0 - \bar{\upsilon}_0$, $u_1 = \upsilon_1 - \bar{\upsilon}_1$, $\bar{u}_0 = \upsilon_0 + \bar{\upsilon}_0$ and $\bar{u}_1 = \upsilon_1 + \bar{\upsilon}_1$.

In $N = 4$ the necessary condition for gordian move is that the dressing matrix should commute with $Q_1 Q_2 Q_3 Q_4$.

2.3.5 Non-minimal, connected, pure $\mathcal{N} = 4$ supermultiplets revisited

In order not to over count the inequivalent supermultiplets, we have to eliminate from (2.10) the supermultiplets which, under a gordian transformation, can be related to a disconnected graph. It is convenient to apply the notion of *pure, connected supermultiplet* only to those pure supermultiplets which do not admit any presentation in terms of a disconnected graph. Therefore, as we have seen, a supermultiplet with a connected graph is not necessarily, according to this definition, a connected supermultiplet.

The non-minimal, connected, pure $\mathcal{N} = 4$ supermultiplets have length $l = 3, 4, 5$ (the length-2 root supermultiplet $(8, 8)$ is not connected, see (2.17)). The connected supermultiplet of length $l = 4, 5$ are given in (2.16) . It is indeed easily proved that no gordian transformation can transform them into a disconnected graph with same length and field content.

The complete list of (dually related under mirror symmetry) non-minimal connected supermultiplets of length $l = 3$ is the restriction of (2.10) given by

$(1, 8, 7)_{conn}$	$\leftrightarrow$	$(7, 8, 1)_{conn}$
$(2, 8, 6)_A$	$\leftrightarrow$	$(6, 8, 2)_A$
$(3, 8, 5)_A$	$\leftrightarrow$	$(5, 8, 3)_A$
$(3, 8, 5)_B$	$\leftrightarrow$	$(5, 8, 3)_B$
	$(4, 8, 4)_A$	
	$(4, 8, 4)_B$	
	$(4, 8, 4)_C$	

(2.19)

(the supermultiplets with $(4, 8, 4)$ field content are self-dual).

It is useful to present a further table describing the decompositions of the above supermultiplets into $\mathcal{N} = 3$ supermultiplets. Indeed, the supermultiplets in (2.19) can be regarded as two minimal $\mathcal{N} = 3$ supermultiplets linked together by a fourth supersymmetry. Since we have 4 supersymmetry transformations that can be singled as the "4^{th}"

supersymmetry, there are 4 ways of decomposing the (2.19) supermultiplets into pairs of $\mathcal{N}=3$ supermultiplets. The following results are obtained

$$
\begin{array}{|ll|}
\hline
 & \mathcal{N}=3 \quad decomposition \\
\hline
(7,8,1): & \mathbf{4}\times\{(3,4,1)+(4,4)\} \\
\hline
(6,8,2)_A: & \mathbf{2}\times\{(3,4,1)+(3,4,1)\} \\
 & \mathbf{2}\times\{(2,4,2)+(4,4)\} \\
\hline
(5,8,3)_A: & \mathbf{3}\times\{(3,4,1)+(2,4,2)\} \\
 & \mathbf{1}\times\{(1,4,3)+(4,4)\} \\
\hline
(5,8,3)_B: & \mathbf{4}\times\{(3,4,1)+(2,4,2)\} \\
\hline
(4,8,4)_A: & \mathbf{4}\times\{(3,4,1)+(1,4,3)\} \\
\hline
(4,8,4)_B: & \mathbf{3}\times\{(2,4,2)+(2,4,2)\} \\
 & \mathbf{1}\times\{(4,4,0)+(0,4,4)\} \\
\hline
(4,8,4)_C: & \mathbf{2}\times\{(2,4,2)+(2,4,2)\} \\
 & \mathbf{2}\times\{(3,4,1)+(1,4,3)\} \\
\hline
\end{array}
\tag{2.20}
$$

An interesting observation is the following. Two cases, $(4,8,4)_B$ and $(4,8,4)_C$, produce decompositions into pairs of $(2,4,2)$ $\mathcal{N}=3$ supermultiplets. For them the coloring plays an important role. The $(2,4,2)$ supermultiplets produced from $(4,8,4)_B$ possess the same coloring, while the supermultiplets produced from $(4,8,4)_C$ possess a different coloring. This property can be stated differently. Starting from two $\mathcal{N}=3$ $(2,4,2)$ supermultiplets of the same coloring there exists a unique connected supermultiplet (given by $\mathcal{N}=4$ $(4,8,4)_B$) that can be obtained by adding a compatible fourth supersymmetry. Starting from two $\mathcal{N}=3$ $(2,4,2)$ supermultiplets of different coloring, the unique supermultiplet that can be obtained is given by $\mathcal{N}=4$ $(4,8,4)_C$.

2.3.6 The $(\mathcal{N}=4)\Rightarrow(\mathcal{N}=5)$ oxidation

Non-minimal $\mathcal{N}=4$ representations (both the reducible but indecomposable and the fully reducible ones acting on 8 bosonic and 8 fermionic fields) can be "oxidized" to $\mathcal{N}=5$

minimal representations, adding a 5^{th} supersymmetry transformation compatible with the 4 previous supersymmetry transformations.

As we stated before, it has been proven that the $\mathcal{N}=4$ non-minimal, pure, disconnected chiral ($\Delta=2$) supermultiplets are not oxidizable to any $\mathcal{N}=5$ supermultiplet.

The minimal $\mathcal{N}=5$ representations have been classified in [63]. There are inequivalent length-3 $\mathcal{N}=5$ representations of field content $(k,8,8-k)$, for $k=2,3,4,5,6$, which are discriminated by their connectivity symbol.

We present a series of tables specifying which minimal $\mathcal{N}=5$ representation (in a column) results from the oxidation of a non-minimal $\mathcal{N}=4$ representation (in a row). A positive answer is marked by an "X". The representations are expressed in terms of their connectivity symbol. The $\mathcal{N}=4$ reducible but indecomposable representations (associated with connected graphs) appear in the upper rows; the $\mathcal{N}=4$ fully reducible, non chiral $\Delta=0$ representations (associated with disconnected graphs) appear in the lower rows.

We get the following oxidation tables, for each given field content $(k,8,8-k)$ with $k=2,3,4,5,6$.

For $(2,8,6)$ we have

	$(\mathcal{N}=4)\Rightarrow(\mathcal{N}=5):$	$2_5+2_4+4_3$	6_4+2_3
Connected:	$2_4+4_3+2_2$	X	X
	8_3		X
Disconnected	4_4+4_2	X	
	8_3		X

(2.21)

For $(3,8,5)$ we have

	$(\mathcal{N}=4)\Rightarrow(\mathcal{N}=5):$	$1_5+3_4+4_2$	$2_4+5_3+1_2$
Connected:	$1_4+3_3+3_2+1_1$	X	X
	4_3+4_2		X
Disconnected:	4_4+4_1	X	
	4_3+4_2		X

(2.22)

For $(4,8,4)$ we have

	$(\mathcal{N}=4) \Rightarrow (\mathcal{N}=5):$	4_4+4_1	$1_4+3_3+3_2+1_1$	4_3+4_2
Connected:	$1_4+6_2+1_0$		X	
	4_3+4_1	X		X
	$2_3+4_2+2_1$		X	X
	8_2			X
Disconnected:	4_4+4_0	X		
	4_3+4_1		X	
	8_2			X

(2.23)

For $(5,8,3)$ we have

	$(\mathcal{N}=4) \Rightarrow (\mathcal{N}=5):$	$4_3+3_1+1_0$	$1_3+5_2+2_1$
Connected:	$1_5+3_2+3_1+1_0$	X	X
	4_2+4_1		X
Disconnected:	4_3+4_0	X	
	4_2+4_1		X

(2.24)

For $(6,8,2)$ we have

	$(\mathcal{N}=4) \Rightarrow (\mathcal{N}=5):$	$4_2+2_1+2_0$	2_2+6_1
Connected:	$2_2+4_1+2_0$	X	X
	8_1		X
Disconnected:	4_2+4_0	X	
	8_1		X

(2.25)

Both the reducible but indecomposable and the fully reducible non-minimal $\mathcal{N}=4$ representations of field content $(8,8)$, $(1,8,7)$ and $(7,8,1)$ are oxidized to the minimal $\mathcal{N}=5$ representations which are uniquely specified by the corresponding field content $(8,8)$, $(1,8,7)$ and $(7,8,1)$.

From these tables, it is clear that inequivalent $\mathcal{N}=4$ graphs that represent the same supermultiplet (presented in the table 2.17) oxidize in the same manner, to the same

$\mathcal{N} = 5$ supermultiplet. Combining these results with the results presented in [69] we get that all non-minimal length-3 representations are oxidized to the maximal number $\mathcal{N}_{max} = 8$ of extended supersymmetries.

The maximal number $\mathcal{N}_{max}$ of supersymmetries operators (oxidized supersymmetries) acting on non-minimal $\mathcal{N} = 4$ representations of length $l = 4, 5$ is given by the table

field content:	$\mathcal{N}_{max}$:
$(1,7,7,1)$	7
$(2,6,6,2)$	6
$(1,6,7,2) \leftrightarrow (2,7,6,1)$	6
$(1,5,7,3) \leftrightarrow (3,7,5,1)$	5
$(1,4,7,4) \leftrightarrow (4,7,4,1)$	4
$(1,4,6,4,1)$	4

(2.26)

2.4 An entangled $\mathcal{N} = 4$ $(3,8,5)$ supermultiplet

In [73] the possibility of realizing supersymmetry transformations which are not *adinkrizable* (in the language of [73]), namely that cannot be expressed through a graphical presentation, was raised.Till very recently no explicit example was produced (so that it was even unclear whether this notion could be applied to a non-empty set). Constructions of non-adinkrizable supermultiplets (in a different context and using different methods from the one proposed here) has been recently discussed in [74, 75]. Explicit examples of supermultiplets which were not described in a way that straightforwardly yielded an Adinkra can be found in [76] and [77], but it was not proven that the component fields could not be rearranged into a form that leads to an Adinkra. This is indeed a hot topic, see e.g. two othar recent papers on the existence of non-adinkrizable supermultiplets [74, 75]. We present here an explicit construction of such type of supermultiplet. We prefer to call it an *entangled* supermultiplet. It is constructed by interpolating two adinkrizable supermultiplets (*pure supermultiplets*, in our language). We believe that the interpolation provides a natural framework to construct entangled supermultiplets.

The specific example of an entangled supermultiplet is given here by interpolating the non-minimal pure $\mathcal{N}=4$ supermultiplets $(3,8,5)_{b,\Delta=0}$ (a non-chiral disconnected supermultiplet, see table (2.7) and the following discussion) and $(3,8,5)_B$ (a connected supermultiplet, presented in (2.10)). The four supercharges acting on $(3,8,5)_{b,\Delta=0}$ can be taken as Q_1, Q_2, Q_3, Q_4. Four supercharges act on $(3,8,5)_B$, with Q_5 replacing Q_4. The entangled supermultiplet $(3,8,5)_\theta$ is constructed in terms of an interpolating angle θ. The four supercharges acting on it are Q_1, Q_2, Q_3 and $Q' = Q_4\cos\theta + Q_5\sin\theta$. It is straightforward to show that, for $\theta \neq \frac{n\pi}{2}$, the $(3,8,5)_\theta$ supermultiplet does not admit a graphical presentation (no recombination of the fields with given engineering dimension allows to do that). The pure supermultiplets $(3,8,5)_{b,\Delta=0}$ and $(3,8,5)_B$ are recovered for, respectively, $\theta=0$ and $\theta=\frac{\pi}{2}$.

Explicitly, the component fields entering the $(3,8,5)_\theta$ supermultiplet can be expressed as

$$(\upsilon_0, \upsilon_1, \bar{\upsilon}_0; \lambda_0, \lambda_1, \lambda_2, \lambda_3, \bar{\lambda}_0, \bar{\lambda}_1, \bar{\lambda}_2, \bar{\lambda}_3; g_2, g_3, \bar{g}_1, \bar{g}_2, \bar{g}_3).$$

The four supersymmetry transformations Q_1, Q_2, Q_3, Q' acting on this set of fields are given by

	Q_1	Q_2	Q_3	$Q' = Q_4 \cos\theta + Q_5 \sin\theta$
υ_0	λ_1	λ_2	λ_3	$\lambda_0 \cos\theta + \bar{\lambda}_0 \sin\theta$
υ_1	$-\lambda_0$	λ_3	$-\lambda_2$	$\lambda_1 \cos\theta + \bar{\lambda}_1 \sin\theta$
λ_0	$-\dot{\upsilon}_1$	$-g_2$	$-g_3$	$\dot{\upsilon}_0 \cos\theta - \dot{\bar{\upsilon}}_0 \sin\theta$
λ_1	$\dot{\upsilon}_0$	$-g_3$	g_2	$\dot{\upsilon}_1 \cos\theta - \bar{g}_1 \sin\theta$
λ_2	g_3	$\dot{\upsilon}_0$	$-\dot{\upsilon}_1$	$g_2 \cos\theta - \bar{g}_2 \sin\theta$
λ_3	$-g_2$	$\dot{\upsilon}_1$	$\dot{\upsilon}_0$	$g_3 \cos\theta - \bar{g}_3 \sin\theta$
g_2	$-\dot{\lambda}_3$	$-\dot{\lambda}_0$	$\dot{\lambda}_1$	$\dot{\lambda}_2 \cos\theta + \dot{\bar{\lambda}}_2 \sin\theta$
g_3	$\dot{\lambda}_2$	$-\dot{\lambda}_1$	$-\dot{\lambda}_0$	$\dot{\lambda}_3 \cos\theta + \dot{\bar{\lambda}}_3 \sin\theta$
$\bar{\upsilon}_0$	$-\bar{\lambda}_1$	$-\bar{\lambda}_2$	$-\bar{\lambda}_3$	$\bar{\lambda}_0 \cos\theta - \lambda_0 \sin\theta$
$\bar{\lambda}_0$	$\bar{g}_1$	$\bar{g}_2$	$\bar{g}_3$	$\dot{\bar{\upsilon}}_0 \cos\theta + \dot{\upsilon}_0 \sin\theta$
$\bar{\lambda}_1$	$-\dot{\bar{\upsilon}}_0$	$\bar{g}_3$	$-\bar{g}_2$	$\bar{g}_1 \cos\theta + \dot{\upsilon}_1 \sin\theta$
$\bar{\lambda}_2$	$-\bar{g}_3$	$-\dot{\bar{\upsilon}}_0$	$\bar{g}_1$	$\bar{g}_2 \cos\theta + g_2 \sin\theta$
$\bar{\lambda}_3$	$\bar{g}_2$	$-\bar{g}_1$	$-\dot{\bar{\upsilon}}_0$	$\bar{g}_3 \cos\theta + g_3 \sin\theta$
$\bar{g}_1$	$\dot{\bar{\lambda}}_0$	$-\dot{\bar{\lambda}}_3$	$\dot{\bar{\lambda}}_2$	$\dot{\bar{\lambda}}_1 \cos\theta - \dot{\lambda}_1 \sin\theta$
$\bar{g}_2$	$\dot{\bar{\lambda}}_3$	$\dot{\bar{\lambda}}_0$	$-\dot{\bar{\lambda}}_1$	$\dot{\bar{\lambda}}_2 \cos\theta - \dot{\lambda}_2 \sin\theta$
$\bar{g}_3$	$-\dot{\bar{\lambda}}_2$	$\dot{\bar{\lambda}}_1$	$\dot{\bar{\lambda}}_0$	$\dot{\bar{\lambda}}_3 \cos\theta - \dot{\lambda}_3 \sin\theta$

(2.27)

Our given example is suitably chosen to simplify the proof that there exists no linear combination of the component fields which guarantees a graphical presentation ("Adinkra") of the interpolated supermultiplet in the interval $0 < \theta < \frac{\pi}{2}$ of the interpolating angle θ. An important observation is that the interpolating mechanism is a general phenomenon and that entangled supermultiplets tend to proliferate for large $\mathcal{N}$ values of the one-dimensional $\mathcal{N}$-Extended Supersymmetry. It is also important to notice that the entangled supermultiplet has dynamical consequences. An $\mathcal{N} = 4$, one-dimensional, off-shell invariant sigma-model with a three-dimensional target is based on it. Its action (3.23) carries an explicit dependence on θ. This model is supersymmetric only under the supertransformations specified by the entangled supermultiplet. Therefore, entangled supermultiplets allow to enlarge the class of supersymmetric actions so far considered.

2.5 Discussion

We have stressed in the Introduction the difference between two types of moves (local and global) acting on graphs and the so-called "gordian transformations" acting on pure supermultiplets. As a result a given pure supermultiplet can be associated with inequivalent (under local and global moves) graphs. In certain cases, in particular, a given supermultiplet can be associated to both a disconnected and a connected graph. In order to avoid overcounting, the notion of *connected pure supermultiplets* (the supermultiplets which are associated to connected graphs only) has been introduced. The classification of the non-minimal, $\mathcal{N} = 4$, pure, connected supermultiplets has been presented in Section **2.3**with the help of information contained in the connectivity symbol, commuting group, the node choice group and its possible inequivalent presentations (colorings) under local moves.

The notion of "coloring", similarly to the notion of "chiral" supermultiplets [61], plays an important role in supersymmetry representations. It is well-known that minimal $\mathcal{N} = 8$ supermultiplets are non-chiral [61], being necessarily obtained by linking together (with extra supertransformations) two minimal $\mathcal{N} = 4$ supermultiplets of opposite chirality. We have shown (see the discussion at the end of Section **2.3.5**) that inequivalent non-minimal $\mathcal{N} = 4$ supermultiplets are obtained by linking together two $\mathcal{N} = 3$ $(2, 4, 2)$ supermultiplets based on the fact that their coloring is either the same or different. This property naturally extends to the construction of non-minimal $\mathcal{N} = 5$ supermultiplets by linking together non-minimal $\mathcal{N} = 4$ supermultiplets (whose respective colorings have been listed here).

The non-minimal $\mathcal{N} = 4$ linear supermultiplets are progressively oxidized to minimal $\mathcal{N} = 5, 6, 7, 8$ linear supermultiplets possessing 8 bosonic and 8 fermionic component fields. Following [69], the word *oxidation* has been here consistently used in a specific and restricted sense, referring to the operation of enlarging the number of extended supersymmetries (from $\mathcal{N}$ to $\mathcal{N}+1$) acting on a supermultiplet with the same number of component fields.

We provided an explicit construction of a supersymmetric one-dimensional entangled supermultiplet (which does not admit a graphical presentation). The possibility of *non-adinkrizable* supermultiplets (here called "entangled") was raised in [73]. Till very recently no explicit example was produced [74, 75]. Our given example (based on the interpolation between two non-minimal $\mathcal{N}=4$ supermultiplets of $(3,8,5)$ field content) was suitably chosen to simplify the proof that there exists no "Adinkra" representation inside the interval $0<\theta<\frac{\pi}{2}$ of the interpolating angle θ. An important observation is that the interpolating mechanism is a general phenomenon and that entangled supermultiplets tend to proliferate for large $\mathcal{N}$ values of the one-dimensional $\mathcal{N}$-Extended Supersymmetry. It is also important to notice that the entangled supermultiplet has dynamical consequences. An $\mathcal{N}=4$, one-dimensional, off-shell invariant sigma-model with a three-dimensional target is based on it, that will be discussed in the next chapter.

Let us close this chapter by pointing out that the present results can be applied to investigate supersymmetry representations in presence of inhomogeneous terms [32], non-linear realizations of supersymmetry [80, 65], D-module representations of superconformal algebras and their associated superconformal mechanics (which are subjects of two chapters in this dissertation). All these extensions (inhomogeneous representations, non-linear realizations, D-module representations) are induced and derived from linear homogeneous supermultiplets, such as those investigated here.

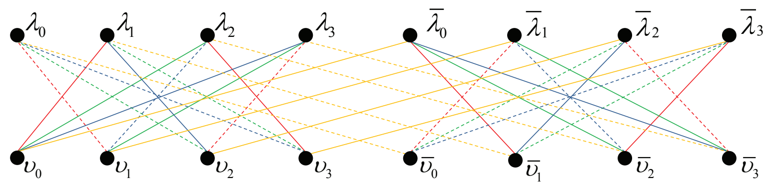

Figure 2.5: $\mathcal{N} = 4$ Connected Root

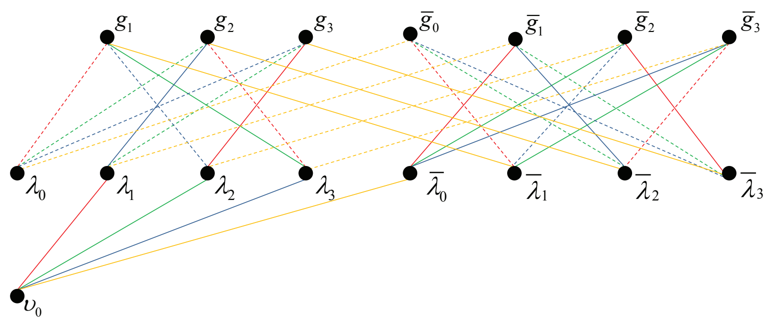

Figure 2.6: $(1, 8, 7)$ Non-minimal $\mathcal{N} = 4$, C.S. $4_4 + 4_3$

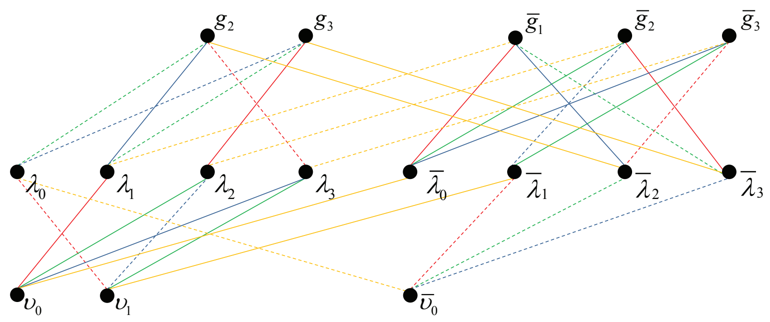

Figure 2.7: $(3, 8, 5)_B$ Non-minimal $\mathcal{N} = 4$, C.S. $4_3 + 4_2$

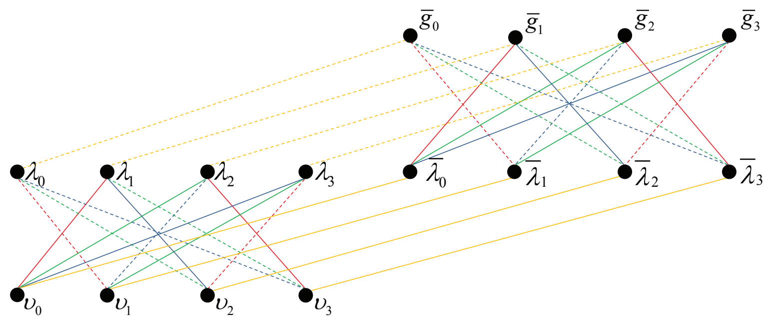

Figure 2.8: $(4, 8, 4)_B$ Non-minimal $\mathcal{N} = 4$, C.S. $4_3 + 4_1$

Chapter 3

Associated Sigma models

This Chapter is an edited version of second parts of the references [66]and [67] written in collaboration with M. Gonzales, K. Iga, and F. Toppan.

3.1 Introduction

This chapter deals with the construction of off-shell, $\mathcal{N} = 4$-invariant supersymmetric σ-models associated with each given non-minimal linear supermultiplet. $1D$ supersymmetric σ-models were first discussed in [81, 82]; minimal $\mathcal{N} = 4$ σ-models were constructed in [83]-[84], while minimal $\mathcal{N} = 8$-invariant σ-models were investigated in [35]. There are also supersymmetric σ-models based on non-linear realizations of the supersymmetry that we are not discussing here (for a partial list of references one can consult [65]).

The fields $x_j(t)$ ($j = 1, \ldots, k$) of lower-dimension in a supermultiplet can be assumed [37] to be bosonic and have 0-mass dimension. They are physically interpreted [70, 71, 65] as the target-coordinates of the associated σ-model. An $\mathcal{N} = 4$-invariant off-shell action $\mathcal{S}$, with the correct dimension of a kinetic term, is obtained [37, 70, 71, 65] through

$$\mathcal{S} = \int dt\mathcal{L} = \frac{1}{m}\int dtQ_1Q_2Q_3Q_4F(\vec{x}), \tag{3.1}$$

where the supersymmetry operators Q_i's act as graded derivatives and F is the prepotential. By construction, the action $\mathcal{S}$ is manifestly $\mathcal{N} = 4$-invariant no matter which is the choice of F (unconstrained prepotential).

In the case of a fully reducible supermultiplet given by the direct sum of two $\mathcal{N}=4$ irreducible supermultiplets (whose 0-mass dimension fields are denoted as $\vec{x}$, $\vec{y}$, respectively) we have that interacting terms involving the fields belonging to the irreducible supermultiplets arise provided that

$$F(\vec{x},\vec{y}) \quad \neq \quad A(\vec{x})+B(\vec{y}). \tag{3.2}$$

As a result, non-trivial interacting Lagrangians can be produced even from fully reducible representations (which are trivial, from the representation theory point of view), therefore justifying the attention we have to pay to them.

The (3.1) manifest $\mathcal{N}=4$ construction has been discussed in [71]. In several cases, however, this construction does not produce the most general $\mathcal{N}=4$ invariant action. The resulting Lagrangian can be of first order and furthermore, in the presence of fermionic sources[1], not all fields belonging to the given supermultiplet enter the Lagrangian.

On the other hand, even in those cases, the existence of a second-order Lagrangian involving all fields of the supermultiplet is known [37, 80]. To systematically construct them a novel approach is here presented (it will be referred as "Construction II", while (3.1) will be referred as "Construction I"). Construction II is outlined as follows. For a reducible length-3 $\mathcal{N}=4$ representation of field content $(k,8,8-k)$, we consider at first its associated root supermultiplet of length-2 (in a different context, the importance of invariant actions induced by the root supermultiplets has also been discussed in [85, 86, 87]). The root supermultiplet contains 8 bosonic fields x_i (the target coordinates) and 8 fermionic fields ψ_i (see Figure *2.5*). A Lagrangian $\mathcal{L}$ for the root supermultiplet is at first constructed by setting, as in (3.1),

$$\mathcal{L} \quad = \quad Q_1Q_2Q_3Q_4\Phi(\vec{x}). \tag{3.3}$$

[1]A fermionic source [88, 89, 63] is a fermionic $\frac{1}{2}$-mass dimension field which, in the graphical presentation of the supermultiplet, does not possess edges connecting it to the 0-mass dimension fields (the bosonic target coordinates). For $\mathcal{N}=4$, the connectivity symbol of a graph with r fermionic sources is expressed as $r_4+\ldots$, see Introduction.

An equivalent, up to a total derivative, Lagrangian $\overline{\mathcal{L}}$ functionally depends on the fields and their first-order time-derivatives alone. Therefore

$$\overline{\mathcal{L}} \equiv \overline{\mathcal{L}}(x_i, \dot{x}_i, \psi_i, \dot{\psi}_i). \tag{3.4}$$

The next step consists in constraining $\overline{\mathcal{L}}$ such that, for $j = k+1, \ldots, 8$, we have

$$\frac{\partial \overline{\mathcal{L}}}{\partial x_j} = 0, \tag{3.5}$$

eliminating its dependence on x_j's. This condition allows us to regard, according to the dressing procedure, the $\dot{x}_j$'s no longer as derivative fields, but as the auxiliary fields g_j of mass-dimension 1 entering the $(k, 8, 8-k)$ supermultiplet. We can therefore set

$$\begin{aligned} g_j &= \dot{x}_j \\ \overline{\mathcal{L}} &\equiv \overline{\mathcal{L}}(x_l, \dot{x}_l, \psi_i, \dot{\psi}_i, g_j), \end{aligned} \tag{3.6}$$

($l = 1, \ldots, k$, while $i = 1, \ldots, 8$ and $j = k+1, \ldots, 8$).

Setting (3.6) is not something innocuous. One has in fact to guarantee that the resulting action, after the "renaming" of the fields, is still $\mathcal{N} = 4$-invariant. Together with (3.5), this requirement produces a constraint on the prepotential $\Phi(x_l)$. In the following we will compare the invariant actions arising from the constructions I and II and discuss the constraints on Φ.

To derive the off-shell invariant actions we implemented a special package for *Maple 11*. For convenience we had to use different (but equivalent) conventions for the presentations of the non-minimal supermultiplets with respect to the explicit construction given in previous chapter. We present in Section **2** and **3** some selected cases which exemplify the general picture . In Section **2** we discuss the construction I. In Section **3** we discuss the construction II. Inequivalent actions are obtained for supermultiplets presenting the same field content, but differing in connectivity symbol.

In Section **4** we present an $\mathcal{N} = 4$-invariant σ model for the entangled supermultiplet $(3, 8, 5)_\theta$ presented in previous chapter, using the construction I. In this section we also show that inequivalent actions are obtained for different supermultiplets, sharing the same field content and connectivity symbol, but differing their commuting group.

In the Discussion we make comments on the obtained results.

3.2 Manifestly $\mathcal{N}=4$ σ-models for non-minimal supermultiplets

For the following length-3, non-minimal supermultiplets, the Construction I (see (3.1)) of the manifestly $\mathcal{N}=4$ off-shell invariant actions produces a first-order Lagrangian:

$$(1,8,7)_{red}, (2,8,6)_A, (3,8,5)_A, (4,8,4)_A, (4,8,4)_B. \tag{3.7}$$

With the only exception of $(4,8,4)_B$, these are the supermultiplets admitting fermionic sources.

In the remaining cases, namely for

$$\begin{gathered}(2,8,6)_B, (3,8,5)_B, (4,8,4)_C, (4,8,4)_D, (5,8,3)_A, \\ (5,8,3)_B, (6,8,2)_A, (6,8,2)_B, (7,8,1)_{red},\end{gathered} \tag{3.8}$$

the Construction I produces second-order Lagrangians.

For each supermultiplet entering (3.8), the Constructions I and II produce, up to a total derivative, the same Lagrangian. Therefore, the corresponding actions are (as a consequence of Construction I) manifestly $\mathcal{N}=4$-invariant and depend on an unconstrained prepotential. We present, for a few selected cases, the explicit computation of the Lagrangian. We write down the invariant Lagrangian, up to a total derivative, for $(4,8,4)_C$ (with connectivity symbol $2_3+4_2+2_1$) and $(2,8,6)_B$. We compare the latter result with the Lagrangian obtained from Construction "I" applied to the fully reducible supermultiplet $(2,8,6)_b$, characterized by the same connectivity symbol, 8_3, as $(2,8,6)_B$ and related by Gordian transformation (see 2.17).

The component fields (the υ's, barred or otherwise, denote the target coordinates, the λ's the fermionic fields and the g's the auxiliary fields) are respectively given by

$(\upsilon_1,\upsilon_2,\upsilon_3,\bar{\upsilon}_1;\lambda_0,\lambda_i,\bar{\lambda}_0,\bar{\lambda}_i;g_0,\bar{g}_0,\bar{g}_2,\bar{g}_3)$, with $i=1,2,3$, for $(4,8,4)_C$ and by $(\upsilon_0,\bar{\upsilon}_0;\lambda_0,\lambda_i,\bar{\lambda}_0,\bar{\lambda}_i;g_i,\bar{g}_i)$, with $i=1,2,3$, for both $(2,8,6)_B$ and $(2,8,6)_b$.

The supersymmetry transformations are here explicitly obtained by dressing the $\mathcal{N}=8$ root supermultiplet expressed in terms of the octonionic structure constants C_{ijk}, see [42, 37]

$$
\begin{aligned}
\widehat{Q}_i(\upsilon_0,\upsilon_j;\lambda_0,\lambda_j) &= (\lambda_i,-\delta_{ij}\lambda_0-C_{ijk}\lambda_k;-\dot{\upsilon}_i,\delta_{ij}\dot{\upsilon}_0+C_{ijk}\dot{\upsilon}_k),\\
\widehat{Q}_8(\upsilon_0,\upsilon_j;\lambda_0,\lambda_j) &= (\lambda_0,\lambda_j;\dot{\upsilon}_0,\dot{\upsilon}_j),
\end{aligned} \tag{3.9}
$$

where $i=1,..,7$ and the fields have been renamed in such a way to respect the $\mathcal{N}=4$ quaternionic subalgebra. We have the following results (the Einstein convention over repeated indexes is understood):

For $(4,8,4)_C$ the Lagrangian is

$$
\begin{aligned}
\mathcal{L} = {} & \Phi(\dot{\upsilon}_1^2+\dot{\upsilon}_2^2+\dot{\upsilon}_3^2+\dot{\bar{\upsilon}}_1^2+g_0^2+\bar{g}_0^2+\bar{g}_2^2+\bar{g}_3^2+\dot{\lambda}_0\lambda_0+\dot{\bar{\lambda}}_0\bar{\lambda}_0+ \\
& \dot{\lambda}_1\lambda_1+\dot{\lambda}_2\lambda_2+\dot{\lambda}_3\lambda_3+\dot{\bar{\lambda}}_1\bar{\lambda}_1+\dot{\bar{\lambda}}_2\bar{\lambda}_2+\dot{\bar{\lambda}}_3\bar{\lambda}_3)+ \\
& \Phi_1[\dot{\upsilon}_3(\bar{\lambda}_0\bar{\lambda}_2+\lambda_2\lambda_0+\bar{\lambda}_1\bar{\lambda}_3+\lambda_1\lambda_3)-\dot{\upsilon}_2(\bar{\lambda}_0\bar{\lambda}_3+\lambda_3\lambda_0-\bar{\lambda}_1\bar{\lambda}_2-\lambda_1\lambda_2)+ \\
& +g_0(\bar{\lambda}_2\bar{\lambda}_3-\lambda_2\lambda_3-\bar{\lambda}_0\bar{\lambda}_1+\lambda_1\lambda_0)+\bar{g}_0(\lambda_0\bar{\lambda}_1+\lambda_1\bar{\lambda}_0+\lambda_2\bar{\lambda}_3-\lambda_3\bar{\lambda}_2)+ \\
& \bar{g}_2(\lambda_2\bar{\lambda}_1+\lambda_1\bar{\lambda}_2-\lambda_0\bar{\lambda}_3+\lambda_3\bar{\lambda}_0)+\bar{g}_3(\lambda_0\bar{\lambda}_2-\lambda_2\bar{\lambda}_0+\lambda_3\bar{\lambda}_1+\lambda_1\bar{\lambda}_3)]+ \\
& \Phi_2[\dot{\upsilon}_3(\bar{\lambda}_0\bar{\lambda}_1+\lambda_1\lambda_0+\bar{\lambda}_2\bar{\lambda}_3+\lambda_2\lambda_3)+\dot{\upsilon}_1(\bar{\lambda}_0\bar{\lambda}_3+\lambda_3\lambda_0-\bar{\lambda}_1\bar{\lambda}_2-\lambda_1\lambda_2)+ \\
& +g_0(\bar{\lambda}_3\bar{\lambda}_1-\lambda_3\lambda_1-\bar{\lambda}_0\bar{\lambda}_2+\lambda_2\lambda_0)+\bar{g}_0(\lambda_0\bar{\lambda}_2+\lambda_2\bar{\lambda}_0+\lambda_2\bar{\lambda}_3-\lambda_3\bar{\lambda}_2)+ \\
& -\bar{g}_2(\lambda_0\bar{\lambda}_0+\lambda_1\bar{\lambda}_1-\lambda_2\bar{\lambda}_2+\lambda_3\bar{\lambda}_3)+\bar{g}_3(\lambda_3\bar{\lambda}_2+\lambda_2\bar{\lambda}_3-\lambda_0\bar{\lambda}_1+\lambda_1\bar{\lambda}_0)]+ \\
& \Phi_3[\dot{\upsilon}_2(\bar{\lambda}_0\bar{\lambda}_1+\lambda_1\lambda_0-\bar{\lambda}_2\bar{\lambda}_3-\lambda_2\lambda_3)-\dot{\upsilon}_1(\bar{\lambda}_0\bar{\lambda}_2+\lambda_2\lambda_0+\bar{\lambda}_1\bar{\lambda}_3+\lambda_1\lambda_3)+ \\
& +g_0(\bar{\lambda}_1\bar{\lambda}_2-\lambda_1\lambda_2-\bar{\lambda}_0\bar{\lambda}_3+\lambda_3\lambda_0)+\bar{g}_0(\lambda_0\bar{\lambda}_3+\lambda_3\bar{\lambda}_0+\lambda_1\bar{\lambda}_2-\lambda_2\bar{\lambda}_1)+ \\
& \bar{g}_2(\lambda_0\bar{\lambda}_1-\lambda_1\bar{\lambda}_0+\lambda_2\bar{\lambda}_3+\lambda_3\bar{\lambda}_2)-\bar{g}_3(\lambda_0\bar{\lambda}_0+\lambda_1\bar{\lambda}_1+\lambda_2\bar{\lambda}_2-\lambda_3\bar{\lambda}_3)]+ \\
& \Phi_{\bar{1}}[\dot{\upsilon}_1(\lambda_0\bar{\lambda}_0-\lambda_1\bar{\lambda}_1+\lambda_2\bar{\lambda}_2+\lambda_3\bar{\lambda}_3)-\dot{\upsilon}_2(\lambda_0\bar{\lambda}_3-\lambda_3\bar{\lambda}_0+\lambda_2\bar{\lambda}_1+\lambda_1\bar{\lambda}_2)+ \\
& \dot{\upsilon}_3(\lambda_0\bar{\lambda}_2-\lambda_2\bar{\lambda}_0-\lambda_3\bar{\lambda}_1-\lambda_1\bar{\lambda}_3)+g_0(\lambda_2\bar{\lambda}_3-\lambda_3\bar{\lambda}_2-\lambda_0\bar{\lambda}_1-\lambda_1\bar{\lambda}_0)+
\end{aligned} \tag{3.10}
$$

$\bar{g}_0(-\bar{\lambda}_0\bar{\lambda}_1 + \lambda_1\lambda_0 - \bar{\lambda}_2\bar{\lambda}_3 + \lambda_2\lambda_3) + \bar{g}_2(\bar{\lambda}_0\bar{\lambda}_3 + \lambda_3\lambda_0 + \bar{\lambda}_2\bar{\lambda}_1 + \lambda_2\lambda_1) +$

$\bar{g}_3(-\bar{\lambda}_0\bar{\lambda}_2 - \lambda_2\lambda_0 + \bar{\lambda}_3\bar{\lambda}_1 + \lambda_3\lambda_1)] +$

$\Phi_{\bar{1}\bar{1}}(\bar{\lambda}_1\lambda_1\lambda_2\bar{\lambda}_2 + \bar{\lambda}_1\lambda_1\lambda_3\bar{\lambda}_3 + \bar{\lambda}_0\lambda_1\bar{\lambda}_2\lambda_3 + \bar{\lambda}_0\lambda_2\bar{\lambda}_3\lambda_1 + \bar{\lambda}_0\lambda_1\bar{\lambda}_2\lambda_3 +$

$\bar{\lambda}_0\lambda_3\bar{\lambda}_1\lambda_2 - \bar{\lambda}_1\bar{\lambda}_2\bar{\lambda}_3\bar{\lambda}_0) +$

$\Phi_{11}(\bar{\lambda}_0\lambda_0\lambda_1\bar{\lambda}_1 + \bar{\lambda}_0\lambda_2\lambda_3\bar{\lambda}_1 + \bar{\lambda}_1\lambda_1\lambda_2\bar{\lambda}_2 + \bar{\lambda}_1\lambda_1\lambda_3\bar{\lambda}_3 + \lambda_0\lambda_2\bar{\lambda}_3\bar{\lambda}_1 + \lambda_0\lambda_3\bar{\lambda}_1\bar{\lambda}_2$

$-\lambda_1\lambda_2\lambda_3\lambda_0) +$

$\Phi_{22}(\bar{\lambda}_0\lambda_0\lambda_2\bar{\lambda}_2 + \bar{\lambda}_0\lambda_3\lambda_1\bar{\lambda}_2 + \bar{\lambda}_2\lambda_2\lambda_1\bar{\lambda}_1 + \bar{\lambda}_2\lambda_2\lambda_3\bar{\lambda}_3 + \lambda_0\lambda_1\bar{\lambda}_2\bar{\lambda}_3 + \lambda_0\lambda_3\bar{\lambda}_1\bar{\lambda}_2 -$

$-\lambda_1\lambda_2\lambda_3\lambda_0) +$

$\Phi_{33}(\bar{\lambda}_0\lambda_0\lambda_3\bar{\lambda}_3 + \bar{\lambda}_0\lambda_1\lambda_2\bar{\lambda}_3 + \bar{\lambda}_3\lambda_3\lambda_1\bar{\lambda}_1 + \bar{\lambda}_3\lambda_3\lambda_2\bar{\lambda}_2 + \lambda_0\lambda_1\bar{\lambda}_2\bar{\lambda}_3 + \lambda_0\lambda_2\bar{\lambda}_3\bar{\lambda}_1$

$-\lambda_1\lambda_2\lambda_3\lambda_0) +$

$\Phi_{\bar{1}1}(\lambda_1\lambda_0\lambda_2\bar{\lambda}_3 - \bar{\lambda}_0\bar{\lambda}_1\bar{\lambda}_2\lambda_3 + \lambda_0\lambda_3\lambda_1\bar{\lambda}_2 + \bar{\lambda}_0\bar{\lambda}_3\bar{\lambda}_1\lambda_2 + \lambda_0\lambda_2\lambda_3\bar{\lambda}_1 - \bar{\lambda}_0\bar{\lambda}_2\bar{\lambda}_3\lambda_1$

$-\lambda_1\lambda_2\lambda_3\bar{\lambda}_0 - \bar{\lambda}_1\bar{\lambda}_2\bar{\lambda}_3\lambda_0) +$

$\Phi_{\bar{1}2}(\lambda_0\lambda_2\lambda_3\bar{\lambda}_2 + \bar{\lambda}_2\bar{\lambda}_1\lambda_0\bar{\lambda}_0 + \bar{\lambda}_0\bar{\lambda}_2\bar{\lambda}_3\lambda_2 + \lambda_0\lambda_3\lambda_1\bar{\lambda}_1 - \bar{\lambda}_0\bar{\lambda}_3\bar{\lambda}_1\lambda_1 - \lambda_1\lambda_2\lambda_3\bar{\lambda}_3$

$-\bar{\lambda}_1\bar{\lambda}_2\lambda_3\bar{\lambda}_3) +$

$\Phi_{\bar{1}3}(\lambda_0\lambda_2\lambda_3\bar{\lambda}_3 + \bar{\lambda}_3\bar{\lambda}_1\lambda_0\bar{\lambda}_0 + \bar{\lambda}_0\bar{\lambda}_2\bar{\lambda}_3\lambda_3 + \lambda_0\lambda_1\lambda_2\bar{\lambda}_1 - \bar{\lambda}_0\bar{\lambda}_1\bar{\lambda}_2\lambda_1 - \lambda_1\lambda_3\lambda_2\bar{\lambda}_2$

$-\bar{\lambda}_1\bar{\lambda}_3\lambda_2\bar{\lambda}_2) +$

$\Phi_{12}(\bar{\lambda}_0\lambda_0\lambda_2\bar{\lambda}_1 - \lambda_0\bar{\lambda}_0\lambda_1\bar{\lambda}_2 + \bar{\lambda}_0\lambda_2\lambda_3\bar{\lambda}_2 - \lambda_0\bar{\lambda}_2\bar{\lambda}_3\lambda_2 + \bar{\lambda}_0\lambda_3\lambda_1\bar{\lambda}_1 - \lambda_0\bar{\lambda}_3\bar{\lambda}_1\lambda_1$

$+\bar{\lambda}_1\lambda_2\lambda_3\bar{\lambda}_3 + \bar{\lambda}_2\lambda_1\lambda_3\bar{\lambda}_3) +$

$\Phi_{13}(\bar{\lambda}_0\lambda_0\lambda_3\bar{\lambda}_1 - \lambda_0\bar{\lambda}_0\lambda_1\bar{\lambda}_3 + \bar{\lambda}_0\lambda_1\lambda_2\bar{\lambda}_1 - \lambda_0\bar{\lambda}_1\bar{\lambda}_2\lambda_1 + \bar{\lambda}_0\lambda_2\lambda_3\bar{\lambda}_3 - \lambda_0\bar{\lambda}_2\bar{\lambda}_3\lambda_3$

$+\bar{\lambda}_1\lambda_3\lambda_2\bar{\lambda}_2 + \bar{\lambda}_3\lambda_1\lambda_2\bar{\lambda}_2) +$

$\Phi_{23}(\bar{\lambda}_0\lambda_0\lambda_3\bar{\lambda}_2 - \lambda_0\bar{\lambda}_0\lambda_2\bar{\lambda}_3 + \bar{\lambda}_0\lambda_1\lambda_2\bar{\lambda}_2 - \lambda_0\bar{\lambda}_1\bar{\lambda}_2\lambda_2 + \bar{\lambda}_0\lambda_3\lambda_1\bar{\lambda}_3 - \lambda_0\bar{\lambda}_3\bar{\lambda}_1\lambda_3$

$+\bar{\lambda}_2\lambda_3\lambda_1\bar{\lambda}_1 + \bar{\lambda}_3\lambda_2\lambda_1\bar{\lambda}_1) +$

$\Omega(\dot{v}_1\dot{\bar{v}}_1 + g_0\bar{g}_0 + \dot{v}_2\bar{g}_2 + \dot{v}_3\bar{g}_3 + \lambda_0\dot{\bar{\lambda}}_0 + \lambda_1\dot{\bar{\lambda}}_1 + \lambda_2\dot{\bar{\lambda}}_2 + \lambda_3\dot{\bar{\lambda}}_3) +$

$\Omega_1(\dot{v}_1\lambda_1\bar{\lambda}_1 + \dot{v}_2\lambda_2\bar{\lambda}_1 + \dot{v}_3\lambda_3\bar{\lambda}_1 + g_0\lambda_0\bar{\lambda}_1 - \bar{g}_0\lambda_2\lambda_3 + \bar{g}_2\lambda_0\lambda_3 - \bar{g}_3\lambda_0\lambda_2) +$

$\Omega_2(\dot{v}_1\lambda_1\bar{\lambda}_2 + \dot{v}_2\lambda_2\bar{\lambda}_2 + \dot{v}_3\lambda_3\bar{\lambda}_2 - \dot{\bar{v}}_1\lambda_0\lambda_3 - g_0\bar{\lambda}_2\lambda_0 - \bar{g}_0\lambda_3\lambda_1 + \bar{g}_3\lambda_0\lambda_1) +$

$$
\begin{aligned}
&\Omega_3(\dot{\upsilon}_1\lambda_1\bar{\lambda}_3+\dot{\upsilon}_2\lambda_2\bar{\lambda}_3+\dot{\upsilon}_3\lambda_3\bar{\lambda}_3+\dot{\bar{\upsilon}}_1\lambda_0\lambda_2-g_0\bar{\lambda}_3\lambda_0-\bar{g}_0\lambda_1\lambda_2-\bar{g}_2\lambda_0\lambda_1)+\\
&\Omega_{\bar{1}}(\dot{\upsilon}_1(\lambda_2\bar{\lambda}_2+\lambda_3\bar{\lambda}_3)+\dot{\upsilon}_2\bar{\lambda}_0\bar{\lambda}_3-\dot{\upsilon}_3\bar{\lambda}_0\bar{\lambda}_2-g_0\bar{\lambda}_2\bar{\lambda}_3+\bar{g}_0\bar{\lambda}_0\lambda_1+\bar{g}_2\bar{\lambda}_2\lambda_1+\bar{g}_3\bar{\lambda}_3\lambda_1)+\\
&\Omega_{\bar{1}1}(\lambda_0\bar{\lambda}_0\lambda_1\bar{\lambda}_1+\lambda_1\bar{\lambda}_1\lambda_2\bar{\lambda}_2+\lambda_1\bar{\lambda}_1\lambda_3\bar{\lambda}_3)+\\
&\Omega_{\bar{1}2}(\lambda_0\bar{\lambda}_0\lambda_1\bar{\lambda}_2+\lambda_1\bar{\lambda}_2\lambda_3\bar{\lambda}_3)+\\
&\Omega_{\bar{1}3}(\lambda_0\bar{\lambda}_0\lambda_1\bar{\lambda}_3+\lambda_1\bar{\lambda}_3\lambda_2\bar{\lambda}_2)+\\
&\Omega_{12}(\lambda_2\lambda_0\lambda_3\bar{\lambda}_2-\lambda_0\lambda_3\lambda_1\bar{\lambda}_1)+\\
&\Omega_{13}(\lambda_1\lambda_0\lambda_2\bar{\lambda}_1-\lambda_0\lambda_2\lambda_3\bar{\lambda}_3)+\\
&\Omega_{23}(\lambda_1\lambda_0\lambda_2\bar{\lambda}_2-\lambda_0\lambda_3\lambda_1\bar{\lambda}_3),
\end{aligned}
$$

where $F(\upsilon_1,\upsilon_2,\upsilon_3,\bar{\upsilon}_1)$ is the unconstrained prepotential, while

$$
\begin{aligned}
\Omega &= \Box F=\partial_{11}F+\partial_{22}F+\partial_{33}F+\partial_{\bar{1}\bar{1}}F,\\
\Phi &= \partial_{1\bar{1}}F \qquad (3.11)
\end{aligned}
$$

and the partial derivative of Ω (Φ) w.r.t. υ_i, $\bar{\upsilon}_1$ is expressed as Ω_i, $\Omega_{\bar{1}}$ (Φ_i, $\Phi_{\bar{1}}$), respectively.

Similarly to the results of [71], the constraints $\Phi=0$ and $\Box\Omega=0$ arise as a consequence of imposing an extra invariance under an $\mathcal{N}=5$-Extended Supersymmetry (under such constraints the resulting off-shell action is also automatically $\mathcal{N}=8$-invariant).

For $(2,8,6)_B$ the associated Lagrangian is

$$
\begin{aligned}
\mathcal{L} \;=\; &\Phi(\dot{\upsilon}_0^2+\dot{\bar{\upsilon}}_0^2+g_i^2+\bar{g}_i^2+\dot{\lambda}_0\lambda_0+\dot{\bar{\lambda}}_0\bar{\lambda}_0+\dot{\lambda}_i\lambda_i+\dot{\bar{\lambda}}_i\bar{\lambda}_i)+ \qquad (3.12)\\
&\Phi_0[\dot{\bar{\upsilon}}_0(\bar{\lambda}_i\lambda_i+\lambda_0\bar{\lambda}_0)+g_i(\bar{\lambda}_0\bar{\lambda}_i-\lambda_i\lambda_0)+\bar{g}_i(\lambda_0\bar{\lambda}_i+\lambda_i\bar{\lambda}_0)\\
&-\frac{\varepsilon_{ijk}}{2}g_i(\bar{\lambda}_j\bar{\lambda}_k-\lambda_j\lambda_k)-\varepsilon_{ijk}\lambda_i\bar{\lambda}_j\bar{g}_k]+\\
&\Phi_{\bar{0}}[\dot{\upsilon}_0(\lambda_i\bar{\lambda}_i+\bar{\lambda}_0\lambda_0)+\bar{g}_i(\bar{\lambda}_0\bar{\lambda}_i-\lambda_i\lambda_0)-g_i(\lambda_0\bar{\lambda}_i+\lambda_i\bar{\lambda}_0)\\
&+\frac{\varepsilon_{ijk}}{2}\bar{g}_i(\bar{\lambda}_j\bar{\lambda}_k-\lambda_j\lambda_k)-\varepsilon_{ijk}\lambda_i\bar{\lambda}_j g_k]+\\
&\Phi_{00}\frac{\varepsilon_{ijk}}{6}[3\bar{\lambda}_0\lambda_i\bar{\lambda}_j\lambda_k+\lambda_0\lambda_i\lambda_j\lambda_k]+\\
&\Phi_{\bar{0}\bar{0}}\frac{\varepsilon_{ijk}}{6}[3\lambda_0\lambda_i\bar{\lambda}_j\bar{\lambda}_k+\bar{\lambda}_0\bar{\lambda}_i\bar{\lambda}_j\bar{\lambda}_k]
\end{aligned}
$$

$$
\begin{aligned}
&-(\Phi_{\bar{0}\bar{0}}+\Phi_{00})\lambda_0\bar{\lambda}_0\lambda_i\bar{\lambda}_i+\\
&\Phi_{0\bar{0}}\frac{\varepsilon_{ijk}}{6}[\lambda_i\lambda_j\lambda_k\bar{\lambda}_0+\bar{\lambda}_i\bar{\lambda}_j\bar{\lambda}_k\lambda_0-3(\lambda_0\lambda_i\lambda_j\bar{\lambda}_k+\bar{\lambda}_0\bar{\lambda}_i\bar{\lambda}_j\lambda_k)]+\\
&\Omega(\dot{\upsilon}_0\dot{\bar{\upsilon}}_0+\lambda_0\dot{\bar{\lambda}}_0+\lambda_i\dot{\bar{\lambda}}_i+g_i\bar{g}_i)\\
&-\Omega_0(\dot{\upsilon}_0\bar{\lambda}_0\lambda_0+g_i\bar{\lambda}_0\lambda_i)+\\
&\Omega_{\bar{0}}(\dot{\bar{\upsilon}}_0\bar{\lambda}_i\lambda_i+\bar{g}_i\bar{\lambda}_i\lambda_0)+\\
&\Omega_{0\bar{0}}\lambda_0\bar{\lambda}_0\lambda_i\bar{\lambda}_i\\
&-\frac{\varepsilon_{ijk}}{6}(\Omega_{00}\bar{\lambda}_0\lambda_i\lambda_j\lambda_k+\Omega_{\bar{0}\bar{0}}\bar{\lambda}_i\bar{\lambda}_j\bar{\lambda}_k\lambda_0),
\end{aligned}
$$

where now we have

$$
\begin{aligned}
\Omega &= \partial_{00}F+\partial_{\bar{0}\bar{0}}F,\\
\Phi &= \partial_{0\bar{0}}F. \qquad (3.13)
\end{aligned}
$$

For $(2,8,6)_b$ the associated Lagrangian is

$$
\begin{aligned}
\mathcal{L} = {} & \Gamma(\dot{\upsilon}_0^2+g_i^2+\dot{\lambda}_0\lambda_0+\dot{\lambda}_i\lambda_i)-\bar{\Gamma}(\dot{\bar{\upsilon}}_0^2+\bar{g}_i^2+\dot{\bar{\lambda}}_0\bar{\lambda}_0+\dot{\bar{\lambda}}_i\bar{\lambda}_i)+ \qquad (3.14)\\
&\bar{\Gamma}_0\dot{\bar{\upsilon}}_0(\lambda_i\bar{\lambda}_i+\lambda_0\bar{\lambda}_0)-g_i(\bar{\Gamma}_0\bar{\lambda}_0\bar{\lambda}_i+\Gamma_0\lambda_i\lambda_0)-\bar{\Gamma}_0\bar{g}_i(\lambda_0\bar{\lambda}_i+\lambda_i\bar{\lambda}_0)+\\
&g_i\frac{\varepsilon_{ijk}}{2}(\bar{\Gamma}_0\bar{\lambda}_i\bar{\lambda}_j+\Gamma_0\lambda_j\lambda_k)+\bar{\Gamma}_0\varepsilon_{ijk}\lambda_i\bar{\lambda}_j\bar{g}_k+\\
&\Gamma_{\bar{0}}\dot{\upsilon}_0(\lambda_i\bar{\lambda}_i+\lambda_0\bar{\lambda}_0)-\bar{g}_i(\bar{\Gamma}_{\bar{0}}\bar{\lambda}_0\bar{\lambda}_i+\Gamma_{\bar{0}}\lambda_i\lambda_0)-\Gamma_{\bar{0}}g_i(\lambda_0\bar{\lambda}_i+\lambda_i\bar{\lambda}_0)\\
&-\bar{g}_i\frac{\varepsilon_{ijk}}{2}(\bar{\Gamma}_{\bar{0}}\bar{\lambda}_i\bar{\lambda}_j+\Gamma_{\bar{0}}\lambda_j\lambda_k)-\Gamma_{\bar{0}}\varepsilon_{ijk}\lambda_i\bar{\lambda}_jg_k\\
&-\frac{\varepsilon_{ijk}}{6}(3\bar{\Gamma}_{00}\bar{\lambda}_0\lambda_i\bar{\lambda}_j\lambda_k-\Gamma_{00}\lambda_0\lambda_i\lambda_j\lambda_k)+\\
&\frac{\varepsilon_{ijk}}{6}(3\Gamma_{\bar{0}\bar{0}}\lambda_0\bar{\lambda}_i\lambda_j\bar{\lambda}_k-\bar{\Gamma}_{\bar{0}\bar{0}}\bar{\lambda}_0\bar{\lambda}_i\bar{\lambda}_j\bar{\lambda}_k)\\
&-\Gamma_{0\bar{0}}\frac{\varepsilon_{ijk}}{6}(\lambda_i\lambda_j\lambda_k\bar{\lambda}_0+3\lambda_0\lambda_i\lambda_j\bar{\lambda}_k)+\bar{\Gamma}_{0\bar{0}}\frac{\varepsilon_{ijk}}{6}(\bar{\lambda}_i\bar{\lambda}_j\bar{\lambda}_k\lambda_0+3\bar{\lambda}_0\bar{\lambda}_i\bar{\lambda}_j\lambda_k),
\end{aligned}
$$

with

$$
\begin{aligned}
\Gamma &= \partial_{00}F(\upsilon_0,\bar{\upsilon}_0),\\
\bar{\Gamma} &= \partial_{\bar{0}\bar{0}}F(\upsilon_0,\bar{\upsilon}_0). \qquad (3.15)
\end{aligned}
$$

Apparently it seems, comparing (3.12) with (3.14), that the two equivalent $\mathcal{N}=4$ supermultiplets $(2,8,6)_B$ and $(2,8,6)_b$ produce inequivalent (for generic values of the functions

$\Omega, \Phi, \Gamma, \bar{\Gamma}$) $\mathcal{N} = 4$ off-shell invariant actions, indeed it is not case and one can easily see that they are related by the corresponding gordian transformation too.

It is interesting to present an explicit example of $\mathcal{N} = 4$-invariant, first-order action derived from Construction I. For the $(4, 8, 4)_B$ non-minimal supermultiplet with connectivity symbol $4_3 + 4_1$ and component fields $(\upsilon_0, \upsilon_i; \lambda_0, \lambda_i, \bar{\lambda}_0, \bar{\lambda}_i; \bar{g}_0, \bar{g}_i)$ $(i = 1, 2, 3)$, we have that the associated lagrangian is

$$\begin{aligned} \mathcal{L} &= \Omega(\dot{\upsilon}_0\bar{g}_0 + \dot{\upsilon}_i\bar{g}_i) + \Omega(\lambda_0\dot{\bar{\lambda}}_0 + \lambda_k\dot{\bar{\lambda}}_k) + \\ &\quad \varepsilon_{ijk}\Omega_j\bar{g}_k\lambda_0\lambda_i - \Omega_i\dot{\upsilon}_0\bar{\lambda}_i\lambda_0 - \Omega_0\dot{\upsilon}_i\bar{\lambda}_0\lambda_i + \\ &\quad \frac{\varepsilon_{ijk}}{2}(\Omega_0\bar{g}_k - \Omega_k\bar{g}_0)\lambda_i\lambda_j - \Omega_j\dot{\upsilon}_i\bar{\lambda}_j\lambda_i - \Omega_0\dot{\upsilon}_0\bar{\lambda}_0\lambda_0 \\ &\quad -\frac{1}{2}\varepsilon_{ijk}\Omega_{0k}\lambda_i\lambda_j\lambda_0\bar{\lambda}_0 - \frac{\varepsilon_{ijk}}{2}\delta_{pq}\Omega_{pk}\lambda_0\lambda_i\lambda_j\bar{\lambda}_q \\ &\quad -\frac{\varepsilon_{ijk}}{6}(\Omega_{00}\bar{\lambda}_0 - \Omega_{0p}\lambda_p)\lambda_i\lambda_j\lambda_k, \end{aligned} \tag{3.16}$$

where

$$\Omega = \partial_{00}F + \partial_{ii}F. \tag{3.17}$$

In the next Section we show how, for this one and the other supermultiplets entering (3.7), by applying Construction II, we obtain a second-order, $\mathcal{N} = 4$-invariant action expressed in terms of a constrained prepotential.

3.3 $\mathcal{N} = 4$-invariant σ-models with a constrained prepotential

The application of Construction II to the non-minimal supermultiplet $(4, 8, 4)_B$ induces an $\mathcal{N} = 4$-invariant theory, obtained from a second-order Lagrangian, which is expressed in terms of the two independent functions $\Phi(\upsilon_0, \upsilon_i)$ and $\Omega(\upsilon_0, \upsilon_i)$. The $\mathcal{N} = 4$-invariance is however recovered if and only if the constraint

$$\Phi_{00} + \Phi_{ii} = 0 \tag{3.18}$$

is satisfied.

The Lagrangian is explicitly given by

$$
\begin{aligned}
\mathcal{L} \;=\;& \Phi(\dot{\upsilon}_0^2+\dot{\upsilon}_i^2+\bar{g}_0^2+\bar{g}_i^2+\dot{\lambda}_0\lambda_0+\dot{\bar{\lambda}}_0\bar{\lambda}_0+\dot{\lambda}_i\lambda_i+\dot{\bar{\lambda}}_i\bar{\lambda}_i)+ \qquad (3.19)\\
& \varepsilon_{ijk}\Phi_k\dot{\upsilon}_j(\bar{\lambda}_0\bar{\lambda}_i+\lambda_i\lambda_0)+(\Phi_0\dot{\upsilon}_i-\Phi_i\dot{\upsilon}_0)(\bar{\lambda}_0\bar{\lambda}_i-\lambda_i\lambda_0)\\
& -\varepsilon_{ijk}\Phi_j\bar{g}_k(\lambda_0\bar{\lambda}_i-\lambda_i\bar{\lambda}_0)+(\Phi_0\bar{g}_i+\Phi_i\bar{g}_0)(\lambda_0\bar{\lambda}_i+\lambda_i\bar{\lambda}_0)\\
& \frac{\varepsilon_{ijk}}{2}(\Phi_i\dot{\upsilon}_0-\Phi_0\dot{\upsilon}_i)(\bar{\lambda}_j\bar{\lambda}_k-\lambda_j\lambda_k)+\frac{1}{2}(\Phi_j\dot{\upsilon}_k-\Phi_k\dot{\upsilon}_j)(\bar{\lambda}_j\bar{\lambda}_k+\lambda_j\lambda_k)\\
& (\Phi_0\bar{g}_0+\Phi_j\bar{g}_j)(\bar{\lambda}_0\lambda_0+\bar{\lambda}_k\lambda_k)-\Phi_i\bar{g}_j(\bar{\lambda}_i\lambda_j-\lambda_i\bar{\lambda}_j)-\Phi_0\bar{g}_0(\bar{\lambda}_0\lambda_0-\lambda_0\bar{\lambda}_0)\\
& \varepsilon_{ijk}\lambda_i\bar{\lambda}_j(\Phi_k\bar{g}_0-\Phi_0\bar{g}_k)+(\varepsilon_{ijk}\Phi_{0k}-\Phi_{ij})\lambda_0\bar{\lambda}_0\lambda_i\bar{\lambda}_j\\
& \Phi_{0j}(\bar{\lambda}_0\lambda_j-\lambda_0\bar{\lambda}_j)\lambda_i\bar{\lambda}_i+\frac{1}{2}\varepsilon_{ijk}\delta_{pq}\Phi_{kp}(\bar{\lambda}_0\lambda_i\lambda_j\bar{\lambda}_q-\lambda_0\bar{\lambda}_i\bar{\lambda}_j\lambda_q)\\
& \Phi_{jk}\bar{\lambda}_j\lambda_k\lambda_i\bar{\lambda}_i-\frac{1}{2}\varepsilon_{ijk}\delta_{pq}\Phi_{0p}\lambda_i\bar{\lambda}_j\lambda_k\bar{\lambda}_q\\
& -\Phi_{00}\lambda_0\bar{\lambda}_0\lambda_i\bar{\lambda}_i+\frac{\varepsilon_{ijk}}{2}[\Phi_{pp}(\lambda_0\lambda_i\bar{\lambda}_j\bar{\lambda}_k)+\Phi_{00}(\bar{\lambda}_0\lambda_i\bar{\lambda}_j\lambda_k)]\\
& -\frac{\varepsilon_{ijk}}{6}[(\Phi_{00}+\Phi_{pp})\lambda_i\lambda_j\lambda_k\lambda_0]\\
& \Omega(\dot{\upsilon}_0\bar{g}_0+\dot{\upsilon}_i\bar{g}_i)+\Omega(\lambda_0\dot{\bar{\lambda}}_0+\lambda_k\dot{\bar{\lambda}}_k)\\
& \varepsilon_{ijk}\Omega_j\bar{g}_k\lambda_0\lambda_i-\Omega_i\dot{\upsilon}_0\bar{\lambda}_i\lambda_0-\Omega_0\dot{\upsilon}_i\bar{\lambda}_0\lambda_i\\
& \frac{\varepsilon_{ijk}}{2}(\Omega_0\bar{g}_k-\Omega_k\bar{g}_0)\lambda_i\lambda_j-\Omega_j\dot{\upsilon}_i\bar{\lambda}_j\lambda_i-\Omega_0\dot{\upsilon}_0\bar{\lambda}_0\lambda_0\\
& -\frac{1}{2}\varepsilon_{ijk}\Omega_{0k}\lambda_i\lambda_j\lambda_0\bar{\lambda}_0-\frac{\varepsilon_{ijk}}{2}\delta_{pq}\Omega_{pk}\lambda_0\lambda_i\lambda_j\bar{\lambda}_q\\
& -\frac{\varepsilon_{ijk}}{6}(\Omega_{00}\bar{\lambda}_0-\Omega_{0p}\lambda_p)\lambda_i\lambda_j\lambda_k
\end{aligned}
$$

Requiring an extra, $\mathcal{N}=5$-invariance for the action implies the further constraint $\Omega=0$. As expected from [71], the resulting action is automatically $\mathcal{N}=8$-invariant.

All actions obtained via Construction II from the supermultiplets entering (3.7) share the same features. In the case of the supermultiplet $(1,8,7)_{red}$, described by the fields $(\upsilon_0;\lambda_0,\lambda_i,\bar{\lambda}_0,\bar{\lambda}_i;g_i,\bar{g}_0,\bar{g}_i)$, the associated Lagrangian is

$$
\begin{aligned}
\mathcal{L} \;=\;& \Phi(\dot{\upsilon}_0^2+\bar{g}_0^2+g_i^2+\bar{g}_i^2+\dot{\lambda}_0\lambda_0+\dot{\bar{\lambda}}_0\bar{\lambda}_0+\dot{\lambda}_i\lambda_i+\dot{\bar{\lambda}}_i\bar{\lambda}_i) \qquad (3.20)\\
& \Phi_0[\bar{g}_0(\lambda_0\bar{\lambda}_0-\lambda_i\bar{\lambda}_i)+g_i(\bar{\lambda}_0\bar{\lambda}_i-\lambda_i\lambda_0)+\bar{g}_i(\lambda_0\bar{\lambda}_i+\lambda_i\bar{\lambda}_0)-
\end{aligned}
$$

$$
\begin{aligned}
&-\varepsilon_{ijk}(\bar{g}_i\bar{\lambda}_j\lambda_k + \frac{g_i}{2}(\bar{\lambda}_j\bar{\lambda}_k - \lambda_j\lambda_k))] \\
&-\Phi_{00}(\lambda_0\bar{\lambda}_0\lambda_i\bar{\lambda}_i + \frac{\varepsilon_{ijk}}{2}\lambda_0\bar{\lambda}_i\bar{\lambda}_j\lambda_k - \frac{\varepsilon_{ijk}}{6}\lambda_0\lambda_i\lambda_j\lambda_k) + \\
&\Omega(\dot{\upsilon}_0\bar{g}_0 + \lambda_0\dot{\bar{\lambda}}_0 + \lambda_i\dot{\bar{\lambda}}_i + g_i\bar{g}_i) + \\
&\Omega_0[(\dot{\upsilon}_0\lambda_0 + g_i\lambda_i)\bar{\lambda}_0 + \frac{1}{2}\varepsilon_{ijk}\bar{g}_i\lambda_j\lambda_k] + \\
&\Omega_{00}\frac{1}{6}\varepsilon_{ijk}\lambda_i\lambda_j\lambda_k\bar{\lambda}_0,
\end{aligned}
$$

where Φ, Ω are fuctions of υ_0 and

$$
\Phi_{00} = 0. \tag{3.21}
$$

Implementing the $\mathcal{N}=5$-invariance gives the constraint $\Omega = 0$ (again, the resulting action is automatically fully $\mathcal{N}=8$-invariant).

The $\mathcal{N}=8$ model based on the supermultiplet of field content $(1,8,7)$ was first obtained in [37]. Unlike the present, more general construction, the action was derived through an "$\mathcal{N}=8$ covariantization Ansatz" which cannot be applied neither to deduce the $\mathcal{N}=4$-invariant action, nor to obtain the invariant actions for the other supermultiplets entering (3.7).

3.4 The σ-model associated to the $\mathcal{N}=4$ entangled supermultiplet

A manifest $\mathcal{N}=4$ off-shell invariant action can be constructed for the entangled supermultiplet $(3,8,5)_\theta$ (presented in previous chapter). The action corresponds to a one-dimensional sigma-model with a three-dimensional target spanned by the bosonic coordinates $\upsilon_0, \upsilon_1, \bar{\upsilon}_0$. The invariant Lagrangian $\mathcal{L}$ is constructed, following [37, 71, 66], through

$$
\mathcal{L} = Q_3Q_2Q_1Q'F(\upsilon_0, \upsilon_1, \bar{\upsilon}_0), \tag{3.22}
$$

where the four supercharges act as odd Leibniz derivatives on an unconstrained function F (the prepotential) of the three target coordinates. The Lagrangian has the correct dimension for a kinetic term. It is explicitly given by

$$\mathcal{L} = \mathcal{L}_1 \cos\theta + \mathcal{L}_2 \sin\theta \tag{3.23}$$

with

$$\begin{aligned}
\mathcal{L}_1 \;=\;& \Gamma(\dot{\upsilon}_0^2 + \dot{\upsilon}_1^2 + g_2^2 + g_3^2) - \bar{\Gamma}(\dot{\bar{\upsilon}}_0^2 + \bar{g}_1^2 + \bar{g}_2^2 + \bar{g}_3^2) \\
& + \Gamma(\dot{\lambda}_0\lambda_0 + \dot{\lambda}_1\lambda_1 + \dot{\lambda}_2\lambda_2 + \dot{\lambda}_3\lambda_3) - \bar{\Gamma}(\dot{\bar{\lambda}}_0\bar{\lambda}_0 + \dot{\bar{\lambda}}_1\bar{\lambda}_1 + \dot{\bar{\lambda}}_2\bar{\lambda}_2 + \dot{\bar{\lambda}}_3\bar{\lambda}_3) \\
& - \Gamma_0(\dot{\upsilon}_1(\lambda_1\lambda_0 + \lambda_3\lambda_2) + g_2(\lambda_2\lambda_0 + \lambda_1\lambda_3) + g_3(\lambda_3\lambda_0 + \lambda_2\lambda_1)) \\
& - \bar{\Gamma}_{\bar{0}}(\bar{g}_1(\bar{\lambda}_0\bar{\lambda}_1 + \bar{\lambda}_2\bar{\lambda}_3) + \bar{g}_2(\bar{\lambda}_0\bar{\lambda}_2 + \bar{\lambda}_3\bar{\lambda}_1) + \bar{g}_3(\bar{\lambda}_0\bar{\lambda}_3 + \bar{\lambda}_1\bar{\lambda}_2)) \\
& - \Gamma_1(\dot{\upsilon}_0(\lambda_0\lambda_1 + \lambda_2\lambda_3) + g_2(\lambda_3\lambda_0 + \lambda_2\lambda_1) + g_3(\lambda_0\lambda_2 + \lambda_3\lambda_1)) \\
& - \bar{\Gamma}_1(\dot{\upsilon}_0(\bar{\lambda}_1\bar{\lambda}_0 + \bar{\lambda}_2\bar{\lambda}_3) + g_2(\bar{\lambda}_1\bar{\lambda}_2 + \bar{\lambda}_3\bar{\lambda}_0) + g_3(\bar{\lambda}_0\bar{\lambda}_2 + \bar{\lambda}_1\bar{\lambda}_3) \\
& + \bar{g}_1(\bar{\lambda}_0\lambda_0 + \lambda_1\bar{\lambda}_1 + \bar{\lambda}_2\lambda_2 + \bar{\lambda}_3\lambda_3) + \bar{g}_2(\bar{\lambda}_3\lambda_0 + \lambda_3\bar{\lambda}_0 + \lambda_2\bar{\lambda}_1 + \lambda_1\bar{\lambda}_2) \\
& + \bar{g}_3(\lambda_0\bar{\lambda}_2 + \bar{\lambda}_0\lambda_2 + \lambda_1\bar{\lambda}_3 + \lambda_3\bar{\lambda}_1) - \dot{\bar{\upsilon}}_0(\bar{\lambda}_0\lambda_1 + \bar{\lambda}_1\lambda_0 + \bar{\lambda}_3\lambda_2 - \bar{\lambda}_2\lambda_3)) \\
& - \bar{\Gamma}_0(\dot{\upsilon}_1(\bar{\lambda}_0\bar{\lambda}_1 + \bar{\lambda}_3\bar{\lambda}_2) + g_2(\bar{\lambda}_0\bar{\lambda}_2 + \bar{\lambda}_1\bar{\lambda}_3) + g_3(\bar{\lambda}_0\bar{\lambda}_3 + \bar{\lambda}_2\bar{\lambda}_1) \\
& + \bar{g}_1(\lambda_0\bar{\lambda}_1 + \lambda_1\bar{\lambda}_0 + \bar{\lambda}_3\lambda_2 + \lambda_3\bar{\lambda}_2) + \bar{g}_2(\lambda_0\bar{\lambda}_2 + \lambda_2\bar{\lambda}_0 + \lambda_1\bar{\lambda}_3 + \bar{\lambda}_1\lambda_3) \\
& + \bar{g}_3(\lambda_0\bar{\lambda}_3 + \lambda_3\bar{\lambda}_0 + \bar{\lambda}_2\lambda_1 + \lambda_2\bar{\lambda}_1) - \dot{\bar{\upsilon}}_0(\bar{\lambda}_0\lambda_0 + \bar{\lambda}_1\lambda_1 + \bar{\lambda}_2\lambda_2 + \bar{\lambda}_3\lambda_3)) \\
& + \Gamma_{\bar{0}}(\dot{\upsilon}_1(\bar{\lambda}_1\lambda_0 + \bar{\lambda}_0\bar{\lambda}_1 + \bar{\lambda}_3\lambda_2 + \lambda_3\bar{\lambda}_2) + g_2(\bar{\lambda}_2\lambda_0 + \bar{\lambda}_0\lambda_2 + \bar{\lambda}_1\lambda_3 + \lambda_1\bar{\lambda}_3) \\
& + g_3(\bar{\lambda}_3\lambda_0 + \bar{\lambda}_0\lambda_3 + \bar{\lambda}_2\lambda_1 + \lambda_2\bar{\lambda}_1) - \dot{\upsilon}_0(\bar{\lambda}_0\lambda_0 + \bar{\lambda}_1\lambda_1 + \bar{\lambda}_2\lambda_2 + \bar{\lambda}_3\lambda_3) \\
& + \bar{g}_1(\lambda_0\lambda_1 + \lambda_3\lambda_2) + \bar{g}_2(\lambda_0\lambda_2 + \lambda_1\lambda_3) + \bar{g}_3(\lambda_0\lambda_3 + \lambda_2\lambda_1)) \\
& + \Gamma_{00}\lambda_0\lambda_1\lambda_2\lambda_3 - \bar{\Gamma}_{00}(\bar{\lambda}_1\lambda_2\bar{\lambda}_3\lambda_0 + \bar{\lambda}_0\lambda_1\lambda_2\bar{\lambda}_3 + \bar{\lambda}_1\bar{\lambda}_2\lambda_3\lambda_0 + \bar{\lambda}_0\lambda_1\bar{\lambda}_2\lambda_3) \\
& + \Gamma_{11}\lambda_0\lambda_1\lambda_2\lambda_3 - \bar{\Gamma}_{11}(\lambda_1\bar{\lambda}_1\bar{\lambda}_3\lambda_3 + \lambda_1\bar{\lambda}_1\bar{\lambda}_2\lambda_2 + \lambda_0\bar{\lambda}_0\bar{\lambda}_2\lambda_2 + \lambda_0\bar{\lambda}_0\lambda_3\bar{\lambda}_3) \\
& + \Gamma_{\bar{0}\bar{0}}(\lambda_0\lambda_1\bar{\lambda}_2\bar{\lambda}_3 - \bar{\lambda}_0\bar{\lambda}_1\lambda_2\lambda_3) - \bar{\Gamma}_{\bar{0}\bar{0}}(\bar{\lambda}_0\bar{\lambda}_1\bar{\lambda}_2\bar{\lambda}_3) \\
& + \Gamma_{0\bar{0}}(\lambda_1\bar{\lambda}_2\lambda_3\lambda_0 + \bar{\lambda}_0\lambda_1\lambda_2\lambda_3 + \bar{\lambda}_1\lambda_2\lambda_3\lambda_0 + \lambda_1\lambda_2\bar{\lambda}_3\lambda_0) \\
& - \bar{\Gamma}_{0\bar{0}}(\lambda_1\bar{\lambda}_2\bar{\lambda}_3\bar{\lambda}_0 + \bar{\lambda}_1\bar{\lambda}_2\lambda_3\bar{\lambda}_0 + \lambda_0\bar{\lambda}_1\bar{\lambda}_2\bar{\lambda}_3 + \bar{\lambda}_1\lambda_2\bar{\lambda}_3\bar{\lambda}_0) \\
& + \Gamma_{1\bar{0}}(\lambda_1\lambda_2\bar{\lambda}_2\lambda_0 - \lambda_1\bar{\lambda}_1\lambda_2\lambda_3 + \lambda_0\bar{\lambda}_0\lambda_2\lambda_3 + \lambda_3\bar{\lambda}_3\lambda_1\lambda_0) \\
& - \bar{\Gamma}_{1\bar{0}}(\lambda_2\bar{\lambda}_2\bar{\lambda}_1\bar{\lambda}_0 + \bar{\lambda}_0\bar{\lambda}_2\bar{\lambda}_3\lambda_0 + \lambda_3\bar{\lambda}_3\bar{\lambda}_1\bar{\lambda}_0 + \bar{\lambda}_2\bar{\lambda}_3\lambda_1\bar{\lambda}_1)
\end{aligned}$$

$$+\bar{\Gamma}_{01}(\lambda_3\bar{\lambda}_3\lambda_0\bar{\lambda}_1 + \lambda_3\bar{\lambda}_3\lambda_1\bar{\lambda}_0 + \lambda_1\bar{\lambda}_1\bar{\lambda}_2\lambda_3 + \lambda_2\bar{\lambda}_3\lambda_1\bar{\lambda}_1$$
$$+\lambda_2\bar{\lambda}_2\lambda_0\bar{\lambda}_1 + \bar{\lambda}_0\lambda_2\bar{\lambda}_3\lambda_0 + \lambda_2\bar{\lambda}_2\lambda_1\bar{\lambda}_0 + \bar{\lambda}_0\bar{\lambda}_2\lambda_3\lambda_0) \quad (3.24)$$

and

$$\begin{aligned}
\mathcal{L}_2 \quad = \quad & -\Phi(\dot{v}_0^2 + \dot{v}_1^2 + \dot{\bar{v}}_0^2 + g_2^2 + g_3^2 + \bar{g}_1^2 + \bar{g}_2^2 + \bar{g}_3^2) \\
& -\Phi(\dot{\lambda}_0\lambda_0 + \dot{\lambda}_1\lambda_1 + \dot{\lambda}_2\lambda_2 + \dot{\lambda}_3\lambda_3 + \dot{\bar{\lambda}}_0\bar{\lambda}_0 + \dot{\bar{\lambda}}_1\bar{\lambda}_1 + \dot{\bar{\lambda}}_2\bar{\lambda}_2 + \dot{\bar{\lambda}}_3\bar{\lambda}_3) \\
& +\Phi_0(-\dot{v}_1(\bar{\lambda}_0\bar{\lambda}_1 - \lambda_1\lambda_0 - \bar{\lambda}_2\bar{\lambda}_3 + \lambda_2\lambda_3) - g_2(\bar{\lambda}_0\bar{\lambda}_2 - \lambda_2\lambda_0 - \bar{\lambda}_3\bar{\lambda}_1 + \lambda_3\lambda_1) \\
& -g_3(\bar{\lambda}_0\bar{\lambda}_3 - \lambda_3\lambda_0 - \bar{\lambda}_1\bar{\lambda}_2 + \lambda_1\lambda_2) - \bar{g}_1(\lambda_0\bar{\lambda}_1 + \lambda_1\bar{\lambda}_0 + \lambda_3\bar{\lambda}_2 - \lambda_2\bar{\lambda}_3) \\
& -\bar{g}_2(\lambda_0\bar{\lambda}_2 + \lambda_2\bar{\lambda}_0 + \lambda_1\bar{\lambda}_3 - \lambda_3\bar{\lambda}_1) - \bar{g}_3(\lambda_0\bar{\lambda}_3 + \lambda_3\bar{\lambda}_0 + \lambda_2\bar{\lambda}_1 - \lambda_1\bar{\lambda}_2) \\
& +\dot{\bar{v}}_0(-\bar{\lambda}_0\lambda_0 + \bar{\lambda}_1\lambda_1 + \bar{\lambda}_2\lambda_2 + \bar{\lambda}_3\lambda_3)) \\
& +\Phi_1(\bar{g}_1(\lambda_0\bar{\lambda}_0 + \lambda_1\bar{\lambda}_1 + \lambda_2\bar{\lambda}_2 + \lambda_3\bar{\lambda}_3) - \bar{g}_2(\lambda_1\bar{\lambda}_2 + \lambda_2\bar{\lambda}_1 - \lambda_0\bar{\lambda}_3 + \lambda_3\bar{\lambda}_0) \\
& +\bar{g}_3(\lambda_2\bar{\lambda}_0 - \lambda_0\bar{\lambda}_2 - \lambda_1\bar{\lambda}_3 - \lambda_3\bar{\lambda}_1) - g_2(\lambda_0\lambda_3 + \bar{\lambda}_3\bar{\lambda}_0 + \lambda_1\lambda_2 + \bar{\lambda}_1\bar{\lambda}_2) \\
& +g_3(\lambda_0\lambda_2 + \bar{\lambda}_3\bar{\lambda}_1 + \lambda_3\lambda_1 + \bar{\lambda}_2\bar{\lambda}_0) + \dot{v}_0(\lambda_0\lambda_1 + \bar{\lambda}_0\bar{\lambda}_1) + \dot{\bar{v}}_0(\lambda_3\bar{\lambda}_2 - \lambda_2\bar{\lambda}_3)) \\
& +\Phi_{\bar{0}}(-\bar{g}_1(\bar{\lambda}_0\bar{\lambda}_1 - \lambda_1\lambda_0 + \bar{\lambda}_2\bar{\lambda}_3 - \lambda_2\lambda_3) - \bar{g}_2(\bar{\lambda}_0\bar{\lambda}_2 - \lambda_2\lambda_0 + \bar{\lambda}_3\bar{\lambda}_1 - \lambda_3\lambda_1) \\
& -\bar{g}_3(\bar{\lambda}_0\bar{\lambda}_3 - \lambda_3\lambda_0 + \bar{\lambda}_1\bar{\lambda}_2 - \lambda_1\lambda_2) - g_2(\lambda_1\bar{\lambda}_3 - \lambda_3\bar{\lambda}_1 - \lambda_0\bar{\lambda}_2 - \lambda_2\bar{\lambda}_0) \\
& -g_3(\lambda_2\bar{\lambda}_1 - \lambda_1\bar{\lambda}_2 - \lambda_0\bar{\lambda}_3 - \lambda_3\bar{\lambda}_0) + \dot{v}_0(\bar{\lambda}_0\lambda_0 - \bar{\lambda}_1\lambda_1 - \bar{\lambda}_2\lambda_2 - \bar{\lambda}_3\lambda_3) \\
& -\dot{v}_1(\lambda_3\bar{\lambda}_2 - \lambda_2\bar{\lambda}_3 - \lambda_0\bar{\lambda}_1 - \lambda_1\bar{\lambda}_0)) \\
& +\Phi_{00}(\lambda_1\lambda_2\lambda_3\lambda_0 - \bar{\lambda}_0\lambda_1\lambda_2\bar{\lambda}_3 + \bar{\lambda}_0\lambda_3\bar{\lambda}_2\lambda_1 - \bar{\lambda}_0\bar{\lambda}_1\lambda_2\lambda_3) \\
& +\Phi_{11}(\lambda_1\bar{\lambda}_1\lambda_2\bar{\lambda}_2 + \lambda_0\bar{\lambda}_3\lambda_2\lambda_1 + \lambda_0\lambda_3\bar{\lambda}_2\bar{\lambda}_1 - \bar{\lambda}_3\lambda_3\lambda_1\bar{\lambda}_1 - \lambda_0\lambda_2\bar{\lambda}_3\bar{\lambda}_1 \\
& -\bar{\lambda}_0\lambda_0\bar{\lambda}_3\lambda_3 - \bar{\lambda}_0\bar{\lambda}_1\lambda_2\lambda_3 + \bar{\lambda}_0\lambda_0\lambda_2\bar{\lambda}_2) \\
& +\Phi_{\bar{0}\bar{0}}(\lambda_0\lambda_3\bar{\lambda}_2\bar{\lambda}_1 + \lambda_0\bar{\lambda}_3\lambda_2\bar{\lambda}_1 + \bar{\lambda}_0\bar{\lambda}_3\bar{\lambda}_2\bar{\lambda}_1 + \lambda_0\bar{\lambda}_3\bar{\lambda}_2\lambda_1) \\
& +\Phi_{01}(\bar{\lambda}_2\lambda_3\lambda_1\bar{\lambda}_1 + \bar{\lambda}_0\lambda_0\lambda_2\bar{\lambda}_3 - \bar{\lambda}_0\lambda_2\bar{\lambda}_2\lambda_1 + \lambda_1\bar{\lambda}_1\lambda_2\bar{\lambda}_3 + \bar{\lambda}_0\bar{\lambda}_3\lambda_3\lambda_1 \\
& +\bar{\lambda}_0\lambda_0\bar{\lambda}_2\lambda_3 + \lambda_0\lambda_2\bar{\lambda}_2\bar{\lambda}_1 - \lambda_0\bar{\lambda}_3\lambda_3\bar{\lambda}_1) \\
& +\Phi_{0\bar{0}}(\lambda_0\bar{\lambda}_1\lambda_2\lambda_3 + \lambda_0\lambda_1\bar{\lambda}_2\lambda_3 + \bar{\lambda}_0\bar{\lambda}_1\bar{\lambda}_2\lambda_3 + \bar{\lambda}_0\lambda_1\bar{\lambda}_2\bar{\lambda}_3 + \bar{\lambda}_0\bar{\lambda}_1\lambda_2\bar{\lambda}_3 \\
& +\lambda_0\lambda_1\lambda_2\bar{\lambda}_3 - \bar{\lambda}_0\lambda_1\lambda_2\lambda_3 + \lambda_0\bar{\lambda}_3\bar{\lambda}_2\bar{\lambda}_1)
\end{aligned}$$

$$
\begin{aligned}
&+\Phi_{1\bar{0}}(\lambda_1\bar{\lambda}_1\bar{\lambda}_3\bar{\lambda}_2+\bar{\lambda}_0\lambda_0\bar{\lambda}_3\bar{\lambda}_2+\lambda_0\lambda_2\bar{\lambda}_2\lambda_1+\bar{\lambda}_0\lambda_2\lambda_2\lambda_1-\bar{\lambda}_0\bar{\lambda}_3\lambda_3\bar{\lambda}_1\\
&-\lambda_0\bar{\lambda}_3\lambda_3\lambda_1+\bar{\lambda}_0\lambda_0\lambda_2\lambda_3+\lambda_1\bar{\lambda}_1\lambda_2\lambda_3)\\
&-(\Gamma+\bar{\Gamma})(\dot{\upsilon}_0\dot{\bar{\upsilon}}_0+\dot{\upsilon}_1\bar{g}_1+g_2\bar{g}_2+g_3\bar{g}_3)\\
&-(\Gamma+\bar{\Gamma})(\lambda_0\dot{\bar{\lambda}}_0+\lambda_1\dot{\bar{\lambda}}_1+\lambda_2\dot{\bar{\lambda}}_2+\lambda_3\dot{\bar{\lambda}}_3)\\
&+(\Gamma+\bar{\Gamma})_0(\dot{\upsilon}_1\bar{\lambda}_0\lambda_1+g_2\bar{\lambda}_0\lambda_2+g_3\bar{\lambda}_0\lambda_3-\bar{g}_1\lambda_2\lambda_3-\bar{g}_2\lambda_3\lambda_1-\bar{g}_3\lambda_1\lambda_2-\dot{\upsilon}_0\bar{\lambda}_0\lambda_0)\\
&-(\Gamma+\bar{\Gamma})_{\bar{0}}(\dot{\upsilon}_1\bar{\lambda}_2\bar{\lambda}_3+g_2\bar{\lambda}_3\bar{\lambda}_1+g_3\bar{\lambda}_1\bar{\lambda}_2+\bar{g}_1\bar{\lambda}_1\lambda_0+\bar{g}_2\bar{\lambda}_2\lambda_0+\bar{g}_3\bar{\lambda}_3\lambda_0\\
&+\dot{\bar{\upsilon}}_0(\bar{\lambda}_1\lambda_1+\bar{\lambda}_2\lambda_2+\bar{\lambda}_3\lambda_3))\\
&-(\Gamma+\bar{\Gamma})_1(g_2\lambda_2\bar{\lambda}_1+g_3\lambda_3\bar{\lambda}_1+\bar{g}_2\lambda_0\lambda_3+\bar{g}_3\lambda_2\lambda_0-\dot{\upsilon}_0\bar{\lambda}_1\lambda_0-\dot{\upsilon}_1\bar{\lambda}_1\lambda_1-\dot{\bar{\upsilon}}_0\lambda_2\lambda_3)\\
&+(\Gamma+\bar{\Gamma})_{1\bar{0}}(\lambda_0\bar{\lambda}_1\bar{\lambda}_2\lambda_2+\lambda_0\bar{\lambda}_3\lambda_3\bar{\lambda}_1)\\
&+(\Gamma+\bar{\Gamma})_{00}\bar{\lambda}_0\lambda_1\lambda_2\lambda_2+(\Gamma+\bar{\Gamma})_{11}\lambda_0\bar{\lambda}_1\lambda_2\lambda_3+(\Gamma+\bar{\Gamma})_{\bar{0}\bar{0}}\lambda_0\bar{\lambda}_1\bar{\lambda}_2\bar{\lambda}_3\\
&+(\Gamma+\bar{\Gamma})_{01}(\lambda_0\bar{\lambda}_0\lambda_2\lambda_3-\lambda_1\bar{\lambda}_1\lambda_2\lambda_3). \qquad (3.25)
\end{aligned}
$$

The functions $\Gamma, \bar{\Gamma}$ and Φ are expressed through the prepotential F as

$$
\Gamma=\partial^2{}_{00}F+\partial^2{}_{11}F=F_{00}+F_{11}, \qquad \bar{\Gamma}=\partial^2{}_{\bar{0}\bar{0}}F=F_{\bar{0}\bar{0}}, \qquad \Phi=\partial^2{}_{0\bar{0}}F=F_{0\bar{0}}. \qquad (3.26)
$$

In the above formulas the partial derivative with respect to υ_0 is denoted with the suffix "0" (and similarly for υ_1 and $\upsilon_{\bar{0}}$).

Comparing (3.24) with (3.25), it turns out that the two inequivalent $\mathcal{N}=4$ supermultiplets $(3,8,5)_B$ and $(3,8,5)_b$ produce inequivalent $\mathcal{N}=4$ off-shell invariant actions, while sharing the same field content and the same connectivity symbol (4_3+4_2), but differing their commuting groups (as well, their node choice groups , see 2.10 and 2.7).

By setting equal to zero all the fermionic fields in the action we obtain the bosonic part $\mathcal{L}_{bos}$ of the Lagrangian

$$
\begin{aligned}
\mathcal{L}_{bos} \;=\;& [\Gamma(\dot{\upsilon}_0^2+\dot{\upsilon}_1^2+g_2^2+g_3^2)-\bar{\Gamma}(\dot{\bar{\upsilon}}_0^2+\bar{g}_1^2+\bar{g}_2^2+\bar{g}_3^2)]\cos\theta \qquad (3.27)\\
&-[\Phi(\dot{\upsilon}_0^2+\dot{\upsilon}_1^2+\dot{\bar{\upsilon}}_0^2+g_2^2+g_3^2+\bar{g}_1^2+\bar{g}_2^2+\bar{g}_3^2)\\
&+(\Gamma+\bar{\Gamma})(\dot{\upsilon}_0\dot{\bar{\upsilon}}_0+\dot{\upsilon}_1\bar{g}_1+g_2\bar{g}_2+g_3\bar{g}_3)]\sin\theta.
\end{aligned}
$$

By solving the algebric equations of motion for the auxiliary fields and up to total derivatives we can write

$$\mathcal{L}_{bos} = g_{ij}\dot{X}^i\dot{X}^j, \qquad i,j = 1,2,3, \tag{3.28}$$

where $\vec{X} = (\upsilon_0, \upsilon_1, \bar{\upsilon}_0)$ and the metric g_{ij} is given by

$$g_{ij} = \begin{pmatrix} \Gamma\cos\theta - \Phi\sin\theta & 0 & -\frac{(\Gamma+\bar{\Gamma})}{2}\sin\theta \\ 0 & (\Gamma - \bar{\Gamma}A^2)\cos\theta + [(\Gamma+\bar{\Gamma})A - \Phi(1+A^2)]\sin\theta & 0 \\ -\frac{(\Gamma+\bar{\Gamma})}{2}\sin\theta & 0 & -\bar{\Gamma}\cos\theta - \Phi\sin\theta \end{pmatrix}, \tag{3.29}$$

with

$$A = \frac{(\Gamma+\bar{\Gamma})\sin\theta}{2(\bar{\Gamma}\cos\theta + \Phi\sin\theta)}. \tag{3.30}$$

The non-vanishing components of the diagonalized metric are

$$\begin{aligned} g_{11} &= \frac{1}{2}(\Gamma - \bar{\Gamma})\cos\theta - \Phi\sin\theta + \frac{(\Gamma+\bar{\Gamma})}{2}, \\ g_{22} &= (\Gamma - \bar{\Gamma}A^2)\cos\theta + [(\Gamma+\bar{\Gamma})A - \Phi(1+A^2)]\sin\theta, \\ g_{33} &= \frac{1}{2}(\Gamma - \bar{\Gamma})\cos\theta - \Phi\sin\theta - \frac{(\Gamma+\bar{\Gamma})}{2}. \end{aligned} \tag{3.31}$$

Extra supersymmetry generators can be consistently applied on the $(3,8,5)_\theta$ supermultiplet so that an $\mathcal{N}=8$ Extended Supersymmetry can be defined on (2.27) (in the language of [71], the $\mathcal{N}=5$ $(3,8,5)_\theta$ supermultiplet can be "oxidized" to an $\mathcal{N}=8$ supermultiplet). This property is a consequence of the fact that both $(3,8,5)_{b,\Delta=0}$ and $(3,8,5)_B$ can be "oxidized" to the same $\mathcal{N}=8$ $(3,8,5)$ supermultiplet, with supersymmetry generators Q_I, $I=1,2,\ldots,8$. The eight supertransformations acting on $(3,8,5)_\theta$, besides Q_1, Q_2, Q_3, Q', are Q'', Q_6, Q_7, Q_8. Q'' is obtained, similarly to Q', by rotating the Q_4, Q_5 plane ($Q'' = Q_4\sin\theta - Q_5\cos\theta$).

Imposing the $\mathcal{N}=8$ invariance for the action given by (3.22) requires constraining the prepotential F. The $\mathcal{N}=8$ invariance implies the constraint

$$\Gamma + \bar{\Gamma} = 0. \tag{3.32}$$

The metric of the $\mathcal{N}=8$-invariant sigma-model is conformally flat,

$$g_{ij} \;=\; \delta_{ij}H, \tag{3.33}$$

with the conformal factor H given by

$$H \;=\; \Gamma\cos\theta - \Phi\sin\theta. \tag{3.34}$$

The $\mathcal{N}=8$-invariant constraint implies that the action given by (3.23) is a function of the conformal factor H (the dependence on Γ, Φ enters only through the conformal factor).

3.5 Discussion

We obtained evidences that supermultiplets, sharing the same field content but differing in connectivity symbol, can induce inequivalent supersymmetric-invariant actions (one should compare, e.g., the actions given in formulas (3.10) and (3.19)). It was known, from the analysis of [88, 89, 63, 69], that inequivalent representations, discriminated by their respective connectivity symbol, can be found. On the other hand, so far, no dynamical characterization was associated to the connectivity symbol. In [71] the $\mathcal{N}=5$-supersymmetric off-shell invariant actions, induced with respect to inequivalent $\mathcal{N}=5$ supermultiplets of a given field content, were proven to coincide and possess an overall $\mathcal{N}=8$ supersymmetry invariance. The crucial feature here is the fact that the inequivalent $\mathcal{N}=4$ off-shell invariant actions are induced by inequivalent *non-minimal* $\mathcal{N}=4$ linear supermultiplets (with the same field content).

As we emphasized in the previous chapter, the non-minimal $\mathcal{N}=4$ linear supermultiplets are progressively oxidized to minimal $\mathcal{N}=5,6,7,8$ linear supermultiplets possessing 8 bosonic and 8 fermionic component fields. It is clear, from these considerations, that the $\mathcal{N}=4$ off-shell invariant actions based on non-minimal supermultiplets are not of mere academic interest. Indeed, an $N=2$, $D=4$ theory, dimensionally reduced to $1D$, produces a supersymmetric model with $\mathcal{N}=8$ extended supersymmetries; on the other hand the partial spontaneous breaking of $N = 2$ into $N = 1$ produces an $\mathcal{N} = 4$ invariant $1D$ supersymmetric model whose component fields belong to $\mathcal{N} = 8$ supermultiplets and are therefore non-minimal supermultiplets w.r.t. the $\mathcal{N} = 4$ invariant supersymmetries. The inequivalent $\mathcal{N} = 4$ non-minimal supermultiplets and their inequivalent $\mathcal{N} = 4$-invariant off-shell actions can therefore be regarded as building blocks for constructing supersymmetric models obtained from dimensional reduction of partial spontaneous supersymmetry breaking of $N = 2$, $D = 4$ supersymmetry.

It is worth mentioning a paper [90] in which the $\mathcal{N} = 4$-invariance for a non-minimal supermultiplet in presence of a Yang monopole is discussed (see also [68]).

We also provided the first explicit construction of a supersymmetric one-dimensional sigma-model based on an entangled supermultiplet (which does not admit a graphical presentation). The entangled supermultiplet has dynamical consequences. An $\mathcal{N} = 4$, one-dimensional, off-shell invariant sigma-model with a three-dimensional target is based on it. Its action (3.23) carries an explicit dependence on θ. This model is supersymmetric only under the supertransformations specified by the entangled supermultiplet. Therefore, entangled supermultiplets allow to enlarge the class of supersymmetric actions so far considered.

In the case of the (3.23) action the dependence on θ can be reabsorbed only if the constraint (3.32), which implies an $\mathcal{N} = 8$ invariance, is imposed. The $\mathcal{N} = 8$ action turns out to be dependent on the conformal factor (3.34). The $\mathcal{N} = 8$ invariance is made possible by the fact that the two $\mathcal{N} = 4$ pure supermultiplets recovered at $\theta = 0$ and $\theta = \frac{\pi}{2}$ can be extended ("oxidized", see [71]) to the same $\mathcal{N} = 8$ $(3, 8, 5)$ supermultiplet. On the other hand, when (3.32) is not satisfied, the action (3.23) is $\mathcal{N} = 4$ supersymmetric and possesses a genuine θ-dependence.

Chapter 4

D-module representations of $\mathcal{N} = 3$, 4, 7, 8 Superconformal Algebras and the Critical scaling dimensions

This Chapter is a slightly edited version of the first part of [91]which was written in collaboration with F. Toppan.

4.1 Introduction

In this chapter we extend the construction of [92] proving that all finite $\mathcal{N} = 7, 8$ superconformal algebras are recovered as D-module representations for a critical value of the scaling dimension λ of some given global supermultiplet. Therefore, together with the results concerning $\mathcal{N} = 4$, a criticality is encountered for D-module representations of the $\mathcal{N} = 4, 7, 8$ finite superconformal algebras. This criticality has deep consequences in constructing and constraining the admissible one-dimensional superconformal mechanical models in a Lagrangian setting.

In one dimension the superconformal invariance can be characterized by the superconformal algebras, which are recovered from the list of finite simple Lie superalgebras [93, 94, 95, 96] with further restrictions. Their even sector $\mathcal{G}_{even}$ is a direct sum $\mathcal{G}_{even} = sl(2) \oplus R$ (where R is known as the R-symmetry), while the odd sector $\mathcal{G}_{odd}$ is spanned by $2\mathcal{N}$ odd generators ($\mathcal{N}$ denotes the value of the extended supersymmetry). The superalgebras are closed under (anti)commutators and satisfy the graded Jacobi identities. The $sl(2)$ generators will be denoted as D, K, H (D is the dilatation operator). Explicitly, they satisfy the commutation relations

$$[D, H] = H, \qquad [D, K] = -K, \qquad [H, K] = 2D. \tag{4.1}$$

A grading is induced by the dilatation operator D, so that ($\mathcal{G}_i$ is the sector of grading i and for any $g_i^\alpha \in \mathcal{G}_i$ the commutator $[D, g_i^\alpha] = i g_i^\alpha$ holds)

$$\mathcal{G}_{even} = \mathcal{G}_{-1} \oplus \mathcal{G}_0 \oplus \mathcal{G}_1, \qquad \mathcal{G}_{odd} = \mathcal{G}_{-\frac{1}{2}} \oplus \mathcal{G}_{\frac{1}{2}}. \tag{4.2}$$

The sector $\mathcal{G}_1$ ($\mathcal{G}_{-1}$) contains a unique generator given by H (K). The odd sectors $\mathcal{G}_{\frac{1}{2}}$ and $\mathcal{G}_{-\frac{1}{2}}$ are spanned by the $\mathcal{N}$ supercharges Q_I's and their $\mathcal{N}$ superconformal partners $\widetilde{Q}_I$'s, respectively. The $\mathcal{G}_0$ sector is given by the union of D and the R-symmetry subalgebra ($\mathcal{G}_0 = \{D\} \bigcup \{R\}$).

We have, in particular, that the following anticommutation relations are satisfied for $I, J = 1, 2, \ldots, \mathcal{N}$:

$$\begin{aligned} \{Q_I, Q_J\} &= 2\delta_{IJ} H, \\ \{\widetilde{Q}_I, \widetilde{Q}_J\} &= -2\delta_{IJ} K, \end{aligned} \tag{4.3}$$

with furthermore

$$[H, Q_I] = [K, \widetilde{Q}_I] = 0. \tag{4.4}$$

It follows that the positive sector $\mathcal{G}_{>0} = \mathcal{G}_{\frac{1}{2}} \oplus \mathcal{G}_1$, spanned by the generators Q_I's and H, is isomorphic to the one-dimensional, $\mathcal{N}$-extended, global supersymmetry, namely the superalgebra underlying the Supersymmetric Quantum Mechanics.

The $sl(2)$ algebra admits a (non-critical) D-module representation, expressed by the differential operators

$$
\begin{aligned}
H &= \frac{d}{dt}, \\
D &= -t\frac{d}{dt} - \lambda, \\
K &= -t^2\frac{d}{dt} - 2\lambda t.
\end{aligned} \tag{4.5}
$$

This representation is non-critical since it closes the (4.1) $sl(2)$ algebra for any value of the scaling dimension λ.

On the other hand, linear D-modules representations for the $\mathcal{N}$-extended global supersymmetry algebra [15] spanned by the generators Q_I's and H have been intensively studied in the last decade [38]-[67]. They are surprisingly rich and intricated. They can be divided into several classes: minimal versus non-minimal representations, homogeneous versus inhomogeneous representations, admitting (or not, see [67]) a graphical presentation, etc. Those features which are relevant were briefly recalled in Introduction.

Following [92], we require the compatibility condition of the (4.5) $sl(2)$ D-module representations with the D-module representations of the $\mathcal{N}$-extended global supersymmetries. The H, D, K generators are assumed to act diagonally on the component fields of the global supermultiplets. The relative scaling dimension of the component fields entering the global supermultiplets are unambiguously fixed by the dimensionality ($[Q_I] = \frac{1}{2}, [H] = 1$) of the generators of the global supersymmetry. A unique free parameter is left. It is the overall scaling dimension λ of the global supermultiplet which, by definition, coincides with the lowest scaling dimension of its component fields.

The finite superconformal algebras introduced above admit the following properties. Their generators can be recovered by repeatedly applying the (anti)commutation relations involving the supercharges Q_I and the conformal generator K. We have therefore the possibility to check, for any given global supermultiplet with $2n$ component fields and any overall assignment λ of the scaling dimension, whether the Ansatz for K expressed by

$$
K = -t^2\frac{d}{dt}\mathbf{1}_{2n} - 2t\Lambda \tag{4.6}
$$

(Λ is a diagonal matrix whose diagonal entries coincide with the scaling dimensions of the component fields so that, for a length-l supermultiplet Λ is given by $\Lambda = diag(\lambda_1 = \lambda, \ldots, \lambda_{2n} = \lambda + \frac{l-1}{2}))$ induces a D-module representation for a finite superconformal algebra (to be determined).

To this end we recall that in a finite superconformal algebra the conformal superpartners $\widetilde{Q}_I$ of the global supercharges Q_I are recovered through the position

$$[K, Q_I] \;=\; \widetilde{Q}_I, \tag{4.7}$$

which can be assumed to be the definition of the $\widetilde{Q}_I$ generators. The closure of the superalgebra requires the introduction of new even generators

$$S_{IJ} \;=\; \{Q_I, \widetilde{Q}_J\}. \tag{4.8}$$

By making use of the Jacobi identities one can easily prove that, for any I, $S_{II} = -2D$ while, for $I \neq J$, the antisymmetric property $S_{IJ} = -S_{JI}$ holds. For $I > J$ the S_{IJ}'s generators commute with H and K,

$$[H, S_{IJ}] = [K, S_{IJ}] = 0, \quad (I > J), \tag{4.9}$$

so that they can be identified with the R-symmetry generators. The S_{IJ} generators are not, in general, linearly independent.

The explicit construction of the differential operators $\widetilde{Q}_I$ and S_{IJ} from repetend (anti)-commutators involving K and the Q_I's does not yet guarantee that we have a D-module representation for a finite $\mathcal{N}$-extended superconformal algebra. We need to verify that the superalgebra of differential operators $H, K, D, Q_I, \widetilde{Q}_I, S_{IJ}$ closes without the introduction of further generators. For this purpose it is sufficient to check whether the commutators involving the S_{IJ}'s generators and the Q_K's close on the global supercharges, namely that

$$[S_{IJ}, Q_K] \;=\; \alpha^L_{IJ,K} Q_L \tag{4.10}$$

is verified for some structure constants $\alpha^L_{IJ,K}$. Indeed, by making use of the Jacobi identities the (4.10) equation implies the closure of the commutators $[S_{IJ}, \widetilde{Q}_K]$ and $[S_{IJ}, S_{MN}]$. We have

$$\begin{aligned} [S_{IJ}, \widetilde{Q}_K] &= \alpha^L_{IJ,K} \widetilde{Q}_L, \\ [S_{IJ}, S_{MN}] &= -\{Q_M, [\widetilde{Q}_N, S_{IJ}]\} + \{\widetilde{Q}_N, [S_{IJ}, Q_M]\} = \alpha^L_{IJ,N} S_{ML} + \alpha^L_{IJ,M} S_{LN} \end{aligned} \tag{4.11}$$

The condition (4.10) on the differential operators will be called the "closure condition". In the cases that we investigated three possibilities are encountered, associated with the choice of the global supermultiplet and its scaling dimension λ:

i) the closure condition is automatically satisfied for any value of λ (no criticality in this case),

ii) the closure condition leads to a linear equation for λ which pinpoints the critical value λ,

iii) the closure condition admits no solution. In this latter case the global supermultiplet cannot be lifted to a D-module representation for a finite superconformal algebra.

To perform the needed computations we developed a package for Mathematica. We report here the results. All four $\mathcal{N} = 8$ finite superconformal algebras are recovered from a D-module representation induced by the $\mathcal{N} = 8$ $(k, 8, 8-k)$ global supermultiplets for $k \neq 4$ at the critical values $\lambda_k = \frac{1}{k-4}$, with the following identifications: $D(4,1)$ for $k = 0, 8$, $F(4)$ for $k = 1, 7$, $A(3,1)$ for $k = 2, 6$ and $D(2,2)$ for $k = 3, 5$. For $k = 4$ the closure condition admits no solution. Furthermore, the unique $\mathcal{N} = 7$ global supermultiplet which cannot be extended to an $\mathcal{N}=8$ supermultiplet (identified by its field content $(1,7,7,1)$ and first introduced in [37]) induces, at the critical value $\lambda=-\frac{1}{4}$, a D-module representation of the exceptional superalgebra $G(3)$.

For $\mathcal{N}=4$ the class of exceptional superalgebras $D(2,1;\alpha)$, parameterized by α, is recovered for the $(k,4,4-k)$ global supermultiplets with the identification $\alpha=(2-k)\lambda$ in terms of the scaling dimension λ. The unique global $\mathcal{N}=3$ supermultiplet which cannot be extended to an $\mathcal{N}=4$ supermultiplet (identified by its field content $(1,3,3,1)$) induces the D-module of the $B(1,1)=osp(3|2)$ $\mathcal{N}=3$ superconformal algebra for any value of the scaling dimension λ (no criticality).

Besides these results we also proved that the inhomogeneous extension of the $\mathcal{N}=8$ global supermultiplet $(3,8,5)$ induces a D-module representation for the $D(2,2)$ superalgebra. The inhomogeneous term is essential, see [**?**, 92], to introduce Calogero-type terms [97] in superconformal mechanics [98]-[101]. We further analyzed a few cases of $\mathcal{N}=6$ global supermultiplets and their induced D-module representations.

The scheme of this chapter is the following. In Section **2** we review the results of [92] about D-module representations of the $\mathcal{N}=4$ superconformal algebras from minimal global $\mathcal{N}=4$ supermultiplets and we present the derivation of the non-critical D-module representation of the $B(1,1)$ simple superalgebra from the $\mathcal{N}=3$ $(1,3,3,1)$ global supermultiplet. In Section **3** the four finite $\mathcal{N}=8$ superconformal algebras are realized as D-module representations from the $\mathcal{N}=8$ global supermultiplets $(k,8,8-k)$ for critical values of the scaling dimension λ associated with $k\neq 4$. The unique $\mathcal{N}=7$ finite superconformal algebra $G(3)$ is recovered, at the critical value $\lambda=-\frac{1}{4}$, from the $(1,7,7,1)$ supermultiplet, namely the unique $\mathcal{N}=7$ minimal supermultiplet which cannot be extended to an $\mathcal{N}=8$ representation. Other cases, both critical and non-critical, of D-module representations for $\mathcal{N}=6$ superconformal algebras are presented in Section **4**.

In the Discussion we make more comments on the obtained results.

4.2 D-module representations of $\mathcal{N} = 3, 4$ Superconformal algebras

We summarize at first the results of [92] concerning the D-module representations of finite $\mathcal{N} = 4$ superconformal algebras induced by the minimal linear representations of the $\mathcal{N} = 4$ global supersymmetry. We postpone to next chapter a detailed discussion of the finite $\mathcal{N} = 4$ superconformal algebras and the interpretation of the results here reported.

The minimal homogeneous linear global $\mathcal{N} = 4$ supermultiplets are expressed in terms of their field content $(k, 4, 4-k)$, with $k = 0, 1, 2, 3, 4$. The $k = 0, 1$ global supermultiplets admit a inhomogeneous extension [92].

For $k = 2$, the D-module representation of the $A(1,1) = sl(2|2)/\mathcal{Z}$ superalgebra is encountered at the $\lambda = 0$ value of the scaling dimension, while a D-module representation of $sl(2|2)$ is found at $\lambda \neq 0$.

For $k \neq 2$, D-module representations of the $\mathcal{N} = 4$ exceptional superalgebras $D(2,1;\alpha)$ are found. The identification between α and the scaling dimension is expressed by

$$\alpha = (2-k)\lambda, \tag{4.12}$$

where the $A(1,1)$ superalgebra is recovered as a degenerate case for $\alpha = 0, -1$ (see Section **5.4**).

The $(0, 4, 4)_{inhom}$ inhomogeneous extension of the $k = 0$ global supermultiplet does not induce D-module representations for finite $\mathcal{N} = 4$ superconformal algebras. The $(1, 4, 3)_{inhom}$ inhomogeneous extension of $k = 1$, on the other hand, induces a D-module representation of the $A(1,1)$ superalgebra. The presence of the inhomogeneous term in the supertransformations forces the scaling dimension of the $(1, 4, 3)_{inhom}$ inhomogeneous supermultiplet to be given by $\lambda = -1$. Since formula (4.12) continues to hold, the $\alpha = -1$ value is recovered.

At this stage we have already encountered, for $\mathcal{N} = 4$, the phenomenon of "critical scaling". Indeed, due to (4.12), inequivalent $D(2,1;\alpha)$ superalgebras are identified for different values of λ (and k). It is worth mentioning that, for $\alpha \in \mathbb{C}\backslash\{0,-1\}$, two $D(2,1;\alpha)$ superalgebras are isomorphic if and only if their parameters are related by one of the six transformations belonging to the S_3 group given by (5.15).

In the next chapter Section **5.4** we will discuss the implications of the $\mathcal{N}=4$ critical scaling for D-module representations of the $\mathcal{N}>4$ superconformal algebras and in constraining the Lagrangian formulation of $\mathcal{N}=4$ superconformal theories.

There exists a unique global $\mathcal{N}=3$ supermultiplet which cannot be extended to an $\mathcal{N}=4$ supermultiplet (see [38]). Its field content is $(1,3,3,1)$ and its component fields will be denoted as x, ψ_i, g_i, ω ($i=1,2,3$). Their scaling dimensions are, respectively, $[x]=\lambda$, $[\psi_i]=\lambda+\frac{1}{2}$, $[g_i]=\lambda+1$, $[\omega]=\lambda+\frac{3}{2}$ (λ is the scaling dimension of the supermultiplet). The fields x, g_i are assumed to be bosonic, while ψ_i, ω are assumed to be fermionic. They can be rearranged in the supermultiplet $|m>$ such that $|m>^T=<m|=<x,g_1,g_2,g_3;\omega,\psi_1,\psi_2,\psi_3|$. The three global supercharges acting on $|m>$ are given by the 8×8 supermatrices Q_i obtained from the "root" supercharges Q_i^R by applying the dressing transformation $S=diag(1,\partial_t,\partial_t,\partial_t,\partial_t,1,1,1)$.

The $(1,3,3,1)$ supermultiplet induces a D-module representation of the $\mathcal{N}=3$ superconformal algebra $B(1,1)=osp(3,2)$, with 6 even and 6 odd generators. Its bosonic subalgebra is $sl(2)\oplus so(3)$. The D-module representation is obtained for any λ, since the "closure condition" (4.10) gives no restriction on λ (there is no critical scaling in this case). The construction outlined in the Introduction produces, besides the generators H, D, K and Q_i's, the superconformal partners $\widetilde{Q}_i$'s and the three independent R-symmetry generators S_{ij}, for $i>j$.

An explicit presentation of the D-module generators of $B(1,1)$ is given by the following supermatrices acting on $(4|4)$ supermultiplets (here and in the following E_{mn} denotes the matrix with entry 1 at the crossing of the m-th row and n-th column and 0 otherwise)

$$H = \mathbf{1}_8 \cdot \partial_t,$$

$$
\begin{array}{rcl}
D &=& -\mathbf{1}_8 \cdot t\partial_t - \Lambda, \\
K &=& -\mathbf{1}_8 \cdot t^2\partial_t - 2t\Lambda, \\
Q_1 &=& (-E_{38} + E_{47} - E_{52} + E_{61})\partial_t + E_{16} - E_{25} + E_{74} - E_{83}, \\
Q_2 &=& (E_{28} - E_{46} - E_{53} + E_{71})\partial_t + E_{17} - E_{35} - E_{64} + E_{82}, \\
Q_3 &=& (-E_{27} + E_{36} - E_{54} + E_{81})\partial_t + E_{18} - E_{45} + E_{63} - E_{72}, \\
\widetilde{Q}_1 &=& (-E_{38} + E_{47} - E_{52} + E_{61})t\partial_t + (E_{16} - E_{25} + E_{74} - E_{83})t + \\
&& -E_{52}(2\lambda + 2) + (-E_{38} + E_{47})(2\lambda + 1) + E_{61}2\lambda, \\
\widetilde{Q}_2 &=& (E_{28} - E_{46} - E_{53} + E_{71})t\partial_t + (E_{17} - E_{35} - E_{64} + E_{82})t + \\
&& -E_{53}(2\lambda + 2) + (E_{28} - E_{46})(2\lambda + 1) + E_{71}2\lambda, \\
\widetilde{Q}_3 &=& (-E_{27} + E_{36} - E_{54} + E_{81})t\partial_t + (E_{18} - E_{45} + E_{63} - E_{72})t + \\
&& -E_{54}(2\lambda + 2) + (-E_{27} + E_{36})(2\lambda + 1) + E_{81}2\lambda, \\
S_1 &=& E_{34} - E_{43} + E_{78} - E_{87}, \\
S_2 &=& -E_{24} + E_{42} - E_{68} + E_{86}, \\
S_3 &=& E_{23} - E_{32} + E_{67} - E_{76},
\end{array}
\tag{4.13}
$$

where Λ is the diagonal matrix $\Lambda = diag(\lambda, \lambda+1, \lambda+1, \lambda+1, \lambda+\frac{3}{2}, \lambda+\frac{1}{2}, \lambda+\frac{1}{2}, \lambda+\frac{1}{2})$ and $S_i = \epsilon_{ijk}S_{jk}$ ($\epsilon_{123} = 1$).

In this basis the (anti)commutation relations of the $B(1,1)$ superalgebra reads as

$$
\begin{array}{rclrcl}
[H, K] &=& 2D, && & \\
[D, H] &=& H, & [D, K] &=& -K, \\
[D, Q_i] &=& \frac{1}{2}Q_i, & [D, \widetilde{Q}_i] &=& -\frac{1}{2}\widetilde{Q}_i, \\
[H, \widetilde{Q}_i] &=& Q_i, & [K, Q_i] &=& \widetilde{Q}_i, \\
\{Q_i, Q_j\} &=& 2\delta_{ij}H, & \{\widetilde{Q}_i, \widetilde{Q}_j\} &=& -2\delta_{ij}K, \\
\{Q_i, \widetilde{Q}_j\} &=& -2\delta_{ij}D + \epsilon_{ijk}S_k, & [S_i, Q_j] &=& -\epsilon_{ijk}Q_k, \\
[S_i, \widetilde{Q}_j] &=& -\epsilon_{ijk}\widetilde{Q}_k, & [S_i, S_j] &=& -\epsilon_{ijk}S_k.
\end{array}
\tag{4.14}
$$

With this $\mathcal{N}=3$ derivation we have shown explicitly our construction. In the following, in order to avoid reproducing too cumbersome formulas, we limit ourselves to report the main results.

It could be instructive to close this Section with an example of a derivation of an $\mathcal{N}=4$ superconformal algebra from a non-minimal global $\mathcal{N}=4$ supermultiplet (whose number of component fields is doubled with respect to the minimal supermultiplets, see [66]). The unique length-5 $\mathcal{N}=4$ supermultiplet is the "enveloping supermultiplet" [37], with field content $(1,4,6,4,1)$. It induces a D-module representation for any λ (the (4.10) closure condition is automatically satisfied) and the derived superalgebra is unique (it does not depend on the scaling dimension λ). It is given by $D(2,1;\alpha=1)$. At this special value of α, $D(2,1;1)$ is also denoted as $D(2,1)$. It corresponds to a superalgebra which belongs to the $D(m,n)=osp(2m|2n)$ classical series [95].

4.3 Critical scaling dimensions and D-module representations of $\mathcal{N}=7,8$ Superconformal Algebras

The minimal homogeneous linear global $\mathcal{N}=8$ supermultiplets are expressed in terms of their field content $(k,8,8-k)$, with $k=0,1,2,\ldots,8$. The $k=0,1,2,3$ global supermultiplets admit a inhomogeneous extension [32]. There exists a unique minimal, global $\mathcal{N}=7$, linear supermultiplet which cannot be extended to $\mathcal{N}=8$. It is a length-4 supermultiplet whose field content is $(1,7,7,1)$, see [37].

Before reporting the results of their D-module induced representations for finite superconformal algebras we recall that, over $\mathbb{C}$, there are four finite $\mathcal{N}=8$ superconformal algebras and one finite $\mathcal{N}=7$ superconformal algebra [95]. The finite $\mathcal{N}=8$ superconformal algebras are:

i) the $A(3,1)=sl(4|2)$ superalgebra, possessing 19 even generators and bosonic sector given by $sl(2)\oplus sl(4)\oplus u(1)$, *ii*) the $D(4,1) = osp(8,2)$ superalgebra, possessing 31 even generators and bosonic sector given by $sl(2) \oplus so(8)$,

iii) the $D(2,2) = osp(4|4)$ superalgebra, possessing 16 even generators and bosonic sector given by $sl(2) \oplus so(3) \oplus sp(4)$,

iv) the $F(4)$ exceptional superalgebra, possessing 24 even generators and bosonic sector given by $sl(2) \oplus so(7)$.

The finite $\mathcal{N} = 7$ superconformal algebra is the exceptional superalgebra $G(3)$, possessing 17 even generators and bosonic sector given by $sl(2) \oplus g_2$.

without loss of generality we used the following conventions for the $\mathcal{N} = 8$ root supermultiplet. The 8 supercharges are given by

$$\begin{aligned} Q_J^R &= \begin{pmatrix} 0 & \gamma_J \\ -\gamma_J \partial_t & 0 \end{pmatrix}, \\ Q_8^R &= \begin{pmatrix} 0 & \mathbf{1}_8 \\ \mathbf{1}_8 \partial_t & 0 \end{pmatrix}, \end{aligned} \quad (4.15)$$

where the γ_J matrices ($J = 1, \ldots, 7$) are the generators of the $Cl(0,7)$ Euclidean Clifford algebra: $\{\gamma_J, \gamma_L\} = -2\delta_{JL}\mathbf{1}_8$. We have, explicitly,

$$\begin{aligned} &\gamma_1 = \tau_2 \otimes \tau_2 \otimes \tau_3, \quad \gamma_2 = \tau_2 \otimes \tau_3 \otimes \mathbf{1}_2, \quad \gamma_3 = \tau_2 \otimes \tau_1 \otimes \tau_3, \\ &\gamma_4 = \tau_3 \otimes \mathbf{1}_2 \otimes \mathbf{1}_2, \quad \gamma_5 = \tau_1 \otimes \mathbf{1}_2 \otimes \tau_3, \quad \gamma_6 = \tau_1 \otimes \tau_3 \otimes \tau_2, \\ &\gamma_7 = \tau_1 \otimes \tau_3 \otimes \tau_1. \end{aligned} \quad (4.16)$$

The three 2×2 matrices τ_i are $\tau_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\tau_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ $\tau_3 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

supermultiplets of higher length are obtained from the given root supermultiplet via a dressing transformation determined by a dressing operator S. S is a diagonal differential operator whose diagonal entries (for the cases here considered) are 1 and $\partial_t = \frac{d}{dt}$.

Let Q_I^R be the supermatrices expressing the supercharges in the root representation. The supermatrices Q_I of a dressed representation are given by

$$Q_I \;=\; SQ_I^R S^{-1}. \qquad (4.17)$$

One should note that S^{-1} is a pseudo-differential operator. The requirement that the supermatrices Q_I are differential operators (see the analysis in [38, 37]) puts a constraint on the admissible dressings and, as a consequence, on the admissible higher length supermultiplets.

The $\mathcal{N} = 8$ minimal global supermultiplets $(k, 8, 8 - k)$ admits an inhomogeneous extension for $k = 0, 1, 2, 3$. Among them, the $(3, 8, 5)_{inhom}$ global inhomogeneous supermultiplet is the only one which induces a D-module representation for an $\mathcal{N} = 8$ superconformal algebra.

The hamiltonian H is $H = \partial_t \cdot \mathbf{1}$, while the conformal generator K is given by formula (4.6). The diagonal matrix Λ entering (4.6) is unambiguously determined in terms of the dressing transformation S and the overall scaling dimension λ. The remaining generators of the superconformal algebras $(\widetilde{Q}_I, D, S_{IJ})$ are obtained from equation (4.7) $(\widetilde{Q}_I)$ and (4.8) (D and S_{IJ}). The closure condition (4.10) admits no solution for the global $\mathcal{N} = 8$ $(4, 8, 4)$ supermultiplet. In the remaining cases it fixes the critical value λ of the scaling dimension.

The results of the D-module representations of the $\mathcal{N} = 8$ finite superconformal algebras present a $(k, 8, 8 - k) \leftrightarrow (8 - k, 8, k)$ duality which is already encountered in the global supersymmetry (see [37]).

The results are the following:

4.3.1 Critical D-module representations of $D(4,1)$ from $(8,8,0)$ at $\lambda = \frac{1}{4}$ and $(0,8,8)$ at $\lambda = -\frac{1}{4}$.

The respective diagonal dressing matrices S (and their associated diagonal matrices Λ with the scaling dimensions of the component fields) are explicitly given by

$$
\begin{aligned}
k=8 \quad : \quad & S = diag(1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1),\\
& \Lambda = diag(\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{3}{4},\frac{3}{4},\frac{3}{4},\frac{3}{4},\frac{3}{4},\frac{3}{4},\frac{3}{4},\frac{3}{4}),\\
k=0 \quad : \quad & S = diag(\partial_t,\partial_t,\partial_t,\partial_t,\partial_t,\partial_t,\partial_t,\partial_t,1,1,1,1,1,1,1,1),\\
& \Lambda = diag(\frac{3}{4},\frac{3}{4},\frac{3}{4},\frac{3}{4},\frac{3}{4},\frac{3}{4},\frac{3}{4},\frac{3}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4}).
\end{aligned}
\tag{4.18}
$$

In both cases the 28 S_{IJ} generators for $I > J$ are all linearly independent (unambiguously identifying the finite superconformal algebra as $D(4,1)$). The non-vanishing values of the structure constants $\alpha^L_{IJ,K}$ entering the closure condition (4.10) are

$$
\alpha^I_{IJ,J} = 1, \qquad \alpha^J_{IJ,I} = -1.
\tag{4.19}
$$

For $k = 8$ we recover the result of [92].

4.3.2 Critical D-module representations of $F(4)$ from $(7,8,1)$ at $\lambda = \frac{1}{3}$ and $(1,8,7)$ at $\lambda = -\frac{1}{3}$.

The respective diagonal dressing matrices S (and their associated diagonal matrices Λ with the scaling dimensions of the component fields) are explicitly given by

$$
\begin{aligned}
k=7 \quad : \quad & S = diag(1,1,1,1,1,1,1,\partial_t,1,1,1,1,1,1,1,1),\\
& \Lambda = diag(\frac{1}{3},\frac{1}{3},\frac{1}{3},\frac{1}{3},\frac{1}{3},\frac{1}{3},\frac{1}{3},\frac{4}{3},\frac{5}{6},\frac{5}{6},\frac{5}{6},\frac{5}{6},\frac{5}{6},\frac{5}{6},\frac{5}{6},\frac{5}{6}).\\
k=1 \quad : \quad & S = diag(1,\partial_t,\partial_t,\partial_t,\partial_t,\partial_t,\partial_t,\partial_t,1,1,1,1,1,1,1,1),\\
& \Lambda = diag(-\frac{1}{3},\frac{2}{3},\frac{2}{3},\frac{2}{3},\frac{2}{3},\frac{2}{3},\frac{2}{3},\frac{2}{3},\frac{1}{6},\frac{1}{6},\frac{1}{6},\frac{1}{6},\frac{1}{6},\frac{1}{6},\frac{1}{6},\frac{1}{6}).
\end{aligned}
\tag{4.20}
$$

In both cases 7 relations reduce the number of linearly independent S_{IJ} generators (for $I > J$) to 21, unambiguously identifying the $\mathcal{N} = 8$ finite superconformal algebra as $F(4)$.

For $(7,8,1)$ we have

$$\begin{aligned}
S_{21} - S_{65} - S_{74} + S_{83} &= 0, \\
S_{31} - S_{64} + S_{75} - S_{82} &= 0, \\
S_{32} + S_{54} + S_{76} + S_{81} &= 0, \\
S_{41} + S_{63} + S_{72} + S_{85} &= 0, \\
S_{42} - S_{53} - S_{71} + S_{86} &= 0, \\
S_{43} + S_{52} - S_{61} - S_{87} &= 0, \\
S_{51} + S_{62} - S_{73} - S_{84} &= 0. \qquad (4.21)
\end{aligned}$$

For $(1,8,7)$ we have

$$\begin{aligned}
S_{21} - S_{65} + S_{74} - S_{83} &= 0, \\
S_{31} - S_{64} - S_{75} + S_{82} &= 0, \\
S_{32} + S_{54} - S_{76} - S_{81} &= 0, \\
S_{41} + S_{63} - S_{72} - S_{85} &= 0, \\
S_{42} - S_{53} + S_{71} - S_{86} &= 0, \\
S_{43} + S_{52} - S_{61} - S_{87} &= 0, \\
S_{51} + S_{62} + S_{73} + S_{84} &= 0. \qquad (4.22)
\end{aligned}$$

These relations have the following general form :

$$\epsilon_1 S_{i_1 j_1} + \epsilon_2 S_{i_2 j_2} + \epsilon_3 S_{i_3 j_3} + \epsilon_4 S_{i_4 j_4} = 0$$

or in a more compact:

$$\Sigma_a \epsilon_a S_{I_a J_a} = 0 \quad a, b = 1, 2, 3, 4$$

with coefficients $\epsilon_a = \pm 1$.

The structure constants of the superconformal algebra are explicitly computed. Non zero structure constants are :

$$\alpha^{I}_{IJ,J} = 1, \qquad \alpha^{J}_{IJ,I} = -1, \tag{4.23}$$

Apart of these cases, another set of non zero constants appear when $I_aJ_a \neq I_bJ_b$ and both I_aJ_a and I_bJ_b appear in one of the above relations, then we have :

$$\alpha^{I_b}_{I_aJ_a,J_b} = -\frac{1}{3}\epsilon_a\epsilon_b, \alpha^{J_b}_{I_aJ_a,I_b} = \frac{1}{3}\epsilon_a\epsilon_b.$$

4.3.3 Critical D-module representations of $A(3,1)$ from $(6,8,2)$ at $\lambda = \frac{1}{2}$ and $(2,8,6)$ at $\lambda = -\frac{1}{2}$.

The respective diagonal dressing matrices S (and their associated diagonal matrices Λ with the scaling dimensions of the component fields) are explicitly given by

$$\begin{aligned}
k=6 \quad &: \quad S = diag(1,1,1,1,1,1,\partial_t,\partial_t,1,1,1,1,1,1,1,1),\\
&\quad\ \ \Lambda = diag(\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{3}{2},\frac{3}{2},1,1,1,1,1,1,1,1).\\
k=2 \quad &: \quad S = diag(1,1,\partial_t,\partial_t,\partial_t,\partial_t,\partial_t,\partial_t,1,1,1,1,1,1,1,1),\\
&\quad\ \ \Lambda = diag(-\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},0,0,0,0,0,0,0,0).
\end{aligned} \tag{4.23}$$

In both cases 12 relations reduce the number of linearly independent S_{IJ} generators (for $I > J$) to 16, unambiguously identifying the $\mathcal{N} = 8$ finite superconformal algebra as $A(3,1)$.

For $(6,8,2)$ we have

$$\begin{aligned}
S_{21} + S_{83} &= 0, \qquad & S_{65} + S_{74} &= 0,\\
S_{31} - S_{82} &= 0, \qquad & S_{64} - S_{75} &= 0,\\
S_{41} + S_{85} &= 0, \qquad & S_{63} + S_{72} &= 0,\\
S_{42} - S_{53} &= 0, \qquad & S_{71} - S_{86} &= 0,\\
S_{43} + S_{52} &= 0, \qquad & S_{61} + S_{87} &= 0,\\
S_{51} - S_{84} &= 0, \qquad & S_{62} - S_{73} &= 0.
\end{aligned} \tag{4.24}$$

For $(2, 8, 6)$ we have

$$
\begin{aligned}
S_{21} - S_{83} &= 0, & S_{65} - S_{74} &= 0,\\
S_{31} + S_{82} &= 0, & S_{64} + S_{75} &= 0,\\
S_{41} - S_{85} &= 0, & S_{63} - S_{72} &= 0,\\
S_{42} - S_{53} &= 0, & S_{71} - S_{86} &= 0,\\
S_{43} + S_{52} &= 0, & S_{61} + S_{87} &= 0,\\
S_{51} + S_{84} &= 0, & S_{62} + S_{73} &= 0.
\end{aligned}
\tag{4.25}
$$

These relations have the following general form :

$$\epsilon_1 S_{i_1 j_1} + \epsilon_2 S_{i_2 j_2} = 0$$

or in a more compact:

$$\Sigma_a \epsilon_a S_{I_a J_a} = 0 a, b = 1, 2.$$

with coefficients $\epsilon_a = \pm 1$.

The structure constants of the superconformal algebra are explicitly computed. Non zero structure constants are either:

$$\alpha^{I}_{IJ,J} = 1, \qquad \alpha^{J}_{IJ,I} = -1, \tag{4.26}$$

or when $I_a J_a \neq I_b J_b$ and both $I_a J_a$ and $I_b J_b$ are connected in one of the above relations, then we have :

$$\alpha^{I_b}_{I_a J_a, J_b} = -\epsilon_a \epsilon_b, \alpha^{J_b}_{I_a J_a, I_b} = \epsilon_a \epsilon_b.$$

.

4.3.4 Critical D-module representations of $D(2,2)$ from $(5,8,3)$ at $\lambda = 1$ and $(3,8,5)$ at $\lambda = -1$.

The respective diagonal dressing matrices S (and their associated diagonal matrices Λ with the scaling dimensions of the component fields) are explicitly given by

$$
\begin{aligned}
k=5 \quad &: \quad S = diag(1,1,1,1,1,\partial_t,\partial_t,\partial_t,1,1,1,1,1,1,1,1),\\
&\quad\ \ \Lambda = diag(1,1,1,2,2,2,2,2,\tfrac{3}{2},\tfrac{3}{2},\tfrac{3}{2},\tfrac{3}{2},\tfrac{3}{2},\tfrac{3}{2},\tfrac{3}{2},\tfrac{3}{2}).\\
k=3 \quad &: \quad S = diag(1,1,1,\partial_t,\partial_t,\partial_t,\partial_t,\partial_t,1,1,1,1,1,1,1,1),\\
&\quad\ \ \Lambda = diag(-1,-1,-1,0,0,0,0,0,-\tfrac{1}{2},-\tfrac{1}{2},-\tfrac{1}{2},-\tfrac{1}{2},-\tfrac{1}{2},-\tfrac{1}{2},-\tfrac{1}{2},-\tfrac{1}{2}).
\end{aligned}
\tag{4.26}
$$

In both cases 15 relations reduce the number of linearly independent S_{IJ} generators (for $I > J$) to 13, unambiguously identifying the $\mathcal{N} = 8$ finite superconformal algebra as $D(2,2)$.

For $(5,8,3)$ we have

$$
\begin{aligned}
S_{21} - S_{65} - S_{74} + S_{83} &= 0,\\
S_{31} + S_{64} - S_{75} - S_{82} &= 0,\\
S_{32} - S_{54} - S_{76} + S_{81} &= 0,
\end{aligned}
\tag{4.27}
$$

together with

$$
\begin{aligned}
&S_{41} = -S_{63} = S_{72} = -S_{85},\\
&S_{42} = S_{53} = -S_{71} = -S_{86},\\
&S_{43} = -S_{52} = S_{61} = -S_{87},\\
&S_{51} = S_{62} = S_{73} = S_{84}.
\end{aligned}
\tag{4.28}
$$

For $(3,8,5)$ we have

$$\begin{aligned} S_{21} - S_{65} + S_{74} - S_{83} &= 0, \\ S_{31} + S_{64} + S_{75} + S_{82} &= 0, \\ S_{32} - S_{54} + S_{76} - S_{81} &= 0, \end{aligned} \tag{4.29}$$

together with

$$\begin{aligned} &S_{41} = -S_{63} = -S_{72} = S_{85}, \\ &S_{42} = S_{53} = S_{71} = S_{86}, \\ &S_{43} = -S_{52} = S_{61} = -S_{87}, \\ &S_{51} = S_{62} = -S_{73} = -S_{84}. \end{aligned} \tag{4.30}$$

The structure constants of the superconformal algebra are explicitly computed. Their expression is too cumbersome to be given by closed formulas, they are explicitly reported in Appendix **A** at the end of this chapter.

4.3.5 Critical D-module representations of $G(3)$ from $(1, 7, 7, 1)$ at $\lambda = -\frac{1}{4}$.

In this case the diagonal dressing matrix S can be chosen, without loss of generality, to be given by

$$S = diag(1, \partial_t, \partial_t, \partial_t, \partial_t, \partial_t, \partial_t, \partial_t, 1, 1, 1, 1, 1, 1, 1). \tag{4.31}$$

After dressing, 7 global supercharges Q_I's ($I = 1, \ldots, 7$) remain as differential operators.

14 out of the 21 generators S_{IJ} (for $I > J$) are linearly independent, due to the 7

relations

$$
\begin{aligned}
S_{21} - S_{65} + S_{74} &= 0, \\
S_{31} - S_{64} - S_{75} &= 0, \\
S_{32} + S_{54} - S_{76} &= 0, \\
S_{41} + S_{63} - S_{72} &= 0, \\
S_{42} - S_{53} + S_{71} &= 0, \\
S_{43} + S_{52} - S_{61} &= 0, \\
S_{51} + S_{62} + S_{73} &= 0. \qquad (4.32)
\end{aligned}
$$

These relations have the following general form :

$$\epsilon_1 S_{i_1 j_1} + \epsilon_2 S_{i_2 j_2} + \epsilon_3 S_{i_3 j_3} = 0$$

or in a more compact:

$$\Sigma_a \epsilon_a S_{I_a J_a} = 0 \quad a, b = 1, 2, 3.$$

with coefficients $\epsilon_a = \pm 1$.

At the critical scaling dimension $\lambda = -\frac{1}{4}$, the closure condition satisfies and one recovers the exceptional superalgebra $G(3)$.

At this critical value the diagonal matrix Λ with the scaling dimensions of the component fields is given by

$$\Lambda \;=\; diag(-\frac{1}{4}, \frac{3}{4}, \frac{3}{4}, \frac{3}{4}, \frac{3}{4}, \frac{3}{4}, \frac{3}{4}, \frac{3}{4}, \frac{5}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}). \qquad (4.33)$$

The structure constants of the closure condition are explicitly computed. Non zero structure constants are either:

$$\alpha^I_{IJ,J} = 1, \qquad \alpha^J_{IJ,I} = -1, \qquad (4.34)$$

or another set of non zero constants appear when $I_aJ_a \neq I_bJ_b$ and both I_aJ_a and I_bJ_b appear in one of the above relations :

$$\alpha^{I_b}_{I_aJ_a,J_b} = -\frac{1}{2}\epsilon_a\epsilon_b, \alpha^{J_b}_{I_aJ_a,I_b} = \frac{1}{2}\epsilon_a\epsilon_b.$$

4.3.6 D-module representation of $D(2,2)$ from the inhomogeneous $(3,8,5)$ supermultiplet.

There exists a unique inhomogeneous global supermultiplet which induces a D-module representation for a finite $\mathcal{N}=8$ superconformal algebra. The presence of the inhomogeneous extension fixes the scaling dimension of the auxiliary fields to be 0 (see [92]). This implies that the overall scaling dimension of the supermultiplet has to be $\lambda=-1$. The homogeneous $(3,8,5)$ supermultiplet is the only one inducing a D-module representation at this critical value of λ, the superconformal algebra being $D(2,2)$. It remains to be checked whether the presence of the inhomogeneous extension is compatible with the D-module representation for $D(2,2)$. The answer is positive. We adapted the construction discussed in [92] for the D-module representation induced by the inhomogeneous $(1,4,3)$ supermultiplet. The eight supercharges act now on a $(9|8)$ supermultiplet $|m>$ such that $<m|^T = (x_i, g_j, 1; \psi_a)$ $(i=1,2,3,\ j=1,\ldots,5,\ a=1,\ldots,8)$. They are explicitly given by

$$\begin{aligned}
Q_1 &= E_{1,11}-E_{2,10}-E_{3,13}+E_{12,4}+E_{14,6}-E_{15,5}-E_{16,8}+E_{17,7}+\\
&\quad (E_{4,12}-E_{5,15}+E_{6,14}+E_{7,17}-E_{8,16}-E_{10,2}+E_{11,1}-E_{13,3})\partial_t,\\
Q_2 &= E_{1,12}+E_{2,13}-E_{3,10}-E_{11,4}+E_{14,7}+E_{15,8}-E_{16,5}-E_{17,6}+\\
&\quad (-E_{4,11}-E_{5,16}-E_{6,17}+E_{7,14}+E_{8,15}-E_{10,3}+E_{12,1}+E_{13,2})\partial_t,\\
Q_3 &= E_{1,13}-E_{2,12}+E_{3,11}-E_{10,4}+E_{14,8}-E_{15,7}+E_{16,6}-E_{17,5}+\\
&\quad (-E_{4,10}-E_{5,17}+E_{6,16}-E_{7,15}+E_{8,14}+E_{11,3}-E_{12,2}+E_{13,1})\partial_t,
\end{aligned}$$

$$
\begin{aligned}
Q_4 &= E_{1,14}+E_{2,15}+E_{3,16}-E_{10,5}-E_{11,6}-E_{12,7}-E_{13,8}+E_{17,4}+cE_{17,9} \\
&\quad (E_{4,17}-E_{5,10}-E_{6,11}-E_{7,12}-E_{8,13}+E_{14,1}+E_{15,2}+E_{16,3})\partial_t, \\
Q_5 &= E_{1,15}-E_{2,14}+E_{3,17}-E_{10,6}+E_{11,5}-E_{12,8}+E_{13,7}-E_{16,4}-cE_{16,9}+ \\
&\quad (-E_{4,16}+E_{5,11}-E_{6,10}+E_{7,13}-E_{8,12}-E_{14,2}+E_{15,1}+E_{17,3})\partial_t, \\
Q_6 &= E_{1,16}-E_{2,17}-E_{3,14}-E_{10,7}+E_{11,8}+E_{12,5}-E_{13,6}+E_{15,4}+cE_{15,9}+ \\
&\quad (E_{4,15}+E_{5,12}-E_{6,13}-E_{7,10}+E_{8,11}-E_{14,3}+E_{16,1}-E_{17,2})\partial_t, \\
Q_7 &= E_{1,17}+E_{2,16}-E_{3,15}-E_{10,8}-E_{11,7}+E_{12,6}+E_{13,5}-E_{14,4}-cE_{14,9}+ \\
&\quad (-E_{4,14}+E_{5,13}+E_{6,12}-E_{7,11}-E_{8,10}-E_{15,3}+E_{16,2}+E_{17,1})\partial_t, \\
Q_8 &= E_{1,10}+E_{2,11}+E_{3,12}+E_{13,4}+E_{14,5}+E_{15,6}+E_{16,7}+E_{17,8}+ \\
&\quad (E_{4,13}+E_{5,14}+E_{6,15}+E_{7,16}+E_{8,17}+E_{10,1}+E_{11,2}+E_{12,3})\partial_t.
\end{aligned} \tag{4.34}
$$

One should notice the presence of the inhomogeneous constant c in the Q_4, Q_5, Q_6, Q_7 transformations.

The conformal generator K (4.6) is uniquely specified by the diagonal matrix Λ given by

$$
\Lambda = diag(-1,-1,-1,0,0,0,0,0,0,-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2}). \tag{4.35}
$$

The $D(2,2)$ superalgebra closes as in the homogeneous case (recovered for $c = 0$). The explicate calculations has been reported in the Appendix **A**, where we use a different basis. Its explicit presentation and the discussion of its properties is left to a forthcoming paper in preparation.

4.4 Some extra, critical and non-critical, D-modules with $\mathcal{N} = 6$

The strategy of searching for D-module representations of superconformal algebras requires, as an input, the supermultiplets of the global $\mathcal{N}$-Extended supersymmetry. For low values of $\mathcal{N}$ these supermultiplets have been classified. In these cases it is therefore possible to look for their possible, associated, superconformal D-modules. We limit here to discuss some further selected cases, besides the ones that we have already presented. To be definite, let us focus on the length-4 $\mathcal{N}=6$ global supermultiplets. Their resulting D-modules are of interest in the light of the analysis, based on their $\mathcal{N}=4$ decomposition, discussed in next chapter .

The three length-4 global $\mathcal{N}=6$ supermultiplets are [37] $(2,6,6,2)$, $(1,6,7,2)$ and $(2,7,6,1)$. In all three cases we obtain a D-module representation for the $A(2,1)=sl(3|2)$ $\mathcal{N}=6$ superconformal algebra, whose R-sector is the bosonic subalgebra $sl(3)\oplus u(1)$. The 15 generators S_{ij} (with $i,j=1,\ldots,6$ and $i>j$) are not linearly independent. 6 of them are determined in terms of the 9 remaining generators, unambiguously identifying the S_{ij}'s as the R-sector of $A(2|1)$. The request that no further odd generator is obtained, besides the 6 global supercharges Q_i's and their 6 conformal superpartners $\widetilde{Q}_i$'s, produces no constraint on λ for the $(2,6,6,2)$ case (no critical scaling dimension). On the other hand the scaling dimensions of $(1,6,7,2)$ and $(2,7,6,1)$ are constrained to the values, respectively, $\lambda=0$ and $\lambda=-\frac{1}{2}$. We can summarize these results in the following table. The $A(2,1)$ $\mathcal{N}=6$ superconformal algebra D-modules are recovered from

$$\begin{aligned}
&(2,6,6,2): \quad A(2,1), \quad \forall\lambda\in\mathbb{R},\\
&(1,6,7,2): \quad A(2,1), \quad for \quad \lambda=0,\\
&(2,7,6,1): \quad A(2,1), \quad for \quad \lambda=-\frac{1}{2}.
\end{aligned} \tag{4.36}$$

4.5 Discussion

In this chapter we constructed a class of D-module representations for one-dimensional superconformal algebras. These representations exhibit, for $\mathcal{N}=4,7,8$, the property of criticality. This means that they only close at critical values of the scaling dimension λ characterizing the supermultiplets of time-dependent component fields. The superalgebras under consideration are a given subclass of finite, simple, Lie superalgebras. Their D-module representations are an extension of the D-module representation (4.5) of the $sl(2)$ algebra (this representation is non-critical, being recovered for any value of λ).

The connection with the D-module representations [37] of the $\mathcal{N}$-Extended global supersymmetry (the superalgebra of the one-dimensional Supersymmetric Quantum Mechanics) is given by the fact that the latter is a subalgebra of the superconformal algebras. Certain minimal global supermultiplets induce, at a given λ, their associated D-module representations of a superconformal algebra. In particular, the exceptional finite Lie superalgebras [93, 95] $D(2,1;\alpha)$, $G(3)$ and $F(4)$ (which are superconformal algebras for $\mathcal{N}=4,7,8$, respectively) admit a D-module representation. Quite interestingly, the D-module representation of $G(3)$ is only induced from the minimal $(1,7,7,1)$ global $\mathcal{N}=7$ supermultiplet, first introduced in [37].

4.6 Appendix A: Superconformal D-module rep. based on inhomogeneous (3,8,5) supermultiplet

We report here for clarity, explicitly the generators and closure condition for a specific and important case, the only $N = 8$ inhomogeneous supermultiplet which admits superconformal representation: inhomogeneous (3,8,5) supermultiplet. Its explicit presentation and the discussion of its properties is left to a forthcoming paper in preparation. Please note that we have used a slightly different basis. The superconformal algebra is closed on under the condition of $\lambda = -1$ which has been applied.

$$
\begin{array}{rcl}
Q_1 & = & E_{1,11} - E_{2,10} - E_{5,15} + E_{12,4} - E_{13,3} + E_{14,6} - E_{16,8} + E_{17,7} \\
& & +(-E_{3,13} + E_{4,12} + E_{6,14} + E_{7,17} - E_{8,16} - E_{10,2} + E_{11,1} - E_{15,5})\partial_t \\
Q_2 & = & E_{1,12} + E_{2,13} - E_{5,16} - E_{10,3} - E_{11,4} + E_{14,7} + E_{15,8} - E_{17,6} - cE_{17,9} \\
& & +(-E_{3,10} - E_{4,11} - E_{6,17} + E_{7,14} + E_{8,15} + E_{12,1} + E_{13,2} - E_{16,5})\partial_t \\
Q_3 & = & E_{1,13} - E_{2,12} - E_{5,17} - E_{10,4} + E_{11,3} + E_{14,8} - E_{15,7} + E_{16,6} + cE_{16,9} \\
& & +(E_{3,11} - E_{4,10} + E_{6,16} - E_{7,15} + E_{8,14} - E_{12,2} + E_{13,1} - E_{17,5})\partial_t \\
Q_4 & = & E_{1,14} + E_{2,15} - E_{5,10} - E_{11,6} - E_{12,7} - E_{13,8} + E_{16,3} + E_{17,4} \\
& & +(E_{3,16} + E_{4,17} - E_{6,11} - E_{7,12} - E_{8,13} - E_{10,5} + E_{14,1} + E_{15,2})\partial_t \\
Q_5 & = & E_{1,15} - E_{2,14} + E_{5,11} - E_{10,6} - E_{12,8} + E_{13,7} - E_{16,4} + E_{17,3} \\
& & +(E_{3,17} - E_{4,16} - E_{6,10} + E_{7,13} - E_{8,12} + E_{11,5} - E_{14,2} + E_{15,1})\partial_t \\
Q_6 & = & E_{1,16} - E_{2,17} + E_{5,12} - E_{10,7} + E_{11,8} - E_{13,6} - cE_{13,9} - E_{14,3} + E_{15,4} \\
& & +(-E_{3,14} + E_{4,15} - E_{6,13} - E_{7,10} + E_{8,11} + E_{12,5} + E_{16,1} - E_{17,2})\partial_t \\
Q_7 & = & E_{1,17} + E_{2,16} + E_{5,13} - E_{10,8} - E_{11,7} + E_{12,6} + cE_{12,9} - E_{14,4} - E_{15,3} \\
& & +(-E_{3,15} - E_{4,14} + E_{6,12} - E_{7,11} - E_{8,10} + E_{13,5} + E_{16,2} + E_{17,1})\partial_t \\
Q_8 & = & E_{1,10} + E_{2,11} + E_{5,14} + E_{12,3} + E_{13,4} + E_{15,6} + E_{16,7} + E_{17,8} \\
& & +(E_{3,12} + E_{4,13} + E_{6,15} + E_{7,16} + E_{8,17} + E_{10,1} + E_{11,2} + E_{14,5})\partial_t
\end{array}
$$

$$
\begin{array}{rcll}
H & = & \partial_t \mathbf{1}_{17}, & (4.37) \\
K & = & -t^2\partial_t \mathbf{1}_{17} - 2t\Lambda, & (4.38) \\
D & = & -t\partial_t \mathbf{1}_{17} - \Lambda. & (4.39)
\end{array}
$$

Where $\Lambda = diag(-1, -1, 0, 0, -1, 0, 0, 0, 0, -1/2, -1/2, -1/2, -1/2, -1/2, -1/2, -1/2, -1/2)$ is a diagonal 17×17 matrix. another generators of the algebra are explicitly are as following: $\tilde{Q}_i = [K, Q_i]$

$$
\begin{aligned}
\tilde{Q_1} &= (E_{1,11} - E_{2,10} - E_{5,15} + E_{12,4} - E_{13,3} + E_{14,6} - E_{16,8} + E_{17,7})t \\
&\quad +(-E_{3,13} + E_{4,12} + E_{6,14} + E_{7,17} - E_{8,16} - E_{10,2} + E_{11,1} - E_{15,5})(t\partial_t - 1) \\
&\quad -(-E_{10,2} + E_{11,1} - E_{15,5}) \\
\tilde{Q_2} &= (E_{1,12} + E_{2,13} - E_{5,16} - E_{10,3} - E_{11,4} + E_{14,7} + E_{15,8} - E_{17,6} - cE_{17,9})t \\
&\quad +(-E_{3,10} - E_{4,11} - E_{6,17} + E_{7,14} + E_{8,15} + E_{12,1} + E_{13,2} - E_{16,5})(t\partial_t - 1) \\
&\quad -(E_{12,1} + E_{13,2} - E_{16,5}) \\
\tilde{Q_3} &= (E_{1,13} - E_{2,12} - E_{5,17} - E_{10,4} + E_{11,3} + E_{14,8} - E_{15,7} + E_{16,6} + cE_{16,9})t \\
&\quad +(E_{3,11} - E_{4,10} + E_{6,16} - E_{7,15} + E_{8,14} - E_{12,2} + E_{13,1} - E_{17,5})(t\partial_t - 1) \\
&\quad -(-E_{12,2} + E_{13,1} - E_{17,5}) \\
\tilde{Q_4} &= (E_{1,14} + E_{2,15} - E_{5,10} - E_{11,6} - E_{12,7} - E_{13,8} + E_{16,3} + E_{17,4})t \\
&\quad +(E_{3,16} + E_{4,17} - E_{6,11} - E_{7,12} - E_{8,13} - E_{10,5} + E_{14,1} + E_{15,2})(t\partial_t - 1) \\
&\quad -(-E_{10,5} + E_{14,1} + E_{15,2}) \\
\tilde{Q_5} &= (E_{1,15} - E_{2,14} + E_{5,11} - E_{10,6} - E_{12,8} + E_{13,7} - E_{16,4} + E_{17,3})t \\
&\quad +(E_{3,17} - E_{4,16} - E_{6,10} + E_{7,13} - E_{8,12} + E_{11,5} - E_{14,2} + E_{15,1})(t\partial_t - 1) \\
&\quad -(-E_{8,12} + E_{11,5} - E_{14,2} + E_{15,1}) \\
\tilde{Q_6} &= (E_{1,16} - E_{2,17} + E_{5,12} - E_{10,7} + E_{11,8} - E_{13,6} - cE_{13,9} - E_{14,3} + E_{15,4})t \\
&\quad +(-E_{3,14} + E_{4,15} - E_{6,13} - E_{7,10} + E_{8,11} + E_{12,5} + E_{16,1} - E_{17,2})(t\partial_t - 1) \\
&\quad -(E_{12,5} + E_{16,1} - E_{17,2}) \\
\tilde{Q_7} &= (E_{1,17} + E_{2,16} + E_{5,13} - E_{10,8} - E_{11,7} + E_{12,6} + cE_{12,9} - E_{14,4} - E_{15,3})t \\
&\quad +(-E_{3,15} - E_{4,14} + E_{6,12} - E_{7,11} - E_{8,10} + E_{13,5} + E_{16,2} + E_{17,1})(t\partial_t - 1) \\
&\quad -(E_{13,5} + E_{16,2} + E_{17,1}) \\
\tilde{Q_8} &= (E_{1,10} + E_{2,11} + E_{5,14} + E_{12,3} + E_{13,4} + E_{15,6} + E_{16,7} + E_{17,8})t \\
&\quad +(E_{3,12} + E_{4,13} + E_{6,15} + E_{7,16} + E_{8,17} + E_{10,1} + E_{11,2} + E_{14,5})(t\partial_t - 1) \\
&\quad -(E_{10,1} + E_{11,2} + E_{14,5})
\end{aligned}
$$

There are another set of generators in bosonic sector which complete the set of generators of the algebra, defined by $S_{ij} := \{Q_i, \tilde{Q_j}\}$. We do not present them explicitly here. Apart of the relations $S_{ij} = -Sji$ and $S_{ii} = -2D$, the following relations are hold between them $(i > j)$:

$$
\begin{aligned}
S_{21} = S_{65} = S_{74} = S_{83}, \\
S_{31} = -S_{64} = S_{75} = -S_{82}, \\
S_{42} = S_{53} = S_{71} = S_{86}, \\
S_{43} = -S_{52} = -S_{61} = S_{87}
\end{aligned}
$$

and:

$$
\begin{aligned}
S_{32} - S_{54} - S_{76} + S_{81} = 0 \\
S_{41} + S_{63} - S_{72} - S_{85} = 0 \\
S_{51} - S_{62} - S_{73} + S_{84} = 0
\end{aligned}
$$

By these relations, only thirteen out of S_{ij}s are genuinely independent. The “closure condition” for this algebra is satisfied only when $\lambda = -1$, as we applied already. We report the “closure condition” result.

$$[S_{21},Q_1]=Q_2,\quad [S_{21},Q_2]=Q_1,\qquad [S_{21},Q_3]=Q_8,\quad [S_{21},Q_4]=Q_7$$
$$[S_{21},Q_5]=Q_6,\quad [S_{21},Q_6]=-Q_5,\qquad [S_{21},Q_7]=-Q_4,\quad [S_{21},Q_8]=-Q_3$$
$$[S_{31},Q_1]=Q_3,\quad [S_{31},Q_2]=-Q_8,\qquad [S_{31},Q_3]=-Q_1,\quad [S_{31},Q_4]=-Q_6$$
$$[S_{31},Q_5]=Q_7,\quad [S_{31},Q_6]=Q_4,\qquad [S_{31},Q_7]=-Q_5,\quad [S_{31},Q_8]=Q_2$$
$$[S_{32},Q_1]=Q_8,\quad [S_{32},Q_2]=Q_3,\qquad [S_{32},Q_3]=-Q_2,\quad [S_{32},Q_4]=-Q_5$$
$$[S_{32},Q_5]=Q_4,\quad [S_{32},Q_6]=3Q_7,\qquad [S_{32},Q_7]=-3Q_6,\quad [S_{32},Q_8]=-Q_1$$
$$[S_{41},Q_1]=Q_4,\quad [S_{41},Q_2]=-Q_7,\qquad [S_{41},Q_3]=Q_6,\quad [S_{41},Q_4]=-Q_1$$
$$[S_{41},Q_5]=3Q_8,\quad [S_{41},Q_6]=-Q_3,\qquad [S_{41},Q_7]=Q_2,\quad [S_{41},Q_8]=-3Q_5$$
$$[S_{42},Q_1]=Q_7,\quad [S_{42},Q_2]=Q_4,\qquad [S_{42},Q_3]=Q_5,\quad [S_{42},Q_4]=-Q_2$$
$$[S_{42},Q_5]=-Q_3,\quad [S_{42},Q_6]=Q_8,\qquad [S_{42},Q_7]=-Q_1,\quad [S_{42},Q_8]=-Q_6$$
$$[S_{43},Q_1]=-Q_6,\quad [S_{43},Q_2]=-Q_5,\qquad [S_{43},Q_3]=Q_4,\quad [S_{43},Q_4]=-Q_3$$
$$[S_{43},Q_5]=Q_2,\quad [S_{43},Q_6]=Q_1,\qquad [S_{43},Q_7]=Q_8,\quad [S_{43},Q_8]=-Q_7$$
$$[S_{51},Q_1]=Q_5,\quad [S_{51},Q_2]=-Q_6,\qquad [S_{51},Q_3]=-Q_7,\quad [S_{51},Q_4]=-3Q_8$$
$$[S_{51},Q_5]=-Q_1,\quad [S_{51},Q_6]=Q_2,\qquad [S_{51},Q_7]=Q_3,\quad [S_{51},Q_8]=3Q_4$$
$$[S_{52},Q_1]=Q_6,\quad [S_{52},Q_2]=Q_5,\qquad [S_{52},Q_3]=-Q_4,\quad [S_{52},Q_4]=Q_3$$
$$[S_{52},Q_5]=-Q_2,\quad [S_{52},Q_6]=-Q_1,\qquad [S_{52},Q_7]=-Q_8,\quad [S_{52},Q_8]=Q_7$$
$$[S_{53},Q_1]=Q_7,\quad [S_{53},Q_2]=Q_4,\qquad [S_{53},Q_3]=Q_5,\quad [S_{53},Q_4]=-Q_2$$
$$[S_{53},Q_5]=-Q_3,\quad [S_{53},Q_6]=Q_8,\qquad [S_{53},Q_7]=-Q_1,\quad [S_{53},Q_8]=-Q_6$$
$$[S_{54},Q_1]=3Q_8,\quad [S_{54},Q_2]=-Q_3,\qquad [S_{54},Q_3]=Q_2,\quad [S_{54},Q_4]=Q_5$$
$$[S_{54},Q_5]=-Q_4,\quad [S_{54},Q_6]=Q_7,\qquad [S_{54},Q_7]=-Q_6,\quad [S_{54},Q_8]=-3Q_1$$
$$[S_{61},Q_1]=Q_6,\quad [S_{61},Q_2]=Q_5,\qquad [S_{61},Q_3]=-Q_4,\quad [S_{61},Q_4]=Q_3$$
$$[S_{61},Q_5]=-Q_2,\quad [S_{61},Q_6]=-Q_1,\qquad [S_{61},Q_7]=-Q_8,\quad [S_{61},Q_8]=Q_7$$
$$[S_{62},Q_1]=-Q_5,\quad [S_{62},Q_2]=Q_6,\qquad [S_{62},Q_3]=-3Q_7,\quad [S_{62},Q_4]=-Q_8$$
$$[S_{62},Q_5]=Q_1,\quad [S_{62},Q_6]=-Q_2,\qquad [S_{62},Q_7]=3Q_3,\quad [S_{62},Q_8]=Q_4$$
$$[S_{63},Q_1]=Q_4,\quad [S_{63},Q_2]=3Q_7,\qquad [S_{63},Q_3]=Q_6,\quad [S_{63},Q_4]=-Q_1$$
$$[S_{63},Q_5]=-Q_8,\quad [S_{63},Q_6]=-Q_3,\qquad [S_{63},Q_7]=-3Q_2,\quad [S_{63},Q_8]=Q_5$$
$$[S_{64},Q_1]=-Q_3,\quad [S_{64},Q_2]=Q_8,\qquad [S_{64},Q_3]=Q_1,\quad [S_{64},Q_4]=Q_6$$
$$[S_{64},Q_5]=-Q_7,\quad [S_{64},Q_6]=-Q_4,\qquad [S_{64},Q_7]=Q_5,\quad [S_{64},Q_8]=-Q_2$$
$$[S_{65},Q_1]=Q_2,\quad [S_{65},Q_2]=-Q_1,\qquad [S_{65},Q_3]=Q_8,\quad [S_{65},Q_4]=Q_7$$
$$[S_{65},Q_5]=Q_6,\quad [S_{65},Q_6]=-Q_5,\qquad [S_{65},Q_7]=-Q_4,\quad [S_{65},Q_8]=-Q_3$$
$$[S_{71},Q_1]=Q_7,\quad [S_{71},Q_2]=Q_4,\qquad [S_{71},Q_3]=Q_5,\quad [S_{71},Q_4]=-Q_2$$
$$[S_{71},Q_5]=-Q_3,\quad [S_{71},Q_6]=Q_8,\qquad [S_{71},Q_7]=-Q_1,\quad [S_{71},Q_8]=-Q_6$$
$$[S_{72},Q_1]=-Q_4,\quad [S_{72},Q_2]=Q_7,\qquad [S_{72},Q_3]=3Q_6,\quad [S_{72},Q_4]=Q_1$$
$$[S_{72},Q_5]=Q_8,\quad [S_{72},Q_6]=-3Q_3,\qquad [S_{72},Q_7]=-Q_2,\quad [S_{72},Q_8]=-Q_5$$
$$[S_{73},Q_1]=-Q_5,\quad [S_{73},Q_2]=-3Q_6,\qquad [S_{73},Q_3]=Q_7,\quad [S_{73},Q_4]=-Q_8$$
$$[S_{73},Q_5]=Q_1,\quad [S_{73},Q_6]=3Q_2,\qquad [S_{73},Q_7]=-Q_3,\quad [S_{73},Q_8]=Q_4$$
$$[S_{74},Q_1]=Q_2,\quad [S_{74},Q_2]=-Q_1,\qquad [S_{74},Q_3]=Q_8,\quad [S_{74},Q_4]=Q_7$$
$$[S_{74},Q_5]=Q_6,\quad [S_{74},Q_6]=-Q_5,\qquad [S_{74},Q_7]=-Q_4,\quad [S_{74},Q_8]=-Q_3$$
$$[S_{75},Q_1]=Q_3,\quad [S_{75},Q_2]=-Q_8,\qquad [S_{75},Q_3]=-Q_1,\quad [S_{75},Q_4]=-Q_6$$
$$[S_{75},Q_5]=Q_7,\quad [S_{75},Q_6]=Q_4,\qquad [S_{75},Q_7]=-Q_5,\quad [S_{75},Q_8]=Q_2$$
$$[S_{76},Q_1]=-Q_8,\quad [S_{76},Q_2]=3Q_3,\qquad [S_{76},Q_3]=-3Q_2,\quad [S_{76},Q_4]=Q_5$$
$$[S_{76},Q_5]=-Q_4,\quad [S_{76},Q_6]=Q_7,\qquad [S_{76},Q_7]=-Q_6,\quad [S_{76},Q_8]=Q_1$$
$$[S_{81},Q_1]=Q_8,\quad [S_{81},Q_2]=Q_3,\qquad [S_{81},Q_3]=-Q_2,\quad [S_{81},Q_4]=3Q_5$$
$$[S_{81},Q_5]=-3Q_4,\quad [S_{81},Q_6]=-Q_7,\qquad [S_{81},Q_7]=Q_6,\quad [S_{81},Q_8]=-Q_1$$

$$
\begin{aligned}
&[S_{82},Q_1]=-Q_3,\quad [S_{82},Q_2]=Q_8, && [S_{82},Q_3]=Q_1,\quad [S_{82},Q_4]=Q_6\\
&[S_{82},Q_5]=-Q_7,\quad [S_{82},Q_6]=-Q_4, && [S_{82},Q_7]=Q_5,\quad [S_{82},Q_8]=-Q_2\\
&[S_{83},Q_1]=Q_2,\quad [S_{83},Q_2]=-Q_1, && [S_{83},Q_3]=Q_8,\quad [S_{83},Q_4]=Q_7\\
&[S_{83},Q_5]=Q_6,\quad [S_{83},Q_6]=-Q_5, && [S_{83},Q_7]=-Q_4,\quad [S_{83},Q_8]=-Q_3\\
&[S_{84},Q_1]=-3Q_5,\quad [S_{84},Q_2]=-Q_6, && [S_{84},Q_3]=-Q_7,\quad [S_{84},Q_4]=Q_8\\
&[S_{84},Q_5]=3Q_1,\quad [S_{84},Q_6]=Q_2, && [S_{84},Q_7]=Q_3,\quad [S_{84},Q_8]=-Q_4\\
&[S_{85},Q_1]=3Q_4,\quad [S_{85},Q_2]=Q_7, && [S_{85},Q_3]=-Q_6,\quad [S_{85},Q_4]=-3Q_1\\
&[S_{85},Q_5]=Q_8,\quad [S_{85},Q_6]=Q_3, && [S_{85},Q_7]=-Q_2,\quad [S_{85},Q_8]=-Q_5\\
&[S_{86},Q_1]=Q_7,\quad [S_{86},Q_2]=Q_4, && [S_{86},Q_3]=Q_5,\quad [S_{86},Q_4]=-Q_2\\
&[S_{86},Q_5]=-Q_3,\quad [S_{86},Q_6]=Q_8, && [S_{86},Q_7]=-Q_1,\quad [S_{86},Q_8]=-Q_6\\
&[S_{87},Q_1]=-Q_6,\quad [S_{87},Q_2]=-Q_5, && [S_{87},Q_3]=Q_4,\quad [S_{87},Q_4]=-Q_3\\
&[S_{87},Q_5]=Q_2,\quad [S_{87},Q_6]=Q_1, && [S_{87},Q_7]=Q_8,\quad [S_{87},Q_8]=-Q_7
\end{aligned}
$$

Chapter 5

Superconformal models

This Chapter is basically a lightly edited version of the second half of [91], written in collaboration with F. Toppan.

5.1 Inroduction

We reviewed the construction of superconformal mechanics (in a Lagrangian setting) recovered from the D-module representations of the finite superconformal algebras. As an example, we proved that the $\mathcal{N} = 8$ global action associated to the $(1, 8, 7)$ supermultiplet, under a homogeneity condition and in presence of a non-trivial interaction (see (5.4) and the following discussion), is invariant under the exceptional $F(4)$ superalgebra. We thus recovered, in a different framework, the finding of [80]. The previously cited $\mathcal{N} = 8$ inhomogeneous $(3, 8, 5)$ supermultiplet induces, on the other hand, a new $D(2, 2)$-invariant superconformal mechanical model that will be presented elsewhere.

In application to classical superconformal mechanics in a Lagrangian framework the scaling dimensions of the component fields have to satisfy a reality condition.

On the other hand the D-module operators can be applied to quantum systems. In this case they should satisfy a Hermiticity condition which depends on a chosen metric η. For the dilatation operator D the chosen metric can be constant (either 1 or depending on the momentum operator p_t) or non-constant. It is well-known that, by assuming the metric to be 1, the Hermiticity condition of the dilatation operator implies that the scaling dimension λ should belong to the critical strip $\lambda = \frac{1}{2} + i\gamma$, with $\gamma \in \mathbb{R}$. This is the critical strip where the non-trivial zeros of the Riemann's zeta function are encountered (the Hermiticity property of D is at the basis of an attempt [79] to prove the Riemann hypothesis). We linked the Hermiticity constraints on the scaling dimensions with the choice (constant and non-constant) of the metric η.

The last part of the chapter is devoted to a thorough investigation of the constraints on $\mathcal{N} = 4$ superconformal mechanics resulting from the $\mathcal{N} = 4$ criticality condition. Multi-particle superconformal mechanics is based on several interacting supermultiplets which carry a representation of the same finite superconformal algebra. The $\mathcal{N} = 4$ exceptional superconformal algebras $D(2, 1; \alpha)$ are isomorphic for values of α which are related by an S_3 group of transformations (see (5.15)). This fact, together with the critical relations between α and the scaling dimension λ for the various global $\mathcal{N} = 4$ supermultiplets, has deep and non-trivial consequences in constraining multiparticle superconformal mechanics. The origin of these constraints are of representation theoretical nature. We derived in particular, see Appendix **B**, the admissible common scaling dimensions λ which allow inequivalent global $\mathcal{N} = 4$ supermultiplets to induce D-module representations for the same superconformal algebra. As an application we find that in certain cases irrational solutions for λ exist. The superconformal models based on these interacting supermultiplets are $\mathcal{N} = 4$ invariant, but cannot be extended to a full $\mathcal{N} = 8$ invariance. One particular superconformal example, obtained from the (5.25) prepotential, involves the interaction of the $(1, 4, 3)$ and the $(3, 4, 1)$ supermultiplets and is based on a scaling dimension related to the golden mean.

The representation theoretical nature of the $\mathcal{N} = 4$ constraints has implications for the critical scaling dimensions of the $\mathcal{N} = 7, 8$ superconformal algebras D-modules. This is due to the fact that the minimal $\mathcal{N} = 7, 8$ supermultiplets admit (at least) one decomposition in terms of two minimal $\mathcal{N} = 4$ supermultiplets. The critical scaling dimensions can be sometimes partly and sometimes completely determined (as it indeed happens for the $\mathcal{N} = 8$ $(5, 8, 3)$ and $(3, 8, 5)$ D-modules) by the $\mathcal{N} = 4$ analysis. In Appendix **B** it is shown how the $\mathcal{N} = 4$ constraints imply the absence of $\mathcal{N} = 8$ superconformal algebras induced by the $(4, 8, 4)$ supermultiplet.

The scheme of this chapter is the following. In Section **2** we review the construction of the superconformal mechanics in a Lagrangian setting derived from D-module representations of superconformal algebras. In Section **3** we analyze the Hermiticity conditions of the D-module representations in association with a constant or non-constant metric. In Section **4** the constraints on (multiparticle) superconformal mechanics are derived from the criticality condition of $\mathcal{N} = 4$ superconformal algebras and the isomorphism of the $D(2, 1; \alpha)$ superalgebras under the S_3 group of transformations acting on the parameter α. The existence of $\mathcal{N} = 4$ multi-particle superconformal mechanics for certain irrational values of the scaling dimension λ of the supermultiplets is pointed out (in the Appendix the admissible real values λ, associated to pairs of $\mathcal{N} = 4$ supermultiplets which carry a D-module representation for the same $\mathcal{N} = 4$ superconformal algebra, are explicitly presented).

In the Discussion we comment about these results.

5.2 Superconformal mechanics in Lagrangian framework

The superconformal algebras that we are dealing with admits the following decomposition in terms of the grading induced by the dilatation operator D ($\mathcal{G}_i$ is the sector of grading i)

$$\mathcal{G} \quad = \quad \mathcal{G}_{-1} \oplus \mathcal{G}_{-\frac{1}{2}} \oplus \mathcal{G}_0 \oplus \mathcal{G}_{\frac{1}{2}} \oplus \mathcal{G}_1. \tag{5.1}$$

The sector $\mathcal{G}_1$ ($\mathcal{G}_{-1}$) contains a unique generator given by H (K). The odd sectors $\mathcal{G}_{\frac{1}{2}}$ and $\mathcal{G}_{-\frac{1}{2}}$ are spanned by the supercharges Q_I's and their superconformal partners $\widetilde{Q}_I$'s, respectively. The $\mathcal{G}_0$ sector is given by the union of D and the R-symmetry subalgebra ($\mathcal{G}_0 = \{D\} \bigcup \{R\}$).

The invariance under the global supercharges Q_I's and the generator K implies the invariance under the full superconformal algebra $\mathcal{G}$.

D-module representations can be employed to induce superconformal mechanics in a Lagrangian setting [92]. Let the $\mathcal{N} = 4$ supermultiplet $(k, 4, 4-k)$ $(k \geq 1)$ being expressed by k component fields x_l (the propagating bosons), $4-k$ auxiliary fields g_m and 4 fermions ψ_n. A global $\mathcal{N} = 4$-invariant action is obtained from the Lagrangian

$$\mathcal{L} \quad = \quad Q_4 Q_3 Q_2 Q_1 [F(x_l)], \tag{5.2}$$

where $F(x_l)$, known as the *prepotential*, is an arbitrary function of the propagating bosons. The $\mathcal{N} = 4$ superconformal invariance is obtained by suitably constraining F, so that the equation

$$K\mathcal{L} \quad = \quad \frac{d}{dt} M \tag{5.3}$$

(M is some function of the component fields and their derivatives) is satisfied.

This approach is straightforwardly extended to the multiparticle superconformal mechanics (based on several $\mathcal{N} = 4$ interacting supermultiplets and such that the prepotential F is a function of all propagating bosons entering the different supermultiplets) and to $\mathcal{N} = 8$ superconformal mechanics. In this case the $\mathcal{N} = 8$ (with global supercharges Q_I, $I = 1, \ldots, 8$) $(k, 8, 8-k)$ supermultiplet is at first decomposed into two $\mathcal{N} = 4$ supermultiplets under Q_i's with $i = 1, 2, 3, 4$ ($(k, 8, 8-k) = (k', 4, 4-k') \oplus (k-k', 4, 4-k+k')$). The global $\mathcal{N} = 8$ invariance is obtained by constraining the Lagrangian to satisfy the equations $Q_j \mathcal{L} = \frac{d}{dt} P_j$, for $j = 5, 6, 7, 8$.

Further details of this approach to the construction of invariant Lagrangians are found in [92]. In Chapter *3* a slightly more general approach than the one here discussed is presented. It allows to construct $\mathcal{N} = 8$-invariant actions for $\mathcal{N} = 4$ decompositions with $k' = 0$ (so that it can be applied to the $(1, 8, 7) = (1, 4, 3) \oplus (0, 4, 4)$ decomposition).

Let us discuss now some applications of the $\mathcal{N}=8$ critical scaling dimensions we obtained in this work. We revisit at first the $\mathcal{N}=8$ $(1,8,7)$ model with a unique propagating boson x, fermions ψ, ψ_j and auxiliary fields g_j $(j=1,2,\ldots,7)$. Its global $\mathcal{N}=8$ action has been derived in [37]. It is given by

$$\mathcal{S} \;=\; \int dt\mathcal{L} = \int dt\{(ax+b)[\dot{x}^2 - \psi\dot{\psi} - \dot{\psi}_j\psi_j + {g_j}^2] + a[\psi\psi_j g_j - \frac{1}{2}C_{ijk}g_j\psi_j\psi_k]\}(5.4)$$

for some real coefficients a, b. The connection between $\mathcal{N}=8$ supersymmetry and octonions implies that, without loss of generality, the totally antisymmetric coupling constants C_{ijk} can be identified with the octonionic structure constants. A consistent choice is $C_{123}=C_{147}=C_{165}=C_{257}=C_{354}=C_{367}=1$.

It was later shown in [80] that the (5.4) model can be made superconformally invariant with respect to the $F(4)$ exceptional superalgebra. The D-module analysis of this model goes as follows. The scale-invariance and the dimensionless of the action requires the homogeneity of the Lagrangian. Therefore, either we have $a=0$ or $b=0$. In the $a=0$ (for $b\neq 0$) case we obtain a constant kinetic term. The scaling dimension λ of x coincides with the scaling dimension of the $(1,8,7)$ supermultiplet. It is given, see formula (5.10) for $\beta=0$, by $\lambda=-\frac{1}{2}$. This value, however, does not coincide with the critical scaling dimension for the $\mathcal{N}=8$ supermultiplet with $k=1$. In the second case ($b=0$ and $a\neq 0$) we obtain a nontrivial Lagrangian, due to the presence of the cubic term. The scaling dimension λ is now recovered from (5.10) with $\beta=1$. We obtain for this value the critical scaling dimension $\lambda=-\frac{1}{3}$ of the $\mathcal{N}=8$ $k=1$ supermultiplet. At this critical value the $(1,8,7)$ supermultiplet induces a D-module representation of the $F(4)$ superconformal algebra. Straightforward computations show that, at $\lambda=-\frac{1}{3}$, the action of the K generator on the (5.4) Lagrangian satisfies the (5.3) condition. We recover, with a different method, the $F(4)$ superconformal invariance of the (5.4) model for $b=0$. Unlike the superspace approach of [80], the $F(4)$ generators act linearly on the $(1,8,7)$ component fields.

The next model we analyze is based on the inhomogeneous $\mathcal{N}=8$ $(2,8,6)$ supermultiplet and was introduced in [32]. In the Lagrangian D-module approach the inhomogeneity (expressed by a real parameter c) is essential to produce Calogero-type terms in the action. The presence of the inhomogeneous term in the global $\mathcal{N}=8$ transformations implies that the 6 auxiliary fields have the same scaling dimension ($=0$) of the real inhomogeneous parameter c. This requirement unambiguosly fixes the scaling dimension of the $(2,8,6)$ inhomogeneous supermultiplet to be $\lambda=-1$. The present analysis proves that $\lambda=-1$ does not coincide with the critical scaling dimension of the $\mathcal{N}=8$ $k=2$ case. As a consequence the scale-invariant, global $\mathcal{N}=8$, model of [32] does not possess a superconformal invariance under a finite superconformal algebra.

Quite a different picture is recovered for the uniquely defined scale-invariant and global $\mathcal{N}=8$ model based on the inhomogeneous $(3,8,5)$ supermultiplet (the only arbitrariness is the value of the inhomogeneous parameter c). For $k=3$, $\lambda=-1$ is a critical scaling dimension. It can be proven that the action of this model, derived from the inhomogeneous $(3,8,5)$ $\mathcal{N}=8$ transformations introduced in subsection **4.3.6** and in the Appendix **A**, is invariant under the $D(2,2)=osp(4|4)$ superconformal algebra. Its explicit presentation and the discussion of its properties is left to a forthcoming paper in preparation.

5.3 Classical versus quantum D-module representations

The D-module representations of finite superconformal algebras that we introduced before can be called "classical representations". Two equivalent viewpoints can be applied to their entries. They can be regarded either as differential operators in the variable t (the "time") or, alternatively, they can be regarded as elements of an abstract Poisson brackets algebra generated by the relation $\{\pi_t, t\}=1$, where π_t (which, as a differential operator, can be identified with $\frac{d}{dt}$) is the conjugate momentum of t.

In Section **2** we presented the construction of classical superconformal mechanics in a Lagrangian formalism from the classical D-module representations that we discussed so far.

The extension to quantum mechanics can be achieved in at least two different ways. The Lagrangian mechanics can be reformulated in the Hamiltonian framework, so that standard methods of quantization can be applied, at least in principle, to the classical Hamiltonian dynamics.

A more direct approach (the one we discuss here) consists in realizing the generators of the D-module representations as Hermitian operators. The entries will be expressed in terms of the Hermitian operators t and $p_t = i\frac{d}{dt}$. We introduce at first the Hermitian generators for the $sl(2)$ diagonal subalgebra and the $\mathcal{N}$ global supercharges Q_i. The hermiticity properties of the remainining generators are determined as a consequence. It is convenient to express the Hermitian $sl(2)$ generators D, H, K acting on a given component field as

$$H = p_t, \quad D = -(tp_t + i\lambda), \quad K = -(t^2 p_t + 2i\lambda t) \tag{5.5}$$

(the constraint on the scaling dimension λ will be determined in the following), while the "quantum" D-module representation for the Q_i's is obtained from the classical one by replacing the $\pm\frac{d}{dt}$ entries ($\pm\frac{d}{dt} \rightarrow \pm p_t$), while leaving unchanged the c-number entries (± 1). We will see that this is the correct prescription to obtain Hermitian global supercharges.

For our purposes here it is sufficient to discuss the hermiticity properties of the dilatation operator D and of the global supercharges Q_i. We require in particular that, acting on given supermultiplets $|m_j>$, the equalities

$$\int dt < m_1|\eta|Dm_2 >= \int dt < Dm_1|\eta|m_2 >, \qquad \int dt < m_1|\eta|Q_i m_2 >= \int dt < Q_i m_1|\eta|m_2 >, \tag{5.6}$$

(with η a given metric to be specified) have to be satisfied. Let us discuss, for simplicity, the $|m_1 >= |m_2 >\equiv |m >$ case and let us take $|m>$ as a $(k, \mathcal{N}, \mathcal{N}-k)$ supermultiplet for $\mathcal{N} = 4, 8$ (the extension to other length-3 supermultiplets for arbitrary values of $\mathcal{N}$ is immediate). The component fields in the $|m>$ supermultiplets are x_l ($l = 1, \ldots, k$),

g_m $(m = 1, \ldots, \mathcal{N} - k)$ and the fermionic (anticommuting) fields ψ_n $(n = 1, \ldots, \mathcal{N})$. A constant metric η can be chosen to be $\eta = diag(1, \ldots, {p_t}^2, \ldots, p_t, \ldots)$ with the 1 entry repeated k times, the ${p_t}^2$ entry repeated $\mathcal{N} - k$ times and the p_t entry repeated $\mathcal{N}$ times. The global supercharges Qi's, recovered from the classical ones with the prescription introduced before, satisfy formula (5.6). For what concerns the dilatation operator D, the requirement of satisfying (5.6) implies constraints on the scaling dimensions λ_x, λ_g and λ_ψ of the component fields x_l, g_m and ψ_n, respectively. We obtain

$$\lambda_x + {\lambda_x}^* = 1, \qquad \lambda_g + {\lambda_g}^* = -1, \qquad \lambda_\psi + {\lambda_\psi}^* = 0. \tag{5.7}$$

The hermiticity condition for the scaling dimension λ_x (associated with the metric $\eta = 1$) implies that λ_x belongs to the critical strip

$$\lambda_x = \frac{1}{2} + i\gamma, \qquad (\gamma \in \mathbb{R}). \tag{5.8}$$

This is the critical strip where the non-trivial zeros of the Riemann's zeta function are encountered. This fact is at the core of a well-known strategy which has been elaborated for proving the Riemann's conjecture by linking it with the hermiticity property of the dilatation operator.

The hermiticity condition implies λ_g, λ_ψ belonging to the strips $\lambda_g = -\frac{1}{2} + i\gamma'$ and $\lambda_\psi = i\gamma''$ (with $\gamma', \gamma'' \in \mathbb{R}$), respectively. By setting the scaling dimensions of the component fields to be real, it turns out that they differ by $\frac{1}{2}$ $(\lambda_\psi = \lambda_g + \frac{1}{2}, \lambda_x = \lambda_\psi + \frac{1}{2})$ as it should be, also in accordance with the classical analysis.

The hermiticity conditions depend on the choice of the metric η, which is not necessarily constant. We illustrate this fact with the example of a single component field $|x>$ with scaling dimension λ. In the classical framework the real action (for β real)

$$\mathcal{S} = \int dt \mathcal{L} = \int dt (x^\beta \dot{x}^2) \tag{5.9}$$

is scale-invariant and dimensionless provided that the scaling dimension λ for the field x satisfies the condition

$$\lambda = -\frac{1}{\beta + 2} \tag{5.10}$$

(the scaling dimension of t is assumed to be $[t] = -1$).

Its quantum counterpart is the hermiticity condition $\int dt < x|\eta|Dx >= \int dt < Dx|\eta|x >$ for a non-constant metric η of the form

$$\eta \quad = \quad A\eta_1 + B\eta_2, \qquad \eta_1 = p_t x^\beta p_t, \qquad \eta_2 = {p_t}^2 x^\beta + x^\beta {p_t}^2, \tag{5.11}$$

with A, B some real constants.

After straightforward computations, one can show that fulfilling the hermiticity condition implies the vanishing of the coefficients a, b multiplying two types of terms (the only ones surviving after integration by parts), given by

$$\left(\int dt < x|\eta|Dx >= \int dt < Dx|\eta|x >\right) \implies \left(a\int dt(x^\beta \dot{x}^2) + b\int dt(tx^{\beta-1}\dot{x}^3) = 0\right) \tag{5.12}$$

The vanishing of b fixes the relative coefficient between A and B to be given by

$$A = -\beta N, \qquad\qquad B = N, \tag{5.13}$$

where N is just a normalization factor.

The vanishing of a requires λ to satisfy the condition

$$\lambda + \lambda^* \quad = \quad -\frac{2}{\beta + 2}. \tag{5.14}$$

As in the previous cases (for a constant metric η) we obtain a critical strip. The classical value for the scaling dimension is recovered by requiring λ to be real. In the non-constant case the metric η has to be conveniently fine-tuned, see formula (5.13), in order to obtain non-empty solutions for the hermiticity condition.

From this analysis we learn that, for any λ, the dilatation operator D can be made Hermitian by suitably choosing the metric η (specified by the real parameter β). For superconformal algebras, the hermiticity conditions can be defined in terms of the admissible metric η's or, alternatively, by quantizing the classical real Lagrangians.

We are now in position to discuss the criticality conditions (the relation between $\mathcal{N} = 4, 7, 8$ superconformal algebras and the scaling dimensions, which coincide with the scaling dimensions of the x_l component fields of their associated global supermultiplets) for Hermitian operators. The $sl(2)$ diagonal operators are expressed in (5.5), while the global supercharges Q_i's are obtained from the $\frac{d}{dt} \to p_t$ prescription discussed above. The remaining hermitian generators (the superconformal partners $\widetilde{Q}_i$'s and the R-symmetry generators) are determined from the (anti)-commutation relations of the previous generators. For the scaling dimension λ defined in (5.5) the criticality conditions coincide with the classical criticality conditions. The $\mathcal{N}=8$ superconformal algebras are recovered at $\lambda = \frac{1}{k-4}$ ($\mathcal{N}=7$ at $\lambda = -\frac{1}{4}$) and the $\mathcal{N}=4$ relation between α and λ is once more given by $\alpha = (2-k)\lambda$.

5.4 The S_3 α-orbit of $D(2,1;\alpha)$ and the constraints on multiparticle superconformal mechanics

The finite $\mathcal{N}=4$ simple superconformal algebras are $A(1,1)$ and the exceptional superalgebras $D(2,1;\alpha)$, for $\alpha \in \mathbb{C}\backslash\{0,-1\}$ [95]. The superalgebras $D(2,1;\alpha)$'s are isomorphic if and only if the parameters α are connected via an S_3 group of transformations generated by the moves $\alpha \mapsto \frac{1}{\alpha}$ and $\alpha \mapsto -(1+\alpha)$. We have therefore at most 6 different α's producing, up to isomorphism, the superconformal algebra $D(2,1;\alpha)$. They are given, explicitly, by

$$\begin{array}{rclrclrcl} \alpha^{(1)} &=& \alpha, & \alpha^{(3)} &=& -(1+\alpha), & \alpha^{(5)} &=& -\frac{1+\alpha}{\alpha}, \\ \alpha^{(2)} &=& \frac{1}{\alpha}, & \alpha^{(4)} &=& -\frac{1}{(1+\alpha)}, & \alpha^{(6)} &=& -\frac{\alpha}{(1+\alpha)}. \end{array} \tag{5.15}$$

It is convenient to regard $A(1,1)$ as a degenerate superalgebra recovered from $D(2,1;\alpha)$ at the special values $\alpha = 0,-1$ (at these special values three even generators decouple from the rest of the generators; the remaining ones close the $A(1,1)$ superalgebra).

The inequivalent $\mathcal{N}=4$ simple superconformal algebras are therefore expressed by the fundamental domain obtained by quotienting the complex plane ($\alpha \in \mathbb{C}$) under the action of the S_3 group. In application to classical superconformal mechanics, α is restricted to be real ($\alpha \in \mathbb{R}$). With this restriction a fundamental domain under the action of the S_3 group can be chosen to be the closed interval

$$\alpha \in [0,1]. \tag{5.16}$$

Some points in the interval are of special significance. We have that

i) - the extremal point $\alpha = 0$ corresponds to the $A(1,1)$ superalgebra,

ii) - the extremal point $\alpha = 1$ correspond to the $D(2,1)$ superalgebra, belonging to the $D(m|n) = osp(2m|2n)$ classical series,

iii) - the midpoint $\alpha = \frac{1}{2}$ corresponds to the $F(4)$ subalgebra $D(2,1;\frac{1}{2}) \subset F(4)$,

iv) - the point $\alpha = \frac{1}{3}$ corresponds to the $G(3)$ subalgebra $D(2,1;\frac{1}{3}) \subset G(3)$.

We will see in the following the special role played by these points.

The combined properties of having different α's producing isomorphic $\mathcal{N} = 4$ superconformal algebras (5.15), together with the set of critical relations between α and the scaling dimension λ of the $(k, 4, 4-k)$ $\mathcal{N} = 4$ supermultiplets (with $k = 0, 1, 2, 3, 4$), given by

$$\alpha = (2-k)\lambda, \tag{5.17}$$

produce highly non-trivial constraints on the admissible $\mathcal{N} = 4$ superconformal mechanics models and their scaling dimension. For the $\alpha = 0, -1$ case, e.g., we have that the solutions are recovered for any real λ for the $(2, 4, 2)$ supermultiplet (for $\lambda \neq 0$ the superalgebra is $sl(2|2)$, see [92]) while, for $k \neq 2$, they are obtained for $\lambda = 0$ or $\lambda = \frac{1}{k-2}$. It is convenient to summarize some results in a table presenting the admissible scaling dimension λ associated to the $(k, 4, 4-k)$ supermultiplets for the above four cases, specified by α_{FD} (the value α in the (5.16) fundamental domain) given by, respectively, $\alpha_{FD} = 0, 1, \frac{1}{2}, \frac{1}{3}$. We have

$\alpha_{FD} = 0$:

k	λ
0	$0, -\frac{1}{2}$
1	$0, -1$
2	$\mathbb{R}$
3	$0, 1$
4	$0, \frac{1}{2}$

$\alpha_{FD} = 1$:

k	λ
0	$-1, -\frac{1}{4}, \frac{1}{2}$
1	$-2, -\frac{1}{2}, 1$
3	$-1, \frac{1}{2}, 2$
4	$-\frac{1}{2}, \frac{1}{4}, 1$

$\alpha_{FD} = \frac{1}{2}$:

k	λ
0	$-\frac{3}{2}, -\frac{3}{4}, -\frac{1}{3}, -\frac{1}{6}, \frac{1}{4}, 1$
1	$-3, -\frac{3}{2}, -\frac{2}{3}, -\frac{1}{3}, \frac{1}{2}, 2$
3	$-2, -\frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{3}{2}, 3$
4	$-1, -\frac{1}{4}, \frac{1}{6}, \frac{1}{3}, \frac{3}{4}, \frac{3}{2}$

$\alpha_{FD} = \frac{1}{3}$:

k	λ
0	$-2, -\frac{2}{3}, -\frac{3}{8}, -\frac{1}{8}, \frac{1}{6}, \frac{3}{2}$
1	$-4, -\frac{4}{3}, -\frac{3}{4}, -\frac{1}{4}, \frac{1}{3}, 3$
3	$-3, -\frac{1}{3}, \frac{1}{4}, \frac{3}{4}, \frac{4}{3}, 4$
4	$-\frac{3}{2}, -\frac{1}{6}, \frac{1}{8}, \frac{3}{8}, \frac{2}{3}, 2$

(5.18)

The $\mathcal{N} = 4$ superconformal invariance for several (at least two, let's say $(k, 4, 4-k)$ and $(k', 4, 4-k')$) interacting supermultiplets requires that they should carry a D-module representation for the same $D(2,1;\alpha)$ superalgebra. Given two supermultiplets with $k' \neq k$, this requirement produces *a)* a constraint on the admissible values for α and *b)* a consequent constraint on the mutual scaling dimensions of the two supermultiplets (both these constraints will be called the "compatibility condition").

The table above shows that, for $\alpha_{FD} = 0, 1, \frac{1}{2}$, two supermultiplets with $k' \neq k$ and the same scaling dimension λ can be found for

$$\begin{array}{llll} \alpha_{FD} = 0: & \lambda = 0 \quad (k, k' \neq 2), & \lambda = 0, \frac{1}{k-2} & (k' = 2), \\ \alpha_{FD} = 1: & \lambda = -1, \frac{1}{2} \quad (k = 0, k' = 3), & \lambda = 1, -\frac{1}{2} & (k = 1, k' = 4), \\ \alpha_{FD} = \frac{1}{2}: & \lambda = -\frac{3}{2}, -\frac{1}{3} \quad (k = 0, k' = 1), & \lambda = \frac{3}{2}, \frac{1}{3} & (k = 3, k' = 4). \end{array} \qquad (5.19)$$

Some conclusions can be drawn from this result. For instance, the decomposition of $\mathcal{N} = 8$ D-module representations into $\mathcal{N} = 4$ supermultiplets can (partly) explain the $\mathcal{N} = 8$ critical scaling dimensions. The $(7, 8, 1)$ supermultiplet gets decomposed into $(4, 4, 0) \oplus (3, 4, 1)$. Its critical scaling dimension is therefore constrained to be either $\lambda = \frac{3}{2}$ or $\lambda = \frac{1}{3}$ (its actual value). The $(4, 4, 0) \oplus (2, 4, 2)$ decomposition of the $(6, 8, 2)$ supermultiplet implies that its critical scaling dimension can only be found at $\lambda = 0$ or $\lambda = \frac{1}{2}$ (its actual value). The $(5, 8, 3)$ supermultiplet admits the decompositions $(4, 4, 0) \oplus (1, 4, 3)$ and $(3, 4, 1) \oplus (2, 4, 2)$. Their combination uniquely implies a possible critical scaling dimension at $\lambda = 1$ (its actual value).

The $\alpha_{FD} = \frac{1}{3}$ case does not admit a common scaling dimension λ if $k \neq k'$. This value of α corresponds to the decomposition of the $\mathcal{N} = 7$ $(1, 7, 7, 1)$ supermultiplet into the $\mathcal{N} = 4$ $(1, 4, 3, 0) \oplus (0, 3, 4, 1)$ supermultiplets. Let us denote with λ_1, λ_3 the respective scaling dimensions of these $\mathcal{N} = 4$ supermultiplets. Clearly it must be $\lambda_3 = \lambda_1 + \frac{1}{2}$. An inspection of the (5.18) table shows that the unique pair of values differing by $\frac{1}{2}$ are recovered for $\lambda_1 = -\frac{1}{4}$ and $\lambda_3 = \frac{1}{4}$. This analysis corroborates the finding of $\lambda = -\frac{1}{4}$ as the critical scaling dimension of the $(1, 7, 7, 1)$ D-module representation of $G(3)$.

We are also able to partly explain the arising of $\mathcal{N} = 6$ superconformal algebras from length-4 supermultiplets, obtained in Section **4.4**. The $(2, 6, 6, 2)$ supermultiplet produces the $\mathcal{N} = 6$ superconformal algebra $A(2, 1)$ for any value of λ (no criticality). Its $\mathcal{N} = 4$ decomposition reads as $(2, 6, 6, 2) = (2, 4, 2, 0) \oplus (0, 2, 4, 2)$. One should note, from table (5.18), that no restriction on λ is put from the $(2, 4, 2)$ supermultiplets. On the other hand the $(1, 6, 7, 2)$ supermultiplet induces the $A(2, 1)$ superalgebra at the critical value $\lambda = 0$.

The $\mathcal{N}=4$ decomposition is expressed as $(1,6,7,2)=(1,4,3,0)\oplus(0,2,4,2)$. The presence of both supermultiplets $(1,4,3)$ and $(2,4,2)$ requires $\alpha=0,-1$. The admissible values of critical λ, obtained from this analysis, are $\lambda=0,-1$. The $(2,7,6,1)$ supermultiplet induces the $A(2,1)$ superalgebra at the critical value $\lambda=-\frac{1}{2}$. Let λ_2, λ_3 be the scaling dimensions of the respective $\mathcal{N}=4$ supermultiplets entering the $\mathcal{N}=4$ decomposition $(2,7,6,1)=(2,4,2,0)\oplus(0,3,4,1)$. λ_2 coincides with the scaling dimension λ of the $(2,7,6,1)$ supermultiplet, while $\lambda_3=\lambda_2+\frac{1}{2}$. The $\alpha=0,-1$ constraint on λ_3 ($\lambda_3=0,1$) implies that in this case the critical λ is constrained (necessary condition) to be $\lambda=\pm\frac{1}{2}$.

Let us deal now with the general case of finding the compatibility conditions on α and the common scaling dimension λ for two $\mathcal{N}=4$ D-module representations with $k\neq k'$. It is sufficient to discuss the $k,k'\neq 2$ restriction, since the remaining cases are immediately recovered from the λ solutions at $\alpha=0,-1$. Without loss of generality we can set $\alpha\equiv\alpha^{(1)}=(2-k)\lambda$ for the k supermultiplet. The α' value obtained as $\alpha'=(2-k')\lambda$ from the k' supermultiplet must coincide with one of the $\alpha^{(i)}$ in the S_3-orbit of α. Let us introduce the ratios

$$N^{(i)} = \frac{\alpha^{(i)}}{\alpha^{(1)}} \tag{5.20}$$

and

$$w_{kk'} = \frac{2-k'}{2-k}=\frac{1}{w_{k'k}}. \tag{5.21}$$

The values obtained by $w_{kk'}$ in varying k,k' with the given constraints are $-1,\pm\frac{1}{2},\pm2$.

For a given pair $[k,k']$ the admissible values α satisfying the compatibility condition are recovered by $\overline{\alpha}$ and its S_3-group orbit (5.15), with $\overline{\alpha}$ a solution of one of the five equations (for $i=2,3,4,5,6$, since $N^{(1)}=w$ has no solution for $w\neq 1$)

$$N^{(i)} = w \tag{5.22}$$

(since no confusion will arise, for simplicity, we set $w\equiv w_{kk'}$).

The compatibility conditions are recovered from the (5.22) system of equations for three inequivalent values of w, given by $w = -1, -2, 2$. This is due to the fact that the transformation $w \leftrightarrow \frac{1}{w}$ reflects the $k \leftrightarrow k'$ exchange.

Two of the (5.22) equations are linear in α, while the three remaining ones are quadratic (producing, in some cases, complex solutions).

The $\overline{\alpha}$ solutions can be divided into three classes: real and rational, real and irrational, complex.

The complex solutions (associated to scaling dimension λ's which do not satisfy the reality condition) are found to be

$$
\begin{aligned}
w = -1 \quad &: \quad \overline{\alpha} = \pm i, \\
w = -2 \quad &: \quad \overline{\alpha} = \pm \frac{i}{\sqrt{2}}, \\
w = 2 \quad &: \quad \overline{\alpha} = \frac{1}{2}(-1 \pm i), -1 \pm i\sqrt{7}.
\end{aligned}
\tag{5.23}
$$

For what concerns the real solutions the following results are obtained.

In the rational case, the unique solutions are encountered for $w = -2$ (the $\overline{\alpha}$ S_3-orbit is specified by $\alpha_{FD} = 1$) and $w = 2$ (with orbit specified by $\alpha_{FD} = \frac{1}{2}$). We therefore recover the solutions already encountered in (5.19) and their corresponding scaling dimension λ's. No further rational solution is found.

For what concerns the irrational case the encountered results are summarized in the table below, which specifies the $[k, k']$ pairs, the value w, the S_3 orbit representative α_{FD} in the fundamental domain and, finally, the compatible scaling dimension λ's. We have

$[k,k']$	w	α_{FD}	λ
$[1,3]$	-1	$-\frac{1}{2}+\frac{\sqrt{5}}{2}$	$-\frac{1}{2}\pm\frac{\sqrt{5}}{2}$, $\frac{1}{2}\pm\frac{\sqrt{5}}{2}$
$[0,4]$	-1	$-\frac{1}{2}+\frac{\sqrt{5}}{2}$	$-1\pm\sqrt{5}$, $1\pm\sqrt{5}$
$[1,4]$	-2	$-\frac{1}{2}+\frac{\sqrt{3}}{2}$	$-\frac{1}{2}\pm\frac{\sqrt{3}}{2}$
$[3,0]$	-2	$-\frac{1}{2}+\frac{\sqrt{3}}{2}$	$\frac{1}{2}\pm\frac{\sqrt{3}}{2}$
$[1,0]$	2	$\frac{1}{\sqrt{2}}$	$\sqrt{2}$
$[3,4]$	2	$\frac{1}{\sqrt{2}}$	$-\sqrt{2}$
$[1,0]$	2	$\sqrt{2}-1$	$-\sqrt{2}$
$[3,4]$	2	$\sqrt{2}-1$	$\sqrt{2}$

(5.24)

These results give the necessary condition for the existence of $\mathcal{N}=4$ superconformal mechanics based on several inequivalent interacting supermultiplets with the same scaling dimension λ.

Unlike the $\alpha_{FD}=1$ with $\lambda=1$ (for $[k,k']=[1,4]$) and $\lambda=-1$ (for $[k,k']=[0,3]$) and $\alpha_{FD}=\frac{1}{2}$ with $\lambda=\frac{1}{3}$ (for $[k,k']=[3,4]$) and $\lambda=-\frac{1}{3}$ (for $[k,k']=[0,1]$), the irrational cases and the remaining rational cases do not allow the extension of the $\mathcal{N}=4$ superconformal invariance to a broader $\mathcal{N}=8$ superconformal invariance.

A particularly interesting case involves the irrational solution of the $k=1$, $k'=3$ supermultiplets. The value α_{FD} coincides with the golden mean conjugate $\Phi=-\frac{1}{2}(1-\sqrt{5})$ (the golden mean $\varphi=\frac{1}{2}(1+\sqrt{5})$ belongs to its S_3-orbit). This case admits four compatible solutions for the scaling dimension λ (given by $\pm\varphi$ and $\pm\Phi$). $\mathcal{N}=4$ superconformal actions, invariant under $D(2,1;\alpha=\varphi)$, are obtained for the pairs of $\mathcal{N}=4$ supermultiplets $(x;\psi,\psi_i;g_i)$ and $(y_i;\xi,\xi_i;h)$, $i=1,2,3$, in terms of the Lagrangians $\mathcal{L}=Q_4Q_3Q_2Q_1F(x,y_i)$. The x,y_i fields are the propagating bosons. A class of solutions, satisfying the (5.3) constraint, is obtained for the prepotential $F(x,y_i)$ given by

$$F(x,y_i) \;=\; Cx^{\beta}r^{\gamma}, \qquad \beta+\gamma=\frac{1}{\varphi} \tag{5.25}$$

(here $r = \sqrt{y_1^2 + y_2^2 + y_3^2}$ and C is an arbitrary constant) and linear combinations thereof.

5.5 Discussion

The action of the generators of a D-module representation on a set of component fields allows to construct superconformal mechanical models in a Lagrangian framework. With our analysis based on D-module representations we have been able to prove the invariance of certain superconformal models, to check that the [32] model is not invariant under a finite simple Lie superalgebra while, on the other hand, we pointed out the existence of a $D(2,2)$-invariant superconformal model based on the inhomogeneous $(3,8,5)$ supermultiplet.

The isomorphism of the $D(2,1;\alpha)$ superalgebras under an S_3 group of transformations acting on the parameter α, together with the relation between α and λ for the $(k,4,4-k)$ $\mathcal{N}=4$ supermultiplets, gives non-trivial restrictions on the admissible $\mathcal{N}=4$ multi-particle superconformal mechanics (based on several interacting supermultiplets). In Appendix **B** we presented the constraints on the scaling dimensions of the superconformal models with interacting supermultiplets. We proved in particular the existence of an $\mathcal{N}=4$ superconformal mechanics (see formula (5.25)) for the interacting $(1,4,3)$ and $(3,4,1)$ supermultiplets, with scaling dimension based on the golden ratio.

The representation theoretical nature of the constraints on multi-particle $\mathcal{N}=4$ superconformal mechanics allows to partly explain the critical scaling dimensions λ encountered at $\mathcal{N}=7$ and $\mathcal{N}=8$. This is due to the fact that the $\mathcal{N}=7,8$ supermultiplets admit (at least one) decomposition into two separate $\mathcal{N}=4$ supermultiplets.

5.6 Appendix B: admissible common real scaling dimension for pairs of $\mathcal{N}=4$ superconformal multiplets

We report here for clarity a table containing the admissible, common, *real* values of the scaling dimension λ for pairs of $\mathcal{N}=4$ superconformal multiplets transforming under the same $\mathcal{N}=4$ superconformal algebra. This information, extracted from the tables (5.18) and (5.24), is important in constraining the admissible multiparticle superconformal mechanics. We have

$$
\begin{array}{lcl}
(4,4,0)\oplus(4,4,0) & : & \lambda\in\mathbb{R},\\
(4,4,0)\oplus(3,4,1) & : & \lambda=\frac{3}{2},\frac{1}{3},\pm\sqrt{2},\\
(4,4,0)\oplus(2,4,2) & : & \lambda=0,\frac{1}{2},\\
(3,4,1)\oplus(3,4,1) & : & \lambda\in\mathbb{R},\\
(4,4,0)\oplus(1,4,3) & : & \lambda=1,-\frac{1}{2},-\frac{1}{2}\pm\frac{\sqrt{3}}{2},\\
(3,4,1)\oplus(2,4,3) & : & \lambda=0,1,\\
(4,4,0)\oplus(0,4,4) & : & \lambda=-1\pm\sqrt{5},1\pm\sqrt{5},\\
(3,4,1)\oplus(1,4,3) & : & \lambda=-\frac{1}{2}\pm\frac{\sqrt{5}}{2},\frac{1}{2}\pm\frac{\sqrt{5}}{2}\\
(2,4,2)\oplus(2,4,2) & : & \lambda\in\mathbb{R},\\
(3,4,1)\oplus(0,4,4) & : & \lambda=-1,\frac{1}{2},\frac{1}{2}\pm\frac{\sqrt{3}}{2},\\
(2,4,2)\oplus(1,4,3) & : & \lambda=0,-1,\\
(2,4,2)\oplus(0,4,4) & : & \lambda=0,-\frac{1}{2},\\
(1,4,3)\oplus(1,4,3) & : & \lambda\in\mathbb{R},\\
(1,4,3)\oplus(0,4,4) & : & \lambda=-\frac{3}{2},-\frac{1}{3},\pm\sqrt{2},\\
(0,4,4)\oplus(0,4,4) & : & \lambda\in\mathbb{R}.
\end{array}
\tag{5.26}
$$

This table allows to explain why there is no $\mathcal{N}=8$ superconformal D-module based on the $(4,8,4)$ supermultiplet. Indeed, depending on the choice of the $\mathcal{N}=4$ subalgebra, this supermultiplet admits the three $\mathcal{N}=4$ decompositions $(4,4,0)\oplus(0,4,4)$, $(3,4,1)\oplus(1,4,3)$ and $(2,4,2)\oplus(2,4,2)$. There is, however, no common scaling dimension λ belonging to both $(4,4,0)\oplus(0,4,4)$ and $(3,4,1)\oplus(1,4,3)$.

One should note that the $(k,n,n-k)\leftrightarrow(n-k,n,k)$ "mirror duality" of the global supermultiplets (see [37]) extends to a duality for the scaling dimension of the superconformal multiplets.

Chapter 6

Conclusion

Extended Supersymmetries in $1D$ can be used to constrain possible higher-dimensional supersymmetric theories (for instance, constraining the number of auxiliary fields in supergravity theories)[56, 70, 65]. A much more ambitious task, would consist in the reconstruction of a higher-dimensional theory from its one-dimensional supersymmetric data [101, 54].

An N-extended supersymmetric theory in the ordinary $D = 4$ Minkowskian spacetime (SuperYang-Mills or supergravity) produces a $1D$ dimensionally-reduced supersymmetric theory with $\mathcal{N} = 4N$ supercharges. On the other hand, $N = 2$ can be obtained from the dimensional reduction of $D = 6$ (SYM or sugra), $N = 4$ from the dimensional reduction of $D = 10$ (SYM or sugra) and $N = 8$ from the dimensional reduction of the $D = 11$ sugra. As a result, a *necessary condition* for higher-dimensional oxidation consists in producing large $\mathcal{N}$-Extended supersymmetric theories in $1D$. The concept of *oxidation*, which is a pun, employed in superstring/M-theory literature, to denote the process opposite to dimensional reduction, see [70].

We started this thesis with investigation non-minimal pure $\mathcal{N} = 4$ supermultiplets using the information contained in connectivity symbol, commuting group, the node choice group and its possible inequivalent presentations (colorings) under local moves. We have distinguished and emphasized the difference between two types of moves (local and global) acting on graphs and the so-called "gordian transformations" acting on pure supermultiplets. As a result a given pure supermultiplet can be associated with inequivalent (under local and global moves) graphs. In certain cases, in particular, a given supermultiplet can be associated to both a disconnected and a connected graph. In order to avoid overcounting, the notion of *connected pure supermultiplets* (the supermultiplets which are associated to connected graphs only) has been introduced. The classification of the non-minimal, $\mathcal{N} = 4$, pure, connected supermultiplets has been presented in chapter **2**.

The notion of "coloring", similarly to the notion of "chiral" supermultiplets [61], plays an important role in supersymmetry representations. It is well-known that minimal $\mathcal{N} = 8$ supermultiplets are non-chiral [61], being necessarily obtained by linking together (with extra supertransformations) two minimal $\mathcal{N} = 4$ supermultiplets of opposite chirality. We have shown (see the discussion at the end of Section **2.3.5**) that inequivalent non-minimal $\mathcal{N} = 4$ supermultiplets are obtained by linking together two $\mathcal{N} = 3$ $(2, 4, 2)$ supermultiplets based on the fact that their coloring is either the same or different. This property naturally extends to the construction of non-minimal $\mathcal{N} = 5$ supermultiplets by linking together non-minimal $\mathcal{N} = 4$ supermultiplets (whose respective colorings have been listed here). A forthcoming paper discusses the issues of the representations of the $\mathcal{N} = 5$ supersymmetry.

We provided an explicit construction of a supersymmetric one-dimensional entangled supermultiplet (which does not admit a graphical presentation). The possibility of *non-adinkrizable* supermultiplets (here called "entangled") was raised in [73]. Till very recently no explicit example was produced (so that it was even unclear whether this notion could be applied to a non-empty set). Constructions of non-adinkrizable supermultiplets (in a different context and using different methods from the one proposed here) has been recently discussed in [74, 75]. Our given example (based on the interpolation between two non-minimal $\mathcal{N} = 4$ supermultiplets of $(3, 8, 5)$ field content) was suitably chosen to simplify the proof that there exists no linear combination of the component fields which guarantees a graphical presentation ("Adinkra") of the interpolated supermultiplet in the interval $0 < \theta < \frac{\pi}{2}$ of the interpolating angle θ. An important observation is that the interpolating mechanism is a general phenomenon and that entangled supermultiplets tend to proliferate for large $\mathcal{N}$ values of the one-dimensional $\mathcal{N}$-Extended Supersymmetry. It is also important to notice that the entangled supermultiplet has dynamical consequences. An $\mathcal{N} = 4$, one-dimensional, off-shell invariant sigma-model with a three-dimensional target is based on it. Its action (3.23) carries an explicit dependence on θ. This model is supersymmetric only under the supertransformations specified by the entangled supermultiplet. Therefore, entangled supermultiplets allow to enlarge the class of supersymmetric actions so far considered.

In the case of the (3.23) action the dependence on θ can be reabsorbed only if the constraint (3.32), which implies an $\mathcal{N} = 8$ invariance, is imposed. The $\mathcal{N} = 8$ action turns out to be dependent on the conformal factor (3.34). The $\mathcal{N} = 8$ invariance is made possible by the fact that the two $\mathcal{N} = 4$ pure supermultiplets recovered at $\theta = 0$ and $\theta = \frac{\pi}{2}$ can be extended ("oxidized", see [71]) to the same $\mathcal{N} = 8$ $(3, 8, 5)$ supermultiplet. On the other hand, when (3.32) is not satisfied, the action (3.23) is $\mathcal{N} = 4$ supersymmetric and possesses a genuine θ-dependence.

We obtained evidences that supermultiplets, sharing the same field content but differing in connectivity symbol, can induce inequivalent supersymmetric-invariant actions (one should compare, e.g., the actions given in formulas (3.10) and (3.19)). It was known, from the analysis of [88, 89, 63, 69], that inequivalent representations, discriminated by their respective connectivity symbol, can be found. On the other hand, so far, no dynamical characterization was associated to the connectivity symbol. In [71] the $\mathcal{N} = 5$-supersymmetric off-shell invariant actions, induced with respect to inequivalent $\mathcal{N} = 5$ supermultiplets of a given field content, were proven to coincide and possess an overall $\mathcal{N} = 8$ supersymmetry invariance. The crucial feature here is the fact that the inequivalent $\mathcal{N} = 4$ off-shell invariant actions are induced by inequivalent *non-minimal* $\mathcal{N} = 4$ linear supermultiplets (with the same field content).

Following [69], the word *oxidation* has been here consistently used in a specific and restricted sense, referring to the operation of enlarging the number of extended supersymmetries (from $\mathcal{N}$ to $\mathcal{N}+1$) acting on a supermultiplet with the same number of component fields. As we emphasized , the non-minimal $\mathcal{N}=4$ linear supermultiplets are progressively oxidized to minimal $\mathcal{N}=5,6,7,8$ linear supermultiplets possessing 8 bosonic and 8 fermionic component fields. It is clear, from these considerations, that the $\mathcal{N}=4$ off-shell invariant actions based on non-minimal supermultiplets are not of mere academic interest. Indeed, an $N=2$, $D=4$ theory, dimensionally reduced to $1D$, produces a supersymmetric model with $\mathcal{N}=8$ extended supersymmetries; on the other hand the partial spontaneous breaking of $N=2$ into $N=1$ produces an $\mathcal{N}=4$ invariant $1D$ supersymmetric model whose component fields belong to $\mathcal{N}=8$ supermultiplets and are therefore non-minimal supermultiplets w.r.t. the $\mathcal{N}=4$ invariant supersymmetries. The inequivalent $\mathcal{N}=4$ non-minimal supermultiplets and their inequivalent $\mathcal{N}=4$-invariant off-shell actions can therefore be regarded as building blocks for constructing supersymmetric models obtained from dimensional reduction of partial spontaneous supersymmetry breaking of $N=2$, $D=4$ supersymmetry.

It is worth mentioning a recent paper [90] in which the $\mathcal{N}=4$-invariance for a non-minimal supermultiplet in presence of a Yang monopole is discussed (see also [68]).

The presented results can be applied to investigate supersymmetry representations in presence of inhomogeneous terms [32], non-linear realizations of supersymmetry [80, 65], D-module representations of superconformal algebras and their associated superconformal mechanics . All these extensions (inhomogeneous representations, non-linear realizations, D-module representations) are induced and derived from linear homogeneous supermultiplets, such as those investigated in this chapter.

Based on available representations of global supersymmetry in one dimension, we constructed a class of D-module representations for one-dimensional superconformal algebras. These representations exhibit, for $\mathcal{N}=4,7,8$, the property of criticality. This means that they only close at critical values of the scaling dimension λ characterizing the supermultiplets of time-dependent component fields. The superalgebras under consideration are a given subclass of finite, simple, Lie superalgebras. Their D-module representations are an extension of the D-module representation (4.5) of the $sl(2)$ algebra (this representation is non-critical, being recovered for any value of λ).

The connection with the D-module representations [37] of the $\mathcal{N}$-Extended global supersymmetry (the superalgebra of the one-dimensional Supersymmetric Quantum Mechanics) is given by the fact that the latter is a subalgebra of the superconformal algebras. Certain minimal global supermultiplets induce, at a given λ, their associated D-module representations of a superconformal algebra. In particular, the exceptional finite Lie superalgebras [93, 95] $D(2,1;\alpha)$, $G(3)$ and $F(4)$ (which are superconformal algebras for $\mathcal{N}=4,7,8$, respectively) admit a D-module representation. Quite interestingly, the D-module representation of $G(3)$ is only induced from the minimal $(1,7,7,1)$ global $\mathcal{N}=7$ supermultiplet, first introduced in [37].

The action of the generators of a D-module representation on a set of component fields allows to construct superconformal mechanical models in a Lagrangian framework. With our analysis based on D-module representations we have been able to prove the invariance of certain superconformal models, to check that the [32] model is not invariant under a finite simple Lie superalgebra while, on the other hand, we pointed out the existence of a $D(2,2)$-invariant superconformal model based on the inhomogeneous $(3,8,5)$ supermultiplet.

An updated review on several issues of superconformal mechanics (including application to test particles moving near black hole horizons, CFT_1/AdS_2 correspondence, etc.) can be found in [101] (see also the references therein).

The isomorphism of the $D(2,1;\alpha)$ superalgebras under an S_3 group of transformations acting on the parameter α, together with the relation between α and λ for the $(k,4,4-k)$ $\mathcal{N}=4$ supermultiplets, gives non-trivial restrictions on the admissible $\mathcal{N}=4$ multi-particle superconformal mechanics (based on several interacting supermultiplets).

We presented the constraints on the scaling dimensions of the superconformal models with interacting supermultiplets. We proved in particular the existence of an $\mathcal{N} = 4$ superconformal mechanics (see formula (5.25)) for the interacting $(1,4,3)$ and $(3,4,1)$ supermultiplets, with scaling dimension based on the golden ratio.

The representation theoretical nature of the constraints on multi-particle $\mathcal{N} = 4$ superconformal mechanics allows to partly explain the critical scaling dimensions λ encountered at $\mathcal{N} = 7$ and $\mathcal{N} = 8$. This is due to the fact that the $\mathcal{N} = 7, 8$ supermultiplets admit (at least one) decomposition into two separate $\mathcal{N} = 4$ supermultiplets.

A natural extension of this work consists in investigating D-module representations for twisted superconformal algebras. A D-module representation for a twisted version of the $\mathcal{N} = 2$ superconformal algebra was constructed in [45]. The investigation of D-modules for larger values of (twisted) $\mathcal{N}$ superconformal algebras seems a promising tool to analyze the dimensional reduction (to one dimension) of the $N = 4$ super-Yang-Mills theory.

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Papers

This dissertation is based on three published papers:

1) M. Gonzales, S. Khodaee and F. Toppan, *On non-minimal* $\mathcal{N} = 4$ *supermultiplets in* $1D$ *and their associated sigma-models*, J. Math. Phys. **52**:013514 (2011); arXiv:1006.4678[hep-th].

2) M. Gonzales, K. Iga, S. Khodaee and F. Toppan, *Pure and entangled* $\mathcal{N} = 4$ *linear supermultiplets and their one-dimensional sigma-models*, J. Math. Phys. **53**, 103513 (2012) [arXiv:1204.5506 [hep-th]].

3) S. Khodaee and F. Toppan, *Critical scaling dimension of D-module representations of* $\mathcal{N} = 4, 7, 8$ *Superconformal Algebras and constraints on Superconformal Mechanics*, J. Math. Phys. **53**, 103518 (2012) [arXiv:1208.3612 [hep-th]].

Some other works are in progress:

4) M. Gonzales,S. Khodaee,O. Lechtenfeld,and F. Toppan, *Target duality in* $\mathcal{N} = 8$ *superconformal mechanics and the coupling of dual pairs*, (2013) [arXiv:1303.6732 [hep-th]]