Realism in Energy Transition Processes: an example from Bohmian Quantum Mechanics

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Abstract

In this paper we study in details a system of two weakly coupled harmonic oscillators. This system may be viewed as a simple model for the interaction between a photon and a photodetector. We obtain exact solutions for the general case. We then compute approximate solutions for the case of a single photon (where one oscillator is initially in its first excited state) reaching a photodetector in its ground state (the other oscillator). The approximate solutions represent the state of both the photon and the photodetector after the interaction, which is not an eigenstate of the individual hamiltonians for each particle, and therefore the energies for each particle do not exist in the Copenhagen interpretation of Quantum Mechanics. We use the approximate solutions that we obtained to compute bohmian trajectories and to study the energy transfer between the two particles. We conclude that even using the bohmian view the energy of each individual particle is not well defined, as the nonlocal quantum potential is not negligible even after the coupling is turned off.

1 Introduction

The discussions about the incompleteness of the wavefunction to describe physical processes dates back to the beginning of quantum mechanics itself. This discussion is closely related to the possibility of describing quantum mechanical systems from an underlying realistic model. In 1952, David Bohm showed that such a realistic model was possible. However, Bohm's theory had the problem of being nonlocal [1, 2]. In 1963 John Bell showed that in order to obtain the same results predicted by quantum mechanics, any realistic theory would have to be nonlocal [17]. Bell's result and the failure of using the Copenhagen interpretation of quantum mechanics to some particular situations, as in for example Quantum Cosmology, lead to a raised interest in Bohm's interpretation and in nonlocal realistic theories [7].

The subject of reallity and nonlocality has been an interest of Patrick Suppes for quite a while [17], in particular for the photon. In fact, one of the authors of this paper co-published with him a series of papers that layed down the foundational analysis of realistic and local model of photons that could explain the double slit experiment, the EPR experiment and other phenomena [18, 19, 20, 21]. The problem with the Suppes and de Barros model was that it did not have a consistent theory of photon-counting for single photons, and therefore could not explain the non-locality of single photons and the GHZ experiment, for example.

In this paper we try to respond, within Bohm's model, the question: what is a photon? We do not follow the standard Bohmian interpretation for bosonic fields (as can be found in [11]). Instead, we use the simple interpretation that "a photon is what a photodetector detects". One may think of a photodetection as a transfer of energy from a quantized mode of the electromagnetic field (the photon) to an atom in its ground state (the quantum photodetector). Therefore, to study this photodetection we will focus on the process of transfer of energy from the photon to the photodetector.

To study the exchange of energy in details, we have to choose between two different and simple models of a photo-detector: a photo-detector with discrete or continuous band [5]. For the purpose of simplicity, we will choose the former. However, since we are only interested in the aspects of energy transfer between the two systems, we will make an even further simplification and consider that the photon and the detector will both be described by a single harmonic oscillator. Furthermore, during some time ΔT_{int}, we will assume that a linear interaction exists between the two oscillators, and that this interaction is weak. This detection model is known as an indirect measurement [4], and has been the subject of intense research lately as it is directly connected to quantum nondemolition experiments. As we will see, this "toy model" will allow us to capture some important features of the entanglement between the two systems.

This paper is organized in the following way. In Section 2 we will quickly review the interaction between two harmonic oscillators for the classical case. This will allow us to understand how the transfer of energy happens in such case. We then compute the exact solutions for the quantum mechanical system with interaction (Section 3). In Section 4 we apply the results of Section 3 to a specific case of exchange of a single quantum of energy and analyze its outcomes. In Section 5 we use Bohm's theory to interpret the results obtained. The conclusions are in Section 6.

2 The Classical Case

Before we go into the details of the quantum mechanical examples, let us begin by analyzing the classical system of two one-dimensional coupled harmonic oscillators with the same mass m, elastic constant k, and coupling constant λ, as shown in Figure 1. The Hamiltonian for this system is given by

$$H = \frac{1}{2m}\left(P_1^2 + P_2^2\right) + \frac{1}{2}k\left((X_1+d)^2 + (X_2-d)^2\right) + \frac{1}{2}\lambda\left(X_1 - X_2 + 2d\right)^2. \tag{1}$$

To simplify the equations of motion and eliminate the undesirable constant d we can make the canonical transformation

$$\begin{aligned} x_1 &= X_1 + d, \\ x_2 &= X_2 - d, \\ p_1 &= P_1, \\ p_2 &= P_2. \end{aligned}$$

With the new variables equation (1) rewrites to

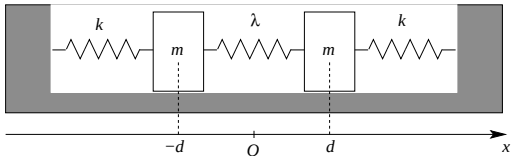

Figure 1: Identical harmonic oscillators coupled by a spring of constant λ.

$$H = \frac{1}{2m}\left(p_1^2 + p_2^2\right) + \frac{1}{2}k\left(x_1^2 + x_2^2\right) + \frac{1}{2}\lambda\left(x_1 - x_2\right)^2. \tag{2}$$

The Hamiltonian equations of motion are

$$\begin{aligned}
\dot{p}_1 &= -\frac{\partial H}{\partial x_1} = -kx_1 - \lambda\left(x_1 - x_2\right), \\
\dot{x}_1 &= \frac{\partial H}{\partial p_1} = \frac{p_1}{m}, \\
\\
\dot{p}_2 &= -\frac{\partial H}{\partial x_2} = -kx_2 + \lambda\left(x_1 - x_2\right), \\
\dot{x}_2 &= \frac{\partial H}{\partial p_2} = \frac{p_2}{m},
\end{aligned}$$

yielding

$$m\left(\ddot{x}_1 + \ddot{x}_2\right) = -k\left(x_1 + x_2\right), \tag{3}$$

and

$$m\left(\ddot{x}_1 - \ddot{x}_2\right) = -\left(k + 2\lambda\right)\left(x_1 - x_2\right). \tag{4}$$

The general solutions to (3) and (4) are

$$\sqrt{2}\xi_+ = x_1 + x_2 = A\cos\left(\sqrt{\frac{k}{m}}\mathrm{t}+\theta\right),$$

$$\sqrt{2}\xi_- = x_1 - x_2 = A'\cos\left(\sqrt{\frac{k+2\lambda}{m}}t + \theta'\right),$$

(ξ_+ and ξ_- are the normal coordinates of the coupled harmonic oscillators) or, equivalently,

$$\begin{aligned}
x_1 &= \frac{A}{2}\cos\left(\sqrt{\frac{k}{m}}t + \theta\right) + \frac{A'}{2}\cos\left(\sqrt{\frac{k+2\lambda}{m}}t + \theta'\right) \\
x_2 &= \frac{A}{2}\cos\left(\sqrt{\frac{k}{m}}t + \theta\right) - \frac{A'}{2}\cos\left(\sqrt{\frac{k+2\lambda}{m}}t + \theta'\right).
\end{aligned}$$

We will assume that the two oscillators are initially at rest the first one in its equilibrium position (null initial energy, $E_1^i = 0$), while the second one is dislocated from its equilibrium position by a distance D (initial energy given by $E_2^i = (1/2)\,kD^2$):

$$\begin{aligned}
\dot{x}_1(0) = \dot{x}_2(0) &= 0, \\
x_1(0) &= 0, \\
x_2(0) &= D.
\end{aligned}$$

The integration constants then read

$$\begin{aligned}
\theta = \theta' &= 0, \\
A &= D, \\
A' &= -D,
\end{aligned}$$

yielding

$$x_1(t) = \frac{D}{2}\left[\cos(\omega t) - \cos(\omega' t)\right] \quad (5)$$

$$x_2(t) = \frac{D}{2}\left[\cos(\omega t) + \cos(\omega' t)\right]. \quad (6)$$

where we defined $\omega \equiv \sqrt{k/m}$ and $\omega' \equiv \omega\sqrt{1+\varepsilon}$, with $\varepsilon = 2\lambda/k$. Equations (5) and (6) can be written in the following suggestive way.

$$x_1(t) = -D\sin\left[\frac{(\omega-\omega')t}{2}\right]\sin\left[\frac{(\omega+\omega')t}{2}\right], \quad (7)$$

$$x_2(t) = D\cos\left[\frac{(\omega-\omega')t}{2}\right]\cos\left[\frac{(\omega+\omega')t}{2}\right]. \quad (8)$$

We will now assume that the interaction constant λ is weak when compared to the elastic constant k, $\varepsilon \ll 1$. Then, we can expand ω' around $\varepsilon = 0$, yielding

$$\omega' = \sqrt{\frac{k+2\lambda}{m}} = \omega\sqrt{1+\varepsilon} = \omega + 2\delta\omega \quad (9)$$

with

$$\delta\omega \equiv \frac{\omega'-\omega}{2} \approx \frac{\lambda}{2\sqrt{km}}. \quad (10)$$

Defining

$$\bar{\omega} \equiv \frac{\omega'+\omega}{2} = \omega + \delta\omega, \quad (11)$$

the solutions can now be written as

$$x_1(t) = D\sin(\delta\omega\, t)\sin[\bar{\omega}\, t], \quad (12)$$

$$x_2(t) = D\cos(\delta\omega\, t)\cos[\bar{\omega}\, t], \quad (13)$$

where the dependence on λ of Eqs. (12) and (13) are present in $\delta\omega$ and $\bar{\omega}$ through (10) and (11).

The movement of both particles is periodic, with two characteristic frequencies $\delta\omega$ and $\bar{\omega}$. The frequencies $\delta\omega$ and $\bar{\omega}$ are known as the normal modes of vibration, with $\bar{\omega}$ being called the higher normal mode and $\delta\omega$ the lower normal mode. Both movements have period $\tau = 2\pi/\bar{\omega}$ and are modulated by a variable amplitude with much greater period given by $\tau = 2\pi/\delta\omega$. They are $\pi/2$ out of phase. We can compute the energy of the two particles, $E_1 = p_1^2/2m + kx_1^2/2$ and $E_2 = p_2^2/2m + kx_2^2/2$. They are

$$E_1(t) = \frac{kD^2}{2}\sin^2(\delta\omega\, t)\left[1 + 4\frac{\delta\omega}{\bar{\omega}}\cos^2(\bar{\omega}\, t)\right] \quad (14)$$

$$E_2(t) = \frac{kD^2}{2}\cos^2(\delta\omega\, t)\left[1 + 4\frac{\delta\omega}{\bar{\omega}}\sin^2(\bar{\omega}\, t)\right] \quad (15)$$

Due to the coupling, the particles exchange energy between themselves periodically, with period $\tau = 2\pi/\delta\omega$. Each of the oscillators achieve its minimum energy value when the other have its maximum value. The maximum value of the energy can be a little bit bigger then $kD^2/2$. This may seem odd, but we must remember that the extra energy is due to the interaction energy $\lambda(x_1 - x_2)^2/2 = k\varepsilon(x_1 - x_2)^2/4$. It is easy to check that if we add this interaction energy to the sum $E_1 + E_2$ we obtain the total energy of the system

$$E_T = \frac{kD^2}{2}\left(1 + 2\frac{\delta\omega}{\bar{\omega}}\right) + O(\delta\omega^2), \quad (16)$$

a value that is constant for the whole movement, as we should expect. For more details, see Refs.[8, 22], where this system and generalizations of it are analyzed with detail. Of course, as the Hamiltonian is time independent, energy is always conserved.

It is also interesting to note that the total energy of the system depends on the coupling constant, as shown by (16). A quick analysis of the origin of the "extra" energy shows us that this happens because of the initial conditions chosen. The initial conditions from which we obtained (16) have the particle represented by x_2 off its equilibrium position, whereas the other particle is at its equilibrium position, with both particles having zero kinetic energy. This initial condition obviously imply that the coupling spring, with elastic coefficient λ, is also stretched from its equilibrium position, and therefore has nonzero potential energy at $t = 0$. If we use other initial conditions, the "extra" energy due to coupling does not appear. For example, we can choose both particles at an initial position where all spring have no potential energy (in our case, $x_1 = x_2 = 0$) and one of the particles has some kinetic energy while the other particle has zero kinetic energy. With this set of initial conditions, the energy transfer from one particle to the other is the same as before, but no coupling energy is present in the total energy.

3 Quantum Evolution: Exact Solutions

Now we want to study the quantized version of the resonant spinless one-dimensional coupled harmonic oscillator presented in the previous Section. First we note that the total Hilbert space $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$ is spanned by $\mathcal{H}_1$ and $\mathcal{H}_2$, the Hilbert spaces for particles 1 and 2, respectively. For example, the two canonical variables describing particle 1 are

$$\hat{X}_1, \hat{P}_1 \in \mathcal{H}_1,$$

with

$$[\hat{X}_1, \hat{P}_1] = i\hbar\hat{1},$$

and are therefore represented as

$$\hat{X}_1 \otimes \hat{1}_2, \hat{P}_1 \otimes \hat{1}_2 \in \mathcal{H},$$

where $\hat{1}_2 \in \mathcal{H}_2$ is the identity operator. In this way, the Hamiltonian operator for particle 1, is written as

$$\hat{H}_1 = \frac{1}{2m}\left(\hat{P}_1 \otimes \hat{1}\right)^2 + \frac{1}{2}k\left(\hat{X}_1 \otimes \hat{1} + d\hat{1} \otimes \hat{1}\right)^2.$$

For shortness of notation, we will drop the tensor product and keep in mind that operators regarding particle 1 act on $\mathcal{H}_1$ whereas operators regarding particle 2 act on $\mathcal{H}_2$.

With the simplified notation, the total quantum Hamiltonian operator for the two oscillators plus the interaction term is

$$\begin{aligned}\hat{H} &= \hat{H}_1 + \hat{H}_2 + \hat{H}_I \\ &= \frac{1}{2m}\hat{P}_1^2 + \frac{1}{2}k\left(\hat{X}_1 + \hat{d}\right)^2 + \frac{1}{2m}\hat{P}_2^2 + \frac{1}{2}k\left(\hat{X}_2 - \hat{d}\right)^2 + \frac{1}{2}\lambda\left(\hat{X}_1 - \hat{X}_2 + 2\hat{d}\right)^2. \qquad (17)\end{aligned}$$

We can now make the following change of variables, similar to the classical case:

$$\begin{aligned}\hat{x}_1 &= \hat{X}_1 + \hat{d}, \\ \hat{x}_2 &= \hat{X}_2 - \hat{d}, \\ \hat{p}_1 &= \hat{P}_1, \\ \hat{p}_2 &= \hat{P}_2.\end{aligned}$$

This change of variables obviously keeps the commutation relations between momenta and positions. Hence, in the coordinate representation we have the Hamiltonian operator

$$\hat{H} = -\frac{\hbar^2}{2m}\left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}\right) + \frac{1}{2}k\left(x_1^2 + x_2^2\right) + \frac{1}{2}\lambda\left(x_1 - x_2\right)^2. \qquad (18)$$

In analogy to the classical case, we work with the normal coordinates

$$\xi_+ = \frac{1}{\sqrt{2}}(x_1 + x_2), \tag{19}$$

$$\xi_- = \frac{1}{\sqrt{2}}(x_1 - x_2). \tag{20}$$

This change of variables has Jacobian one, and does not change the normalization of wavefunctions.

With the normal coordinates, the Hamiltonian is

$$\hat{H} = -\frac{\hbar^2}{2m}\left(\frac{\partial^2}{\partial\xi_+^2} + \frac{\partial^2}{\partial\xi_-^2}\right) + \frac{1}{2}k\xi_+^2 + \frac{1}{2}(k+2\lambda)\xi_-^2, \tag{21}$$

and is now separable, i.e.,

$$\hat{H} = \hat{H}_+ + \hat{H}_-, \tag{22}$$

where

$$\hat{H}_+ = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial\xi_+^2} + \frac{1}{2}k\xi_+^2, \tag{23}$$

$$\hat{H}_- = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial\xi_-^2} + \frac{1}{2}(k+2\lambda)\xi_-^2. \tag{24}$$

Equations (23) and (24) are the well known Hamiltonians for one-dimensional uncoupled harmonic oscillator with frequencies $\sqrt{k/m}$ and $\sqrt{(k+2\lambda)/m}$, respectively.

The Schroedinger equation for the system is

$$\hat{H}\psi(\xi_+,\xi_-,t) = i\hbar\frac{\partial}{\partial t}\psi(\xi_+,\xi_-,t). \tag{25}$$

To solve (25) we need to find its eigenfunctions and eigenvalues since they form a basis for the Hilbert space. The general solution can be written as a superposition of the eigenfunctions. Hence, we need to find the solutions to the time independent Schroedinger equation

$$\hat{H}\psi^{(l)}(\xi_+,\xi_-) = \mathcal{E}_l\psi^{(l)}(\xi_+,\xi_-), \tag{26}$$

where l is an index (perhaps a collective index for both oscillators) for the energy to be determined. Since $\hat{H}$ is separable, we can write (26) as two independent eigenvalue equations

$$\hat{H}_+\phi_+^{(n)}(\xi_+) = E_n\phi_+^{(n)}(\xi_+) \tag{27}$$

and

$$\hat{H}_-\phi_-^{(n')}(\xi_-) = E'_{n'}\phi_-^{(n')}(\xi_-), \tag{28}$$

where we define

$$\psi^{(l)}(\xi_+,\xi_-) = \phi_+^{(n)}(\xi_+)\phi_-^{(n')}(\xi_-), \tag{29}$$

and

$$\mathcal{E}_l = E_n + E'_{n'}.$$

Clearly, l is an index that depends on both n and n', and for that reason we will write $\psi^{(n,n')}(\xi_+,\xi_-)$ instead of $\psi^{(l)}(\xi_+,\xi_-)$. The eigenfunctions of (27) and (28) are well known to be

$$\phi_+^{(n)}(\xi_+) = \left(\frac{\sqrt{mk}}{\pi\hbar 2^{2n}(n!)^2}\right)^{1/4} H_n\left[\left(\frac{\sqrt{mk}}{\hbar}\right)^{1/2}\xi_+\right]\exp\left[-\frac{\sqrt{mk}\xi_+^2}{2\hbar}\right], \tag{30}$$

$$\phi_-^{(n')}(\xi_-) = \left(\frac{\sqrt{m(k+2\lambda)}}{\pi\hbar 2^{2n'}(n'!)^2}\right)^{1/4} H_{n'}\left[\left(\frac{\sqrt{m(k+2\lambda)}}{\hbar}\right)^{1/2}\xi_-\right]\exp\left[-\frac{\sqrt{m(k+2\lambda)}\xi_-^2}{2\hbar}\right] \tag{31}$$

and its corresponding eigenvalues are

$$E_n = \hbar\sqrt{\frac{k}{m}}\left(n+\frac{1}{2}\right) \tag{32}$$

and

$$E'_{n'} = \hbar\sqrt{\frac{k+2\lambda}{m}}\left(n'+\frac{1}{2}\right), \tag{33}$$

where H_n are the Hermite polynomials of order n [3].

The solution to the time dependent Schroedinger equation (25) is obtained applying the time evolution operator

$$\hat{U}(t,t_0) = \exp\left(-i\hat{H}(t-t_0)/\hbar\right)$$

on $\psi(\xi_+,\xi_-,t_0)$. Since $\psi^{(n,n')}(\xi_+,\xi_-) = \phi_+^{(n)}(\xi_+)\phi_-^{(n')}(\xi_-)$ form a basis for $\mathcal{H}$, we have

$$\psi(\xi_+,\xi_-,t_0) = \sum_{n,n'=0}^{\infty} C_{n,n'}\psi^{(n,n')}(\xi_+,\xi_-),$$

$$C_{n,n'} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\phi_+^{(n)}(\xi_+)\phi_-^{(n')}(\xi_-)\psi(\xi_+,\xi_-,t_0)\,d\xi_+ d\xi_-, \tag{34}$$

and we used the reality of $\phi_+^{(n)}(\xi_+)\phi_-^{(n')}(\xi_-)$ in the expression for $C_{n,n'}$. Then,

$$\begin{aligned}
\psi(\xi_+,\xi_-,t) &= \hat{U}(t,t_0)\psi(\xi_+,\xi_-,0) \\
&= \sum_{n,n'=0}^{\infty} C_{n,n'}e^{-iE_nt/\hbar}\phi_+^{(n)}(\xi_+)e^{-iE'_{n'}t/\hbar}\phi_-^{(n')}(\xi_-) \\
&= \sum_{n,n'=0}^{\infty} C_{n,n'}e^{-i\left(E_n+E'_{n'}\right)t/\hbar}\psi^{(n,n')}(\xi_+,\xi_-),
\end{aligned}$$

where $\exp\left(-i\hat{H}t/\hbar\right) = \exp\left(-i\hat{H}_+t/\hbar\right)\exp\left(-i\hat{H}_-t/\hbar\right)$ since $\left[\hat{H}_+,\hat{H}_-\right] = 0$ and we assumed, for simplicity, that $t_0 = 0$.

We can now finally go back to the original coordinate system x_1 and x_2, and the explicit form for the general solution in this coordinate system is

$$\begin{aligned}
\psi(x_1,x_2,t) = &\sqrt{\frac{m}{\pi\hbar}}\sum_{n,n'=0}^{\infty} C_{n,n'}\left(\frac{\omega}{2^{2n}(n!)^2}\right)^{1/4}\left(\frac{\omega'}{2^{2n'}(n'!)^2}\right)^{1/4}\times \\
&H_n\left[\left(\frac{m\omega}{2\hbar}\right)^{1/2}(x_1+x_2)\right]H_{n'}\left[\left(\frac{m\omega'}{2\hbar}\right)^{1/2}(x_1-x_2)\right]\times \\
&\exp\left\{-\frac{m}{4\hbar}\left[\omega\left(x_1+x_2\right)^2+\omega'\left(x_1-x_2\right)^2\right]\right\}\times \\
&\exp\left\{-i\left[\left(n+\frac{1}{2}\right)\omega+\left(n'+\frac{1}{2}\right)\omega'\right]t\right\}.
\end{aligned} \tag{35}$$

where we defined, as before, $\omega = \sqrt{k/m}$ and $\omega' = \sqrt{k+2\lambda/m}$. The wavefunction (35) thus describe spinless one-dimensional coupled harmonic oscillators with no approximation.

4 A Simple Example

We saw in the classical case that two coupled oscillators can transfer energy to each other. This was clear with the example where at $t = 0$ one oscillator had zero mechanical energy while the other one had nonzero potential energy. As time passes, the mechanical energy of the former is transferred to the latter. It is interesting to study the quantum mechanical analogue to this case, i.e., when one quantum oscillator is in an excited state and the other is in its fundamental state.

We will consider as the initial wavefunction the following

$$\psi(x_1, x_2, 0) = \sqrt{\frac{2}{\pi}} \left(\frac{\sqrt{mk}}{\hbar} \right) x_2 \exp\left[-\frac{\sqrt{mk}\,(x_1^2 + x_2^2)}{2\hbar} \right]. \tag{36}$$

The wavefunction (36) is an eigenstate of the Hamiltonian

$$\hat{H} = \hat{H}_1 + \hat{H}_2 \tag{37}$$

without the interaction term $\hat{H}_I$. Clearly, $\psi(x_1, x_2, 0)$ is separable, i.e., it is possible to write $\psi(x_1, x_2, 0) = \varphi_1(x_1, 0)\varphi_2(x_2, 0)$. Since $\hat{H}_1$ ($\hat{H}_2$) acts only in $\varphi_1(x_1, 0)$ ($\varphi_2(x_2, 0)$), the state $\psi(x_1, x_2, 0)$ represents a system where the particle described by x_1 is in the ground state and the particle described by x_2 is in the first excited state. So, we can think of our example as the following. We have initially a system of two harmonic oscillators, one in the ground state and the other in the first excited state. After $t = 0$ we suddenly turn on a interaction between the two oscillators, and as a consequence we expect to have a "transfer of energy" from one oscillator to the other, as it happens in the classical case. We will now proceed to analyze in details this example.

4.1 Approximate Solution

To use equation (35) we need to find the coefficients $C_{nn'}$. It is straightforward to compute the coefficients from (34) by just using the orthogonal properties of the Hermite polynomials and by rewriting (36) in the normal coordinates, yielding

$$\begin{aligned} C_{n,n'} &= \sqrt{\omega} \left(\frac{\omega}{2^{2n}(n!)^2} \right)^{1/4} \left(\frac{\omega'}{2^{2n'}(n'!)^2} \right)^{1/4} \sqrt{\frac{2}{(\omega + \omega')}} \left(\frac{\omega' - \omega}{\omega + \omega'} \right)^j \times \\ &\left[\sqrt{\frac{1}{\omega}} \frac{2j!}{j!} \delta_{n',2j} \delta_{1,n} - \sqrt{\frac{2}{(\omega + \omega')}} \frac{(2j+1)!}{j!} \sqrt{\frac{2\omega'}{\omega + \omega'}} \delta_{n',2j+1} \delta_{0,n} \right], \end{aligned} \tag{38}$$

where δ_{ij} is Kroenecker's delta.

It is interesting to note that there exists infinite terms of $C_{n,n'}$ that are different from zero. Therefore, if we write down the expression for the time evolution of the wavefunction after the interaction we obtain an expression with an infinite number of terms. However, a close look at the $C_{n,n'}$ coefficients may shed light on how to deal with this problem. First we see from (38) that only the terms $C_{0,n'}$ and $C_{1,n'}$ are nonzero. If we compute the ratio between two consecutive nonzero terms, i.e, $C_{0,n'+2}/C_{0,n'}$ and $C_{1,n'+2}/C_{1,n'}$ we obtain

$$\frac{C_{0,n'+2}}{C_{0,n'}} = \left(\frac{\omega' - \omega}{\omega + \omega'} \right) \sqrt{\frac{(n'+2)}{(n'+1)}}, \tag{39}$$

$$\frac{C_{1,n'+2}}{C_{1,n'}} = \left(\frac{\omega' - \omega}{\omega + \omega'} \right) \sqrt{\frac{(n'+1)}{(n'+2)}}. \tag{40}$$

We note that both ratios (39) and (40) are proportional to $\left(\frac{\omega' - \omega}{\omega + \omega'}\right)$. Then, if the coupling constant λ is small compared to k (weak coupling) we can make an expansion of (39) and (40) around $\lambda = 0$

and obtain, up to first order in λ, that

$$\left(\frac{\omega' - \omega}{\omega + \omega'}\right) = \frac{\lambda}{2k} + O\left(\lambda^2\right).$$

We conclude that if λ is small compared to k, as we increase the value of n', the coefficients $C_{n,n'}$ become less important. Therefore, it is justifiable to keep only a finite amount of terms in the expression for $\psi(x_1, x_2, t)$ for small λ. In our example, we will keep only terms up to first order in λ.

Since we will be working with λ small, it is convenient now to introduce the following parameters already used in the classical case

$$\delta\omega = \frac{\omega' - \omega}{2},$$

$$\bar{\omega} = \frac{\omega' + \omega}{2}.$$

Then, if λ is small,

$$\delta\omega = \frac{\omega\lambda}{2k} + O(\lambda^2),$$

and

$$\frac{\delta\omega}{\bar{\omega}} \ll 1.$$

Keeping only terms up to first order in $\frac{\delta\omega}{\bar{\omega}}$, we have

$$\begin{aligned} \psi(x_1, x_2, 0) &= \sum_{n,n'=0}^{\infty} C_{n,n'} \psi^{(n,n')}(x_1, x_2) \\ &\cong C_{1,0}\psi^{(1,0)} + C_{0,1}\psi^{(0,1)} + C_{1,2}\psi^{(1,2)} + C_{0,3}\psi^{(0,3)}, \end{aligned} \tag{41}$$

where

$$C_{10} \cong \frac{\sqrt{2}}{2}, \tag{42}$$

$$C_{01} \cong -\frac{\sqrt{2}}{2}, \tag{43}$$

$$C_{12} \cong \frac{1}{2}\frac{\delta\omega}{\bar{\omega}}, \tag{44}$$

$$C_{03} \cong -\frac{\sqrt{3}}{2}\frac{\delta\omega}{\bar{\omega}}. \tag{45}$$

We are finally in a position to write, up to first order, the time dependent wavefunction for the coupled harmonic oscillators. From (41) and (42)–(45) it is straightforward to obtain

$$\begin{aligned} \psi(x_1, x_2, t) = & \sqrt{\frac{1}{2\pi}}\frac{m\bar{\omega}}{\hbar} \exp\left\{-\frac{m\bar{\omega}}{2\hbar}\left[x_1^2 + x_2^2\right]\right\} \exp\left\{-i2\bar{\omega}t\right\} \times \\ & \left\{2i\left(x_1 + x_2\left[\frac{1}{2} - \frac{m\bar{\omega}}{\hbar}x_1^2\right]\frac{\delta\omega}{\bar{\omega}}\right)\sin\left(\delta\omega t\right) + 2\left(x_2 + x_1\left[\frac{1}{2} - \frac{m\bar{\omega}}{\hbar}x_2^2\right]\frac{\delta\omega}{\bar{\omega}}\right)\cos\left(\delta\omega t\right)\right. \\ & + \frac{1}{2}\frac{\delta\omega}{\bar{\omega}}\left(x_1 + x_2\right)\left[\frac{m\bar{\omega}}{\hbar}\left(x_1 - x_2\right)^2 - 1\right]\exp\left\{-i\left(2\bar{\omega} + \delta\omega\right)t\right\} \\ & \left. - \frac{1}{2}\frac{\delta\omega}{\bar{\omega}}\left(x_1 - x_2\right)\left[\frac{m\bar{\omega}}{\hbar}\left(x_1 - x_2\right)^2 - 3\right]\exp\left\{-i\left(2\bar{\omega} + 3\delta\omega\right)t\right\}\right\} + O(\delta\omega^2). \end{aligned} \tag{46}$$

The wavefunction (46) determines the evolution of the system. We will now proceed to analyze the system using (46).

4.2 Marginal Probabilities

From (46) we compute the joint probability density for x_1 and x_2 as a function of t. The joint density is simply

$$P(x_1, x_2, t) = |\Psi(x_1, x_2, t)|^2,$$

and keeping terms up to first order in $\delta\omega$ we have

$$\begin{aligned} P(x_1,x_2,t) &= \frac{1}{2\pi}\left(\frac{m\bar{\omega}}{\hbar}\right)^2 \exp\left\{-\frac{m\bar{\omega}}{\hbar}\left[x_1^2+x_2^2\right]\right\}\times \\ &\left\{4\left(x_2^2+2x_1x_2\left[\frac{1}{2}-\frac{m\bar{\omega}}{\hbar}x_2^2\right]\frac{\delta\omega}{\bar{\omega}}\right)\cos^2(\delta\omega t)\right. \\ &+4\left(x_1^2+2x_1x_2\left[\frac{1}{2}-\frac{m\bar{\omega}}{\hbar}x_1^2\right]\frac{\delta\omega}{\bar{\omega}}\right)\sin^2(\delta\omega t) \\ &+2x_2\frac{\delta\omega}{\bar{\omega}}(x_1+x_2)\left[\frac{m\bar{\omega}}{\hbar}(x_1-x_2)^2-1\right]\cos(\delta\omega t)\cos\{(2\bar{\omega}+\delta\omega)t\} \\ &-2x_2\frac{\delta\omega}{\bar{\omega}}(x_1-x_2)\left[\frac{m\bar{\omega}}{\hbar}(x_1-x_2)^2-3\right]\cos(\delta\omega t)\cos\{(2\bar{\omega}+3\delta\omega)t\} \\ &-2x_1\frac{\delta\omega}{\bar{\omega}}(x_1+x_2)\left[\frac{m\bar{\omega}}{\hbar}(x_1-x_2)^2-1\right]\sin(\delta\omega t)\sin\{(2\bar{\omega}+\delta\omega)t\} \\ &\left.+2x_1\frac{\delta\omega}{\bar{\omega}}(x_1-x_2)\left[\frac{m\bar{\omega}}{\hbar}(x_1-x_2)^2-3\right]\sin(\delta\omega t)\sin\{(2\bar{\omega}+3\delta\omega)t\}\right\} \quad (47) \end{aligned}$$

It is interesting to see how the marginal probability distributions for x_1 and x_2 behave. Let us recall that the marginals are defined as

$$P(x_1,t) = \int_{-\infty}^{\infty} P(x_1,x_2,t)\,dx_2, \quad (48)$$

and

$$P(x_2,t) = \int_{-\infty}^{\infty} P(x_1,x_2,t)\,dx_1. \quad (49)$$

Therefore, $P(x_1,t)\,dx_1$ represents the probability of measuring the position of particle 1 in the interval (x_1, x_1+dx_1) independently of particle 2. The interpretation for $P(x_2,t)$ is similar.

From (47), (48), and (49) it is tedious but straightforward to compute (once again up to first order in $\delta\omega$) such quantities, which read

$$\begin{aligned} P(x_1,t) &= \sqrt{\frac{m\bar{\omega}}{\hbar\pi}}\exp\left\{-\frac{m\bar{\omega}}{\hbar}x_1^2\right\} \\ &\left\{\cos^2(\delta\omega t)+\frac{2m\bar{\omega}}{\hbar}x_1^2\sin^2(\delta\omega t)\right. \\ &-\frac{\delta\omega}{\bar{\omega}}\left[\left[\frac{1}{4}-\frac{m\bar{\omega}}{2\hbar}x_1^2\right](3\cos((2\bar{\omega}+3\delta\omega)t)-\cos((2\bar{\omega}+\delta\omega)t))\cos(\delta\omega t)\right. \\ &\left.\left.-\frac{m\bar{\omega}}{\hbar}x_1^2\left[\frac{3}{2}-\frac{m\bar{\omega}}{\hbar}x_1^2\right](\sin((2\bar{\omega}+3\delta\omega)t)-\sin((2\bar{\omega}+\delta\omega)t))\sin(\delta\omega t)\right]\right\}, \quad (50) \end{aligned}$$

and

$$\begin{aligned} P(x_2,t) &= \sqrt{\frac{m\bar{\omega}}{\pi\hbar}}\exp\left\{-\frac{m\bar{\omega}}{\hbar}x_2^2\right\} \\ &\left\{\sin^2(\delta\omega t)+\frac{2m\bar{\omega}}{\hbar}x_2^2\cos^2(\delta\omega t)\right. \\ &-\frac{\delta\omega}{\bar{\omega}}\left[\left[\frac{1}{4}-\frac{m\bar{\omega}}{2\hbar}x_2^2\right](3\sin((2\bar{\omega}+3\delta\omega)t)+\sin((2\bar{\omega}+\delta\omega)t))\sin(\delta\omega t)\right. \\ &\left.\left.-\frac{m\bar{\omega}}{\hbar}x_2^2\left[\frac{3}{2}-\frac{m\bar{\omega}}{\hbar}x_2^2\right](\cos((2\bar{\omega}+3\delta\omega)t)+\cos((2\bar{\omega}+\delta\omega)t))\cos(\delta\omega t)\right]\right\}, \quad (51) \end{aligned}$$

We can compute the values of the marginals (50) and (51) at $t = 0$ and find that, after making sure that we use ω as the frequency instead of $\bar{\omega}$, and keeping only terms up to first order in $\delta\omega/\bar{\omega}$, such marginals indeed represent the ones for the ground state HO and the first excited state HO, as one should expect.

To better grasp the behavior of (50) and (51), let us plot them as a function of time. Before plotting, we need to choose the appropriate values for the constants in the equations. If our system is in atomic scale, it is not reasonable, from a computational point of view, to use the MKS system. So, we will measure time in femtoseconds ($1\,\mathrm{fs} = 10^{-15}\,\mathrm{s}$) and distance in Angstroms (1 Å$= 10^{-10}\,\mathrm{m}$). If we say that the particles in the oscillators are electrons, then $m = 1\,\mathrm{m}_e$, where $\mathrm{m_e}$ is the mass of the electron, then we have

$$\hbar \;=\; 10\,\mathrm{m}_e \cdot \mathrm{\AA}^2 \cdot \mathrm{fs}^{-1},$$

and

$$k = 1\,\mathrm{m}_e \cdot \mathrm{fs}^{-2},$$

and, for the harmonic oscillator,

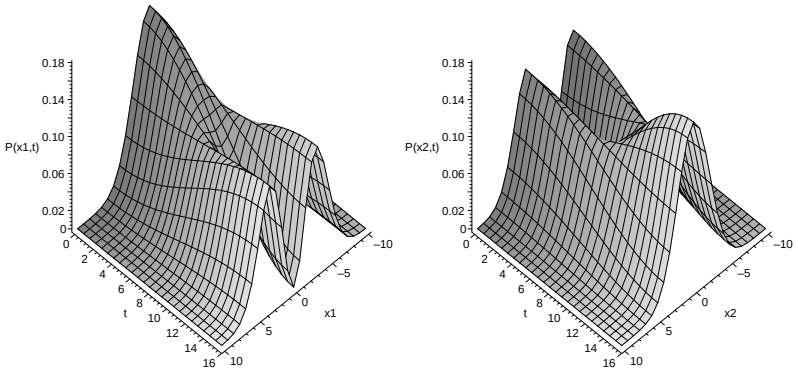

Figure 2: Graphs for the marginal probabilities of x_1 and x_2 as a function of time. In these graphs we used $m = 1\,\mathrm{m_e}$, $\bar{\omega} = 1\,\mathrm{fs}^{-1}$ and $\delta\omega/\bar{\omega} = 1/10$. The scale for time is fs and the scale for distance is Å.

$$\langle (\Delta x)^2 \rangle = \frac{\hbar}{2m_e\omega}.$$

The behavior of the probability density for particles 1 and 2 are found in Figure 2. The time interval chosen for the time axis in the graphs was $\Delta t = \pi/\delta\omega$ as this is the value where $\cos(\delta\omega t) = -1$, which is an extreme in the behavior of the marginal densities. Looking at the graphs we see that particle 1 starts with a marginal density that is mainly a Gaussian function, whereas particle 2 starts from the product of x_2^2 times a Gaussian. This is because particle 1 is at the ground state and particle 2 is at the first excited state at $t = 0$. However, as time passes there is a swap in the roles of particle 1 and 2, in the sense that at $t = \pi/\delta\omega$ the marginal density for particle 1 resembles that of particle 2 for $t = 0$ and vice versa. This is of course due to the interaction between the two particles. We may think of those densities as showing that, at $t = \pi/\delta\omega$ (more generally when $t = (2n+1)\,\pi/\delta\omega$) particle 1 is no longer in the ground state, but in the first excited state, whereas particle 2 is in the ground state.

4.3 Energy Expectations

The densities above suggest that there is an energy transfer from one particle to the other. To see that this is the case, let us compute the energy values for each particle. First we should note

that the system is not in an eigenstate of the Hamiltonian, as we started from a superposition of different energy states. We define the energy or particle 1 as

$$E_1 = \langle \hat{H}_1 \rangle,$$

the energy of particle 2 as

$$E_2 = \langle \hat{H}_2 \rangle,$$

and the total energy as the sum of the two energies plus the interaction energy

$$E_T = E_1 + E_2 + \langle \hat{H}_I \rangle.$$

In coordinate representation we have that

$$\begin{aligned} E_1 &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dx_1 dx_2\, \psi(x_1,x_2,t)^* \hat{H}_1 \psi(x_1,x_2,t) \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dx_1 dx_2\, \psi(x_1,x_2,t)^* \left[-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x_1^2} + \frac{1}{2}kx_1^2\right] \psi(x_1,x_2,t), \end{aligned}$$

and computing this term we obtain, up to second order in $\delta\omega/\bar{\omega}$,

$$\begin{aligned} E_1 &= \hbar\bar{\omega}\left(\frac{1}{2} + \sin^2(\delta\omega t)\right)(1 - \delta\omega/\bar{\omega}) \\ &= \hbar\omega\left(\frac{1}{2} + \sin^2(\delta\omega t)\right). \end{aligned} \tag{52}$$

Similarly, for E_2 we have

$$\begin{aligned} E_2 &= \hbar\bar{\omega}\left(\frac{1}{2} + \cos^2(\delta\omega t)\right)(1 - \delta\omega/\bar{\omega}). \\ &= \hbar\omega\left(\frac{1}{2} + \cos^2(\delta\omega t)\right). \end{aligned} \tag{53}$$

If we compare the quantum energies (52) and (53) to the classical expressions (14) and (15) the resemblance is striking. They are practically the same for $\delta\omega/\bar{\omega} \ll 1$, except for a zero energy factor of $\frac{1}{2}\hbar\bar{\omega}$ present in the quantum mechanical case. In fact, the same conclusions can now be drawn from (52) and (53) , i.e., that due to the coupling, the particles exchange energy between themselves periodically, with period $\tau = 2\pi/\delta\omega$. Each of the oscillators achieve its minimum energy value when the other have its maximum value. For the interaction energy we compute

$$\langle \hat{H}_I \rangle = 2\hbar\delta\omega. \tag{54}$$

Then, it is easy to compute the total mean energy

$$\begin{aligned} E_T &= E_1 + E_2 + \langle \hat{H}_I \rangle \\ &= 2\hbar\bar{\omega} \\ &= 2\hbar\omega + 2\hbar\delta\omega. \end{aligned}$$

This is once again in agreement with the classical case seen above, in the sense that the total energy is the sum of the energy of each oscillator (keeping into account the nonclassical zero point energy) without the interaction term plus an interaction term $2\hbar\delta\omega$.

We just saw that the state we used had a term in the total energy $2\hbar\delta\omega$ that was due to the coupling between the two oscillators. However, if we remember the classical case of Section 2, with different initial conditions — e.g. $x_1 = 0$, $x_2 = 0$, $\dot{x}_1 = v$, $\dot{x}_2 = 0$, at $t = 0$ — no interaction term is present in the total energy. What about the quantum case? Do we always have an interaction term

present, as in (54)? A short computation shows that for any initial state that is a combination of Fock states for the two HO of the form

$$|\psi\rangle = |n_1\rangle \otimes |n_2\rangle,$$

where $|n_1\rangle$ and $|n_2\rangle$ are eigenstates of two uncoupled HO, the value of $\langle \hat{H}_I \rangle_\psi$ (the interaction term) is different from zero.

The question remains as to whether it is possible to find an initial state that has an interaction term that is zero. A good guess would be to take both HO in a coherent state at $t = 0$, since it is a state that has many of the characteristics of a classical system [6]. It is easy to show that it is indeed true that for the state

$$|\psi\rangle = |\alpha\rangle \otimes |\beta\rangle,$$

where

$$|\alpha\rangle = e^{-\frac{|\alpha|^2}{2}} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle,$$

and similar for $|\beta\rangle$, the expected value of the interaction energy at $t = 0$ is zero if α and β have an appropriate phase relation. It is left up to the reader to find out this phase relation and a set of initial conditions for a classical system which reproduces the expectations in the quantum mechanical case.

5 The Bohmian Interpretation

Before we analyze the transfer of energy from a Bohmian point of view, let us quickly review Bohm's interpretation of quantum mechanics. Let us begin with the causal interpretation for the case of the Schrödinger equation describing a single particle. In the coordinate representation, for a non-relativistic particle with Hamiltonian $\hat{H} = \hat{p}^2/2m + V(\hat{x})$, the Schrödinger equation is

$$i\hbar \frac{\partial \Psi(x,t)}{\partial t} = \left[-\frac{\hbar^2}{2m} \nabla^2 + V(x) \right] \Psi(x,t). \tag{55}$$

We can transform this differential equation over a complex field into a pair of coupled differential equations over real fields. We do that by writing $\Psi = R \exp(iS/\hbar)$, where R and S are real functions, and substituting it into (55). We obtain the following equations.

$$\frac{\partial S}{\partial t} + \frac{(\nabla S)^2}{2m} + V - \frac{\hbar^2}{2m} \frac{\nabla^2 R}{R} = 0, \tag{56}$$

$$\frac{\partial R^2}{\partial t} + \nabla \cdot (R^2 \frac{\nabla S}{m}) = 0. \tag{57}$$

The usual probabilistic interpretation, i.e. the Copenhagen interpretation, understands equation (57) as a continuity equation for the probability density R^2 for finding the particle at position x and time t. All physical information about the system is contained in R^2, and the total phase S of the wave function is completely irrelevant. In this interpretation, nothing is said about S and its evolution equation (56).

However, examining equation (57), we can see that $\nabla S/m$ may be interpreted as a velocity field, suggesting the identification $p = \nabla S$. Hence, we can look to equation (56) as a Hamilton-Jacobi equation for the particle with the extra potential term

$$Q = -\frac{\hbar^2}{2m} \frac{\nabla^2 R}{R},$$

where Q is the so called quantum potential. Thus, since Bohm's interpretation identifies p with ∇S, from the differential equation $p = m\dot{x} = \nabla S$ we may compute its solutions and obtain the

trajectory of the quantum particle. Therefore, in Bohm's interpretation both momentum and position are quantities that are ontologically well defined.

For our case of two coupled-HO, the configuration space has two variables, x_1 and x_2, representing the positions of particles 1 and 2, respectively. For two particles, the nonlocality of Bohm's interpretation becomes evident as the Schrödinger equation becomes

$$i\hbar\frac{\partial\Psi(x_1,x_2,t)}{\partial t}=\left[-\frac{\hbar^2}{2m_1}\nabla_1^2-\frac{\hbar^2}{2m_2}\nabla_2^2+V(x_1,x_2)\right]\Psi(x_1,x_2,t), \tag{58}$$

where ∇_i^2 is the laplacian operator with respect to the coordinates of particle i. If we follow the same transformation as before, we can obtain the following equations.

$$\frac{\partial S}{\partial t}+\frac{(\nabla_1 S)^2}{2m_1}+\frac{(\nabla_2 S)^2}{2m_2}+V-\frac{\hbar^2}{2m}\frac{\nabla^2 R}{R}=0, \tag{59}$$

$$\frac{\partial R^2}{\partial t}+\nabla_1\cdot\left(R^2\frac{\nabla_1 S}{m_1}\right)+\nabla_2\cdot\left(R^2\frac{\nabla_2 S}{m_2}\right)=0. \tag{60}$$

The nonlocality comes from the fact that, even if the potential $V(x_1,x_2)$ is local, it is possible that the quantum potential given by

$$Q=-\frac{\hbar^2}{2m_1}\frac{\nabla_1^2 R}{R}-\frac{\hbar^2}{2m_2}\frac{\nabla_2^2 R}{R}$$

is nonlocal, depending on the form of R. This characteristic is necessary, as proved by Bell, if Bohm's theory is to recover all quantum mechanical predictions.

Using (46) it is straightforward to compute the phase $S(x_1,x_2,t)$ from the expression

$$S(x_1,x_2,t)=-\hbar\,\arctan\left[-i\,\frac{\Psi(x_1,x_2,t)-\Psi(x_1,x_2,t)^*}{\Psi(x_1,x_2,t)+\Psi(x_1,x_2,t)^*}\right].$$

After some long and tedious algebra we obtain

$$S(x_1,x_2,t)=-\hbar\,\arctan\left(\frac{S_A(x_1,x_2t)}{S_B(x_1,x_2t)}\right),$$

where

$$\begin{aligned}S_A(x_1,x_2,t) &= 4\cos(2\bar{\omega}t)\left\{\left(x_1^2\sin(\delta\omega t)^2-x_2^2\cos(\delta\omega t)^2\right)\sin(2\bar{\omega}t)\right.\\ &\quad\left.+x_1x_2\sin(\delta\omega t)\cos(\delta\omega t)\right\},\end{aligned}$$

and

$$S_B(x_1,x_2,t) = \left(x_2\cos(2\omega t)\cos(\delta\omega t)+x_1\sin(2\omega t)\sin(\delta\omega t)\right)^2,$$

where we keep all terms in $(\delta\omega/\bar{\omega})\,t$ but we neglected terms in $\delta\omega/\bar{\omega}$.

From $S(x_1,x_2t)$ we obtain the differential equation that describes the trajectories of particles x_1 and x_2 as

$$\frac{\mathrm{d}x_1}{\mathrm{d}t}=\frac{1}{m}\frac{\partial S(x_1,x_2t)}{\partial x_1}=-\frac{\hbar}{m}\frac{x_2\cos(\delta\omega t)\sin(\delta\omega t)}{x_1^2\sin^2(\delta\omega t)+x_2^2\cos^2(\delta\omega t)} \tag{61}$$

and

$$\frac{\mathrm{d}x_2}{\mathrm{d}t}=\frac{1}{m}\frac{\partial S(x_1,x_2t)}{\partial x_2}=\frac{\hbar}{m}\frac{x_1\cos(\delta\omega t)\sin(\delta\omega t)}{x_1^2\sin^2(\delta\omega t)+x_2^2\cos^2(\delta\omega t)}. \tag{62}$$

We can see that the trajectories follow a set of differential equations that are coupled and nonlinear. It is interesting to notice that if $\delta\omega=0$ we recover the standard Bohmian result that in the case

of no interaction each HO is in an eigenstate and therefore both particles are at rest. However, if $\delta\omega \neq 0$, we obtain at once that, after the change of variables

$$t' = \frac{\delta\omega}{\delta\omega'}t, \tag{63}$$

$$x_1' = \sqrt{\frac{\delta\omega}{\delta\omega'}}x_1, \tag{64}$$

$$x_2' = \sqrt{\frac{\delta\omega}{\delta\omega'}}x_2, \tag{65}$$

the differential equations (61) and (62) are form invariant with respect to a change in the coupling constant from $\delta\omega$ to $\delta\omega'$. This invariance is illustrated in Figures 3 and 4, where typical Bohmian trajectories were computed for both particles. The solutions shown in Figures 3 and 4 were obtained numerically using a 7th-8th-order continuous Runge-Kuta method.

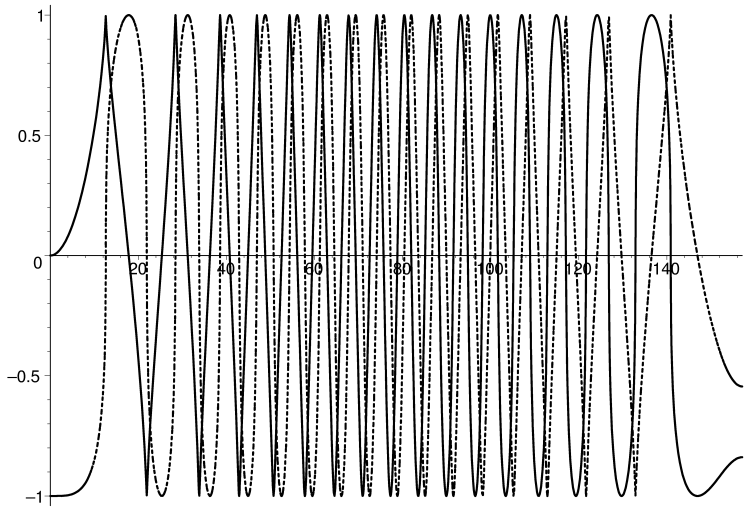

Figure 3: Bohmian trajectories for two CHO. The trajectories correspond to $\bar{\omega} = 1\,\mathrm{fs}^{-1}$, $\delta\omega/\bar{\omega} = 0.01$, $x_1(0) = 0$, and $x_2(0) = -1$. The solid line represents the trajectory of $x_1(t)$ whereas the dashed line represents that of $x_2(t)$. The scale for the ordinates is in Å and the time scale is in fs.

It is important to compute, in Bohmian theory, the quantum potential Q defined as

$$Q = Q_1 + Q_2$$

where

$$Q_1 = -\frac{\hbar^2}{2m}\frac{1}{\sqrt{P(x_1,x_2,t)}}\frac{\partial^2\sqrt{P(x_1,x_2,t)}}{\partial x_1^2}$$

and

$$Q_2 = -\frac{\hbar^2}{2m}\frac{1}{\sqrt{P(x_1,x_2,t)}}\frac{\partial^2\sqrt{P(x_1,x_2,t)}}{\partial x_2^2}.$$

It is straightforward to compute

$$\begin{aligned} Q_1 &= \hbar\bar{\omega} - \frac{1}{2}m\bar{\omega}^2x_1^2 \\ &\quad + \frac{1}{2}\hbar\bar{\omega}\frac{x_1^2\sin^2(\delta\omega t) - x_2^2\cos^2(\delta\omega t)}{x_1^2\sin^2(\delta\omega t) + x_2^2\cos^2(\delta\omega t)} \end{aligned}$$

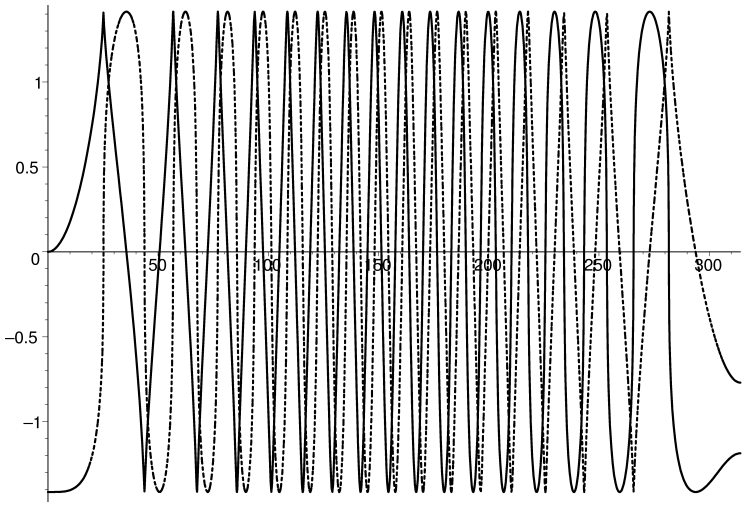

Figure 4: Bohmian trajectories for two CHO. The trajectories correspond to $\bar{\omega} = 1\,\mathrm{fs}^{-1}$, $\delta\omega/\bar{\omega} = 0.005$, $x_1(0) = 0$, and $x_2(0) = -\sqrt{2}$. The solid line represents the trajectory of $x_1(t)$ whereas the dashed line represents that of $x_2(t)$. The scale for the ordinates is in Å and the time scale is in fs. We can observe that the trajectories are identical to the ones shown in the previous Figure, except for the coordinate scales, a result consistent with equations (63)–(65).

$$-\frac{1}{2}\frac{\hbar^2}{m}\frac{x_2^2\cos^2(\delta\omega t)\sin^2(\delta\omega t)}{\left(x_1^2\sin^2(\delta\omega t)+x_2^2\cos^2(\delta\omega t)\right)^2}, \tag{66}$$

and

$$\begin{aligned} Q_2 &= \hbar\bar{\omega} - \frac{1}{2}m\bar{\omega}^2 x_2^2 \\ &\quad + \frac{1}{2}\hbar\bar{\omega}\frac{x_2^2\cos^2(\delta\omega t) - x_1^2\sin^2(\delta\omega t)}{x_2^2\cos^2(\delta\omega t) + x_1^2\sin^2(\delta\omega t)} \\ &\quad - \frac{1}{2}\frac{\hbar^2}{m}\frac{x_1^2\sin^2(\delta\omega t)\cos^2(\delta\omega t)}{\left(x_2^2\cos^2(\delta\omega t)+x_1^2\sin^2(\delta\omega t)\right)^2}, \end{aligned} \tag{67}$$

which yields

$$\begin{aligned} Q(x_1,x_2,t) &= 2\hbar\bar{\omega} - \frac{1}{2}m\bar{\omega}^2\left(x_1^2+x_2^2\right) \\ &\quad - \frac{1}{2}\frac{\hbar^2}{m}\frac{\left(x_1^2+x_2^2\right)\sin^2(\delta\omega t)\cos^2(\delta\omega t)}{\left(x_2^2\cos^2(\delta\omega t)+x_1^2\sin^2(\delta\omega t)\right)^2}. \end{aligned} \tag{68}$$

We are now in a position to compute the total bohmian energy for each one of the particles,

$$\begin{aligned} E_1 &= K_1 + V_1 + Q_1 \\ &= \hbar\bar{\omega} + \frac{1}{2}\hbar\bar{\omega}\frac{x_1^2\sin^2(\delta\omega t) - x_2^2\cos^2(\delta\omega t)}{x_1^2\sin^2(\delta\omega t) + x_2^2\cos^2(\delta\omega t)}, \end{aligned}$$

$$\begin{aligned} E_2 &= K_2 + V_2 + Q_2 \\ &= \hbar\bar{\omega} - \frac{1}{2}\hbar\bar{\omega}\frac{x_1^2\sin^2(\delta\omega t) - x_2^2\cos^2(\delta\omega t)}{x_1^2\sin^2(\delta\omega t) + x_2^2\cos^2(\delta\omega t)}, \end{aligned}$$

where $K_i = \frac{1}{2}m\left(\frac{\mathrm{d}x_i}{\mathrm{d}t}\right)^2$ is the kinetic energy of particle i (obtained from the guidance equations (61) and (62)) and V_i is the potential for particle i (neglecting terms in $\delta\omega/\bar{\omega}$).

The total energy for the system is just the sum of the individiual energies, yielding

$$E_T = E_1 + E_2 = 2\hbar\bar{\omega},$$

the same value as the expected energy of the system.

6 Conclusions and Final Remarks

We see that the expressions obtained for E_1 and E_2 involve an interaction term that makes it impossible to distinguish what part of the energy belongs to the particle x_1 and what part belongs to the particle x_2, except for some particular values of t. In the Copenhagen interpretation of QM it does not make any sense to talk about the energy of each oscillator for all t, as the oscillators are in a quantum superposition and are not in an eigenstate of its hamiltonian operator. In Bohm, it will not make any sense to talk about the energy of each oscillator for all t, since the quantum potential creates an interaction between the two oscillators that is of the same order of the other terms in the hamiltonian. Therefore it does not make any sense in the bohmian theory to say that the energy of the photon was transfered to the photodetector (except for very special values of t).

However, the bohmian interpretation gives an onthological explanation for the indefiniteness of the energy of each particle. Even with the interaction turned off, there is still a quantum nonlocal interaction between the oscillators given by the quantum potential and, in fact, one oscillator is not isolated from the other. This indicates that a real measurement has not yet ocurred. It seems to us that in order for a measurement to take place, a more elaborated description of the photodetection process involving a thermal bath or a macroscopic description must be used. In such case, we expect that the quantum potential will vanish and no further nonlocal interaction will be present after the measurement.

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