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Superconformal mechanics in $SU(2|1)$ superspace

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Using the worldline $SU(2|1)$ superfield approach, we construct $\mathcal{N}=4$ superconformally invariant actions for the $d=1$ multiplets $(\mathbf{1},\mathbf{4},\mathbf{3})$ and $(\mathbf{2},\mathbf{4},\mathbf{2})$. The $SU(2|1)$ superfield framework automatically implies the trigonometric realization of the superconformal symmetry and the harmonic oscillator term in the corresponding component actions. We deal with the general $\mathcal{N}=4$ superconformal algebra $D(2,1;\alpha)$ and its central-extended $\alpha=0$ and $\alpha=-1$ $psu(1,1|2)\oplus su(2)$ descendants. We capitalize on the observation that $D(2,1;\alpha)$ at $\alpha\neq 0$ can be treated as a closure of its two $su(2|1)$ subalgebras, one of which defines the superisometry of the $SU(2|1)$ superspace, while the other is related to the first one through the reflection of μ, the parameter of contraction to the flat $\mathcal{N}=4$, $d=1$ superspace. This closure property and its $\alpha=0$ analog suggest a simple criterion for the $SU(2|1)$ invariant actions to be superconformal: they should be even functions of μ. We find that the superconformal actions of the multiplet $(\mathbf{2},\mathbf{4},\mathbf{2})$ exist only at $\alpha=-1,0$ and are reduced to a sum of the free sigma-model-type action and the conformal superpotential yielding, respectively, the oscillator potential $\sim\mu^2$ and the standard conformal inverse-square potential in the bosonic sector. The sigma-model action in this case can be constructed only on account of nonzero central charge in the superalgebra $su(1,1|2)$.

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I. INTRODUCTION

Recently, essential progress has been achieved in constructing and understanding the rigid supersymmetric theories in curved superspace which attract attention in connection with the general "gauge/gravity" correspondence (see, e.g., Refs. [1–3] and references therein). In Refs. [4,5], two of us elaborated on the simplest $d=1$ analogs of such theories, the $SU(2|1)$ supersymmetric quantum mechanics (SQM) models, proceeding from the $SU(2|1)$ covariant worldline superfield approach. Two types of the worldline $SU(2|1)$ superspace as the proper supercosets of the supergroup $SU(2|1)$ were constructed. Both superspaces are deformations of the standard $\mathcal{N}=4$, $d=1$ superspace (see Ref. [6] and references therein) by a mass parameter m. The off- and on-shell deformed versions of the $\mathcal{N}=4, d=1$ multiplets $(\mathbf{1},\mathbf{4},\mathbf{3})$ and $(\mathbf{2},\mathbf{4},\mathbf{2})$ were studied and proved to possess a number of interesting peculiarities as compared with their "flat" $m=0$ cousins. One such new feature is the necessary presence of the harmonic oscillator terms $\sim m^2$ in the bosonic sectors of the corresponding invariant Lagrangians. The "weak supersymmetry" model of Ref. [7] and the "super Kähler oscillator" models of Refs. [8,9] were recovered as the particular cases of generic $SU(2|1)$ SQM associated, respectively, with the single multiplet $(\mathbf{1},\mathbf{4},\mathbf{3})$ and a few multiplets $(\mathbf{2},\mathbf{4},\mathbf{2})$. It is interesting to inspect the superconformal subclass of the $SU(2|1)$ SQM models. This is the main subject of the present paper.

As was argued in Ref. [10], conformal mechanics [11] can be divided into three classes characterized by the *parabolic*, *trigonometric* and *hyperbolic* realizations of the $d=1$ conformal group $SO(2,1)\sim SL(2,\mathbb{R})$. Earlier, supersymmetric extensions of conformal mechanics corresponding only to the parabolic transformations were mostly addressed [6,12,13]. Motivated by Ref. [10], the classification of superconformal $\mathcal{N}=4$ SQM models was recently extended by the *trigonometric/hyperbolic* type [14]. The basic difference of the trigonometric/hyperbolic superconformal actions from the parabolic ones is the presence of oscillator potentials. The standard $d=1$ Poincaré supercharges present in the superconformal algebras are not squared to the canonical Hamiltonian in such models. The actions of trigonometric/hyperbolic superconformal mechanics cannot be obtained from the standard $\mathcal{N}=4, d=1$ superfield approach, while the parabolic actions are well described just within the latter.[1] It turns out that it is the $SU(2|1)$ superfield approach that is ideally suited for the comprehensive description of the trigonometric $\mathcal{N}=4$ superconformal actions. The hyperbolic actions can be obtained from the trigonometric ones by a simple substitution.

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[1]The possibility of adding an oscillator term to the de Alfaro-Fubini-Furlan action [11] without breaking conformal symmetry was first noticed in Ref. [15]. The $\mathcal{N}=2$ superconformal extensions of such actions were considered in Refs. [16,17].

Our construction is based on the appropriate two-parameter embedding of the superspace supergroup $SU(2|1)$ into the most general $\mathcal{N}=4, d=1$ superconformal group $D(2,1;\alpha)$, with the contraction parameter m being redefined as $m \to -\alpha\mu$ and μ also appearing in the basic anticommutator on its own. At any $\alpha \neq 0$ the whole conformal superalgebra $D(2,1;\alpha)$ can be obtained as a closure of the original superalgebra $su(2|1)$ and its $-\mu$ counterpart, which suggests a simple selection rule for the superconformal $SU(2|1)$ SQM Lagrangians as those depending only on μ^2. At $\alpha = 0$, the basic $su(2|1)$ contracts into some flat $\mathcal{N}=4, d=1$ superalgebra which is still different from the standard $\mathcal{N}=4, d=1$ "Poincaré" superalgebra and involves the parameter μ in such a way that $D(2,1;\alpha=0) \sim psu(1,1|2) \oplus su(2)$ (and its central extension) can be obtained as a closure of this flat superalgebra and its $-\mu$ counterpart as subalgebras of $D(2,1;\alpha=0)$. This important property makes it possible to sort out the superconformal actions in the special $\alpha=0$ case too. Exploiting the closure property just mentioned, we find the universal two-parameter family of the realizations of the conformal supergroup $D(2,1;\alpha)$ on the coordinates of the $SU(2|1)$ superspace, as well as on the superfields representing the off-shell multiplets $(\mathbf{1},\mathbf{4},\mathbf{3})$ and $(\mathbf{2},\mathbf{4},\mathbf{2})$, at all admissible values of the parameter α (for the second multiplet, only $\alpha=-1$ and $\alpha=0$ are allowed). These realizations automatically prove to be trigonometric, while the corresponding superconformal actions necessarily involve the oscillator-type terms $\sim\mu^2$. The parabolic realizations of $D(2,1;\alpha)$ and the corresponding actions are recovered in the limit $\mu=0$, in which both $su(2|1)$ and its $\alpha=0$ analog go over into the standard μ-independent $\mathcal{N}=4, d=1$ Poincaré superalgebra.

The paper is organized as follows: The salient features of the $SU(2|1)$ superspace approach are sketched in Sec. II. In Sec. III, the embedding of $su(2|1)$ in $D(2,1;\alpha)$ is discussed along the lines outlined above, and the relevant $SU(2|1)$ superspace realizations of $D(2,1;\alpha)$ are explicitly presented. The study of the trigonometric models of superconformal mechanics associated with the multiplets $(\mathbf{1},\mathbf{4},\mathbf{3})$ and $(\mathbf{2},\mathbf{4},\mathbf{2})$ is the subject of Secs. IV–VII. We construct the superfield and component off- and on-shell actions for various cases, distinguishing those which admit additional conformal inverse-square potentials in the bosonic sector. The alternative (albeit equivalent) construction of the component superconformally invariant actions, based on the D-module representation techniques, is briefly outlined in Sec. VIII on the example of the multiplet $(\mathbf{2},\mathbf{4},\mathbf{2})$. Section IX is a summary of the basic results of the paper. In the appendixes, we collect some details concerning the central extensions of the superalgebra $D(2,1;\alpha)$ with $\alpha=-1$ (or $\alpha=0$), the generalized chiral $SU(2|1)$ multiplets $(\mathbf{2},\mathbf{4},\mathbf{2})$, as well as the hyperbolic superconformal mechanics.

II. $SU(2|1)$ SUPERSPACE

First of all, we need to define the superalgebra $su(2|1)$. Its standard form is given by the following nonvanishing (anti)commutators:

$$\begin{aligned}
&\{Q^i,\bar{Q}_j\} = 2mI^i_j + 2\delta^i_j\tilde{H}, \quad [I^i_j, I^k_l] = \delta^k_j I^i_l - \delta^i_l I^k_j,\\
&[I^i_j,\bar{Q}_l] = \frac{1}{2}\delta^i_j\bar{Q}_l - \delta^i_l\bar{Q}_j, \quad [I^i_j, Q^k] = \delta^k_j Q^i - \frac{1}{2}\delta^i_j Q^k,\\
&[\tilde{H},\bar{Q}_l] = \frac{m}{2}\bar{Q}_l, \qquad [\tilde{H}, Q^k] = -\frac{m}{2}Q^k. \qquad (2.1)
\end{aligned}$$

The generators satisfy the following rules of the Hermitian conjugation:

$$(Q^k)^\dagger = \bar{Q}_k,\ \ (\bar{Q}_k)^\dagger = Q^k,\ \ (I^k_i)^\dagger = I^i_k,\ \ \tilde{H}^\dagger = \tilde{H}. \qquad (2.2)$$

The generators I^i_j are the $SU(2)$ symmetry generators, while the mass-dimension generator $\tilde{H}$ corresponds to $U(1)$ symmetry. The superalgebra (2.1) can be regarded as a deformation of the flat $\mathcal{N}=4$, $d=1$ "Poincaré" superalgebra by a real mass parameter m. In the limit $m=0$, $\tilde{H}$ becomes the Hamiltonian (alias the time-translation generator), and the generators I^i_j define the outer $SU(2)$ automorphisms.

One can extend (2.1) by an external $U(1)$ automorphism symmetry (R-symmetry) generator F which has nonzero commutation relations only with the supercharges [1]:

$$[F,\bar{Q}_l] = -\frac{1}{2}\bar{Q}_l, \quad [F,Q^k] = \frac{1}{2}Q^k, \quad (F)^\dagger = F. \qquad (2.3)$$

After redefining $\tilde{H} \equiv H - mF$, the extended superalgebra $su(2|1) \oplus u(1)_{\rm ext}$ acquires the form of a centrally extended superalgebra $\hat{su}(2|1)$:

$$\begin{aligned}
&\{Q^i,\bar{Q}_j\} = 2mI^i_j + 2\delta^i_j(H-2mF),\ \ [I^i_j,I^k_l] = \delta^k_j I^i_l - \delta^i_l I^k_j,\\
&[I^i_j,\bar{Q}_l] = \frac{1}{2}\delta^i_j\bar{Q}_l - \delta^i_l\bar{Q}_j,\ \ [I^i_j,Q^k] = \delta^k_j Q^i - \frac{1}{2}\delta^i_j Q^k,\\
&[F,\bar{Q}_l] = -\frac{1}{2}\bar{Q}_l,\ \ [F,Q^k] = \frac{1}{2}Q^k. \qquad (2.4)
\end{aligned}$$

All other (anti)commutators are vanishing. The generator H is the relevant central charge. This extended superalgebra is also a deformation of the $\mathcal{N}=4$, $d=1$ Poincaré superalgebra. In the limit $m=0$, H becomes the Hamiltonian and I^i_j, F turn into the outer $U(2)$ automorphism generators.

In the present paper, we start from the framework of the $SU(2|1)$ superspace constructed in Ref. [4]. The $SU(2|1)$, $d=1$ superspace is identified with the following coset of the extended superalgebra (2.4):

$$\frac{SU(2|1) \rtimes U(1)_{\rm ext}}{SU(2) \times U(1)_{\rm int}} \sim \frac{\{Q^i, \bar{Q}_j, H, F, I^i_j\}}{\{I^i_j, F\}}. \tag{2.5}$$

It is convenient to deal with the superspace coordinates $\zeta := \{t, \theta_i, \bar{\theta}^k\}$ as in Refs. [4,5]. They are related to those in the exponential parametrization of the supercoset (2.5) as

$$g = \exp\left\{\left(1 - \frac{2m}{3}\bar{\theta}^k\theta_k\right)(\theta_i Q^i + \bar{\theta}^j\bar{Q}_j)\right\}\exp\{itH\}, \quad \overline{(\theta_i)} = \bar{\theta}^i. \tag{2.6}$$

The extended supergroup $\hat{SU}(2|1)$ acts as left shifts of the supercoset element (2.6). The corresponding supercharges are realized as

$$\begin{aligned} Q^i &= \frac{\partial}{\partial\theta_i} - 2m\bar{\theta}^i\bar{\theta}^k\frac{\partial}{\partial\bar{\theta}^k} + i\bar{\theta}^i\partial_t - m\bar{\theta}^i\tilde{F} \\ &\quad + m\bar{\theta}^k(1 - m\bar{\theta}^k\theta_k)\tilde{I}^i_k, \\ \bar{Q}_j &= \frac{\partial}{\partial\bar{\theta}^j} + 2m\theta_j\theta_k\frac{\partial}{\partial\theta_k} + i\theta_j\partial_t - m\theta_j\tilde{F} \\ &\quad + m\theta_k(1 - m\bar{\theta}^k\theta_k)\tilde{I}^k_j, \end{aligned} \tag{2.7}$$

and the bosonic generators as

$$\begin{aligned} I^i_j &= \left(\bar{\theta}^i\frac{\partial}{\partial\bar{\theta}^j} - \theta_j\frac{\partial}{\partial\theta_i}\right) - \frac{1}{2}\delta^i_j\left(\bar{\theta}^k\frac{\partial}{\partial\bar{\theta}^k} - \theta_k\frac{\partial}{\partial\theta_k}\right), \\ H &= i\partial_t, \qquad F = \frac{1}{2}\left(\bar{\theta}^k\frac{\partial}{\partial\bar{\theta}^k} - \theta_k\frac{\partial}{\partial\theta_k}\right). \end{aligned} \tag{2.8}$$

Here, $\tilde{I}^k_j$ and $\tilde{F}$ are matrix generators of the $U(2)$ representation by which the given superfield is rotated with respect to its external indices. According to (2.7), the supersymmetric transformations ϵ_i, $\bar{\epsilon}^i = \overline{(\epsilon_i)}$ of the superspace coordinates are given by

$$\begin{aligned} \delta\theta_i &= \epsilon_i + 2m\bar{\epsilon}^k\theta_k\theta_i, \\ \delta\bar{\theta}^i &= \bar{\epsilon}^i - 2m\epsilon_k\bar{\theta}^k\bar{\theta}^i, \\ \delta t &= i(\bar{\epsilon}^k\theta_k + \epsilon_k\bar{\theta}^k). \end{aligned} \tag{2.9}$$

The $SU(2|1)$ invariant integration measure is defined as

$$d\zeta = dt d^2\theta d^2\bar{\theta}(1 + 2m\bar{\theta}^k\theta_k), \qquad \delta d\zeta = 0. \tag{2.10}$$

The covariant derivatives $\mathcal{D}^i$, $\bar{\mathcal{D}}_j$, $\mathcal{D}_{(t)}$ are defined by the expressions[2]

$$\begin{aligned} \mathcal{D}^i &= \left[1 + m\bar{\theta}^k\theta_k - \frac{3m^2}{8}(\theta)^2(\bar{\theta})^2\right]\frac{\partial}{\partial\theta_i} - m\bar{\theta}^i\theta_j\frac{\partial}{\partial\theta_j} \\ &\quad - i\bar{\theta}^i\partial_t + m\bar{\theta}^i\tilde{F} - m\bar{\theta}^j(1 - m\bar{\theta}^k\theta_k)\tilde{I}^i_j, \\ \bar{\mathcal{D}}_j &= -\left[1 + m\bar{\theta}^k\theta_k - \frac{3m^2}{8}(\theta)^2(\bar{\theta})^2\right]\frac{\partial}{\partial\bar{\theta}^j} + m\bar{\theta}^k\theta_j\frac{\partial}{\partial\bar{\theta}^k} \\ &\quad + i\theta_j\partial_t - m\theta_j\tilde{F} + m\theta_k(1 - m\bar{\theta}^k\theta_k)\tilde{I}^k_j, \\ \mathcal{D}_{(t)} &= \partial_t \end{aligned} \tag{2.11}$$

and satisfy, together with $\tilde{I}^k_j, \tilde{F}$, the superalgebra which mimics (2.4). Under the left $\hat{SU}(2|1)$ shifts of the coset element (2.5), the spinor covariant derivatives undergo the induced $SU(2)$ transformations in their doublet indices and an induced F transformation with respect to which $\mathcal{D}^i$ and $\bar{\mathcal{D}}_i$ possess opposite charges. In the limit $m = 0$, the formulas of the standard flat $\mathcal{N} = 4$, $d = 1$ superspace are recovered. The superfields given on the $SU(2|1)$ superspace (2.5) can have external $SU(2)$ indices and $U(1)$ charges on which the proper matrix realizations of the relevant generators act.

There exists an alternative definition of the $SU(2|1)$ superspace, in which the time coordinate is associated as a coset parameter with the total internal $U(1)$ generator $\tilde{H} = H - mF$, while F is still placed into the stability subgroup [5]. As was already mentioned, in the basis $(\tilde{H}, F)$, the generator F is split from other generators, becoming the purely external $U(1)$ automorphism. The relevant supercoset is schematically related to (2.5) just by replacing $H \to \tilde{H}$:

$$\frac{SU(2|1) \rtimes U(1)_{\rm ext}}{SU(2) \times U(1)_{\rm ext}} \sim \frac{\{Q^i, \bar{Q}_j, \tilde{H}, F, I^i_j\}}{\{I^i_j, F\}} \sim \frac{\{Q^i, \bar{Q}_j, \tilde{H}, I^i_j\}}{\{I^i_j\}}. \tag{2.12}$$

The same replacement $H \to \tilde{H}$ should be made in the coset element (2.5), giving rise to the coset element $\tilde{g}$. Due to the relation $\tilde{H} = H - mF$, these two coset elements are related as

$$\tilde{g} = g\exp\{-imtF\}. \tag{2.13}$$

Under the left shifts by the fermionic generators, the coordinates $\zeta = \{t, \theta_i, \bar{\theta}^k\}$ are transformed according to the same formulas (2.9), so they can also be treated as the parameters of the new supercoset. The difference from the first type of the $SU(2|1)$ superspace is the absence of independent constant shift of the time coordinate, which can still be realized under the choice (2.5). The left $\tilde{H}$ shift gives rise to a shift of t accompanied by the proper $U(1)$ rotation of the Grassmann coordinates. The corresponding covariant spinor derivatives differ from (2.11) by the absence of the part $\sim\tilde{F}$ and by some overall phase factor

[2]For Grassmann coordinates and variables we use the following conventions: $(\chi)^2 = \chi_i\chi^i, (\bar{\chi})^2 = \bar{\chi}^i\bar{\chi}_i$.

ensuring them to transform only under induced $SU(2)$ transformations. These modifications can be easily established from the precise relation (2.13). The corresponding superfields can carry only external $SU(2)$ indices.

III. EMBEDDING OF $su(2|1)$ INTO $D(2,1;\alpha)$

The most general $d=1$, $\mathcal{N}=4$ superconformal algebra is $D(2,1;\alpha)$, with α being a real parameter [6,18]. It is spanned by eight fermionic and nine bosonic generators with the following nonvanishing (anti)commutators:

$$\{Q_{\alpha i i'}, Q_{\beta j j'}\} = 2[\epsilon_{ij}\epsilon_{i'j'}T_{\alpha\beta} + \alpha\epsilon_{\alpha\beta}\epsilon_{i'j'}J_{ij} - (1+\alpha)\epsilon_{\alpha\beta}\epsilon_{ij}L_{i'j'}], \tag{3.1}$$

$$\begin{aligned}
[T_{\alpha\beta}, Q_{\gamma i i'}] &= -i\epsilon_{\gamma(\alpha}Q_{\beta)ii'},\\
[T_{\alpha\beta}, T_{\gamma\delta}] &= i(\epsilon_{\alpha\gamma}T_{\beta\delta} + \epsilon_{\beta\delta}T_{\alpha\gamma}),\\
[J_{ij}, Q_{\alpha k i'}] &= -i\epsilon_{k(i}Q_{\alpha j)i'},\\
[J_{ij}, J_{kl}] &= i(\epsilon_{ik}J_{jl} + \epsilon_{jl}J_{ik}),\\
[L_{i'j'}, Q_{\alpha i k'}] &= -i\epsilon_{k'(i'}Q_{\alpha i j')},\\
[L_{i'j'}, L_{k'l'}] &= i(\epsilon_{i'k'}L_{j'l'} + \epsilon_{j'l'}L_{i'k'}).
\end{aligned} \tag{3.2}$$

The bosonic subalgebra is $su(2)\oplus su'(2)\oplus so(2,1)$ with the generators J_{ik}, $L_{i'k'}$ and $T_{\alpha\beta}$, respectively. Switching α as $\alpha \leftrightarrow -(1+\alpha)$ amounts to switching $SU(2)$ generators as $J_{ik} \leftrightarrow L_{i'k'}$.[3] The Hermitian conjugation rules are

$$(Q_{\alpha i i'})^\dagger = \epsilon^{ij}\epsilon^{i'j'}Q_{\alpha j j'}, \qquad (T_{\alpha\beta})^\dagger = T_{\alpha\beta},$$
$$(J_{ij})^\dagger = \epsilon^{ik}\epsilon^{jl}J_{kl}, \qquad (L_{i'j'})^\dagger = \epsilon^{i'k'}\epsilon^{j'l'}L_{k'l'}. \tag{3.3}$$

The $\mathcal{N}=4$, $d=1$ Poincaré superalgebra can be defined as the following subalgebra of $D(2,1;\alpha)$:

$$\{Q_{1ii'}, Q_{1jj'}\} = 2\epsilon_{ij}\epsilon_{i'j'}\hat{H}, \tag{3.4}$$

where $\hat{H}$ is one of the generators of the conformal algebra $so(2,1)$ represented in (3.1) and (3.2) by the generators $T_{\alpha\beta}$. The standard conformal $so(2,1)$ generators are identified as

$$\hat{H} \coloneqq T_{11}, \qquad \hat{K} \coloneqq T_{22}, \qquad \hat{D} \coloneqq T_{12}, \tag{3.5}$$

$$[\hat{D},\hat{H}] = -i\hat{H}, \qquad [\hat{D},\hat{K}] = i\hat{K}, \qquad [\hat{H},\hat{K}] = 2i\hat{D}. \tag{3.6}$$

In the degenerate case $\alpha=-1$, one may retain all eight fermionic generators $Q_{\alpha i i'}$ and only six bosonic generators $T_{\alpha\beta}$, J_{ij} forming together the superalgebra $psu(1,1|2)$ without central charge. The second $SU(2)$ generators $L_{i'j'}$ drop out from the basic anticommutation relation (3.1). Yet, they can be treated as the generators of some extra $SU'(2)$ automorphisms. Taking $\alpha=0$, one can suppress, in the same way, the generators J_{ij} in (3.1), ending up with $SU'(2)$ as the internal group and the first $SU(2)$ as the external automorphism group. Thus, in the cases $\alpha=-1$ and $\alpha=0$ the supergroup $D(2,1;\alpha)$ is reduced to a semidirect product:

$$\alpha=-1,0, \quad D(2,1;\alpha) \cong PSU(1,1|2) \rtimes SU(2)_{\rm ext}, \tag{3.7}$$

with $SU(2)_{\rm ext}$ being generated, respectively, by $L_{i',j'}$ or J_{ij}. Note that in these exceptional cases one can extend the $psu(1,1|2)$ superalgebra by the proper $SU(2)_{\rm ext}$ triplets of central charges [13]. If these central charges are constant, the triplet can be reduced to one central charge, which enlarges $psu(1,1|2)$ to $su(1,1|2)$ and simultaneously breaks $SU(2)_{\rm ext}$ to $U(1)_{\rm ext}$ (see Appendix A).

We will be interested in the most general embedding of the superalgebra $su(2|1)$ into $D(2,1;\alpha)$. To this end, we pass to the new basis in $D(2,1;\alpha)$ through the following linear relations:

$$\begin{aligned}
&\varepsilon^{ik}Q_{1k1'} =: -\frac{1}{2}(S^i + Q^i), \qquad Q_{1j2'} =: -\frac{1}{2}(\bar{S}_j + \bar{Q}_j),\\
&\varepsilon^{ik}Q_{2k1'} =: \frac{i}{\mu}(Q^i - S^i), \qquad Q_{2j2'} =: -\frac{i}{\mu}(\bar{Q}_j - \bar{S}_j),\\
&T_{22} =: \frac{2}{\mu^2}\left[\mathcal{H} - \frac{1}{2}(T+\bar{T})\right],\\
&T_{11} =: \frac{1}{2}\left[\mathcal{H} + \frac{1}{2}(T+\bar{T})\right],\\
&T_{12} = T_{21} =: \frac{i}{2\mu}(T-\bar{T}), \qquad \mu \neq 0,\\
&L_{1'1'} =: -iC, \qquad L_{2'2'} =: i\bar{C}, \qquad L_{1'2'} = L_{2'1'} =: -iF,\\
&J^i_j =: -iI^i_j.
\end{aligned} \tag{3.8}$$

Here μ is a real parameter of the mass dimension. In the new basis, the (anti)commutators (3.1), (3.2) are rewritten as

$$\begin{aligned}
&\{Q^i, \bar{Q}_j\} = -2\alpha\mu I^i_j + 2\delta^i_j[\mathcal{H} + (1+\alpha)\mu F],\\
&\{S^i, \bar{S}_j\} = 2\alpha\mu I^i_j + 2\delta^i_j[\mathcal{H} - (1+\alpha)\mu F],\\
&\{S^i, \bar{Q}_j\} = 2\delta^i_j T, \qquad \{Q^i, \bar{S}_j\} = 2\delta^i_j\bar{T},\\
&\{Q^i, S^k\} = -2(1+\alpha)\mu\varepsilon^{ik}C,\\
&\{\bar{Q}_j, \bar{S}_k\} = 2(1+\alpha)\mu\varepsilon_{jk}\bar{C},
\end{aligned} \tag{3.9}$$

[3]More generally, for a complex form of $D(2,1;\alpha)$ with the bosonic subalgebra $sl(2)\oplus sl'(2)\oplus sl''(2)$, the equivalent superalgebras are related through the substitutions $\alpha \to -(1+\alpha), \alpha^{-1}$, and α can be a complex number. The real form of $D(2,1;\alpha)$ we are dealing with here reveals an equivalence only under the substitution $\alpha \to -(1+\alpha)$, with $\alpha \in \mathbb{R}$.

$$
\begin{aligned}
&[I^i_j, I^k_l] = \delta^k_j I^i_l - \delta^i_l I^k_j,\\
&[I^i_j, \bar{Q}_l] = \frac{1}{2}\delta^i_j \bar{Q}_l - \delta^i_l \bar{Q}_j, \quad [I^i_j, Q^k] = \delta^k_j Q^i - \frac{1}{2}\delta^i_j Q^k,\\
&[I^i_j, \bar{S}_l] = \frac{1}{2}\delta^i_j \bar{S}_l - \delta^i_l \bar{S}_j, \quad [I^i_j, S^k] = \delta^k_j S^i - \frac{1}{2}\delta^i_j S^k,
\end{aligned} \tag{3.10}
$$

$$
\begin{aligned}
&[C, \bar{C}] = 2F, \quad [F, C] = C, \quad [F, \bar{C}] = -\bar{C},\\
&[C, \bar{Q}_j] = -\varepsilon_{jl} S^l, \quad [C, \bar{S}_j] = -\varepsilon_{jl} Q^l,\\
&[\bar{C}, Q^i] = -\varepsilon^{ik} \bar{S}_k, \qquad [\bar{C}, S^i] = -\varepsilon^{ik} \bar{Q}_k,\\
&[F, \bar{Q}_l] = -\frac{1}{2}\bar{Q}_l, \quad [F, Q^k] = \frac{1}{2} Q^k,\\
&[F, \bar{S}_l] = -\frac{1}{2}\bar{S}_l, \qquad [F, S^k] = \frac{1}{2} S^k,
\end{aligned} \tag{3.11}
$$

$$
\begin{aligned}
&[T, \bar{T}] = -2\mu\mathcal{H}, \qquad [\mathcal{H}, T] = \mu T, \qquad [\mathcal{H}, \bar{T}] = -\mu\bar{T},\\
&[T, Q^i] = -\mu S^i, \qquad [T, \bar{S}_j] = -\mu \bar{Q}_j,\\
&[\bar{T}, \bar{Q}_j] = \mu \bar{S}_j, \qquad [\bar{T}, S^i] = \mu Q^i,\\
&[\mathcal{H}, \bar{S}_l] = -\frac{\mu}{2}\bar{S}_l, \qquad [\mathcal{H}, S^k] = \frac{\mu}{2} S^k,\\
&[\mathcal{H}, \bar{Q}_l] = \frac{\mu}{2}\bar{Q}_l, \qquad [\mathcal{H}, Q^k] = -\frac{\mu}{2} Q^k.
\end{aligned} \tag{3.12}
$$

The bosonic sector consisting of the three mutually commuting algebras is now given by the following sets of the generators

$$
\begin{aligned}
&su(2) \oplus su'(2) \oplus so(2,1)\\
&\quad \equiv \{I^i_k\} \oplus \{F, C, \bar{C}\} \oplus \{\mathcal{H}, T, \bar{T}\}.
\end{aligned} \tag{3.13}
$$

According to (3.3) and (3.8), the conjugation rules are as follows:

$$
\begin{aligned}
&(Q^k)^\dagger = \bar{Q}_k, \qquad (S^k)^\dagger = \bar{S}_k, \qquad (F)^\dagger = F,\\
&(C)^\dagger = \bar{C}, \qquad (I^k_i)^\dagger = I^i_k, \qquad \mathcal{H}^\dagger = \mathcal{H}, \qquad (T)^\dagger = \bar{T}.
\end{aligned} \tag{3.14}
$$

Note the relation

$$
\mathcal{H} = \hat{H} + \frac{\mu^2}{4}\hat{K}. \tag{3.15}
$$

In the contraction limit $\mu = 0$, the algebra (3.9)–(3.12) becomes a kind of $\mathcal{N} = 8, d = 1$ Poincaré superalgebra (with the common Hamiltonian $\mathcal{H}$) extended by the central charges $T, \bar{T}$ originating from the $so(2,1)$ generators. The remaining two $su(2)$ subalgebras become outer automorphism algebras which form a semidirect product with this $\mathcal{N} = 8, d = 1$ superalgebra.[4] At any $\mu \neq 0$, the relations (3.8) defining the new basis contain no singularities, and so Eqs. (3.9)–(3.12) yield an equivalent form of the original superalgebra $D(2,1;\alpha)$. After coming back to the original superconformal generators, any dependence of the (anti) commutation relations on μ disappears while it still retains in the realizations of $D(2,1;\alpha)$ on the coordinates of the $SU(2|1)$ superspaces (see below). Taking the $\mu = 0$ limit in this basis gives rise to the standard parabolic realizations of $D(2,1;\alpha)$ in the flat $\mathcal{N} = 4, d = 1$ superspaces.

The $su(2|1)$ basis in $D(2,1;\alpha)$ makes manifest some remarkable properties of this superalgebra which are implicit in the "standard" basis:

(i) It is straightforward to see that the superconformal algebra (3.9)–(3.12) includes as a subalgebra the following superalgebra $su(2|1)$:

$$
\begin{aligned}
&\{Q^i, \bar{Q}_j\} = -2\alpha\mu I^i_j + 2\delta^i_j[\mathcal{H} + (1+\alpha)\mu F],\\
&[I^i_j, \bar{Q}_l] = \frac{1}{2}\delta^i_j \bar{Q}_l - \delta^i_l \bar{Q}_j,\\
&[I^i_j, Q^k] = \delta^k_j Q^i - \frac{1}{2}\delta^i_j Q^k,\\
&[F, \bar{Q}_l] = -\frac{1}{2}\bar{Q}_l, \qquad [F, Q^k] = \frac{1}{2} Q^k,\\
&[\mathcal{H}, \bar{Q}_l] = \frac{\mu}{2}\bar{Q}_l, \qquad [\mathcal{H}, Q^k] = -\frac{\mu}{2} Q^k.
\end{aligned} \tag{3.16}
$$

These relations coincide with (2.4) under the following identifications:

$$
m(\mu) = -\alpha\mu, \tag{3.17}
$$

$$
H(\mu) = \mathcal{H} + \mu F. \tag{3.18}
$$

We observe that the closure of the $SU(2|1)$ supercharges depends on the parameter α, because the $SU(2)$ and $SU'(2)$ generators $J_{ij} = -iI_{ij}$ and $L_{i'j'} \sim \{F, C, \bar{C}\}$ appear in the basic anticommutator (3.1) with the factors α, $1+\alpha$, respectively. The $U(1)$ generator F in (3.16) comes from $su'(2)$, while the first $su(2)$ with the generators I_{ij} is just $su(2) \subset su(2|1)$.

(ii) We see from (3.9)–(3.12) that there exists another $su(2|1) \subset D(2,1;\alpha)$ generated by the generators $S^i, \bar{S}_j$ and corresponding to the identification

$$
m(-\mu) = \alpha\mu, \tag{3.19}
$$

$$
H(-\mu) = \mathcal{H} - \mu F \tag{3.20}
$$

in (2.4). Hence, its (anti)commutation relations are obtained from (3.16) via the substitution $\mu \to -\mu$ and passing to the new independent supercharges $(S_i, \bar{S}^j)$. As follows from (3.9)–(3.12), all the remaining generators of $D(2,1;\alpha)$ (i.e. $T, \bar{T}, C, \bar{C}$) appear in the cross-anticommutators of the

[4] The full automorphism group $SO(8)$ of the $\mathcal{N} = 8, d = 1$ superalgebra is broken down to $SO(4) \sim SU(2) \times SU'(2)$ due to the presence of central charges $T, \bar{T}$.

supercharges $(Q^i, \bar{Q}_j)$ with $(S_i, \bar{S}^j)$. Thus, the superalgebra $D(2,1;\alpha)$ can be represented as a closure of its two $su(2|1)$ supersubalgebras: $su(2|1)$ given by the relations (3.16) and another independent $su(2|1)$ with the (anti)commutation relations obtained from those of the former $su(2|1)$ through the replacement $\mu \to -\mu$. We were not able to find such a statement about the structure of $D(2,1;\alpha)$ in the literature. This property is similar to the property that the $\mathcal{N}=1, d=4$ superconformal group $SU(2,2|1)$ can be viewed as a closure of its two different $OSp(1,4)$ subgroups related to each other through the analogous "reflection" of the anti–de Sitter radius as a parameter of contraction to the flat $\mathcal{N}=1, d=4$ Poincaré supersymmetry [19]. In what follows, this observation will be useful for constructing $D(2,1;\alpha)$ invariant subclasses of the $SU(2|1)$ invariant actions.

(iii) In the cases $\alpha=-1$ and $\alpha=0$, the supergroup $D(2,1;\alpha)$ is reduced to the semidirect product (3.7), with $SU(2)_{\rm ext}$ being generated, respectively, by $L_{i',j'}$ or $J_{ij}=-iI_{ij}$. The remaining $SU(2)$ subgroups enter the relevant $PSU(1,1|2)$ factors. Each of the corresponding superalgebras $psu(1,1|2)$ can still be interpreted as a closure of its two $su(2|1)$ subalgebras, like in the case of $\alpha \in \mathbb{R}\backslash\{0\}, \mathbb{R}\backslash\{-1\}$. In particular, the superalgebra (3.16) at $\alpha=-1$ is identical to (2.1) with $m=\mu$ and $\mathcal{H}$ as the $U(1)$ generator. The generator F splits off as an external automorphism.

(iv) One more peculiarity is associated with the presence of the "composite" deformation parameter $m=-\alpha\mu$ in (3.16). It vanishes not only in the standard contraction limit $\mu=0$, but also at $\alpha=0$ with $\mu \neq 0$. For $\alpha=0$, the superalgebra (3.16) is reduced to the flat $\mathcal{N}=4$ superalgebra

$$\{Q^i, \bar{Q}_j\} = 2\delta^i_j(\mathcal{H}+\mu F), \quad [F, \bar{Q}_l] = -\frac{1}{2}\bar{Q}_l, \quad [F, Q^k] = \frac{1}{2}Q^k, \quad [\mathcal{H}, \bar{Q}_l] = \frac{\mu}{2}\bar{Q}_l, \quad [\mathcal{H}, Q^k] = -\frac{\mu}{2}Q^k. \tag{3.21}$$

This algebra is still a subalgebra of $D(2,1;\alpha=0)$. However, it does not coincide with the standard flat $\mathcal{N}=4, d=1$ Poincaré superalgebra corresponding to the limit $\mu=0$, because the rhs of the anticommutator in (3.21) still involves μ and is a sum of $\mathcal{H}$ and the internal $U(1)$ charge F. The $SU(2)$ generators I^j_i now define automorphisms of both the superalgebra (3.21) and the $\alpha=0$ superalgebra $psu(1,1|2)$, while F is an internal $U(1)$ generator. The whole $D(2,1;\alpha=0)$ superalgebra [including the $so(2,1)$ generators and those of the $su'(2) \sim \{F, C, \bar{C}\}$] can now be treated as a closure of the superalgebra (3.21) and its $\mu \to -\mu$ counterpart.[5]

To avoid a confusion, let us point out that both (3.21) and (3.4) can of course be regarded as the Poincaré $\mathcal{N}=4, d=1$ superalgebras. However, in contrast to (3.4), the superalgebra (3.21) is embedded in the superconformal algebra in a different way, with the Hamiltonian $H=\mathcal{H}+\mu F$ defined in (3.18), instead of the standard $\hat{H}$ in (3.4) [recall Eq. (3.15)]. In the limit $\mu \to 0$, any difference between $\mathcal{H}$, H and $\hat{H}$ disappears.

(v) It is worth noting that the parameter α characterizes only the superconformal mechanics models, while the generic $SU(2|1)$ models lack any dependence on it. So in the case of superconformal models, we deal with the pair of parameters, α and μ. In the particular case $\alpha=-1$, we have $m=\mu$.

(vi) Besides the $SU(2|1)$ superspaces (2.5), (2.12), we can now consider another type of the $SU(2|1)$ superspace defined as the supercoset:

$$\frac{SU(2|1) \rtimes U(1)_{\rm ext}}{SU(2)\times U(1)_{\rm int}} \sim \frac{\{Q^i, \bar{Q}_j, \mathcal{H}, F, I^i_j\}}{\{I^i_j, F\}}. \tag{3.22}$$

According to (3.13), this definition of superspace matches to the proper embedding of $SU(2|1)$ in $D(2,1;\alpha)$ for $\alpha \in \mathbb{R}\backslash\{0\}$:

$$\begin{array}{ccccc} D(2,1;\alpha) & \xrightarrow{\rm bos} & SU(2)\times & SU(2) \times & SO(2,1), \\ \downarrow & & \downarrow I^j_i & \downarrow F & \downarrow \mathcal{H} \\ SU(2|1)\rtimes U(1)_{\rm ext} & \xrightarrow{\rm bos} & SU(2)\times & U(1)_{\rm ext}\times & U(1). \end{array} \tag{3.23}$$

In the case $\alpha=-1$ corresponding to the second line in (3.23), one can omit the generator F in (3.22) since it becomes an external automorphism. So (3.22) is reduced to (2.12) in this case. For generic $\alpha \in \mathbb{R}\backslash\{-1\}$, the coset (3.22) "interpolates" between (2.5) and (2.12), since F appears in the rhs of the anticommutator in (3.16) along with the generator $\mathcal{H}$, and so cannot be decoupled.

[5]In the $\alpha=0$ case, one can still define $su(2|1) \subset D(2,1;\alpha=0)$, which involves $su'(2) \sim \{F, C, \bar{C}\}$ as the internal subalgebra, as well as the proper analog of the $U(1)$ generator $\mathcal{H}$.

(vii) In the limit $\alpha = 0$, the relevant coset is

$$\frac{(\mathcal{N}=4, d=1) \rtimes U(1)_{\rm ext}}{U(1)_{\rm int}} \sim \frac{\{Q^i, \bar{Q}_j, \mathcal{H}, F\}}{\{F\}}, \tag{3.24}$$

where $(\mathcal{N}=4, d=1) \rtimes U(1)_{\rm ext}$ stands for the semidirect product of the supergroup with the algebra (3.21) and the external $U(1)$ automorphism generated by $F \in \{F, C, \bar{C}\}$. We can deal with the coset superspace (3.24) in the standard manner, just substituting $\alpha = 0$ into all the relations of the $SU(2|1)$ superspace formalism pertinent to the choice (3.22).

A. Superconformal generators

Superconformal generators of (3.9)–(3.12) can be naturally realized on the $SU(2|1)$ superspace (3.22). An element of this supercoset is defined as

$$g_1 = \exp\left\{\left(1 + \frac{2\alpha\mu}{3}\bar{\theta}^k\theta_k\right)(\theta_i Q^i + \bar{\theta}^j\bar{Q}_j)\right\}\exp\{it\mathcal{H}\}, \tag{3.25}$$

where the superspace coordinates $\{t, \theta_i, \bar{\theta}^k\}$ coincide with those defined in (2.6). Because of the relation (3.18), the coset elements (3.25) and (2.6) are related as

$$g_1 = g\exp\{-i\mu t F\}. \tag{3.26}$$

In the particular case $\alpha = 0$, the relevant superspace coset (3.24) is parametrized by the flat superspace coordinates $\zeta_{(\alpha=0)} = \{t, \theta_i, \bar{\theta}^k\}$. An element of this coset is obtained by setting $\alpha = 0$ in (3.25).

Dropping matrix parts of generators, one can obtain the $SU(2|1)$ supercharges for generic α just through the substitution $m = -\alpha\mu$ in (2.7):

$$Q^i = \frac{\partial}{\partial\theta_i} + 2\alpha\mu\bar{\theta}^i\bar{\theta}^k\frac{\partial}{\partial\bar{\theta}^k} + i\bar{\theta}^i\partial_t,$$
$$\bar{Q}_j = \frac{\partial}{\partial\bar{\theta}^j} - 2\alpha\mu\theta_j\theta_k\frac{\partial}{\partial\theta_k} + i\theta_j\partial_t. \tag{3.27}$$

They generate the $su(2|1)$ superalgebra (3.16) with the bosonic generators

$$I^i_j = \left(\bar{\theta}^i\frac{\partial}{\partial\bar{\theta}^j} - \theta_j\frac{\partial}{\partial\theta_i}\right) - \frac{1}{2}\delta^i_j\left(\bar{\theta}^k\frac{\partial}{\partial\bar{\theta}^k} - \theta_k\frac{\partial}{\partial\theta_k}\right),$$
$$\mathcal{H} = i\partial_t - \frac{\mu}{2}\left(\bar{\theta}^k\frac{\partial}{\partial\bar{\theta}^k} - \theta_k\frac{\partial}{\partial\theta_k}\right),$$
$$F = \frac{1}{2}\left(\bar{\theta}^k\frac{\partial}{\partial\bar{\theta}^k} - \theta_k\frac{\partial}{\partial\theta_k}\right). \tag{3.28}$$

The extra supercharges of the superconformal algebra $D(2,1;\alpha)$ are defined as

$$S^i = e^{-i\mu t}\left\{\left[1 - (1+2\alpha)\mu\bar{\theta}^k\theta_k - \frac{1}{4}(1+2\alpha)^2\mu^2(\theta)^2(\bar{\theta})^2\right]\frac{\partial}{\partial\theta_i} + 2(1+\alpha)\mu\bar{\theta}^i\theta_k\frac{\partial}{\partial\theta_k} + i\bar{\theta}^i[1 + (1+2\alpha)\mu\bar{\theta}^k\theta_k]\partial_t\right\},$$
$$\bar{S}_j = e^{i\mu t}\left\{\left[1 - (1+2\alpha)\mu\bar{\theta}^k\theta_k - \frac{1}{4}(1+2\alpha)^2\mu^2(\theta)^2(\bar{\theta})^2\right]\frac{\partial}{\partial\bar{\theta}^j} - 2(1+\alpha)\mu\theta_j\bar{\theta}^k\frac{\partial}{\partial\bar{\theta}^k} + i\theta_j[1 + (1+2\alpha)\mu\bar{\theta}^k\theta_k]\partial_t\right\}. \tag{3.29}$$

The anticommutators of (3.27) with (3.29) give the new bosonic generators

$$T = e^{-i\mu t}\left\{i\left[1 - \frac{1}{4}(1+2\alpha)\mu^2(\theta)^2(\bar{\theta})^2\right]\partial_t + \mu[1 - (1+2\alpha)\mu\bar{\theta}^k\theta_k]\theta_i\frac{\partial}{\partial\theta_i}\right\},$$
$$\bar{T} = e^{i\mu t}\left\{i\left[1 - \frac{1}{4}(1+2\alpha)\mu^2(\theta)^2(\bar{\theta})^2\right]\partial_t - \mu[1 - (1+2\alpha)\mu\bar{\theta}^k\theta_k]\bar{\theta}^i\frac{\partial}{\partial\bar{\theta}^i}\right\},$$
$$C = e^{-i\mu t}\varepsilon_{jl}[1 + (1+2\alpha)\mu\bar{\theta}^k\theta_k]\bar{\theta}^j\frac{\partial}{\partial\theta_l},$$
$$\bar{C} = e^{i\mu t}\varepsilon^{jl}[1 + (1+2\alpha)\mu\bar{\theta}^k\theta_k]\theta_j\frac{\partial}{\partial\bar{\theta}^l}. \tag{3.30}$$

Under the ε, $\bar{\varepsilon}$ transformations generated by (3.29),

$$\delta\theta_i = \left[1-(1+2\alpha)\mu\bar{\theta}^k\theta_k - \frac{1}{4}(1+2\alpha)^2\mu^2(\theta)^2(\bar{\theta})^2\right]\varepsilon_i e^{-i\mu t} + 2(1+\alpha)\mu\varepsilon_k\bar{\theta}^k\theta_i e^{-i\mu t},$$
$$\delta\bar{\theta}^i = \left[1-(1+2\alpha)\mu\bar{\theta}^k\theta_k - \frac{1}{4}(1+2\alpha)^2\mu^2(\theta)^2(\bar{\theta})^2\right]\bar{\varepsilon}^i e^{i\mu t} - 2(1+\alpha)\mu\bar{\varepsilon}^k\theta_k\bar{\theta}^i e^{i\mu t},$$
$$\delta t = i(\bar{\varepsilon}^k\theta_k e^{i\mu t} + \varepsilon_k\bar{\theta}^k e^{-i\mu t})[1+(1+2\alpha)\mu\bar{\theta}^k\theta_k], \tag{3.31}$$

the $SU(2|1)$ invariant measure (2.10) is transformed as

$$\delta_\varepsilon d\zeta = 2\mu d\zeta(1-\mu\bar{\theta}^k\theta_k)(\bar{\varepsilon}^i\theta_i e^{i\mu t} - \varepsilon_i\bar{\theta}^i e^{-i\mu t}). \tag{3.32}$$

Starting from the new coset given by (3.25) and taking advantage of the relation (3.26), one can calculate the relevant covariant derivatives

$$\mathcal{D}^i = e^{-\frac{i}{2}\mu t}\left\{\left[1-\alpha\mu\bar{\theta}^k\theta_k - \frac{3}{8}\alpha^2\mu^2(\theta)^2(\bar{\theta})^2\right]\frac{\partial}{\partial\theta_i} + \alpha\mu\bar{\theta}^i\theta_j\frac{\partial}{\partial\theta_j} - i\bar{\theta}^i\partial_t - (1+\alpha)\mu\bar{\theta}^i\tilde{F} + \alpha\mu\bar{\theta}^j(1+\alpha\mu\bar{\theta}^k\theta_k)\tilde{I}^i_j\right\},$$
$$\bar{\mathcal{D}}_j = e^{\frac{i}{2}\mu t}\left\{-\left[1-\alpha\mu\bar{\theta}^k\theta_k - \frac{3}{8}\alpha^2\mu^2(\theta)^2(\bar{\theta})^2\right]\frac{\partial}{\partial\bar{\theta}^j} - \alpha\mu\bar{\theta}^k\theta_j\frac{\partial}{\partial\bar{\theta}^k} + i\theta_j\partial_t + (1+\alpha)\mu\theta_j\tilde{F} - \alpha\mu\theta_k(1+\alpha\mu\bar{\theta}^k\theta_k)\tilde{I}^k_j\right\},$$
$$\mathcal{D}_{(t)} = \partial_t. \tag{3.33}$$

Together with the matrix generators $\tilde{I}^i_k$, $\tilde{F}$ they mimic the superalgebra (3.16). In the particular case $\alpha=-1$, the matrix generator $\tilde{F}$ drops out from (3.33), which is consistent with the superalgebra (3.16) at $\alpha=-1$ [5]. In this case, the supercoset (3.22), (3.25) is reduced to the supercoset (2.12) with $\tilde{H}=\mathcal{H}$ and $m=\mu$. In the case $\alpha=0$, the generators $\tilde{I}^i_k$ drop out (they become the outer automorphism ones). The $\alpha=0$ covariant derivatives correspond to the degenerate supercoset (3.24).

The redefinition (3.17) allows one to avoid singularities at $\alpha=0$. Taking $\alpha=0$ in the superconformal generators (3.27)–(3.30), one can naturally pass to the generators corresponding to the coset space (3.24) with the relevant algebra (3.21). Thus, within the $SU(2|1)$ superspace defined as the supercoset (3.22) with the elements (3.25), the superspace realization of superconformal generators has been written in the universal form consistent with both choices $\alpha=0$ and $\alpha\neq0$, i.e., with any choice of $\alpha\in\mathbb{R}$. This refers to the covariant derivatives (3.33) as well.

Any dependence of the superalgebra relations (3.9)–(3.12) on the dimensionful parameter μ naturally disappears after passing to the original basis (3.1), (3.2). However, in the realization of the generators (3.8) on the superspace coordinates, the dependence on μ is still retained. Thus, the parameter μ is a deformation parameter of the particular *superspace realization* of (3.1), (3.2). This new deformed realization corresponds to the trigonometric type of $\mathcal{N}=4$ superconformal mechanics [14]. Sending $\mu\to0$ in these realizations (and in the corresponding realizations on the $d=1$ fields) reduces the deformed superconformal models to the standard superconformal mechanics models of the parabolic type [6,12,13].

To be more precise, the trigonometric form of the conformal generators $\{\mathcal{H},T,\bar{T}\}$,

$$\mathcal{H}=i\partial_t, \qquad T=ie^{-i\mu t}\partial_t, \qquad \bar{T}=ie^{i\mu t}\partial_t, \tag{3.34}$$

is obtained as the bosonic truncations of the generators defined by Eqs. (3.28), (3.30) (or an alternative realization of these generators given in the next subsection). The standard $so(2,1)$ generators $\hat{H},\hat{K}$ and $\hat{D}$ defined in (3.5) and (3.8) are expressed, respectively, as

$$\hat{H}=\frac{i}{2}(1+\cos\mu t)\partial_t, \qquad \hat{K}=\frac{2i}{\mu^2}(1-\cos\mu t)\partial_t,$$
$$\hat{D}=\frac{i}{\mu}\sin\mu t\partial_t, \qquad \mu\neq0. \tag{3.35}$$

These generators satisfy the conventional relations of the $d=1$ conformal algebra:

$$[\hat{D},\hat{H}]=-i\hat{H}, \quad [\hat{D},\hat{K}]=i\hat{K}, \quad [\hat{H},\hat{K}]=2i\hat{D}. \tag{3.36}$$

Thus, the definition of conformal superalgebra by Eqs. (3.9)–(3.12) automatically provides the trigonometric form for the conformal algebra $so(2,1)$ [10].

In the limit $\mu\to0$, the generators (3.35) turn into the standard parabolic generators

$$\hat{H}=i\partial_t, \qquad \hat{D}=it\partial_t, \qquad \hat{K}=it^2\partial_t. \tag{3.37}$$

The same properties are inherent to the total set of the $D(2,1;\alpha)$ generators (3.8) for $\mu\neq0$. Thus, we treat the superspace realization of the superconformal symmetry generators found in this paper as a trigonometric deformation of the parabolic $\mathcal{N}=4, d=1$ superconformal generators constructed in Refs. [6,12,13].

The main reason for considering the basis (3.34) is that the generator $\mathcal{H}=\hat{H}+\frac{1}{4}\mu^2\hat{K}$ is directly given by the time derivative, $\mathcal{H}=i\partial_t$ [10]. Another peculiarity of this basis concerns the Cartan generator (diagonal generator) of conformal algebra [11]. In (3.34) we have the Hamiltonian $\mathcal{H}$ as the Cartan generator, while in the parabolic basis (3.37) the Cartan generator of $so(2,1)$ is associated with the dilatation generator $\hat{D}$. Thus, the relevant quantum mechanical system must be solved in terms of eigenvalues and eigenstates of the quantum Hamiltonian $\mathcal{H}=\hat{H}+\frac{1}{4}\mu^2\hat{K}$, which just coincides with the "improved" Hamiltonian of the $d=1$ conformal mechanics [11], ensuring the energy spectrum to be bounded from below.[6] In the next subsection, we will demonstrate that there is a basis in the $SU(2|1)$ superspace in which the full generator $\mathcal{H}$ defined in (3.28) (not only its bosonic truncation) becomes just $i\partial_t$.

B. An alternative realization of superconformal generators

According to (3.9), the supercharges Q form the $su(2|1)$ superalgebra with the deformation parameter μ, while the supercharges S form the $su(2|1)$ superalgebra with $-\mu$. Analogously, in the case $\alpha=0$, the relevant deformed superalgebras are (3.21) and its $-\mu$ counterpart. In the limit $\mu=0$ both sets of supercharges reproduce the same flat $\mathcal{N}=4, d=1$ supercharges.

Here we demonstrate that, after the appropriate redefinition of the $SU(2|1)$ superspace coordinates, the whole set of the superconformal generators can be constructed in terms of the pair of deformed supercharges $Q(\mu)$ and $S(\mu)\equiv Q(-\mu)$. This explains why the $SU(2|1)$ and superconformal transformations of the component fields obtained below for the multiplets $(\mathbf{1},\mathbf{4},\mathbf{3})$, $(\mathbf{2},\mathbf{4},\mathbf{2})$ can be represented as deformations of the standard $\mathcal{N}=4$, $d=1$ transformations of component fields, with the deformation parameters μ and $-\mu$, respectively.

The new coordinates $\{t,\tilde{\theta}_j,\bar{\tilde{\theta}}^i\}$ represent the same supercoset (3.22) and are related to the previously employed supercoordinates as

$$\tilde{\theta}_j = e^{\frac{i}{2}\mu t}\theta_j\left[1+\frac{1}{2}(1+2\alpha)\mu\bar{\theta}^k\theta_k\right],\qquad \bar{\tilde{\theta}}^i=\overline{(\tilde{\theta}_j)}=e^{-\frac{i}{2}\mu t}\bar{\theta}^i\left[1+\frac{1}{2}(1+2\alpha)\mu\bar{\theta}^k\theta_k\right]. \tag{3.38}$$

The supercharges (3.27) are rewritten as

$$\begin{aligned}Q^i &= e^{\frac{i}{2}\mu t}\Big\{\left[1+\frac{1}{2}(1+2\alpha)\mu\bar{\tilde{\theta}}^k\tilde{\theta}_k-\frac{1}{16}(1+2\alpha)\mu^2(\tilde{\theta})^2(\bar{\tilde{\theta}})^2\right]\frac{\partial}{\partial\tilde{\theta}_i}-(1+\alpha)\mu\bar{\tilde{\theta}}^i\tilde{\theta}_k\frac{\partial}{\partial\tilde{\theta}_k}\\&\quad+\alpha\mu\bar{\tilde{\theta}}^i\bar{\tilde{\theta}}^k\frac{\partial}{\partial\bar{\tilde{\theta}}^k}+i\bar{\tilde{\theta}}^i\left[1-\frac{1}{2}(1+2\alpha)\mu\bar{\tilde{\theta}}^k\tilde{\theta}_k\right]\partial_t\Big\},\\ \bar{Q}_j &= e^{-\frac{i}{2}\mu t}\Big\{\left[1+\frac{1}{2}(1+2\alpha)\mu\bar{\tilde{\theta}}^k\tilde{\theta}_k-\frac{1}{16}(1+2\alpha)\mu^2(\tilde{\theta})^2(\bar{\tilde{\theta}})^2\right]\frac{\partial}{\partial\bar{\tilde{\theta}}^j}+(1+\alpha)\mu\tilde{\theta}_j\bar{\tilde{\theta}}^k\frac{\partial}{\partial\bar{\tilde{\theta}}^k}\\&\quad-\alpha\mu\tilde{\theta}_j\tilde{\theta}_k\frac{\partial}{\partial\tilde{\theta}_k}+i\tilde{\theta}_j\left[1-\frac{1}{2}(1+2\alpha)\mu\bar{\tilde{\theta}}^k\tilde{\theta}_k\right]\partial_t\Big\}.\end{aligned} \tag{3.39}$$

The new form of the supercharges (3.29) is given by

$$\begin{aligned}S^i &= e^{-\frac{i}{2}\mu t}\Big\{\left[1-\frac{1}{2}(1+2\alpha)\mu\bar{\tilde{\theta}}^k\tilde{\theta}_k-\frac{1}{16}(1+2\alpha)\mu^2(\tilde{\theta})^2(\bar{\tilde{\theta}})^2\right]\frac{\partial}{\partial\tilde{\theta}_i}+(1+\alpha)\mu\bar{\tilde{\theta}}^i\tilde{\theta}_k\frac{\partial}{\partial\tilde{\theta}_k}\\&\quad-\alpha\mu\bar{\tilde{\theta}}^i\bar{\tilde{\theta}}^k\frac{\partial}{\partial\bar{\tilde{\theta}}^k}+i\bar{\tilde{\theta}}^i\left[1+\frac{1}{2}(1+2\alpha)\mu\bar{\tilde{\theta}}^k\tilde{\theta}_k\right]\partial_t\Big\},\\ \bar{S}_j &= e^{\frac{i}{2}\mu t}\Big\{\left[1-\frac{1}{2}(1+2\alpha)\mu\bar{\tilde{\theta}}^k\tilde{\theta}_k-\frac{1}{16}(1+2\alpha)\mu^2(\tilde{\theta})^2(\bar{\tilde{\theta}})^2\right]\frac{\partial}{\partial\bar{\tilde{\theta}}^j}-(1+\alpha)\mu\tilde{\theta}_j\bar{\tilde{\theta}}^k\frac{\partial}{\partial\bar{\tilde{\theta}}^k}\\&\quad+\alpha\mu\tilde{\theta}_j\tilde{\theta}_k\frac{\partial}{\partial\tilde{\theta}_k}+i\tilde{\theta}_j\left[1+\frac{1}{2}(1+2\alpha)\mu\bar{\tilde{\theta}}^k\tilde{\theta}_k\right]\partial_t\Big\}.\end{aligned} \tag{3.40}$$

[6]The orthogonal combination $\mathcal{H}_{\rm h}=\hat{H}-\frac{1}{4}\mu^2\hat{K}$ corresponds to the hyperbolic case discussed in Appendix C. It yields a non-unitary model.

We observe that they are obtained from the supercharges (3.39) just through the change of the sign of μ, $S(\mu) \equiv Q(-\mu)$. The bosonic generators (3.28) of $SU(2|1)$ are written as

$$I^i_j = \left(\bar{\tilde{\theta}}^i \frac{\partial}{\partial \bar{\tilde{\theta}}^j} - \tilde{\theta}_j \frac{\partial}{\partial \tilde{\theta}_i}\right) - \frac{1}{2}\delta^i_j \left(\bar{\tilde{\theta}}^k \frac{\partial}{\partial \bar{\tilde{\theta}}^k} - \tilde{\theta}_k \frac{\partial}{\partial \tilde{\theta}_k}\right),$$
$$F = \frac{1}{2}\left(\bar{\tilde{\theta}}^k \frac{\partial}{\partial \bar{\tilde{\theta}}^k} - \tilde{\theta}_k \frac{\partial}{\partial \tilde{\theta}_k}\right), \qquad \mathcal{H} = i\partial_t. \tag{3.41}$$

In this new realization the Hamiltonian $\mathcal{H}$ takes the correct form as the time-translation generator. The rest of the bosonic generators (3.30) is rewritten as

$$T = e^{-i\mu t}\left\{ i\left[1 - \frac{1}{4}(1+2\alpha)\mu^2(\tilde{\theta})^2(\bar{\tilde{\theta}})^2\right]\partial_t + \frac{\mu}{2}\left(\bar{\tilde{\theta}}^k \frac{\partial}{\partial \bar{\tilde{\theta}}^k} + \tilde{\theta}_k \frac{\partial}{\partial \tilde{\theta}_k}\right) + \frac{1}{2}(1+2\alpha)\mu^2 \bar{\tilde{\theta}}^i \tilde{\theta}_i \left(\bar{\tilde{\theta}}^k \frac{\partial}{\partial \bar{\tilde{\theta}}^k} - \tilde{\theta}_k \frac{\partial}{\partial \tilde{\theta}_k}\right)\right\},$$
$$\bar{T} = e^{i\mu t}\left\{ i\left[1 - \frac{1}{4}(1+2\alpha)\mu^2(\tilde{\theta})^2(\bar{\tilde{\theta}})^2\right]\partial_t - \frac{\mu}{2}\left(\bar{\tilde{\theta}}^k \frac{\partial}{\partial \bar{\tilde{\theta}}^k} + \tilde{\theta}_k \frac{\partial}{\partial \tilde{\theta}_k}\right) + \frac{1}{2}(1+2\alpha)\mu^2 \bar{\tilde{\theta}}^i \tilde{\theta}_i \left(\bar{\tilde{\theta}}^k \frac{\partial}{\partial \bar{\tilde{\theta}}^k} - \tilde{\theta}_k \frac{\partial}{\partial \tilde{\theta}_k}\right)\right\},$$
$$C = \varepsilon_{jl}\bar{\tilde{\theta}}^j \frac{\partial}{\partial \tilde{\theta}_l}, \qquad \bar{C} = \varepsilon^{jl}\tilde{\theta}_j \frac{\partial}{\partial \bar{\tilde{\theta}}^l}. \tag{3.42}$$

Note that the supercharges (3.39), (3.40) acquired the exponential factors $\sim e^{\pm \frac{i}{2}\mu t}$, which are needed for ensuring the correct commutation relations with $\mathcal{H} = i\partial_t$. Also note that the $su(2)$ and $su'(2)$ generators now include no μ dependence at all, while the $so(2,1)$ generators T and $\bar{T}$ are just related by the reflection $\mu \leftrightarrow -\mu, T(-\mu) = \bar{T}(\mu)$. So the property that the whole superalgebra $D(2,1;\alpha)$ is contained in the closure of $(Q_i(\mu), \bar{Q}^j(\mu))$ and $(S_i(\mu) = Q_i(-\mu), \bar{S}^j(\mu) = \bar{Q}^j(-\mu))$ becomes manifest in the new parametrization of the $SU(2|1)$ superspace.

For further use, we give the new basis form of the $SU(2|1)$ invariant measure (2.10):

$$d\tilde{\zeta} = dt d^2\tilde{\theta} d^2\bar{\tilde{\theta}}(1 + \mu \bar{\tilde{\theta}}^k \tilde{\theta}_k). \tag{3.43}$$

Under the ε, $\bar{\varepsilon}$ transformations generated by (3.40), it is transformed as

$$\delta_\varepsilon d\tilde{\zeta} = 2\mu d\tilde{\zeta}\left[1 - \frac{1}{2}(3+2\alpha)\mu \bar{\tilde{\theta}}^k \tilde{\theta}_k\right](\bar{\varepsilon}^i \tilde{\theta}_i e^{\frac{i}{2}\mu t} - \varepsilon_i \bar{\tilde{\theta}}^i e^{-\frac{i}{2}\mu t}). \tag{3.44}$$

IV. THE MULTIPLET (1, 4, 3)

A. Constraints

The multiplet $(\mathbf{1},\mathbf{4},\mathbf{3})$ was described in Ref. [4] in the framework of the $SU(2|1)$ superspace (2.5). It is represented by the real neutral superfield G satisfying the $SU(2|1)$ covariantization of the standard $(\mathbf{1},\mathbf{4},\mathbf{3})$ multiplet constraints

$$\varepsilon^{lj}\bar{\mathcal{D}}_l\bar{\mathcal{D}}_j G = \varepsilon_{lj}\mathcal{D}^l\mathcal{D}^j G = 0, \qquad [\mathcal{D}^i, \bar{\mathcal{D}}_i]G = 4mG. \tag{4.1}$$

They are solved by

$$G = [1 - m\bar{\theta}^k\theta_k + m^2(\theta)^2(\bar{\theta})^2]x + \frac{\ddot{x}}{4}(\theta)^2(\bar{\theta})^2 - i\bar{\theta}^k\theta_k(\theta_i\dot{\psi}^i + \bar{\theta}^j\dot{\bar{\psi}}_j) + (1 - 2m\bar{\theta}^k\theta_k)(\theta_i\psi^i - \bar{\theta}^j\bar{\psi}_j) + \bar{\theta}^j\theta_i B^i_j, \qquad B^k_k = 0. \tag{4.2}$$

For studying superconformal properties of this $SU(2|1)$ supermultiplet, it will be more convenient to reformulate it in the superspace (3.22). By rewriting the constraints (4.1) through the covariant derivatives (3.33) as

$$\varepsilon^{lj}\bar{\mathcal{D}}_l\bar{\mathcal{D}}_j G = \varepsilon_{lj}\mathcal{D}^l\mathcal{D}^j G = 0, \qquad [\mathcal{D}^i, \bar{\mathcal{D}}_i]G = -4\alpha\mu G, \tag{4.3}$$

we obtain

$$G = x[1 + \alpha\mu\bar{\theta}^k\theta_k + \alpha^2\mu^2(\theta)^2(\bar{\theta})^2] + \frac{\ddot{x}}{4}(\theta)^2(\bar{\theta})^2 - i\bar{\theta}^k\theta_k(\theta_i\dot{\psi}^i e^{\frac{i}{2}\mu t} + \bar{\theta}^j\dot{\bar{\psi}}_j e^{-\frac{i}{2}\mu t}) + \left[1 + \frac{1}{2}(1+4\alpha)\mu\bar{\theta}^k\theta_k\right](\theta_i\psi^i e^{\frac{i}{2}\mu t} - \bar{\theta}^j\bar{\psi}_j e^{-\frac{i}{2}\mu t}) + \bar{\theta}^j\theta_i B^i_j, \tag{4.4}$$

where we have redefined

$$\psi^i \to \psi^i e^{\frac{i}{2}\mu t}, \qquad \bar{\psi}_j \to \bar{\psi}_j e^{-\frac{i}{2}\mu t}. \tag{4.5}$$

This field redefinition makes the $U(1)$ generator F act only on fermionic fields and ensures that the operator $\mathcal{H}$ is realized on the component fields as the pure time derivative $i\partial_t$ without additional $U(1)$ rotation terms. We see that the irreducible set of the off-shell component fields is $x(t)$, $\psi^i(t), \bar{\psi}_i(t), B^i_j(t)$ ($B^k_k = 0$), i.e., G reveals just the $(\mathbf{1},\mathbf{4},\mathbf{3})$ content. In the contraction limit $\mu = 0$, it is reduced to the ordinary $(\mathbf{1},\mathbf{4},\mathbf{3})$ superfield.

As the most important requirement, the constraints (4.3) [rewritten through the covariant derivatives (3.33)] must be covariant under the superconformal symmetry $D(2,1;\alpha)$. From this requirement, one can actually restore the supercharges (3.29) and the bosonic generators (3.30) as the differential operators acting on the superspace (3.22). Moreover, it implies that these extra generators for the multiplet $(\mathbf{1},\mathbf{4},\mathbf{3})$ should be extended by the proper weight terms. The supercharges (3.29) are extended as

$$\begin{aligned}
S^i &= e^{-i\mu t}\Big\{\Big[1-(1+2\alpha)\mu\bar{\theta}^k\theta_k-\frac{1}{4}(1+2\alpha)^2\mu^2(\theta)^2(\bar{\theta})^2\Big]\frac{\partial}{\partial\theta_i}+2(1+\alpha)\mu\bar{\theta}^i\theta_k\frac{\partial}{\partial\theta_k}\\
&\quad+i\bar{\theta}^i[1+(1+2\alpha)\mu\bar{\theta}^k\theta_k]\partial_t+2\alpha\mu\bar{\theta}^i(1-\mu\bar{\theta}^k\theta_k)\Big\},\\
\bar{S}_j &= e^{i\mu t}\Big\{\Big[1-(1+2\alpha)\mu\bar{\theta}^k\theta_k-\frac{1}{4}(1+2\alpha)^2\mu^2(\theta)^2(\bar{\theta})^2\Big]\frac{\partial}{\partial\bar{\theta}^j}-2(1+\alpha)\mu\theta_j\bar{\theta}^k\frac{\partial}{\partial\bar{\theta}^k}\\
&\quad+i\theta_j[1+(1+2\alpha)\mu\bar{\theta}^k\theta_k]\partial_t-2\alpha\mu\theta_j(1-\mu\bar{\theta}^k\theta_k)\Big\}.
\end{aligned} \tag{4.6}$$

Respectively, the bosonic generators are modified as

$$\begin{aligned}
T &= e^{-i\mu t}\Big\{i\Big[1-\frac{1}{4}(1+2\alpha)\mu^2(\theta)^2(\bar{\theta})^2\Big]\partial_t+\mu[1-(1+2\alpha)\mu\bar{\theta}^k\theta_k]\theta_i\frac{\partial}{\partial\theta_i}\Big\}+\alpha\mu e^{-i\mu t}\Big[1-\mu\bar{\theta}^k\theta_k+\frac{1}{4}(1-2\alpha)\mu^2(\theta)^2(\bar{\theta})^2\Big],\\
\bar{T} &= e^{i\mu t}\Big\{i\Big[1-\frac{1}{4}(1+2\alpha)\mu^2(\theta)^2(\bar{\theta})^2\Big]\partial_t-\mu[1-(1+2\alpha)\mu\bar{\theta}^k\theta_k]\bar{\theta}^i\frac{\partial}{\partial\bar{\theta}^i}\Big\}-\alpha\mu e^{i\mu t}\Big[1-\mu\bar{\theta}^k\theta_k+\frac{1}{4}(1-2\alpha)\mu^2(\theta)^2(\bar{\theta})^2\Big],\\
C &= e^{-i\mu t}\varepsilon_{jl}[1+(1+2\alpha)\mu\bar{\theta}^k\theta_k]\bar{\theta}^j\frac{\partial}{\partial\theta_l}+\alpha\mu(\bar{\theta})^2e^{-i\mu t},\\
\bar{C} &= e^{i\mu t}\varepsilon^{jl}[1+(1+2\alpha)\mu\bar{\theta}^k\theta_k]\theta_j\frac{\partial}{\partial\bar{\theta}^l}-\alpha\mu(\theta)^2e^{i\mu t}.
\end{aligned} \tag{4.7}$$

These modifications of the additional $D(2,1;\alpha)$ generators imply the following "passive" transformation law for the superfield G under the $\varepsilon_i, \bar{\varepsilon}^j$ transformations:

$$\delta_\varepsilon G = 2\alpha\mu(1-\mu\bar{\theta}^k\theta_k)(\bar{\varepsilon}^i\theta_i e^{i\mu t}-\varepsilon_i\bar{\theta}^i e^{-i\mu t})G. \tag{4.8}$$

All other transformations are produced by commuting (4.8) with the odd $SU(2|1)$ transformations which are generated by the pure differential operators (3.27). It is worth pointing out once more that all additional weight terms in the $D(2,1;\alpha)$ generators are necessary for the $D(2,1;\alpha)$ covariance of the $(\mathbf{1},\mathbf{4},\mathbf{3})$ constraints (4.3) and, in fact, can be deduced from requiring this covariance. Making the bosonic truncation of the conformal generators with the weight terms,

$$\mathcal{H} = i\partial_t,\qquad T = e^{-i\mu t}(i\partial_t+\alpha\mu),\qquad \bar{T} = e^{i\mu t}(i\partial_t-\alpha\mu), \tag{4.9}$$

one observes that α can be identified with the scaling dimension parameter λ_D for the multiplet $(\mathbf{1},\mathbf{4},\mathbf{3})$ [14].

Digression.—In Sec. III B, we showed that, after passing to the new superspace basis $\{t,\tilde{\theta}_j,\bar{\tilde{\theta}}^i\}$, the differential parts of the $D(2,1;\alpha)$ supercharges in the μ representation satisfy the relation $S(\mu) = Q(-\mu)$, thus making manifest the property that $D(2,1;\alpha)$ is the closure of two its $su(2|1)$ subalgebras, one defined at μ and the other at $-\mu$. Due to the presence of the additional weight terms, the supercharges (4.6) written in the new basis no longer exhibit this nice correspondence. To restore it, one needs to make the appropriate θ-dependent rescaling of the superfield G,

$$G = AG_0, \tag{4.10}$$

and to pick up the factor A in such a way that the extra weight terms acquired by the supercharges $Q^i(\mu)$ and $S^i(\mu)$ when acting on G_0 ensure the needed relation. The factor A is defined up to a freedom associated with a real parameter β:

$$A(\tilde{\theta}) = 1+\alpha\mu\bar{\tilde{\theta}}^k\tilde{\theta}_k-\frac{1}{2}\beta\mu^2(\tilde{\theta})^2(\bar{\tilde{\theta}})^2, \tag{4.11}$$

$$G_0(t,\tilde\theta) = \left[1 + \frac{1}{2}(\beta-\alpha)\mu^2(\tilde\theta)^2(\bar{\tilde\theta})^2\right]x + \frac{\ddot x}{4}(\tilde\theta)^2(\bar{\tilde\theta})^2 + \tilde\theta_i\psi^i - \bar{\tilde\theta}^j\bar\psi_j - i\bar{\tilde\theta}^k\tilde\theta_k(\tilde\theta_i\dot\psi^i + \bar{\tilde\theta}^j\dot{\bar\psi}_j) + \bar{\tilde\theta}^j\tilde\theta_i B^i_j. \tag{4.12}$$

The ϵ and ε variations of G_0 are related just through the substitution $\mu\to-\mu$,

$$\delta_\epsilon G_0 = -\mu\left[\alpha - \frac{1}{2}(4\beta-3\alpha)\mu\bar{\tilde\theta}^k\tilde\theta_k\right](\bar\epsilon^i\tilde\theta_i e^{-\frac{i}{2}\mu t} - \epsilon_i\bar{\tilde\theta}^i e^{\frac{i}{2}\mu t})G_0,$$
$$\delta_\varepsilon G_0 = \mu\left[\alpha + \frac{1}{2}(4\beta-3\alpha)\mu\bar{\tilde\theta}^k\tilde\theta_k\right](\bar\varepsilon^i\tilde\theta_i e^{\frac{i}{2}\mu t} - \varepsilon_i\bar{\tilde\theta}^i e^{-\frac{i}{2}\mu t})G_0, \tag{4.13}$$

and imply the following expressions for the total $D(2,1;\alpha)$ generators in the realization on G_0:

$$Q^i = e^{\frac{i}{2}\mu t}\Big\{\left[1+\frac{1}{2}(1+2\alpha)\mu\bar{\tilde\theta}^k\tilde\theta_k - \frac{1}{16}(1+2\alpha)\mu^2(\tilde\theta)^2(\bar{\tilde\theta})^2\right]\frac{\partial}{\partial\tilde\theta_i} - (1+\alpha)\mu\bar{\tilde\theta}^i\tilde\theta_k\frac{\partial}{\partial\tilde\theta_k}$$
$$+ \alpha\mu\bar{\tilde\theta}^i\bar{\tilde\theta}^k\frac{\partial}{\partial\bar{\tilde\theta}^k} + i\bar{\tilde\theta}^i\left[1-\frac{1}{2}(1+2\alpha)\mu\bar{\tilde\theta}^k\tilde\theta_k\right]\partial_t - \alpha\mu\bar{\tilde\theta}^i + \frac{1}{2}(4\beta-3\alpha)\mu^2\bar{\tilde\theta}^i\bar{\tilde\theta}^k\tilde\theta_k\Big\},$$
$$\bar Q_j = e^{-\frac{i}{2}\mu t}\Big\{\left[1+\frac{1}{2}(1+2\alpha)\mu\bar{\tilde\theta}^k\tilde\theta_k - \frac{1}{16}(1+2\alpha)\mu^2(\tilde\theta)^2(\bar{\tilde\theta})^2\right]\frac{\partial}{\partial\bar{\tilde\theta}^j} + (1+\alpha)\mu\tilde\theta_j\bar{\tilde\theta}^k\frac{\partial}{\partial\bar{\tilde\theta}^k}$$
$$- \alpha\mu\tilde\theta_j\tilde\theta_k\frac{\partial}{\partial\tilde\theta_k} + i\tilde\theta_j\left[1-\frac{1}{2}(1+2\alpha)\mu\bar{\tilde\theta}^k\tilde\theta_k\right]\partial_t + \alpha\mu\tilde\theta_i - \frac{1}{2}(4\beta-3\alpha)\mu^2\tilde\theta_i\bar{\tilde\theta}^k\tilde\theta_k\Big\},$$
$$S^i(\mu) = Q^i(-\mu), \qquad \bar S_j(\mu) = \bar Q_j(-\mu). \tag{4.14}$$

One can directly check that their (anti)commutators form the superalgebra $D(2,1;\alpha)$. The $so(2,1)$ generators T, $\bar T$ are given by

$$T = e^{-i\mu t}\Big\{i\left[1-\frac{1}{4}(1+2\alpha)\mu^2(\tilde\theta)^2(\bar{\tilde\theta})^2\right]\partial_t + \frac{\mu}{2}\left(\bar{\tilde\theta}^k\frac{\partial}{\partial\bar{\tilde\theta}^k} + \tilde\theta_k\frac{\partial}{\partial\tilde\theta_k}\right)$$
$$+\frac{1}{2}(1+2\alpha)\mu^2\bar{\tilde\theta}^i\tilde\theta_i\left(\bar{\tilde\theta}^k\frac{\partial}{\partial\bar{\tilde\theta}^k} - \tilde\theta_k\frac{\partial}{\partial\tilde\theta_k}\right) + \alpha\mu\left[1+\left(\frac{3}{4}-\frac{\beta}{\alpha}\right)\mu^2(\tilde\theta)^2(\bar{\tilde\theta})^2\right]\Big\},$$
$$\bar T = e^{i\mu t}\Big\{i\left[1-\frac{1}{4}(1+2\alpha)\mu^2(\tilde\theta)^2(\bar{\tilde\theta})^2\right]\partial_t - \frac{\mu}{2}\left(\bar{\tilde\theta}^k\frac{\partial}{\partial\bar{\tilde\theta}^k} + \tilde\theta_k\frac{\partial}{\partial\tilde\theta_k}\right)$$
$$+\frac{1}{2}(1+2\alpha)\mu^2\bar{\tilde\theta}^i\tilde\theta_i\left(\bar{\tilde\theta}^k\frac{\partial}{\partial\bar{\tilde\theta}^k} - \tilde\theta_k\frac{\partial}{\partial\tilde\theta_k}\right) - \alpha\mu\left[1+\left(\frac{3}{4}-\frac{\beta}{\alpha}\right)\mu^2(\tilde\theta)^2(\bar{\tilde\theta})^2\right]\Big\}. \tag{4.15}$$

The rest of the bosonic generators contain no weight terms.

The parameter β appears neither in the structure constants of $D(2,1;\alpha)$ nor in the superconformal component actions (see following subsections), so it can be chosen at will. One choice is $\beta = \frac{3}{4}\alpha$, which ensures the simplest structure of the weight terms in (4.14), (4.15), (4.13). Another possible choice is $\beta = \alpha$, under which the superfield G_0 in (4.12) contains no μ dependence at all. In this case, the $SU(2|1)$ constraints (4.3) are reduced to the linear combination of the flat constraints:

$$\varepsilon_{ik}D^iD^kG_0 = \varepsilon^{ik}\bar D_i\bar D_kG_0 = 0, \qquad [D^i,\bar D_i]G_0 = 0, \tag{4.16}$$

$$D^i = \frac{\partial}{\partial\tilde\theta_i} - i\bar{\tilde\theta}^i\partial_t, \qquad \bar D_j = -\frac{\partial}{\partial\bar{\tilde\theta}^j} + i\tilde\theta_j\partial_t. \tag{4.17}$$

These constraints are still covariant under the relevant trigonometric realization of $D(2,1;\alpha)$ [with $\beta=\alpha$ in (4.14), (4.15), (4.13)]. The corresponding superconformal actions of G_0 written as integrals over the $SU(2|1)$ superspace do not coincide with the standard ones constructed as integrals over flat $\mathcal{N}=4, d=1$ superspace.

As a final remark, we note that the constraints (4.3) can be generalized as

$$\varepsilon^{lj}\bar{\mathcal{D}}_l\bar{\mathcal{D}}_j\tilde G = \varepsilon_{lj}\mathcal{D}^l\mathcal{D}^j\tilde G = 0, \qquad [\mathcal{D}^i,\bar{\mathcal{D}}_i]\tilde G = -4\alpha\mu\tilde G - 4c. \tag{4.18}$$

Their solution is

$$\tilde{G}(x,\psi,\bar{\psi},B) = G(x,\psi,\bar{\psi},B) + c\bar{\theta}^j\theta_j(1+2\alpha\mu\bar{\theta}^k\theta_k), \tag{4.19}$$

where $G(x,\psi,\bar{\psi},B)$ was defined in (4.4). Once again, this solution can be adapted to the supercoset (3.22). We observe that the superconformal covariance of the corresponding version of the constraints (4.18) implies the additional condition

$$c\mathcal{D}_l\mathcal{D}^l(1-\mu\bar{\theta}^k\theta_k)(\bar{\varepsilon}^i\theta_i e^{i\mu t} - \varepsilon_i\bar{\theta}^i e^{-i\mu t}) = 0. \tag{4.20}$$

Substituting the explicit expressions (3.33) for the covariant derivatives, one can show that at $c\neq 0$ the condition (4.20) is satisfied only for $\alpha = -1$. Then the superfield $\tilde{G}$ transforms as

$$\delta_\varepsilon\tilde{G} = -2\mu(1-\mu\bar{\theta}^k\theta_k)(\bar{\varepsilon}^i\theta_i e^{i\mu t} - \varepsilon_i\bar{\theta}^i e^{-i\mu t})\tilde{G}. \tag{4.21}$$

Thus, at $c\neq 0$ the relevant superconformal group is reduced to the supergroup $PSU(1,1|2)\rtimes U(1)$. At $c=0$, any $\alpha\neq 0$ is admissible, including $\alpha=-1$.[7] In what follows, the special case $\alpha=0$ will be considered separately.

B. $SU(2|1)$ invariant Lagrangians

One can construct the general Lagrangian and action for the $SU(2|1)$ multiplet $(\mathbf{1},\mathbf{4},\mathbf{3})$ as

$$S(\tilde{G}) = \int dt\mathcal{L} = -\int d\zeta f(\tilde{G}). \tag{4.22}$$

We consider the invariant Lagrangians for the superfield $\tilde{G}$ satisfying the generalized constraints (4.18) with $c\neq 0$. The action for the superfield G subject to the constraints (4.3) can then be obtained by setting $c=0$.

Any action with an arbitrary Lagrangian function $f(\tilde{G})$ is $SU(2|1)$ invariant and provides a deformation of the standard $(\mathbf{1},\mathbf{4},\mathbf{3})$ models. Substituting the expression (4.19) for $\tilde{G}$ into (4.22) and doing there the Berezin integration, we obtain the component off-shell Lagrangian

$$\begin{aligned}\mathcal{L} = {}& \dot{x}^2 g(x) + i(\bar{\psi}_i\dot{\psi}^i - \dot{\bar{\psi}}_i\psi^i)g(x) + \frac{1}{2}B^i_jB^j_i g(x) - B^i_j\left(\frac{1}{2}\delta^j_i\bar{\psi}_k\psi^k - \bar{\psi}_i\psi^j\right)g'(x) \\ & - \frac{1}{4}(\psi)^2(\bar{\psi})^2 g''(x) - [(1+2\alpha)g(x) + \alpha x g'(x)]\mu\bar{\psi}_i\psi^i - \alpha^2\mu^2x^2g(x) - cg'(x)\bar{\psi}_i\psi^i - 2c\alpha\mu x g(x) - c^2g(x),\end{aligned} \tag{4.23}$$

where $g\coloneqq f''$ and primes mean differentiation in x, $f' = \partial_x f$, etc. The parameter c produces new additional potential-type terms in the Lagrangian.

The ϵ, $\bar{\epsilon}$ transformation law of (4.19),

$$\delta\tilde{G} = -[\epsilon_i Q^i + \bar{\epsilon}^j\bar{Q}_j, \tilde{G}], \tag{4.24}$$

implies the following $SU(2|1)$ transformation laws for the component fields:

$$\begin{aligned}\delta x &= \bar{\epsilon}^k\bar{\psi}_k e^{-\frac{i}{2}\mu t} - \epsilon_k\psi^k e^{\frac{i}{2}\mu t}, \qquad \delta\psi^i = e^{-\frac{i}{2}\mu t}(i\bar{\epsilon}^i\dot{x} + \alpha\mu\bar{\epsilon}^i x + c\bar{\epsilon}^i + \bar{\epsilon}^k B^i_k), \\ \delta B^i_j &= -2i\left[\epsilon_j\dot{\psi}^i e^{\frac{i}{2}\mu t} + \bar{\epsilon}^i\dot{\bar{\psi}}_j e^{-\frac{i}{2}\mu t} - \frac{1}{2}\delta^i_j(\epsilon_k\dot{\psi}^k e^{\frac{i}{2}\mu t} + \bar{\epsilon}^k\dot{\bar{\psi}}_k e^{-\frac{i}{2}\mu t})\right] \\ &\quad - (1+2\alpha)\mu\left[\bar{\epsilon}^i\bar{\psi}_j e^{-\frac{i}{2}\mu t} - \epsilon_j\psi^i e^{\frac{i}{2}\mu t} - \frac{1}{2}\delta^i_j(\bar{\epsilon}^k\bar{\psi}_k e^{-\frac{i}{2}\mu t} - \epsilon_k\psi^k e^{\frac{i}{2}\mu t})\right].\end{aligned} \tag{4.25}$$

We can simplify the Lagrangian (4.23) by passing to the new bosonic field $y(x)$ with the free kinetic term. From the equality

$$\dot{x}^2 g(x) = \frac{1}{2}\dot{y}^2, \tag{4.26}$$

we find the equation

$$y'(x) = \sqrt{2g(x)}, \qquad y'(x) = \frac{1}{x'(y)} \tag{4.27}$$

[7] At $c=0,\alpha=-1$, the whole automorphism $SU(2)_{\rm ext}$ is a symmetry of the superfield constraints. It is reduced to $U(1)_{\rm ext}$ only at $c\neq 0$.

and define

$$\chi^i = \psi^i y'(x), \qquad \tilde{B}^i_j = \frac{1}{2} B^i_j y'(x), \qquad V(y) = \frac{x(y)}{x'(y)}. \tag{4.28}$$

Solving the last of the equations in (4.28) as

$$x(y) = \exp\left\{\int^y \frac{d\tilde{y}}{V(\tilde{y})}\right\}, \tag{4.29}$$

we can cast the Lagrangian (4.23) in the form

$$\begin{aligned}\mathcal{L} = {}& \frac{\dot{y}^2}{2} + \frac{i}{2}(\bar{\chi}_i\dot{\chi}^i - \dot{\bar{\chi}}_i\chi^i) + \tilde{B}^i_j\tilde{B}^j_i - \frac{V'(y)-1}{V(y)}\tilde{B}^j_i(\delta^i_j\bar{\chi}_k\chi^k - 2\bar{\chi}_j\chi^i) \\ & - \frac{V''(y)V(y) + [2V'(y)-3][V'(y)-1]}{4V^2(y)}(\chi)^2(\bar{\chi})^2 \\ & - \partial_y\left[\alpha\mu V(y) + cV(y)\exp\left\{-\int^y \frac{d\tilde{y}}{V(\tilde{y})}\right\}\right]\bar{\chi}_i\chi^i - \frac{\mu}{2}\bar{\chi}_i\chi^i \\ & - \frac{1}{2}\left[\alpha\mu V(y) + cV(y)\exp\left\{-\int^y \frac{d\tilde{y}}{V(\tilde{y})}\right\}\right]^2.\end{aligned} \tag{4.30}$$

Here, $V(y)$ can be regarded as an arbitrary function due to the arbitrariness of $g(x)$ in (4.27). Thus, we have finally obtained the $SU(2|1)$ Lagrangian involving an arbitrary function and extended by additional terms which depend on the parameter c. In the new representation, the supersymmetry transformations acquire the form

$$\begin{aligned}\delta y &= \bar{\epsilon}^k\bar{\chi}_k e^{-\frac{i}{2}\mu t} - \epsilon_k\chi^k e^{\frac{i}{2}\mu t}, \\ \delta\chi^i &= e^{-\frac{i}{2}\mu t}\left[i\bar{\epsilon}^i\dot{y} + \alpha\mu\bar{\epsilon}^i V(y) + c\bar{\epsilon}^i V(y)\exp\left\{-\int^y \frac{d\tilde{y}}{V(\tilde{y})}\right\} + 2\bar{\epsilon}^k\tilde{B}^i_k + \chi^i(\bar{\epsilon}^k\bar{\chi}_k - \epsilon_k\chi^k e^{i\mu t})\frac{V'(y)-1}{V(y)}\right], \\ \delta\tilde{B}^i_j &= -i\left[\epsilon_j\dot{\chi}^i e^{\frac{i}{2}\mu t} + \bar{\epsilon}^i\dot{\bar{\chi}}_j e^{-\frac{i}{2}\mu t} - \frac{1}{2}\delta^i_j(\epsilon_k\dot{\chi}^k e^{\frac{i}{2}\mu t} + \bar{\epsilon}^k\dot{\bar{\chi}}_k e^{-\frac{i}{2}\mu t})\right] \\ & \quad - \frac{\mu}{2}(1+2\alpha)\left[\bar{\epsilon}^i\bar{\chi}_j e^{-\frac{i}{2}\mu t} - \epsilon_j\chi^i e^{\frac{i}{2}\mu t} - \frac{1}{2}\delta^i_j(\bar{\epsilon}^k\bar{\chi}_k e^{-\frac{i}{2}\mu t} - \epsilon_k\chi^k e^{\frac{i}{2}\mu t})\right] \\ & \quad + \tilde{B}^i_j(\bar{\epsilon}^k\bar{\chi}_k e^{-\frac{i}{2}\mu t} - \epsilon_k\chi^k e^{\frac{i}{2}\mu t})\frac{V'(y)-1}{V(y)} \\ & \quad + i\dot{y}\left[\epsilon_j\chi^i e^{\frac{i}{2}\mu t} + \bar{\epsilon}^i\bar{\chi}_j e^{-\frac{i}{2}\mu t} - \frac{1}{2}\delta^i_j(\epsilon_k\chi^k e^{\frac{i}{2}\mu t} + \bar{\epsilon}^k\bar{\chi}_k e^{-\frac{i}{2}\mu t})\right]\frac{V'(y)-1}{V(y)}.\end{aligned} \tag{4.31}$$

In the particular case $c = 0$, the models described by these transformations and the Lagrangian (4.30) correspond to the off-shell form of "weak supersymmetry" models [7].

C. Superconformal mechanics with $c = 0$

The superconformal $(\mathbf{1}, \mathbf{4}, \mathbf{3})$ action with $c = 0$ can be written in the superfield formulation as

$$S^{(\alpha)}_{\rm sc}(G) = -\int d\zeta f^{(\alpha)}_{\rm sc}(G), \tag{4.32}$$

where the corresponding superfield function $f(G)$ is given by

$$f^{(\alpha)}_{\rm sc}(G) = \begin{cases}\frac{1}{8(\alpha+1)}G^{-\frac{1}{\alpha}} & \text{for } \alpha \neq -1, 0, \\ \frac{1}{8}G\ln G & \text{for } \alpha = -1.\end{cases} \Rightarrow g(x) = \frac{x^{-\frac{1}{\alpha}-2}}{8\alpha^2}. \tag{4.33}$$

Using (4.8) and (3.32), one can check that the action (4.32) is indeed invariant with respect to the superconformal group $D(2,1;\alpha)$.

We can consider a few special cases, e.g., $\alpha = -1$, $\alpha = -1/2$. As we will see in the next subsection, in the case $\alpha = -1$ the action (4.32) can be generalized to incorporate the nonzero parameter c defined in (4.18). The case $\alpha = -1/2$ corresponds to the free action. We cannot treat the $\alpha = 0$ case as a particular case of the $SU(2|1)$ models under consideration, since we defined the $SU(2|1)$ superspace for $\alpha \neq 0$, while passing to $\alpha = 0$ amounts to contraction of the original $SU(2|1)$ supergroup into the supergroup with the flat algebra (3.21). Nevertheless, as we will see soon, the $\alpha = 0$ superconformal action can still be constructed within the properly modified superfield approach based on the contracted supergroup.

Taking a θ integral in the superfield action (4.32) and making the redefinition (3.17), we calculate the superconformal Lagrangian as[8]

$$\mathcal{L}^{(\alpha)}_{\rm sc} = \dot{x}^2 g(x) + i(\bar{\psi}_i\dot{\psi}^i - \dot{\bar{\psi}}_i\psi^i)g(x) + \frac{1}{2}B^i_jB^j_i g(x) - B^i_j\left(\frac{1}{2}\delta^j_i\bar{\psi}_k\psi^k - \bar{\psi}_i\psi^j\right)g'(x) - \frac{1}{4}(\psi)^2(\bar{\psi})^2 g''(x) - \alpha^2\mu^2x^2g(x). \tag{4.34}$$

We observe that it depends only on μ^2, not on μ. Taking advantage of the redefinitions just mentioned, one can conveniently rewrite the transformations (4.25) as

$$\delta x = \bar{\epsilon}^k\bar{\psi}_k e^{-\frac{i}{2}\mu t} - \epsilon_k\psi^k e^{\frac{i}{2}\mu t}, \qquad \delta\psi^i = e^{-\frac{i}{2}\mu t}(i\bar{\epsilon}^i\dot{x} + \alpha\mu\bar{\epsilon}^i x + \bar{\epsilon}^k B^i_k),$$
$$\delta B^i_j = -2i\left[\epsilon_j\dot{\psi}^i e^{\frac{i}{2}\mu t} + \bar{\epsilon}^i\dot{\bar{\psi}}_j e^{-\frac{i}{2}\mu t} - \frac{1}{2}\delta^i_j(\epsilon_k\dot{\psi}^k e^{\frac{i}{2}\mu t} + \bar{\epsilon}^k\dot{\bar{\psi}}_k e^{-\frac{i}{2}\mu t})\right] - (1+2\alpha)\mu\left[\bar{\epsilon}^i\bar{\psi}_j e^{-\frac{i}{2}\mu t} - \epsilon_j\psi^i e^{\frac{i}{2}\mu t} - \frac{1}{2}\delta^i_j(\bar{\epsilon}^k\bar{\psi}_k e^{-\frac{i}{2}\mu t} - \epsilon_k\psi^k e^{\frac{i}{2}\mu t})\right]. \tag{4.35}$$

The Lagrangian (4.34) is invariant under the second $SU(2|1)$ transformations with the parameters ε, $\bar{\varepsilon}$,

$$\delta x = \bar{\varepsilon}^k\bar{\psi}_k e^{\frac{i}{2}\mu t} - \varepsilon_k\psi^k e^{-\frac{i}{2}\mu t}, \qquad \delta\psi^i = e^{\frac{i}{2}\mu t}(i\bar{\varepsilon}^i\dot{x} - \alpha\mu\bar{\varepsilon}^i x + \bar{\varepsilon}^k B^i_k),$$
$$\delta B^i_j = -2i\left[\varepsilon_j\dot{\psi}^i e^{-\frac{i}{2}\mu t} + \bar{\varepsilon}^i\dot{\bar{\psi}}_j e^{\frac{i}{2}\mu t} - \frac{1}{2}\delta^i_j(\varepsilon_k\dot{\psi}^k e^{-\frac{i}{2}\mu t} + \bar{\varepsilon}^k\dot{\bar{\psi}}_k e^{\frac{i}{2}\mu t})\right] + (1+2\alpha)\mu\left[\bar{\varepsilon}^i\bar{\psi}_j e^{\frac{i}{2}\mu t} - \varepsilon_j\psi^i e^{-\frac{i}{2}\mu t} - \frac{1}{2}\delta^i_j(\bar{\varepsilon}^k\bar{\psi}_k e^{\frac{i}{2}\mu t} - \varepsilon_k\psi^k e^{-\frac{i}{2}\mu t})\right], \tag{4.36}$$

which correspond to the supercharges (4.6). We see that (4.35) and (4.36) are related by the replacement $\mu \to -\mu$ in accord with the structure of $D(2,1;\alpha)$ as the closure of these two $su(2|1)$ subalgebras.

The parabolic transformations of the $(\mathbf{1},\mathbf{4},\mathbf{3})$ component fields can be obtained from the trigonometric transformations (4.35), (4.36) in two steps. First, one passes to the new pair $\{\epsilon',\bar{\epsilon}'\}$, $\{\varepsilon',\bar{\varepsilon}'\}$ of infinitesimal parameters with opposite dimensions by redefining the old parameters as

$$\epsilon_i = \frac{1}{2}\epsilon'_i + \frac{i}{\mu}\varepsilon'_i, \qquad \varepsilon_i = \frac{1}{2}\epsilon'_i - \frac{i}{\mu}\varepsilon'_i, \qquad \text{and c.c.} \tag{4.37}$$

This redefinition just corresponds to passing to the original basis for the $D(2,1;\alpha)$ supercharges, in which the super Poincaré supercharges and those of the superconformal boosts have the opposite dimensions. Only after that can we send $\mu \to 0$ and obtain the parabolic transformations. This procedure is universal and can be performed for the transformations of superfields, component fields and superspace coordinates, regardless of the type of the realization of $D(2,1;\alpha)$. In this way one can, e.g., deduce the parabolic transformations of the integration measure (2.10), which becomes the standard flat measure $dtd^2\theta d^2\bar{\theta}$ in the limit $\mu = 0$.

Using (4.26)–(4.28), we can calculate the function $V(y)$ corresponding to (4.33):

$$V(y) = -\frac{y}{2\alpha}, \qquad V'(y) = -\frac{1}{2\alpha}, \qquad \frac{V'(y)-1}{V(y)} = \frac{1+2\alpha}{y}. \tag{4.38}$$

As a result, we obtain the superconformal Lagrangian in the form

[8] The term $\sim\bar{\psi}\psi$ vanishes because of the identity $(-\frac{1}{\alpha} - 2)g(x) = xg'(x)$ for (4.33).

$$\mathcal{L}^{(\alpha)}_{\rm sc}=\frac{\dot{y}^2}{2}+\frac{i}{2}(\bar{\chi}_i\dot{\chi}^i-\dot{\bar{\chi}}_i\chi^i)+\tilde{B}^i_j\tilde{B}^j_i-\frac{1+2\alpha}{y}\tilde{B}^j_i(\delta^i_j\bar{\chi}_k\chi^k-2\bar{\chi}_j\chi^i)\\ -\frac{(1+3\alpha)(1+2\alpha)}{2y^2}(\chi)^2(\bar{\chi})^2-\frac{\mu^2}{8}y^2. \quad (4.39)$$

It is invariant (modulo a total derivative) under the following $SU(2|1)$ odd transformations:

$$\delta y=\bar{\epsilon}^k\bar{\chi}_k e^{-\frac{i}{2}\mu t}-\epsilon_k\chi^k e^{\frac{i}{2}\mu t},$$
$$\delta\chi^i=e^{-\frac{i}{2}\mu t}\left[i\bar{\epsilon}^i\dot{y}-\frac{\mu}{2}\bar{\epsilon}^i y+2\bar{\epsilon}^k\tilde{B}^i_k+\chi^i(\bar{\epsilon}^k\bar{\chi}_k-\epsilon_k\chi^k e^{i\mu t})\frac{1+2\alpha}{y}\right],$$
$$\delta\tilde{B}^i_j=-i\left[\epsilon_j\dot{\chi}^i e^{\frac{i}{2}\mu t}+\bar{\epsilon}^i\dot{\bar{\chi}}_j e^{-\frac{i}{2}\mu t}-\frac{1}{2}\delta^i_j(\epsilon_k\dot{\chi}^k e^{\frac{i}{2}\mu t}+\bar{\epsilon}^k\dot{\bar{\chi}}_k e^{-\frac{i}{2}\mu t})\right]-\frac{\mu}{2}(1+2\alpha)\left[\bar{\epsilon}^i\bar{\chi}_j e^{-\frac{i}{2}\mu t}-\epsilon_j\chi^i e^{\frac{i}{2}\mu t}-\frac{1}{2}\delta^i_j(\bar{\epsilon}^k\bar{\chi}_k e^{-\frac{i}{2}\mu t}-\epsilon_k\chi^k e^{\frac{i}{2}\mu t})\right]\\ +\tilde{B}^i_j(\bar{\epsilon}^k\bar{\chi}_k e^{-\frac{i}{2}\mu t}-\epsilon_k\chi^k e^{\frac{i}{2}\mu t})\frac{1+2\alpha}{y}+i\dot{y}\left[\epsilon_j\chi^i e^{\frac{i}{2}\mu t}+\bar{\epsilon}^i\bar{\chi}_j e^{-\frac{i}{2}\mu t}-\frac{1}{2}\delta^i_j(\epsilon_k\chi^k e^{\frac{i}{2}\mu t}+\bar{\epsilon}^k\bar{\chi}_k e^{-\frac{i}{2}\mu t})\right]\frac{1+2\alpha}{y}. \quad (4.40)$$

Changing μ in these transformations as $\mu\to-\mu$, one obtains the transformations associated with the extra generators $S(\mu)=Q(-\mu)$. Since the Lagrangian (4.39) depends only on μ^2 like (4.34), it is automatically invariant under these S^i transformations and, hence, under the full $D(2,1;\alpha)$.

Thus, in the present case we deal with the superconformal mechanics corresponding to the trigonometric transformations [14]. Another type of superconformal mechanics is that associated with the parabolic transformations, and its superfield description is based on the standard $\mathcal{N}=4$, $d=1$ superspace. The only difference is that the trigonometric-type action (4.39) has an additional oscillator term. Thus, by sending $\mu\to0$, the parabolic type of superconformal mechanics can be restored.

The property that the component superconformal trigonometric actions are even functions of the parameter μ can be established already at the superfield level. One should pass to the $SU(2|1)$ superspace basis $\{t,\tilde{\theta}_j,\bar{\tilde{\theta}}^i\}$, in which the property $S^i(\mu)=Q^i(-\mu)$ is valid and the integration measure is defined by (3.43), and express the superfield G through G_0 according to Eqs. (4.10)–(4.12):

$$S^{(\alpha)}_{\rm sc}(G)=-\int dtd^2\tilde{\theta}d^2\bar{\tilde{\theta}}(1+\mu\bar{\tilde{\theta}}^k\tilde{\theta}_k)G^{-\frac{1}{\alpha}}\\ =-\int dtd^2\tilde{\theta}d^2\bar{\tilde{\theta}}\left[1+\frac{1}{4}\left(\alpha-1+\frac{2\beta}{\alpha}\right)\mu^2(\tilde{\theta})^2(\bar{\tilde{\theta}})^2\right](G_0)^{-\frac{1}{\alpha}},\qquad \alpha\neq-1,0 \quad (4.41)$$

and

$$S^{(\alpha=-1)}_{\rm sc}(G)=-\int dtd^2\tilde{\theta}d^2\bar{\tilde{\theta}}(1+\mu\bar{\tilde{\theta}}^k\tilde{\theta}_k)G\ln G\\ =-\int dtd^2\tilde{\theta}d^2\bar{\tilde{\theta}}\left\{\left[1-\frac{1}{2}(1+\beta)\mu^2(\tilde{\theta})^2(\bar{\tilde{\theta}})^2\right]G_0\ln G_0-\frac{\mu^2}{4}(1+2\beta)(\tilde{\theta})^2(\bar{\tilde{\theta}})^2G_0\right\}, \quad (4.42)$$

where one should take into account that

$$\int dtd^2\tilde{\theta}d^2\bar{\tilde{\theta}}(\mu\bar{\tilde{\theta}}^k\tilde{\theta}_k G_0)=0.$$

All terms with the manifest θ's in (4.41), (4.42), equally as the superfield G_0, depend only on μ^2. Also, it is easy to show that all β-dependent terms in these actions are canceled among themselves. For any other trigonometric superconformal action treated below (e.g., in the $\alpha=-1$, $c\neq0$ case), it is possible to show in a similar way that, in the appropriate superfield formulation, they depend only on μ^2 like in the component field formulations.

D. The model with $\alpha=-1$, $c\neq0$

Let us consider the case of $c\neq0$ for which the superconformal invariance requires that $\alpha=-1$ ($m=\mu$).

The corresponding supergroup is $D(2,1;\alpha=-1) = PSU(1,1|2) \rtimes SU(2)_{\rm ext}$, but the constraints (4.18) are covariant only with respect to $PSU(1,1|2) \rtimes U(1)_{\rm ext}$.

The corresponding superfield action is

$$S_{\rm sc}^{(\alpha=-1)}(\tilde{G}) = -\int d\zeta \tilde{G} \ln \tilde{G}. \tag{4.43}$$

Starting from the general $SU(2|1)$ invariant component Lagrangian (4.23) with $c \neq 0$ and substituting there $f(\tilde{G}) \to \tilde{G} \ln \tilde{G}$, we obtain, up to an additive constant, the following $c \neq 0, \alpha = -1$ generalization of the superconformal Lagrangian (4.34):

$$\begin{aligned}\mathcal{L}_{\rm sc}^{(\alpha=-1,c)} &= \frac{\dot{x}^2}{x} + \frac{i}{x}(\bar{\psi}_i\dot{\psi}^i - \dot{\bar{\psi}}_i\psi^i) \\ &+ \frac{B^i_jB^j_i}{2x} + \frac{B^i_j}{x^2}\left(\frac{1}{2}\delta^j_i\bar{\psi}_k\psi^k - \bar{\psi}_i\psi^j\right) \\ &- \frac{1}{2x^3}(\psi)^2(\bar{\psi})^2 - \mu^2 x + \frac{c\bar{\psi}_i\psi^i}{x^2} - \frac{c^2}{x}.\end{aligned} \tag{4.44}$$

Here, the new term $\sim\bar{\psi}\psi$ is responsible for reducing superconformal symmetry to $PSU(1,1|2) \rtimes U(1)$. This action is invariant under the supersymmetry transformations, with $\delta\psi^i$ being a generalization of the relevant transformations in (4.35), (4.36):

$$\begin{aligned}\delta\psi^i &= e^{-\frac{i}{2}\mu t}(i\bar{\epsilon}^i\dot{x} - \mu\bar{\epsilon}^i x + \bar{\epsilon}^k B^i_k + c\bar{\epsilon}^i x) \\ &+ e^{\frac{i}{2}\mu t}(i\bar{\varepsilon}^i\dot{x} + \mu\bar{\varepsilon}^i x + \bar{\varepsilon}^k B^i_k + c\bar{\varepsilon}^i x).\end{aligned} \tag{4.45}$$

Transformations of the bosonic fields are the same as in (4.35), (4.36).

Passing to the action with free kinetic terms, we find the relevant function $V(y)$ to be

$$V(y) = \frac{y}{2}, \qquad V'(y) = \frac{1}{2}, \qquad \frac{V'(y)-1}{V(y)} = -\frac{1}{y}. \tag{4.46}$$

In accordance with (4.30), we also should take into account additional terms involving c. Thus the superconformal Lagrangian (4.39) is generalized to this special case as

$$\begin{aligned}\mathcal{L}_{\rm sc}^{(\alpha=-1,c)} &= \frac{\dot{y}^2}{2} + \frac{i}{2}(\bar{\chi}_i\dot{\chi}^i - \dot{\bar{\chi}}_i\chi^i) + \tilde{B}^i_j\tilde{B}^j_i \\ &+ \frac{\tilde{B}^j_i}{y}(\delta^i_j\bar{\chi}_k\chi^k - 2\bar{\chi}_j\chi^i) - \frac{1}{y^2}(\chi)^2(\bar{\chi})^2 \\ &+ \frac{c}{2y^2}\bar{\chi}_i\chi^i - \frac{\mu^2y^2}{8} - \frac{c^2}{8y^2}.\end{aligned} \tag{4.47}$$

The relevant on-shell Lagrangian,

$$\begin{aligned}\mathcal{L}_{\rm sc}^{(\alpha=-1,c)} &= \frac{\dot{y}^2}{2} + \frac{i}{2}(\bar{\chi}_i\dot{\chi}^i - \dot{\bar{\chi}}_i\chi^i) - \frac{1}{4y^2}(\chi)^2(\bar{\chi})^2 \\ &+ \frac{c}{2y^2}\bar{\chi}_i\chi^i - \frac{\mu^2y^2}{8} - \frac{c^2}{8y^2},\end{aligned} \tag{4.48}$$

as a superconformal Lagrangian, was previously found in Ref. [17].[9] The $SU(2|1)$ superspace approach allowed us to find the off-shell superfield form of (4.48).

E. The $\alpha = 0$ model

Inspecting the Lagrangian (4.34), we observe that the limit $\alpha \to -0$ is divergent and the opposite limit $\alpha \to +0$ yields $\mathcal{L}_{\rm sc}^{(\alpha=0)} = 0$. Nevertheless, we can unambiguously define this limit for the Lagrangian (4.34) by introducing an inhomogeneity parameter ρ [14].

The limit $\alpha \to 0$ can be obtained, if we redefine the Lagrangian (4.34) by shifting the field x as

$$x \to x + \frac{\rho}{\alpha}. \tag{4.49}$$

The homogeneous Lagrangian (4.34) is rewritten as

$$\begin{aligned}\mathcal{L}_{\rm sc}^{(\alpha,\rho)} &= \frac{\alpha^{\frac{1}{\alpha}}}{8}(\alpha x + \rho)^{-\frac{1}{\alpha}-2}\left[\dot{x}^2 + i(\bar{\psi}_i\dot{\psi}^i - \dot{\bar{\psi}}_i\psi^i) + \frac{1}{2}B^i_jB^j_i\right] \\ &+ \frac{\alpha^{\frac{1}{\alpha}}(1+2\alpha)}{8}B^i_j\left(\frac{1}{2}\delta^j_i\bar{\psi}_k\psi^k - \bar{\psi}_i\psi^j\right)(\alpha x + \rho)^{-\frac{1}{\alpha}-3} \\ &- \frac{\alpha^{\frac{1}{\alpha}}(1+2\alpha)(1+3\alpha)}{32}(\psi)^2(\bar{\psi})^2(\alpha x + \rho)^{-\frac{1}{\alpha}-4} \\ &- \frac{\alpha^{\frac{1}{\alpha}}\mu^2}{8}(\alpha x + \rho)^{-\frac{1}{\alpha}}.\end{aligned} \tag{4.50}$$

Detaching the divergent factor $\sim(\frac{\alpha}{\rho})^{\frac{1}{\alpha}}$ and sending $\alpha \to 0$ in the remainder, we obtain the Lagrangian $\mathcal{L}_{\rm sc}^{(\alpha=0,\rho)}$ as

$$\begin{aligned}\mathcal{L}_{\rm sc}^{(\alpha=0,\rho)} &= e^{-\frac{x}{\rho}}\left[\dot{x}^2 + i(\bar{\psi}_i\dot{\psi}^i - \dot{\bar{\psi}}_i\psi^i) + \frac{1}{2}B^i_jB^j_i\right] \\ &+ \frac{B^i_j}{\rho}\left(\frac{1}{2}\delta^j_i\bar{\psi}_k\psi^k - \bar{\psi}_i\psi^j\right)e^{-\frac{x}{\rho}} \\ &- \frac{1}{4\rho^2}(\psi)^2(\bar{\psi})^2e^{-\frac{x}{\rho}} - \mu^2\rho^2e^{-\frac{x}{\rho}}.\end{aligned} \tag{4.51}$$

Following the same procedure as in (4.26)–(4.28), we can obtain the Lagrangian which coincides with (4.39) at $\alpha = 0$ [14]. For ensuring the superconformal invariance in this case, one needs to extend the transformations (4.35), (4.36) for $\alpha = 0$ by the inhomogeneous parts

[9] One needs to perform a redefinition of fields in order to show the coincidence of these two Lagrangians.

$$\delta_{(\rho)}\psi^i = \rho\mu(\bar{\epsilon}^i e^{-\frac{i}{2}\mu t} - \bar{\varepsilon}^i e^{\frac{i}{2}\mu t}), \qquad \delta_{(\rho)}x = \delta_{(\rho)}B^i_k = 0. \tag{4.52}$$

This modification entails the appearance of inhomogeneous pieces in the conformal transformations of x,

$$Tx = e^{-i\mu t}(i\dot{x} + \rho\mu), \qquad \bar{T}x = e^{i\mu t}(i\dot{x} - \rho\mu). \tag{4.53}$$

The standard conformal $so(2,1)$ generators defined in (3.5), (3.8) act on x as

$$\hat{H}x = \frac{i}{2}(1+\cos\mu t)\dot{x} - \frac{i}{2}\rho\mu\sin\mu t,$$
$$\hat{K}x = \frac{2i}{\mu^2}(1-\cos\mu t)\dot{x} + \frac{2i}{\mu}\rho\sin\mu t,$$
$$\hat{D}x = \frac{i}{\mu}\sin\mu t\dot{x} + i\rho\cos\mu t. \tag{4.54}$$

The superconformal superfield action (4.32) is not defined at $\alpha = 0$. Nevertheless, the superfield description of (4.51) can be given in the framework of the supercoset (3.24) associated with the $\alpha = 0$ superalgebra (3.21). According to (4.4), the superfield G is written as

$$G = x + \frac{\ddot{x}}{4}(\theta)^2(\bar{\theta})^2 - i\bar{\theta}^k\theta_k(\theta_i\dot{\psi}^i e^{\frac{i}{2}\mu t} + \bar{\theta}^j\dot{\bar{\psi}}_j e^{-\frac{i}{2}\mu t})$$
$$+ \left(1 + \frac{\mu}{2}\bar{\theta}^k\theta_k\right)(\theta_i\psi^i e^{\frac{i}{2}\mu t} - \bar{\theta}^j\bar{\psi}_j e^{-\frac{i}{2}\mu t}) + \bar{\theta}^j\theta_i B^i_j \tag{4.55}$$

and satisfies the standard "flat" $(\mathbf{1},\mathbf{4},\mathbf{3})$ constraints

$$\varepsilon^{lj}\bar{\mathcal{D}}_l\bar{\mathcal{D}}_j G = \varepsilon_{lj}\mathcal{D}^l\mathcal{D}^j G = 0, \qquad [\mathcal{D}^i, \bar{\mathcal{D}}_i]G = 0, \tag{4.56}$$

where the covariant derivatives are[10]

$$\mathcal{D}^i = e^{-\frac{i}{2}\mu t}\left(\frac{\partial}{\partial\theta_i} - i\bar{\theta}^i\partial_t - \mu\bar{\theta}^i\tilde{F}\right),$$
$$\bar{\mathcal{D}}_j = e^{\frac{i}{2}\mu t}\left(-\frac{\partial}{\partial\bar{\theta}^j} + i\theta_j\partial_t + \mu\theta_j\tilde{F}\right), \tag{4.57}$$

$$\mathcal{D}_{(t)} = \partial_t, \qquad \{\mathcal{D}^i, \bar{\mathcal{D}}_j\} = 2\delta^i_j(i\partial_t + \mu\tilde{F}). \tag{4.58}$$

Then the component Lagrangian (4.51) is reproduced from the superfield action

$$S^{(\alpha=0,\rho)}_{\rm sc}(G) = \int dt\mathcal{L}^{(\alpha=0,\rho)}_{\rm sc} = -\rho^2\int dtd^2\theta d^2\bar{\theta}e^{-\frac{G}{\rho}-\mu\bar{\theta}^k\theta_k}. \tag{4.59}$$

The "passive" superfield infinitesimal transformation of G involves only the inhomogeneous piece

$$\delta_{(\rho)}G = -\rho\mu(\bar{\varepsilon}^k\theta_k - \epsilon_k\bar{\theta}^k)$$
$$+ \rho\mu(1+\mu\bar{\theta}^k\theta_k)(\bar{\varepsilon}^k\theta_k e^{i\mu t} - \varepsilon_k\bar{\theta}^k e^{-i\mu t}), \tag{4.60}$$

since its standard homogeneous part (4.8) vanishes at $\alpha = 0$.

Note that the superfield G at $\alpha = 0$, though being defined in fact on the flat $\mathcal{N} = 4$ superspace, still possesses an unusual inhomogeneous transformation law (4.60) under the $\mathcal{N} = 4, d = 1$ Poincaré supersymmetry to which, at $\alpha = 0$, the $\epsilon, \bar{\epsilon}$ transformations are reduced. We can reformulate this model in terms of the superfield u having the standard homogeneous transformation law under the $\mathcal{N} = 4, d = 1$ Poincaré supersymmetry

$$u = G + \rho\mu\bar{\theta}^k\theta_k, \qquad \delta_\epsilon u = 0, \tag{4.61}$$

$$\varepsilon^{lj}\bar{\mathcal{D}}_l\bar{\mathcal{D}}_j u = \varepsilon_{lj}\mathcal{D}^l\mathcal{D}^j u = 0, \qquad [\mathcal{D}^i, \bar{\mathcal{D}}_i]u = -4\rho\mu. \tag{4.62}$$

The inhomogeneity of the full odd superconformal transformation law of u is retained only in the part $\sim\varepsilon_i, \bar{\varepsilon}^k$ associated with the generators $S^i, \bar{S}_k$:

$$\delta_{(\rho)}u = 2\rho\mu(1-\mu\bar{\theta}^k\theta_k)(\bar{\varepsilon}^k\theta_k e^{i\mu t} - \varepsilon_k\bar{\theta}^k e^{-i\mu t}). \tag{4.63}$$

The action (4.59) is rewritten in the form in which it does not involve explicit θ:

$$S^{(\alpha=0,\rho)}_{\rm sc}(u) = -\rho^2\int dtd^2\theta d^2\bar{\theta}e^{-\frac{u}{\rho}}. \tag{4.64}$$

We also note that the μ dependence in the solution (4.55) is fake because it can be removed by the inverse phase transformation of fermionic fields as $\psi^i \to \psi^i e^{-\frac{i}{2}\mu t}$. Then the whole μ dependence in the component actions (4.50), (4.51) is generated by the $\theta =$ dependent term in (4.59) or the θ-dependent additional term in u defined in (4.61) (if one prefers the u representation (4.64) for the superconformal action). The definition of the fermionic fields as in (4.55) is convenient since it ensures the absence of the fermionic "mass terms" $\sim\mu\psi^i\bar{\psi}_i$ in (4.50), (4.51). Despite the fact that at $\alpha = 0$ we deal with the standard flat $\mathcal{N} = 4$ superfield u, the superconformal transformations (4.63) still correspond to the trigonometric realization of the conformal subgroup $SO(2,1)$, as well as that of the full $PSU(1,1|2)$. The parabolic realization is achieved by redefining the fermionic parameters as in (4.37) and then

[10]Though the superfield G has no external $U(1)$ charge and the generator $\tilde{F}$ yields zero on G, it is nonvanishing when acting on the covariant derivative itself. Nevertheless, it is direct to check that in the $\alpha = 0$ constraints (4.56) such contributions are canceled against terms coming from the phase factors in the definition (4.57).

sending $\mu \to 0$ in the resulting transformations, like in other cases.

As a final remark, we notice that the $\alpha = 0$ analog of the superconformal action (4.43) with $c \neq 0$ and $\alpha = -1$ can be obtained [20] by considering the superfield action dual to (4.43):

$$S_{\rm sc}^{(\alpha=0,\rho,\tilde{c})}(G) = \int dt d^2\theta d^2\bar{\theta}[-\rho^2 e^{-\frac{G}{\rho}-\mu\bar{\theta}^k\theta_k} + \tilde{c}(\bar{\theta}^1\theta_1 - \bar{\theta}^2\theta_2)G]$$
$$= \int dt d^2\theta d^2\bar{\theta}[-\rho^2 e^{-u} + \tilde{c}(\bar{\theta}^1\theta_1 - \bar{\theta}^2\theta_2)u]. \tag{4.65}$$

It can be checked that the relevant component Lagrangian coincides with the off-shell Lagrangian (4.47), modulo the replacements of all $SU(2)$ indices by the $SU'(2)$ indices (on which the generators $\{F, C, \bar{C}\}$ act) and the substitution $c \to \tilde{c}$.

V. THE MULTIPLET (2, 4, 2)

A. Chiral $SU(2|1)$ superfields

In this section, we will consider the multiplet $(\mathbf{2}, \mathbf{4}, \mathbf{2})$, proceeding from the superspace (2.5). Also, in Ref. [5] the multiplet $(\mathbf{2}, \mathbf{4}, \mathbf{2})$ was generalized by exploiting the superspace coset (2.12). Such a generalization will be addressed in the next section.

Employing the covariant derivatives (2.11), the standard form of the chiral and antichiral conditions is as follows:

$$\text{(a) } \bar{\mathcal{D}}_i\Phi = 0, \qquad \text{(b) } \mathcal{D}^i\bar{\Phi} = 0. \tag{5.1}$$

This implies the existence of the left and right chiral subspaces [4]:

$$(t_L, \theta_i), \qquad (t_R, \bar{\theta}^i), \tag{5.2}$$

where

$$t_L = t + i\bar{\theta}^k\theta_k - \frac{i}{2}m(\theta)^2(\bar{\theta})^2, \quad \text{and c.c.} \tag{5.3}$$

These coordinate sets are closed under the $SU(2|1)$ transformations

$$\delta\theta_i = \epsilon_i + 2m\bar{\epsilon}^k\theta_k\theta_i, \qquad \delta t_L = 2i\bar{\epsilon}^k\theta_k, \quad \text{and c.c.} \tag{5.4}$$

One can require that the complex superfield Φ with the minimal field contents $(\mathbf{2}, \mathbf{4}, \mathbf{2})$ possess a fixed overall $U(1)$ charge:

$$\tilde{F}\Phi = 2\kappa\Phi, \qquad \tilde{I}^i_k\Phi = 0. \tag{5.5}$$

The general solution of (5.1) for an arbitrary real κ reads

$$\Phi(t,\theta,\bar{\theta}) = (1 + 2m\bar{\theta}^k\theta_k)^{-\kappa}\Phi_L(t_L,\theta),$$
$$\Phi_L(t_L,\theta) = z + \sqrt{2}\theta_i\xi^i + (\theta)^2B, \qquad \overline{(\xi^i)} = \bar{\xi}_i. \tag{5.6}$$

The chiral superfield Φ transforms as

$$\delta\Phi = 2\kappa m(\bar{\epsilon}^i\theta_i + \epsilon_i\bar{\theta}^i)\Phi, \qquad \delta\Phi_L = 4\kappa m\bar{\epsilon}^i\theta_i\Phi_L. \tag{5.7}$$

This transformation law implies the following off-shell $SU(2|1)$ transformations of the component fields in (5.6):

$$\delta z = -\sqrt{2}\epsilon_k\xi^k, \qquad \delta\xi^i = \sqrt{2}\bar{\epsilon}^i(i\dot{z} - 2\kappa m z) - \sqrt{2}\epsilon^i B,$$
$$\delta B = -\sqrt{2}\bar{\epsilon}_k\left[i\dot{\xi}^k - \left(2\kappa - \frac{1}{2}\right)m\xi^k\right]. \tag{5.8}$$

As in case of the multiplet $(\mathbf{1}, \mathbf{4}, \mathbf{3})$, for analyzing the superconformal properties of the multiplet $(\mathbf{2}, \mathbf{4}, \mathbf{2})$ it will be convenient to pass to the supercoset (3.22), in which the time-translation generator is $\mathcal{H} \in so(2,1)$. Imposing the constraints (5.1) with the covariant derivatives defined in (3.33) and choosing $\kappa = 0$, we come to the left chiral subspace parametrized by the same coordinates (t_L, θ_i) as before, with the definition (5.3) being valid. It is straightforward to check that this set of coordinates is closed under the superconformal transformations generated by (3.27) and (3.29) only for $\alpha = -1$. The relevant coordinate transformations read

$$\delta\theta_i = \epsilon_i + 2\mu\bar{\epsilon}^k\theta_k\theta_i + \varepsilon_i e^{-i\mu t_L},$$
$$\delta t_L = 2i\bar{\epsilon}^k\theta_k + 2i\bar{\varepsilon}^k\theta_k e^{i\mu t_L}. \tag{5.9}$$

This agrees with the observation that under the action of the generators C, $\bar{C}$ (3.30) belonging to the group $SU'(2)$ the constraints (5.1) are not covariant. Thus, the chiral subspaces are closed, and, respectively, the chirality constraints are covariant, only for the conformal supergroup $D(2,1;\alpha=-1) = PSU(1,1|2) \rtimes U(1)_{\rm ext}$. Note that the chiral superfields in the flat $\mathcal{N}=4$ superspace are also known to preserve the superconformal $D(2,1;\alpha)$ covariance only for the values $\alpha = -1, 0$ [12,13].

At $\alpha = -1$, the overall $U(1)$ charge operator $\tilde{F}$ drops out from the covariant derivatives (3.33), so the only solution of (5.1) in this case, i.e. the solution consistent with the superconformal covariance, corresponds to the choice $\kappa = 0$ in (5.6). As follows from (5.7), the $Q^i, \bar{Q}_k$ transformations of Φ_L at $\kappa = 0$ (i.e. those with $\epsilon_i, \bar{\epsilon}^i$) do not involve any weight terms. Since in the appropriate basis $S^i(\mu) = Q^i(-\mu), \bar{S}_i(\mu) = \bar{Q}_i(-\mu)$, the same should be true for the $\varepsilon_i, \bar{\varepsilon}^i$ transformations, i.e.

$$\delta_\epsilon\Phi_L = \delta_\varepsilon\Phi_L = 0. \tag{5.10}$$

On the other hand, the measure of integration over the $SU(2|1)$ superspace $d\zeta$ is not superconformally invariant at any α [recall (3.32)], so it is impossible to construct a homogeneous superconformally invariant action out of the superfield Φ_L transforming as in (5.10).

One way to construct the superconformal action is to pass to its inhomogeneous version, as was done for the $\alpha = 0$ case of the multiplet $(\mathbf{1}, \mathbf{4}, \mathbf{3})$ in Sec. IV E. This will be performed in Sec. V C. Another way which allows one to construct a more general class of superconformal actions is to start from the embedding of $SU(2|1)$ into a central-charge extension of $PSU(1,1|2)$, i.e., the supergroup $SU(1,1|2)$ with the superalgebra given in Appendix A, Eqs. (A4)–(A6). The corresponding $su(2|1)$ subalgebra is specified by the anticommutator

$$\{Q^i, \bar{Q}_j\} = 2\mu I^i_j + 2\delta^i_j(\mathcal{H} - \mu Z_1), \tag{5.11}$$

where the central charge generator Z_1 *commutes* with all other generators. The natural modification of the supercoset (3.22) for $\alpha = -1$ is as follows:

$$\frac{SU(2|1) \rtimes U(1)_{\text{ext}}}{SU(2) \times U(1)_{\text{int}} \times U(1)_{\text{ext}}} \sim \frac{\{Q^i, \bar{Q}_j, \mathcal{H}, Z_1, F, I^i_j\}}{\{I^i_j, Z_1, F\}} \sim \frac{\{Q^i, \bar{Q}_j, \mathcal{H}, Z_1, I^i_j\}}{\{I^i_j, Z_1\}}, \tag{5.12}$$

where $SU(2|1)$ in the numerator is defined through the Z_1 extended anticommutation relation (5.11) and we placed Z_1 into the stability subgroup. Recall that the former internal generator F becomes an outer automorphism generator at $\alpha = -1$ and is completely split from the remaining $su(1,1|2)$ generators.

An element of the coset (5.12) coincides with (3.25). However, due to the appearance of the new generator Z_1 in the stability subgroup and the modification of the basic $SU(2|1)$ anticommutator as in (5.11), the covariant spinor derivatives (3.33) at $\alpha = -1$ should be extended as

$$\mathcal{D}^i \Rightarrow \mathcal{D}^i_Z = \mathcal{D}^i + \mu e^{-\frac{i}{2}\mu t}\bar{\theta}^i Z_1,$$
$$\bar{\mathcal{D}}_j \Rightarrow \bar{\mathcal{D}}_{Zj} = \bar{\mathcal{D}}_j - \mu e^{\frac{i}{2}\mu t}\theta_j Z_1. \tag{5.13}$$

Now we can require that the superfield Φ have a nonzero charge with respect to Z_1:

$$Z_1\Phi = b\Phi. \tag{5.14}$$

Then, imposing the chirality condition (5.1) with the modified covariant derivative (5.13), i.e.,

$$\bar{\mathcal{D}}_{Zj}\Phi = (\bar{\mathcal{D}}_j - \mu e^{\frac{i}{2}\mu t}\theta_j Z_1)\Phi = 0, \tag{5.15}$$

one obtains the solution

$$\Phi(t,\theta,\bar{\theta}) = (1 + 2\mu\bar{\theta}^k\theta_k)^{-\frac{b}{2}}\Phi_L(t_L,\theta),$$
$$\Phi_L(t_L,\theta) = z + \sqrt{2}\theta_i\xi^i e^{\frac{i}{2}\mu t_L} + (\theta)^2 B e^{i\mu t_L}, \tag{5.16}$$

which looks like (5.6), with $b = 2\kappa$ and the fields redefined as

$$\xi^i(t) \to \xi^i(t)e^{\frac{i}{2}\mu t}, \qquad B(t) \to B(t)e^{i\mu t}, \quad \text{and c.c.} \tag{5.17}$$

To preserve this $b \neq 0$ chirality, the holomorphic chiral superfield Φ_L should have the following ϵ and ε transformation laws:

$$\delta_\epsilon\Phi_L = 2b\mu\bar{\epsilon}^i\theta_i\Phi_L, \quad \delta_\varepsilon\Phi_L = -2b\mu\bar{\varepsilon}^i\theta_i e^{i\mu t_L}\Phi_L, \tag{5.18}$$

or, in terms of the superfield Φ,

$$\delta_\epsilon\Phi = b\mu(\bar{\epsilon}^i\theta_i + \epsilon_i\bar{\theta}^i)\Phi,$$
$$\delta_\varepsilon\Phi = -b\mu(3\bar{\varepsilon}^i\theta_i e^{i\mu t} - \varepsilon_i\bar{\theta}^i e^{-i\mu t})(1 - \mu\bar{\theta}^k\theta_k)\Phi. \tag{5.19}$$

Under the odd transformations (5.9), (5.18), the component fields in (5.16) are transformed as

$$\delta z = -\sqrt{2}\epsilon_k\xi^k e^{\frac{i}{2}\mu t} - \sqrt{2}\varepsilon_k\xi^k e^{-\frac{i}{2}\mu t},$$
$$\delta\xi^i = \sqrt{2}\bar{\epsilon}^i(i\dot{z} - b\mu z)e^{-\frac{i}{2}\mu t} - \sqrt{2}\epsilon^i B e^{\frac{i}{2}\mu t} + \sqrt{2}\bar{\varepsilon}^i(i\dot{z} + b\mu z)e^{\frac{i}{2}\mu t} - \sqrt{2}\varepsilon^i B e^{-\frac{i}{2}\mu t},$$
$$\delta B = -\sqrt{2}\bar{\epsilon}_k\left[i\dot{\xi}^k - \left(b - \frac{1}{2}\right)\mu\xi^k\right]e^{-\frac{i}{2}\mu t} - \sqrt{2}\bar{\varepsilon}_k\left[i\dot{\xi}^k + \left(b - \frac{1}{2}\right)\mu\xi^k\right]e^{\frac{i}{2}\mu t}. \tag{5.20}$$

To avoid a possible confusion, let us point out that, leaving aside the issues of superconformal covariance, the $SU(2|1)$ chirality based on the coset (2.5) and the covariant derivatives defined in (2.11) [Eqs. (5.1)–(5.8)] is equivalent to that based on the coset (5.12) and the covariant derivatives (5.13). Indeed, using the relation $\mathcal{H} = H - \mu F$, one can rewrite (5.11) as

$$\{Q^i, \bar{Q}_j\} = 2\mu I^i_j + 2\delta^i_j[H - \mu(F + Z_1)],$$

which has the same form as the anticommutator in (2.4), with $m=\mu$ and the substitution $F\to F+Z_1$. The generator $F+Z_1$ cannot be distinguished from F, since Z_1 commutes with anything and does not act on the superspace coordinates. Then one can start from the supercoset (2.5), make the shift $F\to F+Z_1$, and impose, instead of (5.5), the condition $(\tilde{F}+Z_1)\Phi=2\kappa\Phi$, which can be realized either with $\tilde{F}\Phi=2\kappa\Phi, Z_1\Phi=0$ or with $\tilde{F}\Phi=0$, $Z_1\Phi=2\kappa\Phi, b\equiv 2\kappa$. The relevant covariant derivatives (2.11) and (5.13), equally as the solutions (5.6) and (5.16), have the same form for both options. The difference between F and Z_1 is displayed at the full superconformal level: In the basis $(\mathcal{H},F)$, the generator F entirely splits from all other superconformal generators, while there is no way to make Z_1 not appear on the right-hand sides of the relevant anticommutators [see Eqs. (A4)–(A6) for the case $Z_2=Z_3=0$].

B. Superconformal Lagrangian

The general $SU(2|1)$ invariant action of the chiral superfields is defined as

$$S(\Phi)=\int dtL=\frac{1}{4}\int d\zeta f(\Phi,\bar{\Phi}), \qquad (5.21)$$

where $f(\Phi,\bar{\Phi})$ is a Kähler potential. The corresponding component Lagrangian reads

$$\begin{aligned} L&=g\dot{\bar{z}}\,\dot{z}+\frac{i}{2}g(\bar{\xi}_i\dot{\xi}^i-\dot{\bar{\xi}}_i\xi^i)-\frac{i}{2}\bar{\xi}_k\xi^k(\dot{\bar{z}}g_{\bar{z}}-\dot{z}g_z)-\frac{1}{2}(\xi)^2\bar{B}g_z\\ &-\frac{1}{2}(\bar{\xi})^2Bg_{\bar{z}}+g\bar{B}B+\frac{1}{4}(\xi)^2(\bar{\xi})^2g_{z\bar{z}}+ib\mu(\dot{\bar{z}}z-\dot{z}\,\bar{z})g\\ &-\frac{i}{2}\mu(\dot{\bar{z}}f_{\bar{z}}-\dot{z}f_z)-\mu\bar{\xi}_k\xi^kU-\mu^2V, \end{aligned} \qquad (5.22)$$

where

$$\begin{aligned} V&=\frac{b}{2}(\bar{z}\partial_{\bar{z}}+z\partial_z)f-\frac{b^2}{4}(\bar{z}\partial_{\bar{z}}+z\partial_z)^2f,\\ U&=\frac{b}{2}(\bar{z}\partial_{\bar{z}}+z\partial_z)g+(b-1)g+\frac{g}{2}. \end{aligned} \qquad (5.23)$$

Here, the lowercase indices denote the differentiation in $z,\bar{z}$, $f_{z\bar{z}}=\partial_z\partial_{\bar{z}}f$, and $g\coloneqq f_{z\bar{z}}$ is the metric on a Kähler manifold. Performing the redefinition (5.17) in (5.22) and choosing $b=2\kappa$, one can see that this Lagrangian coincides with the chiral $SU(2|1)$ Lagrangian given in Ref. [4] on the basis of the supercoset (2.5), in accord with the equivalency of two definitions of chirality, as was discussed in the end of the previous subsection.

According to (5.19), in order to render the action (5.21) superconformal, one needs to define the Kähler potential as

$$f^{(b)}_{\text{sc}}(\Phi,\bar{\Phi})=(\Phi\bar{\Phi})^{\frac{1}{2b}}. \qquad (5.24)$$

Then the Lagrangian

$$\begin{aligned} L^{(b)}_{\text{sc}}&=\frac{(z\bar{z})^{\frac{1}{2b}-1}}{4b^2}\left[\dot{\bar{z}}\,\dot{z}+\frac{i}{2}(\bar{\xi}_i\dot{\xi}^i-\dot{\bar{\xi}}_i\xi^i)+\bar{B}B\right]\\ &+\frac{(2b-1)^2}{64b^4}(z\bar{z})^{\frac{1}{2b}-2}(\xi)^2(\bar{\xi})^2+\frac{2b-1}{8b^3}(z\bar{z})^{\frac{1}{2b}-2}\\ &\times\left[\frac{i}{2}\bar{\xi}_k\xi^k(\dot{\bar{z}}z-\dot{z}\,\bar{z})+\frac{1}{2}(\xi)^2\bar{B}\,\bar{z}+\frac{1}{2}(\bar{\xi})^2Bz\right]-\frac{\mu^2}{4}(z\bar{z})^{\frac{1}{2b}} \end{aligned} \qquad (5.25)$$

is invariant under the superconformal transformations (5.20).

The simplest case of (5.25) corresponding to the choice $b=1/2$ and yielding the free action,

$$L^{(b=1/2)}_{\text{sc}}=\dot{\bar{z}}\,\dot{z}+\frac{i}{2}(\bar{\xi}_i\dot{\xi}^i-\dot{\bar{\xi}}_i\xi^i)+\bar{B}B-\frac{\mu^2}{4}z\bar{z}, \qquad (5.26)$$

was previously worked out in Ref. [4].[11]

Thus we observe that the superconformal sigma-model-type action for the multiplet $(\mathbf{2},\mathbf{4},\mathbf{2})$ exists only for the nonzero central charge Z_1; i.e., the relevant invariance supergroup is $SU(1,1|2)$, not its quotient $PSU(1,1|2)$. It is worth noting that the action (5.21) with the superfield Lagrangian (5.24), at any $b\neq 0$, is in fact related to the free bilinear action through the field redefinition

$$\begin{aligned} &(\Phi_L)^{\frac{1}{2b}}=\hat{\Phi}_L,\\ &S^{(b)}_{\text{sc}}(\Phi)=S^{(b=1/2)}_{\text{sc}}(\hat{\Phi})=\frac{1}{4}\int d\zeta(1+2\mu\bar{\theta}^k\theta_k)^{-\frac{1}{2}}\hat{\Phi}_L\bar{\hat{\Phi}}_R. \end{aligned} \qquad (5.27)$$

In other words, without loss of generality, we can always choose $b=1/2$ and deal with the Lagrangian (5.26). The same equivalence to the free actions is valid also for other types of the superconformal sigma-model term of the multiplet $(\mathbf{2},\mathbf{4},\mathbf{2})$.

C. Conformal superpotential

One can define the chiral superspace measure $d\zeta_L$ which is invariant under the superconformal transformations (5.9):

$$d\zeta_L=dt_Ld^2\theta e^{-i\mu t_L}, \qquad \delta_\epsilon(d\zeta_L)=\delta_\varepsilon(d\zeta_L)=0. \qquad (5.28)$$

Taking into account the explicit form of the superconformal transformations with $b\neq 0$, Eq. (5.18), the only superpotential term respecting superconformal invariance is

[11]These actions become identical after choosing $\kappa=1/4$ and making the redefinition (5.17) in the action of Ref. [4], which eliminates there the term $\sim\bar{\xi}\xi$. Note that the $su(2|2)$ symmetry found in this problem in Ref. [4] appears only at the quantum level and is not related to the superconformal symmetry $SU(1,1|2)\rtimes U(1)$ which is present already at the classical level.

$$S_{\rm sc}^{\rm pot}(\Phi) = \nu \int d\zeta_L \ln \Phi_L + {\rm c.c.} \qquad (5.29)$$

The corresponding superconformal Lagrangian reads

$$L_{\rm sc}^{\rm pot} = \nu\left(\frac{2B}{z} + \frac{\xi_i \xi^i}{z^2}\right) + {\rm c.c.} \qquad (5.30)$$

After summing it with (5.26) and eliminating the auxiliary fields, the on-shell superconformal trigonometric Lagrangian acquires the standard conformal potential

$$-4\frac{|\nu|^2}{z\bar z}, \qquad (5.31)$$

in addition to the oscillator term $-\frac{\mu^2}{4} z\bar z$. Thus, the nontrivial dynamics in the Lagrangian of the multiplet $(\mathbf{2},\mathbf{4},\mathbf{2})$ invariant under the trigonometric realization of the superconformal group arises solely due to the superpotential term (5.29). For the parabolic realization, the same statement can be traced back to Ref. [21].

D. Inhomogeneous superconformal action at $b = 0$

As was already mentioned, at $b = 0$ (or, equivalently, at $\kappa = 0$) we encounter difficulties when trying to construct the superconformal action. It is still possible to define the inhomogeneous superconformal action with $b = 0$ by resorting to the same procedure as in Sec. IV E. Indeed, the parameter b can be identified with a central charge of $su(1,1|2)$, and therefore one can identify $-b$ with the scaling dimension λ_D of the chiral multiplet [14,22,23]. Making the redefinition

$$z \to z + \frac{\rho}{b}, \qquad \bar z \to \bar z + \frac{\rho}{b}, \qquad (5.32)$$

detaching the singular factors, and finally sending $b \to 0$, we obtain the Lagrangian

$$\begin{aligned} L_{\rm sc}^{(b=0,\rho)} &= e^{\frac{z+\bar z}{2\rho}}\left[\dot{\bar z}\,\dot z + \frac{i}{2}(\bar\xi_i \dot\xi^i - \dot{\bar\xi}_i \xi^i) + \bar B B\right] \\ &\quad - \frac{i}{4\rho}\bar\xi^k \xi_k(\dot{\bar z} - \dot z)e^{\frac{z+\bar z}{2\rho}} - \frac{1}{4\rho}[(\xi)^2 \bar B + (\bar\xi)^2 B]e^{\frac{z+\bar z}{2\rho}} \\ &\quad + \frac{1}{16\rho^2}e^{\frac{z+\bar z}{2\rho}}(\xi)^2(\bar\xi)^2 - \mu^2\rho^2 e^{\frac{z+\bar z}{2\rho}}. \end{aligned} \qquad (5.33)$$

It can be derived from the following $SU(2|1)$ superfield action:

$$\begin{aligned} S_{\rm sc}^{(b=0,\rho)}(\Phi) &= \int dt L_{\rm sc}^{(b=0,\rho)} \\ &= \rho^2 \int d\zeta (1 + 2\mu\bar\theta^k\theta_k)^{-\frac12} e^{\frac{\Phi_L + \bar\Phi_R}{2\rho}}. \end{aligned} \qquad (5.34)$$

The relevant supersymmetric transformations (5.20) with $b = 0$ should be extended by the inhomogeneous pieces

$$\begin{aligned} &\delta_{(\rho)}\xi^i = -\sqrt{2}\rho\mu(\bar\epsilon^i e^{-\frac{i}{2}\mu t} - \bar\varepsilon^i e^{\frac{i}{2}\mu t}), \\ &\delta_{(\rho)} z = \delta_{(\rho)} B = 0. \end{aligned} \qquad (5.35)$$

This is equivalent to saying that, at $b = 0$, the "passive" variation of the holomorphic chiral superfield Φ_L under both supersymmetries involves only the inhomogeneous parts

$$\delta_{(\rho)}\Phi_L = 2\rho\mu(\bar\epsilon^k\theta_k - \bar\varepsilon^k\theta_k e^{i\mu t_L}). \qquad (5.36)$$

It can be obtained from the transformation (5.18), where Φ_L is shifted as

$$\Phi_L \to \Phi_L + \frac{\rho}{b} \qquad (5.37)$$

in conjunction with the shift (5.32). Then we can write the invariant superpotential term as

$$\begin{aligned} &S'^{\rm pot}_{\rm sc}(\Phi) = \nu \int d\zeta_L \Phi_L + {\rm c.c.} \\ &\Rightarrow L'^{\rm pot}_{\rm sc} = 2\nu B + 2\bar\nu\,\bar B\,. \end{aligned} \qquad (5.38)$$

The action (5.34), like its $b \neq 0$ counterpart, can be reduced to the bilinear action by means of the redefinition

$$e^{\frac{\Phi_L}{2\rho}} \sim \hat\Phi_L, \qquad \Phi_L \sim \ln \hat\Phi_L.$$

Then the full $b = 0$ superconformal superfield action amounts to a sum of the free kinetic action and the logarithmic superconformal potential.

Note that the action (5.34) can be rewritten as

$$S_{\rm sc}^{(b=0,\rho)}(\Phi) = \rho^2 \int d\zeta e^{\frac{\Phi + \bar\Phi}{2\rho}}, \qquad (5.39)$$

where

$$\Phi(t,\theta,\bar\theta) = \Phi_L(t_L,\theta) - \rho\mu\bar\theta^k\theta_k(1 - \mu\bar\theta^i\theta_i), \qquad (5.40)$$

and

$$\begin{aligned} \delta\Phi &= \rho\mu(\bar\epsilon^i\theta_i + \epsilon_i\bar\theta^i) \\ &\quad - \rho\mu(1 - \mu\bar\theta^k\theta_k)(3\bar\varepsilon^k\theta_k e^{i\mu t} - \varepsilon_k\bar\theta^k e^{-i\mu t}). \end{aligned} \qquad (5.41)$$

The superfield (5.40) can be regarded as a solution of the chirality condition (5.1a) with the covariant derivative (5.13), in which the central charge Z_1 acts on Φ as the pure shift

$$Z_1\Phi = \rho. \qquad (5.42)$$

In this way, the parameter $\rho \neq 0$ activates a nonvanishing central charge in $su(1,1|2)$. Thus, the superconformal sigma-model-type action at $b = 0$ exists only on account of a nonzero central charge in $su(1,1|2)$, like in the $b \neq 0$ case.

E. The limit $\mu = 0$

As an instructive example, we consider the parabolic chiral model obtained in the limit $\mu = 0$.

In this limit, the superconformally invariant action of the chiral multiplet becomes

$$S_{\rm sc}^{(\mu=0)}(\Phi) = \frac{1}{4}\int dt d^2\theta d^2\bar{\theta}(\Phi\bar{\Phi})^{\frac{1}{2b}}. \qquad (5.43)$$

The chiral superfield Φ transforms under the superconformal charges as

$$\delta\Phi = -4ib\bar{\varepsilon}'^i\theta_i\Phi, \qquad (5.44)$$

while transforming as a scalar under the $d = 1$ Poincaré supersymmetry with the parameters $\epsilon'^i, \bar{\varepsilon}'_i$. The whole amount of superconformal transformations is derived from the trigonometric ones according to the procedure (4.37). The parameter b is still interpreted as the central charge of $su(1,1|2)$. Then the superconformal component off-shell Lagrangian

$$L_{\rm sc}^{(\mu=0)} = \frac{(z\bar{z})^{\frac{1}{2b}-1}}{4b^2}\left[\dot{\bar{z}}\,\dot{z} + \frac{i}{2}(\bar{\xi}_i\dot{\xi}^i - \dot{\bar{\xi}}_i\xi^i) + \bar{B}B\right] + \frac{(2b-1)^2}{64b^4}(z\bar{z})^{\frac{1}{2b}-2}(\xi)^2(\bar{\xi})^2$$
$$+ \frac{2b-1}{8b^3}(z\bar{z})^{\frac{1}{2b}-2}\left[\frac{i}{2}\bar{\xi}_k\xi^k(\dot{\bar{z}}z - \dot{z}\,\bar{z}) + \frac{1}{2}(\xi)^2\bar{B}\,\bar{z} + \frac{1}{2}(\bar{\xi})^2Bz\right] \qquad (5.45)$$

is invariant under both the Poincaré and the superconformal $\mathcal{N} = 4, d = 1$ transformations

$$\delta z = -\sqrt{2}\epsilon'_k\xi^k + \sqrt{2}t\epsilon'_k\xi^k,$$
$$\delta\xi^i = \sqrt{2}i\bar{\epsilon}'^i\dot{z} - \sqrt{2}\epsilon'^iB - \sqrt{2}i\bar{\epsilon}'^i(t\dot{z} - 2bz) + \sqrt{2}t\epsilon'^iB,$$
$$\delta B = -\sqrt{2}i\bar{\epsilon}'_k\dot{\xi}^k + \sqrt{2}i\bar{\epsilon}'_k[t\dot{\xi}^k - (2b-1)\xi^k]. \qquad (5.46)$$

The inhomogeneous superconformal Lagrangian at $b = 0$ reads

$$L_{\rm sc}^{(\mu=0,b=0,\rho)} = e^{\frac{z+\bar{z}}{2\rho}}\left[\dot{\bar{z}}\,\dot{z} + \frac{i}{2}(\bar{\xi}_i\dot{\xi}^i - \dot{\bar{\xi}}_i\xi^i) + \bar{B}B\right]$$
$$- \frac{i}{4\rho}\bar{\xi}^k\xi_k(\dot{\bar{z}} - \dot{z})e^{\frac{z+\bar{z}}{2\rho}} - \frac{1}{4\rho}[(\xi)^2\bar{B} + (\bar{\xi})^2B]e^{\frac{z+\bar{z}}{2\rho}}$$
$$+ \frac{1}{16\rho^2}e^{\frac{z+\bar{z}}{2\rho}}(\xi)^2(\bar{\xi})^2, \qquad (5.47)$$

and it can be deduced from the superfield action

$$S_{\rm sc}^{(\mu=0,b=0,\rho)}(\Phi) = \int dt L_{\rm sc}^{(\mu=0,b=0,\rho)}$$
$$= \rho^2\int dt d^2\theta d^2\bar{\theta}e^{\frac{\Phi+\bar{\Phi}}{2\rho}}. \qquad (5.48)$$

In the inhomogeneous case, the superconformal transformation of the superfield Φ involves only the inhomogeneous piece

$$\delta_{(\rho)}\Phi = -4i\rho\bar{\varepsilon}'^k\theta_k. \qquad (5.49)$$

Since the superpotential terms (5.30) and (5.38) do not depend on μ, their form is preserved in the parabolic limit $\mu = 0$. The only peculiarity is that the invariant chiral integration measure (5.28) turns into the flat measure $dt_Ld^2\theta$. Obviously, the kinetic superfield term (5.48) is reduced to the free one after the appropriate holomorphic redefinition of Φ.

VI. GENERALIZED CHIRAL MULTIPLET

A. Another type of chiral $SU(2|1)$ superspace

In Ref. [5], there was defined a different kind of $SU(2|1)$ chiral superfield. Let us consider the general coset (5.12). The chiral condition (5.1) can be generalized as

$$\text{(a)}\ \tilde{\bar{\mathcal{D}}}_i\varphi = 0, \qquad \text{(b)}\ \tilde{\mathcal{D}}^i\bar{\varphi} = 0, \qquad (6.1)$$

where the spinor derivatives $\tilde{\mathcal{D}}^i, \tilde{\bar{\mathcal{D}}}_i$ are the following linear combinations of the covariant derivatives defined in (3.33):

$$\tilde{\bar{\mathcal{D}}}_i = \cos\lambda\bar{\mathcal{D}}_i - \sin\lambda\mathcal{D}_i,$$
$$\tilde{\mathcal{D}}^i = \cos\lambda\mathcal{D}^i + \sin\lambda\bar{\mathcal{D}}^i. \qquad (6.2)$$

One can treat such combinations as the result of particular rotation by an extra $SU'(2)$ group with the generators $\{C, \bar{C}, F\}$. In general, the $SU'(2)$ transformations break the covariance of the constraints (6.1). The latter remain covariant only under the special combination of the $SU'(2)$ generators,

$$F' = F\cos 2\lambda + \frac{1}{2}(C+\bar{C})\sin 2\lambda. \tag{6.3}$$

Thus, the constraints (6.1) are covariant under the superconformal group $D(2,1;\alpha)$ only for $\alpha=-1$, when it is reduced to the supergroup $PSU(1,1|2)$, and under the external automorphism $U(1)$ group with the generator F' (6.3). The Hamiltonian $\mathcal{H}$ is identified with the whole internal $U(1)$ generator of the nonextended subalgebra $su(2|1)\subset psu(1,1|2)$ for $\alpha=-1$, $m=\mu$.

The conditions (6.1) amount to the existence of the left and right chiral subspaces:

$$(\hat{t}_L,\hat{\theta}_i),\qquad (\hat{t}_R,\hat{\bar{\theta}}^i), \tag{6.4}$$

where

$$\hat{t}_L = t + i\hat{\bar{\theta}}^k\hat{\theta}_k,$$
$$\hat{\theta}_i = (\cos\lambda\theta_i e^{\frac{i}{2}\mu t} + \sin\lambda\bar{\theta}_i e^{-\frac{i}{2}\mu t})\left(1-\frac{\mu}{2}\bar{\theta}^k\theta_k\right). \tag{6.5}$$

As expected, the coordinate set $(\hat{t}_L,\hat{\theta}_i)$ is closed under the $SU(2|1)$ transformations

$$\delta\hat{\theta}_i = \cos\lambda(\epsilon_i e^{\frac{i}{2}\mu\hat{t}_L} + \mu\bar{\epsilon}^k\hat{\theta}_k\hat{\theta}_i e^{-\frac{i}{2}\mu\hat{t}_L}) + \sin\lambda(\bar{\epsilon}_i e^{-\frac{i}{2}\mu\hat{t}_L} + \mu\epsilon^k\hat{\theta}_k\hat{\theta}_i e^{\frac{i}{2}\mu\hat{t}_L}),$$
$$\delta\hat{t}_L = 2i\cos\lambda\bar{\epsilon}^k\hat{\theta}_k e^{-\frac{i}{2}\mu\hat{t}_L} - 2i\sin\lambda\epsilon^k\hat{\theta}_k e^{\frac{i}{2}\mu\hat{t}_L}. \tag{6.6}$$

The second $SU(2|1)$ transformations

$$\delta\hat{\theta}_i = \cos\lambda(\varepsilon_i e^{-\frac{i}{2}\mu\hat{t}_L} - \mu\bar{\varepsilon}^k\hat{\theta}_k\hat{\theta}_i e^{\frac{i}{2}\mu\hat{t}_L}) + \sin\lambda(\bar{\varepsilon}_i e^{\frac{i}{2}\mu\hat{t}_L} - \mu\varepsilon^k\hat{\theta}_k\hat{\theta}_i e^{-\frac{i}{2}\mu\hat{t}_L}),$$
$$\delta\hat{t}_L = 2i\cos\lambda\bar{\varepsilon}^k\hat{\theta}_k e^{\frac{i}{2}\mu\hat{t}_L} - 2i\sin\lambda\varepsilon^k\hat{\theta}_k e^{-\frac{i}{2}\mu\hat{t}_L} \tag{6.7}$$

are generated by (3.29) for $\alpha=-1$ and also leave the left chiral subspace invariant. The chiral subspace (6.5) is not closed under the $SU'(2)$ transformations generated by $\{C,\bar{C},F\}$, except those generated by the $U(1)$ generator (6.3).

Since at $\alpha=-1$ the superconformal group admits the central extension, in what follows we will assume that the $\alpha=-1$ spinor covariant derivatives in the definition (6.2) are replaced by the central-extended ones $\mathcal{D}^i_Z, \bar{\mathcal{D}}_{Zi}$ (5.13), i.e. in the chirality constraints (6.1), we will use

$$\tilde{\bar{\mathcal{D}}}_i = \cos\lambda\bar{\mathcal{D}}_{Zi} - \sin\lambda\mathcal{D}_{Zi},$$
$$\tilde{\mathcal{D}}^i = \cos\lambda\mathcal{D}^i_Z + \sin\lambda\bar{\mathcal{D}}^i_Z. \tag{6.8}$$

Assuming that the central charge acts on the superfield as[12]

$$Z_1\varphi = b\cos 2\lambda\varphi, \tag{6.9}$$

the solution of (6.1) is given by

$$\varphi(t,\hat{\theta},\hat{\bar{\theta}}) = e^{-b\mu\cos 2\lambda\hat{\bar{\theta}}^k\hat{\theta}_k}\varphi_L(\hat{t}_L,\hat{\theta}),$$
$$\varphi_L(\hat{t}_L,\hat{\theta}) = z + \sqrt{2}\hat{\theta}_k\xi^k + (\hat{\theta})^2 B. \tag{6.10}$$

As we will see, the parameter $|b|$ is associated with the norm of the triplet of central charges like in the previous section, since in the case under consideration the superalgebra $psu(1,1|2)$ turns out to be extended by three constant central charges. This is consistent with the limit $\cos 2\lambda = 1$ in the generalized conditions (6.1).

The transformations of the superfield φ are given by

$$\begin{aligned}
\delta_\epsilon\varphi &= b\mu\cos 2\lambda\bar{\epsilon}^i e^{-\frac{i}{2}\mu t}\left[\cos\lambda\hat{\theta}_i\left(1+\frac{\mu}{2}\hat{\bar{\theta}}^k\hat{\theta}_k\right) - \sin\lambda\hat{\bar{\theta}}_i\left(1-\frac{\mu}{2}\hat{\bar{\theta}}^k\hat{\theta}_k\right)\right]\varphi \\
&\quad + b\mu\cos 2\lambda\epsilon_i e^{\frac{i}{2}\mu t}\left[\cos\lambda\hat{\bar{\theta}}^i\left(1+\frac{\mu}{2}\hat{\bar{\theta}}^k\hat{\theta}_k\right) + \sin\lambda\hat{\theta}^i\left(1-\frac{\mu}{2}\hat{\bar{\theta}}^k\hat{\theta}_k\right)\right]\varphi, \\
\delta_\varepsilon\varphi &= b\mu\cos 2\lambda\bar{\varepsilon}^i e^{\frac{i}{2}\mu t}\left[\cos\lambda\hat{\theta}_i\left(1-\frac{\mu}{2}\hat{\bar{\theta}}^k\hat{\theta}_k\right) - \sin\lambda\hat{\bar{\theta}}_i\left(1+\frac{\mu}{2}\hat{\bar{\theta}}^k\hat{\theta}_k\right)\right]\varphi \\
&\quad + b\mu\cos 2\lambda\varepsilon_i e^{-\frac{i}{2}\mu t}\left[\cos\lambda\hat{\bar{\theta}}^i\left(1-\frac{\mu}{2}\hat{\bar{\theta}}^k\hat{\theta}_k\right) + \sin\lambda\hat{\theta}^i\left(1+\frac{\mu}{2}\hat{\bar{\theta}}^k\hat{\theta}_k\right)\right]\varphi \\
&\quad - 4b\mu\left[\cos\lambda\bar{\varepsilon}^i\hat{\theta}_i\left(1-\frac{\mu}{2}\hat{\bar{\theta}}^k\hat{\theta}_k\right)e^{\frac{i}{2}\mu t} + \sin\lambda\varepsilon^i\hat{\theta}_i\left(1+\frac{\mu}{2}\hat{\bar{\theta}}^k\hat{\theta}_k\right)e^{-\frac{i}{2}\mu t}\right]\varphi.
\end{aligned} \tag{6.11}$$

[12]The eigenvalue of the central charge Z_1 in this case is not obliged to be the same b as in Sec. V. We hope that denoting it also by b will not give rise to any confusion.

The relevant "passive" transformations of the holomorphic superfield φ_L are

$$\delta_\epsilon\varphi_L = 2b\mu\cos 2\lambda(\cos\lambda\bar{\epsilon}^i e^{-\frac{i}{2}\mu\hat{t}_L} - \sin\lambda\epsilon^i e^{\frac{i}{2}\mu\hat{t}_L})\hat{\theta}_i\varphi_L,$$
$$\delta_\varepsilon\varphi_L = 2b\mu\cos 2\lambda(\cos\lambda\bar{\varepsilon}^i e^{\frac{i}{2}\mu\hat{t}_L} - \sin\lambda\varepsilon^i e^{-\frac{i}{2}\mu\hat{t}_L})\hat{\theta}_i\varphi_L - 4b\mu(\cos\lambda\bar{\varepsilon}^i e^{\frac{i}{2}\mu\hat{t}_L} + \sin\lambda\varepsilon^i e^{-\frac{i}{2}\mu\hat{t}_L})\hat{\theta}_i\varphi_L. \qquad (6.12)$$

Then the full set of the off-shell transformations of the component fields is generated by (6.12) and by the coordinate transformations (6.6), (6.7):

$$\delta z = -\sqrt{2}\cos\lambda\epsilon_k\xi^k e^{\frac{i}{2}\mu t} - \sqrt{2}\sin\lambda\bar{\epsilon}_k\xi^k e^{-\frac{i}{2}\mu t},$$
$$\delta\xi^i = \sqrt{2}\bar{\epsilon}^i(i\cos\lambda\dot{z} - b\mu\cos 2\lambda\cos\lambda z - \sin\lambda B)e^{-\frac{i}{2}\mu t} - \sqrt{2}\epsilon^i(i\sin\lambda\dot{z} - b\mu\cos 2\lambda\sin\lambda z + \cos\lambda B)e^{\frac{i}{2}\mu t},$$
$$\delta B = -\sqrt{2}\cos\lambda\bar{\epsilon}_k\left[i\dot{\xi}^k + \frac{\mu}{2}(1-2b\cos 2\lambda)\xi^k\right]e^{-\frac{i}{2}\mu t} + \sqrt{2}\sin\lambda\epsilon_k\left[i\dot{\xi}^k - \frac{\mu}{2}(1+2b\cos 2\lambda)\xi^k\right]e^{\frac{i}{2}\mu t}, \qquad (6.13)$$

$$\delta z = -\sqrt{2}\cos\lambda\varepsilon_k\xi^k e^{-\frac{i}{2}\mu t} - \sqrt{2}\sin\lambda\bar{\varepsilon}_k\xi^k e^{\frac{i}{2}\mu t},$$
$$\delta\xi^i = \sqrt{2}\bar{\varepsilon}^i\left[i\cos\lambda\dot{z} + 2b\mu\cos\lambda\left(1-\frac{1}{2}\cos 2\lambda\right)z - \sin\lambda B\right]e^{\frac{i}{2}\mu t} - \sqrt{2}\varepsilon^i\left[i\sin\lambda\dot{z} - 2b\mu\sin\lambda\left(1+\frac{1}{2}\cos 2\lambda\right)z + \cos\lambda B\right]e^{-\frac{i}{2}\mu t},$$
$$\delta B = -\sqrt{2}\cos\lambda\bar{\varepsilon}_k\left[i\dot{\xi}^k - \frac{\mu}{2}\xi^k + 2b\mu\left(1-\frac{1}{2}\cos 2\lambda\right)\xi^k\right]e^{\frac{i}{2}\mu t} + \sqrt{2}\sin\lambda\varepsilon_k\left[i\dot{\xi}^k + \frac{\mu}{2}\xi^k - 2b\mu\left(1+\frac{1}{2}\cos 2\lambda\right)\xi^k\right]e^{-\frac{i}{2}\mu t}. \qquad (6.14)$$

The new set of the transformations (6.13), (6.14) closes on the centrally extended superalgebra (A4)–(A6) with the central charges

$$Z_1 = b\cos 2\lambda, \qquad Z_2 = b\sin 2\lambda,$$
$$Z_3 = -b\sin 2\lambda, \qquad (Z_1)^2 - Z_2 Z_3 = b^2. \qquad (6.15)$$

The precise realization of the central charges on the superfields $\varphi, \bar{\varphi}$ is given by the following transformations:

$$\delta\varphi = 2ib\mu(a_1\cos 2\lambda + a_2\sin 2\lambda)\varphi, \qquad (6.16)$$

where a_1, a_2 are infinitesimal parameters associated with Z_1 and $Z_2 = -Z_3$.

B. The superconformal Lagrangian

The most general sigma-model part of the $SU(2|1)$ invariant action of the generalized chiral superfields $\varphi(t, \hat{\theta}, \bar{\hat{\theta}})$ is specified by an arbitrary Kähler potential $f(\varphi, \bar{\varphi})$:

$$S(\varphi) = \int dt\tilde{L} = \frac{1}{4}\int d\hat{\zeta} f(\varphi, \bar{\varphi}), \qquad (6.17)$$

where the $SU(2|1)$ invariant measure is

$$d\hat{\zeta} = dtd^2\hat{\theta}d^2\bar{\hat{\theta}}\left[1 + \mu\cos 2\lambda\bar{\hat{\theta}}^k\hat{\theta}_k - \frac{\mu}{2}\sin 2\lambda(\bar{\hat{\theta}})^2 - \frac{\mu}{2}\sin 2\lambda(\hat{\theta})^2\right]. \qquad (6.18)$$

This measure is not invariant under the second-type $SU(2|1)$ transformations (with $\mu \to -\mu$).

The transformations of $d\hat{\zeta}$ can be canceled, using the inhomogeneity of the chiral superfield φ transformation (6.11) for $b \neq 0$. One can check that the superconformal action is uniquely specified by the following Kähler potential:

$$f^{(b)}_{\rm sc}(\varphi, \bar{\varphi}) = (\varphi\bar{\varphi})^{\frac{1}{2b}}. \qquad (6.19)$$

The corresponding full superconformally invariant off-shell component Lagrangian reads

$$\tilde{L}^{(b)}_{\rm sc} = \frac{(z\bar{z})^{\frac{1}{2b}-1}}{4b^2}\left[\dot{\bar{z}}\dot{z} + \frac{i}{2}(\bar{\xi}_i\dot{\xi}^i - \dot{\bar{\xi}}_i\xi^i) + \bar{B}B\right] + \frac{(2b-1)^2}{64b^4}(z\bar{z})^{\frac{1}{2b}-2}(\xi)^2(\bar{\xi})^2 + \frac{2b-1}{8b^3}(z\bar{z})^{\frac{1}{2b}-2}\left[\frac{i}{2}\bar{\xi}_k\xi^k(\dot{\bar{z}}z - \dot{z}\bar{z}) + \frac{1}{2}(\xi)^2\bar{B}\bar{z} + \frac{1}{2}(\bar{\xi})^2 Bz\right] - \frac{2b-1}{16b^2}(z\bar{z})^{\frac{1}{2b}-2}\mu\sin 2\lambda[\bar{z}^2(\xi)^2 + z^2(\bar{\xi})^2] - \frac{(z\bar{z})^{\frac{1}{2b}-1}}{2b}\left[\frac{\mu}{2}\sin 2\lambda(\bar{B}z + B\bar{z}) + \frac{b\mu^2}{2}\cos^2 2\lambda z\bar{z}\right]. \qquad (6.20)$$

In the particular case $\cos\lambda = 1$, one comes back to the Lagrangian (5.25).

C. Remark

Let us make the following redefinition in (6.20):

$$B = \tilde{B} + b\mu\sin 2\lambda z, \quad \text{and c.c.} \qquad (6.21)$$

The redefined superconformal Lagrangian (6.20) exactly coincides with the previously constructed superconformal Lagrangian (5.25) (with $B \to \tilde{B}$). However, it is invariant under the following modified transformations:

$$\begin{aligned}
\delta z &= -\sqrt{2}(\cos\lambda\epsilon_k + \sin\lambda\bar{\epsilon}_k)\xi^k e^{\frac{i}{2}\mu t} \\
&\quad - \sqrt{2}(\sin\lambda\bar{\epsilon}_k + \cos\lambda\varepsilon_k)\xi^k e^{-\frac{i}{2}\mu t}, \\
\delta\xi^i &= \sqrt{2}(\cos\lambda\bar{\epsilon}^i - \sin\lambda\varepsilon^i)(i\dot{z} - b\mu z)e^{-\frac{i}{2}\mu t} \\
&\quad - \sqrt{2}(\sin\lambda\bar{\epsilon}^i + \cos\lambda\varepsilon^i)\tilde{B}e^{-\frac{i}{2}\mu t} \\
&\quad - \sqrt{2}(\sin\lambda\epsilon^i - \cos\lambda\bar{\varepsilon}^i)(i\dot{z} + b\mu z)e^{\frac{i}{2}\mu t} \\
&\quad - \sqrt{2}(\cos\lambda\epsilon^i + \sin\lambda\bar{\varepsilon}^i)\tilde{B}e^{\frac{i}{2}\mu t}, \\
\delta\tilde{B} &= -\sqrt{2}(\cos\lambda\bar{\epsilon}_k - \sin\lambda\varepsilon_k)\left[i\dot{\xi}^k - \left(b - \frac{1}{2}\right)\mu\xi^k\right]e^{-\frac{i}{2}\mu t} \\
&\quad + \sqrt{2}(\sin\lambda\epsilon_k - \cos\lambda\bar{\varepsilon}_k)\left[i\dot{\xi}^k + \left(b - \frac{1}{2}\right)\mu\xi^k\right]e^{\frac{i}{2}\mu t},
\end{aligned} \tag{6.22}$$

which are just (6.13), (6.14) rewritten in terms of $\tilde{B}$ defined in (6.21). These transformations are induced by (6.6), (6.7) and the superfield transformations

$$\begin{aligned}
\delta\tilde{\varphi}_L &= 2b\mu[(\cos\lambda\bar{\epsilon}^i - \sin\lambda\varepsilon^i)\hat{\theta}_i e^{-\frac{i}{2}\mu\hat{t}_L} \\
&\quad - (\cos\lambda\bar{\varepsilon}^i - \sin\lambda\epsilon^i)\hat{\theta}_i e^{\frac{i}{2}\mu\hat{t}_L}]\tilde{\varphi}_L.
\end{aligned} \tag{6.23}$$

The newly defined chiral superfield $\tilde{\varphi}_L$ encompasses the field set $(z, \xi^k, \tilde{B})$ and is related to (6.10) as

$$\begin{aligned}
\varphi_L(\hat{t}_L, \hat{\theta}) &= [1 + b\mu\sin 2\lambda(\hat{\theta})^2]\tilde{\varphi}_L(\hat{t}_L, \hat{\theta}), \\
\tilde{\varphi}_L(\hat{t}_L, \hat{\theta}) &= z + \sqrt{2}\hat{\theta}_k\xi^k + (\hat{\theta})^2\tilde{B}.
\end{aligned} \tag{6.24}$$

Note that the $\varepsilon_i, \bar{\varepsilon}^k$ transformations in (6.22), (6.23) are obtained from the $\epsilon_i, \bar{\epsilon}^k$ ones just by the replacement $\mu \to -\mu$ in the latter, in agreement with the general statement of Sec. III.

After passing to the new independent linear combinations of the infinitesimal parameters $\{\epsilon, \bar{\epsilon}, \varepsilon, \bar{\varepsilon}\}$ as

$$\begin{aligned}
\tilde{\epsilon}_k &= \cos\lambda\epsilon_k + \sin\lambda\bar{\varepsilon}_k, \\
\tilde{\varepsilon}_k &= \cos\lambda\varepsilon_k + \sin\lambda\bar{\epsilon}_k, \quad \text{and c.c.,}
\end{aligned} \tag{6.25}$$

the above transformations take just the form of (5.20). These new combinations of the parameters correspond to the following redefinition of the $D(2,1;\alpha = -1)$ supercharges:

$$\begin{aligned}
\tilde{Q}^i &= \cos\lambda Q^i - \sin\lambda\bar{S}^i, \\
\tilde{S}^i &= \cos\lambda S^i - \sin\lambda\bar{Q}^i, \quad \text{and c.c.}
\end{aligned} \tag{6.26}$$

The redefined supercharges close on the superalgebra (A4)–(A6) with the *single* central charge $Z_1 = b$, i.e., the superconformal models of the generalized chiral multiplet prove to be equivalent to the superconformal models associated with the standard chiral multiplet. One can check that the generator (6.3) is the $U(1)$ automorphism generator of the $su(1,1|2)$ superalgebra with the supercharges $\tilde{Q}^i$, $\tilde{S}^i$. Thus, as far as the superconformal $SU(2|1)$ mechanics is concerned, the generalized $SU(2|1)$ chiral multiplet does not give rise to new models compared to the "standard" chiral multiplet.

More details on connection between the standard and generalized $SU(2|1)$ chiralities from the superspace point of view are given in Appendix B.

VII. THE "MIRROR" MULTIPLET (2, 4, 2)

The $\alpha = 0$ version of the chirality conditions (5.1) or (6.1) is not covariant under the full second $SU'(2) \propto \{F, C, \bar{C}\}$ and, hence, under the superconformal group $D(2,1;\alpha = 0)$ which necessarily contains $SU'(2)$ as a subgroup.

However, one can define the "mirror" chiral multiplet $(\mathbf{2}, \mathbf{4}, \mathbf{2})$ which respects the covariance under the $\alpha = 0$ superconformal group realized in the coset (3.24). Using the $\alpha = 0$ covariant derivatives (4.57), we may impose the relevant chiral conditions as

$$\bar{\mathcal{D}}_1\tilde{\Phi} = \mathcal{D}^2\tilde{\Phi} = 0. \tag{7.1}$$

It is straightforward to show that at $\alpha = 0$ these conditions are covariant with respect to the superconformal symmetry $PSU(1,1|2) \rtimes U(1)_{\rm ext}$, with the internal $SU(2)$ group generated by $\{F, C, \bar{C}\}$ and $\mathcal{H}$ as the Hamiltonian. The generator $I_1^1 = -I_2^2$ plays the role of an external automorphism $U(1)_{\rm ext}$ generator, while the generators I_1^2, I_2^1 violate the covariance of (7.1) and so should be thrown away. Since the $SU'(2)$ generators $\{F, C, \bar{C}\}$ form a subalgebra of $psu(1,1|2)$, allowing the chiral superfield to have an external $U(1)$ charge with respect to $\tilde{F}$ would entail the necessity to attach the whole $SU'(2)$ index to $\tilde{\Phi}$. This would result in extension of the field contents of $\tilde{\Phi}$. In order to deal with the chiral multiplet possessing the minimal field contents $(\mathbf{2}, \mathbf{4}, \mathbf{2})$, we are so led to require that

$$\tilde{F}\,\tilde{\Phi} = 0. \tag{7.2}$$

The conditions (7.1) amount to the existence of the chiral subspace $(t_L, \theta_1, \bar{\theta}^2)$, where

$$t_L = t + i\bar{\theta}^1\theta_1 - i\bar{\theta}^2\theta_2. \tag{7.3}$$

It is closed under the superconformal transformations

$$\begin{aligned}
\delta t_L &= 2i(\bar{\epsilon}^1\theta_1 + \epsilon_2\bar{\theta}^2 + \bar{\varepsilon}^1\theta_1 e^{i\mu t_L} + \varepsilon_2\bar{\theta}^2 e^{-i\mu t_L}), \\
\delta\theta_1 &= \epsilon_1 + \varepsilon_1 e^{-i\mu t_L} + 2\mu\varepsilon_2\bar{\theta}^2\theta_1 e^{-i\mu t_L}, \\
\delta\bar{\theta}^2 &= \bar{\epsilon}^2 + \bar{\varepsilon}^2 e^{i\mu t_L} - 2\mu\bar{\varepsilon}^1\theta_1\bar{\theta}^2 e^{i\mu t_L}.
\end{aligned} \tag{7.4}$$

As in the case of the $\alpha = -1$ chiral multiplets, we extend the algebra (3.21) by the central charge generator:

$$\{Q^i, \bar{Q}_j\} = 2\delta^i_j(\mathcal{H} + \mu F) + 2\mu(\sigma_3)^i_j V,$$
$$[F, \bar{Q}_l] = -\frac{1}{2}\bar{Q}_l, \qquad [F, Q^k] = \frac{1}{2}Q^k,$$
$$[\mathcal{H}, \bar{Q}_l] = \frac{\mu}{2}\bar{Q}_l, \qquad [\mathcal{H}, Q^k] = -\frac{\mu}{2}Q^k. \tag{7.5}$$

The superfield $\tilde{\Phi}$ can have a nonzero charge under V:

$$V\tilde{\Phi} = a\tilde{\Phi}. \tag{7.6}$$

Then the extended algebra (7.5) is embedded in the $\alpha = 0$ counterpart of (A4)–(A6). Like in the coset (5.12), we place the central charge in the stability subgroup

$$\frac{\{Q^i, \bar{Q}_j, \mathcal{H}, F, V\}}{\{F, V\}}. \tag{7.7}$$

The modified covariant derivatives are as follows:

$$\mathcal{D}^1 = e^{-\frac{i}{2}\mu t}\left(\frac{\partial}{\partial\theta_1} - i\bar{\theta}^1\partial_t - \mu\bar{\theta}^1\tilde{F} - \mu\bar{\theta}^1 V\right),$$
$$\bar{\mathcal{D}}_2 = e^{\frac{i}{2}\mu t}\left(-\frac{\partial}{\partial\bar{\theta}^2} + i\theta_2\partial_t + \mu\theta_2\tilde{F} - \mu\theta_2 V\right). \tag{7.8}$$

Keeping in mind the condition (7.2), the solution of (7.1) can be written as

$$\tilde{\Phi}(t, \theta_k, \bar{\theta}^k) = e^{-a\mu\bar{\theta}^k\theta_k}[z(t_L) + \sqrt{2}\theta_1\eta^1(t_L)e^{\frac{i}{2}\mu t_L} + \sqrt{2}\bar{\theta}^2\bar{\eta}_2(t_L)e^{-\frac{i}{2}\mu t_L} - 2\theta_1\bar{\theta}^2 B(t_L)]. \tag{7.9}$$

Thus the number a is an analog of the charge b, and it can be identified with the central charge of the conformal superalgebra $su(1,1|2)$ of the $\alpha = 0$ case.

The $\alpha = 0$ chirality-preserving odd transformations of $\tilde{\Phi}$ read

$$\delta_\epsilon\tilde{\Phi} = a\mu(\bar{\epsilon}^1\theta_1 - \epsilon_2\bar{\theta}^2)\tilde{\Phi} - a\mu(\bar{\epsilon}^2\theta_2 - \epsilon_1\bar{\theta}^1)\tilde{\Phi},$$
$$\delta_\varepsilon\tilde{\Phi} = -3a\mu(\bar{\varepsilon}^1\theta_1 e^{i\mu t} - \varepsilon_2\bar{\theta}^2 e^{-i\mu t})\left(1 - \frac{\mu}{3}\bar{\theta}^k\theta_k\right)\tilde{\Phi} - a\mu(\bar{\varepsilon}^2\theta_2 e^{i\mu t} - \varepsilon_1\bar{\theta}^1 e^{-i\mu t})(1 - 3\mu\bar{\theta}^k\theta_k)\tilde{\Phi}. \tag{7.10}$$

They generate the off-shell transformations of the component fields

$$\delta z = -\sqrt{2}\epsilon_1\eta^1 e^{\frac{i}{2}\mu t} - \sqrt{2}\bar{\epsilon}^2\bar{\eta}_2 e^{-\frac{i}{2}\mu t} - \sqrt{2}\varepsilon_1\eta^1 e^{-\frac{i}{2}\mu t} - \sqrt{2}\bar{\varepsilon}^2\bar{\eta}_2 e^{\frac{i}{2}\mu t},$$
$$\delta\eta^1 = \sqrt{2}\bar{\epsilon}^1(i\dot{z} - a\mu z)e^{-\frac{i}{2}\mu t} + \sqrt{2}\bar{\epsilon}^2 B e^{-\frac{i}{2}\mu t} + \sqrt{2}\bar{\varepsilon}^1(i\dot{z} + a\mu z)e^{\frac{i}{2}\mu t} + \sqrt{2}\bar{\varepsilon}^2 B e^{\frac{i}{2}\mu t},$$
$$\delta\bar{\eta}_2 = \sqrt{2}\epsilon_2(i\dot{z} + a\mu z)e^{\frac{i}{2}\mu t} - \sqrt{2}\epsilon_1 B e^{\frac{i}{2}\mu t} + \sqrt{2}\varepsilon_2(i\dot{z} - a\mu z)e^{-\frac{i}{2}\mu t} - \sqrt{2}\varepsilon_1 B e^{-\frac{i}{2}\mu t},$$
$$\delta B = -\sqrt{2}\epsilon_2\left[i\dot{\eta}^1 + \left(a - \frac{1}{2}\right)\mu\eta^1\right]e^{\frac{i}{2}\mu t} - \sqrt{2}\varepsilon_2\left[i\dot{\eta}^1 - \left(a - \frac{1}{2}\right)\mu\eta^1\right]e^{-\frac{i}{2}\mu t} + \sqrt{2}\bar{\epsilon}^1\left[i\dot{\bar{\eta}}_2 - \left(a - \frac{1}{2}\right)\mu\bar{\eta}_2\right]e^{-\frac{i}{2}\mu t} + \sqrt{2}\bar{\varepsilon}^1\left[i\dot{\bar{\eta}}_2 + \left(a - \frac{1}{2}\right)\mu\bar{\eta}_2\right]e^{\frac{i}{2}\mu t}. \tag{7.11}$$

The superconformally invariant superfield action

$$S^{(\alpha=0,a)}_{\rm sc}(\tilde{\Phi}) = \int dt L^{(\alpha=0,a)}_{\rm sc} = -\frac{1}{4}\int dt d^2\theta d^2\bar{\theta}(\tilde{\Phi}\,\bar{\tilde{\Phi}})^{\frac{1}{2a}} \tag{7.12}$$

yields the following component superconformal Lagrangian:

$$L^{(\alpha=0,a)}_{\rm sc} = \frac{(z\bar{z})^{\frac{1}{2a}-1}}{4a^2}\left[\dot{\bar{z}}\,\dot{z} + \frac{i}{2}(\bar{\eta}_i\dot{\eta}^i - \dot{\bar{\eta}}_i\eta^i) + \bar{B}B\right] - \frac{(2a-1)^2}{64a^4}(z\bar{z})^{\frac{1}{2a}-2}(\eta)^2(\bar{\eta})^2 + \frac{2a-1}{8a^3}(z\bar{z})^{\frac{1}{2a}-2}\left[\frac{i}{2}(\bar{\eta}_1\eta^1 - \bar{\eta}_2\eta^2)(\dot{\bar{z}}z - \dot{z}\,\bar{z}) + \bar{\eta}_2\eta^1\bar{B}\,\bar{z} + \bar{\eta}_1\eta^2 B z\right] - \frac{\mu^2}{4}(z\bar{z})^{\frac{1}{2a}}. \tag{7.13}$$

One can cast it into the form of the Lagrangian (5.25) by passing to the fermions $\xi^{i'}$ with the *primed* doublet indices as

$$\eta^1 = \xi^{1'}, \quad \bar{\eta}_1 = \bar{\xi}_{1'}, \quad \eta^2 = \bar{\xi}_{2'}, \quad \bar{\eta}_2 = \xi^{2'}, \tag{7.14}$$

$$\overline{(\xi^{i'})} = \bar{\xi}_{i'}, \qquad \overline{(\eta^i)} = \bar{\eta}_i. \tag{7.15}$$

This redefinition makes manifest the property that the fermionic fields are transformed as doublets of the $SU'(2)$ group with the generators $\{F, C, \bar{C}\}$.

As in the case of $\alpha = -1$, we can add the superconformal superpotential term

$$S_{\rm sc}^{{\rm pot}(\alpha=0)}(\tilde{\Phi}) = s\int dt_L d\theta_1 d\bar{\theta}^2 \ln\tilde{\Phi}_L + \text{c.c.}$$
$$\Rightarrow L_{\rm sc}^{{\rm pot}(\alpha=0)} = 2s\left(\frac{B}{z}+\frac{\bar{\eta}_2\eta^1}{z^2}\right)+\text{c.c.}, \quad (7.16)$$

which yields on shell the standard conformal mechanics potential in addition to the oscillator-type term $\sim\mu^2$ coming from the superconformal superfield kinetic term. The latter can be reduced to the free one as in the previous cases.

Thus, the superconformal action at $\alpha = 0$ can be constructed using the superfield approach associated with the $\alpha = 0$ supercoset (3.24), while the $\alpha = -1$ action (5.25) was based on the $SU(2|1)$ supercoset. In the parabolic limit $\mu = 0$, both supercosets are reduced to the standard flat $\mathcal{N} = 4, d = 1$ superspace.

VIII. *D*-MODULE REPRESENTATION APPROACH

Here we sketch a different approach to the $d = 1$ superconformal actions based solely on the component field considerations [14,22,23].

A. The $\mathcal{N} = 4$ linear supermultiplets

As a preamble, it is instructive, following Ref. [14], to give a concise account of the general superconformal properties of the set of linear $\mathcal{N} = 4$ supermultiplets $(\mathbf{k}, \mathbf{4}, \mathbf{4}-\mathbf{k})$ for $k = 0, 1, 2, 3, 4$, despite the fact that in the present paper we deal with the cases $k = 1, 2$ only.

The linear supermultiplets $(\mathbf{k}, \mathbf{4}, \mathbf{4}-\mathbf{k})$ for $k = 0, 1, 2, 3, 4$ exist in parabolic and hyperbolic/trigonometric variants [14]. The parabolic variant leads to actions which both are superconformally invariant and show up the manifest Poincaré supersymmetry. The hyperbolic/trigonometric variants lead to superconformally invariant actions in which the $d = 1$ Poincaré supersymmetry is implicit (the corresponding supercharges are not a "square root" of the canonical Hamiltonian as the time-translation generator), so they look nonsupersymmetric or weakly supersymmetric. The potentials are bounded from below in the trigonometric version (i.e., they are well behaved). They are unbounded (badly behaved) in the hyperbolic version. In the parabolic case, the Hamiltonian is a Cartan generator of the conformal $so(2, 1)$ subalgebra. In the hyperbolic/trigonometric case, the canonical Hamiltonian is a root generator of $so(2, 1)$.

The connection of these $\mathcal{N} = 4$ linear supermultiplets with the $\mathcal{N} = 4$ superconformal algebras and the corresponding scaling dimensions λ_D is as follows [14,22,23]:

(i) $(\mathbf{0}, \mathbf{4}, \mathbf{4})$: $D(2, 1; \alpha = 2\lambda_D)$.
(ii) $(\mathbf{1}, \mathbf{4}, \mathbf{3})$: $D(2, 1; \alpha = \lambda_D)$. At $\alpha = -1$ and $\alpha = 0$, the extra inhomogeneous constant parameter c is allowed.
(iii) $(\mathbf{2}, \mathbf{4}, \mathbf{2})$: $su(1, 1|2)$. The scaling dimension λ_D is associated with a central charge of $su(1, 1|2)$ as $\lambda_D = -b$.
(iv) $(\mathbf{3}, \mathbf{4}, \mathbf{1})$: $D(2, 1; \alpha = -\lambda_D)$.
(v) $(\mathbf{4}, \mathbf{4}, \mathbf{0})$: $D(2, 1; \alpha = -2\lambda_D)$.

For all multiplets except $(\mathbf{2}, \mathbf{4}, \mathbf{2})$, the superconformal algebra at $\alpha = -1, 0$ can be reduced to the superalgebra $psu(1, 1|2)$.

Another type of inhomogeneous linear transformation [14] is only present at $\lambda_D = 0$. The inhomogeneous parameter is ρ. The supermultiplets $(\mathbf{k}, \mathbf{4}, \mathbf{4}-\mathbf{k})_\rho$ carry a representation of $psu(1, 1|2)$ for $k = 0, 1, 3, 4$ and its central-extended version $su(1, 1|2)$ for $k = 2$. One should note that the superconformal actions based on $(\mathbf{k}, \mathbf{4}, \mathbf{4}-\mathbf{k})$ at a given λ_D are not defined at $\lambda_D = 0$. On the other hand, the superconformal actions based on $(\mathbf{k}, \mathbf{4}, \mathbf{4}-\mathbf{k})_\rho$ are well defined.

B. Superconformally invariant $(\mathbf{2}, \mathbf{4}, \mathbf{2})$ actions from the *D*-module approach

In Ref. [14], all hyperbolic *D*-module representations for the $\mathcal{N} = 4$ linear multiplets $(\mathbf{k}, \mathbf{4}, \mathbf{4}-\mathbf{k})$ were obtained, and the trigonometric *D*-module representations can be easily derived from the hyperbolic representations. Then one can construct the hyperbolic/trigonometric superconformal actions proceeding from the *D*-module representations. The method of construction is described in Ref. [22]. Some superconformal actions of the supermultiplet $(\mathbf{1}, \mathbf{4}, \mathbf{3})$ were found in this way in Ref. [14]. Here we present the realization of the $\mathcal{N} = 4$ superconformal algebras and perform the construction of the superconformal actions for the supermultiplet $(\mathbf{2}, \mathbf{4}, \mathbf{2})$ in this alternative approach.

We use the same notation and definitions for the component fields and superconformal generators as in the previous sections. The action of generators of the conformal algebra is given below:

$$\mathcal{H}z = i\dot{z}, \qquad \mathcal{H}\xi^i = i\dot{\xi}^i, \qquad \mathcal{H}B = i\dot{B},$$
$$Tz = e^{-i\mu t}(i\dot{z} - b\mu z), \qquad T\xi^i = e^{-i\mu t}\left[i\dot{\xi}^i - \left(b-\frac{1}{2}\right)\mu\xi^i\right],$$
$$TB = e^{-i\mu t}[i\dot{B} - (b-1)\mu B],$$
$$\bar{T}z = e^{i\mu t}(i\dot{z} + b\mu z), \qquad \bar{T}\xi^i = e^{i\mu t}\left[i\dot{\xi}^i + \left(b-\frac{1}{2}\right)\mu\xi^i\right],$$
$$\bar{T}B = e^{i\mu t}[i\dot{B} + (b-1)\mu B]. \quad (8.1)$$

The fermionic generators are specified by

$$Q^i z = -\sqrt{2}\xi^i e^{\frac{i}{2}\mu t}, \qquad Q^i\xi^k = \sqrt{2}\varepsilon^{ik}Be^{\frac{i}{2}\mu t}, \qquad Q^i B = 0,$$
$$\bar{Q}_i z = 0, \qquad \bar{Q}_i\xi^k = -\sqrt{2}\delta_i^k(i\dot{z} - b\mu z)e^{-\frac{i}{2}\mu t},$$
$$\bar{Q}_i B = -\sqrt{2}\varepsilon_{ik}\left[i\dot{\xi}^k - \left(b-\frac{1}{2}\right)\mu\xi^k\right]e^{-\frac{i}{2}\mu t},$$
$$S^i z = -\sqrt{2}\xi^i e^{-\frac{i}{2}\mu t}, \qquad S^i\xi^k = \sqrt{2}\varepsilon^{ik}Be^{-\frac{i}{2}\mu t}, \qquad S^i B = 0,$$
$$\bar{S}_i z = 0, \qquad \bar{S}_i\xi^k = -\sqrt{2}\delta_i^k(i\dot{z} + b\mu z)e^{\frac{i}{2}\mu t},$$
$$\bar{S}_i B = -\sqrt{2}\varepsilon_{ik}\left[i\dot{\xi}^k + \left(b-\frac{1}{2}\right)\mu\xi^k\right]e^{\frac{i}{2}\mu t}. \quad (8.2)$$

Since all $(\mathbf{2},\mathbf{4},\mathbf{2})$ Lagrangians can be reduced to the free Lagrangian (5.26), it is enough to consider the free case $b = 1/2$. Then the superconformally invariant action is generated from the *prepotential* $f(z,\bar{z})$ by acting with the supercharges Q_i on the propagating bosons $z, \bar{z}$ as

$$L_{\rm sc}^{(b=1/2)} = \frac{1}{16} Q_i Q^i \bar{Q}^k \bar{Q}_k f(z,\bar{z}). \qquad (8.3)$$

The prepotential $f(z,\bar{z})$ can be found from the constraint that the action of conformal generators on the Lagrangian produces a total time derivative:

$$T L_{\rm sc}^{(b=1/2)} = \frac{d}{dt} M, \qquad \bar{T} L_{\rm sc}^{(b=1/2)} = \frac{d}{dt} \bar{M}, \qquad (8.4)$$

where the explicit form of M is of no interest for our purposes. Solving these constraints, we obtain the *prepotential*

$$f(z,\bar{z}) = z\bar{z}. \qquad (8.5)$$

The corresponding superconformal action (8.3) generated from the D-module representations can be shown to coincide with the superconformal action (5.26) derived from the $SU(2|1)$ superspace approach.

The superpotential term (5.30) can also be equivalently constructed using the D-module approach. We define

$$L_{\rm sc}^{\rm pot} = \frac{1}{2} Q_i Q^i h(z) + \frac{1}{2} \bar{Q}^i \bar{Q}_i \bar{h}(\bar{z}) \qquad (8.6)$$

and impose the conformal constraints in the same way as for (8.4). As their solution we uniquely obtain

$$h(z) = -\nu \ln z, \qquad \bar{h}(\bar{z}) = -\bar{\nu} \ln \bar{z}. \qquad (8.7)$$

It is direct to check that (8.6) for such $h(z)$ coincides with (5.30).

Note that the superfield and D-module approaches can be regarded as complementary to each other. The second method directly yields the component off-shell Lagrangians. On the other hand, the superfield techniques bring to light some properties which are hidden in the component formulations. For instance, the reducibility of the general sigma-model-type action of the multiplet $(\mathbf{2},\mathbf{4},\mathbf{2})$ to the free one is immediately seen, when using the chiral $SU(2|1)$ superfield language, as in Secs. V–VII.

IX. SUMMARY AND OUTLOOK

In this paper, we presented the superspace realization of the trigonometric-type $\mathcal{N}=4, d=1$ superconformal symmetry. This realization can be given in terms of the $SU(2|1)$ superspace at $\alpha \neq 0$ or in terms of the $U(1)$ deformed flat $\mathcal{N}=4, d=1$ superspace at $\alpha = 0$. In the contraction limit $\mu = 0$, the relevant superconformal models are reduced to the standard models of the parabolic superconformal mechanics, with the superconformal Lagrangians constructed out of the standard $\mathcal{N}=4,\ d=1$ superfields. The main advantage of the $SU(2|1)$ superfield approach (or its degenerate $\alpha = 0$ version) is that it automatically yields the trigonometric-type realization of the superconformal symmetry, with the correct-sign harmonic oscillator term $\sim\mu^2$ in the component actions.

Our construction is based on the new observation that the most general $\mathcal{N}=4, d=1$ superconformal algebra $D(2,1;\alpha)$ at $\alpha \neq 0$ in the $SU(2|1)$ superspace realizations can be represented as a closure of its two $su(2|1)$ subalgebras, one of which defines the superisometry of the underlying $SU(2|1)$ superspace while the other is obtained from the first one by the reflection of the contraction parameter as $\mu \to -\mu$. This suggests the simple selection rule for singling out the superconformally invariant actions in the general set of the $SU(2|1)$ invariant actions constructed in Refs. [4,5]. The superconformal $SU(2|1)$ actions are those which are even functions of μ. The superalgebra $D(2,1;\alpha=0) \sim psu(1,1|2) \oplus su(2)$ (and its central extensions) admit a similar closure structure, this time in terms of two μ-dependent $U(1)$ deformed flat $\mathcal{N}=4, d=1$ superalgebras.

We gave an off-shell superfield formulation of the trigonometric superconformal actions of the multiplet $(\mathbf{1},\mathbf{4},\mathbf{3})$, some of which were constructed earlier at the component level in Ref. [14], and presented new trigonometric superconformal actions for the chiral multiplet $(\mathbf{2},\mathbf{4},\mathbf{2})$. For the latter multiplet, the superconformal actions exist only for $\alpha = -1$ and $\alpha = 0$, and they are always reduced to a sum of the free kinetic (sigma-model-type) $SU(2|1)$ superfield action and the superconformal superfield potential, yielding, in the bosonic component sector, a sum of the standard conformal mechanics potential $\sim\frac{1}{|z|^2}$ and the oscillator term $\sim\mu^2|z|^2$. The $SU(2|1)$ superfield approach provides a simple proof of this notable property. Another feature easily revealed in the $SU(2|1)$ superfield approach is that the superconformal $\alpha = -1$ models corresponding to the generalized $(\mathbf{2},\mathbf{4},\mathbf{2})$ chirality [5] proved to be equivalent to the superconformal models associated with the standard chiral $SU(2|1)$ multiplet. The common property of all superconformal sigma-model type $(\mathbf{2},\mathbf{4},\mathbf{2})$ actions (at $\alpha = -1$ and $\alpha = 0$) is that they exist only on account of nonzero central charge in the corresponding superconformal algebras $su(1,1|2)$. We also presented an alternative way of deriving the component superconformal $(\mathbf{2},\mathbf{4},\mathbf{2})$ actions, based on the D-module representation approach developed in Refs. [14,22,23], and found a nice agreement with the superfield considerations.

It would be interesting to use the $SU(2|1)$ superspace approach to construct analogous models with the trigonometric realization of superconformal symmetry for other off-shell $SU(2|1)$ supermultiplets, with the field contents $(\mathbf{3},\mathbf{4},\mathbf{1})$ and $(\mathbf{4},\mathbf{4},\mathbf{0})$, as well as the multiparticle generalizations of all such models (including those studied in the present paper). Also, it seems important to better understand the relationship between the $SU(2|1)$ superfield approach and the component approach based on the D-module representations of the superconformal symmetries, including $D(2,1;\alpha)$. Finding out the possible links with the superconformal structures in the higher-dimensional theories based on curved analogs of flat rigid supersymmetries (see, e.g., Ref. [24]) is also an urgent subject for future study.

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APPENDIX A: CENTRAL EXTENSION OF SUPERCONFORMAL ALGEBRA

At $\alpha=-1$ (or $\alpha=0$) it is possible to extend the superalgebra $D(2,1;\alpha)$ by additional central charges. In this particular case, the (anti)commutators (3.1), (3.2) can be cast in the form

$$\begin{aligned}
&\{Q_{\alpha i i'},Q_{\beta j j'}\}=2(\epsilon_{ij}\epsilon_{i'j'}T_{\alpha\beta}-\epsilon_{\alpha\beta}\epsilon_{i'j'}J_{ij}-\epsilon_{\alpha\beta}\epsilon_{ij}C_{i'j'}),\\
&[T_{\alpha\beta},Q_{\gamma i i'}]=-i\epsilon_{\gamma(\alpha}Q_{\beta)ii'},\quad [T_{\alpha\beta},T_{\gamma\delta}]=i(\epsilon_{\alpha\gamma}T_{\beta\delta}+\epsilon_{\beta\delta}T_{\alpha\gamma}),\\
&[J_{ij},Q_{\alpha k i'}]=-i\epsilon_{k(i}Q_{\alpha j)i'},\quad [J_{ij},J_{kl}]=i(\epsilon_{ik}J_{jl}+\epsilon_{jl}J_{ik}),
\end{aligned}\tag{A1}$$

where the central charges $C_{i'j'}$ commute with all other generators. They form a vector with respect to the automorphism $SU'(2)_{\rm ext}$ transformations acting on the indices i', j'. The norm of the vector $C_{i'j'}$ of central charges,

$$|C|^2 \coloneqq \frac{1}{2}C^{i'k'}C_{i'k'},\tag{A2}$$

is an invariant of these $SU'(2)_{\rm ext}$ transformations. Hence, in the case of constant central charges, we can choose the $SU'(2)_{\rm ext}$ frame in such a way that only one nonvanishing central charge remains, e.g., its third component:

$$C_{1'2'}\neq 0,\qquad C_{1'1'}=C_{2'2'}=0.\tag{A3}$$

Simultaneously, $SU'(2)_{\rm ext}$ is reduced to the automorphism $U(1)_{\rm ext}$.

One can equivalently rewrite the superalgebra (A1) as the appropriate extension of (3.9)–(3.12) at $\alpha=-1$:

$$\begin{aligned}
&\{Q^i,\bar Q_j\}=2\mu I^i_j+2\delta^i_j(\mathcal{H}-\mu Z_1),\\
&\{S^i,\bar S_j\}=-2\mu I^i_j+2\delta^i_j(\mathcal{H}+\mu Z_1),\\
&\{S^i,\bar Q_j\}=2\delta^i_j T,\qquad \{Q^i,\bar S_j\}=2\delta^i_j\bar T,\\
&\{Q^i,S^k\}=2\mu\epsilon^{ik}Z_2,\qquad \{\bar Q_j,\bar S_k\}=2\mu\epsilon_{jk}Z_3,
\end{aligned}\tag{A4}$$

$$\begin{aligned}
&[I^i_j,I^k_l]=\delta^k_jI^i_l-\delta^i_lI^k_j,\\
&[I^i_j,\bar Q_l]=\frac{1}{2}\delta^i_j\bar Q_l-\delta^i_l\bar Q_j,\quad [I^i_j,Q^k]=\delta^k_jQ^i-\frac{1}{2}\delta^i_jQ^k,\\
&[I^i_j,\bar S_l]=\frac{1}{2}\delta^i_j\bar S_l-\delta^i_l\bar S_j,\quad [I^i_j,S^k]=\delta^k_jS^i-\frac{1}{2}\delta^i_jS^k,
\end{aligned}\tag{A5}$$

$$\begin{aligned}
&[T,\bar T]=-2\mu\mathcal{H},\qquad [\mathcal{H},T]=\mu T,\qquad [\mathcal{H},\bar T]=-\mu\bar T,\\
&[\mathcal{H},\bar S_l]=-\frac{\mu}{2}\bar S_l,\qquad [\mathcal{H},S^k]=\frac{\mu}{2}S^k,\\
&[\mathcal{H},\bar Q_l]=\frac{\mu}{2}\bar Q_l,\qquad [\mathcal{H},Q^k]=-\frac{\mu}{2}Q^k,\\
&[T,Q^i]=-\mu S^i,\qquad [T,\bar S_j]=-\mu\bar Q_j,\\
&[\bar T,\bar Q_j]=\mu\bar S_j,\qquad [\bar T,S^i]=\mu Q^i.
\end{aligned}\tag{A6}$$

According to (3.8), the central charges appearing here are related to the central charges defined in (A1) as

$$\begin{aligned}
&C_{1'2'}=C_{2'1'}=iZ_1,\qquad C_{1'1'}=iZ_2,\\
&C_{2'2'}=iZ_3,\qquad |C|^2=(Z_1)^2-Z_2Z_3.
\end{aligned}\tag{A7}$$

APPENDIX B: MORE ON $SU(2|1)$ CHIRALITIES

As was demonstrated, the superconformal $SU(2|1)$ models of the $(\mathbf{2},\mathbf{4},\mathbf{2})$ superfield defined by the generalized (central-charge-extended) chirality condition (6.1) are in fact equivalent to those constructed on the basis of the superfield subjected to the "standard" chirality condition (5.1) [or its central-charge-extended version (5.15)]. So in the superconformal case, the parameter λ entering (6.1), (6.2), (6.8) is unessential. This is in contrast with the pure $SU(2|1)$ invariant models in which λ is a physical parameter specifying a new class of such models [5].

Let us discuss the interplay between two types of the $SU(2|1)$ chirality in more detail, based upon the superspace considerations. It will be useful to pass to the coordinates $\{t,\tilde\theta_j,\bar{\tilde\theta}^i\}$ defined by the relations (3.38). Being specialized to $\alpha=-1$, these relations read

$$\begin{aligned}
&\tilde\theta_j=e^{\frac{i}{2}\mu t}\theta_j\Big(1-\frac{\mu}{2}\bar\theta^k\theta_k\Big)=e^{\frac{i}{2}\mu t_L}\theta_j,\qquad t_L=t+i\bar{\tilde\theta}^i\tilde\theta_i,\\
&\bar{\tilde\theta}^i=e^{-\frac{i}{2}\mu t}\bar\theta^i\Big(1-\frac{\mu}{2}\bar\theta^k\theta_k\Big)=e^{-\frac{i}{2}\mu t_R}\bar\theta^i,\qquad t_R=t-i\bar{\tilde\theta}^i\tilde\theta_i.
\end{aligned}\tag{B1}$$

The $SU(2|1)$ supercharges (3.39) are rewritten as

$$Q^i = e^{\frac{i}{2}\mu t}\left\{\left[1-\frac{\mu}{2}\bar{\tilde{\theta}}^k\tilde{\theta}_k-\frac{\mu^2}{16}(\tilde{\theta})^2(\bar{\tilde{\theta}})^2\right]\frac{\partial}{\partial\tilde{\theta}_i}-\mu\bar{\tilde{\theta}}^i\bar{\tilde{\theta}}^k\frac{\partial}{\partial\bar{\tilde{\theta}}^k}+i\bar{\tilde{\theta}}^i\left(1+\frac{\mu}{2}\bar{\tilde{\theta}}^k\tilde{\theta}_k\right)\partial_t\right\},$$
$$\bar{Q}_j = e^{-\frac{i}{2}\mu t}\left\{\left[1-\frac{\mu}{2}\bar{\tilde{\theta}}^k\tilde{\theta}_k-\frac{\mu^2}{16}(\tilde{\theta})^2(\bar{\tilde{\theta}})^2\right]\frac{\partial}{\partial\bar{\tilde{\theta}}^j}+\mu\tilde{\theta}_j\tilde{\theta}_k\frac{\partial}{\partial\tilde{\theta}_k}+i\tilde{\theta}_j\left(1+\frac{\mu}{2}\bar{\tilde{\theta}}^k\tilde{\theta}_k\right)\partial_t\right\}. \quad \text{(B2)}$$

The extra generators S_i completing $SU(2|1)$ to $D(2,1;\alpha=-1)$ are represented in this basis as $S(\mu)=Q(-\mu)$. The covariant derivatives (5.13) take the form

$$\mathcal{D}^i_Z=\left[1+\frac{\mu}{2}\bar{\tilde{\theta}}^k\tilde{\theta}_k-\frac{\mu^2}{16}(\tilde{\theta})^2(\bar{\tilde{\theta}})^2\right]\left(\frac{\partial}{\partial\tilde{\theta}_i}-i\bar{\tilde{\theta}}^i\partial_t+\mu\bar{\tilde{\theta}}^iZ_1\right),$$
$$\bar{\mathcal{D}}_{Zj}=\left[1+\frac{\mu}{2}\bar{\tilde{\theta}}^k\tilde{\theta}_k-\frac{\mu^2}{16}(\tilde{\theta})^2(\bar{\tilde{\theta}})^2\right]\left(-\frac{\partial}{\partial\bar{\tilde{\theta}}^j}+i\tilde{\theta}_j\partial_t-\mu\tilde{\theta}_jZ_1\right). \quad \text{(B3)}$$

We ignore the matrix $SU(2)$ generators $\tilde{I}^l_j$ in $\mathcal{D}^i,\bar{\mathcal{D}}_j$, because the generalized chiral superfields defined by (6.1) cannot carry external $SU(2)$ indices owing to the compatibility relation

$$\{\bar{\tilde{\mathcal{D}}}_k,\bar{\tilde{\mathcal{D}}}_j\}=-2\mu\sin 2\lambda\tilde{I}_{ij},\quad \text{and c.c.}$$

Using the explicit expressions (B3), the generalized chirality condition (6.1) with $\bar{\tilde{\mathcal{D}}}_j$ defined according to (6.8) can be rewritten in the basis $\{t,\tilde{\theta}_j,\bar{\tilde{\theta}}^i\}$ as

$$\bar{\tilde{\mathcal{D}}}_j\varphi=\left[1+\frac{\mu}{2}\bar{\tilde{\theta}}^k\tilde{\theta}_k-\frac{\mu^2}{16}(\tilde{\theta})^2(\bar{\tilde{\theta}})^2\right]$$
$$\times\left[\cos\lambda\left(-\frac{\partial}{\partial\bar{\tilde{\theta}}^j}+i\tilde{\theta}_j\partial_t-\mu\tilde{\theta}_jZ_1\right)\right.$$
$$\left.-\varepsilon_{ji}\sin\lambda\left(\frac{\partial}{\partial\tilde{\theta}_i}-i\bar{\tilde{\theta}}^i\partial_t+\mu\bar{\tilde{\theta}}^iZ_1\right)\right]\varphi=0. \quad \text{(B4)}$$

It is easy to check that the coordinates $\hat{\theta}_i$ defined in (6.5) and parametrizing the left chiral superspace (6.4) can be represented, for generic λ, as a particular $SU(2)$ rotation of the coordinates $\tilde{\theta}_j$, $\bar{\tilde{\theta}}^i$:

$$\hat{\theta}_i=\cos\lambda\tilde{\theta}_i+\sin\lambda\bar{\tilde{\theta}}_i,\qquad \bar{\hat{\theta}}^i=\cos\lambda\bar{\tilde{\theta}}^i-\sin\lambda\tilde{\theta}^i. \quad \text{(B5)}$$

In the basis $\{t,\hat{\theta}_j,\bar{\hat{\theta}}^i\}$ the condition (B4) becomes

$$\bar{\tilde{\mathcal{D}}}_j\varphi=\left[1+\frac{\mu}{2}\cos 2\lambda\bar{\hat{\theta}}^k\hat{\theta}_k-\frac{\mu}{4}\sin 2\lambda(\bar{\hat{\theta}}^k\bar{\hat{\theta}}_k+\hat{\theta}_k\hat{\theta}^k)-\frac{\mu^2}{16}(\hat{\theta})^2(\bar{\hat{\theta}})^2\right]$$
$$\times\left(-\frac{\partial}{\partial\bar{\hat{\theta}}^j}+i\hat{\theta}_j\partial_t-\mu\hat{\theta}_jZ_1\right)\varphi=0. \quad \text{(B6)}$$

Comparing (B6) with the "standard" chirality constraint (5.15) written through $\bar{\mathcal{D}}_{Zj}$ from (B3), we see that they have the same form, up to an unessential nonsingular scalar factor and the change of Grassmann coordinates as $\tilde{\theta}\leftrightarrow\hat{\theta}$.

One can define the new supercharges

$$\tilde{Q}^i=\cos\lambda Q^i-\sin\lambda\bar{S}^i,\quad \text{and c.c.}, \quad \text{(B7)}$$

and check that they coincide with the generators (B2) in which the same substitution $(\tilde{\theta},\bar{\tilde{\theta}})\to(\hat{\theta},\bar{\hat{\theta}})$ has been performed. The same applies to the $\tilde{S}^i$ supercharges

$$\tilde{S}^i=\cos\lambda S^i-\sin\lambda\bar{Q}^i,\quad \text{and c.c.}, \quad \text{(B8)}$$

and the corresponding conformal subgroup generators. We also observe that the $U(1)$ generator (6.3) takes the form

$$F'=F\cos 2\lambda+\frac{1}{2}(C+\bar{C})\sin 2\lambda$$
$$=\frac{1}{2}\left(\bar{\hat{\theta}}^k\frac{\partial}{\partial\bar{\hat{\theta}}^k}-\hat{\theta}_k\frac{\partial}{\partial\hat{\theta}_k}\right), \quad \text{(B9)}$$

which, up to the coordinate change just mentioned, coincides with the definition (3.41) of F.

As was shown in Sec. VI C, the transformations (6.22) of the component fields under the supercharges (B7), (B8) with the parameters $\tilde{\epsilon}_i,\tilde{\varepsilon}_i$ defined in (6.25) have the same form as the original (Q,S) transformations (5.20) with the parameters ϵ_i,ε_i. Accordingly, the superfield (Q,S) transformations (5.18) of Φ_L can be given the same form as the transformations (6.23) of the superfield $\tilde{\varphi}_L(\hat{t}_L,\hat{\theta})$ under the supercharges (B7), (B8) by rewriting (5.18) through the coordinates $(t_L,\tilde{\theta})$:

$$\delta\Phi_L(t_L,\tilde{\theta})=2b\mu(\bar{\varepsilon}^i\tilde{\theta}_ie^{-\frac{i}{2}\mu t_L}-\bar{\varepsilon}^i\tilde{\theta}_ie^{\frac{i}{2}\mu t_L})\Phi_L(t_L,\tilde{\theta}). \quad \text{(B10)}$$

Thus we observe the full similarity between Φ_L and $\tilde{\varphi}_L$ modulo the change $(t_L,\tilde{\theta})\leftrightarrow(\hat{t}_L,\hat{\theta})$.

This phenomenon can be summarized as follows: In the basis $\{t,\hat{\theta}_j,\bar{\hat{\theta}}^i\}$, the rotated superconformal generators (B7), (B8) have the same form as the original supercharges Q^i,S^i in the basis $\{t,\tilde{\theta}_j,\bar{\tilde{\theta}}^i\}$. The superconformal subclass of the actions of the generalized multiplet $(\mathbf{2},\mathbf{4},\mathbf{2})$ is invariant under both Q and S supersymmetries, hence it is invariant under their $\tilde{Q}$ and $\tilde{S}$ realizations as well.

The generalized chiral $SU(2|1)$ superfield defined for the Q, S realization of the superconformal group looks just like the standard chiral $SU(2|1)$ superfield with respect to the equivalent $\tilde{Q}, \tilde{S}$ realization. So the superconformal $(\mathbf{2},\mathbf{4},\mathbf{2})$ actions actually cannot distinguish on which kind of the chiral $SU(2|1)$ superfield they are built and, respectively, cannot involve any dependence on the parameter λ.

To make the latter property manifest, let us proceed from the superconformal action of generalized chiral superfield $\varphi(t,\hat{\theta},\bar{\hat{\theta}})$ as the solution (6.10) of (B6):

$$S^{(b)}_{\rm sc}(\varphi) = \frac{1}{4}\int d\hat{\zeta}(\varphi\bar{\varphi})^{\frac{1}{2b}}, \tag{B11}$$

where the integration measure $d\hat{\zeta}$ in the $SU(2|1)$ superspace basis $\{t,\hat{\theta},\bar{\hat{\theta}}\}$ is defined in (6.18). Since the component action (6.20) has no dependence on λ, the λ dependence of the superfield action (B11) is also expected to be fake. Using the relations (6.10) and (6.18), we rewrite (B11) through the (anti)holomorphic superfields φ_L, $\bar{\varphi}_R$ as

$$S^{(b)}_{\rm sc}(\varphi) = \frac{1}{4}\int dt d^2\hat{\theta} d^2\bar{\hat{\theta}}\left[1-\frac{\mu^2}{4}(\bar{\hat{\theta}})^2(\hat{\theta})^2\right]\left[1-\frac{\mu}{2}\sin 2\lambda(\hat{\theta})^2\right] \times\left[1-\frac{\mu}{2}\sin 2\lambda(\bar{\hat{\theta}})^2\right](\varphi_L\bar{\varphi}_R)^{\frac{1}{2b}}. \tag{B12}$$

One can absorb the (anti)holomorphic factors in this action into the redefinition of $\bar{\varphi}_R$, φ_L as in (6.24) and cast (B12) in the following final form:

$$S^{(b)}_{\rm sc}(\varphi) = \frac{1}{4}\int dt d^2\hat{\theta} d^2\bar{\hat{\theta}}(1+\mu\bar{\hat{\theta}}^k\hat{\theta}_k)(\tilde{\varphi}\bar{\tilde{\varphi}})^{\frac{1}{2b}}. \tag{B13}$$

Here, the newly introduced superfield $\tilde{\varphi}$ is a solution of (B6) with $Z_1\tilde{\varphi}_L = b\tilde{\varphi}_L$:

$$\bar{\hat{\mathcal{D}}}_j\tilde{\varphi} = 0, \Rightarrow \tilde{\varphi}(t,\hat{\theta},\bar{\hat{\theta}}) = e^{-b\mu\bar{\hat{\theta}}^k\hat{\theta}_k}\tilde{\varphi}_L(\hat{t}_L,\hat{\theta}), \tag{B14}$$

and it does not display any λ dependence, equally as the action (B13). Comparing it with the superconformal action (5.21), (5.24) rewritten in the basis $\{t,\tilde{\theta}_i,\bar{\tilde{\theta}}^k\}$,

$$S^{(b)}_{\rm sc}(\Phi) = \frac{1}{4}\int dt d^2\tilde{\theta} d^2\bar{\tilde{\theta}}(1+\mu\bar{\tilde{\theta}}^k\tilde{\theta}_k)(\Phi\bar{\Phi})^{\frac{1}{2b}},$$
$$\Phi(t,\tilde{\theta},\bar{\tilde{\theta}}) = e^{-b\mu\bar{\tilde{\theta}}^i\tilde{\theta}_i}\Phi_L(t_L,\tilde{\theta}), \tag{B15}$$

where the expression (3.43) for the $d\tilde{\zeta}$ integration measure was used, we observe its identity with (B13), up to the interchange $\tilde{\theta}\leftrightarrow\hat{\theta}$, as was anticipated above. Note that the integration measure in (B13),

$$dt d^2\hat{\theta} d^2\bar{\hat{\theta}}(1+\mu\bar{\hat{\theta}}^k\hat{\theta}_k), \tag{B16}$$

is invariant with respect to $SU(2|1)$ generated by the rotated supercharges (B7).

The nonconformal $SU(2|1)$ invariant chiral actions are invariant under the transformations generated by Q_i and $\bar{Q}^i$, but not under the $\tilde{Q}_i, \bar{\tilde{Q}}^i$ transformations, since the definition of the latter involve the superconformal generators S_i and $\bar{S}^i$. Hence, they differ for the standard and generalized chiral $(\mathbf{2},\mathbf{4},\mathbf{2})$ multiplets and depend on λ as an essential parameter. It labels nonequivalent $SU(2|1)$ actions and the corresponding SQM models [5].

APPENDIX C: HYPERBOLIC SUPERCONFORMAL MECHANICS

The hyperbolic superconformal mechanics can be obtained by substituting the deformation parameter in the trigonometric models as $\mu\to i\mu$. One can see that the superconformal generators defined in Sec. III B go over to the new generators

$$Q^i \longrightarrow \Pi^i, \quad \bar{Q}_k \longrightarrow \bar{\Theta}_k, \quad S^i \longrightarrow \Theta^i, \quad \bar{S}_k \longrightarrow \bar{\Pi}_k,$$
$$T \longrightarrow T_2, \quad \bar{T}\longrightarrow T_1, \quad \mathcal{H}\longrightarrow\mathcal{H}_{\rm h}, \tag{C1}$$

which behave under the Hermitian conjugation as

$$(\Pi^k)^\dagger = \bar{\Pi}_k, \quad (\Theta^k)^\dagger = \bar{\Theta}_k$$
$$\Rightarrow (T_2)^\dagger = T_2, \quad (T_1)^\dagger = T_1, \quad (\mathcal{H}_{\rm h})^\dagger = \mathcal{H}_{\rm h}. \tag{C2}$$

In this basis, the basic anticommutation relations of $D(2,1;\alpha)$ can be rewritten as

$$\{\Pi^i,\bar{\Theta}_j\} = -2i\alpha\mu I^i_j + 2\delta^i_j[\mathcal{H}_{\rm h} + i(1+\alpha)\mu F],$$
$$\{\Theta^i,\bar{\Pi}_j\} = 2i\alpha\mu I^i_j + 2\delta^i_j[\mathcal{H}_{\rm h} - i(1+\alpha)\mu F],$$
$$\{\Theta^i,\bar{\Theta}_j\} = 2\delta^i_j T_2, \quad \{\Pi^i,\bar{\Pi}_j\} = 2\delta^i_j T_1,$$
$$\{\Pi^i,\Theta^k\} = -2i(1+\alpha)\mu\varepsilon^{ik}C,$$
$$\{\bar{\Theta}_j,\bar{\Pi}_k\} = 2i(1+\alpha)\mu\varepsilon_{jk}\bar{C}. \tag{C3}$$

The bosonic truncation of the corresponding conformal group generators (3.34) yields their hyperbolic realization:

$$\mathcal{H}_{\rm h} = i\partial_t, \quad T_1 = ie^{-\mu t}\partial_t, \quad T_2 = ie^{\mu t}\partial_t. \tag{C4}$$

The corresponding hyperbolic realization of (3.35) now reads

$$\hat{H} = \frac{i}{2}(1+\cosh\mu t)\partial_t, \quad \hat{K} = -\frac{2i}{\mu^2}(1-\cosh\mu t)\partial_t,$$
$$\hat{D} = \frac{i}{\mu}\sinh\mu t\partial_t, \quad \mu\neq 0. \tag{C5}$$

In contrast to the trigonometric case, the time-translation generator $\mathcal{H}_{\rm h}$ is now

$$\mathcal{H}_{\rm h} = \hat{H} - \frac{\mu^2}{4}\hat{K}. \tag{C6}$$

Due to the minus sign before $\mu^2\hat{K}$, we face the quantum mechanical problem in which the potentials accompanying the kinetic terms are not bounded from below, like in the parabolic case [11]. This difficulty could, of course, be cured in a similar way by passing to

$$\mathcal{H}_{\rm trig} = \hat{H} + \frac{\mu^2}{4}\hat{K} = \cosh\mu t\partial_t \tag{C7}$$

as the correct time-evolution operator. The discrete energy spectrum with the canonical Hamiltonian can be obtained only in the trigonometric models of superconformal mechanics. Note that $D(2,1;\alpha)$ contains no *self-conjugated* subalgebra with four real supercharges, in which $\mathcal{H}_{\rm h}$ would appear on the rhs of the basic anticommutator, in contrast to the parabolic and trigonometric cases.

1. Example

As an instructive example, we consider the simplest free case $b = 1/2$ of the multiplet $(\mathbf{2},\mathbf{4},\mathbf{2})$

$$L^{(b=1/2)}_{\rm sc} = \dot{\bar{z}}\,\dot{z} + \frac{i}{2}(\bar{\xi}_i\dot{\xi}^i - \dot{\bar{\xi}}_i\xi^i) + \bar{B}B - \frac{\mu^2}{4}z\bar{z}, \tag{C8}$$

and the relevant superconformal transformations

$$\begin{aligned}
\delta z &= -\sqrt{2}\epsilon_k\xi^k e^{\frac{i}{2}\mu t} - \sqrt{2}\varepsilon_k\xi^k e^{-\frac{i}{2}\mu t},\\
\delta\xi^i &= \sqrt{2}\bar{\epsilon}^i\left(i\dot{z} - \frac{\mu}{2}z\right)e^{-\frac{i}{2}\mu t} - \sqrt{2}\epsilon^i B e^{\frac{i}{2}\mu t}\\
&\quad + \sqrt{2}\bar{\varepsilon}^i\left(i\dot{z} + \frac{\mu}{2}z\right)e^{\frac{i}{2}\mu t} - \sqrt{2}\varepsilon^i B e^{-\frac{i}{2}\mu t},\\
\delta B &= -\sqrt{2}i\bar{\epsilon}_k\dot{\xi}^k e^{-\frac{i}{2}\mu t} - \sqrt{2}i\bar{\varepsilon}_k\dot{\xi}^k e^{\frac{i}{2}\mu t}.
\end{aligned} \tag{C9}$$

These transformations correspond to the superalgebra (A4)–(A6) with $Z_1 = 1/2$.

After the change $\mu \to i\mu$ in (C8), (C9), we obtain the hyperbolic mechanics Lagrangian as

$$L^{(b=1/2)}_{\rm sc\,(h)} = \dot{\bar{z}}\,\dot{z} + \frac{i}{2}(\bar{\xi}_i\dot{\xi}^i - \dot{\bar{\xi}}_i\xi^i) + \bar{B}B + \frac{\mu^2}{4}z\bar{z}, \tag{C10}$$

and the superconformal transformations as

$$\begin{aligned}
\delta z &= -\sqrt{2}\upsilon_k\xi^k e^{-\frac{1}{2}\mu t} - \sqrt{2}\varsigma_k\xi^k e^{\frac{1}{2}\mu t},\\
\delta\xi^i &= \sqrt{2}i\bar{\upsilon}^i\left(\dot{z} + \frac{\mu}{2}z\right)e^{-\frac{1}{2}\mu t} - \sqrt{2}\upsilon^i B e^{-\frac{1}{2}\mu t}\\
&\quad + \sqrt{2}i\bar{\varsigma}^i\left(\dot{z} - \frac{\mu}{2}z\right)e^{\frac{1}{2}\mu t} - \sqrt{2}\varsigma^i B e^{\frac{1}{2}\mu t},\\
\delta B &= -\sqrt{2}i\bar{\upsilon}_k\dot{\xi}^k e^{-\frac{1}{2}\mu t} - \sqrt{2}i\bar{\varsigma}_k\dot{\xi}^k e^{\frac{1}{2}\mu t}.
\end{aligned} \tag{C11}$$

The parameters $\upsilon, \bar{\upsilon}$ and $\varsigma, \bar{\varsigma}$ correspond to the supercharges $\Pi, \bar{\Pi}$ and $\Theta, \bar{\Theta}$, respectively. Note that the original $SU(2|1)$ transformations are embedded in (C11) as

$$\begin{aligned}
\delta z &= -\epsilon_k\xi^k(e^{-\frac{1}{2}\mu t} + ie^{\frac{1}{2}\mu t}), \qquad \delta B = -i\bar{\epsilon}_k\dot{\xi}^k(e^{-\frac{1}{2}\mu t} - ie^{\frac{1}{2}\mu t}),\\
\delta\xi^i &= i\bar{\epsilon}^i[\dot{z}(e^{-\frac{1}{2}\mu t} - ie^{\frac{1}{2}\mu t}) + \frac{\mu}{2}z(e^{-\frac{1}{2}\mu t} + ie^{\frac{1}{2}\mu t})]\\
&\quad - \epsilon^i B(e^{-\frac{1}{2}\mu t} + ie^{\frac{1}{2}\mu t}),
\end{aligned} \tag{C12}$$

where $\epsilon_k \coloneqq \frac{1}{\sqrt{2}}(\upsilon_k - i\varsigma_k)$.

[1] G. Festuccia and N. Seiberg, Rigid supersymmetric theories in curved superspace, J. High Energy Phys. 06 (2011) 114.

[2] T. T. Dumitrescu, G. Festuccia, and N. Seiberg, Exploring curved superspace, J. High Energy Phys. 08 (2012) 141.

[3] I. B. Samsonov and D. Sorokin, Superfield theories on S^3 and their localization, J. High Energy Phys. 04 (2014) 102; Gauge and matter superfield theories on S^2, J. High Energy Phys. 09 (2014) 097.

[4] E. Ivanov and S. Sidorov, Deformed supersymmetric mechanics, Classical Quantum Gravity **31**, 075013 (2014).

[5] E. Ivanov and S. Sidorov, Super Kähler oscillator from $SU(2|1)$ superspace, J. Phys. A **47**, 292002 (2014).

[6] S. Fedoruk, E. Ivanov, and O. Lechtenfeld, Superconformal mechanics, J. Phys. A **45**, 173001 (2012).

[7] A. V. Smilga, Weak supersymmetry, Phys. Lett. B **585**, 173 (2004).

[8] S. Bellucci and A. Nersessian, (Super)oscillator on CP(N) and constant magnetic field, Phys. Rev. D **67**, 065013 (2003).

[9] S. Bellucci and A. Nersessian, Supersymmetric Kähler oscillator in a constant magnetic field, arXiv:hep-th/0401232.

[10] G. Papadopoulos, New potentials for conformal mechanics, Classical Quantum Gravity **30**, 075018 (2013).

[11] V. de Alfaro, S. Fubini, and G. Furlan, Conformal invariance in quantum mechanics, Nuovo Cimento A **34**, 569 (1976).

[12] E. Ivanov, S. Krivonos, and O. Lechtenfeld, New variant of $N = 4$ superconformal mechanics, J. High Energy Phys. 03 (2003) 014.

[13] E. Ivanov, S. Krivonos, and O. Lechtenfeld, $N = 4$, $d = 1$ supermultiplets from nonlinear realizations of $D(2,1;\alpha)$, Classical Quantum Gravity **21**, 1031 (2004).

[14] N. L. Holanda and F. Toppan, Four types of (super)conformal mechanics: D-module reps and invariant actions, J. Math. Phys. (N.Y.) **55**, 061703 (2014).

[15] V. P. Akulov, S. Catto, O. Cebecioglu, and A. Pashnev, On the time dependent oscillator and the nonlinear realizations of the Virasoro group, Phys. Lett. B **575**, 137 (2003).

[16] F. Delduc and E. Ivanov, Gauging $\mathcal{N} = 4$ supersymmetric mechanics, Nucl. Phys. **B753**, 211 (2006).

[17] S. Bellucci and S. Krivonos, Potentials in $N = 4$ superconformal mechanics, Phys. Rev. D **80**, 065022 (2009).

[18] L. Frappat, A. Sciarrino, and P. Sorba, *Dictionary on Lie Algebras and Superalgebras* (Academic Press, New York, 2000).

[19] E. Ivanov and A. Sorin, The structure of representations of conformal supergroup in the $OSp(1,4)$ basis, Teor. Mat. Fiz. **45**, 30 (1980) [The structure of representations of conformal supergroup in the $OSp(1,4)$ basis, Theor. Math. Phys. **45**, 862 (1980)].

[20] E. A. Ivanov, S. O. Krivonos, and A. I. Pashnev, Partial supersymmetry breaking in $N = 4$ supersymmetric quantum mechanics, Classical Quantum Gravity **8**, 19 (1991).

[21] E. A. Ivanov, S. O. Krivonos, and V. M. Leviant, Geometric superfield approach to superconformal mechanics, J. Phys. A **22**, 4201 (1989).

[22] Z. Kuznetsova and F. Toppan, D-module representations of $N = 2, 4, 8$ superconformal algebras and their superconformal mechanics, J. Math. Phys. (N.Y.) **53**, 043513 (2012).

[23] S. Khodaee and F. Toppan, Critical scaling dimension of D-module representations of $N = 4, 7, 8$ superconformal algebras and constraints on superconformal mechanics, J. Math. Phys. (N.Y.) **53**, 103518 (2012).

[24] S. M. Kuzenko and D. Sorokin, Superconformal structures on the three-sphere, J. High Energy Phys. 10 (2014) 80.