On the stability of phantom K-essence theories

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We show that phantom dark energy, if it is described by a K-essence theory, has three fundamental problems: first, its hamiltonian is unbounded from below. Second, classical stability precludes the equation of state from crossing the "Lambda-barrier", $w_\Lambda = -1$. Finally, both the equation of state and the sound speed are unbounded — the first, from below, the second, from above — if the kinetic term is not bounded by dynamics.

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I. INTRODUCTION

Observations of Type 1A Supernovae [1], of the cosmic microwave background radiation [3] and of large-scale structure [4] indicate that the universe is currently experiencing a stage of accelerated expansion. When the SN data is considered independently of any other osbervations, the resulting constraints on dark energy parameters allow (indeed, prefer) values for the equation of state $w = p_{de}/\rho_{de} \lesssim -1$, where p_{de} and ρ_{de} are the cosmological pressure and energy density of the dark energy component [2]. However, when cosmic microwave background and large-scale-structure data are considered as well, the constraints become much tighter, and there is not much space to wiggle around $w \simeq -1$. Nevertheless, much thought has been devoted to the possibility that superacceleration ($w < -1$) may rule our universe in the near future [5] and even cause a future spacelike singularity ("big-rip") [6].

In this paper we show that, in the realm of General Relativity, a non-interacting phantom matter field is a very tough sell indeed. If we try to build phantom matter out of canonical scalar fields, we are led to consider a negative kinetic term in the Lagrangian [5], which makes the theory unstable — classically *and* quantum-mechanically [7, 8]. On the other hand, if we enlarge the class of scalar field Lagrangians to include those of the type of K-essence [9], then, as we will presently show, phantom K-matter [10] suffers from at least three basic problems.

First, phantom K-matter is quantum-mechanically unstable. This means that there is always a region in phase space where hard (UV) excitations of the field possess negative energies. Since there is nothing that can prevent positive-energy modes from decaying into negative-energy modes, all matter fields in the universe would decay instantly through tunneling into the negative-energy modes of such a scalar field. The theory is sick, and there is no cure.

Second, the equation of state is unbounded from below ($w \to -\infty$) if the kinetic energy is not bounded. Moreover, as the equation of state diverges, so does the sound speed, $c_s^2 \to \infty$. This means that, besides potential problems with causality, the frequency of field oscillations diverges near the big-rip singularity. The only way to stop running into these divergences is if relativistic dynamics intervenes to stop the field from running to $w = -\infty$. We will show in Sec. IV that this is indeed possible.

Third, we give a simple proof of a known result [11] which shows that phantom K-matter, with $w < -1$, cannot cross the "phantom barrier" at $w = -1$ if one assumes classical stability, $c_s^2 \geq 0$. This means that if the equation of state is found to cross this barrier, then dark energy must be described by other fluid-like models [12] or through non-minimal couplings to gravity [13].

II. PHANTOM DARK ENERGY WITH K-ESSENCE MATTER

A minimal generalization of the usual canonical Lagrangian is the so-called K-Lagrangian, introduced initially to enlarge the class of inflationary models [14, 15]. Later, they were also studied as dark energy models (see, e.g., [9, 16].)

Any sensible phantom k-essence matter that can drive the super-acceleration of the Universe should satisfy three criteria:

a) It should be a phantom field: $w \equiv p/\rho < -1$ in some spacetime volume.

b) Classical solutions should be stable: the sound speed c_s^2 of classical perturbations of the field around homogeneous FLRW solutions cannot be negative.

c) Quantum stability: the Hamiltonian must be limited from below. Evidently, any system

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with a Hamiltonian which is unbounded from below would be instantly destroyed by quantum tunelling of positive-energy particles into the negative-energy particles. For theories with non-canonical kinetic terms, the quantum stability is a nontrivial issue.

In what follows we will assume that the matter sector is represented by a K-essence scalar field Lagrangian, which has the form:

$$\mathcal{L} = \sqrt{-g}\, F(X)V(\phi) \; , \tag{1}$$

where:

$$X \equiv \frac{1}{2}\, \partial^\mu \phi\, \partial_\mu \phi \; , \tag{2}$$

and we use the timelike metric signature, $\eta_{\mu\nu} = \mathrm{diag}(+,-,-,-)$. For a canonical scalar field theory, $L = X - V(\phi)$.

The energy-momentum tensor of this fluid reads:

$$T_{\mu\nu} = \frac{2}{\sqrt{-g}} \frac{\partial \mathcal{L}}{\partial g^{\mu\nu}} = VF' \partial_\mu \phi \partial_\nu \phi - g_{\mu\nu} VF \; , \tag{3}$$

where a prime denotes a derivative with respect to X. In gaussian coordinates, which can be constructed in finite regions in general and globally if the topology of spacetime is $M^4 = R \bigotimes M^3$, one can write:

$$X \equiv \frac{1}{2}\, g^{00} \partial_0 \phi\, \partial_0 \phi - \frac{1}{2} |g^{ij}| \partial_i \phi\, \partial_j \phi \equiv X_t - X_s \; , \tag{4}$$

where X_t and X_s are both non-negative. The Hamiltonian then reads:

$$\mathcal{H} = T^0_0 = V\, \left(2F'[\Pi, X_s, \phi] X_t[\Pi, X_s, \phi] - F[\Pi, X_s, \phi] \right) \; , \tag{5}$$

where $\Pi \equiv F' V \partial^0 \phi$ is the conjugated momentum.

In homogeneous and isotropic spacetimes $X_s = 0$, and we obtain:

$$\rho = T^0_0 = V(2F'X - F) \; , \tag{6}$$

and:

$$p = -\frac{1}{3} T^i_i = VF \; , \tag{7}$$

implying that:

$$w = \frac{p}{\rho} = -\frac{1}{F^2 \left(\frac{X}{F^2} \right)'} . \tag{8}$$

Hence, condition (a), $w < -1$, yields:

$$0 < 2X \frac{F'}{F} < 1 \; , \tag{9}$$

or, equivalently:

$$\begin{aligned} F' > 0 \quad &, \quad F - 2XF' > 0 \quad \text{if} \quad F > 0 \; , \\ F' < 0 \quad &, \quad F - 2XF' < 0 \quad \text{if} \quad F < 0 \; . \end{aligned} \tag{10}$$

The sound speed, c_s^2, is the function appearing before spatial gradients in the scalar field equation-of-motion, $\ddot{\phi} + c_s^2 \nabla^2 \phi + \ldots = 0$, and for perturbations around homogeneous solutions, it is a function of time. The sound speed expresses the phase velocity of the inhomogeneous perturbations of the scalar field. Therefore, to avoid exponentially growing solutions and thus ensure classical stability, we must have $c_s^2 \geq 0$. On the other hand, to ensure causality in the usual sense the condition would be $c_s^2 \leq 1$. However, since the theories we consider are perfectly Lorentz invariant (the superluminal propagation being just a consequence of the nonlinearity of the theory [15]), we will only impose the first condition (classical stability), as superluminal propagation cannot be ruled out by observations if the scalar field is dark, i.e., if it does not interact with normal matter.

For K-essence models the sound speed takes the simple expression [15]:

$$c_s^2 = \frac{p'}{\rho'} = \frac{F'^2}{(F'^2 X)'} \; . \tag{11}$$

Condition (b), that $c_s^2 \geq 0$, implies:

$$(F'^2 X)' \geq 0 \; . \tag{12}$$

In the subsequent subsections we will use the following relation between c_s^2 and w coming from Eqs. (8) and (11):

$$1 - \frac{w}{c_s^2} = \frac{2Xw'}{1+w} \; . \tag{13}$$

Finally, condition (c) implies that $\mathcal{H} = V(2F'X_t - F)$ must be bounded from below. As we will show in the next section, this is impossible given the two conditions above, Eqs. (10) and (12).

III. GENERAL RESULTS

A. The general Hamiltonian is not bounded from below

Suppose initially that $V > 0$, so the function $\mathcal{H}/V$ must be shown to be bounded from below. The conditions in Eqs. (10) are constraints on the function $F(X)$ and its derivative, that do not depend on how X is obtained from the phantom field (or whether it is homogeneous or not). That condition must be satisfied whenever $X > 0$ (which is the only constraint the restriction to homogeneous fields imposes on the possible values of the real variable X), otherwise the homogeneous model would not yield a super-accelerated expansion, and it should be discarded. As $\rho+3p = 2V(F'X+F) < 0$, in order to have acceleration in the homogeneous background (where $X > 0$), one must take from condition (10) the alternative $F < 0$ and $F' < 0$ because we are now considering $V > 0$. Hence, in this case, one must have from Eq. (10) that $F'_0 = F'(X_0) < 0$ and $F_0 = F(X_0) < 0$ for any given value $X = X_0 > 0$. We will show below that the subset of fluctuations that keeps $X > 0$ constant, while varying X_t and X_s, is such that the Hamiltonian is unbounded from below. This suffices to prove that the system is unstable.

Consider, then, quantum fluctuations such that X is kept fixed, $X = X_t - X_s = X_0 > 0$, but where X_t can vary freely. In order for the Hamiltonian to be bounded from below, one must have $2X_tF'_0 - F_0 > C$, where C is some finite constant. However, since $F'_0 < 0$ and $F_0 < 0$ are fixed, the stronger the quantum fluctuation, the larger X_t and therefore the Hamiltonian is unbounded from below.

On the regions where $V < 0$ the proof is similar. Using the same reasoning, one now has that $F'_0 > 0$ and $F_0 > 0$, and hence $\mathcal{H} = V(2F'X_t - F)$ is unbounded from below because here V is negative. Notice that for $VF < 0$, in the case of FLRW backgrounds, the condition (9) implies that $\rho = -VF^3(X/F^2)' > 0$. Notice also that, within this class of Lagrangians, the Hamiltonians of the canonical and the Born-Infeld cases, respectively $F = X$ and $F = -\sqrt{1-2X}$, *are* bounded from below. However, these cases do not represent phantom fields.

B. The equation of state and sound speed are unbounded if $w < -1$ and $c_s^2 \geq 0$

From Eq. (13) one has:

$$c_s^2 = \frac{|w||1+w|}{2X|w'| - |1+w|} \ . \tag{14}$$

In order for $0 \leq c_s^2 < \infty$ we must have:

$$\lim_{X\to\infty}[-2Xw'(X)] > \lim_{X\to\infty}|1+w| > 0 \ . \tag{15}$$

We will prove by contradiction that one cannot have $w(X) = w_\infty + f(X)$, where $-\infty < w_\infty < -1$ is a constant and $\lim_{X\to\infty} f(X) = 0$. Notice that:

$$\begin{aligned}\lim_{X\to\infty}[-2Xf(X)]' &= 2\lim_{X\to\infty}[-f(X) - Xf'(X)] \\ &= \lim_{X\to\infty}[-2Xf'(X)] \\ &> |1+w_\infty| \ ,\end{aligned}$$

where we used Eq.(15) and $\lim_{X\to\infty} w(X) = w_\infty$ in the last step. Hence:

$$\begin{aligned}\lim_{X\to\infty} f(X) &= -\frac{1}{2}\lim_{X\to\infty}[\frac{1}{X}\int(-2Xf'(X))dX] \\ &< -\frac{1}{2}|1+w_\infty| < 0 \ ,\end{aligned}$$

contradicting the hypothesis that $\lim_{X\to\infty} f(X) = 0$. Hence, for $w < -1$ and $0 \leq c_s^2 < \infty$ we must have that:

$$\lim_{X\to\infty} w(X) \to -\infty \ . \tag{16}$$

Similarly, we can prove that the sound speed, c_s^2, is also unbounded (from above) in phantom K-matter models. Here it is useful to write Eq. (13) as:

$$c_s^{-2} = \frac{1}{w} - \frac{2Xw'}{1+w} \ . \tag{17}$$

One can see that c_s^2 does not diverge if and only if the expression above does not goes to zero when $X \to \infty$, where, as we have proven above, $w \to -\infty$. Hence, for very large values of X we have:

$$\frac{1}{w} - \frac{2Xw'}{1+w} \approx 2\left(\frac{X}{w}\right)' = h(X) = h_{min} > 0 \ , \tag{18}$$

where h_{min} is the minimum value of $h(X)$ for X very large. However, from Eq.(18), one could then write:

$$\lim_{X\to\infty}\frac{2}{w} = \frac{1}{X}\int h(X)dX > h_{min} \ , \tag{19}$$

which is contradiction with the previous result that $\lim_{X\to\infty} w(X) \to -\infty$. Hence, $\lim_{X\to\infty} c_s^2(X) \to \infty$.

C. The equation of state cannot cross the value -1 if $c_s^2 \geq 0$

Let us first consider cosmological solutions, for which $X = X_t > 0$. If $w < -1$, Eq.(13) reads:

$$1 + \frac{|w|}{c_s^2} = -\frac{2Xw'}{|1+w|} \ , \tag{20}$$

and $w' < 0$. This means that, as a function of X the equation of state is a monotonically decreasing function.

On the other hand, if $-1 < w < 0$ then Eq.(13) reads:

$$1 + \frac{|w|}{c_s^2} = \frac{2Xw'}{|1+w|} \ , \tag{21}$$

and $w' > 0$. In this case, w as a function of X is a monotonically increasing function. Hence, w cannot cross the value -1 if the sound speed is non-negative [11].

IV. TOY MODEL

In this section we present an example of a K-essence Lagrangian where conditions (a) and (b) of Sec. II are satisfied, and in which no divergences of w and c_s^2 appear. The kinetic function $F(X)$ reads:

$$F(X) = 1 + aX(1+X) - b(1-X^n)^n \ , \tag{22}$$

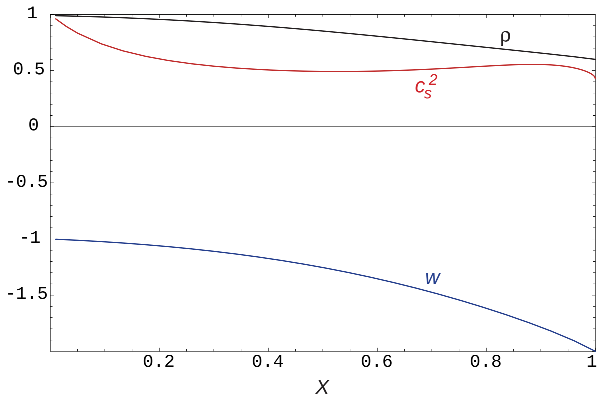

FIG. 1: The three curves are, from top to bottom: ρ/V (black), the sound speed c_s^2 (red), and the equation of state w (blue).

where a and b are some constants, n is some positive half-integer, and $V(\phi)$ is left arbitrary. General-relativistic dynamics constrain X to the interval $0 < X < 1$, because at $X = 0$ and $X = 1$ there are singularities in the derivatives of the curvature tensor. With $n = 5/2$, $a = 0.1$ and $b = 0.01$, for example, the equation of state remains bounded, $-1 < w < -2$, and the sound speed is always positive but finite, $0.43 < c_s^2 < 1$ — see Fig. 1. However, the parameter subspace where this occurs seems very small when compared to the full parameter space of this toy model.

V. DISCUSSION

We have shown in this paper that one cannot obtain K-essence phantom models without quantum instabilities. This is a direct consequence of the phantom imposition, namely, $w = p/\rho \leq -1$ for some range of $X \equiv \partial^\mu\phi\partial_\mu\phi/2$, which in fact must be true for all values of X if one imposes classical stability. Hence, lagrangians of the type of Eq. (1) cannot be considered as fundamental descriptions of phantom fields, being at most effective lagrangians aplicable to the cosmological set-up.

We have also shown that classically stable K-essence phantom models present divergences of w in the negative direction (w is bounded from above by $w = -1$, but unbounded from below), and of c_s^2 in the positive direction. These divergences occur unless $X \to \infty$ is dynamically forbidden – either by limiting the range of X in the lagrangian or by adjusting the potential $V(\phi)$. We have exhibited an example of a phantom K-essence lagrangian with the first property.

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