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Abstract

We introduce *color* Heisenberg-Lie (super)algebras graded by the abelian groups $\mathbb{Z}_3^2$, $\mathbb{Z}_2^p \times \mathbb{Z}_3^2$ for $p = 1, 2, 3$, and investigate the properties of their associated multi-particle quantum paraoscillators.

In the Rittenberg-Wyler's color Lie (super)algebras framework the above abelian groups are the simplest ones which induce *mixed brackets* interpolating commutators and anticommutators. These mixed brackets allow to accommodate two types of parastatistics: one based on the permutation group (beyond bosons and fermions in any space dimension) and an anyonic parastatistics based on the braid group. In both such cases the two broad classes of paraparticles are given by parabosons and parafermions.

Mixed-bracket parafermions are created by nilpotent operators; they satisfy a generalized Pauli exclusion principle leading to roots-of-unity truncations in their multi-particle energy spectrum (braided Majorana qubits and their Gentile-type parastatistics are recovered in this color Lie superalgebra setting).

Mixed-bracket parabosons do not admit truncations of the spectrum; the minimal detectable signature of their parastatistics is encoded in the measurable probability density of two indistinguishable parabosonic oscillators in a given energy eigenstate.

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1 Introduction

Ordinary Lie (super)algebras were extended by Rittenberg and Wyler in [1, 2] to "*Color Lie (super)algebras*" defined in terms of a general abelian-group grading. Quite sooon this new mathematical structure was systematically analyzed by Scheunert in [3].

The simplest examples of this class of theories are obtained for gradings based on the $\mathbb{Z}_2^n$ groups. For these gradings the defining brackets are ordinary commutators/anticommutators organized in peculiar ways. They imply a class of parastatistics which can be referred to, see [4], as "*n-bit*" parastatistics. The resulting paraparticles are exchanged under the permutation group and can exist in any space dimension.

Beyond the $\mathbb{Z}_2^n$ groups, new possibilities are opened. Already in the original [1] paper, Rittenberg-Wyler produced, for the $\mathbb{Z}_3 \times \mathbb{Z}_3 := \mathbb{Z}_3^2$ group, a color Lie algebra whose defining graded brackets consist of nontrivial linear combinations of commutators/anticommutators (from now on we will apply the term *mixed-bracket* to such nontrivial linear combinations).

At the beginning, some limited attention was given to the physical applications of $\mathbb{Z}_2^2$-graded color Lie (super)algebras, see e.g. [5,6]; on the other hand, the recent decade experienced a boom (more on that later) of papers devoted to mathematical investigations and physical applications of $\mathbb{Z}_2^n$-graded color Lie (super)algebras.

What about the more general case, namely the possible physical applications offered by *mixed-bracket* color Lie (super)algebras? Strangely enough, almost nothing has been done. Perhaps the most notable exception consists in the $\mathbb{Z}_4 \times \mathbb{Z}_4$-graded extensions of the Poincaré algebra introduced in [7,8].

In this paper we introduce the simplest mixed-bracket *color* extensions of the Heisenberg-Lie (super)algebras (their grading abelian groups are given by $\mathbb{Z}_3^2$ and $\mathbb{Z}_2^p \times \mathbb{Z}_3^2$ for $p = 1, 2, 3$). We construct their associated multi-particle quantum (para)oscillators and investigate their physical properties. The two broad classes of mixed-bracket paraparticles are given by parabosons and parafermions.

Mixed-bracket parafermions are created by nilpotent operators and satisfy a generalized Pauli exclusion principle which implies roots-of-unity truncations in their multi-particle energy spectrum. Their simplest physical application consists in reformulating the multi-particle braided Majorana qubits [9, 10] in the color Lie superalgebra framework; the roots-of-unity truncations lead to an implementation of a Gentile-type parastatistics [11].

Mixed-bracket parabosons do not admit truncations of the spectrum. We show that the minimal detectable signature of their parastatistics is encoded in the measurable probability density of two indistinguishable parabosonic oscillators in a given energy eigenstate.

On general grounds we also point out in this paper that, unlike the *n-bit* parastatistics, *mixed-bracket* color Lie (super)algebras can accommodate not only the exchange of paraparticles under the permutation group (i.e., beyond bosons and fermions in any space dimension), but also an anyonic parastatistics with paraparticles exchanged via the braid group.

We postpone to the Conclusions a more detail summary of the main results of the paper, together with a list of further investigations and main open questions concerning both physical applications and the relation with other mathematical structures (notably, quantum group reps at roots of unity) of mixed-bracket color Lie (super)algebras.

The remaining part of this Introduction is devoted to briefly detail three main topics which are related to this work. The first topic is a concise state-of-the-art account about mathematics and physical applications of color Lie (super)algebras; the second topic summarizes the applications of the $\mathbb{Z}_3$-grading to ternary structures and ternary physics; the last topic mentions the possible relevance of the braided Majorana qubits in the light of the Kitaev's proposal [12] of Topological Quantum Computation offering protection from decoherence.

1 - *On the color Lie (super)algebras state-of-the-art.*

At the beginning, it was mostly the mathematical properties of color Lie (super)algebras which started being investigated, see e.g. [13–15]. Their parastatistics was analyzed in a series of papers [16–21]. Despite of that, the applications of color Lie (super)algebras to physics was hampered by a widely widespread (wrong) assumption that their parastatatistics could always be recovered by ordinary bosons/fermions (a more detailed discussion of this point is given in the Conclusions). This situation radically changed in the last decade, with several works investigating different aspects and applications of $\mathbb{Z}_2^n$-graded color Lie (super)algebras. It was recognized [22, 23] that they appear as symmetries of known physical systems describing nonrelativistic spinors; a $\mathbb{Z}_2^2$-graded invariant quantum Hamiltonian which prompted further investigations was proposed [24]; classical [25, 26] and quantum [27] systems started being systematically analyzed; superspace formulation [28] and integrable systems [29, 30] were presented and so on (the list is growing). It was finally proved in [31] that $\mathbb{Z}_2^2$-graded parafermions are theoretically detectable, presenting a distinct signature which discriminates them from ordinary fermions; soon after, the detectability of $\mathbb{Z}_2^2$-graded parabosons was established in [32]. More works on $\mathbb{Z}_2^n$-graded parastatistics followed [4, 33, 34].

As already mentioned, this recent activity on physical applications focused on $\mathbb{Z}_2^n$-graded color Lie (super)algebras, leaving aside the most general class of gradings which induce *mixed brackets*. Concerning the mathematical structure, a recent paper [35] presents the updated state of the art of the mathematical properties of color Lie (super)algebras and Hopf algebras graded by arbitrary abelian groups.

2 - *About $\mathbb{Z}_3$-grading and ternary structures.*

It is pointed out in Section **2** that no nontrivial color Lie algebra is obtained from the $\mathbb{Z}_3$ grading group. In the literature the $\mathbb{Z}_3$ grading group has been applied to the so-called "ternary structures" producing, e.g., the cubic root of the Dirac equation [36]. In ternary mathematics, besides quadratic multiplications, cubic relations are imposed (see, for instance, the papers [37, 38] and the references therein); their physical applications are discussed, e.g., in [39, 40]. This ternary, $\mathbb{Z}_3$-graded, mathematics/physics is not directly related to color Lie (super)algebras. Nevertheless, $\mathbb{Z}_3$-graded matrices as the ones presented in [41] can be used as building blocks (see Section **3**) to construct matrix representations of color Lie (super)algebras. Furthermore, ternary extensions of color Lie (super)algebras were introduced in [42].

3 - *On Topological Quantum Computation and some recent developments.*

The notion of *$\mathbb{Z}_2$-graded Majorana qubits* was introduced in [9] (see also [10]) and their braiding properties analyzed. In the $\mathbb{Z}_2$-graded qubit the excited state is fermionic and coincides with its own antiparticle (hence, it is a Majorana fermion).

The main idea behind [9, 10] was to make contact with the Kitaev's program [12] of Topological Quantum Computation implemented by emergent braided Majorana particles; it offers topological protection from the quantum decoherence of "ordinary" quantum computers. The Kitaev's program was also discussed in [43,44]; the "knot logic" behind topological quantum computation with Majorana fermions was presented in [45]. As advocated in [46], the manipolation of Majorana qubits could offer a minimal setting to implement topological quantum computers, paralleling the manipulation of bits for ordinary computers and of qubits for the present-day quantum computers. An exciting advance is the recent Microsoft's announcement [47] of the first quantum chip powered by a topological architecture. It leads, see the [48] roadmap to fault tolerant quantum computation, to a practical implementation of the Kitaev's proposal.

Further comments about the difference between permutation-group versus braid-group (anyonic) parastatistics are left to the **Conclusions**.

The structure of the paper is the following:

The paper is dividided in two parts separated by an *Intermezzo* (Section **6**).

The first part (Sections **2-5**) details the foundation of the theory. Section **2** reviews, following [3], the basic notions of color Lie (super)algebras. Section **3** presents several $\mathbb{Z}_3$-graded matrices used as building blocks in the construction of the color Lie (super)algebras later introduced. Section **4** presents the two inequivalent $\mathbb{Z}_3 \times \mathbb{Z}_3$-graded color Lie algebras and the four inequivalent $\mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_3$ color Lie (super)algebras. Graded para-oscillators (both parabosonic and parafermionic) are introduced in Section **5**.

The second part (Sections **7,8**) describes applications. Section **7** points out which is the minimal signature for mixed-bracket indistinguishable parabosons: it is given by multi-particle probability densities in certain energy eigenstates. In Section **8** it is shown that a signature of mixed-bracket indistinguishable parafermions is given by truncations of the energy spectra (these truncations generalize the Pauli exclusion principle of ordinary fermions). The connection with braided Majorana qubits (at the $s = 3, 6$ roots of unity levels) is made.

The Section **6** *Intermezzo* discusses two scenarios for constructing mixed-bracket indistinguishable multi-particle sectors; the first scenario is based on graded Hopf algebras and their coproducts, while the second one is based on the symmetrization of the creation operators. It is further pointed out in *Intermezzo* that mixed-bracket color Lie (super)algebras can be applied to two types of parastatistics (both the parastatistics associated with the permutation group and the anyonic parastatistics associated with the braid group).

The important notion of *roots-of-unity level* is defined in Appendix **A**.

Several comments are presented in the **Conclusions**. Besides summarizing the main results of the paper, we present a list of further investigations and main open questions concerning physical applications of mixed-bracket color Lie (super)algebras and their relation with other mathematical structures.

2 Review of color Lie (super)algebras

For self-consistency of the paper we introduce definitions and basic features of color Lie (super)algebras. We follow Scheunert's presentation [3] which is more convenient for our purposes than Rittenberg-Wyler's [1,2] approach. On the other hand, since it is more widely accepted in the physical literature, we refer to these structures by using the term "color Lie (super)algebras" coined by Rittenberg-Wyler, rather than "ε Lie algebras" introduced by Scheunert.

Let Γ be an Abelian group and V a Γ-graded vector space $V = \oplus_\gamma V_\gamma$, $\gamma \in \Gamma$, so that the homogeneous elements belong to V_γ.

A linear mapping $g : V \to W$ between Γ-graded spaces is homogeneous of degree $\gamma \in \Gamma$ if $g(V_\alpha) \subset W_{\alpha+\gamma}$.

For three Γ-graded spaces U, V, W let $h : U \rightarrow V$, $g : V \rightarrow W$ be homogeneous of degree $deg(h) = \alpha$, $deg(g) = \beta$; it follows that the composition map $g \circ h : U \rightarrow W$ is homogeneous of degree $deg(g \circ h) = \alpha + \beta$.

An algebra S is Γ-graded if $S = \oplus_{\gamma \in \Gamma} S_\gamma$ as a vector space and $S_\alpha S_\beta \subset S_{\alpha+\beta}$ for any $\alpha, \beta \in \Gamma$. Let T be a second Γ-graded algebra. An $S \rightarrow T$ homomorphism of Γ-graded algebras is an homomorphism of S to T which is a homogeneous mapping of degree zero.

Definition of commutation factor ε on abelian group Γ: it is a bilinear map $\varepsilon : \Gamma \times \Gamma \rightarrow C^*$ (taking values in the punctured complex plane $\mathbb{C}^* \equiv \mathbb{C} \backslash \{0\}$) satisfying, for any α, β, γ in Γ, the properties:

$$
\begin{array}{rl}
i): & \varepsilon(\alpha,\beta)\cdot\varepsilon(\beta,\alpha) = 1, \\
ii): & \varepsilon(\alpha,\beta+\gamma) = \varepsilon(\alpha,\beta)\cdot\varepsilon(\alpha,\gamma), \\
iii): & \varepsilon(\alpha+\beta,\gamma) = \varepsilon(\alpha,\gamma)\cdot\varepsilon(\beta,\gamma).
\end{array} \tag{1}
$$

A Γ-graded ε-Lie algebra $L = \oplus_\gamma L_\gamma$ (*color Lie (super)algebra* in [1, 2]) is defined by the brackets $\langle ., . \rangle : L \times L \rightarrow L$ which satisfy the conditions, for any $A, B, C \in L$ and respective α, β, γ gradings:

$$
\begin{array}{lll}
i) & \langle A,B\rangle = -\varepsilon(\alpha,\beta)\langle B,A\rangle & \text{(the } \varepsilon\text{-skew symmetry) and} \\
ii) & \varepsilon(\gamma,\alpha)\langle A,\langle B,C\rangle\rangle + cyclic = 0 & \text{(the } \varepsilon\text{-Jacobi identities).}
\end{array} \tag{2}
$$

It follows, from the first relation in (1), that $\varepsilon(\alpha, \alpha) = \pm 1$. Motivated by physical applications, Rittenberg-Wyler introduced the term *color Lie algebra* if the sign $+1$ is valid for any $\alpha \in \Gamma$; if there exists at least one $\alpha \in \Gamma$ such that $\varepsilon(\alpha, \alpha) = -1$, the corresponding algebra is called *color Lie superalgebra.*

For any associative Γ-graded space S the introduction of a bracket $\langle ., . \rangle$ defined by

$$
\langle A, B \rangle := AB - \varepsilon(\alpha, \beta) BA \tag{3}
$$

induces, on the graded vector space S, a color Lie (super)algebra satisfying the equations (2).

Remark 1: at the special value $\varepsilon(\alpha, \beta) = +1$ the graded bracket $\langle A, B \rangle \equiv AB - BA = [A, B]$ defines an ordinary commutator while, at the special value $\varepsilon(\alpha, \beta) = -1$, the graded bracket $\langle A, B \rangle \equiv AB + BA = \{A, B\}$ defines an anticommutator.

Remark 2: in this work the abelian groups Γ under consideration belong to the (direct product of) multiplicative groups $\mathbb{Z}_n$ of integers modulo n. For special cases, such as the group $\mathbb{Z}_2^p := \mathbb{Z}_2 \times \ldots \times \mathbb{Z}_2$ (taken p times), all graded brackets are either commutators or anticommutators (namely, $\varepsilon(\alpha, \beta) = \pm 1$ for any $\alpha, \beta \in \mathbb{Z}_2^p$). Only commutators/anticommutators are also encountered for the direct product group $\mathbb{Z}_2 \times \mathbb{Z}_3$, while $\mathbb{Z}_3$ admits only commutators (one gets, $\forall \alpha, \beta \in \mathbb{Z}_3$, $\varepsilon(\alpha, \beta) = +1$). The group $\mathbb{Z}_3 \times \mathbb{Z}_3$ provides, see [1], the simplest nontrivial example of consistent graded brackets which satisfy the (1) properties and do not reduce to (anti)commutators (this means that $\varepsilon(\alpha, \beta) \neq \pm 1$ for at least a pair of α, β values). These types of brackets are the focus of the present work. We consider, in particular, the direct product groups $\mathbb{Z}_2^p \times \mathbb{Z}_3^q$ for integer values $p, q = 0, 1, 2, \ldots$). Throughout the paper the $\mathbb{Z}_2$-grading will be denoted, following [4], by one bit $(0, 1)$, while the $\mathbb{Z}_3$-grading will be denoted by a trit $(\underline{0}, \underline{1}, \underline{2})$. The integers in a trit are underlined to avoid confusion with the $0, 1$ values entering the $\mathbb{Z}_2$-grading.

2.1 Iterated commutation factors

We present a construction which induces a consistent subclass of commutation factors (satisfying the (1) properties) for the direct groups $\mathbb{Z}_2 \times \Gamma$ and $\mathbb{Z}_3 \times \Gamma$ once a consistent choice of ε commutation factors of an n-dimensional abelian group Γ is given. Let U be a table presenting the commutation factors of Γ in a $n \times n$ array (throughout the paper the $\varepsilon(\alpha, \beta)$ commutation factor is expressed as the entry associated with the α row and β column). Then, commutation factors are obtained for:

a) the $\mathbb{Z}_2 \times \Gamma$ *abelian group*: two sets of commutation factors are obtained for $\delta = \pm 1$. They are presented in a $2n \times 2n$ array according to the scheme

$$|U| \quad \Rightarrow \quad \begin{vmatrix} U & U \\ U & \delta U \end{vmatrix}; \tag{4}$$

b) the $\mathbb{Z}_3 \times \Gamma$ *abelian group*: a set of commutation factors is given by the following $3n \times 3n$ array

$$|U| \quad \Rightarrow \quad \begin{vmatrix} U & U & U \\ U & U & U \\ U & U & U \end{vmatrix}. \tag{5}$$

As an example of a), an ordinary Lie algebra with commutation factor 1 produces

$$\text{for } \delta = +1 \quad : \quad |1| \Rightarrow \begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix}, \quad \text{i.e., an ordinary Lie algebra endowed with a } \mathbb{Z}_2 \text{ grading,}$$

$$\text{for } \delta = -1 \quad : \quad |1| \Rightarrow \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix}, \quad \text{i.e., an ordinary Lie superalgebra.} \tag{6}$$

As an example of b), an ordinary Lie algebra with commutation factor 1 produces

$$|1| \quad \Rightarrow \quad \begin{vmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{vmatrix}, \quad \text{i.e., an ordinary Lie algebra endowed with a } \mathbb{Z}_3 \text{ grading.} \tag{7}$$

The above one is the only consistent $\mathbb{Z}_3$-graded color Lie (super)algebra (which turns out to be a trivial ordinary Lie algebra). By inserting the gradings of the rows/columns, the above 3×3 array reads as

	$\underline{0}$	$\underline{1}$	$\underline{2}$
$\underline{0}$	1	1	1
$\underline{1}$	1	1	1
$\underline{2}$	1	1	1

(8)

By applying the a) iteration to the above commutation factors we obtain the only consistent

$\mathbb{Z}_2 \times \mathbb{Z}_3$-graded color Lie (super)algebras, expressed by the δ-dependent 6×6 arrays

$$\begin{array}{|c|ccc|ccc|}\hline & 0\underline{0} & 0\underline{1} & 0\underline{2} & 1\underline{0} & 1\underline{1} & 1\underline{2} \\ \hline 0\underline{0} & 1 & 1 & 1 & 1 & 1 & 1 \\ 0\underline{1} & 1 & 1 & 1 & 1 & 1 & 1 \\ 0\underline{2} & 1 & 1 & 1 & 1 & 1 & 1 \\ \hline 1\underline{0} & 1 & 1 & 1 & \delta & \delta & \delta \\ 1\underline{1} & 1 & 1 & 1 & \delta & \delta & \delta \\ 1\underline{2} & 1 & 1 & 1 & \delta & \delta & \delta \\ \hline \end{array} \qquad (9)$$

The rows/columns are expressed in terms of the $0, 1$ bit notation for $\mathbb{Z}_2$ and the $\underline{0}, \underline{1}, \underline{2}$ trit notation for $\mathbb{Z}_3$.

The two inequivalent cases are recovered for:
$\delta = +1$, producing an ordinary Lie algebra and
$\delta = -1$, producing an ordinary Lie superalgebra.

3 Building blocks: $\mathbb{Z}_3$-graded matrices

In the construction of $\mathbb{Z}_3 \times \mathbb{Z}_3$-graded color Lie algebras we use, as building blocks, $\mathbb{Z}_3$-graded 3×3 matrices. We present here their main features and introduce some sets of $\mathbb{Z}_3$-graded matrices which are relevant for this work.

The non-vanishing entries of $\mathbb{Z}_3$-graded 3×3 matrices $\mathbf{M}_{\underline{k}}$, with $\underline{k} = \underline{0}, \underline{1}, \underline{2}$, are accommodated according to

$$\mathbf{M}_{\underline{0}} = \begin{pmatrix} * & 0 & 0 \\ 0 & * & 0 \\ 0 & 0 & * \end{pmatrix}, \qquad \mathbf{M}_{\underline{1}} = \begin{pmatrix} 0 & * & 0 \\ 0 & 0 & * \\ * & 0 & 0 \end{pmatrix}, \qquad \mathbf{M}_{\underline{2}} = \begin{pmatrix} 0 & 0 & * \\ * & 0 & 0 \\ 0 & * & 0 \end{pmatrix}. \qquad (10)$$

Under matrix multiplication $\mathbf{M}_{\underline{k}} \cdot \mathbf{M}_{\underline{k}'}$ has grading $\underline{k + k'} = \underline{k} + \underline{k}'$ mod 3.

The following sets of $\mathbb{Z}_3$-graded 3×3 matrices are used in this paper. Their entries are $0, 1, j, j^2$, where j is a third root of unity satisfying $j^3 = 1$. We have:

i) the $\underline{0}$-graded diagonal matrices

$$N_{\underline{0}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad N'_{\underline{0}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & j & 0 \\ 0 & 0 & j^2 \end{pmatrix}, \qquad N''_{\underline{0}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & j^2 & 0 \\ 0 & 0 & j \end{pmatrix}; \qquad (11)$$

ii) the commuting matrices

$$N_{\underline{0}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad N_{\underline{1}} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \qquad N_{\underline{2}} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \qquad (12)$$

satisfying, for any $\underline{k}, \underline{k}'$, $N_{\underline{k}} N_{\underline{k}'} = N_{\underline{k}'} N_{\underline{k}}$;

iii) the $\underline{1}$-graded matrices,

$$Q_{+1} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & j \\ j^2 & 0 & 0 \end{pmatrix}, \qquad Q_{+2} = \begin{pmatrix} 0 & j & 0 \\ 0 & 0 & 1 \\ j^2 & 0 & 0 \end{pmatrix}, \qquad Q_{+3} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \tag{13}$$

and their hermitian conjugates $Q_{+i}^\dagger := Q_{-i}$, given by

iv) the $\underline{2}$-graded matrices

$$Q_{-1} = \begin{pmatrix} 0 & 0 & j \\ 1 & 0 & 0 \\ 0 & j^2 & 0 \end{pmatrix}, \qquad Q_{-2} = \begin{pmatrix} 0 & 0 & j \\ j^2 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \qquad Q_{-3} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}. \tag{14}$$

4 Some relevant color Lie (super)algebras

For an abelian group Γ of small dimension n it is computationally feasible to impose the (1) properties on an $n \times n$ array, resulting in the whole set of inequivalent, admissible commutation factors. We present here the results for $\mathbb{Z}_3 \times \mathbb{Z}_3$ and $\mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_3$. All resulting commutation factors are given by roots of unity. For $\mathbb{Z}_3 \times \mathbb{Z}_3$ we recover the color Lie algebra introduced in [1].

4.1 The $\mathbb{Z}_3 \times \mathbb{Z}_3$-graded color Lie algebra

The consistent commutation factors of the $\mathbb{Z}_3 \times \mathbb{Z}_3$-graded abelian group are presented as entries of the following 9×9 array. The entries are given by the numbers $1, j, j^2$, where j is a third root of unity satisfying $j^3 = 1$. The rows and columns are labeled by the two-trit notation. We have

	$\underline{00}$	$\underline{01}$	$\underline{02}$	$\underline{10}$	$\underline{11}$	$\underline{12}$	$\underline{20}$	$\underline{21}$	$\underline{22}$
$\underline{00}$	1	1	1	1	1	1	1	1	1
$\underline{01}$	1	1	1	j^2	j^2	j^2	j	j	j
$\underline{02}$	1	1	1	j	j	j	j^2	j^2	j^2
$\underline{10}$	1	j	j^2	1	j	j^2	1	j	j^2
$\underline{11}$	1	j	j^2	j^2	1	j	j	j^2	1
$\underline{12}$	1	j	j^2	j	j^2	1	j^2	1	j
$\underline{20}$	1	j^2	j	1	j^2	j	1	j^2	j
$\underline{21}$	1	j^2	j	j^2	j	1	j	1	j^2
$\underline{22}$	1	j^2	j	j	1	j^2	j^2	j	1

(15)

Three consistent sets of commutation factors are recovered for each one of the three solutions of $j^3 = 1$, given by $j_1 = e^{\frac{2\pi i}{3}}$, $j_2 = e^{\frac{4\pi i}{3}}$, $j_3 = 1$.

As discussed in Appendix **A**, j_1, j_2 are level-3 roots of unity which, in particular, satisfy the equations $1 + j_1 + j_1^2 = 1 + j_2 + j_2^2 = 0$. On the other hand j_3 is a level-1 root of unity which satisfies $1 + j_3 + j_3^2 = 3$. Setting $j_3 = 1$ in the above table produces an ordinary Lie algebra, while setting j_1, j_2 produces a non-trivial color Lie algebra whose (3) graded brackets are not all given by commutators/anticommutators.

Remark: The $\mathbb{Z}_3 \times \mathbb{Z}_3$-graded color Lie algebras induced by j_1, j_2 are isomorphic. This results from the fact that the j_2 array in (15) can be recovered from the j_1 array under the $\underline{k1} \leftrightarrow \underline{k2}$ permutation (for any $\underline{k} = \underline{0}, \underline{1}, \underline{2}$) of the graded sectors entering the rows/columns.

Corollary: there are two inequivalent choices for consistent $\mathbb{Z}_3 \times \mathbb{Z}_3$-graded commutation factors. We have

$$\begin{array}{lll} i: & j = j_3 = +1, & \text{producing an ordinary Lie algebra and} \\ ii: & j = j_1, & \text{giving a nontrivial color Lie algebra.} \end{array} \tag{16}$$

Observation: The computation of all sets of admissible commutation factors (the $\varepsilon(\underline{ij}, \underline{kl})$ entries in the (15) array) is straightforward. The first equation in (1) implies 9 diagonal plus 36 upper triangular independent parameters. They are constrained by the two remaining equations in (1). By setting $\alpha = \underline{00}$ and $\gamma = \underline{ij}$ we get, from the third equation, $\varepsilon(\underline{00}, \underline{ij}) = 1$ for any choice of $\underline{i}, \underline{j}$. The other parameters are determined by different relations. For instance $\varepsilon(\underline{12}, \underline{ij}) = \varepsilon(\underline{21}, \underline{ij})^{-1}$ is obtained from from $1 = \varepsilon(\underline{00}, \underline{ij}) = \varepsilon(\underline{12}, \underline{ij}) \cdot \varepsilon(\underline{21}, \underline{ij})$.
The presence of the $j^3 = 1$ constraint is induced, e.g., from the series of relations $\varepsilon(\underline{02}, \underline{11}) = \varepsilon(\underline{01}, \underline{11}) \cdot \varepsilon(\underline{01}, \underline{11})$ and $1 = \varepsilon(\underline{00}, \underline{11}) = \varepsilon(\underline{02}, \underline{11}) \cdot \varepsilon(\underline{01}, \underline{11})$ which imply $\varepsilon(\underline{01}, \underline{11})^3 = 1$.
Implementing the complete set of relations produces the (15) array.

4.2 The $\mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_3$ graded color Lie (super)algebras

Direct computations give the whole class of inequivalent commutation factors, satisfying the (1) properties, for the $\mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_3$ grading. The results are summarized in the following 18×18 table with rows/columns labeled by 1 bit and 2 trits:

	$0\underline{00}$	$0\underline{01}$	$0\underline{02}$	$0\underline{10}$	$0\underline{11}$	$0\underline{12}$	$0\underline{20}$	$0\underline{21}$	$0\underline{22}$	$1\underline{00}$	$1\underline{01}$	$1\underline{02}$	$1\underline{10}$	$1\underline{11}$	$1\underline{12}$	$1\underline{20}$	$1\underline{21}$	$1\underline{22}$
$0\underline{00}$	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
$0\underline{01}$	1	1	1	j^2	j^2	j^2	j	j	j	1	1	1	j^2	j^2	j^2	j	j	j
$0\underline{02}$	1	1	1	j	j	j	j^2	j^2	j^2	1	1	1	j	j	j	j^2	j^2	j^2
$0\underline{10}$	1	j	j^2	1	j	j^2	1	j	j^2	1	j	j^2	1	j	j^2	1	j	j^2
$0\underline{11}$	1	j	j^2	j^2	1	j	j	j^2	1	1	j	j^2	j^2	1	j	j	j^2	1
$0\underline{12}$	1	j	j^2	j	j^2	1	j^2	1	j	1	j	j^2	j	j^2	1	j^2	1	j
$0\underline{20}$	1	j^2	j	1	j^2	j	1	j^2	j	1	j^2	j	1	j^2	j	1	j^2	j
$0\underline{21}$	1	j^2	j	j^2	j	1	j	1	j^2	1	j^2	j	j^2	j	1	j	1	j^2
$0\underline{22}$	1	j^2	j	j	1	j^2	j^2	j	1	1	j^2	j	j	1	j^2	j^2	j	1
$1\underline{00}$	1	1	1	1	1	1	1	1	1	δ	δ	δ	δ	δ	δ	δ	δ	δ
$1\underline{01}$	1	1	1	j^2	j^2	j^2	j	j	j	δ	δ	δ	δj^2	δj^2	δj^2	δj	δj	δj
$1\underline{02}$	1	1	1	j	j	j	j^2	j^2	j^2	δ	δ	δ	δj	δj	δj	δj^2	δj^2	δj^2
$1\underline{10}$	1	j	j^2	1	j	j^2	1	j	j^2	δ	δj	δj^2	δ	δj	δj^2	δ	δj	δj^2
$1\underline{11}$	1	j	j^2	j^2	1	j	j	j^2	1	δ	δj	δj^2	δj^2	δ	δj	δj	δj^2	δ
$1\underline{12}$	1	j	j^2	j	j^2	1	j^2	1	j	δ	δj	δj^2	δj	δj^2	δ	δj^2	δ	δj
$1\underline{20}$	1	j^2	j	1	j^2	j	1	j^2	j	δ	δj^2	δj	δ	δj^2	δj	δ	δj^2	δj
$1\underline{21}$	1	j^2	j	j^2	j	1	j	1	j^2	δ	δj^2	δj	δj^2	δj	δ	δj	δ	δj^2
$1\underline{22}$	1	j^2	j	j	1	j^2	j^2	j	1	δ	δj^2	δj	δj	δ	δj^2	δj^2	δj	δ

(17)

The above table is expressed in terms of a third root of unity j satisfying $j^3 = 1$ and of a $\delta = \pm 1$ sign. It turns out that all inequivalent $\mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_3$-graded commutation factors are recovered from the iteration (4) applied to the (15) array.

Remark: Setting $\delta = -1$ in the above table induces, following the Rittenberg-Wyler's terminology, a *color Lie superalgebra*. By repeating the previous Subsection analysis it results that, out of the $6 = 3 \times 2$ arrays obtained by setting $j = j_1, j_2, j_3$ and $\delta = \pm 1$, four of them produce inequivalent commutation factors.

Summary: there are 4 inequivalent $\mathbb{Z}_2\times\mathbb{Z}_3\times\mathbb{Z}_3$-graded color Lie (super)algebras, given by

$$\begin{array}{rlll} i: & j_3=+1, & \delta=+1 & \text{(ordinary Lie algebra)},\\ ii: & j_3=+1, & \delta=-1 & \text{(ordinary Lie superalgebra)},\\ iii: & j_1=e^{\frac{2\pi i}{3}}, & \delta=+1 & \text{(color Lie algebra)},\\ iv: & j_1=e^{\frac{2\pi i}{3}}, & \delta=-1 & \text{(color Lie superalgebra)}. \end{array} \quad (18)$$

4.3 A graded-abelian $\mathbb{Z}_3\times\mathbb{Z}_3$ color Lie algebra

The $\mathbb{Z}_3$-graded 3×3 matrices (11,12,13,14) introduced in Section **3** allow, used as building blocks, to construct matrix representations of $\mathbb{Z}_3\times\mathbb{Z}_3$-graded color Lie algebras. In particular, the following 9×9 matrix representation of the graded-abelian $\mathbb{Z}_3\times\mathbb{Z}_3$ color Lie algebra with one generator in each graded sector (for a total number of 9 generators), is obtained. The matrices are denoted as $C_{\underline{ij}}$; the suffix indicates the $\mathbb{Z}_3\times\mathbb{Z}_3$ grading. We have

$$\begin{array}{lll} C_{\underline{00}}=N_{\underline{0}}\otimes N_{\underline{0}}, & C_{\underline{01}}=N'_{\underline{0}}\otimes N_{\underline{1}}, & C_{\underline{02}}=N''_{\underline{0}}\otimes N_{\underline{0}},\\ C_{\underline{10}}=Q_{+1}\otimes N_{\underline{0}}, & C_{\underline{11}}=Q_{+2}\otimes N_{\underline{1}}, & C_{\underline{12}}=Q_{+3}\otimes N_{\underline{2}},\\ C_{\underline{20}}=Q_{-1}\otimes N_{\underline{0}}, & C_{\underline{21}}=Q_{-3}\otimes N_{\underline{1}}, & C_{\underline{22}}=Q_{-2}\otimes N_{\underline{2}}. \end{array} \quad (19)$$

It is easily realized that the above matrices are graded-commutative. Indeed, their graded commutative brackets defined by

$$\langle C_{\underline{ij}}, C_{\underline{kl}}\rangle \;:=\; C_{\underline{ij}}C_{\underline{kl}}-\varepsilon(\underline{ij},\underline{kl})C_{\underline{kl}}C_{\underline{ij}} \quad (20)$$

satisfy, for any choice of $\underline{i},\underline{j},\underline{k},\underline{l}$, the vanishing relations

$$\langle C_{\underline{ij}}, C_{\underline{kl}}\rangle \;=\; 0. \quad (21)$$

The commutation factors $\varepsilon(\underline{ij},\underline{kl})$ entering (20) are given in table (15).

5 Graded para-oscillators from color Lie (super)algebras

The graded bracket (3) introduced in terms of the commutation factors $\varepsilon(\alpha,\beta)$ can be recasted into a linear combination of commutators/anticommutators as

$$\langle A,B\rangle = AB-\varepsilon(\alpha,\beta)BA=\tfrac{1}{2}\left(1+\varepsilon(\alpha,\beta)\right)\cdot[A,B]+\tfrac{1}{2}\left(1-\varepsilon(\alpha,\beta)\right)\cdot\{A,B\}. \quad (22)$$

$\langle A,B\rangle$ is, for $\varepsilon(\alpha,\beta)\neq\pm 1$, a genuine *mixed-bracket* which does not coincide (up to a normalization) with an ordinary $[A,B]$ commutator or $\{A,B\}$ anticommutator. In the following the term *mixed-bracket* will be employed for color Lie (super)algebras which present, for at least one choice of α,β, a commutation factor $\varepsilon(\alpha,\beta)\neq\pm 1$. The notion of mixed-bracket applies, in particular, to $\varepsilon(\alpha,\beta)$ given by roots-of-unity of level-k with $k>2$ (see Appendix **A** for the notion root-of-unity level).

In this Section we introduce mixed-bracket para-oscillators from color Lie (super)algebras; they are given by (color) mixed-bracket generalizations of the ordinary Heisenberg-Lie (super)algebras. It is important, for the physical applications discussed in the following, to point out the differences between two broad classes of para-oscillators: the parabosonic oscillators versus the parafermionic ones. The color Lie (super)algebras under consideration in this paper have $\mathbb{Z}_2^p \times \mathbb{Z}_3^q$ gradings for integer values of p, q. We introduce here specific examples of mixed-brackets para-oscillators which, in the following, will be used to illustrate the physical implications of the parastatistics derived from these color Lie (super)algebras.

Before proceeding with the construction of mixed-bracket para-oscillators we present two important remarks which clarify the framework.

Remark 1: The mixed-bracket parabosonic and parafermionic oscillators are not directly related to the notion of parabosons and parafermions defined, following [49, 50], by the trilinear relations. They are defined in terms of a different mathematical structure.

Remark 2: The color Heisenberg-Lie (super)algebras under consideration act as spectrum-generating (super)algebras to determine energy eigenstates and spectrum of quantum models. Contrary to parabosonic oscillators, the creation operators of the parafermionic oscillators are nilpotent. Due to that, the parafermionic oscillators obey a generalization of the Pauli exclusion principle leading, as in the models discussed in the following, to truncations of the energy spectra. Parafermionic creation operators are associated with a -1 diagonal commutation factor (therefore, see also the comment after formula (2), they require a color Lie superalgebra), while parabosonic creation operators are associated with a $+1$ commutation factor.

5.1 Parabosonic oscillators from color Lie algebras

We introduce at first four bosonic oscillators $a_I, a_I^\dagger := (a_I)^\dagger$ with $I = 1, 2, 3, 4$, spanning together with the central charge c the bosonic Heisenberg-Lie algebra $\mathfrak{h}_{bos}(4)$.

$$\begin{array}{l} \mathfrak{h}_{bos}(4) \;:\quad \{a_I, a_I^\dagger, c\}, \qquad \text{where} \\ [a_I, a_J^\dagger] = \delta_{IJ}\cdot c, \qquad [c, a_I] = [c, a_I^\dagger] = [a_I, a_J] = [a_I^\dagger, a_J^\dagger] = 0 \qquad \forall I, J. \end{array} \tag{23}$$

With the help of the $\mathbb{Z}_3 \times \mathbb{Z}_3$ graded matrices (19) $C_{\underline{ij}}$ we can introduce the parabosonic color Heisenberg-Lie algebra $\mathfrak{h}_{pb}(4) = \{A_I, A_I^\dagger, C\}$ spanned by 9 generators given by

$$\begin{array}{ll} A_1 = C_{\underline{20}}\cdot a_1, & A_1^\dagger = C_{\underline{10}}\cdot a_1^\dagger, \\ A_2 = C_{\underline{22}}\cdot a_2, & A_2^\dagger = C_{\underline{11}}\cdot a_2^\dagger, \\ A_3 = C_{\underline{21}}\cdot a_3, & A_3^\dagger = C_{\underline{12}}\cdot a_3^\dagger, \\ A_4 = C_{\underline{02}}\cdot a_1, & A_4^\dagger = C_{\underline{01}}\cdot a_4^\dagger, \\ C\ = C_{\underline{00}}. & \end{array} \tag{24}$$

Their respective $\mathbb{Z}_3 \times \mathbb{Z}_3$ gradings are expressed by the $\underline{ij}$ suffix in the right hand side.
It follows that $\mathfrak{h}_{pb}(4)$ is a color Lie algebra satisfying the graded brackets:

$$\begin{array}{l} \mathfrak{h}_{pb}(4) \;:\quad \{A_I, A_I^\dagger, C\}, \qquad \text{where} \\ \langle A_I, A_J^\dagger\rangle = \delta_{IJ}\cdot C, \qquad \langle C, A_I\rangle = \langle C, A_I^\dagger\rangle = \langle A_I, A_J\rangle = \langle A_I^\dagger, A_J^\dagger\rangle = 0 \qquad \forall I, J. \end{array} \tag{25}$$

The graded brackets are recovered from equation (3) by inserting the $\varepsilon(\underline{ij}, \underline{kl})$ commutation factors presented in table (15). The special choice $j = j_1 = e^{\frac{2\pi i}{3}}$ gives to $\mathfrak{h}_{pb}(4)$ the status of a mixed-bracket color Heisenberg-Lie algebra.

One should note that the four pairs of creation/annihilation operators are constructed in terms of the pairings

$$\underline{01} \leftrightarrow \underline{02}, \qquad \underline{10} \leftrightarrow \underline{20}, \qquad \underline{11} \leftrightarrow \underline{22}, \qquad \underline{12} \leftrightarrow \underline{21} \tag{26}$$

which lead to a $\underline{00}$-graded sector and whose commutation factors satisfy

$$\varepsilon(\underline{01},\underline{02}) = \varepsilon(\underline{10},\underline{20}) = \varepsilon(\underline{11},\underline{22}) = \varepsilon(\underline{12},\underline{21}) = 1. \tag{27}$$

Remark: the paraparticles created by the parabosonic creation oscillators do not obey the Pauli exclusion principle satisfied by ordinary fermions. The Pauli exclusion principle (or its generalization which leads to truncations of the energy spectra) is obeyed by para-fermions which, to be introduced, require a color Lie superalgebra. A mixed-bracket color Heisenberg-Lie superalgebra which contains (besides parabosons) parafermions is introduced in the next Subsection.

5.2 Parafermionic oscillators from color Lie superalgebras

The simplest example of a color Lie superalgebra which introduces mixed-bracket parafermionic oscillators is obtained from the $\mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_3$ grading whose commutation factors are presented in table (17). The presence of nontrivial mixed brackets requires to set $j = j_1 = e^{\frac{2\pi i}{3}}$, while the presence of parafermions is implied by setting $\delta = -1$. This choice of parameters produces the *iv* case in the (18) list of (super)algebras.

We now proceed to introduce the mixed-bracket color Heisenberg-Lie superalgebra $\mathfrak{h}_{pf}(4|4)$. The following ingredients are used in its construction:

1) the $\mathbb{Z}_3 \times \mathbb{Z}_3$ graded matrices $C_{\underline{ij}}$ given in (19);
2) the 2×2, $\mathbb{Z}_2$-graded matrices I and Y, of respective grading 0 and 1, given by

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \in [0], \qquad Y = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \in [1];$$

3) the four ordinary bosonic oscillators $a_I, a_I^\dagger$ (for $I = 1,2,3,4$) and central charge c which have already been introduced in (23);
4) four ordinary fermionic oscillators $f_I, f_I^\dagger$ (for $I = 1,2,3,4$); together with the central charge c they close the fermionic Heisenberg-Lie superalgebra $\mathfrak{h}_{fer}(4)$ given by

$$\begin{array}{ll} \mathfrak{h}_{fer}(4) \;:\; \{f_I, f_I^\dagger, c\}, & \text{where} \\ \{f_I, f_J^\dagger\} = \delta_{IJ}\cdot c, & [c, f_I] = [c, f_I^\dagger] = \{f_I, f_J\} = \{f_I^\dagger, f_J^\dagger\} = 0 \qquad \forall I, J. \end{array} \tag{28}$$

The bosonic and the fermionic oscillators mutually commute:

$$[a_I, f_J] = [a_I, f_J^\dagger] = [a_I^\dagger, f_J] = [a_I^\dagger, f_J^\dagger] = 0 \qquad \text{for all } I, J. \tag{29}$$

The mixed-bracket color Heisenberg-Lie superalgebra $\mathfrak{h}_{pf}(4|4)$ is spanned by 17 generators $(C, A_I, A_I^\dagger, F_I, F_I^\dagger$ for $I = 1,2,3,4)$. The subalgebra spanned by $C, A_I, A_I^\dagger$ is parabosonic and isomorphic to $\mathfrak{h}_{pb}(4)$. The subalgebra spanned by $C, F_I, F_I^\dagger$ defines four pairs of parafermionic oscillators. A realization of the $\mathfrak{h}_{pf}(4|4)$ generators is obtained through the following positions:

$$
\begin{array}{ll}
C \ = I \otimes C_{\underline{00}}, & \\
A_1 = I \otimes C_{\underline{20}} \cdot a_1, & A_1^\dagger = I \otimes C_{\underline{10}} \cdot a_1^\dagger, \\
A_2 = I \otimes C_{\underline{22}} \cdot a_2, & A_2^\dagger = I \otimes C_{\underline{11}} \cdot a_2^\dagger, \\
A_3 = I \otimes C_{\underline{21}} \cdot a_3, & A_3^\dagger = I \otimes C_{\underline{12}} \cdot a_3^\dagger, \\
A_4 = I \otimes C_{\underline{02}} \cdot a_1, & A_4^\dagger = I \otimes C_{\underline{01}} \cdot a_4^\dagger, \\
F_1 = Y \otimes C_{\underline{20}} \cdot f_1, & F_1^\dagger = Y \otimes C_{\underline{10}} \cdot f_1^\dagger, \\
F_2 = Y \otimes C_{\underline{22}} \cdot f_2, & F_2^\dagger = Y \otimes C_{\underline{11}} \cdot f_2^\dagger, \\
F_3 = Y \otimes C_{\underline{21}} \cdot f_3, & F_3^\dagger = Y \otimes C_{\underline{12}} \cdot f_3^\dagger, \\
F_4 = Y \otimes C_{\underline{02}} \cdot f_4, & F_4^\dagger = Y \otimes C_{\underline{01}} \cdot f_4^\dagger.
\end{array}
\tag{30}
$$

Since no confusion will arise, for simplicity we employed for the $\mathfrak{h}_{pb}(4)$ subalgebra generators the same symbols as those introduced in the set of equations (24).

In the 1 bit - 2 trits notation the grading assignment of the (30) generators is:

$$
\begin{array}{l}
C \in [0\underline{00}], \\
A_1^\dagger \in [0\underline{10}],\ A_1 \in [0\underline{20}],\ A_2^\dagger \in [0\underline{11}],\ A_2 \in [0\underline{22}],\ A_3^\dagger \in [0\underline{12}],\ A_3 \in [0\underline{21}],\ A_4^\dagger \in [0\underline{01}],\ A_4 \in [0\underline{02}], \\
F_1^\dagger \in [1\underline{10}],\ F_1 \in [1\underline{20}],\ F_2^\dagger \in [1\underline{11}],\ F_2 \in [1\underline{22}],\ F_3^\dagger \in [1\underline{12}],\ F_3 \in [1\underline{21}],\ F_4^\dagger \in [1\underline{01}],\ F_4 \in [1\underline{02}].
\end{array}
\tag{31}
$$

The color Heisenberg-Lie superalgebra $\mathfrak{h}_{pf}(4|4)$ closes the following set of graded brackets which are defined in terms of the commutation factors entering table (17). We have

$$
\begin{array}{l}
\mathfrak{h}_{pf}(4|4) \ : \quad \{A_I, A_I^\dagger, F_I, F_I^\dagger, C\}, \qquad \text{where}, \quad \forall I, J, \\
\langle C, C\rangle = \langle C, A_I\rangle = \langle C, A_I^\dagger\rangle = \langle C, F_I\rangle = \langle C, F_I^\dagger\rangle = 0, \\
\langle A_I, A_J^\dagger\rangle = \delta_{IJ}\cdot C, \qquad \langle A_I, A_J\rangle = \langle A_I^\dagger, A_J^\dagger\rangle = 0, \\
\langle F_I, F_J^\dagger\rangle = \delta_{IJ}\cdot C, \qquad \langle F_I, F_J\rangle = \langle F_I^\dagger, F_J^\dagger\rangle = 0, \\
\langle A_I, F_J^\dagger\rangle = \langle A_I, F_J^\dagger\rangle = \langle A_I^\dagger, F_J\rangle = \langle A_I^\dagger, F_J^\dagger\rangle = 0.
\end{array}
\tag{32}
$$

From formula (32) one should note, in particular, the nilpotency of the parafermionic creation operators induced by the $\langle F_I^\dagger, F_I^\dagger\rangle = 0$ bracket. Indeed, we get

$$
(F_I^\dagger)^2 \ = \ 0 \qquad \text{for any } I = 1, 2, 3, 4. \tag{33}
$$

Remark: one should note that one of the 18 graded sectors of $\mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_3$ is left empty ($\emptyset \in [1\underline{11}]$), while each one of the remaining 17 graded sectors accommodate a single generator of $\mathfrak{h}_{pf}(4|4)$. It would be tempting to add to $[1\underline{11}]$ an extra generator such as $\overline{C} = Y \otimes C_{\underline{00}}$. The introduction of $\overline{C}$ would require to further introduce extra generators in the algebra since, e.g., $\langle \overline{C}, F_1\rangle = \{\overline{C}, F_1\} = 2 \cdot I \otimes F_1 \in [0\underline{20}]$ while, on the other hand, $\langle \overline{C}, F_1\rangle \not\propto A_1 \in [0\underline{20}]$.

The classification of the *minimal*, inequivalent $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded color Lie algebras and color Lie superalgebras was presented in [51]. In this context the notion of *minimal* refers to the (super)algebras being spanned by one and only one generator in each graded sector. The superalgebra $\mathfrak{h}_{pf}(4|4)$ is not minimal due to its empty sector [11]. On the other hand, the 9-generator parabosonic Heisenberg-Lie algebra $\mathfrak{h}_{pb}(4)$ previously introduced in (24,25) is an example of a *minimal* $\mathbb{Z}_3 \times \mathbb{Z}_3$-graded color Lie algebra.

6 Intermezzo: braidings and multi-particle states of colored para-oscillators

In this Section we anticipate some relevant features of colored para-oscillators; the aim is to provide an introduction to the more detailed discussions (presented in the construction of specific models) which will be given in the following.

The parabosonic $A_I^\dagger$ and the parafermionic $F_I^\dagger$ creation operators, respectively introduced in (24) and (30), belong to a larger class of non-(anti)commutative creation operators recovered from color Lie (super)algebras whose graded brackets (22) are of mixed type (i.e., admitting commutation factors $\varepsilon(\alpha,\beta) \neq \pm 1$). Let's denote with $D_J^\dagger$ the creation operators belonging to this broader class. We get, in particular, for $J \neq J'$:

$$D_J^\dagger D_{J'}^\dagger \quad = \quad a_{J,J'} \cdot D_{J'}^\dagger D_J^\dagger, \tag{34}$$

where the nonvanishing $a_{J,J'}$ parameters encode the non (anti)commutative properties of the creation operators. By specializing to three operators ($J, J' = 1,2,3$) we can set, e.g.,

$$D_1^\dagger D_2^\dagger = a \cdot D_2^\dagger D_1^\dagger, \qquad D_2^\dagger D_3^\dagger = b \cdot D_3^\dagger D_2^\dagger, \qquad D_3^\dagger D_1^\dagger = c \cdot D_1^\dagger D_3^\dagger, \tag{35}$$

where the nonvanishing parameters a, b, c are given by certain commutation factors. By identifying the $D_J^\dagger$'s with three creation operators from (24) and (30) we respectively have

$$\begin{array}{llll} D_J^\dagger \equiv A_J^\dagger & \quad \text{for} \quad J = 1,2,3, & \quad \text{implying that} & \quad a = b = c = j, \\ D_J^\dagger \equiv F_J^\dagger & \quad \text{for} \quad J = 1,2,3, & \quad \text{implying that} & \quad a = b = c = -j. \end{array} \tag{36}$$

The three creation operators $D_J^\dagger$ from (35) can either be assumed to be para-bosonic (hence, $(D_J^\dagger)^2 \neq 0$) or para-fermionic; in the latter case they are nilpotent: $(D_J^\dagger)^2 = 0$ for $J = 1,2,3$. Due to the nilpotency, in the parafermionic case the powers of their linear combinations vanish for $n \geq 4$, so that

$$(D_1^\dagger + D_2^\dagger + D_3^\dagger)^n \quad = \quad 0 \qquad \text{for } n \geq 4. \tag{37}$$

At the special $n = 3$ value we have $(D_1^\dagger + D_2^\dagger + D_3^\dagger)^3 \propto D_1^\dagger D_2^\dagger D_3^\dagger$, with normalizing factor $w(a,b,c)$ given by

$$\begin{array}{rcl} (D_1^\dagger + D_2^\dagger + D_3^\dagger)^3 & = & w(a,b,c) \cdot D_1^\dagger D_2^\dagger D_3^\dagger, \qquad \text{where} \\ w(a,b,c) & = & (1 + a^{-1} + b^{-1} + ca^{-1} + cb^{-1} + ca^{-1}b^{-1}). \end{array} \tag{38}$$

The right hand side is nonvanishing apart from a, b, c values producing the "miraculous cancellation"

$$w(a,b,c) \quad = \quad 0. \tag{39}$$

The cancellation is obviously satisfied for $a=b=c=-1$, i.e. ordinary fermions; for these values, due to the Pauli exclusion principle, even the $n=2$ power vanishes: $(D_1^\dagger+D_2^\dagger+D_3^\dagger)^2=0$.

A special (39) cancellation is obtained from a,b,c being primitive third roots of unity given by $a=b=j_1$, $c=j_1^2$. Indeed, we get:

$$w(j_1,j_1,j_1^2) \quad = \quad 0 \qquad \text{for } j_1=e^{\frac{2\pi i}{3}}. \tag{40}$$

This cancellation is responsible, as detailed in the following, for the truncation of the multi-particle energy spectrum of braided Majorana qubits at the third root of unity.

Remark: The $n=3$ cancellation is not observed for the colored $F_J^\dagger$ parafermionic creation operators introduced in (30) and whose a,b,c commutation factors are given in (36): $a=b=c=-j$ for $j\equiv j_1$. Indeed we get, for those parafermions:

$$w(-j_1,-j_1,-j_1) \quad = \quad 3(1-j_1^2)\neq 0. \tag{41}$$

The absence, in this case, of the $n=3$ truncation is implied by $-j_1$ being a sixth root of unity satisfying $(-j_1)^6=1$.

We present in Section **8** the mixed-bracket parafermionic color Lie superalgebra whose j_1,j_1,j_1^2 commutation factors lead to the (40) truncation.

6.1 Braiding properties

We illustrate the braiding properties of the mixed-bracket colored para-oscillators by analyzing the example of three parabosonic creation operators $A_J^\dagger$ introduced in (24) and whose mutual commutation factors are given in (36). We set $j\equiv j_1=e^{\frac{2\pi i}{3}}$.

The 6-generator $\mathbf{S}_3$ permutation group acts with generators S_{12}, S_{23} defined as

$$S_{12}: \;\; A_1^\dagger\leftrightarrow A_2^\dagger, \quad A_3^\dagger\mapsto A_3^\dagger, \qquad\qquad S_{23}: \;\; A_1^\dagger\mapsto A_1^\dagger, \quad A_2^\dagger\leftrightarrow A_3^\dagger. \tag{42}$$

The generators S_{12}, S_{23} satisfy

$$S_{12}^2=S_{23}^2=\mathbf{1}, \qquad S_{12}\cdot S_{23}\cdot S_{12}=S_{23}\cdot S_{12}\cdot S_{23}. \tag{43}$$

The action on symmetric polynomials such as $(A_1^\dagger+A_2^\dagger+A_3^\dagger)^n$ obviously produces the identity

$$S_{12}\left((A_1^\dagger+A_2^\dagger+A_3^\dagger)^n\right)=S_{23}\left((A_1^\dagger+A_2^\dagger+A_3^\dagger)^n\right)=(A_1^\dagger+A_2^\dagger+A_3^\dagger)^n. \tag{44}$$

On the other hand, the action on non-symmetric polynomials such as $A_1^\dagger A_2^\dagger$, $A_1^\dagger A_2^\dagger A_3^\dagger$ induces a braid group representation with $S_{12}\to B_{12}$, $S_{23}\to B_{23}$. Indeed, we get

$$B_{12}(A_1^\dagger A_2^\dagger)=A_2^\dagger A_1^\dagger=j_1^2\cdot A_1^\dagger A_2^\dagger, \qquad B_{12}^2(A_1^\dagger A_2^\dagger)=j_1\cdot A_1^\dagger A_2^\dagger, \qquad B_{12}^3(A_1^\dagger A_2^\dagger)=A_1^\dagger A_2^\dagger \tag{45}$$

and

$$\begin{array}{rlcl} B_{12}(A_1^\dagger A_2^\dagger A_3^\dagger)=B_{23}(A_1^\dagger A_2^\dagger A_3^\dagger)=j_1^2\cdot A_1^\dagger A_2^\dagger A_3^\dagger, & & \text{so that } B_{12}\equiv B_{23} & \\ \text{with} \qquad B_{12}\cdot B_{23}\cdot B_{12}=B_{23}\cdot B_{12}\cdot B_{23} & & \text{and} & B_{12}^3=B_{23}^3=\mathbf{1}. \end{array} \tag{46}$$

Formula (46) gives a trivial representation, at the third root of unity, of the braid group $\mathbf{B}_3$ (whose defining property is the first equation of the second line).

Remark: the para-oscillators defined by $\mathbb{Z}_2^n$-graded color Lie (super)algebras induce representations of the permutation groups when applied to both symmetric and non-symmetric polynomials. On the other hand, the *mixed-bracket* para-oscillators defined by more general abelian groups (such as $\mathbb{Z}_3 \times \mathbb{Z}_3$) induce, when applied to symmetric polynomials, representations of the permutation group and, when applied to non-symmetric polynomials, representations of the braid group. An important corollary follows.

Corollary:

i) the $\mathbb{Z}_2^n$-graded color Lie (super)algebras define parastatistics based on the permutation group; therefore, they can be applied to paraparticles beyond bosons and fermions which live in any space dimension;

ii) for more general abelian groups presenting *mixed-bracket* commutation factors, a case by case analysis of a specific application to model-construction should be made. It has to specify whether the paraparticles introduced by these color Lie (super)algebras obey a parastatistics based on the permutation group or, alternatively, on the braid group. In the latter case they are suitable to describe $(1 + 2)$-dimensional anyons.

6.2 On the construction of multi-particle states

A color Heisenberg-Lie (super)algebra $\mathfrak{h}$ induces a color Universal Enveloping Algebra $U \equiv \mathcal{U}(\mathfrak{h})$ which has a status of a graded Hopf algebra endowed with a braided tensor product (see [35] and references therein for a recent account on color Hopf algebras). A natural framework to induce multi-particle quantization from single-particle quantum Hamiltonians consists in constructing multi-particle states in terms of Hopf algebra (co)structures; a particular role is played by the $\Delta : U \to U \otimes U$ coproduct. In [4, 31, 32] this framework was applied to $\mathbb{Z}_2^n$-graded color Lie (super)algebras; it was used to prove the existence of signatures of n-bit parastatistics, i.e., of measurements whose results cannot be reproduced from ordinary bosons/fermions.

The coassociativity of the coproduct, expressed as

$$\begin{array}{rcll} \Delta^{(N+1)} & := & (\Delta \otimes \mathbf{1})\Delta^{(N)} = (\mathbf{1} \otimes \Delta)\Delta^{(N)} & (\text{where } \Delta^{(1)} \equiv \Delta), \quad \text{implies} \\ \Delta^{(N)} & : & U \to U^{\otimes^{N+1}} = U \otimes \ldots \otimes U & (\text{the tensor product of } N+1 \text{ spaces}). \end{array} \tag{47}$$

Starting from a single-particle Hilbert space $\mathcal{H}^{(1)}$, spanned by the $D_J^\dagger$ creation operators of $\mathfrak{h}$ applied to the single-particle vacuum $|vac\rangle_{(1)}$, the coproduct allows to construct the First-Quantized N-particle Hilbert space $\mathcal{H}^{(N)}$. We get

$$\begin{array}{ll} \mathcal{H}^{(N)} \subset \mathcal{H}^{(1)^{\otimes^N}}, \qquad & \text{with } \mathcal{H}^{(N)} \text{ spanned by the } \Delta^{(N-1)}(D_J^\dagger) \text{ creation operators applied} \\ & \text{to the } N\text{-particle vacuum } |vac\rangle_{(N)} = |vac\rangle_{(1)} \otimes \ldots \otimes |vac\rangle_{(1)}. \end{array} \tag{48}$$

The consistency of this approach is implied by the braided tensor product. We illustrate it with the example of the parabosonic creation oscillators $A_1^\dagger, A_2^\dagger$ introduced in (24); they satisfy $A_1^\dagger A_2^\dagger = j_1 A_2^\dagger A_1^\dagger$ and possess a vanishing graded bracket: $\langle A_1^\dagger, A_2^\dagger \rangle = 0$. The braided tensor product gives

$$\begin{array}{rcl} (\mathbf{1} \otimes A_1^\dagger) \cdot (A_2^\dagger \otimes \mathbf{1}) & = & j_1 (A_2^\dagger \otimes \mathbf{1}) \cdot (\mathbf{1} \otimes A_1^\dagger) = j_1 A_2^\dagger \otimes A_1^\dagger, \\ (\mathbf{1} \otimes A_2^\dagger) \cdot (A_1^\dagger \otimes \mathbf{1}) & = & j_1^2 (A_1^\dagger \otimes \mathbf{1}) \cdot (\mathbf{1} \otimes A_2^\dagger) = j_1^2 A_1^\dagger \otimes A_2^\dagger. \end{array} \tag{49}$$

Straightforward computations guarantee covariant formulas for the 2-particle creation operators induced by the coproduct. It is easily shown that, introducing the 2-particle graded bracket $\langle\Delta(A_1^\dagger),\Delta(A_2^\dagger)\rangle_{(2)}$ as

$$\langle\Delta(A_1^\dagger),\Delta(A_2^\dagger)\rangle_{(2)} \quad := \quad \Delta(A_1^\dagger)\Delta(A_2^\dagger)-j_1\Delta(A_2^\dagger)\Delta(A_1^\dagger), \tag{50}$$

the following relation is implied:

$$\langle A_1^\dagger,A_2^\dagger\rangle=0 \quad\Rightarrow\quad \langle\Delta(A_1^\dagger),\Delta(A_2^\dagger)\rangle_{(2)}=\Delta(\langle A_1^\dagger,A_2^\dagger\rangle)=0. \tag{51}$$

We end up with the following (first) construction of multi-particle states.

Construction 1: A consistent framework to introduce First-Quantized multi-particle sectors from color Heisenberg-Lie (super)algebras makes use of their associated graded Hopf algebras endowed with a braided tensor product. This construction was already presented in [4, 31, 32] for the special case of $\mathbb{Z}_2^n$-graded color Lie (super)algebras. It can be extended to *mixed bracket* color Lie (super)algebras by taking into account, see [35], the general properties of colored graded Hopf algebras for arbitrary abelian groups. Therefore, we get

$$\boxed{\textbf{1:}\ \textit{multi-particle sectors from graded Hopf algebras endowed with a braided tensor product.}} \tag{52}$$

A different scheme for the construction of multi-particle states is also available. This scheme was already employed in [10] to recover the multi-particle quantization of braided Majorana qubits which was first presented (using another framework) in [9]. We illustrate the main features of this scheme by taking, as an example, three parabosonic creation operators $A_J^\dagger$ $(J=1,2,3)$ from (24) whose noncommutative relations are given in (36). At first they can be applied to construct a single-particle Hilbert space $\mathcal{H}^{(1)}$ which is spanned by the vectors

$$\begin{array}{l}(A_1^\dagger)^{n_1}(A_2^\dagger)^{n_2}(A_3^\dagger)^{n_3}|vac\rangle\in\mathcal{H}^{(1)} \qquad \text{for } n_1,n_2,n_3=0,1,2,\ldots,\\ \text{with } |vac\rangle \text{ being a Fock's vacuum state satisfying } A_J|vac\rangle=0 \text{ for } J=1,2,3.\end{array} \tag{53}$$

This choice implies that the excited states created by $A_1^\dagger,A_2^\dagger,A_3^\dagger$ are *distinguishable*; they create distinct particles.

Next, an *indistinguishability* requirement is imposed following [10] (see in that paper Section **6**: *Indistinguishability as a superselection*). The multi-particle state is created by a symmetrized polynomial of the creation operators acting on the $|vac\rangle$ vacuum.

For the example under consideration a Hilbert space $\mathcal{H}_{ind}$ of indistinguishable, symmetrized, 3-particle states is obtained as

$$\mathcal{H}_{ind}\subset\mathcal{H}^{(1)}, \qquad \text{where} \qquad (A_1^\dagger+A_2^\dagger+A_3^\dagger)^n|vac\rangle\in\mathcal{H}_{ind} \qquad \text{for } n=0,1,2,\ldots. \tag{54}$$

In this framework the parabosonic creation operators $A_J^\dagger$ are building blocks to construct the symmetrized multi-particle states. Further details of this construction are discussed in [10].

Remark: In connection with the $\mathbb{Z}_3\times\mathbb{Z}_3$ grading (respectively given for $A_1^\dagger,\ A_2^\dagger,\ A_3^\dagger$ by the graded sectors $[\underline{10}]$, $[\underline{11}]$, $[\underline{12}]$) it is worth pointing out that:

i) the index $\underline{1}$ of the first $\mathbb{Z}_3$ grading defines the grading of the symmetrized $(A_1^\dagger + A_2^\dagger + A_3^\dagger)$ creation operator, while
ii) the index of the second $\mathbb{Z}_3$ grading ($\underline{0}$, $\underline{1}$ or $\underline{2}$) denotes the position of the three respective particles.

Construction 2: A second framework to introduce multi-particle states from color Heisenberg-Lie (super)algebras consists in constructing a superselected Hilbert space obtained by imposing the symmetrization of the creation oscillators; this construction is illustrated by the (54) example. Therefore, we get

$$\boxed{\textbf{2:}\ \textit{multi-particle sectors from superselected Hilbert space.}} \quad (55)$$

Comment: In this work we present the (55) second construction to multi-particle quantization induced by color Heisenberg-Lie (super)algebras. We prove, within this framework, how detectable signatures of parastatistics can be measured. The main motivation of working with the second construction consists in relating the known [9, 10] multi-particle quantization of braided Majorana qubits with the mixed-bracket color Lie (super)algebra framework.

The investigation of multi-particle quantization from mixed-bracket color Hopf algebras (the (52) first construction) is left for future works.

7 The minimal $\mathbb{Z}_3 \times \mathbb{Z}_3$-graded parabosonic oscillator model

We present in this Section the simplest $\mathbb{Z}_3 \times \mathbb{Z}_3$-graded parabosonic oscillator model derived from the mixed-bracket color Heisenberg-Lie algebra $\mathfrak{h}_{pb}(4)$ introduced in (24). We introduce at first the single-particle Hilbert space; we then introduce the superselected Hilbert space of *indistinguishable* particles according to the scheme outlined, see (55), in the previous Section.

We set the four $a_I, a_I^\dagger$ bosonic oscillators in (23) to be expressed as

$$a_I = \frac{1}{\sqrt{2}}(x_I + \partial_{x_I}), \qquad a_I^\dagger = \frac{1}{\sqrt{2}}(x_I - \partial_{x_I}) \qquad \text{for} \quad I = 1, 2, 3, 4, \tag{56}$$

where the x_I's are real space coordinates. Then, the (24) generators of $\mathfrak{h}_{pb}(4)$ are 9×9 matrix differential operators acting on 9-component column vectors. Let v_i (for $i = 1, 2, \ldots, 9$) denote the column vector with entry 1 in the i-th position and 0 otherwise. If we assume v_1 to be $[\underline{00}]$-graded we get, due to the (19) definition of the $C_{\underline{ij}}$ matrices, that the $\mathbb{Z}_3 \times \mathbb{Z}_3$ grading of the v_i's vectors is given by

$$v_1 \in [\underline{00}],\ v_2 \in [\underline{02}],\ v_3 \in [\underline{01}],\ v_4 \in [\underline{20}],\ v_5 \in [\underline{22}],\ v_6 \in [\underline{21}],\ v_7 \in [\underline{10}],\ v_8 \in [\underline{12}],\ v_9 \in [\underline{11}]. \tag{57}$$

The $[\underline{00}]$-graded Hamiltonian $H_{4d;osc}$ of a four-dimensional matrix quantum oscillator is introduced, for $\hbar = m = \omega = 1$, as

$$H_{4d;osc} \;=\; \sum_{I=1}^{4} A_I^\dagger A_I = \frac{1}{2}\left(-(\partial_{x_1}^2 + \partial_{x_2}^2 + \partial_{x_3}^2 + \partial_{x_4}^2) + x_1^2 + x_2^2 + x_3^2 + x_4^2 - 4\right) \cdot \mathbb{I}_9, \tag{58}$$

where $A_I^\dagger, A_I$ are the $\mathbb{Z}_3 \times \mathbb{Z}_3$-graded creation/annihilation oscillators from (24) and $\mathbb{I}_9$ denotes the 9×9 identity matrix.

The operators $A_I^\dagger$ create an excited state of energy $E = 1$:

$$[H_{4d;osc}, A_I^\dagger] \;=\; A_I^\dagger. \tag{59}$$

A Fock Hilbert space $\mathcal{H}_{4d}$ is introduced in terms of a normalized, $[\underline{00}]$-graded vacuum state $\psi_{4d;vac}(x_1, x_2, x_3, x_4) \equiv |0,0,0,0\rangle$:

$$\begin{aligned} A_I\psi_{4d;vac}(x_1,x_2,x_3,x_4) &= 0, \qquad \text{for } I=,1,2,3,4, \quad \text{implying that} \\ \psi_{4d;vac}(x_1,x_2,x_3,x_4) &= \frac{1}{\pi}e^{-\frac{1}{2}(x_1^2+x_2^2+x_3^2+x_4^2)}\cdot v_1. \end{aligned} \tag{60}$$

The Fock Hilbert space $\mathcal{H}_{4d}$ is spanned by the vectors $|n_1, n_2, n_3, n_4\rangle$ (for n_I's non-negative integers) of energy $E = n_1 + n_2 + n_3 + n_4$:

$$\begin{aligned} &|n_1,n_2,n_3,n_4\rangle \in \mathcal{H}_{4d}, \quad \text{where} \quad |n_1,n_2,n_3,n_4\rangle = (A_1^\dagger)^{n_1}(A_2^\dagger)^{n_2}(A_3^\dagger)^{n_3}(A_4^\dagger)^{n_4}|0,0,0,0\rangle \\ &\text{and} \qquad H_{4d;osc}|n_1,n_2,n_3,n_4\rangle = (n_1+n_2+n_3+n_4)|n_1,n_2,n_3,n_4\rangle. \end{aligned} \tag{61}$$

The (58) model can be simplified in order to introduce the minimal setting which shows a signature of a $\mathbb{Z}_3 \times \mathbb{Z}_3$ parastatistics. It is sufficient to introduce the color Heisenberg-Lie subalgebra $\mathfrak{h}_{pb}(2) \subset \mathfrak{h}_{pb}(4)$ with generators $\{A_1, A_2, A_1^\dagger, A_2^\dagger, C\} \in \mathfrak{h}_{pb}(2)$. The 4-dimensional oscillator $H_{4d;osc}$ is replaced by the 2-dimensional oscillator $H_{2d;osc}$ which, for $x \equiv x_1$, $y \equiv x_2$, is given by

$$H_{2d;osc} \;=\; \sum_{I=1}^{2} A_I^\dagger A_I = \frac{1}{2}\left(-(\partial_x^2+\partial_y^2)+x^2+y^2-2\right)\cdot \mathbb{I}_9. \tag{62}$$

The normalized 2-dimensional vacuum state $\psi_{2d;vac}(x,y) \equiv |0,0\rangle$, satisfying

$$\begin{aligned} A_1\psi_{2d;vac}(x,y) = A_2\psi_{2d;vac}(x,y) = 0, \qquad &\text{is given by} \\ \psi_{2d;vac}(x,y) = \frac{1}{\sqrt{\pi}}e^{-\frac{1}{2}(x^2+y^2)}\cdot v_1. & \end{aligned} \tag{63}$$

The Fock Hilbert space $\mathcal{H}_{2d}$ is spanned by the orthonormal vectors $|n,m\rangle$:

$$\begin{aligned} &|n,m\rangle \in \mathcal{H}_{2d}, \qquad \text{where} \quad n,m=0,1,2,\ldots; \\ &|n,m\rangle = \frac{1}{\sqrt{n!m!}}(A_1^\dagger)^n(A_2^\dagger)^m|0,0\rangle, \quad \text{with} \quad \langle n',m'|n,m\rangle = \delta_{nn'}\cdot\delta_{mm'}. \end{aligned} \tag{64}$$

The $E_{n,m}$ energy eigenvalues are given by

$$H_{2d;osc}|n,m\rangle \;=\; E_{n,m}|n,m\rangle, \qquad \text{where} \qquad E_{n,m} = n+m. \tag{65}$$

The 2-dimensional oscillator $\mathcal{H}_{2d}$ produces the minimal signature of a $\mathbb{Z}_3\times\mathbb{Z}_3$-graded parastatistics when introducing *indistinguishable* 2-particle states from the (second) construction outlined in (55).

7.1 A minimal signature of $\mathbb{Z}_3\times\mathbb{Z}_3$-graded parastatistics

We introduce, following the (55) scheme, the indistinguishable Hilbert space $\mathcal{H}_{ind}$ as a subspace of $H_{2d;osc}$:

$$\mathcal{H}_{ind}\subset H_{2d;osc}\qquad \text{is spanned by the vectors}\qquad (A_1^\dagger+A_2^\dagger)^n|0,0\rangle\in\mathcal{H}_{in}. \tag{66}$$

The energy eigenvalues of the energy eigenstates $(A_1^\dagger+A_2^\dagger)^n|0,0\rangle$ coincide with those of a ordinary harmonic oscillator with vacuum energy set to 0:

$$H_{2d;osc}(A_1^\dagger+A_2^\dagger)^n|0,0\rangle \;=\; E_n(A_1^\dagger+A_2^\dagger)^n|0,0\rangle, \quad \text{where}\quad E_n=n \;\;\text{for}\;\; n=0,1,2,\ldots. \tag{67}$$

Up to $n\leq 2$ the normalized energy eigenstates $|\psi_n\rangle\propto(A_1^\dagger+A_2^\dagger)^n|0,0\rangle$, given in terms of the 9-component vectors v_i, are

$$\begin{array}{llcl}
E_0=0: & |\psi_0\rangle &=& \frac{1}{\sqrt{\pi}}e^{-\frac{1}{2}(x^2+y^2)}\cdot v_1.\\
E_1=1: & |\psi_1\rangle &=& j^2\frac{1}{\sqrt{\pi}}e^{-\frac{1}{2}(x^2+y^2)}(x\cdot v_7+y\cdot v_9).\\
E_2=2: & |\psi_2\rangle &=& N_j\frac{1}{\sqrt{\pi}}e^{-\frac{1}{2}(x^2+y^2)}\left((2x^2-1)\cdot v_4+j^2(2y^2-1)\cdot v_5+2xy(1+j^2)\cdot v_6\right),\\
 & & & \text{with}\quad N_j=\frac{1}{\sqrt{z+5}}\quad\text{for}\quad z=1+j+j^2.
\end{array} \tag{68}$$

One should note in the right hand side the presence of the third root of unity j ($j^3=1$). As pointed out in Section **4**, see (16), two inequivalent cases are recovered from $j\equiv j_3=1$ (ordinary statistics of indistinguishable bosonic particles) and $j\equiv j_1=e^{\frac{2\pi i}{3}}$ (colored parastatistics of indistinguishable parabosonic particles). The key issue is whether these two cases can be discriminated, producing a detectable signature of the colored parastatistics with respect to the bosonic statistics.

For the model under consideration (the $H_{2d;osc}$ Hamiltonian of two indistinguishable particles), the signature *is not given* by the energy spectrum, since the bosonic and the parabosonic energy spectra coincide. On the other hand, a distinct signature is provided by the probability density $p(x,y)$ of finding the particles around the x,y coordinates.

The minimal signature: Let's prepare the 2-particle system in an $E=n$ energy eigenstate. The respective bosonic $p_{bos;n}(x,y)$ and parabosonic $p_{pb;n}(x,y)$ probability densities are given by

$$\begin{array}{rcll}
p_{bos;n}(x,y) &\equiv& p_n(x,y)_{|j=j_3} \quad\text{for}\quad (j_3=1) & \quad\text{and}\\
p_{pb;n}(x,y) &\equiv& p_n(x,y)_{|j=j_1} \quad\text{for}\quad (j_1=e^{\frac{2\pi i}{3}}), & \text{where}\\
p_{j;n}(x,y) &=& |\langle\psi_n|\psi_n\rangle|^2=||\psi_n||. &
\end{array} \tag{69}$$

The probability densities are normalized:

$$\iint_{-\infty}^{+\infty}dxdy\cdot p_{j;n}(x,y)=1. \tag{70}$$

At the $E=0,1$ energy levels no distinct signature of parastatistics is encountered, since

$$\begin{array}{rcll}
p_{pb;0}(x,y) &=& p_{bos;0}(x,y) & \quad\text{and}\\
p_{pb;1}(x,y) &=& p_{bos;1}(x,y). &
\end{array} \tag{71}$$

On the other hand, $E = 2$ is the lowest energy level presenting a distinct signature of parastatistics, due to

$$p_{pb;2}(x,y) \quad \neq \quad p_{bos;2}(x,y). \tag{72}$$

The measurable differences of these two $E = 2$ probability densities are analyzed in the next Subsection.

Remark: the (72) inequality has a physical significance leading to measurable consequences. By preparing a system of two indistinguishable (para)oscillators in the second excited state (energy level $E_2 = 2$), with repeated measurements one can determine whether the system under investigation is composed by ordinary bosons or by colored parabosons.

This is the minimal theoretical signature produced by the colored $\mathbb{Z}_3\times\mathbb{Z}_3$-graded parastatistics. It can be spotted by measuring the probability density of two indistinguishable (para)oscillators at energy level $E = 2$.

7.2 Parabosonic versus bosonic probability densities

The measurable differences of the parabosonic versus bosonic probability densities are respectively due to the noncommutativity/commutativity of the $A_1^\dagger, A_2^\dagger$ creation operators.
The two inequivalent ($j^3 = 1$, with $j \equiv j_1, j_2, j_3$ for $j_1 = e^{\frac{2\pi i}{3}}, j_2 = j_1^2, j_3 = 1$) third root of unity cases are:

i) the bosonic case:	$A_2^\dagger A_1^\dagger = A_1^\dagger A_2^\dagger;$	for $j_3 = 1$;
ii) the parabosonic case:	$A_2^\dagger A_1^\dagger = j_1^2 A_1^\dagger A_2^\dagger,$	for $j_1 = e^{\frac{2\pi i}{3}}$ a primitive third root of unity.

The $(A_1^\dagger + A_2^\dagger)^n$ powers for $n = 0, 1, 2, \ldots$, can be expressed by two respective Pascal's triangles, the ordinary one and a noncommutative one. In order to proceed to their construction and spot their difference we introduce the convenient parameter $z \equiv z|_j$ through the algebraic relation

$$\begin{gathered} z_{|_j} := 1 + j + j^2 \quad \Leftrightarrow \quad \text{either} \quad z_{j_3} = 3 \quad \text{or} \quad z_{j_1} = z_{j_2} = 0. \\ \text{Therefore, } z \text{ denotes either } z_{bos} = 3 \text{ (bosons) or } z_{pb} = 0 \text{ (parabosons).} \end{gathered} \tag{73}$$

The $(A_1^\dagger + A_2^\dagger)^n$ powers are given by

$$(A_1^\dagger + A_2^\dagger)^n \quad = \quad \sum_{k=0}^{n} V_{n;k}, \quad \text{where} \quad V_{n;k} = c_{n;k}(A_1^\dagger)^{n-k}(A_2^\dagger)^k \quad \text{for} \quad k = 0, 1, \ldots, n. \tag{74}$$

The $c_{n;k}$ coefficients are recursively determined from $c_{0;0} = 1$:

$$\begin{aligned} V_{n+1;k} &= A_1^\dagger V_{n;k} + A_2^\dagger V_{n;k-1}, \qquad \text{implying} \\ c_{n+1;k} &= c_{n;k} + j^{2(n-k+1)} c_{n;k-1} \end{aligned} \tag{75}$$

(in the above formula $c_{n;k} = 0$ when k is outside the $k = 0, \ldots, n$ range).

The following triangle, which encompasses both bosonic and parabosonic cases, is derived; the $c_{n;k}$ coefficients (presented as entries of the n^{th} row and $k = 0, 1, \ldots$ row) are given in terms of j, z. We get:

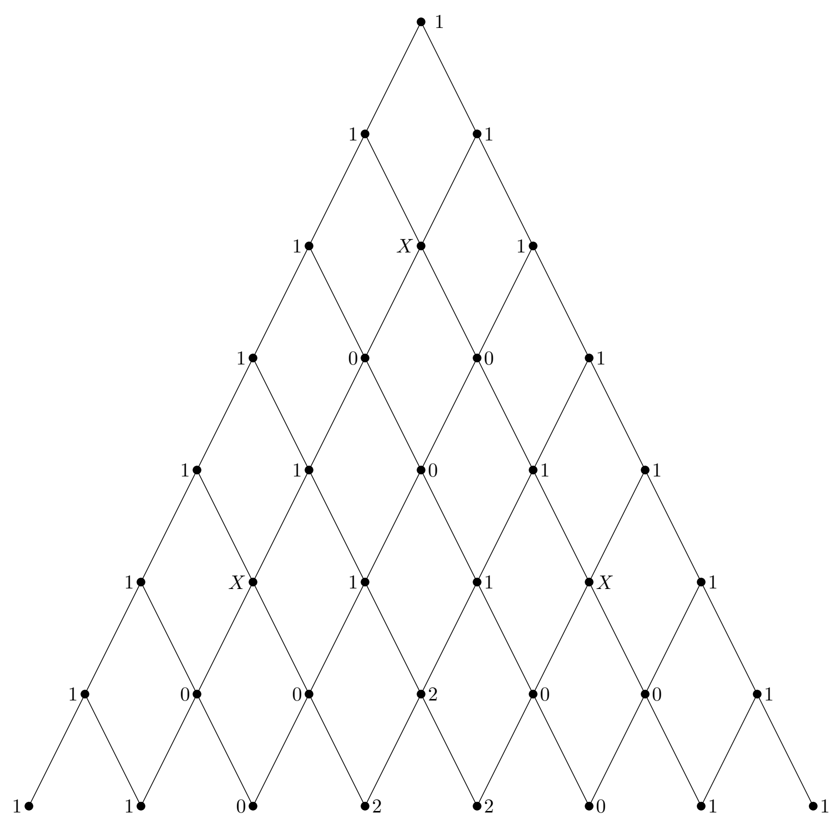
$$\begin{array}{lccccccc}
n=0: & 1 & & & & & & \\
n=1: & 1 & 1 & & & & & \\
n=2: & 1 & z-j & 1 & & & & \\
n=3: & 1 & z & z & 1 & & & \\
n=4: & 1 & z+1 & z(j+1) & z+1 & 1 & & \\
n=5: & 1 & 2z-j & z(2j+1)+1 & z(2j+1)+1 & 2z-j & 1 & \\
n=6: & 1 & 2z & z(2z-j) & z^2+z(j+2)+2 & z(2z-j) & 2z & 1 \\
\ldots: & \ldots & & & & & &
\end{array} \tag{76}$$

By specializing to the $j=1, z=3$ bosonic values we recover the ordinary Pascal triangle:

$$\begin{array}{ccccccc}
1 & & & & & & \\
1 & 1 & & & & & \\
1 & 2 & 1 & & & & \\
1 & 3 & 3 & 1 & & & \\
1 & 4 & 6 & 4 & 1 & & \\
1 & 5 & 10 & 10 & 5 & 1 & \\
1 & 6 & 15 & 20 & 15 & 6 & 1 \\
\ldots & & & & & &
\end{array} \tag{77}$$

The noncommutative parabosonic triangle is recovered from inserting the $j = j_1,\ z = 0$ values. This noncommutative version is presented in the following figure.

Figure 1. Noncommutative Pascal's triangle obtained from the $j_1 = e^{\frac{2\pi i}{3}}$ primitive third root of unity. It finds applications to the corresponding parabosonic statistics. The letter "X" denotes a $X = -j_1$ coefficient. The presence of the 0 coefficients is due to the relation $z_{|_{j_1}} = 1 + j_1 + j_1^2 = 0$:



The difference in the commutative versus noncommutative $c_{n;k}$ coefficients entering the respective triangles has physical implications. It implies that, for $n = 2, 3, \ldots$, different normalized $|\psi_n\rangle \propto (A_1^\dagger + A_2^\dagger)^n |0,0\rangle$ symmetrized energy eigenstates are obtained. As discussed in the previous Subsection, this difference is reflected in the measurable probability densities $p_{j;n}(x,y)$ introduced in (69).

The $n = 2$ normalized probability density $p_{j;2}(x,y)$ (associated with the $E_2 = 2$ energy eigenvalue) can be expressed, in terms of z, as

$$p_{j;2}(x,y) \;=\; \frac{1}{(z+5)\pi} e^{-(x^2+y^2)} \left((2x^2-1)^2 + (2y^2-1)^2 + 4(z+1)x^2y^2 \right). \tag{78}$$

The bosonic/parabosonic probability densities $p_{bos;2}(x,y) = p_{j;2}(x,y)_{|j=1}$ and $p_{pb;2}(x,y) = p_{j;2}(x,y)_{|j=j_1}$ are respectively recovered from $z = 3$ and $z = 0$, so that:

$$\begin{aligned} p_{bos;2}(x,y) &= \frac{1}{8\pi} e^{-(x^2+y^2)} \left((2x^2-1)^2 + (2y^2-1)^2 + 16x^2y^2 \right) \qquad \text{and} \\ p_{pb;2}(x,y) &= \frac{1}{5\pi} e^{-(x^2+y^2)} \left((2x^2-1)^2 + (2y^2-1)^2 + 4x^2y^2 \right). \end{aligned} \tag{79}$$

The (72) difference between these two normalized probability densities can be visualized in the following three-dimensional plots, realized with Mathematica.

Figure 2. Normalized bosonic probability density $p_{bos;2}(x,y)$ in the $-2 \leq x, y \leq 2$ range:

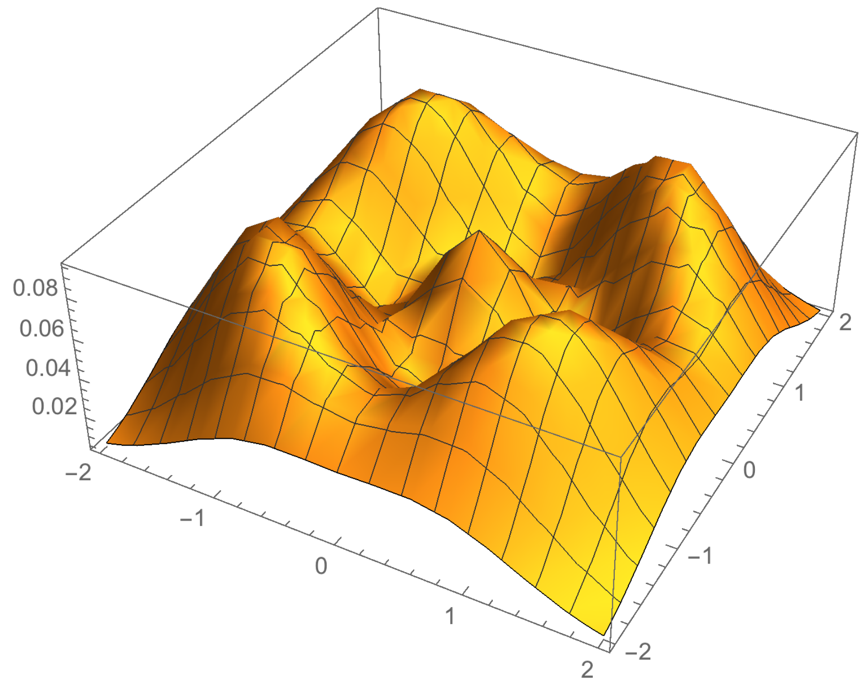
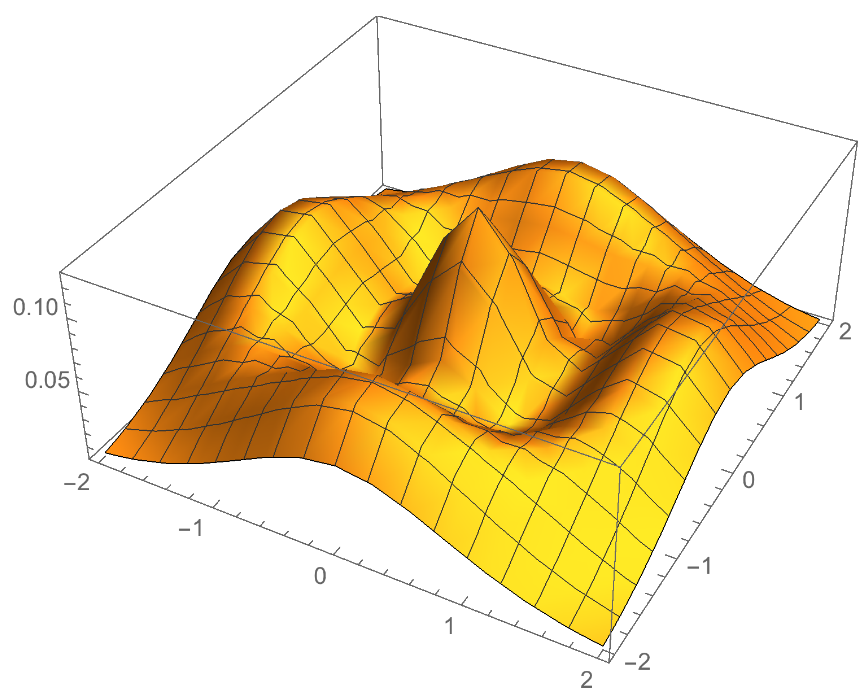


Figure 3. Normalized parabosonic probability density $p_{pb;2}(x,y)$ in the $-2 \leq x, y \leq 2$ range:



One can observe the different shapes of the two probability densities. Both of them admit 5 local maxima. We have

i) in the bosonic case, for $p_{bos}(x,y)$:
- one maximum at the origin, given by $p_{bos}(0,0) = \frac{1}{4\pi} \approx 0.079577$ and
- 4 maxima given by $p_{bos}(\overline{x},\overline{y}) = p_{bos}(\overline{x},-\overline{y}) = p_{bos}(-\overline{x},\overline{y}) = p_{bos}(-\overline{x},-\overline{y}) \approx 0.098055$,

where $\overline{x} = \overline{y} \approx 1.05244$.

ii) in the parabosonic case, for $p_{pb}(x,y)$:
- one maximum at the origin, given by $p_{pb}(0,0) = \frac{2}{5\pi} \approx 0.127324$ and
- 4 maxima given by $p_{pb}(\overline{u},0) = p_{pb}(-\overline{u},0) = p_{pb}(0,\overline{u}) = p_{pb}(0,-\overline{u}) \approx 0.089194$,

where $\overline{u} \approx 1.53819$.

In particular it should be noted that, contrary to $p_{bos}(x,y)$, the parabosonic probability density $p_{pb}(x,y)$ has an absolute maximum at the origin. The pair of two indistinguishable parabosons tends to be more concentrated in the origin than the two indistinguishable bosons. This feature has measurable consequences. By preparing a system of two indistinguishable (para)oscillators in the second excited state (of energy level $E_2 = 2$), by repeated measurements one can determine whether the system is composed by ordinary bosons or by parabosons at the primitive third root of unity.

8 Colored parafermionic oscillators and braided Majorana qubits

In this Section we present two 3-particle colored parafermionic quantum models; the first one is obtained from a $\mathbb{Z}_2 \times \mathbb{Z}_3^2$-grading and the second one from a $\mathbb{Z}_2^3 \times \mathbb{Z}_3^2$-grading. We prove that these colored superalgebra models recover (at the respective special values $s = 6$ and $s = 3$ which are explained below) the 3-particle braided Majorana qubits introduced in [9, 10]. This Section is organized as follows. At first we introduce the parafermionic models, then we briefly review the previous different formulations of multi-particle braided Majorana qubits; after that, the $s = 3, 6$ reconstruction formulas in the colored superalgebra formalism are given and some further comments are presented.

8.1 The $s = 3$ and $s = 6$ colored parafermionic quantum models

Multi-particle colored parafermionic quantum models can be straightforwardly generalized to describe any number N of particles. We limit here to discuss the $N = 3$ construction, since it allows to illustrate the phenomenon of truncation at a primitive third root of unity.

We presented in (28) the $\mathfrak{h}_{fer}(4)$ algebra of 4 ordinary fermionic oscillators. A 3-oscillator subalgebra can be realized by 8×8 matrices through the positions

$$\begin{gathered} f_1^\dagger = \gamma \otimes I \otimes I, \quad f_2^\dagger = X \otimes \gamma \otimes I, \quad f_3^\dagger = X \otimes X \otimes \gamma, \\ f_1 = \beta \otimes I \otimes I, \quad f_2 = X \otimes \beta \otimes I, \quad f_3 = X \otimes X \otimes \beta, \\ c = I \otimes I \otimes I, \end{gathered} \tag{80}$$

where I, X, β, γ are 2×2 matrices given by

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad X = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \beta = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad \gamma = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}. \tag{81}$$

In particular, one should note that the (28) (anti)commutators are recovered since $\{X,\beta\}=\{X,\gamma\}=0$.

In terms of the (80) matrices and of the (30) positions for $\mathfrak{h}_{pf}(4|4)$, we can realize an $\mathfrak{h}_{pf}(0|3)\subset\mathfrak{h}_{pf}(4|4)$ subalgebra as 72×72 matrices given by the following assignments:

$$
\begin{array}{ll}
C = \ c\otimes C_{\underline{00}}, & \\
F_1 = f_1\otimes C_{\underline{20}}, & F_1^\dagger = f_1^\dagger\otimes C_{\underline{10}},\\
F_2 = f_2\otimes C_{\underline{22}}, & F_2^\dagger = f_2^\dagger\otimes C_{\underline{11}},\\
F_3 = f_3\otimes C_{\underline{02}}, & F_3^\dagger = f_3^\dagger\otimes C_{\underline{01}},
\end{array}
\qquad (82)
$$

where the $\mathbb{Z}_3^2$-graded matrices $C_{\underline{ij}}$ have been introduced in (19).

For $j\equiv j_1=e^{\frac{2\pi i}{3}}$, the (82) matrices realize a colored parafermionic superalgebra where, in particular, the non-anticommutative properties of the three nilpotent creation operators $F_I^\dagger$ are given by

$$
\begin{array}{c}
(F_I^\dagger)^2=0 \qquad \text{for} \quad I=1,2,3 \quad \text{and}\\
F_1^\dagger F_2^\dagger=-j_1\cdot F_2^\dagger F_1^\dagger, \qquad F_2^\dagger F_3^\dagger=-j_1\cdot F_3^\dagger F_2^\dagger, \qquad F_1^\dagger F_3^\dagger=-j_1\cdot F_3^\dagger F_1^\dagger.
\end{array}
\qquad (83)
$$

With the above assignments of creation/annihilation operators the common $(-j_1)$ normalization factor enters the right hand sides of the equations in the second line. This normalization factor is a *primitive* 6^{th} root of unity satisfying $(-j_1)^3=1$.

Following the Appendix **A** conventions, the 6 different 6^{th} roots of unity are split into:

the level-1 root 1, the level-2 root -1, two level-3 roots given by j_1, j_1^2 and, finally, two primitive level-6 roots given by $-j_1,-j_1^2$.

Comment: the $s=6$ label associated with the colored parafermionic superalgebra (82) indicates the presence of a primitive 6^{th} root of unity in the (83) non-anticommutativity of the parafermionic creation operators. We can therefore set $\mathfrak{h}_{pf}(0|3)\equiv\mathfrak{h}_{pf;s6}(3)$.

The $s=6$, three-particle oscillator Hamiltonian $H_{s6,osc}$ is defined to be

$$
H_{s6,osc} \ := \ F_1^\dagger F_1+F_2^\dagger F_2+F_3^\dagger F_3. \qquad (84)
$$

The Hilbert space $\mathcal{H}_{s6}$ is spanned by the vectors

$$
(F_1^\dagger)^{n_1}(F_2^\dagger)^{n_2}(F_3^\dagger)^{n_3}|vac\rangle_{s6}\in\mathcal{H}_{s6} \qquad \text{for } n_1,n_2,n_3=0,1. \qquad (85)
$$

The Fock vacuum $|vac\rangle_{s6}$ satisfies

$$
F_I|vac\rangle_{s6}=0 \qquad \text{for any } I=1,2,3. \qquad (86)
$$

The vacuum is given by $|vac\rangle_{s6}=v_1$, the 72-component vector with entry 1 in the first position and 0 otherwise.

The energy eigenvalues of the model are given by $E=0,1,2,3$; their respective degeneracies are $(1,3,3,1)$. One gets a total number of $1+3+3+1=8$ energy eigenvectors which span the $\mathcal{H}_{s6}$ Hilbert space.

We can now apply the superselection outlined in (55) to introduce a Hilbert space $\mathcal{H}_{s6;ind}$ of three indistinguishable particles. We get that

$$\mathcal{H}_{s6;ind}\subset\mathcal{H}_{s6}\quad\text{is spanned by}\quad (F_1^\dagger+F_2^\dagger+F_3^\dagger)^n|vac\rangle_{s6}\quad\text{for}\quad n=0,1,2,3. \tag{87}$$

The following result is obtained for $s=6$:

$$\boxed{\begin{array}{l}\textit{The indistinguishable Hilbert space } \mathcal{H}_{s6;ind} \textit{ is spanned by } 4 \textit{ energy eigenvectors;}\\ \textit{the energy eigenvalues, given by } E=0,1,2,3, \textit{ are nondegenerate.}\end{array}} \tag{88}$$

The introduction of a non-anticommutativity expressed by a primitive third root of unity requires a $\mathbb{Z}_2^3\times\mathbb{Z}_3^2$-graded color Lie superalgebra. This produces an $s=3$ parafermionic model. The first step in its construction consists in replacing the set of three fermionic oscillators from (80) with a set of three $\mathbb{Z}_2^3$-graded parafermionic oscillators defined by the following 8×8 matrices:

$$\begin{array}{lll} p_1^\dagger=\gamma\otimes I\otimes I, & p_2^\dagger=I\otimes\gamma\otimes I, & p_3^\dagger=I\otimes I\otimes\gamma,\\ p_1=\beta\otimes I\otimes I, & p_2=I\otimes\beta\otimes I, & p_3=I\otimes I\otimes\beta,\\ & c=I\otimes I\otimes I, & \end{array} \tag{89}$$

Their $\mathbb{Z}_2^3$-grading can be assigned, e.g., from table 3_5 in reference [4] as:

$$p_1,p_1^\dagger\in[100],\quad p_2,p_2^\dagger\in[010],\quad p_3,p_3^\dagger\in[001],\quad c\in[000].$$

The important point is that the $\mathbb{Z}_2^3$-graded Lie superalgebra $\mathfrak{h}^*_{pf}(3)=\{p_1,p_1^\dagger,p_2,p_2^\dagger,p_3,p_3^\dagger,c\}$ obeyed by the (89) generators is defined in terms of the (anti)commutators:

$$\begin{array}{lll} \{p_i,p_i\}=\{p_i^\dagger,p_i^\dagger\}=0\quad\text{and}\quad\{p_i,p_i^\dagger\}=c & \text{for any} & i=1,2,3,\\ [p_i,p_j]=[p_i,p_i^\dagger]=[p_i^\dagger,p_j^\dagger]=0 & \text{for any} & i\neq j,\\ [c,w]=0 & \text{for any} & w\in\mathfrak{h}^*_{pf}(3). \end{array} \tag{90}$$

We are now in the position to realize the $s=3$, $\mathbb{Z}_2^3\times\mathbb{Z}_3^2$-graded parafermionic superalgebra $\mathfrak{h}_{pf;s3}(3)=\{P_1,P_1^\dagger,P_2,P_2^\dagger,P_3,P_3^\dagger,C\}$ in terms of 72×72 matrices through the positions

$$\begin{array}{ll} C=\ c\otimes C_{\underline{00}}, & \\ P_1=p_1\otimes C_{\underline{20}}, & P_1^\dagger=p_1^\dagger\otimes C_{\underline{10}},\\ P_2=p_2\otimes C_{\underline{22}}, & P_2^\dagger=p_2^\dagger\otimes C_{\underline{11}},\\ P_3=p_3\otimes C_{\underline{02}}, & P_3^\dagger=p_3^\dagger\otimes C_{\underline{01}}. \end{array} \tag{91}$$

In the 3 bits - 2 trits notation the grading of the $\mathfrak{h}_{pf;s3}(3)$ generators is given by

$$C\in[000\underline{00}],\ P_1\in[100\underline{20}],\ P_1^\dagger=\ [100\underline{10}],\ P_2=[010\underline{22}],\ P_2^\dagger=[010\underline{11}],\ P_3\in[001\underline{02}],\ P_3^\dagger=[001\underline{01}]. \tag{92}$$

The non-anticommutative properties of the three nilpotent creation operators $P_I^\dagger$ are given by

$$\begin{array}{lll} (P_I^\dagger)^2=0 & \text{for}\quad I=1,2,3\quad\text{and} & \\ P_1^\dagger P_2^\dagger=j_1\cdot P_2^\dagger P_1^\dagger, & P_2^\dagger P_3^\dagger=j_1\cdot P_3^\dagger P_2^\dagger, & P_3^\dagger P_1^\dagger=j_1\cdot P_1^\dagger P_3^\dagger. \end{array} \tag{93}$$

We can proceed as before, mimicking the previous construction.
The $s=3$, three-particle oscillator Hamiltonian $H_{s3,osc}$ is defined to be

$$H_{s3,osc} \quad := \quad P_1^\dagger P_1 + P_2^\dagger P_2 + P_3^\dagger P_3. \tag{94}$$

The Hilbert space $\mathcal{H}_{s3}$ is spanned by the vectors

$$(P_1^\dagger)^{n_1}(P_2^\dagger)^{n_2}(P_3^\dagger)^{n3}|vac\rangle_{s3} \in \mathcal{H}_{s3} \qquad \text{for } n_1, n_2, n_3 = 0, 1. \tag{95}$$

The Fock vacuum $|vac\rangle_{s3}$ satisfies

$$P_I|vac\rangle_{s3} = 0 \qquad \text{for any } I = 1, 2, 3. \tag{96}$$

The vacuum is given by $|vac\rangle_{s3} = v_1$, the 72-component vector with entry 1 in the first position and 0 otherwise.

The energy eigenvalues of the model are given by $E = 0, 1, 2, 3$, with respective degeneracies $(1, 3, 3, 1)$. One gets, as in the $s = 6$ case, a total number of $1+3+3+1 = 8$ energy eigenvectors which span the $\mathcal{H}_{s3}$ Hilbert space.

Comment 1: Up to this moment, the physical properties of the $s = 6$ and $s = 3$ cases coincide. For three distinct (para)fermionic oscillators no measurable observable can discriminate the respective quantum models. No signature can detect the difference in the physical properties of:

i) three *distinguishable* ordinary fermions (80),
ii) three *distinguishable* $s = 6$ parafermions (82) and
iii) three *distinguishable* $s = 3$ parafermions (91).

A discriminating signature is detected for the Hilbert spaces of three *indistinguishable* particles recovered from the superselection outlined in (55). For $s = 3$, the Hilbert space $\mathcal{H}_{s3;ind}$ of three indistinguishable particles is defined as

$$\mathcal{H}_{s3;ind} \subset \mathcal{H}_{s3}, \quad \text{spanned by} \quad (P_1^\dagger + P_2^\dagger + P_3^\dagger)^n|vac\rangle_{s3} \quad \text{for} \quad n = 0, 1, 2, 3. \tag{97}$$

The following result is obtained for $s = 3$:

$$\boxed{\begin{array}{l}\textit{The indistinguishable Hilbert space } \mathcal{H}_{s3;ind} \textit{ is spanned by } \underline{\textit{only}}\ 3 \textit{ energy eigenvectors}\\ \textit{whose energy eigenvalues, given by } E = 0, 1, 2, \textit{ are nondegenerate.}\end{array}} \tag{98}$$

The reason is the truncation of the energy spectrum, at a primitive third root of unity, due to the miraculous cancellation (39) obtained, see (40), for $w(j_1, j_1, j_1^2) = 0$.

Comment 2: For *indistinguishable* (para)fermions, the discriminating signature is the energy spectrum, given by:

i) $E = 0, 1$ for the three *indistinguishable* $(s = 2)$ ordinary fermions (80),
ii) $E = 0, 1, 2, 3$ for the three *indistinguishable* $s = 6$ parafermions (82) and
iii) $E = 0, 1, 2$ for the three *indistinguishable* $s = 3$ parafermions (91).

Remark: a straightforward combinatorics presented in [9, 10] extrapolates this result for N nilpotent creation operators $\overline{F}_I^\dagger$ labeled by $I = 1, 2, \ldots, N$. Let they satisfy

$$
\begin{array}{rcll}
(\overline{F}_I^\dagger)^2 &=& 0 & \text{for any } I, \text{ together with} \\
\overline{F}_I^\dagger \overline{F}_J^\dagger &=& s_k \cdot \overline{F}_J^\dagger \overline{F}_I^\dagger & \text{for } I < J, \text{ where} \\
s_k &=& e^{\frac{2\pi i}{k}} & \text{is a primitive } k^{th} \text{ root of unity for } k = 2, 3, \ldots.
\end{array}
\tag{99}
$$

Then, the following properties are satisfied for $n = 0, 1, 2, \ldots$ integer powers:

$$
\begin{array}{rcll}
\left(\sum_{I=1}^N \overline{F}_I^\dagger\right)^n &\neq& 0 & \text{for } \; n < k \; \text{ and} \\
\left(\sum_{I=1}^N \overline{F}_I^\dagger\right)^n &=& 0 & \text{for } \; n \geq k.
\end{array}
\tag{100}
$$

The physical implications of this combinatorics for multi-particle braided Majorana qubits are discussed in the next Subsection.

8.2 Brief summary of multi-particle braided Majorana qubits

A Majorana qubit is introduced, see [9], as a $\mathbb{Z}_2$-graded qubit expressed by a bosonic vacuum state $|0\rangle$ and a fermionic excited state $|1\rangle$ created by the fermionic operator γ acting on the vacuum:

$$
|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \qquad \text{with} \quad \gamma = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.
\tag{101}
$$

The multi-particle braiding is introduced, following the [52] prescription, in terms of a braided tensor product $\otimes_{br}$ satisfying

$$
(\mathbb{I}_2 \otimes_{br} \gamma) \cdot (\gamma \otimes_{br} \mathbb{I}_2) \;=\; B_t \cdot (\gamma \otimes_{br} \mathbb{I}_2) \cdot (\mathbb{I}_2 \otimes_{br} \gamma) \equiv B_t \cdot (\gamma \otimes \gamma),
\tag{102}
$$

where the t-dependent 4×4 constant matrix B_t is invertible for $t \neq 0$. It coincides with the R-matrix of the Alexander-Conway polynomial in the linear crystal rep on exterior algebra [53] and is given by

$$
B_t \;=\; \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1-t & t & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -t \end{pmatrix}.
\tag{103}
$$

The matrix B_t satisfies the braid relation

$$
(B_t \otimes \mathbb{I}_2) \cdot (\mathbb{I}_2 \otimes B_t) \cdot (B_t \otimes \mathbb{I}_2) \;=\; (\mathbb{I}_2 \otimes B_t) \cdot (B_t \otimes \mathbb{I}_2) \cdot (\mathbb{I}_2 \otimes B_t).
\tag{104}
$$

When t is a root of unity ($t = t_s$, labeled by $s = 2, 3, 4, \ldots$) we have

$$
(B_{t_s})^s \;=\; \mathbb{I}_4 \qquad \text{for} \qquad t_s = e^{\pi i (\frac{2}{s} - 1)}.
\tag{105}
$$

Acting, see formula (47), with the $\Delta^{(N-1)}(\gamma)$ graded coproducts of the creation operator γ on the N-particle vacuum $|0\rangle_N = |0\rangle \otimes |0\rangle \otimes \ldots \otimes |0\rangle$ (taken N times) we can compute the energy levels of the N-particle t_s-braided Majorana qubits. The N-particle Hamiltonian is $\Delta^{(N-1)}(H)$, where $H = \left(\begin{array}{cc} 0 & 0 \\ 0 & 1 \end{array}\right)$.

The following results, recovered from the (100) combinatorics, are obtained. The N-particle energy levels E, given by integer numbers, are not degenerate. In terms of s we get

$$\begin{array}{rclcl} E &=& 0,1,\ldots,N & \text{for} & N<s, \\ E &=& 0,1,\ldots,s-1 & \text{for} & N\geq s; \end{array} \tag{106}$$

a plateau is reached at the maximal energy level $s-1$, which corresponds to the maximal number of braided Majorana fermions that can be accommodated in a multi-particle Hilbert space. This implies that braided Majorana qubits, at any given level s, implement a Gentile-type [11] parastatistics. In the $s\rightarrow\infty$ limit ($t_\infty=-1$) no plateau is reached and the maximal energy eigenvalues grow linearly with N:

$$E \;=\; 0,1,\ldots,N \qquad \text{for any given} \quad N. \tag{107}$$

Further analysis presented in [10] proved two extra features of braided Majorana qubits:

i) The (106) level-s truncations of the energy spectra are recovered by superselected reps, at roots of unity, of the quantum superalgebra $\mathcal{U}_q(\mathfrak{osp}(1|2))$ introduced in [54] and

ii) With the help of the 2×2, level-s intertwiner operator W_{t_s} for t_s, the $\otimes_{br}$ braided tensor product can be expressed as an ordinary tensor product via the positions:

$$\begin{array}{lcll} (\gamma\ \otimes_{br}\mathbb{I}_2)\mapsto\gamma\otimes\mathbb{I}_2, & & (\mathbb{I}_2\ \otimes_{br}\gamma)\mapsto W_{t_s}\otimes\gamma, & \text{so that} \\ (\mathbb{I}_2\ \otimes_{br}\gamma)\cdot(\gamma\ \otimes_{br}\mathbb{I}_2) & \mapsto & (W_{t_s}\otimes\gamma)\cdot(\gamma\otimes\mathbb{I}_2)=(W_{t_s}\gamma)\otimes\gamma & \\ (\gamma\ \otimes_{br}\mathbb{I}_2)\cdot(\mathbb{I}_2\ \otimes_{br}\gamma) & \mapsto & (\gamma\otimes\mathbb{I}_2)\cdot(W_{t_s}\otimes\gamma)=(\gamma W_{t_s})\otimes\gamma. & \end{array} \tag{108}$$

At $t\equiv t_s$ the (102) relation is recovered for W_{t_s} satisfying

$$W_{t_s}\cdot\gamma \;=\; (-t_s)\gamma\cdot W_{t_s}. \tag{109}$$

A solution of this equation, expressed in terms of s entering t_s in (105), is given by

$$W_{t_s} \;=\; \cos(\frac{\pi}{s})\cdot\left(\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array}\right) - i\sin(\frac{\pi}{s})\cdot\left(\begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array}\right). \tag{110}$$

Under the (108) positions, the multi-particle creation/annihilation operators of the braided Majorana qubits are mapped into ordinary matrices. For integer $s>2$ they satisfy [10] mixed-bracket Heisenberg-Lie superalgebras; their mixed brackets are nontrivial linear combinations of commutators and anticommutators. The 2-particle case is described in the next Subsection.

8.3 Reconstructing the $s=3,6$ braided Majorana qubits in color framework

Braided Majorana qubits induce [10] mixed-bracket Heisenberg-Lie superalgebras defined in terms of $\theta_{x,y}$ angles for any pair of X,Y generators. The corresponding round brackets $(\cdot,\cdot)$ are introduced as

$$(X,Y)_{\theta_{x,y}} \;:=\; i\sin(\theta_{x,y})[X,Y]+\cos(\theta_{x,y})\{X,Y\} \;=\; e^{i\theta_{x,y}}\left(XY+e^{-2i\theta_{x,y}}YX\right). \tag{111}$$

The connection with the color Lie (super)algebra brackets (3) defined by

$$\langle X,Y\rangle \;=\; XY-\varepsilon(x,y)YX \tag{112}$$

requires the identification

$$\langle X,Y\rangle \;=\; e^{-i\theta_{x,y}}(X,Y) \qquad \text{for} \quad \varepsilon(x,y)=-e^{-2i\theta_{x,y}}, \tag{113}$$

which relates $\theta_{x,y}$ angles and commutant factors.

The mixed brackets of the braided Majorana qubits, labeled by s entering formula (108), are expressed by the s-dependent angles ϑ_s defined as

$$\vartheta_s \;=\; \frac{s+2}{2s}\pi \qquad \text{for} \quad s=2,3,4,\ldots. \tag{114}$$

Some interesting cases are

$$\begin{array}{rcll}
s=2 & : & \text{with} \quad \vartheta_{s=2}=\pi & \text{and} \quad (X,Y)_{\vartheta_{s=2}}=-\{X,Y\},\\
s=3 & : & \text{with} \quad \vartheta_{s=3}=\frac{5}{6}\pi, & \\
\ldots & & & \\
s=6 & : & \text{with} \quad \vartheta_{s=6}=\frac{2}{3}\pi, & \\
\ldots & & & \\
s\to\infty & : & \text{with} \; \vartheta_{s=+\infty}=\frac{\pi}{2} & \text{and} \quad (X,Y)_{\vartheta_{s=+\infty}}=i[X,Y].
\end{array} \tag{115}$$

For 2-particle braided Majorana qubits at level s, the 5-generator $(G_0,G_{\pm1},G_{\pm2})$ Heisenberg-Lie superalgebra possesses, see [10], the following $(\cdot,\cdot)$ round brackets:

$$\begin{array}{ll}
\text{for } i=1,2, & (G_0,G_{\pm i})_{\theta=\pm\frac{\pi}{2}}=(G_{\pm i},G_0)_{\theta=\pm\frac{\pi}{2}}=(G_0,G_0)_{\theta=\frac{\pi}{2}}=0,\\
& (G_{\pm i},G_{\mp i})_{\theta=0}=G_0, \qquad (G_{\pm i},G_{\pm i})_{\theta=0}=0,\\
& (G_{\pm1},G_{\pm2})_{\vartheta_s} \;=(G_{\pm2},G_{\pm1})_{-\vartheta_s}=0,\\
& (G_{\pm1},G_{\mp2})_{-\vartheta_s}=(G_{\pm2},G_{\mp1})_{\vartheta_s}=0.
\end{array} \tag{116}$$

We now prove that, at $s=3,6$, these round brackets can be reconstructed from the color brackets (3) via the (113) mapping.

At $s=3$ the (116) brackets are mapped to the $\langle\cdot,\cdot\rangle$ mixed brackets of the color Lie superalgebra defined by the five generators $C,P_1,P_2,P_1^\dagger,P_2^\dagger$ introduced in (91); the identification is

$$C\equiv G_0, \quad P_1^\dagger\equiv G_{+2}, \quad P_1\equiv G_{-2}, \quad P_2^\dagger\equiv G_{+1}, \quad P_2\equiv G_{-1}. \tag{117}$$

The $\vartheta_{s=3}$ angle induces, from (113), the commutant factor $\varepsilon(\vartheta_{s=3})=-e^{-2i\frac{5}{6}\pi}=e^{\frac{4}{3}\pi i}=j_2=j_1^2$. The resulting commutant factor is a primitive third root of unity.

At $s=6$ the (116) brackets are mapped to the $\langle\cdot,\cdot\rangle$ mixed brackets of the color Lie superalgebra defined by the five generators $C,F_1,F_2,F_1^\dagger,F_2^\dagger$ introduced in (82); the identification is

$$C\equiv G_0, \quad F_1^\dagger\equiv G_{+1}, \quad F_1\equiv G_{-1}, \quad F_2^\dagger\equiv G_{+2}, \quad F_2\equiv G_{-2}. \tag{118}$$

The $\vartheta_{s=6}$ angle induces, from (113), the commutant factor $\varepsilon(\vartheta_{s=6}) = -e^{\frac{2}{3}i\pi} = -j_1$.
The resulting commutant factor is a primitive sixth root of unity.

Comment: the proved result is both a restriction and a generalization of the multi-particle braided quantization of Majorana qubits:

i) It is a restriction because it was produced for the $s = 3, 6$ roots of unity levels and not the general integer braided values $s = 3, 4, 5, 6, 7, \ldots$. This restriction is due to the fact that we worked with the grading abelian groups $\mathbb{Z}_2^p \times \mathbb{Z}_3^q$. A natural expectation is that the construction of color Lie (super)algebras by tensoring more general multiplicative groups ($\mathbb{Z}_m$ for arbitrary integer values m) will recover the general case of level-s truncations. The analysis of this general case is left for a future work;

ii) It is a generalization because (for $s = 3, 6$) it allows to construct braided multi-particle quantizations of both parabosonic *and* parafermionic oscillators as seen, e.g., in formula (32). It is worth mentioning that the three different frameworks presented in [9, 10] only apply to parafermionic oscillators.

It should be pointed out that a $\mathbb{Z}_3$-graded matrix representation of creation/annihilation operators was presented in Appendix **B** of [10]; on the other hand, the connection with a color Lie superalgebra could not be established in that paper for the reasons recalled in the Introduction: the presence of a $\mathbb{Z}_3$ grading is necessary but not sufficient to implement a Rittenberg-Wyler color Lie (super)algebra.

A striking similarity of properties of braided Majorana qubits with those of *Volichenko algebras* introduced in [55] (see also [56]) was pointed out in [10]. The exception was the notion of *metaabelianess* satisfied by Volichenko algebras: for these algebras any three generators X, Y, Z satisfy the trilinear relation $[X, [Y, Z]] = 0$ for *ordinary* commutators. This relation is not satisfied even for a single fermionic oscillator. On the other hand, the mixed bracket color Heisenberg-Lie (super)algebras satisfy, for any three generators X, Y, Z, the relation $\langle X, \langle Y, Z\rangle\rangle = 0$ in terms of the graded brackets $\langle \cdot, \cdot \rangle$. This "graded version of the metaabelianess condition" implies that these color Lie (super)algebras are nilpotent of order 3.

9 Conclusions

In this paper we introduced *mixed-bracket* colored Heisenberg-Lie (super)algebras based on the $\mathbb{Z}_3^2$, $\mathbb{Z}_2 \times \mathbb{Z}_3^2$ and $\mathbb{Z}_2^3 \times \mathbb{Z}_3^2$ abelian groups gradings. They define parabosonic and parafermionic oscillators. We pointed out which minimal detectable signatures of their parastatistics are recovered in the multi-particle sectors:

i) for parabosons the creation operators are noncommutative; this implies a noncommutative version of the Pascal triangle, see **Figure 1**. The physical consequence is encoded in the probability densities of some multi-particle energy eigenstates. The minimal signature is recovered in the 2-particle sector at the second excited energy level, see the comparison of bosonic statistics versus parabosons in, respectively, **Figures 2** and **3**;

ii) for parafermions the nilpotency of their creation operators implies a generalized Pauli exclusion principle. It leads to truncations of their multi-particle energy spectra which implement a Gentile-type [11] parastatatistics with at most, see Subsection **8.1**, $s-1$ excited states. We have $s = 3$ for $\mathbb{Z}_2^3 \times \mathbb{Z}_3^2$ and $s = 6$ for $\mathbb{Z}_2 \times \mathbb{Z}_3^2$. The respective truncations are a consequence of formula (100).

In Section **6** we presented two general arguments concerning the relevance of *mixed-bracket* color Lie (super)algebras:

i) About their braiding properties. The n-bit parastatistics implied by $\mathbb{Z}_2^n$-graded color Lie (super)algebras are only based on the permutation groups. Mixed-bracket color Lie (super)algebras, on the other hand, accommodate two different types of parastatistics: beyond bosons/fermions in any space dimension which are exchanged under the permutation group, as well as anyonic paraparticles exchanged under the braid group. The difference, see Subsection **6.1**, is whether symmetric polynomials or non-symmetric polynomials of the creation operators are taken into account. The indistinguishable Hilbert spaces investigated in Sections **7** and **8** are obtained from symmetric polynomials of the creation operators; therefore they imply a parastatistics based on the permutation group;

ii) Two different constructions for mixed-bracket multiparticle states have been outlined in Section **6**.
The first one is based on the graded Hopf algebras coproducts. It is the extension of the approach applied in [4, 31, 32] to $\mathbb{Z}_2^n$-graded color Lie (super)algebras (the connection of the [52] graded Hopf algebra approach to parastatistics and the traditional [49, 50] framework based on trilinear relations has been discussed in [57, 58]).
The second construction induces multi-particle indistinguishability from the symmetrization of the creation operators. It is the extension of the [10] approach to introduce multi-particle braidings of Majorana qubits.
In order to reproduce the braided Majorana qubits in the framework of colored parafermions, only the second construction has been applied in this paper. The first approach to multi-particle states, based on graded colored Hopf algebras, will be presented in forthcoming works.

Mixed-bracket color Lie (super)algebras are at a crossroad of several important recent developments in physics (both fundamental questions and technological applications) and in mathematics. It is long overdue that this class of theories needs to be systematically investigated; basically, the last decade investigations of the more restricted $\mathbb{Z}_2^n$-graded (n-bit) color Lie (super)algebras should be extended to this more general case.

We present here a list of main points and open questions which are related with mixed-bracket color Lie (super)algebras:

i) *About permutation-group parastatistics* (*beyond bosons/fermions in any space dimension*).

It was commonly believed, see the [59] *conventionality of parastatistics argument* based on a localization principle such as [60], that paraparticles exchanged under the permutation group could not be experimentally detected (basically, under a localization hypothesis, all their measurements can be reproduced by ordinary bosons/fermions). It was recently proved that this is not the case. The first test producing a signature for the theoretical detectability of paraparticles exchanged under the permutation group was presented in [31]; further tests were presented, in chronological order, in [32], [61] and [4]. In all these papers the localization hypothesis is evaded (with different mechanisms at work). As recalled in [62], the experimental detectability of paraparticles is still an open challenge; the experimentalists have methods to engineer paraparticles in the laboratory, see e.g., [63, 64]. What is lacking, so far, is to put to experimental test the models (such as those presented in [4, 31, 32, 61]) which present a clear theoretical signature of parastatistics.

About connections with color Lie (super)algebras: the [4, 31, 32] theoretical tests are for $\mathbb{Z}_2^n$-graded parastatistics. In [61] four classes of theories, with R-matrices squaring to the Identity, are investigated. The first class corresponds, see the [61] Supplementary Informations, to the $\mathbb{Z}_2^2$-graded parafermions investigated in [31]. The three remaining classes and their related R-matrices do not have a clear mathematical interpretation. It is pointed out in [35] that R-matrices squaring to the Identity are recovered from color Lie (super)algebras graded by general abelian groups. This opens the possibility that the three extra classes in [61] could be related to mixed-bracket color Lie (super)algebras. It is a line of investigation which deserves being pursued.

In a recent parallel development, a permutation-group parastatistics defined by a primitive third root of unity is discussed in [65].

Concerning applications, recently the possibility of using $\mathbb{Z}_2^2$-graded parafermions for quantum computations has been advocated in [66]. On the other hand, as recalled in the Introduction, the Topological Quantum Computation proposed [12] by Kitaev requires Majorana particles which are exchanged by an anyonic braid statistics.

ii) On anyons and Topological Quantum Computations.

The possibility in low space dimensions of a parastatistics based on the braid group was first pointed out in [67]. These paraparticles were named *anyons* in [68]. For a historical account of their discovery and the relevant bibliography one can consult [69].

Unlike the paraparticles exchanging under the permutation groups, anyons have been experimentally detected. The first experimental evidence was presented in [70]; the first evidence of non-abelian anyons transforming under higher dimensional representations of the braid group was given in [71]. One-dimensional anyons have also been experimentally detected, see [72].

As mentioned in the Introduction, the braid statistics is crucial to implement the Kitaev's program of Topological Quantum Computation offering protection from quantum decoherence. The recent technological advances announced in [47, 48] are quite promising.

As advocated in [46], braided Majorana qubits are a natural minimal setting for this program. The reconstruction of multi-particle braided Majorana qubits in terms of mixed-bracket color Lie superalgebras puts these theories on a central stage. To discuss the general case the analysis here conducted has to be extended to the grading provided the abelian groups defined by tensoring the multiplicative groups $\mathbb{Z}_m$ for arbitrary integer numbers m.

iii) Further comments.

- In [10] the truncations of the multi-particle spectra of the braided Majorana qubits have been linked to representations at roots of unity of the quantum superalgebra $\mathcal{U}_q(\mathfrak{osp}(1|2))$. The special properties of the representations of quantum groups at roots of unity, discussed in [73, 74], are well known. Our present results suggest the existence of a possible connection between quantum group reps at roots of unity and mixed-bracket color Lie (super)algebras; a mathematical investigation in this direction seems to be worth.

- The color Lie (super)algebras based on the abelian groups $\mathbb{Z}_3^2$ and $\mathbb{Z}_2^p \times \mathbb{Z}_3^2$ (for $p = 1, 2, 3$) do not require the introduction of cubic, ternary structures as the ones discussed in the Introduction, see [36–41]. On the other hand, once their associated colored quantum models are introduced, the possibility of ternary structures appearing as *hidden symmetries* should be duly investigated.

- In his influential 1979 paper [3] Scheunert proved an important theorem which states that, for any finitely generated abelian group, the associated color Lie (super)algebra can be mapped, via a multiplier, to an ordinary Lie (super)algebra (sometimes this map is referred to as "decoloring"). This result generated for many years a negative attitude towards color Lie (super)algebras, dismissing their significance with respect to ordinary Lie (super)algebras. This negative attitude is not justified in the light of several recent results (some of them recalled in this paper) concerning the role of colored graded symmetries. In particular, we discussed here at length the relevance of the colored parastatistics with respect to the ordinary bosons/fermions statistics in multi-particle sectors. A proper interpretation of the seemingly negative result by Scheunert requires a careful analysis. For instance, investigating its implications beyond color Lie (super)algebras to the realm of color Hopf algebras and their costructures.

Appendix A: on roots of unity and their levels

We present here the notions of roots of unity and their level.

Over the field of complex numbers, for any integer $n \in \mathbb{N}$, an n^{th}-root of unity denotes any one of the n distinct roots of the $x^n = 1$ algebraic equation. The level k of a given n^{th}-root, where k takes value in the set $k \in \{1, 2, \ldots, n\}$, is determined as follows:

$$\text{A root of unity } \overline{x} \text{ is defined of level-}k \text{ if } k \text{ is the minimal positive integer such that } \overline{x}^k = 1. \tag{A.1}$$

We get, in particular:

$$\begin{array}{lll} n=1: & x^1 = 1 \Rightarrow 1 & \text{solution, } (x_1 = 1, \text{ which is a level-1 root of unity});\\ n=2: & x^2 = 1 \Rightarrow 2 & \text{solutions } (x_1 = 1 \text{ and } x_2 = -1, \text{ with } x_2 \text{ a level-2 root of unity});\\ n=3: & x^3 = 1 \Rightarrow 3 & \text{solutions } (x_1 = 1,\ x_2 = e^{\frac{2\pi i}{3}},\ x_3 = e^{\frac{4\pi i}{3}}, \text{ with } x_2, x_3 \text{ of level-3});\\ n=4: & x^4 = 1 \Rightarrow 4 & \text{solutions } (x_1 = 1,\ x_2 = i,\ x_3 = -1,\ x_4 = -i \text{ with } x_2, x_4 \text{ of level-4}) \end{array}$$

$$\text{and so on for arbitrary values of } n. \tag{A.2}$$

An n^{th}-root of unity of level n is also called a *primitive* root of unity; this term is employed in OEIS - The On-Line Encyclopedia of Integer Sequences, which lists as $A000010$ sequence the number s_n of primitive n^{th} roots of unity. Up to $n = 8$, the $s_1, s_2, s_3, \ldots$ sequence is given by

$$A000010: \qquad s_1 = 1,\ s_2 = 1,\ s_3 = 2,\ s_4 = 2,\ s_5 = 4,\ s_6 = 2,\ s_7 = 6,\ s_8 = 4, \ldots. \tag{A.3}$$

The physical importance of the notion of "level-k root of unity" was stressed in [9, 10]. It was shown that, in the multi-particle quantization of braided Majorana qubits, a level-k root of unity implies a parastatistics such that at most $k - 1$ excited states are accommodated in any multi-particle sector. This is a specific implementation of a Gentile-type parastatistics [11] which extends the Pauli's exclusion principle for ordinary fermions; the ordinary fermionic statistics is recovered from $k = 2$.

Remark: in certain physical applications, both those presented in [9,10] and the ones discussed in this paper, inequivalent physics is only determined by the root-of-unity level; for each level k, each one of its s_k distinct primitive roots produces the same, physically equivalent, results. A corollary of this statement can be seen from the (15) array, where the level-3 roots of unity j_1, j_2 induce isomorphic $\mathbb{Z}_3 \times \mathbb{Z}_3$-graded color Lie algebras.

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