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Braided Majorana qubits as a minimal setting for Topological Quantum Computation?

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Abstract

I point out that a possible minimal setting to realize Kitaev's proposal of a Topological Quantum Computation which offers topological protection from decoherence could in principle be realized by braided Majorana qubits. Majorana qubits and their braiding were introduced in Nucl. Phys. B 980, 115834 (2022) and further analyzed in J. Phys. A: Math. Theor. 57, 435203 (2024). Braided Majorana qubits implement a Gentile-type parastatistics with at most $s-1$ excited states accommodated in a multiparticle sector (the integer $s=2,3,4,\ldots$ labels quantum group reps at roots of unity). It is argued that braided Majorana qubits could play, for topological quantum computers, the same role as standard bits for ordinary computers and as qubits for "ordinary" quantum computers.

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1 Introduction

One of the fascinating areas of research is based on the Kitaev's proposal [1] (see also [2,3]) to use emergent Majorana particles to encode the properties of a topological quantum computation which offers topological protection under quantum decoherence. The logic of topological quantum computation with Majorana fermions has been investigated in [4] and theoretical models have been proposed, see e.g. the "Fibonacci particles" presented in [5]. The recent Microsoft's announcement [6] of the first quantum chip powered by a topological architecture points towards a practical implementation of the Kitaev's proposal (see the [7] roadmap to fault tolerant quantum computation and [8,9] for the production of devices admitting a topological phase with Majorana zero-modes).

Anyonic braid statistics can only apply to emergent particles living in 2 space dimensions. In order to fulfill Kitaev's program a physical model should satisfy the following three conditions:

i) at the theoretical level it should be able to accommodate a braid statistics,
ii) also at theoretical level, it should be able to encode the logic operations of topological quantum computation and, finally,
iii) it should be reproduced on a physical media/device, just like ordinary chips (or, in the old days, transistors) encode the Boolean logic operations.

A relevant question which could be addressed is the following: which is the minimal theoretical setting for realizing the program of a Topological Quantum Computation (TQC for short)? To give perspective, it makes sense to address this question within a broader picture, putting it in comparison with standard computers and "ordinary" quantum computers (here, "ordinary" stands for not topologically protected). We know that

• standard computers manipulate bits of information via Boolean logic gates, while
• what can be called the "ordinary quantum computers" operate at the level of qubits, considered as the minimal building blocks. Non-minimal settings based, instead of 2-state qubits, on qutris and more general qudits (see, e.g., [10]) are also possible; they received, nevertheless, much less attention for the obvious reason that it is much easier to perform the basic quantum operations on "minimal qubits".

The above question can therefore be rephrased as: which is the analogous, for Topological Quantum Computation, of ordinary bits for Boolean logic and of qubits for operations performed by quantum computers?

As a possible candidate, the minimal building blocks could be realized by (braided) Majorana qubits which satisfy the above condition *i*). Majorana qubits and their braiding (based on the [11] R-matrix of the Alexander-Conway polynomial in the linear crystal rep on exterior algebra) were introduced in [12] (for a short presentation, one can see [13]). Braided Majorana qubits implement a Gentile-type parastatistics [14] with at most $s-1$ excited states accommodated in a multiparticle sector (for an integer $s = 2, 3, 4, \ldots$ which labels quantum group reps at roots of unity, with the $s = 2$ case corresponding to ordinary fermions transforming under the permutation group). A significant feature of braided Majorana qubits is that they are described [15] by new mathematical structures (like a generalization of *Volichenko algebras*, more on that below) which have yet to be fully investigated.

Braided Majorana qubits satisfy, for the braid statistics, the minimality criterium which applies to bits and ordinary qubits. They are therefore a natural playground to put to test ideas and different mathematical frameworks to be applied to the braid statistics of quantum models. Braided Majorana qubits pass the *i*) condition. One of the purposes of this paper is to draw the attention to this novel minimal physical model in order to start addressing its properties concerning the conditions *ii*) and *iii*) mentioned before.

This paper presents in a concise and unified framework the three mathematical structures that are known to describe the braided Majorana qubits. It is based on [12] and the new results in [15]. The three mathematical structures under consideration are:
• the braiding induced by a graded Hopf algebra endowed with a braided tensor product,
• the (superselected) reps of the quantum superalgebra $\mathcal{U}_q(\mathfrak{osp}(1|2))$ truncated at roots of unity,
• the generalization of *Volichenko algebras* inducing "mixed-bracket" (i.e., interpolating ordinary commutators/anticommutators) Heisenberg-Lie algebras.

In order to focus on the main ideas, this paper illustrates the main results by skipping the demonstrations presented in the references cited in the text. The paper is structured as follows: Section **2** introduces the notion of a $\mathbb{Z}_2$-graded Majorana qubit, while the three mathematical structures defining the braiding are respectively presented in Sections **3**, **4** and **5**. Further comments (about the new mathematical structures describing braided Majorana qubits, the minimal setting and the possible application to the logic of topological quantum computation) will be given in the Conclusions.

2 $\mathbb{Z}_2$-graded Majorana qubits

A $\mathbb{Z}_2$-graded Majorana qubit corresponds [12] to a bosonic vacuum state $|0\rangle$ and a fermionic excited state $|1\rangle$:

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{1}$$

The following operators, acting on the $\mathbb{Z}_2$-graded qubit, close the Lie superalgebra $\mathfrak{gl}(1|1)$ [16]:

$$\alpha = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \beta = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \gamma = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad \delta = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}. \tag{2}$$

The defining brackets, given by (anti)commutators, are

$$\begin{array}{lllll} [\alpha,\beta]=\beta, & [\alpha,\gamma]=-\gamma, & [\alpha,\delta]=0, & [\delta,\beta]=-\beta, & [\delta,\gamma]=\gamma, \\ & \{\beta,\beta\}=\{\gamma,\gamma\}=0, & & \{\beta,\gamma\}=\alpha+\delta. & \end{array} \tag{3}$$

The diagonal operators α, δ are even, while β, γ are odd (γ being the fermionic creation operator).

The $\mathbb{Z}_2$-grading is given by

$$\mathfrak{gl}(1|1) = \mathfrak{gl}(1|1)_{[0]} \oplus \mathfrak{gl}(1|1)_{[1]}, \qquad \text{with} \quad \alpha,\delta \in \mathfrak{gl}(1|1)_{[0]} \quad \text{and} \quad \beta,\gamma \in \mathfrak{gl}(1|1)_{[1]}. \tag{4}$$

The admissible nonvanishing entries of the even/odd $\mathfrak{gl}(1|1)$ generators are expressed by the "$*$" symbol entering the respective 2×2 matrices:

$$\mathfrak{gl}(1|1)_{[0]} \equiv \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix}, \qquad \mathfrak{gl}(1|1)_{[1]} \equiv \begin{pmatrix} 0 & * \\ * & 0 \end{pmatrix}. \tag{5}$$

The excited state $|1\rangle$ is a Majorana fermion since it coincides with its own antiparticle.

The $\mathbb{Z}_2$ grading makes (1) to differ from an ordinary qubit. In particular, bosons/fermions satisfy a superselection rule which implies that they cannot be superposed (linearly combined).

In the case of an ordinary qubit the inequivalent configurations, determined by the ray vectors, are expressed by the Bloch sphere $\mathbf{S}^2$. In the case of the Majorana qubit its analogous Bloch sphere, determined by the ray vectors, is expressed by $\mathbf{Z}_2$, namely just one bit of information. The identification goes as follows: $0 \equiv |0\rangle$, $1 \equiv |1\rangle$.

3 First scenario of braided Majorana qubits: Multiparticle First Quantization (graded Hopf algebra with braided tensors).

The first scenario of braiding the $\mathbb{Z}_2$-graded Majorana qubits consists in introducing [12] a multiparticle First Quantization defined by the graded Hopf algebra $\mathcal{U}(\mathfrak{gl}(1|1))$ (the Universal Enveloping Algebra of $\mathfrak{gl}(1|1)$) endowed with, following the [17] prescription, a braided tensor product $\otimes_{br}$ which is compatible with the $\mathfrak{gl}(1|1)$ superalgebra.

In terms of the 4×4 matrix B_t, parametrized by a nonvanishing complex parameter t, the creation operator γ defined in (2) is assumed to satisfy

$$(\mathbb{I}_2 \otimes_{br} \gamma)\cdot(\gamma\otimes_{br}\mathbb{I}_2) \;=\; B_t\cdot(\gamma\otimes_{br}\mathbb{I}_2)\cdot(\mathbb{I}_2\otimes_{br}\gamma) \equiv B_t\cdot(\gamma\otimes_{br}\gamma), \qquad (6)$$

where the "$\cdot$" symbol denotes the standard matrix multiplication and "$\mathbb{I}_n$" denotes the $n\times n$ identity matrix. The matrix B_t (the R-matrix of the Alexander-Conway polynomial in the linear crystal rep on exterior algebra [11]) is given by

$$B_t \;=\; \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1-t & t & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -t \end{pmatrix}. \qquad (7)$$

The $t\neq 0$ condition ensures that B_t is invertible.

The $\otimes_{br}$ braided tensor product defined in (6) satisfies the required [17] compatibility condition since B_t obeys the braid relation

$$(B_t\otimes\mathbb{I}_2)\cdot(\mathbb{I}_2\otimes B_t)\cdot(B_t\otimes\mathbb{I}_2) \;=\; (\mathbb{I}_2\otimes B_t)\cdot(B_t\otimes\mathbb{I}_2)\cdot(\mathbb{I}_2\otimes B_t). \qquad (8)$$

3.1 Construction of the multiparticle sectors

Following [12] the N-particle Hilbert space $\mathcal{H}_N$ is a subset of N tensor products of the single-particle Hilbert space $\mathcal{H}$ spanned by $|0\rangle, |1\rangle$ entering (1):

$$\mathcal{H}_N \;\subset\; \mathcal{H}^{\otimes N}. \qquad (9)$$

$\mathcal{H}_N$ is constructed by repeatedly applying, on the N-particle vacuum state $|0\rangle_N$ given by

$$|0\rangle_N \;=\; |0\rangle\otimes\ldots\otimes|0\rangle \qquad\qquad (N \text{ times}), \qquad (10)$$

the Hopf algebra coproducts of the creation operator γ, so that $\mathcal{H}_N$ is spanned by the normalized vectors $|n\rangle_{t,N}$, where the integer n labels the n-th excited state:

$$|n\rangle_{t,N} \;\propto\; \left(\widehat{\Delta^{(N-1)}(\gamma)}\right)^n|0\rangle_N, \qquad \text{for } n=0,1,2,\ldots. \qquad (11)$$

Some comments are in order. The hat in the r.h.s. denotes the evaluation of the coproduct on the given representation of the Universal Enveloping Algebra $\mathcal{U}=\mathcal{U}(\mathfrak{gl}(1|1))$.
The Hopf algebra coproduct map Δ satisfies

$$\Delta \;:\; \mathcal{U}\to\mathcal{U}\otimes_{br}\mathcal{U}, \qquad \Delta^{(N+1)} := (\Delta\otimes_{br} id)\Delta^{(N)} = (id\otimes_{br}\Delta)\Delta^{(N)} \in \mathcal{U}^{\otimes_{br}N}. \qquad (12)$$

The property

$$\Delta(U_AU_B) \quad = \quad \Delta(U_A)\Delta(U_B) \qquad \text{for any} \quad U_A, U_B \in \mathcal{U} \tag{13}$$

implies that the action on any given $U \in \mathcal{U}(\mathfrak{g}l(1|1))$ is recovered from the action of the coproduct on the Hopf algebra unit $\mathbf{1}$ and on the primitive elements $\zeta \in \mathfrak{gl}(1|1)$, respectively given by

$$\Delta(\mathbf{1}) = \mathbf{1} \otimes_{br} \mathbf{1}, \qquad \Delta(\zeta) = \mathbf{1} \otimes_{br} \zeta + \zeta \otimes_{br} \mathbf{1}. \tag{14}$$

The N-particle Hamiltonian H_N can be expressed in terms of the single-particle Hamiltonian H_1 defined as $H_1 := \delta = diag(0,1)$; we have

$$H_N \quad := \quad \widehat{\Delta^{(N-1)}}(H_1). \tag{15}$$

With these positions we have all the ingredients to compute the braided multiparticle spectra for any $t \neq 0$.

3.2 The multiparticle spectra

We just limit here to present the results, referring to [12] and [15] for their derivation.

Truncations of the energy spectra of the multiparticle braided Majorana qubits are recovered for special root-of-unity values of t. For all other $t \neq 0$ cases the energy spectra are untruncated.

In a convenient parametrization for $|t| = 1$, the root of unity truncations are recovered from the position

$$t = -e^{2i\pi g}, \qquad \text{with} \quad g = \tfrac{r}{s} \quad \text{and } r, s \text{ mutually prime integers.} \tag{16}$$

The physics does not depend on the specific value of t, but only on the “root of unity level” specified by the integer $s = 2, 3, 4, \ldots$. In the limit $s \to \infty$ one obtains $t = -1$ which produces an untruncated spectrum (a generic $t \neq 0$ which does not coincide with a root of unity produces the same untruncated spectrum). Depending on s, the following classes of N-particle energy spectra are recovered (the energy eigenvalues are given by integer numbers and are not degenerate):

- truncated L_s level, the N-particle energy eigenvalues E are

$$\begin{array}{lcll} E &=& 0, 1, \ldots, N & \text{for} \quad N < s, \\ E &=& 0, 1, \ldots, s-1 & \text{for} \quad N \geqslant s; \end{array} \tag{17}$$

in this case a plateau is reached at the maximal energy level $s - 1$; this is the maximal number of braided Majorana fermions that can be accommodated in a multi-particle Hilbert space;

- untruncated ($t = -1$) L_∞ level, the N-particle energy eigenvalues E are

$$E \quad = \quad 0, 1, \ldots, N \qquad \text{for any given} \quad N; \tag{18}$$

in this case there is no plateau and the maximal energy eigenvalues grow linearly with N.

As mentioned before, the level-s root of unity implements a Gentile-type [14] parastatistics. The special point $t = 1$, which is the level-2 root of unity, gives the ordinary total antisymmetrization of the fermionic wavefunctions and encodes the Pauli exclusion principle of ordinary fermions.

4 Second scenario of braided Majorana qubits: Roots of unity truncations from superselected quantum group reps.

The truncations of the multi-particle spectra at roots of unity are reminiscent of the well-known special features of the quantum groups representations at roots of unity, see [18] and [19]. On the other hand the approach of [12] does not directly use quantum group data since the compatible braided tensor product is applied to the $\mathfrak{gl}(1|1)$ superalgebra, not its quantum counterpart.

The open question of deriving the spectrum of multiparticle braided Majorana qubits from quantum group data was solved in [15]; we briefly illustrate here the construction which makes use of the quantum superalgebra $\mathcal{U}_q(\mathfrak{osp}(1|2))$. Following [20] this quantum superalgebra is a deformation of the $\mathfrak{osp}(1|2)$ Lie superalgebra, recovered in the special limit $\eta \rightarrow 0$, where η is a complex deformation parameter and q can be expressed as $q = e^{\eta}$.

$\mathcal{U}_q(\mathfrak{osp}(1|2))$ is generated by the three elements $H, F_{\pm}$ satisfying, in terms of the complex parameter $\eta \neq 0$, the (anti)commutation relations

$$\begin{aligned} [H, F_{\pm}]_{\eta} &= \pm\frac{1}{2}F_{\pm}, \\ \{F_+, F_-\}_{\eta} &= \frac{\sinh(\eta H)}{\sinh(2\eta)} = \frac{e^{\eta H} - e^{-\eta H}}{e^{2\eta} - e^{-2\eta}}. \end{aligned} \tag{19}$$

In the limit when η goes to zero one recovers the ordinary $\mathfrak{osp}(1|2)$ anticommutator $\{F_+, F_-\}$ of the $osp(1|2)$ odd generators:

$$\lim_{\eta\rightarrow 0}\{F_+, F_-\}_{\eta} = \{F_+, F_-\} = \frac{1}{2}H. \tag{20}$$

The quantum superalgebra $\mathcal{U}_q(\mathfrak{osp}(1|2))$ has the structure of a graded Hopf superalgebra where, in particular, the following relations for the coproduct hold:

$$\begin{aligned} \Delta(H) &= H \otimes \mathbf{1} + \mathbf{1} \otimes H, \\ \Delta(F_{\pm}) &= F_{\pm} \otimes e^{\frac{\eta}{2}H} + e^{-\frac{\eta}{2}H} \otimes F_{\pm}. \end{aligned} \tag{21}$$

A single-particle Hilbert space $\mathcal{H}_{\eta}$ can be expressed as a lowest-weight representation of $\mathcal{U}_q(\mathfrak{osp}(1|2))$, defined by the Fock vacuum $|0\rangle_{\eta}$ such that

$$\begin{aligned} H|0\rangle_{\eta} &= \lambda|0\rangle_{\eta}, \\ F_-|0\rangle_{\eta} &= 0, \end{aligned} \tag{22}$$

where λ is a given "vacuum energy" and the Hilbert space $\mathcal{H}_{\eta}$ is spanned by the (possibly infinite) series of vectors $|n\rangle_{\eta}$:

$$|n\rangle_{\eta} = F_+^n|0\rangle_{\eta}, \qquad \text{where} \quad n = 0, 1, 2, 3, \ldots. \tag{23}$$

A non-vanishing vector $|n\rangle_{\eta}$ is an eigenvector of H with $\lambda + \frac{n}{2}$ eigenvalue:

$$H|n\rangle_{\eta} = (\lambda + \frac{n}{2})|n\rangle_{\eta}. \tag{24}$$

The connection with the energy spectrum of the single-particle Majorana qubit is performed in two steps. At first one sets $\lambda = 0$ and $H_1 := 2H$ as the normalized single-particle Hamiltonian.

Next, in order to recover from $\mathcal{U}_q(\mathfrak{osp}(1|2))$ a 2-dimensional Hilbert space, a suitable projector P ($P^2 = P$) should be applied. The two-dimensional finite Hilbert space $\mathcal{H}_\eta^{(1)}$ is obtained from $\mathcal{H}_\eta$ by applying the projector $P = diag(1,1,0,0,0\ldots)$, defined as

$$P|0\rangle_\eta = |0\rangle_\eta, \quad P|1\rangle_\eta = |1\rangle_\eta, \quad \text{while} \quad P|n\rangle_\eta = 0 \quad \text{for} \quad n \geqslant 2. \tag{25}$$

It follows that $\mathcal{H}_\eta^{(1)} \subset \mathcal{H}_\eta$ is spanned by the P eigenvectors with $+1$ eigenvalue, so that $|0\rangle_\eta, |1\rangle_\eta \in \mathcal{H}_\eta^{(1)}$.

In the multi-particle sectors the $(N+1)$-particle Hilbert space $\mathcal{H}_\eta^{(N+1)}$ is spanned by the vectors

$$(P \otimes P \otimes \ldots \otimes P) \cdot \widehat{\Delta^{(N)}(F_+^n)} \cdot (|0\rangle_\eta \otimes |0\rangle_\eta \otimes \ldots \otimes |0\rangle_\eta), \tag{26}$$

where the $\widehat{\Delta^{(N)}}$ coproduct acts on the tensor product of $N+1$ spaces.

This construction allows to reproduce, see [15] for details, the multiparticle spectrum of the braided Majorana qubits with the identification, for $t = -e^{2i\pi g}$ given in (16),

$$t \;=\; e^{-\frac{\eta}{2}}, \qquad \text{so that} \qquad \eta = -2\pi i(2g-1). \tag{27}$$

5 Third scenario of braided Majorana qubits: A new type of "Volichenko metasymmetry".

A third scenario for the multiparticle sector of braided Majorana qubits is realized when the braided tensor product $\otimes_{br}$, instead of simply (following [17]) *being declared* to be braided, is reconstructed from an ordinary $\otimes$ tensor product via the introduction of intertwining operators.

The basic example is given by the mappings

$$(\gamma \;\otimes_{br} \mathbb{I}_2) \mapsto \gamma \otimes \mathbb{I}_2, \qquad (\mathbb{I}_2 \;\otimes_{br} \gamma) \mapsto W_t \otimes \gamma, \tag{28}$$

which allow to recover the $\otimes_{br}$ braiding relation (6)

$$\begin{aligned} (\mathbb{I}_2 \;\otimes_{br} \gamma)\cdot(\gamma \;\otimes_{br} \mathbb{I}_2) \;&\mapsto\; (W_t \otimes \gamma)\cdot(\gamma \otimes \mathbb{I}_2) = (W_t\gamma) \otimes \gamma, \\ (\gamma \;\otimes_{br} \mathbb{I}_2)\cdot(\mathbb{I}_2 \;\otimes_{br} \gamma) \;&\mapsto\; (\gamma \otimes \mathbb{I}_2)\cdot(W_t \otimes \gamma) = (\gamma W_t) \otimes \gamma, \end{aligned} \tag{29}$$

provided that the 2×2 intertwining operator W_t satisfies the consistency condition

$$W_t\gamma \;=\; (-t)\gamma W_t. \tag{30}$$

A solution, expressed for the level-s root of unity in terms of the $t_s = -e^{\frac{2i\pi}{s}}$ position, is given by

$$W_{t_s} \;=\; \cos(\frac{-\pi}{s})\cdot \mathbb{I}_2 + i\sin(\frac{-\pi}{s})\cdot X, \qquad \text{where } X = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{31}$$

Indeed, we get

$$W_{t_s}\gamma \;=\; e^{\frac{2\pi i}{s}}\gamma W_{t_s}. \tag{32}$$

Following this prescription, the braided 2-particle ($2P$) and 3-particle ($3P$) creation operators of the Majorana qubits can be respectively expressed (see [15] for details) as

$$\begin{aligned} 2P \;&:\; A_1^\dagger := \gamma\otimes\mathbb{I}_2, \qquad A_2^\dagger := W_{t_s}\otimes\gamma; \\ 3P \;&:\; B_1^\dagger := \gamma\otimes\mathbb{I}_2\otimes\mathbb{I}_2, \quad B_2^\dagger := W_{t_s}\otimes\gamma\otimes\mathbb{I}_2, \quad B_3^\dagger := W_{t_s}\otimes W_{t_s}\otimes\gamma. \end{aligned} \tag{33}$$

For the 2-particle sector, the braided fermionic creation building blocks $A_1^\dagger, A_2^\dagger$ and their respective conjugate matrices A_1, A_2 are 4×4 matrices given by

$$A_1^\dagger = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \qquad A_2^\dagger = \begin{pmatrix} 0 & 0 & 0 & 0 \\ e^{\frac{-i\pi}{s}} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & e^{\frac{i\pi}{s}} & 0 \end{pmatrix},$$
$$A_1 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad A_2 = \begin{pmatrix} 0 & e^{\frac{i\pi}{s}} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & e^{\frac{-i\pi}{s}} \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{34}$$

Together with an even 4×4 central charge c defined as

$$c \quad = \quad diag(1,1,1,1). \tag{35}$$

they close, as shown below, a generalized "mixed-bracket" Heisenberg-Lie algebra.

This construction allows to make a connection with the notion of "symmetries wider than supersymmetry" presented by Leites and Serganova in [21] (see also [22]); this notion concerns the existence of statistics-changing maps which do not preserve the $\mathbb{Z}_2$-grading of ordinary Lie superalgebras. Leites-Serganova introduced the concepts of "metamanifolds" (as an extension of supermanifolds), "metaspace" (as an extension of superspace) and "metasymmetry" (as an extension of supersymmetry). As a concrete implementation of their proposal they investigated statistics-changing maps induced by nonhomogeneous subspaces of Lie superalgebras closed under the superbrackets. This leads to the notion of *Volichenko algebras* (after a theorem proved by Volichenko) which satisfy a condition known as metaabelianess; this means that, for any x, y, z triple of operators, the identity

$$[x,[y,z]] \quad = \quad 0, \tag{36}$$

which involves ordinary commutators, is satisfied.

Concerning Volichenko algebras, historical motivations and references leading to their introduction are found in [21, 22]. A construction of Volichenko algebras as algebras of differential operators is found in [23], while a recent account with updated references is [24].

An intriguing example of a class of Volichenko algebras was presented in [22]. Denoted as $\mathfrak{vgl}_\mu(p|q)$, they are given by the $\mathfrak{gl}(\mathfrak{p}|\mathfrak{q})$ space of $(p+q)\times(p+q)$ supermatrices. Under the $\mathbb{Z}_2$-decomposition $\mathfrak{gl}(\mathfrak{p}|\mathfrak{q}) = \mathfrak{h}_0 \oplus \mathfrak{h}_1$ into even/odd sectors, by fixing $\mu = a/b \in \mathbb{C}P^1$, a multiplication $\mathfrak{h}_1 \times \mathfrak{h}_1 \to \mathfrak{h}_0$ is introduced through the formula

$$(x,y)_\mu \quad = \quad a[x,y] + b\{x,y\} \qquad \text{for any } x, y \in \mathfrak{h}_1. \tag{37}$$

For $ab \neq 0$ the $(.,.)_\mu$ bracket is an interpolation of ordinary commutators and anticommutators.

This particular example was the starting point to explore the possibility of introducing "mixed brackets" to describe the parastatistics of the braided Majorana qubits. There were two main reasons for that:
i) a single-particle Majorana qubit is created/annihilated by the γ, β generators in the odd sector of the $\mathfrak{gl}(1|1)$ superalgebra, as shown in formulas (2, 3) and
ii) the building blocks of the 2-particle creation/annihilation operators are accommodated in the odd sector of a $\mathfrak{gl}(2|2)$ superalgebra.

These considerations led to the introduction in [15] of "mixed-bracket" generalized Heisenberg-Lie algebras which close generalized Jacobi identities. We present here the simplest construction for the level-s 2-particle braided Majorana qubits obtained from the five 4×4 matrices presented in (34,35).

5.1 A generalized mixed-bracket Heisenberg-Lie algebra

Let X, Y be two operators. Their mixed-bracket, defined in terms of a ϑ_{XY} angle and denoted as $(X,Y)_{\vartheta_{XY}}$, is an interpolation of the ordinary $[X,Y]$ commutator and $\{X,Y\}$ anticommutator. We can set

$$(X,Y)_{\vartheta_{XY}} \quad := \quad i\sin(\vartheta_{XY})\cdot[X,Y]+\cos(\vartheta_{XY})\cdot\{X,Y\}, \tag{38}$$

where ϑ_{XY} belongs, $mod\ 2\pi$, to the interval $\vartheta_{XY}\in[-\pi,\pi[$.

For the special case of the five 4×4 matrices introduced in (34,35), a consistent generalized mixed-bracket 2-particle Heisenberg-Lie algebra is introduced as follows.

It is convenient at first to rename the generators as

$$G_0 := c = diag(1,1,1,1),\quad G_{+1}:=A_1^\dagger,\quad G_{-1}:=A_1,\quad G_{+2}:=A_2^\dagger,\quad G_{-2}:=A_2. \tag{39}$$

The 5 generators $(G_{\pm1}, G_{\pm2}, G_0)$ 2-oscillator algebra has the only nonvanishing brackets given by

$$(G_{\pm1},G_{\mp1}) = (G_{\pm2},G_{\mp2}) = G_0. \tag{40}$$

The mixed-bracket formulas, with the explicit insertion of the ϑ_{IJ} angle dependence, are

$$\begin{array}{ll}(G_{+1},G_{-1})_0=(G_{-1},G_{+1})_0=G_0, & (G_{+1},G_{+1})_0=(G_{-1},G_{-1})_0=0,\\ (G_{+2},G_{-2})_0=(G_{-2},G_{+2})_0=G_0, & (G_{+2},G_{+2})_0=(G_{-2},G_{-2})_0=0,\end{array} \tag{41}$$

together with

$$\begin{array}{l}(G_{+1},G_{+2})_{+\frac{s+2}{2s}\pi}=(G_{+2},G_{+1})_{-\frac{s+2}{2s}\pi}=0,\\ (G_{+1},G_{-2})_{-\frac{s+2}{2s}\pi}=(G_{-2},G_{+1})_{+\frac{s+2}{2s}\pi}=0,\\ (G_{-1},G_{+2})_{-\frac{s+2}{2s}\pi}=(G_{+2},G_{-1})_{+\frac{s+2}{2s}\pi}=0,\\ (G_{-1},G_{-2})_{+\frac{s+2}{2s}\pi}=(G_{-2},G_{-1})_{-\frac{s+2}{2s}\pi}=0.\end{array} \tag{42}$$

At $s=2$ these brackets define an ordinary, 2 fermionic oscillators, Heisenberg-Lie algebra.

For any given $s=3,4,5,\ldots$, they are a mixed bracket generalization (a nontrivial interpolation of commutators/anticommutators) of the Heisenberg-Lie algebra which encodes the braid statistics of level s.

In the $s\to\infty$ untruncated limit one recovers again ordinary, i.e. not interpolated, commutators/anticommutators. In that limit, on the other hand, one obtains the Heisenberg-Lie algebra of 2 parafermionic oscillators, see [15] for details.

Since G_0 is a central element, the level$-s$ mixed-bracket algebras not only satisfy generalized Jacobi identities; they also satisfy a "metaabelianess condition with respect to the mixed brackets". This means that

$$(G_I,(G_J,G_K)) \quad = \quad 0 \qquad \text{for any} \quad I,J,K=0,\pm1,\pm2\ . \tag{43}$$

The ordinary metaabelianess condition is not satisfied by $G_{\pm 1}, G_{\pm 2}, G_0$. One can explicitly check, e.g., that

$$[G_{+1},[G_{+2},G_{-2}]] \quad \neq \quad 0. \tag{44}$$

One of the consequences is that braided Majorana qubits enlarge the notion of "Volichenko algebras" as defined by Leites-Serganova. A key observation is that the Leites-Serganova construction is *classical* since their notion of metaspace is applied to classical supergeometry and classical Lie superalgebras. On the other hand, the construction here discussed is related to quantum groups at roots of unity. This suggests the possibility of extending the notion of "metaspace" to a "quantum metaspace".

It is worth mentioning that the construction of the "mixed-bracket" Heisenberg-Lie algebras is performed for any N-particle sector and that the mixing angles ϑ_{IJ} which induce a closed algebraic structure are determined.

6 Conclusions

Section **2** presents the notion of $\mathbb{Z}_2$*-graded Majorana qubit*, while Sections **3**, **4** and **5** succintly summarize the results of [12, 15] concerning braiding properties and mathematical structures describing them. The main features can be itemized as follows:

- The multiparticle quantization depends on a discrete control parameter $s = 2, 3, 4, \ldots$ which labels a Gentile-type parastatistics where at most $s-1$ excited states, as seen from the (17) energy spectra, are accommodated in an N-particle sector.
- The multiparticle quantization is determined by a graded Hopf algebra endowed with a braided tensor product.
- An alternative formulation, recovered from superselected reps of the quantum superalgebra $\mathcal{U}_q(\mathfrak{osp}(1|2))$, "explains" the truncations of the (17) energy spectra.
- Another alternative formulation is recovered from mixed-bracket generalizations of Heisenberg-Lie algebras, where the mixed brackets are, see formulas (38) and (42), interpolations of ordinary commutators and anticommutators.
- The mixed-bracket structure is a quantum group generalization of the Leites-Serganova notion of *Volichenko algebras*, the classical notion of metaabelianess based, see (36), on ordinary commutators being now replaced by a metaabelianess with respect to the mixed brackets, see (43).

I can now present a weaker form of the question asked in the Title. Let's reformulate it as:

Is there a minimal setting for a braided quantum model and, if so, which? (45)

I claim that the first part of this question has a clear affirmative answer and that, for the second part, the braided Majorana qubits provide a minimal braided quantum model. The justification is presented here.

- It is well known that Leibniz, inspired by the hexagrams of the Chinese classic *I Ching*, introduced in the West (*Explication de l'arithmétique binaire*, published in 1703 under the spelling "Leibnitz" in Mémoires de l'Académie royale des Sciences, pages 83-89) the binary code represented by the digits 0 and 1. Leibniz advocated the advantages, due to its simplicity, of performing computations in this notation. Positional number systems work with any base (number of digits) greater than one, like our 10 notation, the Babylonian 60, etc.. The base 2 favoured by Leibniz is clearly the minimal choice since it involves the minimal number, 2, of admissible digits. This simple minimal choice is the reason why ordinary computers manipulate bits and are engineered to work with Boolean logic.

- For "ordinary" quantum computers there is not much to add to what already stated in the Introduction. A minimal quantum framework is based on the manipulation (and entanglement) of 2-component states, the qubits.

- Concerning braided quantum mechanics satisfying a braided (anyonic) statistics, the Majorana qubits play the same role as ordinary qubits. It is sufficient to pinpoint similarities/differences of ordinary qubits versus $\mathbb{Z}_2$-graded Majorana qubits. Let's have a closer look.

The ordinary qubit is described by the 2-state vector

$$\text{qubit:} \qquad \begin{pmatrix} a \\ b \end{pmatrix}, \quad \text{with} \quad a, b \in \mathbb{C} \quad \text{and} \quad |a|^2 + |b|^2 > 0. \tag{46}$$

The inequivalent physical configurations, determined by the ray vector, are obtained from the normalization condition $|a|^2 + |b|^2 = 1$ plus a phase invariance. They are expressed by the Bloch sphere $\mathbf{S}^2$.

The $\mathbb{Z}_2$-graded Majorana qubit presents both bosonic (even) and fermionic (odd) states which, under a superselection rule, *cannot be linearly combined*. It is therefore described by the set of states

$$\mathbb{Z}_2\text{-graded qubit:} \qquad \text{either even} \begin{pmatrix} a \\ 0 \end{pmatrix} \text{ or odd } \begin{pmatrix} 0 \\ b \end{pmatrix}, \quad \text{with} \quad a, b \in \mathbb{C} \quad \text{and} \quad |a|^2 + |b|^2 > 0. \tag{47}$$

As recalled at the end of Section **2**, its inequivalent physical configurations are determined by the ray vectors and correspond to $\mathbf{Z}_2$, i.e. one bit of information.

$\mathbb{Z}_2$-graded Majorana qubits share, with ordinary qubits, the same minimalistic properties; they can even be regarded as a simpler version which allows to introduce a non-trivial braiding for $s > 2$. The (7) braiding matrix B_t, parametrized by the inequivalent values $t \equiv t_s = e^{\frac{-2\pi i}{s}}$, satisfy idempotent relations in terms of powers of s:

$$B_{t_s}^s = \mathbb{I}_4. \tag{48}$$

The $s = 2$ case corresponds to an ordinary bosons/fermions representation of the permutation group. In the mixed-bracket approach the nontrivial braiding is realized by nontrivial interpolations of commutators/anticommutators, see (42). Somewhat unexpectedly, in the $s \to \infty$ limit the interpolation of commutators/anticommutators disappears. In that limit one recovers [15] a Rittenberg-Wyler type of parafermions [25, 26]. Unlike the anyonic braid statistics, this class of paraparticles can live in any space dimension, see [27]. It has been shown in [28, 29] that Rittenberg-Wyler paraparticles are theoretically detectable (see also the account in [30]).
The nontrivial braiding, which can in principle be applied to implement the Kitaev's proposal of topological quantum computation, is realized for any finite, integer value of s in the range $s = 3, 4, 5, \ldots$.
In the multiparticle sectors the inequivalent physical configurations of braided Majorana qubits are determined, see [12], by generalized Bloch spheres; for a 3-component energy level graded

Hilbert space the physical bosonic subspace is described by an $\mathbf{S}^2$ sphere and the fermionic subspace by a single point, while for a 4-component energy level graded Hilbert space both bosonic and fermionic physical subspaces are described by an $\mathbf{S}^2$ sphere. Higher components imply more complicated structures.

Braided Majorana qubits, albeit minimalistic, have a rich structure. They offer an ideal playground to test several mathematical frameworks applied in the construction of braided quantum models. Some of these frameworks, like the mixed-bracket Heisenberg-Lie algebras which generalize the notion of Volichenko algebras, have yet to be fully mathematically investigated. Different frameworks present different advantages. Indeed, one can note that the construction detailed in Section **3** involving a graded Hopf algebra endowed with a braided tensor product is more direct; the quantum group reps connection explained in Section **4**, on the other hand, clarifies the nature of the spectrum truncations observed in (17).

Let's now come back to the (slightly rephrased) question asked in the Title:

Are braided Majorana qubits a minimal setting for Topological Quantum Computation? (49)

For the moment we cannot give a definite answer. Braided Majorana qubits are quite a novel quantum model and, up to now, the investigations focused on their interrelated, underlying, mathematical structures. The proper analysis of the knot logic, see [4], which could be encoded within braided Majorana qubits has yet to be started. This presentation is intended, in the light of the advocated minimalistic viewpoint, to advertise the relevance of forthcoming investigations in this topic.

Having identified a minimal model satisfying the consistency condition *i*) stated in the Introduction, the next two conditions *ii*) and *iii*) should be seriously addressed in forthcoming investigations.

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