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New aspects of the $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded $1D$ superspace: induced strings and $2D$ relativistic models

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Abstract

A novel feature of the $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded supersymmetry which finds no counterpart in ordinary supersymmetry is the presence of 11-graded exotic bosons (implied by the existence of two classes of parafermions). Their interpretation, both physical and mathematical, presents a challenge. The role of the "exotic bosonic coordinate" was not considered by previous works on the one-dimensional $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded superspace (which was restricted to produce point-particle models). By treating this coordinate at par with the other graded superspace coordinates new consequences are obtained.
The graded superspace calculus of the $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded worldline super-Poincaré algebra induces two-dimensional $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded relativistic models; they are invariant under a new $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded $2D$ super-Poincaré algebra which differs from the previous two $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded $2D$ versions of super-Poincaré introduced in the literature. In this new superalgebra the second translation generator and the Lorentz boost are 11-graded. Furthermore, if the exotic coordinate is compactified on a circle $\mathbf{S}^1$, a $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded closed string with periodic boundary conditions is derived.
The analysis of the irreducibility conditions of the $2D$ supermultiplet implies that a larger (β-deformed, where $\beta \geqslant 0$ is a real parameter) class of point-particle models than the ones discussed so far in the literature (recovered at $\beta = 0$) is obtained. While the spectrum of the $\beta = 0$ point-particle models is degenerate (due to its relation with an $\mathcal{N} = 2$ supersymmetry), this is no longer the case for the $\beta > 0$ models.

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1 Introduction

This paper presents a careful investigation of the properties of the graded superspace associated with the $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded worldline super-Poincaré algebra. The main focus is the role of the "exotic bosonic coordinate" which was not considered by previous works on $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded superspace since it was constrained to produce point-particle models. The 11-graded exotic bosonic coordinate (the superalgebra generators and superspace coordinates belong to the 00, 10, 01 and 11 graded sectors) is the novel feature which finds no counterpart in the ordinary (i.e., $\mathbb{Z}_2$-graded) superspace. Therefore its interpretation, both mathematical and physical, is a non-trivial challenge. In this paper we treated the exotic coordinate at par with the other graded superspace coordinates (they are given by an ordinary boson which plays the role of time and two parafermions). We anticipate the main consequences. The calculus implied by the extra, exotic, bosonic coordinate induces two-dimensional $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded relativistic models. They are invariant under a new two-dimensional $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded super-Poincaré algebra; unlike the two previous $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded two-dimensional super-Poincaré algebras introduced in [1] and in [2], for this new superalgebra the second translation generator and the Lorentz boost are 11-graded. Furthermore, if the exotic coordinate is compactified on a circle $\mathbf{S}^1$, a $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded closed string with periodic boundary conditions is derived.

Before introducing in more detail the further results obtained in the paper, we briefly sketch the state-of-the-art of the investigations on $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded Lie superalgebras and their physical applications.

$\mathbb{Z}_2 \times \mathbb{Z}_2$-graded Lie algebras and superalgebras were introduced in [3–5] as the simplest $\mathbb{Z}_2^n$-graded extensions of ordinary Lie (super)algebras. The $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded Lie algebras involve parabosons, see e.g. [6]. The $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded Lie superalgebras on the other hand contain as subalgebras the $\mathbb{Z}_2$-graded Lie superalgebras presented in [7]; they involve (para)fermions and for this reason they can be naturally applied to generalizations of supersymmetric theories. Early works discussing physical applications of $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded Lie superalgebras are [8–10].

In recent years $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded Lie superalgebras found a renewed attention due to progresses in different directions. It became clear that superalgebras of this class are symmetries of well-known physical models such as, see [11,12], the non-relativistic Lévy-Leblond spinors. Systematic investigations of $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded invariant theories, both classical [13–16] and quantum [17,18], appeared; conformally invariant models and more general $\mathbb{Z}_2^n$-graded invariant theories, for $n \geqslant 2$, have started been investigated [19,20]. It was further shown, see [21,22], that the paraparticles implied by $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded theories are theoretically detectable and lead to physical consequences (for previous works on $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded parastatistics see [23–25]).

On the mathematical side various studies of algebraic and geometric aspects of $\mathbb{Z}_2^n$-graded structures have been investigated since their introduction. Here we mention only very recent works of algebraic studies which discuss structures and representations [26–29] (further references on algebraic aspects are found in [26]). The differential geometry on $\mathbb{Z}_2^n$-graded manifolds, which has a close kinship with the $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded superspace formulation, is also a field of extensive study; for details one can see the concise reviews [30,31].

Concerning the $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded superspace formulation, this topic has been investigated in [15,32,33].

We highlight some further results, besides the ones already mentioned at the beginning, of this work. We point out in particular the extension of the [32] analysis of the irreducibility of the supermultiplets. As a consequence we derive β-deformed (in terms of a real parameter β) worldline D-module representations of the $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded $1D$ super-Poincaré algebra. The representations investigated in the literature in [32] (which lead to invariant models possessing a supersymmetric spectrum) are recovered at the special, undeformed, $\beta = 0$ point. We also introduce a convenient matrix representation of the graded supercoordinates; it allows to reconstruct the $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded calculus from matrices (which encode the $\mathbb{Z}_2 \times \mathbb{Z}_2$ grading) coupled to the Berezin's calculus [34]. This representation simplifies the construction of invariant actions. More comments on this work and the future perspectives are given in the Conclusions.

The scheme of the paper is the following. In Section **2** the basic features of $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded Lie superalgebras are recalled. The graded superspace is introduced in Section **3**. Section **4** presents the graded superfields. The recovering of irreducible supermultiplets is discussed in Section **5**. The three types of D-module representations of the $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded worldline super-Poincaré algebra are given in Section **6**. Section **7** outlines the construction of the graded invariant actions. The matrix representation of the superspace is given in Section **8**. The induced two-dimensional $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded super-Poincaré algebra is derived in Section **9**. Two-dimensional relativistic actions, invariant under the $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded super-Poincaré algebra, are obtained in Section **10**. In Section **11** it is shown that the compactification on a $\mathbf{S}^1$ circle of the exotic bosonic coordinate leads to a $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded closed string with periodic boundary conditions. The last Section presents a summary of the results, outlining the main open questions and further lines of investigation. In the Appendix the irreducible, β-deformed worldline generators are derived.

2 $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded superalgebras

In this Section we recall the basic features of $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded superalgebras and introduce the relevant cases (the one-dimensional $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded super-Poincaré algebra and the graded abelian superalgebra) that are discussed in the following.

A $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded Lie superalgebra $\mathcal{G}$ (for our purposes the graded superalgebra is defined over the $\mathbb{R}$, $\mathbb{C}$ fields of real or complex numbers) is an extension of an ordinary superalgebra which is decomposed according to

$$\mathcal{G} \;=\; \mathcal{G}_{00} \oplus \mathcal{G}_{01} \oplus \mathcal{G}_{10} \oplus \mathcal{G}_{11}; \tag{1}$$

the grading $\vec{\alpha} = deg(a)$ of a generator $a \in \mathcal{G}$ is given by the pair $\vec{\alpha}^T = (\alpha_1, \alpha_2)$, with $\alpha_1, \alpha_2 \in \{0,1\}$. The bracket $(\cdot,\cdot) : \mathcal{G} \times \mathcal{G} \to \mathcal{G}$ is introduced on homogeneous generators a, b of grading $\vec{\alpha}, \vec{\beta}$ through the position

$$(a,b) \;=\; ab - (-1)^{\vec{\alpha}\cdot\vec{\beta}} ba, \qquad \text{where} \quad \vec{\alpha}\cdot\vec{\beta} = \alpha_1\beta_1 + \alpha_2\beta_2 \quad mod\ 2. \tag{2}$$

Depending on the $(-1)^{\vec{\alpha}\cdot\vec{\beta}}$ sign, the above bracket coincides with a commutator or an anticommutator; when specified, throughout the paper commutators (anticommutators) are denoted as $[\cdot,\cdot]$ (and, respectively, $\{\cdot,\cdot\}$).

The grading of the (a,b) generator is given by

$$deg((a,b)) \;=\; \vec{\alpha} + \vec{\beta} \quad mod\ 2. \tag{3}$$

For any three elements $a, b, c \in \mathcal{G}$ of respective grading $\vec{\alpha}, \vec{\beta}, \vec{\gamma}$ the following graded Jacobi identity is satisfied:

$$(-1)^{\vec{\gamma}\cdot\vec{\alpha}}(a,(b,c)) + (-1)^{\vec{\alpha}\cdot\vec{\beta}}(b,(c,a)) + (-1)^{\vec{\beta}\cdot\vec{\gamma}}(c,(a,b)) \;=\; 0. \tag{4}$$

The graded superalgebras under consideration here are:

i) the one-dimensional $\mathbb{Z}_2 \times \mathbb{Z}_2$*-graded super-Poincaré algebra* $\mathcal{P}$ which is defined, see [13,14,17], in terms of 4 generators whose grading assignment is $H \in \mathcal{P}_{00},\ Q_{10} \in \mathcal{P}_{10},\ Q_{01} \in \mathcal{P}_{01},\ Z \in \mathcal{P}_{11}$ and whose nonvanishing (anti)commutators are

$$\{Q_{10}, Q_{10}\} = \{Q_{01}, Q_{01}\} = 2H, \qquad [Q_{10}, Q_{01}] = iZ \tag{5}$$

(as shown in Section **9**, the addition of an extra 11-graded "Lorentz-boost" generator consistently defines a two-dimensional $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded super-Poincaré algebra $\mathcal{P}_{d=2}$);
ii) the graded abelian algebra $\mathcal{A}$ of the superspace coordinates, whose 4 generators are denoted as $t \in \mathcal{A}_{00},\ \theta_{10} \in \mathcal{A}_{10},\ \theta_{01} \in \mathcal{A}_{01},\ z \in \mathcal{A}_{11}$. All its (anti)commutators defined by (2) are vanishing; this implies, in particular, that θ_{10}, θ_{01} are nilpotent ($\theta_{10}^2 = \theta_{01}^2 = 0$).

iii) the larger graded superalgebra $\mathcal{S}$, spanned by

$$H, t \in \mathcal{S}_{00}, \quad Q_{10}, \theta_{10} \in \mathcal{S}_{10}, \quad Q_{01}, \theta_{01} \in \mathcal{S}_{01}, \quad Z, z \in \mathcal{S}_{11}. \tag{6}$$

The superalgebras $\mathcal{P}, \mathcal{A}$ are recovered as subalgebras ($\mathcal{P}, \mathcal{A} \subset \mathcal{S}$); in $\mathcal{S}$ the (anti)commutators (a, p) for $a \in \mathcal{A},\ p \in \mathcal{P}$ are defined to be all vanishing. The further superalgebra extension $\mathcal{S}'$ accommodates the derivatives $\partial_t, \partial_{10}, \partial_{01}, \partial_z$ of the (respective) superspace coordinates. The grading of these derivatives, whose action is defined in the next Section, is given by

$$\partial_t \in \mathcal{S}'_{00}, \quad \partial_{10} \in \mathcal{S}'_{10}, \quad \partial_{01} \in \mathcal{S}'_{01}, \quad \partial_z \in \mathcal{S}'_{11}. \tag{7}$$

The operation of star conjugation $*$, which allows to define the hermitian operators, is defined for graded generators a, b and $\lambda \in \mathbb{C}$ to satisfy

$$(ab)^* = b^* a^*, \quad (a^*)^* = a, \quad (\lambda a)^* = \lambda^* a^*, \tag{8}$$

where λ^* denotes the complex conjugation of λ.

Throughout the paper we assume the generators of $\mathcal{S}$ to be hermitian, so that

$$H^* = H, \ \ Q_{10}^* = Q_{10}, \ \ Q_{01}^* = Q_{01}, \ \ Z^* = Z, \ \ t^* = t, \ \ \theta_{10}^* = \theta_{10}, \ \ \theta_{01}^* = \theta_{01}, \ \ z^* = z. \tag{9}$$

The construction here presented is a particular case of the differential calculus [35] on $\mathbb{Z}_2^n$-supermanifolds.

3 The graded superspace

The graded superspace of the $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded one-dimensional super-Poincaré algebra (5) is recovered by introducing a group element $g \in G$ by setting

$$g = \exp(itH - \theta_{10}Q_{10} - \theta_{01}Q_{01} + izZ) \tag{10}$$

in terms of the graded super-Poincaré generators and superspace coordinates entering (6). It follows, by taking into account the hermiticity conditions (9), that g is unitary

$$g^{-1} = g^*. \tag{11}$$

A transformation $g \mapsto g' \in G$ is derived from the left action

$$g' = g_\epsilon \cdot g, \qquad \text{with} \qquad g_\epsilon = \exp(i\epsilon_{00}H - \epsilon_{10}Q_{10} - \epsilon_{01}Q_{01} + i\epsilon_{11}Z), \tag{12}$$

for the infinitesimal ϵ_{ij} ($\epsilon_{ij}^* = \epsilon_{ij}$) graded coordinates.
Let us collectively denote with $X \in \{t, \theta_{10}, \theta_{01}, z\}$ the graded superspace coordinates, so that $g \equiv g(X)$ and $g' \equiv g(X')$ for $X' = X + \delta X$. The transformations of the graded superspace coordinates are given by

$$\begin{aligned} \delta t &= \epsilon_{00} + i\epsilon_{10}\theta_{10} + i\epsilon_{01}\theta_{01}, & \delta z &= \epsilon_{11} + \frac{1}{2}(\epsilon_{10}\theta_{01} - \epsilon_{01}\theta_{10}), \\ \delta\theta_{10} &= \epsilon_{10}, & \delta\theta_{01} &= \epsilon_{01}. \end{aligned} \tag{13}$$

The above transformations are obtained from the Baker-Campbell-Hausdorff formula. They can also be recovered from the left action of differential operators of the graded superspace coordinates (7). The nonvanishing left action of the derivatives on the graded superspace coordinates are given by

$$\partial_{10} \cdot \theta_{10} = 1, \qquad \partial_{01} \cdot \theta_{01} = 1, \qquad \partial_z \cdot z^k = kz^{k-1}, \tag{14}$$

while $\partial_t \equiv \frac{\partial}{\partial t}$ is the ordinary partial derivative of the ordinary real coordinate t.

By equating (13) with the transformations expressed by

$$\delta X \quad = \quad (-i\epsilon_{00}\widehat{H} + \epsilon_{10}\widehat{Q}_{10} + \epsilon_{01}\widehat{Q}_{01} - i\epsilon_{11}\widehat{Z})X \qquad \text{for} \quad X \in \{t, \theta_{10}, \theta_{01}, z\}, \tag{15}$$

one gets

$$\begin{aligned} \widehat{H} &= i\partial_t, & \widehat{Z} &= i\partial_z, \\ \widehat{Q}_{10} &= \partial_{10} + i\theta_{10}\partial_t + \frac{1}{2}\theta_{01}\partial_z, & \widehat{Q}_{01} &= \partial_{01} + i\theta_{01}\partial_t - \frac{1}{2}\theta_{10}\partial_z. \end{aligned} \tag{16}$$

The above set of operators gives a differential representation of the graded super-Poincaré algebra (5) where, in particular, the nonvanishing (anti)commutators are

$$\{\widehat{Q}_{10}, \widehat{Q}_{10}\} = \{\widehat{Q}_{01}, \widehat{Q}_{01}\} = 2\widehat{H}, \qquad [\widehat{Q}_{10}, \widehat{Q}_{01}] = i\widehat{Z}. \tag{17}$$

By construction, the above operators are hermitian:

$$\widehat{H}^\dagger = \widehat{H}, \qquad \widehat{Q}_{10}^\dagger = \widehat{Q}_{10}, \qquad \widehat{Q}_{01}^\dagger = \widehat{Q}_{01}, \qquad \widehat{Z}^\dagger = \widehat{Z}. \tag{18}$$

The (para)fermionic covariant derivatives $\widehat{D}_{10}, \widehat{D}_{01}$ are obtained from the right actions. They are given by

$$\widehat{D}_{10} = \partial_{10} - i\theta_{10}\partial_t - \frac{1}{2}\theta_{01}\partial_z, \qquad \widehat{D}_{01} = \partial_{01} - i\theta_{01}\partial_t + \frac{1}{2}\theta_{10}\partial_z. \tag{19}$$

The covariant derivatives satisfy the (anti)commutators

$$\{\widehat{D}_{10}, \widehat{D}_{10}\} = \{\widehat{D}_{01}, \widehat{D}_{01}\} = -2\widehat{H}, \qquad [\widehat{D}_{10}, \widehat{D}_{01}] = -i\widehat{Z} \tag{20}$$

and

$$\{\widehat{D}_{10}, \widehat{Q}_{10}\} = \{\widehat{D}_{01}, \widehat{Q}_{01}\} = [\widehat{D}_{10}, \widehat{Q}_{01}] = [\widehat{D}_{01}, \widehat{Q}_{10}] = 0. \tag{21}$$

4 $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded superfields

A real, $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded superfield $\Phi(t, \theta_{10}, \theta_{01}, z)$ admits a decomposition, by taking into account the nilpotency of θ_{10}, θ_{01} and the special properties of z, in terms of 8 component fields denoted as φ_{00}, $\widetilde{\varphi}_{00}$, φ_{11}, $\widetilde{\varphi}_{11}$, ψ_{10}, $\widetilde{\psi}_{10}$, ψ_{01}, $\widetilde{\psi}_{01}$ (the suffix indicates their respective gradings). The component fields are functions of the real coordinate t and of the real parameter x introduced as

$$x = z^2 \qquad (x \geqslant 0 \ \text{ due to the hermiticity of } z). \tag{22}$$

The superfield decomposition is

$$\begin{aligned}\Phi(t, \theta_{10}, \theta_{01}, z) \;=\;& 1 \cdot \big(\varphi_{00}(t,x) + z\widetilde{\varphi}_{11}(t,x)\big) + \theta_{10} \cdot \big(i\psi_{10}(t,x) + z\widetilde{\psi}_{01}(t,x)\big) + \\ & \theta_{01} \cdot \big(i\psi_{01}(t,x) + z\widetilde{\psi}_{10}(t,x)\big) + \theta_{10}\theta_{01} \cdot \big(\varphi_{11}(t,x) + z\widetilde{\varphi}_{00}(t,x)\big).\end{aligned} \tag{23}$$

By taking into account the reality properties (9), the reality condition

$$\Phi(t, \theta_{10}, \theta_{01}, z)^* \;=\; \Phi(t, \theta_{10}, \theta_{01}, z) \tag{24}$$

implies that the 8 component fields entering (23) are all real.

By normalizing the scaling dimension to be $[H] = 1$, it follows from (5) and (10) that the scaling dimensions of the one-dimensional $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded super-Poincaré generators and their superspace coordinates are given by

$$[H] = [Z] = 1, \qquad [Q_{10}] = [Q_{01}] = \frac{1}{2}, \qquad [t] = [z] = -1, \qquad [\theta_{10}] = [\theta_{01}] = -\frac{1}{2}. \tag{25}$$

It turns out, once assumed $[\Phi(t, \theta_{10}, \theta_{01}, z)] = s$, that the scaling dimensions of the component fields are given by

$$\begin{aligned} s \;&:\; \varphi_{00}, \\ s + \frac{1}{2} \;&:\; \psi_{10},\ \psi_{01}, \\ s + 1 \;&:\; \varphi_{11},\ \widetilde{\varphi}_{11}, \\ s + \frac{3}{2} \;&:\; \widetilde{\psi}_{10},\ \widetilde{\psi}_{01}, \\ s + 2 \;&:\; \widetilde{\varphi}_{00}, \end{aligned} \tag{26}$$

while the scaling dimensions of their coordinates are

$$[t] = -1, \qquad [x] = -2. \tag{27}$$

Let us set, as before, $X \equiv t, \theta_{10}, \theta_{01}, z$; the scalar property of the superfield $\Phi(X)$,

$$\Phi'(X') \;=\; \Phi(X), \qquad \text{for} \quad X' = X + \delta X, \tag{28}$$

where δX is given in (15), implies

$$\delta\Phi(X) = \Phi(X') - \Phi(X) = (i\epsilon_{00}\widehat{H} - \epsilon_{10}\widehat{Q}_{10} - \epsilon_{01}\widehat{Q}_{01} + i\epsilon_{11}\widehat{Z})\Phi(X). \tag{29}$$

The transformations of the component fields can be read from the action of the $\widehat{H}$, $\widehat{Q}_{10}$, $\widehat{Q}_{01}$, $\widehat{Z}$ operators presented in (16). When applied to the 8-dimensional vector v given by

$$v^T \;=\; (\varphi_{00}, \widetilde{\varphi}_{00}, \varphi_{11}, \widetilde{\varphi}_{11}, \psi_{10}, \widetilde{\psi}_{10}, \psi_{01}, \widetilde{\psi}_{01}) \tag{30}$$

the (16) operators produce a 8×8 differential matrix representation in t,x of the graded superalgebra (17). The corresponding matrix operators, denoted with a regular instead of the italic font used in (16), are

$$
\begin{array}{rcl}
\widehat{\mathrm{H}} &=& i\partial_t\cdot\mathbb{I}_8,\\
\widehat{\mathrm{Q}}_{10} &=& \left(\begin{array}{cccccccc}
0&0&0&0&i&0&0&0\\
0&0&0&0&-i\partial_x&i\partial_t&0&0\\
0&0&0&0&0&0&-\partial_t&-\frac{1}{2}-x\partial_x\\
0&0&0&0&0&0&0&1\\
\partial_t&0&0&0&0&0&0&0\\
\partial_x&1&0&0&0&0&0&0\\
0&0&-i&-\frac{i}{2}-ix\partial_x&0&0&0&0\\
0&0&0&i\partial_t&0&0&0&0
\end{array}\right),\\
\widehat{\mathrm{Q}}_{01} &=& \left(\begin{array}{cccccccc}
0&0&0&0&0&0&i&0\\
0&0&0&0&0&0&i\partial_x&i\partial_t\\
0&0&0&0&-\partial_t&\frac{1}{2}+x\partial_x&0&0\\
0&0&0&0&0&1&0&0\\
0&0&-i&\frac{i}{2}+ix\partial_x&0&0&0&0\\
0&0&0&i\partial_t&0&0&0&0\\
\partial_t&0&0&0&0&0&0&0\\
-\partial_x&1&0&0&0&0&0&0
\end{array}\right),\\
\widehat{\mathrm{Z}} &=& \left(\begin{array}{cccccccc}
0&0&0&i+2ix\partial_x&0&0&0&0\\
0&0&2i\partial_x&0&0&0&0&0\\
0&i+2ix\partial_x&0&0&0&0&0&0\\
2i\partial_x&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&-1-2x\partial_x\\
0&0&0&0&0&0&2\partial_x&0\\
0&0&0&0&0&-1-2x\partial_x&0&0\\
0&0&0&0&2\partial_x&0&0&0
\end{array}\right)
\end{array}
\qquad (31)
$$

(we denote, here and in the following, the $n\times n$ Identity matrix with the symbol $\mathbb{I}_n$).

The diagonal operator $\widehat{\mathrm{Z}}^2$,

$$
\widehat{\mathrm{Z}}^2 \quad=\quad -(2\partial_x+4x\partial_x^2)\cdot\mathbb{I}_8-4\partial_x\cdot(E_{22}+E_{44}+E_{66}+E_{88}), \qquad (32)
$$

is a Casimir operator of the (5) superalgebra (here and in the following E_{ij} denotes the matrix with entries 1 at the crossing of the i-th row with the j-th column and 0 otherwise).

The representations are labeled by the eigenvalues of $\widehat{\mathrm{Z}}^2$. Since these eigenvalues are non-negative, they can be expressed as λ^2, in terms of a real parameter λ which, without loss of generality, can be assumed to be $\lambda\geqslant 0$. The scaling dimension of λ is

$$
[\lambda] \quad=\quad 1. \qquad (33)
$$

In physical applications the parameter t is identified with the time, while $\widehat{\mathrm{H}}$ is the Hamiltonian operator. The physical interpretation of x as en extra dimension induced by the exotic bosonic coordinate will be discussed in the following. The component fields entering (30) are defined in $(1+1)$-dimensions; the graded superfield can be associated with the symbol below which specifies the numbers of component fields, see formula (26), of respective scaling dimension $s, s+\frac{1}{2}, s+1, s+\frac{3}{2}, s+2$:

$$
(1;2;2;2;1). \qquad (34)
$$

Analogous symbols have been employed to describe representations of one-dimensional supermechanics [36,37] and of one-dimensional $\mathbb{Z}_2\times\mathbb{Z}_2$-graded mechanics [13].

The $(1+1)$-dimensional extended $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded superspace generalizes the previous constructions of one-dimensional $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded superspace presented in [30, 32]. The results of these works are recovered by suitably constraining the extended supermultiplets. These constraints, which are based on the notion of irreducible supermultiplet, are only applicable when point-particle mechanics can be derived. In the following we present a more detailed analysis of the admissible point-particle constraints.

The Hamiltonian $\widehat{\mathrm{H}}$ is a Casimir operator. Its energy eigenvalue $E \geqslant 0$, with scaling dimension $[E] = 1$, produces together with $\lambda \geqslant 0$ the pair of values

$$(E, \lambda) \tag{35}$$

which label, see [32, 33], the irreducible representations. They are given by

$$\begin{array}{rcll} I & : & (E \geqslant 0, \lambda = 0) & \quad \text{and} \\ II & : & (E = \alpha\lambda, \lambda > 0). & \end{array} \tag{36}$$

As shown in Appendix **A** the parameter α (which in principle is real and non-negative) has to satisfy the constraint

$$\alpha \;\geqslant\; \frac{1}{2}. \tag{37}$$

5 Irreducible graded supermultiplets

The general forms of the 8 eigenfunctions corresponding to the $\widehat{\mathrm{Z}}^2 = \lambda^2$ eigenvalue are

at $\quad \lambda = 0:$

$$\begin{array}{lll} f_A(t) + f_B(t)\sqrt{x} & \quad \text{for} & \varphi_{00}, \varphi_{11}, \psi_{10}, \psi_{01}, \\ f_A(t) + f_B(t)\frac{1}{\sqrt{x}} & \quad \text{for} & \widetilde{\varphi}_{00}, \widetilde{\varphi}_{11}, \widetilde{\psi}_{10}, \widetilde{\psi}_{01}; \end{array}$$

at $\quad \lambda \neq 0:$

$$\begin{array}{rll} f_A(t)\cos(\lambda\sqrt{x}) + f_B(t)\sin(\lambda\sqrt{x}) & \quad \text{for} & \varphi_{00}, \varphi_{11}, \psi_{10}, \psi_{01}, \\ \frac{1}{\sqrt{x}}\big(f_A(t)\cos(\lambda\sqrt{x}) + f_B(t)\sin(\lambda\sqrt{x})\big) & \quad \text{for} & \widetilde{\varphi}_{00}, \widetilde{\varphi}_{11}, \widetilde{\psi}_{10}, \widetilde{\psi}_{01}. \end{array} \tag{38}$$

In the above formulas $f_A(t),\ f_B(t)$ are generic functions of t.

1) The $\lambda = 0$ case.

At $\lambda = 0$ the restriction

$$\widehat{\mathrm{Z}} \;=\; 0 \tag{39}$$

can be consistently imposed. It is easily checked that this condition (which corresponds to the constraint imposed in [15]) implies:

i) the $\widetilde{\varphi}_{00}, \widetilde{\varphi}_{11}, \widetilde{\psi}_{10}, \widetilde{\psi}_{01}$ fields are all vanishing and
ii) the $\varphi_{00}, \varphi_{11}, \psi_{10}, \psi_{01}$ fields have no dependence on x.

The action of $\widehat{\mathrm{Q}}_{10}, \widehat{\mathrm{Q}}_{01}$ on the restricted vector $v_r^T = \big(\varphi_{00}(t), \varphi_{11}(t), \psi_{10}(t), \psi_{01}(t)\big)$ produces the minimal 4×4 matrix differential representation, see [13,17], of the $Z=0$ Beckers-Debergh [38] algebra. The restricted operators $\widehat{\mathrm{Q}}_{10}^{(r)}, \widehat{\mathrm{Q}}_{01}^{(r)}$ are

$$\widehat{\mathrm{Q}}_{10}^{(r)} = \begin{pmatrix} 0 & 0 & i & 0 \\ 0 & 0 & 0 & -\partial_t \\ \partial_t & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{pmatrix}, \qquad \widehat{\mathrm{Q}}_{01}^{(r)} = \begin{pmatrix} 0 & 0 & 0 & i \\ 0 & 0 & -\partial_t & 0 \\ 0 & -i & 0 & 0 \\ \partial_t & 0 & 0 & 0 \end{pmatrix}. \tag{40}$$

By setting $\widehat{\mathrm{H}}^{(r)} = i\partial_t \cdot \mathbb{I}_4$ we get

$$\{\widehat{\mathrm{Q}}_{10}^{(r)}, \widehat{\mathrm{Q}}_{10}^{(r)}\} = \{\widehat{\mathrm{Q}}_{01}^{(r)}, \widehat{\mathrm{Q}}_{01}^{(r)}\} = 2\widehat{\mathrm{H}}^{(r)}, \qquad [\widehat{\mathrm{H}}^{(r)}, \widehat{\mathrm{Q}}_{10}^{(r)}] = [\widehat{\mathrm{H}}^{(r)}, \widehat{\mathrm{Q}}_{01}^{(r)}] = [\widehat{\mathrm{Q}}_{10}^{(r)}, \widehat{\mathrm{Q}}_{10}^{(r)}] = 0. \tag{41}$$

A few comments are in order:

I - the scaling dimension of λ is the same as the one of a mass-term. On the other hand a mass term m can be introduced in a model even when $\lambda = 0$;

II - the reality condition (24) for the superfield $\Phi(X)$ is valid at the *classical* level. In the quantum case, in order to impose the eigenvalue equation for the Hamiltonian, i.e.

$$i\partial_t \mathbf{\Psi} = \widehat{\mathrm{H}}\mathbf{\Psi} = E\mathbf{\Psi} \tag{42}$$

where E is the energy, the time coordinate t entering the eigenfunctions (38) should be Wick-rotated ($t \to -it$). In the supersymmetric case the distinction bewteen classical versus quantum D-module reps is discussed, e.g., in [39], while the extension to the $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded invariant setting has been presented in [14].

2) The $\lambda \neq 0$ case.

At $\lambda \neq 0$ a representation satisfying the $Z^2 = \lambda^2$ Casimir is recovered by the action of the (31) operators on the 8-dimensional vectors expressed, by introducing the separation of the t, x coordinates, as

$$v_\lambda^T \;=\; \big(g_{00}(t)h(x), \widetilde{g}_{00}(t)\widetilde{h}(x), g_{11}(t)h(x), \widetilde{g}_{11}(t)\widetilde{h}(x), g_{10}(t)h(x), \widetilde{g}_{10}(t)\widetilde{h}(x), g_{01}(t)h(x), \widetilde{g}_{01}(t)\widetilde{h}(x)\big). \tag{43}$$

In the above formula $h(x)$, $\widetilde{h}(x)$ denote the dimensionless functions

$$h(x) = e^{i\lambda\sqrt{x}}, \qquad \widetilde{h}(x) = \frac{1}{\lambda\sqrt{x}} e^{i\lambda\sqrt{x}}, \tag{44}$$

while $g_{ij}(t)$, $\widetilde{g}_{ij}(t)$ are functions of the time coordinate t.

The identities

$$\partial_x h(x) = \frac{i\lambda^2}{2}\widetilde{h}(x), \qquad (\frac{1}{2} + x\partial_x)\widetilde{h}(x) = \frac{i}{2}h(x) \tag{45}$$

imply that, restricting the operators on the (43) vectors, one obtains an 8×8 differential matrix realization of the (5) superalgebra which only depends on the time coordinate t.

The resulting operators are

$$
\begin{aligned}
\widehat{\mathrm{H}}^{(R)} &= i\partial_t \cdot \mathbb{I}_8,\\
\widehat{\mathrm{Q}}_{10}^{(R)} &= \begin{pmatrix}
0 & 0 & 0 & 0 & i & 0 & 0 & 0\\
0 & 0 & 0 & 0 & \frac{\lambda^2}{2} & i\partial_t & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & -\partial_t & -\frac{i}{2}\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1\\
\partial_t & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
\frac{i\lambda^2}{2} & 1 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & -i & \frac{1}{2} & 0 & 0 & 0 & 0\\
0 & 0 & 0 & i\partial_t & 0 & 0 & 0 & 0
\end{pmatrix},\\
\widehat{\mathrm{Q}}_{01}^{(R)} &= \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & i & 0\\
0 & 0 & 0 & 0 & 0 & 0 & -\frac{\lambda^2}{2} & i\partial_t\\
0 & 0 & 0 & 0 & -\partial_t & \frac{i}{2} & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0\\
0 & 0 & -i & -\frac{1}{2} & 0 & 0 & 0 & 0\\
0 & 0 & 0 & i\partial_t & 0 & 0 & 0 & 0\\
\partial_t & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
-\frac{i\lambda^2}{2} & 1 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix},\\
\widehat{\mathrm{Z}}^{(R)} &= \begin{pmatrix}
0 & 0 & 0 & -1 & 0 & 0 & 0 & 0\\
0 & 0 & -\lambda^2 & 0 & 0 & 0 & 0 & 0\\
0 & -1 & 0 & 0 & 0 & 0 & 0 & 0\\
-\lambda^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -i\\
0 & 0 & 0 & 0 & 0 & 0 & i\lambda^2 & 0\\
0 & 0 & 0 & 0 & 0 & -i & 0 & 0\\
0 & 0 & 0 & 0 & i\lambda^2 & 0 & 0 & 0
\end{pmatrix}.
\end{aligned}
\tag{46}
$$

The (E, λ) representation is obtained by imposing the energy eigenvalue, which means to substitute $i\partial_t \mapsto E$ in the entries of the above matrices. We set, as in formula (36),

$$
E = \alpha\lambda, \tag{47}
$$

where the real parameter α is nondimensional.

The irreducible representations of the $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded super-Poincaré algebra $\mathcal{P}$ were investigated in [32] (see [33] for the $\mathcal{N} = 2$ extension of the graded super-Poincaré algebra). The 4-dimensional, minimal, $(\alpha\lambda, \lambda)$ irreducible representation is recovered by consistently constraining the $g_{ij}(t), \widetilde{g}_{ij}(t)$ fields entering (43) in terms of 4 time-dependent fields $f_{00}(t), f_{10}(t), f_{01}(t), f_{11}(t)$ of respective scaling dimensions $s, s + \frac{1}{2}, s + \frac{1}{2}, s + 1$; the resulting representation will therefore be symbolically expressed as $(1; 2; 1)$.

By taking into account the dimensionality of the fields, we are looking for a constrained vector v_{constr} given by

$$
v_{constr}^T = (f_{00},\ k_0\lambda^2 f_{00},\ k_1 f_{11},\ k_2 f_{11},\ f_{10},\ k_3\lambda f_{10},\ f_{01},\ k_4\lambda f_{01}); \tag{48}
$$

the identifications are $g_{00}(t) = f_{00}(t),\ \widetilde{g}_{00}(t) = k_0\lambda^2 f_{00}(t)$ and so on. The dimensionless constants k_0, k_1, k_2, k_3 have to be selected in order to guarantee the compatibility of the transformations obtained from the left actions of (46). The compatibility conditions imply:

$$
\begin{aligned}
k_0 &= \frac{1}{2}\sqrt{4\alpha^2-1}, \\
k_1 &= k_2\frac{1-4\alpha^2+i\sqrt{4\alpha^2-1}}{2i-2\sqrt{4\alpha^2-1}}, \\
k_3 &= \frac{-1+i\sqrt{4\alpha^2-1}}{2\alpha}, \\
k_4 &= \frac{1+i\sqrt{4\alpha^2-1}}{2\alpha}.
\end{aligned}
\tag{49}
$$

The $k_2\neq 0$ constant is arbitrary. Without loss of generality it can be selected to be

$$
k_2 = 1. \tag{50}
$$

The resulting 4-dimensional irreducible representation is given by

$$
\begin{aligned}
\widehat{\mathrm{Q}}_{10}^{(\alpha)} &= \begin{pmatrix} 0 & 0 & i & 0 \\ 0 & 0 & 0 & \frac{1+i\sqrt{4\alpha^2-1}}{2\alpha}\lambda \\ -i\alpha\lambda & 0 & 0 & 0 \\ 0 & \frac{2\alpha^2}{1+i\sqrt{4\alpha^2-1}} & 0 & 0 \end{pmatrix}, \\
\widehat{\mathrm{Q}}_{01}^{(\alpha)} &= \begin{pmatrix} 0 & 0 & 0 & i \\ 0 & 0 & \frac{-1+i\sqrt{4\alpha^2-1}}{2\alpha}\lambda & 0 \\ 0 & -\frac{1-2\alpha^2+i\sqrt{4\alpha^2-1}}{1+i\sqrt{4\alpha^2-1}} & 0 & 0 \\ -i\alpha\lambda & 0 & 0 & 0 \end{pmatrix}, \\
\widehat{\mathrm{Z}}^{(\alpha)} &= \begin{pmatrix} 0 & -1 & 0 & 0 \\ -\lambda^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{-i+\sqrt{4\alpha^2-1}}{2\alpha}\lambda \\ 0 & 0 & \frac{2\alpha}{-i+\sqrt{4\alpha^2-1}}\lambda & 0 \end{pmatrix},
\end{aligned}
\tag{51}
$$

together with $\widehat{\mathrm{H}}^{(\alpha)} = \alpha\lambda\cdot\mathbb{I}_4$.

As shown in Appendix **A**, α is constrained to satisfy $\alpha\geqslant\frac{1}{2}$. The second order Casimir C_2, introduced through the position

$$
C_2 := (\widehat{\mathrm{H}}^{(\alpha)})^2-\frac{1}{4}(\widehat{\mathrm{Z}}^{(\alpha)})^2 \equiv (\alpha^2-\frac{1}{4})\lambda^2, \tag{52}
$$

is such that it recovers

$$
C_2 = 0 \qquad \text{at the special “boundary” value} \quad \alpha=\tfrac{1}{2}. \tag{53}
$$

This special point is of particular importance because it produces, as discussed in Appendix **A**, an $(\mathcal{N}=2)$ supersymmetry. The following anticommutators (which are not defined as $\mathbb{Z}_2\times\mathbb{Z}_2$-graded operations, but can nevertheless being computed) are nonvanishing:

$$
\{\widehat{\mathrm{Q}}_{10}^{(R)},\widehat{\mathrm{Q}}_{01}^{(R)}\} \neq 0 \qquad \text{for the } 8\times 8 \text{ matrices entering (46) and} \tag{54}
$$

$$
\{\widehat{\mathrm{Q}}_{10}^{(\alpha>\frac{1}{2})},\widehat{\mathrm{Q}}_{01}^{(\alpha>\frac{1}{2})}\} \neq 0 \qquad \text{for the } 4\times 4 \text{ matrices entering (51) with } \alpha>\frac{1}{2}. \tag{55}
$$

On the other hand, at $\alpha=\frac{1}{2}$, the 4×4 matrices $\widehat{\rm Q}_{10}^{(\alpha=\frac{1}{2})}$, $\widehat{\rm Q}_{01}^{(\alpha=\frac{1}{2})}$ satisfy

$$\{\widehat{\rm Q}_{10}^{(\alpha=\frac{1}{2})},\widehat{\rm Q}_{01}^{(\alpha=\frac{1}{2})}\} \;=\; 0. \tag{56}$$

The spectrum of the $\mathbb{Z}_2\times\mathbb{Z}_2$-graded invariant models presented in [13,14,17] is supersymmetric since these models were constructed for this special $\alpha=\frac{1}{2}$ value. The $\alpha>\frac{1}{2}$ representations induce non supersymmetric generalizations of $\mathbb{Z}_2\times\mathbb{Z}_2$-graded invariant theories.

6 Three types of D-module representations

The differential operators (31) induce three types of D-module representations of the one-dimensional $\mathbb{Z}_2\times\mathbb{Z}_2$-graded super-Poincaré algebra. They are

i) the 8-dimensional representation, symbolically expressed as $(1;2;2;2;1)$, defined for a $1+1$ spacetime described by the time coordinate t and a space coordinate y of scaling dimensions $[t]=[y]=-1$. As discussed in the following, the $1+1$-dimensional spacetime can be assumed to be Minkowski. This D-module representation is applied to construct invariant relativistic 2-dimensional sigma models and string actions;

ii) the class of 4-dimensional representations, labeled by a real parameter $\beta\geqslant 0$, which can be symbolically expressed as $(1;2;1)_\beta$. They depend on the time coordinate t and find application in the construction of invariant world-line sigma models and point particle quantum mechanics;

iii) the further class of world-line 4-dimensional representations, also labeled by $\beta\geqslant 0$, which can be symbolically expressed as $(2;2;0)_\beta$. With respect to the previous class, these differential operators act on component fields with different scaling dimensions (respectively given by $s,s,s+\frac{1}{2},s+\frac{1}{2}$).

The three types of D-module representations are here presented.

6.1 The $(1;2;2;2;1)$ D-module representation for the $1+1$ Minkowski spacetime

The $(1;2;2;2;1)$ D-module representation for the $1+1$ Minkowski spacetime is recovered from the (31) operators after setting, for $x>0$ and in a convenient normalization,

$$x=\frac{1}{4}y^2, \qquad \text{where} \quad y\in\mathbb{R}. \tag{57}$$

The resulting operators, acting on

$$\overline{\mathbf{v}}^T \;=\; (\varphi_{00}(t,y),\widetilde{\varphi}_{00}(t,y),\varphi_{11}(t,y),\widetilde{\varphi}_{11}(t,y),\psi_{10}(t,y),\widetilde{\psi}_{10}(t,y),\psi_{01}(t,y),\widetilde{\psi}_{01}(t,y)) \tag{58}$$

and denoted in boldface as $\overline{\mathbf{M}}$, are

$$
\begin{array}{rcl}
\overline{\mathbf{H}} &=& i\partial_t\cdot\mathbb{I}_8,\\
\overline{\mathbf{Q}}_{10} &=& \left(\begin{array}{cccccccc}
0&0&0&0&i&0&0&0\\
0&0&0&0&-\frac{2i}{y}\partial_y&i\partial_t&0&0\\
0&0&0&0&0&0&-\partial_t&-\frac{1}{2}-\frac{y}{2}\partial_y\\
0&0&0&0&0&0&0&1\\
\partial_t&0&0&0&0&0&0&0\\
\frac{2}{y}\partial_y&1&0&0&0&0&0&0\\
0&0&-i&-\frac{i}{2}-\frac{iy}{2}\partial_y&0&0&0&0\\
0&0&0&i\partial_t&0&0&0&0
\end{array}\right),\\
\overline{\mathbf{Q}}_{01} &=& \left(\begin{array}{cccccccc}
0&0&0&0&0&0&i&0\\
0&0&0&0&0&0&\frac{2i}{y}\partial_y&i\partial_t\\
0&0&0&0&-\partial_t&\frac{1}{2}+\frac{y}{2}\partial_y&0&0\\
0&0&0&0&0&1&0&0\\
0&0&-i&\frac{i}{2}+\frac{iy}{2}\partial_y&0&0&0&0\\
0&0&0&i\partial_t&0&0&0&0\\
\partial_t&0&0&0&0&0&0&0\\
-\frac{2}{y}\partial_y&1&0&0&0&0&0&0
\end{array}\right),\\
\overline{\mathbf{Z}} &=& \left(\begin{array}{cccccccc}
0&0&0&i+iy\partial_y&0&0&0&0\\
0&0&\frac{4i}{y}\partial_y&0&0&0&0&0\\
0&i+iy\partial_y&0&0&0&0&0&0\\
\frac{4i}{y}\partial_y&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&-1-y\partial_y\\
0&0&0&0&0&0&\frac{4}{y}\partial_y&0\\
0&0&0&0&0&-1-y\partial_y&0&0\\
0&0&0&0&\frac{4}{y}\partial_y&0&0&0
\end{array}\right).
\end{array}
\quad (59)
$$

6.2 The $(1;2;1)_\beta$ worldline D-module representations

The $(1;2;1)_\beta$ worldline D-module representations are directly obtained from the 4×4 matrices (51) by taking into account that one can restore the dependence of the differential operator ∂_t in the $i\partial_t\mapsto\alpha\lambda$ mapping. One can therefore set

$$
\lambda\equiv\frac{i}{\alpha}\partial_t,\qquad \text{plus the position}\quad \alpha=\beta+\frac{1}{2},\qquad \text{where}\quad \beta\geqslant 0. \quad (60)
$$

Following the presentation (A.10) in Appendix **A**, the shifted parameter β is introduced in order to get at $\beta=0$ the supersymmetric critical point from equation (56).

The resulting operators, acting on

$$
\widehat{\mathbf{v}}^T \;=\; \left(\widehat{f}_{00}(t),\widehat{f}_{11}(t),\widehat{f}_{10}(t),\widehat{f}_{01}(t)\right) \quad (61)
$$

and denoted in boldface as $\widehat{\mathbf{M}}^\beta$, are

$$
\begin{array}{rcl}
\widehat{\mathbf{H}}^\beta &=& i\partial_t\cdot\mathbb{I}_4,\\
\widehat{\mathbf{Q}}^\beta_{10} &=& \begin{pmatrix} 0 & 0 & i & 0\\ 0 & 0 & 0 & \frac{2i-4\sqrt{\beta(\beta+1)}}{(2\beta+1)^2}\partial_t\\ \partial_t & 0 & 0 & 0\\ 0 & -\frac{i(2\beta+1)^2}{-2i+4\sqrt{\beta(\beta+1)}} & 0 & 0\end{pmatrix},\\
\widehat{\mathbf{Q}}^\beta_{01} &=& \begin{pmatrix} 0 & 0 & 0 & i\\ 0 & 0 & -\frac{2(i+2\sqrt{\beta(\beta+1)})}{(2\beta+1)^2}\partial_t & 0\\ 0 & \frac{i-4i\beta-4i\beta^2-4\sqrt{\beta(\beta+1)}}{-2i+4\sqrt{\beta(\beta+1)}} & 0 & 0\\ \partial_t & 0 & 0 & 0\end{pmatrix},\\
\widehat{\mathbf{Z}}^\beta &=& \begin{pmatrix} 0 & -1 & 0 & 0\\ \frac{4}{(2\beta+1)^2}\partial_t^2 & 0 & 0 & 0\\ 0 & 0 & 0 & \frac{2+4i\sqrt{\beta(\beta+1)}}{(2\beta+1)^2}\partial_t\\ 0 & 0 & \frac{2i}{-i+2\sqrt{\beta(\beta+1)}}\partial_t & 0\end{pmatrix}.
\end{array}
\tag{62}
$$

6.3 The $(2;2;0)_\beta$ worldline D-module representations

The $(2;2;0)_\beta$ worldline D-module representations are obtained from the $(1;2;1)_\beta$ representations by applying a dressing transformation. The $(2;2;0)_\beta$ differential matrices, denoted in boldface as $\breve{\mathbf{M}}^\beta$, can be expressed from the corresponding $(1;2;1)_\beta$ operators $\widehat{\mathbf{M}}^\beta$ as

$$
\widehat{\mathbf{M}}^\beta \quad\mapsto\quad \breve{\mathbf{M}}^\beta = \mathbf{D}^{-1}\widehat{\mathbf{M}}^\beta\mathbf{D}, \qquad \text{for}\quad \mathbf{D} = diag(1, i\partial_t, 1, 1). \tag{63}
$$

The dressing transformation was introduced in [36], see also [37], for the one-dimensional supermechanics and in [13] for the $\mathbb{Z}_2\times\mathbb{Z}_2$-graded case at the special $\beta = 0$ point.

The $\breve{\mathbf{M}}^\beta$ operators acting on

$$
\breve{\mathbf{v}}^T \;=\; \big(\breve{f}_{00}(t), \breve{f}_{11}(t), \breve{f}_{10}(t), \breve{f}_{01}(t)\big) \tag{64}
$$

are

$$
\begin{array}{rcl}
\check{\mathbf{H}}^{\beta} &=& i\partial_t\cdot\mathbb{I}_4,\\
\check{\mathbf{Q}}^{\beta}_{10} &=& \left(\begin{array}{cccc} 0 & 0 & i & 0\\ 0 & 0 & 0 & -i\frac{2i-4\sqrt{\beta(\beta+1)}}{(2\beta+1)^2}\\ \partial_t & 0 & 0 & 0\\ 0 & -i\frac{i(2\beta+1)^2}{-2i+4\sqrt{\beta(\beta+1)}}\partial_t & 0 & 0\end{array}\right),\\
\check{\mathbf{Q}}^{\beta}_{01} &=& \left(\begin{array}{cccc} 0 & 0 & 0 & i\\ 0 & 0 & i\frac{2(i+2\sqrt{\beta(\beta+1)})}{(2\beta+1)^2} & 0\\ 0 & i\frac{i-4i\beta-4i\beta^2-4\sqrt{\beta(\beta+1)}}{-2i+4\sqrt{\beta(\beta+1)}}\partial_t & 0 & 0\\ \partial_t & 0 & 0 & 0\end{array}\right),\\
\check{\mathbf{Z}}^{\beta} &=& \left(\begin{array}{cccc} 0 & -i\partial_t & 0 & 0\\ -i\frac{4}{(2\beta+1)^2}\partial_t & 0 & 0 & 0\\ 0 & 0 & 0 & \frac{2+4i\sqrt{\beta(\beta+1)}}{(2\beta+1)^2}\partial_t\\ 0 & 0 & \frac{2i}{-i+2\sqrt{\beta(\beta+1)}}\partial_t & 0\end{array}\right).
\end{array}
\qquad (65)
$$

Comment: for all three types of D-module representations the respective covariant derivatives, obtained from (19) and satisfying (20,21), can be constructed.

7 On the construction of $\mathbb{Z}_2\times\mathbb{Z}_2$-graded invariant actions

Before presenting in the next Section the calculus for the graded superspace of the one-dimensional $\mathbb{Z}_2\times\mathbb{Z}_2$-graded super-Poincaré algebra, it is worth pointing out some features of the construction of invariant classical action terms.

In formula (31) we can replace $x\geqslant 0$ with

$$
x=\overline{y}^r,\qquad \overline{y}=x^{\frac{1}{r}}\qquad \text{so that}\quad [x]=-2,\quad [\overline{y}]=-\tfrac{2}{r}. \qquad (66)
$$

For $r=2$ the coordinate proportional to $\overline{y}$, as shown in (57), has the same scaling dimension of the time t ($[t]=[\overline{y}]=-1$) and can be used in a relativistic construction.

By replacing the ∂_x and $\frac{1}{2}+x\partial_x$ entries in formulas (31) with

$$
\begin{array}{rcl}
\partial_x &\mapsto& \frac{1}{r}\overline{y}^{1-r}\partial_{\overline{y}},\\
\frac{1}{2}+x\partial_x &\mapsto& \frac{1}{2}+\frac{1}{r}\overline{y}\partial_{\overline{y}},
\end{array}
\qquad (67)
$$

one can check under which condition the integrand

$$
a\varphi_{11}+b\widetilde{\varphi}_{11} \qquad (68)
$$

transforms as a total derivative under the action of $\widehat{\mathrm{Q}}^{(R)}_{10}$, $\widehat{\mathrm{Q}}^{(R)}_{01}$, so that

$$
\textstyle\int\int d\overline{y}dt\ \left(a\varphi_{11}(t,\overline{y})+b\widetilde{\varphi}_{11}(t,\overline{y})\right) \qquad (69)
$$

is invariant. We get, up to partial integration terms,

$$\begin{aligned}
\delta_{Q_{10}}\big(a\varphi_{11}(t,\overline{y})+b\widetilde{\varphi}_{11}(t,\overline{y})\big) &\approx -a\dot{\psi}_{01}-\frac{a}{r}(\overline{y}\widetilde{\psi}_{01})'+\frac{(2-r)a+2rb}{2r}\widetilde{\psi}_{01},\\
\delta_{Q_{01}}\big(a\varphi_{11}(t,\overline{y})+b\widetilde{\varphi}_{11}(t,\overline{y})\big) &\approx -a\dot{\psi}_{10}-\frac{a}{r}(\overline{y}\widetilde{\psi}_{10})'+\frac{(r-2)a+2rb}{2r}\widetilde{\psi}_{10},
\end{aligned} \tag{70}$$

where the dot and the prime respectively denote the partial derivatives ∂_t, $\partial_{\overline{y}}$. A total derivative in the right hand side requires the vanishing of both last terms presented in the above equations. The solution

$$b=0,\qquad r=2, \tag{71}$$

implies that the invariant contribution to the action can only be constructed for the homogeneous spacetime $[t]=[\overline{y}]=-1$ which gives the D-module representation (59). We denote as y, see (57), the corresponding normalized space coordinate. The constraint $b=0$ corresponds to the so-called integrability condition introduced in [30] and discussed in [32].

If we assume y to be compactified on a circle $y\in[0,2\pi[$, with $\varphi_{11}(t,y)$ even function of y and $\widetilde{\varphi}_{11}(t,y)$ odd function of y, we can set

$$\begin{aligned}
\frac{1}{2\pi}\oint dy\int dt\varphi_{11}(t,y) &= \sum_{n=0}^{\infty}\varphi_{11}^{n}(t)\cos(ny), \qquad \text{so that}\\
\frac{1}{2\pi}\oint dy\int dt\varphi_{11}(t,y) &= \int dt\varphi_{11}^{0}(t)
\end{aligned} \tag{72}$$

reproduces the integration on the $\mathbb{Z}_2\times\mathbb{Z}_2$-graded superspace introduced in [30]. We are able to go further.

By assuming $y\in\mathbb{R}$ and the y-dependent component functions to be in $\mathcal{L}^2(\mathbb{R})$, the invariant term

$$\textstyle\iint_{-\infty}^{+\infty}dy\ dt\ \big(\varphi_{11}(t,y)\big) \tag{73}$$

generalizes the [30] integration.

For the $(1;2;1)_\beta$ worldline supermultiplet (61) the invariant term can be expressed as

$$\textstyle\int dt\widehat{f}_{11}(t). \tag{74}$$

Both (73) and (74) are 11-graded.

The invariant terms of the $(2;2;0)_\beta$ worldline supermultiplet (65) are

$$\int dt\check{f}_{10}(t)\qquad\text{and}\qquad\int dt\check{f}_{01}(t). \tag{75}$$

They are 10- and 01-graded.

8 Matrix representations of the graded superspace

The calculus (derivations, integrations) on the $\mathbb{Z}_2\times\mathbb{Z}_2$-graded superspace is easily described in terms of a matrix representation of the graded superspace coordinates (6), realized by two real parameters t, y and two real Grassmann numbers θ, ξ. It follows, in particular, that the Berezin calculus is one of the ingredients entering the $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded calculus.

The $\mathbb{Z}_2 \times \mathbb{Z}_2$ grading can be accommodated, see [13,14], in 4×4 matrices whose nonvanishing entries of the $[ij]$-sectors are given by

$$M_{00} \equiv \begin{pmatrix} * & 0 & 0 & 0 \\ 0 & * & 0 & 0 \\ 0 & 0 & * & 0 \\ 0 & 0 & 0 & * \end{pmatrix}, \; M_{11} \equiv \begin{pmatrix} 0 & * & 0 & 0 \\ * & 0 & 0 & 0 \\ 0 & 0 & 0 & * \\ 0 & 0 & * & 0 \end{pmatrix}, \; M_{10} \equiv \begin{pmatrix} 0 & 0 & * & 0 \\ 0 & 0 & 0 & * \\ * & 0 & 0 & 0 \\ 0 & * & 0 & 0 \end{pmatrix}, \; M_{01} \equiv \begin{pmatrix} 0 & 0 & 0 & * \\ 0 & 0 & * & 0 \\ 0 & * & 0 & 0 \\ * & 0 & 0 & 0 \end{pmatrix}. \tag{76}$$

One can introduce, as an example of a $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded algebra of matrices, the (complexified) quaternions, see [6]; they are expressed by $\mathbb{I}_4$ and M_i for $i = 1, 2, 3$. In terms of the Pauli matrices σ_i,

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{77}$$

the M_i's are defined through the positions

$$M_1 = \mathbb{I}_2 \otimes \sigma_1, \qquad M_2 = \sigma_1 \otimes \sigma_2, \qquad M_3 = \sigma_1 \otimes \sigma_3. \tag{78}$$

They satisfy the relations

$$M_i M_j = \delta_{ij} \mathbb{I}_4 + i \varepsilon_{ijk} M_k \qquad (M_i^\dagger = M_i), \tag{79}$$

where ε_{ijk} is the totally antisymmetric tensor with normalization $\varepsilon_{123} = 1$. The complexification of the ordinary quaternions imply that the matrices M_i are hermitian.

The vanishing (anti)commutators of the graded superspace coordinates $t, z, \theta_{10}, \theta_{01}$ are reproduced from the matrix representation

$$t \mapsto \overline{t} = t \cdot \mathbb{I}_4, \quad \theta_{10} \mapsto \overline{\theta}_{10} = \theta \cdot M_1, \quad \theta_{01} \mapsto \overline{\theta}_{01} = \xi \cdot M_2, \quad z \mapsto \overline{z} = \tfrac{1}{2} y \cdot M_3 \tag{80}$$

(as mentioned before, t, y are real coordinates while θ, ξ are real Grassmann numbers).

The matrix representation of the graded derivatives (14) is expressed by the positions

$$\partial_{\overline{t}} = \mathbb{I}_4 \cdot \partial_t, \quad \partial_{\overline{z}} = 2M_3 \cdot \partial_y, \quad \partial_{\overline{\theta}_{10}} = M_1 \cdot \partial_\theta, \quad \partial_{\overline{\theta}_{01}} = M_2 \cdot \partial_\xi. \tag{81}$$

It follows that the matrix representation of the $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded super-Poincaré generators (16) and the covariant derivatives (19) are respectively given by

$$\widehat{H} = \widetilde{H} \cdot \mathbb{I}_4, \quad \widehat{Q}_{10} = \widetilde{Q}_1 \cdot M_1, \quad \widehat{Q}_{01} = \widetilde{Q}_2 \cdot M_2, \quad \widehat{Z} = \widetilde{Z} \cdot M_3 \tag{82}$$

and

$$\widehat{D}_{10} = \widetilde{D}_1 \cdot M_1, \qquad \widehat{D}_{01} = \widetilde{D}_2 \cdot M_2, \tag{83}$$

where

$$\begin{aligned} \widetilde{H} &= i\partial_t, & \widetilde{Z} &= 2i\partial_y, \\ \widetilde{Q}_1 &= \partial_\theta + i\theta\partial_t + i\xi\partial_y, & \widetilde{Q}_2 &= \partial_\xi + i\xi\partial_t + i\theta\partial_y, \\ \widetilde{D}_1 &= \partial_\theta - i\theta\partial_t - i\xi\partial_y, & \widetilde{D}_2 &= \partial_\xi - i\xi\partial_t - i\theta\partial_y. \end{aligned} \tag{84}$$

In this way the $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded operators (82,83) are reconstructed in terms of the matrices M_i (which provide the $\mathbb{Z}_2 \times \mathbb{Z}_2$ grading) and the differential operators (84). The latter operators close a $\mathbb{Z}_2$-graded superalgebra $\widetilde{\mathcal{G}} = \widetilde{\mathcal{G}}_0 \oplus \widetilde{\mathcal{G}}_1$ whose even (odd) elements are respectively

$$\widetilde{H}, \widetilde{Z} \in \widetilde{\mathcal{G}}_0, \qquad \widetilde{Q}_1, \widetilde{Q}_2, \widetilde{D}_1, \widetilde{D}_2 \in \widetilde{\mathcal{G}}_1. \tag{85}$$

The only nonvanishing (anti)commutators of $\widetilde{\mathcal{G}}$ are

$$\begin{array}{ll}\{\widetilde{Q}_1,\widetilde{Q}_1\}=\{\widetilde{Q}_2,\widetilde{Q}_2\}=2\widetilde{H}, & \{\widetilde{Q}_1,\widetilde{Q}_2\}=\widetilde{Z},\\ \{\widetilde{D}_1,\widetilde{D}_1\}=\{\widetilde{D}_2,\widetilde{D}_2\}=-2\widetilde{H}, & \{\widetilde{D}_1,\widetilde{D}_2\}=-\widetilde{Z}.\end{array} \tag{86}$$

The introduction of the matrices M_i allows to make the passage from (86) to the $\mathbb{Z}_2\times\mathbb{Z}_2$-graded algebra (17,20,21).

9 The induced $(1+1)$-Minkowski graded super-Poincaré algebra

One of the consequences of the graded superspace of the worldline $\mathbb{Z}_2\times\mathbb{Z}_2$-graded super-Poincaré algebra $\mathcal{P}$ is that it accommodates a two-dimensional Minkowski $\mathbb{Z}_2\times\mathbb{Z}_2$-graded super-Poincaré algebra $\mathcal{P}_{d=2}$ which presents the addition of an extra generator (the Lorentz boost). The exotic bosonic coordinate is the responsible for the extra space coordinate y given in (80).

The induced 5-generator $\mathcal{P}_{d=2}$ superalgebra is a new $\mathbb{Z}_2\times\mathbb{Z}_2$-graded extension of the two-dimensional Poincaré algebra which *differs* from the two previous 2D $\mathbb{Z}_2\times\mathbb{Z}_2$-graded super-Poincaré algebras that have been discussed in the literature. In [2] a $\mathbb{Z}_2\times\mathbb{Z}_2$-graded extension with 11-graded translations was presented; a different $\mathbb{Z}_2\times\mathbb{Z}_2$-graded 2D super-Poincaré algebra with two 00-graded translations was introduced in [15].
The assignment of the $\mathbb{Z}_2\times\mathbb{Z}_2$-gradings of the $\mathcal{P}_{d=2}$ generators is, on the other hand, given by

$$\begin{array}{ll}\text{00-graded sector:} & \text{1 translation,}\\ \text{11-graded sector:} & \text{1 translation and 1 Lorentz boost,}\\ \text{10-graded sector:} & \text{1 parafermion,}\\ \text{01-graded sector:} & \text{1 parafermion.}\end{array} \tag{87}$$

In $\mathcal{P}_{d=2}$ the 00-translation is proportional to $\widehat{H}$, the 11-translation is proportional to $\widehat{Z}$, while the parafermions are $\widehat{Q}_{10}$ and $\widehat{Q}_{01}$. Besides these four (82) operators belonging to $\mathcal{P}$, the extra generator, the 11-graded Lorentz boost $\widehat{L}$, is introduced as

$$\widehat{L}=M_3\cdot\widetilde{L},\qquad \text{with}\quad \widetilde{L}=-iy\partial_t-it\partial_y-\tfrac{i}{2}\theta\partial_\xi-\tfrac{i}{2}\xi\partial_\theta. \tag{88}$$

The closure of the $\mathbb{Z}_2\times\mathbb{Z}_2$-graded $\mathcal{P}_{d=2}\supset\mathcal{P}$ superalgebra extension is guaranteed by the extra (anti)commutators involving $\widehat{L}$; they are given by

$$[\widehat{L},\widehat{P}_0]=i\widehat{P}_1,\quad [\widehat{L},\widehat{P}_1]=i\widehat{P}_0,\quad \{\widehat{L},\widehat{Q}_{10}\}=-\tfrac{1}{2}\widehat{Q}_{01},\quad \{\widehat{L},\widehat{Q}_{01}\}=\tfrac{1}{2}\widehat{Q}_{10}. \tag{89}$$

In the above formulas we set as $\widehat{P}_0=\widehat{H}$ and $\widehat{P}_1=\frac{1}{2}\widehat{Z}$ the generators of the (00-graded and, respectively, 11-graded) translations.

This new $\mathbb{Z}_2\times\mathbb{Z}_2$-graded two-dimensional super-Poincaré algebra $\mathcal{P}_{d=2}$ is a genuine extension of the $2D$ Poincaré algebra which is reproduced by the 00-graded and 11-graded sectors.

$\mathcal{P}_{d=2}$ is spanned by the hermitian generators

$$\widehat{P}_0\in\mathcal{P}_{d=2;00},\qquad \widehat{P}_1,\widehat{L}\in\mathcal{P}_{d=2;11},\qquad \widehat{Q}_{10}\in\mathcal{P}_{d=2;10},\qquad \widehat{Q}_{01}\in\mathcal{P}_{d=2;01}. \tag{90}$$

It is worth pointing out that, in the construction of the classical $\mathbb{Z}_2\times\mathbb{Z}_2$-graded invariant actions, the use of the covariant derivatives (83) automatically implies the invariance under the worldline $\mathcal{P}$ superalgebra. On the other hand, the invariance under the full two-dimensional $\mathcal{P}_{d=2}$ superalgebra (that is, the extra generator $\widehat{L}$) has to be imposed as an extra constraint to be satisfied by the Lagrangian $\mathcal{L}$; the requirement is that $\widehat{L}\cdot\mathcal{L}\approx 0$ up to total derivatives.

10 Invariant actions in superspace matrix representation

Graded superfields and invariant actions are nicely derived in terms of the matrix representation of the $\mathbb{Z}_2\times\mathbb{Z}_2$-graded superspace. The graded superfield $\Phi(t,\theta_{10},\theta_{01},z)$ given in (23) is mapped into a matrix-valued superfield by taking into account:

i) the (80) matrix representation of the graded superspace coordinates and
ii) the $\mathbb{Z}_2\times\mathbb{Z}_2$ grading of the $\varphi_{00},\widetilde{\varphi}_{00},\varphi_{11},\widetilde{\varphi}_{11},\psi_{10},\widetilde{\psi}_{10},\psi_{01},\widetilde{\psi}_{01}$ component fields entering the (23) decomposition.

In the matrix representation one can consistently set

$$\begin{array}{llll}\varphi_{00}=\mathbb{I}_4\cdot\varphi_0, & \widetilde{\varphi}_{00}=\mathbb{I}_4\cdot\widetilde{\varphi}_0, & \varphi_{11}=M_3\cdot\varphi_3, & \widetilde{\varphi}_{11}=M_3\cdot\widetilde{\varphi}_3,\\ \psi_{10}=M_1\cdot\psi_1, & \widetilde{\psi}_{10}=M_1\cdot\widetilde{\psi}_1, & \psi_{10}=M_2\cdot\psi_2, & \widetilde{\psi}_{01}=M_2\cdot\widetilde{\psi}_2\end{array}\tag{91}$$

in terms of ordinary $(t,x=\frac{1}{4}y^2)$-dependent functions; four of them are bosonic $(\varphi_0,\widetilde{\varphi}_0,\varphi_3,\widetilde{\varphi}_3)$ and four of them are fermionic $(\psi_1,\widetilde{\psi}_1,\psi_2,\widetilde{\psi}_2)$. By further setting

$$\varphi:=\varphi_0+\tfrac{1}{2}y\widetilde{\varphi}_3,\quad g:=-\varphi_3-\tfrac{1}{2}\widetilde{\varphi}_0,\quad \psi:=-\psi_1+\tfrac{1}{2}\widetilde{\psi}_2,\quad \chi:=-\psi_2-\tfrac{1}{2}\widetilde{\psi}_1\tag{92}$$

we have that $\Phi(t,\theta_{10},\theta_{01},z)$ is mapped into the 4×4 identity matrix times an ordinary $\mathcal{N}=2$ bosonic superfield $B(t,y,\theta,\xi)$. We have

$$\begin{array}{rcl}\Phi(t,\theta_{10},\theta_{01},z) & \mapsto & \overline{\Phi}(\overline{t},\overline{\theta}_{10},\overline{\theta}_{01},\overline{z})=\mathbb{I}_4\cdot B(t,y,\theta,\xi),\qquad \text{where}\\ & & B(t,y,\theta,\xi)=\varphi(t,y)-i\theta\psi(t,y)-i\xi\chi(t,y)-i\theta\xi g(t,y).\end{array}\tag{93}$$

After setting the infinitesimal parameters (12) of the $\mathbb{Z}_2\times\mathbb{Z}_2$-graded worldline super-Poincaré algebra to be

$$\varepsilon_{00}=\varepsilon_0\cdot\mathbb{I}_4,\qquad \varepsilon_{10}=\varepsilon_1\cdot M_1,\qquad \varepsilon_{01}=\varepsilon_2\cdot M_2,\qquad \varepsilon_{11}=\varepsilon_3\cdot M_3\tag{94}$$

and the infinitesimal parameter ε_L of the $-i\varepsilon_L\widehat{L}$ Lorentz boost (88) to be

$$\varepsilon_L \;=\; \varepsilon_4\cdot M_3,\tag{95}$$

the infinitesimal transformations of the $\mathcal{P}_{d=2}$ superalgebra (90) on the φ,ψ,ξ,g component fields read as follows:

$$\begin{array}{rcl}\delta\varphi &=& \varepsilon_0\dot{\varphi}-i\varepsilon_1\psi-i\varepsilon_2\chi+2\varepsilon_3\varphi'-\varepsilon_4(y\dot{\varphi}+t\varphi'),\\ \delta\psi &=& \varepsilon_0\dot{\psi}-\varepsilon_1\dot{\varphi}-\varepsilon_2(g+\varphi')+2\varepsilon_3\psi'-\varepsilon_4(y\dot{\psi}+t\psi'+\dfrac{1}{2}\chi),\\ \delta\chi &=& \varepsilon_0\dot{\chi}+\varepsilon_1(g-\varphi')-\varepsilon_2\dot{\varphi}+2\varepsilon_3\chi'-\varepsilon_4(y\dot{\chi}+t\chi'+\dfrac{1}{2}\psi),\\ \delta g &=& \varepsilon_0\dot{g}+i\varepsilon_1(\dot{\chi}-\psi')-i\varepsilon_2(\dot{\psi}-\chi')+2\varepsilon_3\varphi'-\varepsilon_4(y\dot{g}+tg').\end{array}\tag{96}$$

The infinitesimal parameters $\varepsilon_0,\varepsilon_3,\varepsilon_4$ are real while $\varepsilon_1,\varepsilon_2$ are Grassmann numbers; in the above formulas $f'\equiv\partial_y f$.

The graded superspace integration can be introduced in the matrix representation through the position

$$\int dtdzd\theta_{01}d\theta_{10} \;=\; \frac{1}{8}Tr\int dtdyd\xi d\theta\cdot M_3M_2M_1 = -\frac{i}{8}Tr\int dtdyd\xi d\theta\cdot \mathbb{I}_4. \tag{97}$$

This integration is equivalent to the prescription (73) that was discussed in Section **7**. It follows that, by construction, the actions constructed with potential terms and the covariant derivatives $\widehat{D}_{10}, \widehat{D}_{01}$ are invariant under the $\mathbb{Z}_2\times\mathbb{Z}_2$-graded worldline super-Poincaré algebra.

An invariant action $\mathcal{S}$ is given by

$$\begin{aligned}\mathcal{S} &= \int dtdzd\theta_{01}d\theta_{10}\cdot\mathcal{L}, \qquad \text{where}\\ \mathcal{L} &= \gamma_{11}\widehat{D}_{10}\Phi\widehat{D}_{01}\Phi - V(\Phi).\end{aligned} \tag{98}$$

The constant γ_{11} is assumed to be 11-graded in order to have a 00-graded action $\mathcal{S}$. By setting $\gamma_{11} = \gamma M_3$, with $\gamma\in\mathbb{R}$, the matrix representation of the action $\mathcal{S}$ is given by

$$\mathcal{S} \;=\; -\frac{i}{2}\int dtdyd\xi d\theta\big(i\gamma\widetilde{D}_1B\cdot\widetilde{D}_2B - V(B)\big). \tag{99}$$

In terms of the component fields, after integrating in ξ,θ and taking the $\gamma=-1$ value, we get

$$\begin{aligned}\mathcal{S} &= \int dtdy\big(\mathcal{L}_{kin}-\mathcal{L}_{pot}\big), \qquad \text{where}\\ \mathcal{L}_{kin} &= \frac{1}{2}\big(\dot{\varphi}^2-(\varphi')^2+g^2+i\psi\dot{\psi}+i\chi\dot{\chi}-i\psi\chi'-i\chi\psi'\big),\\ \mathcal{L}_{pot} &= -\frac{1}{2}\big(gV_\varphi+i\psi\chi V_{\varphi\varphi}\big), \qquad \text{for } \;V_\varphi\equiv\frac{\partial V}{\partial\varphi}.\end{aligned} \tag{100}$$

The Euler-Lagrange equations are

$$\begin{aligned}&\ddot{\varphi}-\varphi''-\frac{1}{2}gV_{\varphi\varphi}-\frac{i}{2}\psi\chi V_{\varphi\varphi\varphi}=0, \qquad && g+\frac{1}{2}V_\varphi=0,\\ &\dot{\psi}-\chi'+\frac{1}{2}\chi V_{\varphi\varphi}=0, && \dot{\chi}-\psi'-\frac{1}{2}\psi V_{\varphi\varphi}=0.\end{aligned} \tag{101}$$

It follows that $g(t,y)$ is an auxiliary field whose equation of motion can be algebraically solved.

As remarked at the end of Section **9**, the action (98) is invariant by construction under the worldline $\mathbb{Z}_2\times\mathbb{Z}_2$-graded super-Poincaré algebra $\mathcal{P}$. In order for the system to describe a two-dimensional relativistic theory, the action (98) should also be invariant under the Lorentz boost $\widehat{L}$. It is easily proved, with lengthy but straightforward computations, that this is indeed the case for any choice of the potential $V(\Phi)$. Therefore, the action (98) is invariant under the full two-dimensional $\mathbb{Z}_2\times\mathbb{Z}_2$-graded $\mathcal{P}_{d=2}$ super-Poincaré algebra (90). The demonstration requires to check that applying the Lorentz boost $\widetilde{L}$ from (88) to the Lagrangian $\mathcal{L}=\mathcal{L}_{kin}-\mathcal{L}_{pot}$ of (100), a total derivative is produced. We just present here the sketch of the demonstration.

The computation

$$\widetilde{L}\cdot\mathcal{L} \;\approx\; 0 \qquad \text{(up to total derivatives)} \tag{102}$$

is split into different terms which should separately verify (102): the purely bosonic part, the part which contains bilinear in ψ contributions, the part with the bilinear in χ contributions and, finally, the part with "mixed" ψ, χ fermionic contributions. As an example, the bilinear in ψ term from $\widetilde{L} \cdot \mathcal{L}_{kin}$ is proportional to $2t\psi'\dot{\psi} - \psi\dot{\psi}'$; this term is equivalent to $t\psi\dot{\psi}' - t\dot{\psi}\psi'$ after integrating by part and, furthermore, to $\partial_y(t\psi\dot{\psi})$. A similar analysis, conducted on the other terms, prove the validity of (102) for any choice of the potential.

By applying standard methods, see e.g. [40], we can present the conserved currents and Noether charges associated with the 5 invariant symmetry operators $\widehat{H}, \widehat{Z}, \widehat{L}, \widehat{Q}_{10}, \widehat{Q}_{01}$ of the (98) action $\mathcal{S}$. Under the (93) mapping and the (82,88) positions, the conserved currents and charges are reconstructed from the corresponding ones obtained from the invariant operators $\widetilde{H}, \widetilde{Z}, \widetilde{L}, \widetilde{Q}_1, \widetilde{Q}_2$ of (100) acting on the $B(t, y, \theta, \xi)$ superfield.

The five conserved currents J_*^μ (for $* \equiv \widetilde{H}, \widetilde{Z}, \widetilde{L}, \widetilde{Q}_1, \widetilde{Q}_2$ and $\mu \equiv t, y$) are defined as

$$J_*^\mu = \Lambda^\mu - \delta\Phi_A \frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi_A)}, \qquad \text{where} \quad \delta\mathcal{L} = \partial_\mu \Lambda^\mu \quad \text{and} \quad \Phi_A \equiv \varphi, \psi, \chi, g. \tag{103}$$

By taking into account the equations of motion, the currents satisfy the conserved equations

$$\partial_t J_*^t + \partial_y J_*^y = 0. \tag{104}$$

The computation of the conserved currents gives

$$\begin{aligned}
J^t_{\widetilde{H}} &= -\frac{1}{2}\big(\dot{\varphi}^2 + (\varphi')^2 + i\psi\chi' + i\chi\psi' + \frac{1}{4}V_\varphi^2 - i\psi\chi V_{\varphi\varphi}\big),\\
J^y_{\widetilde{H}} &= \dot{\varphi}\varphi' + \frac{i}{2}(\chi\dot{\psi} + \psi\dot{\chi});\\
J^t_{\widetilde{Z}} &= -2\dot{\varphi}\varphi' - i\psi\psi' - i\chi\chi',\\
J^y_{\widetilde{Z}} &= \dot{\varphi}^2 + (\varphi')^2 + i\psi\dot{\psi} + i\chi\dot{\chi} - \frac{1}{4}V_\varphi^2 + i\psi\chi V_{\varphi\varphi};\\
J^t_{\widetilde{L}} &= \frac{y}{2}\big(\dot{\varphi}^2 + (\varphi')^2 + i(\psi\chi' + \chi\psi') + \frac{1}{4}V_\varphi^2 - i\psi\chi V_{\varphi\varphi}\big) + t\big(\dot{\varphi}\varphi' + \frac{i}{2}(\psi\psi' + \chi\chi')\big),\\
J^y_{\widetilde{L}} &= \frac{t}{2}\big(-\dot{\varphi}^2 - (\varphi')^2 - i(\psi\dot{\psi} + \chi\dot{\chi}) + \frac{1}{4}V_\varphi^2 - i\psi\chi V_{\varphi\varphi}\big) + y\big(-\dot{\varphi}\varphi' - \frac{i}{2}(\psi\dot{\chi} + \chi\dot{\psi})\big);\\
J^t_{\widetilde{Q}_1} &= -\dot{\varphi}\psi - \varphi'\chi - \frac{1}{2}\chi V_\varphi,\\
J^y_{\widetilde{Q}_1} &= \dot{\varphi}\chi + \varphi'\psi + \frac{1}{2}\psi V_\varphi;\\
J^t_{\widetilde{Q}_2} &= -\dot{\varphi}\chi - \varphi'\psi + \frac{1}{2}\psi V_\varphi,\\
J^y_{\widetilde{Q}_2} &= \dot{\varphi}\psi + \varphi'\chi - \frac{1}{2}\chi V_\varphi.
\end{aligned} \tag{105}$$

As usual, the Noether charges are obtained from the integration $\int dy J_*^t$.

Remarks. The following remarks summarize the results of this Section:

1 - the superspace of the worldline $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded super-Poincaré algebra induces a two-dimensional relativistic model which is invariant under the two-dimensional $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded Poincaré algebra $\mathcal{P}_{d=2}$. The action $\mathcal{S}$ in (98) is invariant under this larger algebra;

2 - under the (93) mapping the $\mathbb{Z}_2\times\mathbb{Z}_2$-graded model is mapped into an ordinary two-dimensional relativistic ($\mathcal{N}=1$) theory defined for the scalar bosonic superfield $B(t,y,\theta,\xi)$. The $\mathbb{Z}_2\times\mathbb{Z}_2$-graded invariant operators and conserved charges are reconstructed from the corresponding supersymmetric invariant operators and charges.

11 The $\mathbb{Z}_2\times\mathbb{Z}_2$-graded closed string

The compactification of the y coordinate on the $\mathbf{S}^1$ circle parametrized by $y\in[0,2\pi[$, coupled with the (72) integration, produces a $\mathbb{Z}_2\times\mathbb{Z}_2$-graded closed string. We present its construction.

We note at first that the $\mathbb{Z}_2\times\mathbb{Z}_2$ grading allows both periodic (P) and/or antiperiodic (A) boundary conditions. By assuming the ordinary 00-graded bosons to be periodic, the consistency of the *mod* 2 grading addition given in (3) requires the following (anti)periodicity of the remaining graded sectors. One of the three alternatives are in principle admissible in association with the corresponding (00/10/01/11) graded sectors:

$$
i:(P/P/P/P),\qquad ii:(P/A/A/P),\qquad iii:(P/P/A/A)\equiv(P/A/P/A). \tag{106}
$$

Even if the three above alternatives are admissible by the $\mathbb{Z}_2\times\mathbb{Z}_2$ grading, it does not mean that they are necessarily implemented in a specific model. As an example, inspecting the transformations parametrized by $\varepsilon_1,\varepsilon_2$, the matching of left and right hand sides periodicities in (96) requires the fermions ψ,χ to be periodic. Similarly, the auxiliary field g needs to be periodic. It follows that the implementation of the $\mathbb{Z}_2\times\mathbb{Z}_2$-graded worldline super-Poincaré algebra for a closed string requires all component fields to be periodic.

This analysis does not rule out the possibility of antiperiodic boundary conditions for the y-compactified closed string recovered from the (100) Lagrangian. By inspecting the (101) equations of motion one can for instance check that, for a vanishing potential $V(\varphi)=0$, the $g=0$ equation is compatible with an antiperiodic boundary condition for the auxiliary field g.

We limit here to discuss the closed string with $(P/P/P/P)$ boundary conditions since this is the case which is directly derived from the $\mathbb{Z}_2\times\mathbb{Z}_2$-graded superspace formalism (comments about the two other possible options in (106) are given later).

Let us assume, for the (98) action which leads to (100), the periodicity conditions for the component fields:

$$
\varphi(y+2\pi)=\varphi(y),\quad \psi(y+2\pi)=\psi(y),\quad \chi(y+2\pi)=\chi(y),\quad g(y+2\pi)=g(y). \tag{107}
$$

The given component field $f(t,y)$ (where f denotes any of the above ϕ,ψ,χ,g fields) is real and mode-expanded according to

$$
f(t,y) \;=\; \sum_{n=-\infty}^{+\infty} f_n(t)e^{iny},\qquad f_n^*=f_{-n}. \tag{108}
$$

The y-integration in (100) is then defined as

$$
\int dy \;\equiv\; \frac{1}{2\pi}\int_0^{2\pi} dy. \tag{109}
$$

After performing the integration in y the action (100) reads as

$$
\mathcal{S} \;=\; \int dt\big(\overline{\mathcal{L}}_{kin}-\overline{\mathcal{L}}_{pot}\big). \tag{110}
$$

The kinetic term is

$$\overline{\mathcal{L}}_{kin} \quad = \quad \frac{1}{2}\sum_{n=-\infty}^{+\infty}\big(|\dot{\varphi}_n|^2 - n^2|\varphi_n|^2 + |g_n|^2 + i\psi_n\dot{\psi}_n^* + i\chi_n\dot{\chi}_n^* + 2n\psi_n^*\chi_n\big). \tag{111}$$

If we take the potential term to be obtained from a quadratic $V(\Phi)\propto\Phi^2$ in (98) we get

$$\overline{\mathcal{L}}_{pot} \quad = \quad \sum_{n=-\infty}^{+\infty} k\big(g_n\varphi_n^* + i\psi_n\chi_n^*\big) \qquad \text{for} \quad k \in \mathbb{R}. \tag{112}$$

The Euler-Lagrange equations of motion are

$$\begin{aligned} &\ddot{\varphi}_n + n^2\varphi_n + kg_n = 0, \qquad && g_n - k\varphi_n = 0, \\ &i\dot{\psi}_n + n\chi_n - ik\chi_n = 0, \qquad && i\dot{\chi}_n + n\psi_n + ik\psi_n = 0. \end{aligned} \tag{113}$$

We get in particular, after solving the equations of motion for the auxiliary fields $g_n(t)$:

$$\ddot{\varphi}_n = -(n^2+k^2)\varphi_n, \qquad \ddot{\psi}_n = -(n^2+k^2)\psi_n, \qquad \ddot{\chi}_n = -(n^2+k^2)\chi_n. \tag{114}$$

The following three scenarios apply:

i) the free closed string (for $V(\Phi) = 0 \Rightarrow k = 0$) corresponds to an infinite set of graded harmonic oscillators for $n \neq 0$ plus the zero-modes recovered from $\varphi_0, \psi_0, \chi_0$;

ii) for the quadratic potential ($k \neq 0$) the energy of the harmonic oscillators is shifted. The fields $\varphi_0, \psi_0, \chi_0$ no longer describe zero-modes, but harmonic oscillators with energy $|k|$;

iii) for an arbitrary potential $V(\Phi)$ the $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded invariant action produces an infinite set of interacting particles.

In scenarios *i* and *ii* one can consistently describe the dynamics of the n-th mode component fields alone. This situation corresponds to restrict the two-dimensional 8×8 matrix D-module representation (59) to the 4×4 worldline D-module representation (62). This is no longer the case in scenario *iii* for an arbitrary $V(\Phi)$. The models with interacting modes cannot be restricted to the irreducible time-dependent 4-component fields, so that in the interacting case the full D-module representation (59) is required. This leads to a new class of $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded invariant models not previously considered in the literature.

We conclude this Section by pointing out some open questions, left for future investigations, concerning the (anti)periodic boundary conditions for the $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded closed string. In [41] a para-Grassmann string (different from our model) was presented for both Ramond (periodic, "R") and Neveu-Schwarz (antiperiodic, "NS") boundary conditions, the two versions being related by a Klein transformation. This suggests that, for the free closed string model (110) with $\mathcal{L}_{pot} = 0$, the analysis of the invariance should be conducted for the separate left-right mover sectors given by $x_\pm = t \pm y$ in all four cases: R-R, R-NS, NS-R, NS-NS. This analysis, which will be presented in a forthcoming paper, should also lead to $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded extensions of the Virasoro algebra.

12 Conclusions

Here we summarize and comment some of the results of the paper.

It has been shown that the superspace of the worldline $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded super-Poincaré algebra induces D-module representations acting on 8 two-dimensional component fields. Rather unexpectedly the derived classical actions, see e.g. formula (98), are invariant under a two-dimensional relativistic $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded super-Poincaré algebra. This two-dimensional superalgebra and the associated relativistic models have not been previously considered in the literature. Furthermore, the compactification of the second (space) coordinate on a circle produces a $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded closed string theory with periodic boundary conditions. A potential term, producing interacting modes, can be consistently added.

The irreducibility conditions of the $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded supermultiplets have been analyzed. We proved that the previously known 4-component worldline supermultiplets admit (see the presentations in Section **6**) a β-deformation, the original worldline supermultiplets being recovered at $\beta = 0$; this is a special supersymmetric point (see formula (56), where $\alpha = \frac{1}{2} + \beta$).

A useful presentation of the $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded calculus, in terms of matrices encoding the $\mathbb{Z}_2 \times \mathbb{Z}_2$-grading coupled with the Berezinian calculus, has also been introduced in Section **8**.

These results imply that a larger class of $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded invariant models (both two-dimensional and one-dimensional) than the ones so far considered, becomes available. These models fall into the realm of $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded parastatistics with, see [21, 22], detectable consequences for the presence of the paraparticles.

It prompts to further investigations of these theories in both classical and quantum settings. For closed string models, see the remarks at the end of Section **11**, this means investigating $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded strings with Ramond and Neveu-Schwarz boundary conditions and $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded extensions of the Virasoro algebra. The construction of $\beta \neq 0$ deformed worldline models is left for a forthcoming paper. Unlike the $\beta = 0$ theories of references [13, 14, 17], the energy spectrum of these deformed theories is not related by a supersymmetry transformation.

Appendix A: irreducible representations and β-deformation of the supersymmetric spectrum

The irreducible representations of the $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded one-dimensional super-Poincaré algebra (5) on a $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded space are 4-dimensional. On the other hand, in physical applications, in order to derive the energy spectrum of the invariant models is also important to consider representations on $\mathbb{Z}_2$-graded spaces. The irreducible ones are 2-dimensional. The analysis goes as follows.

The four hermitian generators of (5) are H, Z, Q_{10}, Q_{01}. We set, for simplicity,

$$Q_1 := Q_{10}, \qquad Q_2 := Q_{01}. \tag{A.1}$$

In the enveloping algebra we can introduce the operator W, hermitian under the $*$ conjugation defined in (8), which is expressed by the anticommutator

$$W \quad := \quad \{Q_1, Q_2\}, \qquad\qquad (W^* = W). \tag{A.2}$$

A simple computation shows that

$$[W, Q_1] = [W, Q_2] = 0. \tag{A.3}$$

It is convenient to set

$$Q_\pm := \frac{1}{\sqrt{2}}(Q_1 \pm Q_2) \qquad \text{and} \qquad H_\pm := H \pm \frac{1}{2}W. \tag{A.4}$$

The $\mathbb{Z}_2$-graded 4-generator superalgebra with $Q_\pm$ odd and $H_\pm$ even elements is recovered:

$$\{Q_\pm, Q_\pm\} = 2H_\pm, \qquad \{Q_+, Q_-\} = 0, \qquad [H_\pm, G] = 0 \quad \text{for any} \quad G = H_\pm, Q_\pm. \quad \text{(A.5)}$$

The above superalgebra gives two independent copies (for Q_+, H_+ and Q_-, H_-, respectively) of the one-dimensional $\mathcal{N} = 1$ supersymmetry. For $W = 0$ one gets the 3-generator one-dimensional $\mathcal{N} = 2$ supersymmetry defined by $Q_\pm$ and $H_{susy} \equiv H_+ = H_-$:

$$\{Q_\pm, Q_\pm\} = 2H_{susy}, \qquad [H_{susy}, Q_\pm] = 0. \tag{A.6}$$

Following the presentation of Section **5** we can parametrize the Z^2 Casimir operator of (5) as $Z^2 = \lambda^2$ by assuming $\lambda > 0$. The energy eigenvalue, corresponding to the Casimir operator H, can be parametrized, see (47), as $\alpha\lambda$ where α is a nondimensional real parameter. Since H is the square of hermitian operators, α is a non-negative real number. The following analysis proves that α is restricted to satisfy

$$\alpha \;\; \geqslant \;\; \frac{1}{2}. \tag{A.7}$$

The relation

$$W^2 = -Z^2 + 4H^2 \quad \text{implies that} \quad W^2 = (4\alpha^2 - 1)\lambda^2. \tag{A.8}$$

By taking into account the hermiticity of W, the (A.7) constraint on α follows.
The eigenvalues of W are $\lambda\sqrt{4\alpha^2 - 1}$; therefore the degenerate $E_\pm$ eigenvalues of $H_\pm$ are

$$E_\pm \;\; = \;\; \lambda\big(\alpha \pm \frac{1}{2}\sqrt{4\alpha^2 - 1}\big). \tag{A.9}$$

Since the $W = 0$ condition is obtained for $\alpha = \frac{1}{2}$, it is convenient to express α in terms of the shifted parameter β:

$$\alpha = \beta + \frac{1}{2}, \qquad \text{where} \quad \beta \geqslant 0. \tag{A.10}$$

The boundary point $\beta = 0$ corresponds to the critical value which produces the enhanced $\mathcal{N} = 2$ supersymmetry (A.6).

In terms of β the $E_\pm$ eigenvalues are

$$E_\pm \;\; = \;\; \lambda\big(\frac{1}{2} + \beta \pm \sqrt{\beta^2 + \beta}\big). \tag{A.11}$$

The $E_\pm(\beta)$ functions are strictly monotonic (respectively, crescent/decrescent) in the $\beta \geqslant 0$ domain, with

$$E_\pm(\beta = 0) = \frac{1}{2}, \qquad \text{while} \qquad \lim_{\beta\to+\infty} E_+(\beta) = +\infty, \qquad \lim_{\beta\to+\infty} E_-(\beta) = 0. \tag{A.12}$$

A 4-dimensional representation of the $\mathbb{Z}_2$-graded superalgebra (5), labeled by the pair of eigenvalues (E_+, E_-) given by

$$H_\pm|vac\rangle \;\; = \;\; E_\pm|vac\rangle, \tag{A.13}$$

is spanned by the 4 vectors

$$|vac\rangle, \qquad Q_+|vac\rangle, \qquad Q_-|vac\rangle, \qquad Q_+Q_-|vac\rangle. \tag{A.14}$$

This representation is reducible. By taking into account the $\mathbb{Z}_2$-graded structure, a 2-dimensional irreducible representation is recovered by constraining

$$Q_+Q_-|vac\rangle = b|vac\rangle, \qquad Q_-|vac\rangle = fQ_+|vac\rangle. \tag{A.15}$$

The constants b, f are determined by the consistency conditions $H_-|vac\rangle = Q_-Q_-|vac\rangle = fQ_-Q_+|vac\rangle$ and $H_+|vac\rangle = Q_+Q_+|vac\rangle = \frac{1}{f}Q_+Q_-|vac\rangle$. One gets

$$b = i\sqrt{E_+E_-}, \qquad f = i\sqrt{\frac{E_-}{E_+}}. \tag{A.16}$$

The action of $Q_\pm$ on the 2-component vector $v^T = (|vac\rangle, Q_+|vac\rangle)$ gives the 2×2 matrix representation

$$\overline{H}_\pm = E_\pm \cdot \mathbb{I}_2, \qquad \overline{Q}_+ = \left(\begin{array}{cc} 0 & E_+ \\ 1 & 0 \end{array}\right), \qquad \overline{Q}_- = \left(\begin{array}{cc} 0 & -i\sqrt{E_+E_-} \\ i\sqrt{\frac{E_-}{E_+}} & 0 \end{array}\right). \tag{A.17}$$

The above operators produce, by inserting the (A.11) relations for $E_\pm(\beta)$, a class of β-dependent 2-dimensional representations.

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