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A Deformed Bose-Einstein Gas Near $q = 1$

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Abstract

We show that the Bose-Einstein condensation occurs, on the region near $q = 1$, for a modified ideal deformed gas.

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Recently, a great deal of interest has been devoted to the study of quantum groups [1–4] and deformed quantum q-gases. Chaichian *et al* [5] investigated the deformed Bose-Einstein gas near $q = 1$ and found that in this limit some divergencies appear in the series expansion of the thermodynamic functions. This fact did not allow the study of the Bose-Einstein condensation (BEC). Later on, R-Monteiro *et al* [6, 7] showed that, for highly deformed q-gases, BEC is present and the specific heat has a λ-point transition behavior.

We propose here a new type of deformed system in which the parameter q is a function of the momentum, p. This procedure is implemented by means of a system where each oscillator is attached to a different value of q, and for particular choices it is possible to regulate the thermodynamic functions. When the Bose-Einstein gas is deformed this way, we show that it is possible to approach the region near $q = 1$ and BEC occurs on this system.

The Hamiltonian for an ideal q-gas is taken as [5, 6]

$$H = \sum_i \hbar\omega_i a_i^\dagger a_i = \sum_i \hbar\omega_i [N_i], \tag{1}$$

where a_i, $a_i^\dagger$, N_i are anihilation, creation and occupation operators, respectively, of particles in the state i. These operators have the following commutation relations:

$$a_i a_j^\dagger - ((q_i - 1)\delta_{ij} + 1) a_j^\dagger a_i = \delta_{ij} q^{-N_i}, \quad [N_i, a_j] = -\delta_{ij} a_j, \quad [N_i, a_j] = \delta_{ij} a_j^\dagger. \tag{2}$$

We also have

$$[N_i] = (q_i^{N_i} - q_i^{-N_i})/(q_i - q_i^{-1}). \tag{3}$$

As in the non-deformed case, the total energy can be written as a sum of single-particle energies :

$$E = \sum_i E_i(n_i) = \sum_i \hbar\omega_i [n]. \tag{4}$$

For real $q_i = exp(t_i)$ we have for the state i

$$E_i(n) = \hbar\omega_i \frac{sinh(nt_i)}{sinh(t_i)}. \tag{5}$$

The grand canonical partition function of the system is

$$Z(z, V, T) = Tr\{exp[-\beta(H - \mu N)]\} = \sum_{N=0}^{\infty} z^N Z_N(V, T), \tag{6}$$

where $z = exp(\beta\mu)$ is the fugacity, μ is the chemical potential, N is the total number operator and Z_N is the canonical partition function :

$$Z_N = \sum_{n_i} exp(-\beta\hbar \sum_i \omega_i [n_i]) \tag{7}$$

Further calculations provide that

$$Z = \prod_i \sum_{n=0}^{\infty} z^n exp(-\beta\hbar\omega_i[n]) \tag{8}$$

So, the grand canonical potential is given by

$$\Omega = -\frac{1}{\beta} lnZ = -\frac{1}{\beta}\sum_i ln\left(\sum_{n=0}^{\infty} z^n exp(-\beta\hbar\omega_i[n])\right) \tag{9}$$

Making the assumption that all t_i are small enough, and then expanding equation (5) to order t_i^2, we have :

$$E_i = \hbar\omega_i(n + \frac{1}{6}n(n+1)(n-1)t_i^2 + \cdots). \tag{10}$$

The substitution of this in (8) gives

$$z^n Z_N(V,T) \simeq \sum_{n=0}^{\infty} e^{\beta(\mu-\hbar\omega)n}\left(1 - \frac{1}{6}n(n+1)(n-1)\beta\hbar\omega t_i^2\right), \tag{11}$$

and makes it possible to perform the sum over n :

$$z^n Z_N(V,T) \simeq Z_0(1 - x_i z^2 e^{-2x_i} Z_0^3 t_i^2), \tag{12}$$

where $x_i = \beta\hbar\omega_i$ and $Z_0 = 1/(1 - ze^{-x_i})$. Finally, the grand canonical potential is, to second order in t_i,

$$\beta\Omega \simeq -\sum_i lnZ_0 + z^2\sum_i t_i^2 x_i e^{-2x_i} Z_0^3 \tag{13}$$

The next step is to enclose the system in a 3-dimensional volume V and replace the sum over levels by an integral over $\vec{p}$-space:

$$\sum_i \rightarrow \frac{4\pi V}{h^3}\int_0^{\infty} p^2 dp, \tag{14}$$

and make t a function of p by choosing

$$t(p,\epsilon) = \sqrt{\epsilon\frac{p^\eta}{(\Lambda)^\eta} e^{-\beta p^2/2m}}, \tag{15}$$

where $\Lambda = (2m/\beta)^{1/2}$, η is an integer,$\epsilon << 1$ and $\hbar\omega = p^2/2m$ defines the dispersion law for non-relativistic particles.

Remembering that $p = -\Omega/V$, we have

$$\beta p = \lambda^{-3}(g_{5/2}(z) - \epsilon G_{5/2}(\eta, z)), \tag{16}$$

where $\lambda = h/\sqrt{2\pi mkT}$, and the g_α and G functions are integrals of the form [8]

$$I(r,\alpha,\nu) = \int_0^{\infty} dx z^r \frac{x^{\alpha-1}}{(e^x - z)^\nu} = \frac{\Gamma(\alpha)}{(\nu-1)!}\sum_{k=0}^{\infty}\frac{(k+\nu-1)!}{k!}\frac{z^k}{(k+\nu)^\alpha} \tag{17}$$

For g_α, we have

$$g_\alpha = \frac{1}{\Gamma(\alpha)} I(1, \alpha, 1). \tag{18}$$

These functions have the following property:

$$z\frac{dg_\alpha(z)}{dz} = g_{\alpha-1}(z). \tag{19}$$

The G functions do not have the above property, as $\alpha = (\eta+5)/2$ always. However, we can, for the sake of simplicity, give them the same subscript of the g functions. Then, the G functions obey (18) in a formal manner, and are defined by

$$\begin{aligned} G_{5/2}(\eta,z) &= \tfrac{2}{\sqrt{\pi}} I(2, \tfrac{\eta+5}{2}, 3), \\ G_{3/2}(\eta,z) &= \tfrac{2}{\sqrt{\pi}}[2I(2, \tfrac{\eta+5}{2}, 3) + 3I(3, \tfrac{\eta+5}{2}, 4)], \\ G_{1/2}(\eta,z) &= \tfrac{2}{\sqrt{\pi}}[4I(2, \tfrac{\eta+5}{2}, 3) + 15I(3, \tfrac{\eta+5}{2}, 4) + 12I(4, \tfrac{\eta+5}{2}, 5)] \end{aligned} \tag{20}$$

Chaichian *et al* [5] showed that the expansion in t, eq.(12), suffers a breakdown when $z \to 1$ and $x \to 0$, as the effective expansion parameter, t^2 times a positive power of Z_0, grows large. It is not the case if we use eq.(15), making t a function of p, or, in terms of x, $t^2 = \epsilon x^{\eta/2} e^{-x}$. Doing so, we have a modified deformed system where the effective expansion parameter has a finite value when $z \to 1$ and $x \to 0$, if $\eta \geq 4$. In our calculations, we used $\eta = 4$.

The density of the system is

$$n = z\left(\frac{\partial(\beta p)}{\partial z}\right)_\beta, \tag{21}$$

and we get

$$n = \lambda^{-3}(g_{3/2}(z) - \epsilon G_{3/2}(\eta,z)). \tag{22}$$

The internal energy U and $(\partial z/\partial T)_V$ are, respectively, equal to

$$\frac{3}{2}\frac{T(g_{5/2}(z) - \epsilon G_{5/2}(\eta,z))}{g_{3/2}(z) - \epsilon G_{3/2}(\eta,z)} \tag{23}$$

and

$$-\frac{3}{2}\frac{z g_{3/2}(z)}{T g_{1/2}(z)} + \frac{3}{2}\frac{\epsilon z G_{3/2}(\eta,z)}{T g_{1/2}(z)}. \tag{24}$$

Using these equations, we find the specific heat of the system, for $T \geq T_c$:

$$\begin{aligned} \frac{C_V}{Nk} = &-\frac{15}{4}\frac{\epsilon G_{5/2}(\eta,z)}{g_{3/2}(z) - \epsilon G_{3/2}(\eta,z)} + \frac{9}{2}\frac{\epsilon g_{3/2}(z) G_{3/2}(\eta,z)}{g_{1/2}(z)(g_{3/2}(z) - \epsilon G_{3/2}(\eta,z))} \\ &-\frac{9}{4}\frac{\epsilon g_{5/2}(z) G_{1/2}(\eta,z)}{g_{1/2}(z)(g_{3/2}(z) - \epsilon G_{3/2}(\eta,z))} \\ &+\frac{15}{4}\frac{g_{5/2}(z)}{g_{3/2}(z) - \epsilon G_{3/2}(\eta,z)} - \frac{9}{4}\frac{g_{3/2}(z)^2}{g_{1/2}(z)(g_{3/2}(z) - \epsilon G_{3/2}(\eta,z))} \end{aligned} \tag{25}$$

For $z = 1$ or $T \leq T_c$, we have

$$\frac{C_V}{Nk} = \frac{15}{4}\frac{g_{5/2}(1) - \epsilon G_{5/2}(\eta, 1)}{g_{3/2}(1) - \epsilon G_{3/2}(\eta, 1)}\left(\frac{T}{T_c}\right)^{3/2} \tag{26}$$

We note that, as $\epsilon \to 0$, we recover the ideal Bose-Einstein gas specific heat. In figure 1 we see that, as $z \to 0$ or $T \to \infty$, the specific heat limit is also 3/2. The critical temperature, T_c, is

$$T_c = \frac{h^2}{2\pi mk}\left(\frac{n}{g_{3/2}(z) - \epsilon G_{3/2}(\eta, z)}\right)^{2/3}. \tag{27}$$

When $\epsilon \to 0$, we recover the critical temperature of the Bose-Einstein ideal gas.

Although our system dynamics is different from the ones in [5] and [7], there are some similarities. For example, our T_c is higher than that of the ideal Bose-Einstein gas, as in [7]. However, our specific heat does not show any discontinuity at $T = T_c$, as we see in the figure. The reason to that is the fact that our system averages the thermodynamic functions for values of q such that $1 \leq q \leq 1 + \epsilon$. Notice, however, that for increasing ϵ, the trend of the curves is compatible with the appearence of a λ - point transition at large q. Finally, we point out that the type of deformation proposed here makes the deformation parameter continuous. This fact enabled us to study BEC near $q = 1$, which has not been achieved for the usual deformed Bose-Einstein gas previously investigated.

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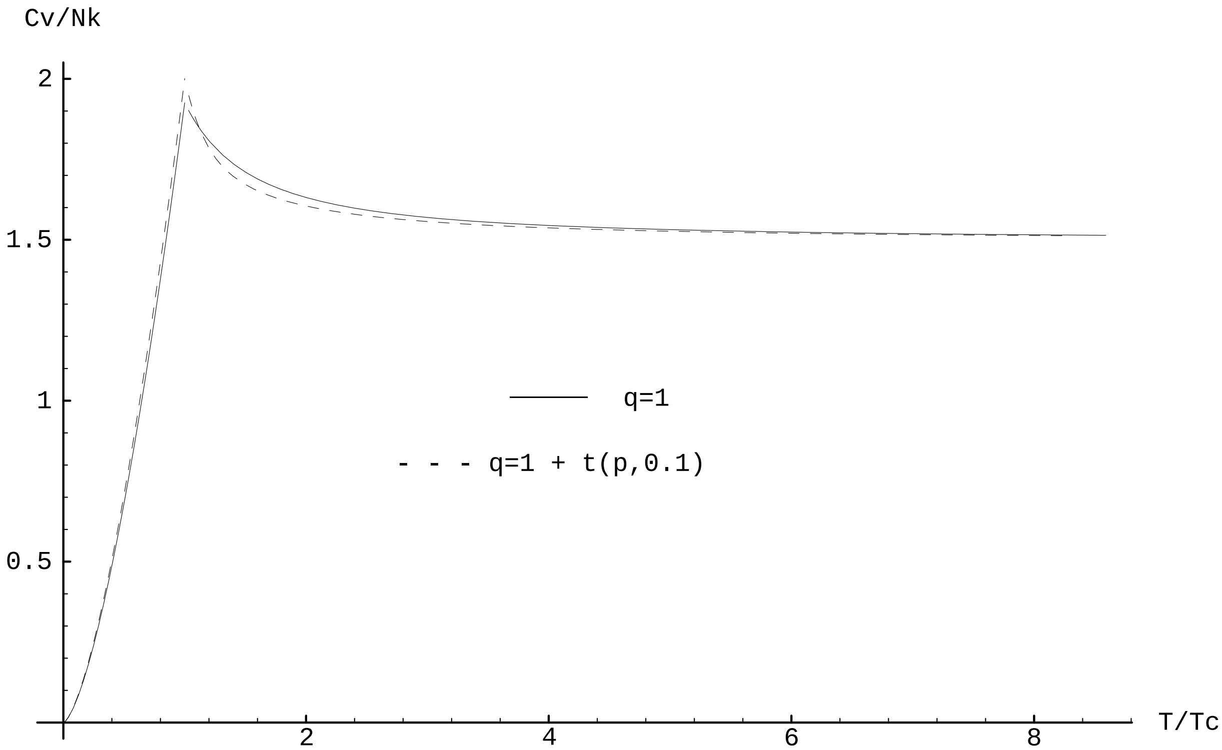

FIG.1. Specific heat of the ideal (q = 1) and deformed (q=1+t(p,0.1))Bose - Einstein gas.

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