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Extended Convergence of Normal Forms Around Unstable Equilibria

by

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Abstract

There is strong numerical evidence that the convergence of normal forms around saddle points of Hamiltonian systems should extend beyond the region originally established by Moser. We show that these normal forms do converge along a neighbourhood of the stable and unstable manifolds emanating from Moser's region if the Hamiltonian is analytical. This allows in principle a fully analytical calculation of homoclinic orbits.

Key-words: Normal forms; Homoclinic orbits.

The Hamiltonian flow around both stable and unstable equilibria is simplified by the normal form transformations derived formally by Birkhoff [1]. Though the former have become a useful tool, it is only for the latter that Moser proved convergence [2]. Even so, Moser's normal forms (MNF) would appear to have little practical relevance, since most orbits spend very little time within the small region of convergence around the equilibrium, because of its instability. Contrary to this assumption, in a recent paper [3] we have used MNF to locate homoclinic orbits emanating from an unstable equilibrium, as well as other features of the surrounding chaotic structure, for a Hamiltonian system with two freedoms (see e.g. [4] for details). In spite of the difficulty of calculating the coefficients to arbitrarily high order, there is clear numerical evidence that the series converge along a neighbourhood of both the stable and the unstable manifold far beyond the limits of Moser's original proof.

The purpose of this paper is to show that Moser's region can indeed be analytically continued in the direction of the flow, if the Hamiltonian is itself analytical. It is true that the width of this region decreases exponentially with the distance from the equilibrium, but it should still allow, in principle, a fully analytical calculation if homoclinic orbits and the periodic orbits that accumulate on then, beyond what was achieved in [3].

We now summarize Moser's theorem for the case of an autonomous hamiltonian system of two degrees of freedom with a saddle point at the origin (see ref. [3] for more details). Let the hamiltonian $H(\underline{r})$ be a real function of $\underline{r} \equiv (q_1, q_2, p_1, p_2)$, where q_1, q_2 are real coordinates and p_1, p_2 are the respective conjugate momenta. $H(\underline{r})$ is analytic in a neighbourhood of the saddle point and it is given there by

$$H(\underline{r}) = \frac{\omega}{2}\,(p_1^2 + q_1^2) + \frac{\lambda}{2}\,(p_2^2 - q_2^2) + N(\underline{r})\ , \tag{1}$$

where $N(\underline{r})$ contains only nonlinear terms (i.e. of order ≥ 3 in the variables) and $\pm i\omega$ and $\pm\lambda$ are the eigenvalues of the linear part of the differential system generated by $H(\underline{r})$.

After performing the symplectic transformation $\sqrt{2}q_1 = x_1 + ix_3$, $-i\sqrt{2}p_1 = x_1 - ix_3$, $\sqrt{2}q_2 = x_2 + x_4$ and $-\sqrt{2}p_2 = x_2 - x_4$, Moser's theorem states that there exists a convergent coordinate transformation, namely MNF, of the form

$$x_i = y_i + \sum_{\ell=2}^{\infty} X(i, \underline{\ell}) \underline{y}^{\underline{\ell}} \quad , \quad i = 1, \cdots, 4 \tag{2}$$

with $\underline{\ell} = (\ell_1, \ell_3, \ell_2, \ell_4)$, $\ell = \ell_1 + \ell_3 + \ell_2 + \ell_4$ and $\underline{y}^{\underline{\ell}} = \underline{y}^{\underline{\ell}} = y_1^{\ell_1} y_3^{\ell_3} y_2^{\ell_2} y_4^{\ell_4}$, such that the normalized flow (i.e. the flow in the coordinates $\underline{y} = (y_1, y_3, y_2, y_4)$) can be immediately integrated to give

$$\begin{aligned} y_1(t) &= \rho \exp[i(\theta_0 + \Omega t)] \\ y_3(t) &= -i\rho \exp[-i(\theta_0 + \Omega t)] \\ y_2(t) &= y_{20} \exp[\Lambda t] \\ y_4(t) &= y_{40} \exp[-\Lambda t] \quad , \quad y_{20} y_{40} = \in \ , \end{aligned} \tag{3}$$

where

$$\begin{aligned} \Omega &= \omega - i \sum_{m_1+m_2=2}^{\infty} m_1 K(m_1, m_2)(-i\rho^2)^{m_1-1}(\in)^{m_2} \\ \Lambda &= \lambda + \sum_{m_1+m_2=2}^{\infty} m_2 K(m_1 m_2)(-\rho^2)^{m_1}(\in)^{m_2-1} \ . \end{aligned} \tag{4}$$

$\rho \leq 0, \in, \Omega, \Lambda, \theta_0, y_{20}$ and y_{40} are real constants and the coefficients $X(i, \underline{\ell})$ and $K(m_1, m_2)$ are consistently determined through appropriate recurrence relations. x_2 and x_4 are also real and $x_3 = i\overline{x}_1$, where $\overline{x}_1$ is the complex conjugate of x_1.

We recall that the conditions $y_{20} = y_{40} = 0$ and $\rho = 0$ parametrize the family of unstable periodic orbits $\{\tau\}$ lying entirely in the plane y_1, y_3. The stable/unstable manifolds originating either from the saddle point ($\rho = 0$) or from an orbit τ are parametrized by $y_{20} \neq 0$, $y_{40} = 0/y_{20} = 0$, $y_{40} \neq 0$. For the saddle point these manifolds are the proper axes y_2/y_4, while for the orbit τ they are semi-infinite cylinders originating from τ. Finally, for $\in \neq 0$ we have a family of orbits either lying entirely in the plane y_2, y_4 (hyperbolae $\rho = 0$) or in an infinite cylinder ($\rho \neq 0$). In any case we see that the motion is separable in the planes y_1, y_3 and y_2, y_4. To this "rectified" motion near the saddle point there corresponds an analogous, yet distorted, motion in the original hamiltonian flow.

Moser's theorem guarantees convergence of the series (2) and (4) within a region

$$(y_1) + (y_3) + (y_2) + (y_4) < a \ , \tag{5}$$

where a is a small positive number. This condition necessarily restricts the constant radius of rotation, $\rho < a/2$. In fact the pair of parametes ε and ρ in (3) are bounded by

$$\rho + \varepsilon^{\frac{1}{2}} < a/2 \ . \tag{6}$$

Now, it is important to note that the series (4) only depend on y through this pair of parameters, so that in effect, if we consider Ω and Λ as functions of y, these series converge for all y such that (6) is satisfied. That is, if there is some y_0 such that (5) holds, then (4) converges for all $y(t)$ in (3) with this initial value, even though $y_2(t)$ or $y_4(t)$ are far out of the range of (3). It is only the series (2) whose convergence we will now try to extend beyond Moser's original range.

Consider the function $r = \chi(y)$, resulting from the composition of (2) with the linear transformation between x and r. This function is analytic within Moser's region (5). Let us now compose this function with the flow f_t generated by the Hamiltonian (1) (i.e. $f_t(r_0) = r(r_0, t), f_0(r_0) = r_0$) and the normalized flow corresponding to (3) and (4) $(F_t(y_0) = y(y_0, t), F_0(y_0) = y_0)$. The total transformation $\mathcal{F}_t = f_t \circ \chi \circ F_{-t}$ maps any point $y(t)$ for which (6) holds onto corresponding points $r(t)$ whose orbits had initial values $r_0 = \chi(y_0)$. If t is sufficiently small, such that $y(t)$ still lies in Moser's region (5), then $\mathcal{F}_t(y(t)) = \chi(y(t))$. Beyond this range, $\mathcal{F}_t$ can be viewed as an extension of χ.

Since F_t is analytic in (6) and χ is analytic in (1), $\mathcal{F}_t$ will also be analytic if f_t is analytic. When dealing directly with maps, this property is manifestly true or false, so that in the former case we can immediately use analytic continuation [5]. Here, we must rely on the theorem of existence and uniqueness of the solutions of analytic differential systems [6, 7]. This establishes that the phase space coordinates are analytic functions $r(t)$ for a finite period, within a region where the vector field is itself analytic. Cartan [6] then proves the corollary that, for a sufficiently small time t, the flow $r(0) \to r(t)$ is also analytic.

Evidently, we may extend the range of analyticity by repeatedly iterating Cartan's theorem as long as the flow starting in a given region only visits finite regions where the Hamiltonian vector field is analytic and bounded. (The difficulties with a possible escape of the motion after a finite time only arise as the motion moves into regions where the field is unbounded). Since the branches of the stable and unstable manifolds that intersect in primary homoclinic points are thus limited, we can assume that the mapping generated by the flow, taking any neighbourhood of Moser's region near the origin to a specified neighbourhood of the unstable manifold, is analytic. Of course, a full proof would require a strict discussion of the bounds in the application of the existence and uniqueness theorem.

Consider now the complex analytic function $\mathcal{F}_t$ defined in the range (6), where $\varepsilon = (y_2 y_4)$ and $\rho = (y_1) = (y_3)$. Any point in this range can be included in a polydisc centred on the origin, within which the Taylor series for $\mathcal{F}_t$ converges. Since this polydisc overlaps with Moser's region (5), where $\mathcal{F}_t = \chi$, the Taylor series for $\mathcal{F}_t$ must coincide with MNF. Not only can the coordinate transformation defined by the MNF be analytically continued beyond Moser's original region, but we have now shown that the region of convergence of the series itself about the origin can be likewise extended.

We are now able to understand the reason for the usefulness of the MNF [3], far beyond the region where it was guaranteed to converge. All indications are that, for sufficiently analytic potentials, we can obtain the stable and unstable manifolds emanating from the unstable equilibrium and from the family of unstable periodic orbits with slightly higher energies. These can be followed out to their homoclinic intersections, allowing us in principle to calculate analytically the homoclinic orbits and the periodic orbits of arbitrary large period that accumulate on them. The difficulty, as evidenced in [3], is that the further out one proceeds along each of these manifolds, the more terms must be calculated for the MNF. A fully analytical computation of typical chaotic structures is thus only feasible with sophisticated computational techniques.

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