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A DIRECT SUM IS HOLOMORPHICALLY BORNOLOGICAL WITH THE TOPOLOGY INDUCED BY A CARTESIAN PRODUCT

by

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Dedicated to the memory of Antonio Aniceto Ribeiro Monteiro, for his valuable contribution to progress of Logic and Mathematics in Latin America and Portugal.

ABSTRACT. Let the direct sum $\mathbb{C}^{(I)}$ have the topology induced by the cartesian product $\mathbb{C}^{I}$, where I is a set. It is shown that $\mathbb{C}^{(I)}$ is holomorphically bornological, hence holomorphically infrabarreled; but it is holomorphically barreled if and only if I is finite. It is known [1], [5] that, if $\mathbb{C}^{(I)}$ has the inductive limit topology, then it is holomorphically bornological, or holomorphically infrabarreled, or holomorphically barreled, if and only if I is countable.

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INTRODUCTION 0. Letting I be a set, it is known that $\mathbb{C}^I$ with its cartesian product topology, is bornological if and only if the following equivalent conditions hold: (HN) I is a Hewitt-Nachbin space; (U) there is no Ulam measure on I. See [9], [3], [6], [4]. These conditions are known to be satisfied if the cardinal number of I is not too big. The problem whether they hold in general seems to be unsolved. If a space is holomorphically bornological, then it is bornological, but not conversely. See [7], [8], [1]. Thus the above equivalent conditions (HN) and (U) are necessary for $\mathbb{C}^I$ to be holomorphically bornological. On the other hand, the direct sum $\mathbb{C}^{(I)}$ with its inductive limit topology is always bornological. It is holomorphically bornological if and only if I is countable. See [1], [5]. In the present note, we want to show that $\mathbb{C}^{(I)}$ is always holomorphically bornological with the topology induced by $\mathbb{C}^I$. This is an offshoot of an attempt by us to prove that $\mathbb{C}^I$ is always bornological. What we prove in this article is not the most general result corresponding to its title (rather the simplest such one), but it is enough to illustrate the idea of the proof. On the other hand, we plan later to reconsider this matter, in the more general setting that an inductive limit is holomorphically bornological with the topology induced by a projective limit.

NOTATION 1. We let $fs(I)$ be the set of all finite subsets of I. If $J \subset I$, then $p_J : \mathbb{C}^I \to \mathbb{C}^I$ is the projection defined as follows: $p_J(x) \in \mathbb{C}^I$ has all i-coordinate equal to the i-coordinate of $x \in \mathbb{C}^I$ for $i \in J$, and equal to 0 for $i \in I-J$. By D_i we denote an i-partial derivative of first order for $i \in I$. Let F be a complex Hausdorff locally convex space, and U be open and nonvoid in some complex locally convex space. By $\mathcal{H}(U;F)$ we denote the vector space of all holomorphic mappings of U to F; and by $\mathcal{H}_a(U;F)$ the vector space of all algebraically holomorphic mappings of U to F.

LEMMA 2. Let K be compact in $\mathbb{C}^I$. Then $L = \bigcup_{J \subset I} p_J(K)$ is compact in $\mathbb{C}^I$.

PROOF. We consider the continuous pointwise multiplication mapping $(x,y) \in \mathbb{C}^I \times \mathbb{C}^I \mapsto xy \in \mathbb{C}^I$. If $J \subset I$, let $\mathbf{1}_J$ be the characteristic function of J in I. The set C of all such characteristic functions is compact in $\mathbb{C}^I$. Note that $p_J(x) = \mathbf{1}_J x$ if $J \subset I$, $x \in \mathbb{C}^I$, hence L is the image of $C \times K$ by that continuous mapping. QED

LEMMA 3. Let $\mathbb{C}^{(I)} \subset S \subset \mathbb{C}^I$, where S is a vector subspace of $\mathbb{C}^I$ endowed with the induced topology. If $U \subset S$ is open, nonvoid and connected, F has some continuous norm, and $\mathfrak{X} \subset \mathcal{H}(U;F)$ is bounded on every compact subset of U, then there is $H \in fs(I)$ such that $D_i f = 0$ for every $f \in \mathfrak{X}$, $i \in I - H$.

The proof is the same as the known one when $S = \mathbb{C}^I$. See [2].

LEMMA 4. Let $\mathbb{C}^{(I)}$ be endowed with the topology induced by that of $\mathbb{C}^I$. If $U \subset \mathbb{C}^{(I)}$ is open, nonvoid and connected, F has some continuous norm, and $\mathfrak{X} \subset \mathcal{H}_a(U;F)$ is bounded on every compact subset of U, then there is $H \in fs(I)$ such that $D_i f = 0$ for every $f \in \mathfrak{X}$, $i \in I - H$.

PROOF. Without loss of generality, we may assume that $0 \in U$. If $f : U \to F$, $J \subset I$, then $f \circ p_J$ is defined precisely on $p_J^{-1}(U)$. Let V be a connected open subset of $\mathbb{C}^{(I)}$ such that $0 \in V \subset U$, and $p_J(V) \subset V$ for every $J \subset I$. We then have $V \subset p_J^{-1}(U)$ if $J \subset I$, because $p_J(V) \subset V \subset U$. Thus $f \circ p_J$ is defined at least on V if $f : U \to F$,

$J \subset I$. Now, if $f \in \mathcal{H}_a(U;F)$ and $J \in fs(I)$, we have that $(f \circ p_J)|V \in \mathcal{H}(V;F)$. Every compact subset K of V is contained in some compact subset L of V such that $p_J(L) \subset L$ for every $J \subset I$, by Lemma 2. It follows that the family $(f \circ p_J)|V$ $(f \in \mathfrak{X},\ J \in fs(I))$ is bounded on every compact subset of V. By Lemma 3, there is $H \in fs(I)$ such that

1) $D_i(f \circ p_J)|V = 0$

if $f \in \mathfrak{X}$, $J \in fs(I)$, $i \in I - H$. Call V_J the connected component of $p_J^{-1}(U)$ containing V, for $J \subset I$. By uniqueness of holomorphic continuation, (1) gives

2) $D_i(f \circ p_J)|V_J = 0$

if $f \in \mathfrak{X}$, $J \in fs(I)$, $i \in I - H$. If $x \in U$, we can join 0 to x by a finite polygonal Γ contained in U. The union J_0 of the supports of all $t \in \Gamma$ is finite (the support of $t \in \mathbb{C}^{(I)}$ being the smallest finite subset of I outside which all coordinates of t do vanish). Therefore, if $t \in \Gamma$, $J_0 \subset J \subset I$, we have $p_J(t) = t$; since $t \in \Gamma \subset U$, then $p_J(t) \in U$, that is $t \in p_J^{-1}(U)$. Hence, $\Gamma \subset p_J^{-1}(U)$ if $J_0 \subset J \subset I$. It follows that $\Gamma \subset V_J$, thus $x \in V_J$, if $J_0 \subset J \subset I$. If we then choose $J \in fs(I)$, $J_0 \subset J$, $p_J(x) = x$ because $x \in \mathbb{C}^{(I)}$, we get from 2) that

$$(D_i f)(x) = (D_i f)[p_J(x)] = 0$$

if $f \in \mathfrak{X}$, $i \in I - H$, that is $D_i f = 0$ on U if $f \in \mathfrak{X}$, $i \in I - H$. QED

PROPOSITION 5. $\mathbb{C}^{(I)}$ is holomorphically bornological when it is endowed with the topology induced by $\mathbb{C}^I$.

PROOF. Let U be a nonvoid open subset of $\mathbb{C}^{(I)}$, and $f \in \mathcal{H}_a(U;F)$ be bounded on every compact subset of U. We want to conclude that $f \in \mathcal{H}(U;F)$. Without loss of generality we may assume U connected and F normed. By Lemma 4, there is $H \in fs(I)$ such that $D_i f = 0$ if $i \in I - H$. It follows that f is continuous, hence $f \in \mathcal{H}(U;F)$. QED

REMARK 6. Under the assumption of Proposition 5, it follows from it that $\mathbb{C}^{(I)}$ is holomorphically infrabarreled [7], [8], [1]. However, $\mathbb{C}^{(I)}$ is holomorphically barreled [7], [8], [1] if and only if I is finite, because it is not barreled if I is infinite; in fact, if ϕ_i is the continuous linear form on $\mathbb{C}^{(I)}$ assigning to each point its i-coordinate for $i \in I$, then the family $(\phi_i)_{i \in I}$ is pointwise bounded, but it is not equicontinuous, on $\mathbb{C}^{(I)}$.

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