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One-dimensional Potts model with long-range interactions: a renormalization group approach

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The one-dimensional q-state Potts model with ferromagnetic pair interactions which decay with the distance r as $1/r^{\alpha}$ is considered. We calculate, through a real-space renormalization group technique using Kadanoff blocks of length b, the critical temperature $T_c(b, q, \alpha)$ and the correlation-length critical exponent $\nu(b, q, \alpha)$ as a function of α for different values of q. Some of the very few known rigorous results for general q are reproduced by our approach. Several asymptotic behaviours are derived analytically for $q = 2$ and 3 in the $b \to \infty$ limit. We also obtain extrapolated critical temperatures ($b = \infty$) for arbitrary values of $\alpha > 1$ and for $q = 2, 3$ and 4, which we believe approximate well the exact ones. In particular, we conjecture that the **exact** critical temperature $T_c(q, \alpha = 2)$ is the same for **any value of** q. Furthermore, we verified that $T_c(q, \alpha \to 1) \propto (\alpha - 1)^{-1}$ $\forall q$, which is consistent with a recent conjecture of Tsallis.

Key-words: Potts model; Long-range interactions; Renormalization group.

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I. INTRODUCTION

It has been known for a long time that one-dimensional spin models can present an ordered state at low temperatures if the microscopic interactions fall off slowly enough with the distance [1]$^-$ [4]. Moreover, the thermodynamic properties of this kind of systems near the critical point present frequently new behaviours, which are absent in short-range (SR) models. Hence, it is of interest the study of such properties in order to gain a deeper comprehension of the general theory of critical phenomena.

Besides their fundamental theoretical interest in physics, microscopic models with long-range (LR) interactions are of interest nowadays in view of their relationship with neural systems modeling [5], where far away localized neurons interact through an action potential which decays slowly along the axon. Other related problems are, for example, spin systems with RKKY like interactions ($1/r_{ij}^{\alpha}\ cos(ar_{ij})$) which are present in spin glasses [6], critical phenomena in highly ionic systems [7], Casimir forces between inert uncharged particles immersed in a fluid near the critical point [8] and the kinetic Ising model with random spin exchanges (Lévy flights) [9].

In this paper we address the q-state Potts model [10] with LR interactions, *i.e.*, we consider the Hamiltonian:

$$H = -J \sum_{(i,j)} \frac{1}{r_{ij}^{\alpha}}\, \delta(\sigma_i \sigma_j) \quad (\sigma_i = 1, 2, \ldots, q,\ \forall\ i;\ J > 0;\ \alpha > 0) \tag{1}$$

where r_{ij} is the distance (in crystal units) between sites i and j (*i.e.*, $r_{ij} =\mid i - j \mid = 1, 2, 3 \cdots$), and where the sum $\sum_{(i,j)}$ runs over all distinct pairs of sites of a one-dimensional lattice of N sites. The $\alpha \to \infty$ limit corresponds to the first-neighbor model,while the $\alpha = 0$ limit corresponds to the infinite range ferromagnet which, after a rescaling $J \to J/N$, yields basically the Mean Field Approach.

This model, in its plain formulation ($\alpha \to \infty$ of Eq.(1)) or in a more general one with many-body interactions, is at the heart of a complex network of relations between geometrical and/or thermal statistical models, like for example various types of percolation, vertex models, generalized resistor and diode network problems, classical spin models, etc (see [11] and references therein).

On the other hand, the one-dimensional Potts model with LR interactions has been much less studied. In particular, very few rigorous results for general q are known. Let us summarize some of the most relevant results up to the present: *i)* this model exhibits long-range order at *finite* temperatures [12] $T \leq T_c(q, \alpha)$ for $1 < \alpha \leq 2$; for $\alpha \to 1$ the critical temperature diverges and for $\alpha \leq 1$ the thermodynamic limit is not defined and the system becomes non-extensive; *ii)* for $\alpha > 2$ (short-range interactions) it has no phase transition at *finite* temperatures [12] for all $q \geq 1$, more precisely, $T_c = 0$; *iii)* it has been proved that for $\alpha = 2$ the order parameter is discontinuous at $T = T_c \neq 0$ for any q [12]; *iv)* for $q = 1$ the percolation threshold satisfies $1/p_c \leq 2\zeta(\alpha)$ for $1 < \alpha \leq 2$, where $\zeta(\alpha)$ is the Riemann Zeta function [13].

The following additional results correspond all to the $q = 2$ case, which is, up to now, the best studied one: *v)* for $1 < \alpha < 1.5$ the critical exponents are classical [14]; *vi)* the region $1.5 < \alpha < 2$ shows non-trivial critical exponents, which are not known exactly. Approximate results in the latter region were obtained by different methods such as (among others): series expansions [15], finite range scaling approximations [16], coherent anomaly method [17], real-space renormalization group [18], ϵ-expansions [2,4] around $\alpha = 1.5$ and $\alpha = 2$ where the critical behaviour is of essential singularity type [19].

Some approximate results for the critical temperature and the correlation length critical exponent ν were obtained for a wide range of values of q using finite-range-scaling calculations [20].

The $\alpha = 2$ (*i.e.*, the $1/r^2$ potential) case is of particular interest because for $q = 2$ it can be mapped into the spin 1/2-Kondo problem [21] (which is related with recent developments in high temperature superconductivity [22]) and for a general value of $q > 2$ it may be related to higher spin generalizations of the Kondo problem [19].

In order to calculate the critical temperature and the critical exponent ν of the q-state LR Potts model in the extensive region $1 < \alpha \leq 2$ we use a real-space renormalization group method (RG), based on a construction of Kadanoff-blocks using the majority-rule. In a recent work [18] , one of us adapted the well known Niemeijer-van Leeuwen's RG recipes [23] to the one-dimensional LR Ising model. In the case of $q = 2$ states the tie-breaking problem in the majority rule can be easily avoided by considering only blocks with an odd number of sites; for $q \geq 3$ this ansatz does not work. Hence, in this paper we generalize the above technique by introducing an *equally-probable tie-breaking majority rule*. We expect that this method gives good results for the Potts model, as far as the phase transition is a second order one [11]. The outline of this paper is as follows. The general RG formalism is described in Sec.II. In Sec.III we present our results, which recover those of Ref.[18] for $q = 2$. Finally, the conclusions are given in Sec.IV.

II. THE RG FORMALISM

We start by constructing Kadanoff-blocks of length $b > 1$, as shown in Fig. 1 for the particular case $b = 3$; we will consider, for simplicity, only odd values of b herein. The parameter b characterizes the *rescaling length* of the RG transformation. The blocks will be numbered by capital letters. We will assign a block-spin variable $\sigma'_I = 1, 2, \ldots, q$ to every block I. Let us denote by $\sigma_i^I = 1, 2, \ldots, q$ ($i = 1, 2, \cdots, b$; $I = 1, 2, \cdots, N/b$) the spin state at the i^{th} site of the block I. Then, defining the dimensionless Hamiltonian

$$\mathcal{H} \equiv -K \sum_{I=1}^{N/b} \sum_{J=1}^{N/b} \sum_{i\in I} \sum_{j\in J} \frac{1}{r_{ij}^{\alpha}}\, \delta(\sigma_i^I, \sigma_j^J) \quad (i \neq j) \tag{2}$$

with $K \equiv \beta J$ ($\beta = 1/k_B T$; hereafter we take $k_B = 1$), a renormalized (block) Hamiltonian is determined by the following RG transformation:

$$e^{-(\mathcal{H}'+\mathcal{C})} = \mathrm{Tr}_{\{\sigma_i^I\}} \left\{ P\left(\{\sigma_i^I\}, \{\sigma'_I\}\right) e^{-\mathcal{H}} \right\}. \tag{3}$$

The symbol $\mathrm{Tr}_{\{\sigma_i^I\}}$ denotes a sum over all the configurations of site-spins σ_i^I, $\mathcal{C}$ is a spin independent constant and

$$P\left(\{\sigma_i^I\}, \{\sigma'_I\}\right) = \prod_{I=1}^{N/b} P_I\left(\{\sigma_i^I\}, \sigma'_I\right) \tag{4}$$

is a weight function which characterizes the majority-rule with **equally probable tie-breaking**, that means:

$$P_I = \begin{cases} 1/m & \text{if one of the } m \text{ major subgroups of } \left\{\sigma_i^I\right\} \text{ is in the state } \sigma'_I \\ 0 & \textit{otherwise} \end{cases} . \tag{5}$$

For instance, in the case $b = 5$, $q = 4$ with $\left\{\sigma_i^I\right\} = \{1, 1, 4, 4, 3\}$ and $\sigma'_I = 4$ then $P_I = 1/2$.

The Hamiltonian $\mathcal{H}$ can be divided into two parts: $\mathcal{H} = \mathcal{H}_0 + V$, where $\mathcal{H}_0 = \sum_I \mathcal{H}_0^I$ and $V = \sum_{(I,J)} V_{IJ}$; $\mathcal{H}_0^I$ includes only the interactions between spins *inside* the block I, whereas V_{IJ} includes the interactions between spins belonging to *different* blocks I and J. Introducing the intra-block expectation values:

$$\langle \mathcal{O} \rangle_0 \equiv \frac{1}{Z_0} \mathrm{Tr}_{\{\sigma_i^I\}} \left\{ P\left(\{\sigma_i^I\}, \{\sigma'_I\}\right) exp\left[-\mathcal{H}_0\left(\{\sigma_i^I\}\right)\right] \mathcal{O} \right\} \tag{6}$$

with

$$Z_0 \equiv \mathrm{Tr}_{\{\sigma_i^I\}} P\left(\{\sigma_i^I\}, \{\sigma'_I\}\right) exp\left[-\mathcal{H}_0\left(\{\sigma_i^I\}\right)\right] \tag{7}$$

we can rewrite Eq.(3) as:

$$e^{-(\mathcal{H}'+\mathcal{C})} = Z_0 \ \left\langle e^{-V} \right\rangle_0 . \tag{8}$$

Using a cumulant expansion of $\left\langle e^{-V} \right\rangle_0$, a *first order* approximation of $\mathcal{H}'$ can be obtained through:

$$\mathcal{H}' \approx \langle V \rangle_0 \big|_{sdp} = \sum_{(I,J)} \langle V_{IJ} \rangle_0 \big|_{sdp} \tag{9}$$

where *sdp* refers to the spin dependent part on $\{\sigma'_I\}$ of the resulted average.

Let r_{IJ} be the distance between the center sites of the blocks I and J (see Fig. 1), measured *in units of the rescaling length b*. For $r_{IJ} >> 1$ we can approximate [18]

$$r_{ij} \approx b\, r_{IJ}. \tag{10}$$

Then

$$\langle V_{IJ} \rangle_0 \big|_{sdp} \approx \ -\frac{K}{b^\alpha \ r_{IJ}^\alpha} \sum_{i\in I} \sum_{j \in J} \left\langle \delta(\sigma_i^I, \sigma_j^J) \right\rangle_0 \Big|_{sdp} \tag{11}$$

Since the expectation value (6) is carried out with a block-independent probability distribution it follows that

$$\begin{aligned} \left\langle \delta(\sigma_i^I, \sigma_j^J) \right\rangle_0 &= \sum_{l=1}^{q} \left\langle \delta(\sigma_i^I, l) \delta(\sigma_j^J, l) \right\rangle_0 \qquad (12) \\ &= \sum_{l=1}^{q} \left\langle \delta(\sigma_i^I, l) \right\rangle_0 \left\langle \delta(\sigma_j^J, l) \right\rangle_0 \qquad (13) \end{aligned}$$

.

On the other hand, by symmetry, one has that:

$$\left\langle \delta(\sigma_i^I, l) \right\rangle_0 = a_i(K, \alpha, q) \delta(\sigma'_I, l) + b_i(K, \alpha) \tag{14}$$

where a_i and b_i are block-independent functions of K, α, q and the site i. Combining Eq.(14) with Eqs.(9), (11) and (13), and using that $\sum_{l=1}^{q} \delta(\sigma'_I, l) = 1$ and $\sum_{l=1}^{q} \delta(\sigma'_I, l) \delta(\sigma'_J, l) = \delta(\sigma'_I, \sigma'_J)$ one gets that:

$$\mathcal{H}' = -K'_b(K, q, \alpha) \sum_{(I,J)} \frac{1}{r_{IJ}^\alpha} \ \delta(\sigma'_I, \sigma'_J) \tag{15}$$

where

$$K'_b(K,q,\alpha) = \frac{K}{b^\alpha}\left[\sum_{i=1}^{b} a_i(K,\alpha,q)\right]^2 \qquad (16)$$

is our RG recurrence equation. Using Eq.(14) we can express $a_i(K,\alpha,q)$ as:

$$a_i(K,\alpha,q) = \frac{1}{q-1}\left[q\,\left\langle\delta(\sigma_i^I,1)\right\rangle_0\Big|_{\sigma'_I=1} - 1\right]. \qquad (17)$$

Since the n^{th} cumulant of $\langle exp(-V)\rangle$ is of order $1/b^{n\alpha}$, the approximation (9) can be seen as the leading term in a series expansion [18] of Eq.(3) in powers of $1/b^\alpha$. Therefore, it is expected that the results will be systematically improved for increasingly high values of b.

III. RESULTS

A. Analysis of the recurrence equation

We now analyze the recurrence equation (16) and its fixed points $K^* = K'_b(K^*,q,\alpha)$ as a function of α for different values of $q \geq 2$. The typical structure of Eq.(16) is as follows. It always shows two trivial fixed points: $K = 0$ ($T = \infty$) and $K = \infty$ ($T = 0$). From Eq.(17) we found that $a_i(K,\alpha,q) \sim 1$ $\forall i,q,\alpha$ for $K >> 1$ ($T \to 0$); hence, from Eq.(16) we obtain the asymptotic behaviour $K'_b(K,q,\alpha) \sim b^{2-\alpha}K$ $\forall q$. For low values of α the slope of $K'_b(K,q,\alpha)$ at $K = 0$ is greater than one and it does not present a (non-trivial) fixed point for finite values of K. In this case the fixed point $K = 0$ is repulsive and therefore $T_c = \infty$. For intermediate values of α, $K'_b(K,q,\alpha)$ possess a non-trivial fixed point at finite $K = K_c(b,q,\alpha) \equiv J/T_c(b,q,\alpha)$. For $\alpha > 2$ the slope of $K'_b(K,q,\alpha)$ is less than one and there is again no fixed point at finite K. In this case, however, the fixed point $K = 0$ is attractive and therefore $T_c = 0$ for all values of b, recovering the exact result. Summarizing: there exists some value $\alpha_1(b,q)$ such that *i)* $T_c = \infty$ for $\alpha \leq \alpha_1(b,q)$; *ii)* there is a phase transition at finite temperature $T_c(b,q,\alpha)$ for $\alpha_1(b,q) < \alpha < 2$ and *iii)* $T_c = 0$ for $\alpha > 2$.

The borderline value $\alpha_1(b,q)$ is determined by the condition $dK'_b/dK\mid_{K=0}= 1$. This equation can be solved by noting that

$$a_i(0,\alpha,q) = \gamma(b,q) \equiv \frac{1}{q-1}\left[q^{2-b}\sum_{m=1}^{m_{max}}\frac{A_m(b,q)}{m} - 1\right] \qquad (\forall i). \qquad (18)$$

The coefficient A_m gives the number of configurations of b spins (where each one can be in the states $\sigma_i^I = 1,2,\ldots,q$) of a block where one of the m major subgroups of $\{\sigma_i^I\}$ is in a fixed state, say 1, and $\sigma_1^I = 1$.

From Eq.(16) we obtain that $dK'_b/dK\mid_{K=0}= b^{2-\alpha}\gamma(b,q)^2$ and therefore

$$\alpha_1(b,q) = 2\left[1+\frac{ln\ \gamma(b,q)}{ln\ b}\right].$$

For $q=2$ we have that [18]

$$\gamma(b,2) = \frac{(b-1)!}{2^{b-1}\ \left(\frac{b-1}{2}\ !\right)^2}$$

which for $b >> 1$ behaves as $\gamma(b,2) \sim 2\sqrt{2/\pi}\ b^{-1/2}$ and we recover the exact result $\alpha_1(b,2) \to 1$ in the limit $b\to\infty$.

For higher values of q the calculation of the quantities A_m involves a lot of combinatorial analysis. For $q=3$ we found the following expression:

$$\gamma(b,3) = \frac{1}{2}\left\{3^{2-b}\left[\sum_{l=0}^{X}\binom{b-1}{l}2^l + \sum_{j=1}^{Int(X/3)}\sum_{j_1=2j}^{X-j}\binom{b-1}{X+j}\binom{X+j}{j_1}\right.\right.$$
$$\left.\left. + \sum_{l=0}^{Int(\frac{X-2}{3})}\binom{b-1}{X+l+1}\binom{X+l+1}{X-l} + \frac{1}{3}\binom{b-1}{2b/3}\binom{2b/3}{b/3}\delta(b,3n)\right] - 1\right\} \quad (19)$$

where $n=1,2,\ldots$, $X\equiv(b-1)/2$ and *Int(...)* represents the integer part of its argument. This form can be easily evaluated numerically for values up to $b\sim 200$. An analysis of the log-log plot of γ *vs* b shows that the asymptotic regime is attained for low values of $b\sim 7$ and clearly $\gamma(b,3)\sim b^{-1/2}$ for $b\to\infty$. Therefore, $\alpha_1(b,3)$ also reproduces the exact result in such a limit. For values of $q\geq 4$ the combinatorial problem becomes very hard. However, we performed a numerical calculation of $\gamma(b,q)$ for $q=4,5$ and $b=3,5,7,9$, finding again $\gamma(b,q)\sim b^{-1/2}$. All these results suggest that $\alpha_1(b,q)\to 1$ for $b\to\infty$ *for all values of* $q\geq 2$.

Closed forms of the function $K'_b(K,q,\alpha)$ can be obtained analytically for low values of q and b with the aid of symbolic computer languages. With these expressions the critical temperature $T_c(b,q,\alpha)$ can be calculated numerically as a function of α for fixed values of q and b. The correlation length critical exponent can also be calculated from the expression

$$\nu(b,q,\alpha) = \frac{ln\ b}{ln\left(\frac{dK'_b}{dK}(K,q,\alpha)\mid_{K_c}\right)}. \quad (20)$$

In Fig.2 we show our results for different values of q and $b=3$ fixed, while in Fig.3 we kept $q=3$ fixed and varied b. The corresponding curves for other values of q are qualitatively similar.

B. $\alpha \to 2^-$ asymptotic results

For $\alpha \to 2^-$ we see that $K_c \to \infty$ ($T_c \to 0$). The asymptotic behaviour of the recurrence equation (16) in such limit can be obtained by adding an external field h into the Hamiltonian (2), *i.e.*, $\mathcal{H}_0^I \to \mathcal{H}_0^I + h\sum_{i=1}^b \delta(\sigma_i^I, 1)$. Then, in the $h \to 0$ limit, it is easy to prove that

$$\sum_{i=1}^{b} \left\langle \delta(\sigma_i^I, 1) \right\rangle_0 = \left. \frac{\partial ln\ Z_0^I}{\partial h} \right|_{h=0} \tag{21}$$

For $K \to \infty$ we can expand

$$Z_0^I(K,h) \sim e^{B_1(b,\alpha)K + bh} \left[1 + 2(q-1)e^{-B(b,\alpha)K - h} + \cdots \right] \tag{22}$$

where $KB_1(b,\alpha)$ and $KB(b,\alpha)$ are the respective energy of the ground state and the energy difference between the ground state and the first excited state of $\mathcal{H}_0^I$. These are given by

$$B_1(b,\alpha) = \sum_{(k,j)} \frac{1}{r_{kj}^\alpha} = \sum_{n=1}^{b-1} \frac{b-n}{n^\alpha} \tag{23}$$

$$B(b,\alpha) = \sum_{n=1}^{b-1} \frac{1}{n^\alpha} \tag{24}$$

Then, from Eqs.(21) and (22) we obtain in the $K \to \infty$ limit, that

$$\sum_{i=1}^{b} \left\langle \delta(\sigma_i^I, 1) \right\rangle_0 \approx b - 2(q-1)e^{-B(b,\alpha)K} \tag{25}$$

which combined with Eqs.(16) and (17) leads to

$$K_b'(K,q,\alpha) \sim \frac{K}{b^\alpha} \left[b - 2q\ e^{-B(b,\alpha)\ K} \right]^2 \qquad (K \to \infty). \tag{26}$$

Therefore, for $\alpha \to 2^-$ the asymptotic behaviour of $T_c(b,q,\alpha)$ is given by the Cauchy function:

$$2 - \alpha \sim D(q,b)\ e^{-B(b,2)\ J/T_c} \tag{27}$$

with $D(q,b) = 4q/(b\ ln\ b)$. In the $b \to \infty$ limit we have $B(b,2) \to \zeta(2) = \pi^2/6$ and $D(q,b) \to 0$. Since $D(q,b)$ determines the region around $\alpha = 2$ in which the asymptotic regime (27) holds, the shrinking of such a region in the $b \to \infty$ limit suggests a non-uniform convergence to a *finite* value $T_c(\alpha = 2) \neq 0$, consistently with the exact result for $q = 1, 2$.

Such value can be estimated, for fixed q, as the value of $T_c(b,q,\alpha)$ at the inflection point of the Cauchy function (27) for finite b and then taking the $b \to \infty$ limit. This procedure gives the value

$$T_c(q,\alpha=2)/J = B(\infty,2)/2 = \pi^2/12 \tag{28}$$

for all values of $q \geq 2$. It is worth noting that there exists some degree of arbitrariness in the choice of the inflection point for the estimation of $T_c(q,\alpha=2)$. Actually, the present procedure is almost the same as the one introduced in Ref.[18] for the Ising model, the only difference being such a criterium. A careful comparison between (28) and the corresponding values obtained by others methods showed that the choice of the inflection point is better than the previous one [24]. In particular, for the Ising model $q=2$ (remember that $(T_c/J)^{Ising} = 2\ \left.(T_c/J)^{Potts}\right|_{q=2}$) we have $T_c(2,\alpha=2)/J = \pi^2/6 \approx 1.64$, which compares well with other results (renormalization group [21]: 1.57; series expansions [15]: 1.63; finite range scaling [16]: 1.68; ζ function [25]: 1.69).

C. High temperature asymptotic results

For $\alpha \to \alpha_1^+(b,q)$ we see that $K_c \to 0$. Then, expanding the fixed point equation $K_c = K_b'(K_c,q,\alpha)$ around $K_c=0$ and $\alpha=\alpha_1$ up to first order in K_c and in $(\alpha-\alpha_1)$ we find after some algebra:

$$\alpha - \alpha_1 \sim C(b,q)\ K_c$$

where

$$C(b,q) = \frac{2\ B_1(b,\alpha_1)}{b\ lnb}\ \frac{G(b,q)\ q^{2-b} - \frac{b}{q}\left[1+(q-1)\gamma(b,q)\right]}{(q-1)\gamma(b,q)} \tag{29}$$

and

$$G(b,q) = \mathrm{Tr}_{\{\sigma_i^I\}}\, P_I\left(\{\sigma_i^I\},1\right)\,\delta(\sigma_k^I,\sigma_j^I)\sum_{i=1}^{b}\delta(\sigma_i^I,1). \tag{30}$$

For $q=2$ the above expression reduces to

$$C(b,2) = \frac{B_1(b,\alpha_1)}{b\ ln\ b}\ \frac{\begin{pmatrix} b-2 \\ \frac{b-3}{2}\end{pmatrix}}{\gamma(b,2)\ 2^{b-2}}$$

Since $B_1(b,\alpha_1) \sim b\, lnb$ for $b \to \infty$, we obtain that $\lim_{b\to\infty} C(b,2) = 1$. Therefore, we find in the limit $b \to \infty$ that

$$T_c(2,\alpha)/J \sim \frac{1}{\alpha - 1} \tag{31}$$

which reproduces known results(see Ref.[18] and references therein). It is worth stressing that expression (31) recovers asymptotically the mean field one [26].

For $q = 3$ a closed form of $G(b,q)$ (and therefore of $C(b,q)$) can also be analytically obtained. The detailed form of it can be seen in the Appendix. We found numerically that $C(b,3) \to constant \approx 0.67 \approx 2/3$ for $b \to \infty$. All these results suggest that $C(\infty,q) = 2/q$ and that the asymptotic behaviour

$$T_c(q,\alpha)/J \sim \frac{2/q}{\alpha - 1} \tag{32}$$

for $\alpha \to 1$ holds **for all values of** $q \geq 2$, provided that the phase transition is a second order one. Eventually it might hold also for $q = 1$, which would be consistent with Schulman's bound [13] $1/p_c \leq 2\zeta(\alpha)$ ($\zeta(\alpha) \sim 1/(\alpha - 1)$ for $\alpha \to 1$). It is interesting to compare this result with the mean field one, which predicts a first order phase transition for $q \geq 3$. We calculated the corresponding critical temperature following along the lines of Mittag and Stephen [27], namely

$$T_c^{MF}/J = \frac{q-2}{q-1}\frac{1}{ln(q-1)}\zeta(\alpha)$$

which for $q \gg 1$ and $\alpha \to 1$ behaves as

$$T_c^{MF}/J \sim \frac{1}{ln(q)}\frac{1}{\alpha - 1}$$

which differs from expression (32).

It is worth stressing that the asymptotic behaviour (32) agrees with Tsallis' proposal [28] for unifying in a single picture both short- and long-range interaction systems. This proposal has been recently verified for Lennard-Jones like potential systems [29,30] as well as for ferromagnetic Ising models [26].

Finally, using the same expansion of $K_c = K'_b(K_c, q, \alpha)$ around $K_c = 0$ and $\alpha = \alpha_1$, and combining it with Eq.(20) we find that

$$\nu(b,q,\alpha) \sim \frac{1}{\alpha - \alpha_1} \quad , \quad (\forall b, q). \tag{33}$$

For $q = 2$ the mean field behaviour $\nu = 1/(\alpha - 1)$ holds for $1 < \alpha < 1.5$ exactly [4,14]. Our results suggest that such behaviour holds, at least asymptotically for $\alpha \to 1$, **for all values of** $q \geq 2$. The results of Glumac and Uzelac [20] also suggests such a behaviour for all $q \leq 1$ and $1 < \alpha \leq 4/3$. So, eventually this asymptotic behaviour for $\alpha \to 1$ might be true for all q, provided the phase transition is a continuous one.

D. $b \to \infty$ extrapolations

Now, we can use the asymptotic behaviours derived in the preceeding section to extrapolate the full curves $T_c(b, q, \alpha)$ *vs* α for $b \to \infty$ as follows [18]. First, we define the rescaled variables $x_q \equiv (2-\alpha)/(2-\alpha_1(b,q))$ and $y_q \equiv T_c(b,q,\alpha)\ (2-\alpha_1(b,q))/J\ C(b,q)$, so that $y_q(x_q) \sim 1/(1-x_q)$ for $x_q \to 1\ \forall\ b, q$. In Fig. 4 we plotted, for $q = 3$, $y_q(x_q)$ *vs* x_q for different values of b. This figure shows clearly data collapse for $b > 5$ (represented by a solid line in Fig. 4). This fact appears also for other values of q. Hence, such curves are expected to be good estimates of the $b = \infty$ ones. Using the results $C(\infty, q) = 2/q$ and $\alpha_1(\infty, q) = 1$ (which are exact at least for $q = 2, 3$), we transform back such curves into the (T_c, α) variables. The results, which are expected to be good estimates of the exact critical temperatures $T_c(q, \alpha)$ for $\alpha \in (1, 2)$, are shown in Fig.5a. In table I we compare our results with those obtained by Glumac and Uzelac [20] using Finite Range Scaling (FRS) for some typical values of α and q (as far as we know, these are the only results available for $q > 2$ in the literature). We see that both results show a good agreement for values of $\alpha \sim 1$ (the percentual discrepancy is below 11% for $\alpha < 1.4$), but the difference increases for $\alpha \to 2$.

The same extrapolation procedure can be applied to the critical exponent ν, using the asymptotic behaviour (33). The numerical results are depicted in Fig. 5b for $q = 2, 3$ and 4; the exact value [14] for $q = 2$ and $1 < \alpha < 1.5$ ($\nu_2 = 1/(\alpha - 1)$) and the asymptotic result from Kosterlitz [2] ($\nu_K \sim (2(2-\alpha))^{-1/2}$) for $\alpha \to 2$ and $q = 2$ are also shown by dashed lines. For $q > 2$ all numerical curves are quite indistinguishable within the resolution of the plot and with a little departure from the $q = 2$ case, suggesting that the critical exponent may be independent of q for all $1 < \alpha < 2$. In table II we compare our results for ν with the corresponding ones obtained by FRS [20] for $q = 2$ (our results for $q > 2$ show a little difference with ours for $q = 2$). Although both results also compare well only for $\alpha \sim 1$, the main departure occurs for $q > 2$, where the FRS results show a strong dependency on q.

IV. CONCLUSIONS

The approach adopted here gives an estimate of some critical properties of the LR Potts model as a function of α for different values of $q \geq 2$, based on an extrapolation of a systematic series of RG calculations. This method allows us to obtain analytically several important quantities as a function of the rescaling parameter b. This fact, in turn, permits to take the $b \to \infty$ limit, where the results are expected to be highly accurate and perhaps reproduce the exact ones. This last assumption is supported by the recovering of several known results for $q = 2$ and some of the very few rigorous results available for general q, giving confidence in the validity of the method. Therefore, we believe that the obtained curves for the critical temperatures $T_c(q, \alpha)$ approximate with high precision the exact ones. In particular, the new result predicted by our method, namely that the critical temperature at $\alpha = 2$ is discontinuous with the **same value** for all $q \geq 2$ is remarkable. We also believe that the asymptotic functional form $T_c(q, \alpha)/J \propto (\alpha - 1)^{-1}$ for $\alpha \to 1$ might be **exact**. Notice that this is consistent with the recent conjectured scalings for generalized thermodynamics which allow for an unification of extensive ($\alpha > 1$) and non-extensive ($0 \leq \alpha \leq 1$) regimes [28].

Some other predictions for arbitrary q and continuous phase transitions, such as the asymptotic behaviour $\nu(\alpha, q)$ for $\alpha \to 1$ and its possible q-independence for all $1 < \alpha < 2$, are also of interest. It would be worth testing our conjectures and predictions by other techniques, such as the recent Monte Carlo method for long-range spin models [31].

Finally, one point which requires some discussion is the possible appearance of a first-order transition for some finite $q > q_c$ (for the two dimensional SR case it is known exactly [32] that $q_c = 4$). For the LR (as well as for the SR) case it was proved [33] that the mean field theory becomes exact (and therefore the transition is of first order) in the limit $q \to \infty$. We have not found any evidence of a first-order transition, but it is also known that the present kind of RG approach does not detect this type of transition in the 2D SR Potts model [34]. As far as we know, this question remains open since the FRS results [20] are also inconclusive with this respect. However, this problem could be solved by introducing appropriately some dilution in the RG formalism [34] and it would be interesting to apply this ansatz to the present case. Other possible extensions of the present paper concern higher dimensional systems, where the crossover between short and long-range regimes could be of interest for some real problems [7]. It could also be used to treat more complex interactions like the RKKY one. Some calculations along these lines are in progress and will be published elsewhere.

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APPENDIX A: Derivation of $G(b, 3)$

Since the expression (30) is independent of the pair of sites k, j it can be written as:

$$G(b,q) = 2G^1(b,q) + (b-2)G^2(b,q) \tag{A1}$$

where

$$G^1(b,q) \equiv \mathrm{Tr}_{\{\sigma_i^I\}}\, P_I\left(\{\sigma_i^I\},1)\right)\delta(\sigma_1^I,\sigma_2^I),\delta(\sigma_1^I,1) \tag{A2a}$$

$$G^2(b,q) \equiv \mathrm{Tr}_{\{\sigma_i^I\}}\, P_I\left(\{\sigma_i^I\},1)\right)\delta(\sigma_2^I,\sigma_3^I)\delta(\sigma_1^I,1) \tag{A2b}$$

which can be written as:

$$G^i(b,q) = \sum_{m=1}^{m_{max}} \frac{G^i_m(b,q)}{m} \qquad (i=1,2). \tag{A3}$$

where

- $G^1_m \equiv$ number of configurations of b spins ($\sigma_i^I = 1, 2, \ldots, q$) of a block where one of the m major subgroups of $\{\sigma_i^I\}$ is in the state 1, and the spins $\sigma_1^I = \sigma_2^I = 1$.
- $G^2_m \equiv$ number of configurations of b spins ($\sigma_i^I = 1, 2, \ldots, q$) of a block where one of the m major subgroups of $\{\sigma_i^I\}$ is in the state 1, and the spins $\sigma_1^I = 1$ and $\sigma_2^I = \sigma_3^I$.

We found that

$$G_1^1(b,3) = \sum_{l=0}^{X} \begin{pmatrix} b-2 \\ l \end{pmatrix} 2^l + \sum_{j=1}^{Int(X/3)} \sum_{j_1=2j}^{X-j} \begin{pmatrix} b-2 \\ X+j \end{pmatrix} \begin{pmatrix} X+j \\ j_1 \end{pmatrix} \tag{A4a}$$

$$G_2^1(b,3) = 2 \sum_{l=0}^{Int(\frac{X-2}{3})} \begin{pmatrix} b-2 \\ X+l+1 \end{pmatrix} \begin{pmatrix} X+l+1 \\ X-l \end{pmatrix} \tag{A4b}$$

$$G_3^1(b,3) = \begin{pmatrix} b-2 \\ 2b/3 \end{pmatrix} \begin{pmatrix} 2b/3 \\ b/3 \end{pmatrix} \delta(b,3n) \tag{A4c}$$

$$\begin{aligned} G_1^2(b,3) = & \sum_{l=0}^{X} \begin{pmatrix} b-3 \\ l \end{pmatrix} 2^l + \sum_{j=1}^{Int(X/3)} \sum_{j_1=2j}^{X-j} \begin{pmatrix} b-3 \\ X+j \end{pmatrix} \begin{pmatrix} X+j \\ j_1 \end{pmatrix} \\ & +2 \sum_{l=2}^{X} \begin{pmatrix} b-3 \\ l-2 \end{pmatrix} 2^{l-2} + 2 \sum_{j=1}^{Int(X/3)} \sum_{j_1=2j}^{X-j} \begin{pmatrix} b-3 \\ X+j-2 \end{pmatrix} \begin{pmatrix} X+j-2 \\ j_1 \end{pmatrix} \end{aligned} \tag{A4d}$$

$$\begin{aligned} G_2^2(b,3) = & \, 2 \sum_{l=0}^{Int(\frac{X-2}{3})} \begin{pmatrix} b-3 \\ X+l+1 \end{pmatrix} \begin{pmatrix} X+l+1 \\ X-l \end{pmatrix} + 2 \sum_{l=0}^{Int(\frac{X-2}{3})} \begin{pmatrix} b-3 \\ X+l-1 \end{pmatrix} \begin{pmatrix} X+l-1 \\ X-l-2 \end{pmatrix} \\ & +2 \sum_{l=1}^{Int(\frac{X-2}{3})} \begin{pmatrix} b-3 \\ X+l-2 \end{pmatrix} \begin{pmatrix} X+l-2 \\ X-l-1 \end{pmatrix} \end{aligned} \tag{A4e}$$

$$G_3^2(b,3) = \left[\begin{pmatrix} b-3 \\ 2b/3 \end{pmatrix} \begin{pmatrix} 2b/3 \\ b/3 \end{pmatrix} + 2 \begin{pmatrix} b-3 \\ 2b/3-2 \end{pmatrix} \begin{pmatrix} 2b/3-2 \\ b/3-2 \end{pmatrix} \right] \delta(b,3n) \tag{A4f}$$

where $n = 1, 2, \ldots$ and $X \equiv (b-1)/2$. Combination of expressions (A1), (A3) and (A4) lead to $G(b,3)$.

FIGURES

FIG. 1. Renormalization group transformation using Kadanoff-blocks of length $b = 3$ in the one dimensional lattice; r_{IJ} is the distance between the blocks I and J.

FIG. 2. Numerical calculations for $b = 3$ and different values of q. (a) Critical temperature $T_c(b, q, \alpha)/J$ *vs* α; (b) correlation length critical exponent $\nu(b, q, \alpha)$ *vs* α.

FIG. 3. Numerical calculations for $q = 3$ and different values of the rescaling length b. (a) Critical temperature $T_c(b, q, \alpha)/J$ *vs* α; (b) correlation lenght critical exponent $\nu(b, q, \alpha)$ *vs* α.

FIG. 4. Rescaled critical temperature $y_q \equiv T_c(b, q, \alpha) \quad (2 - \alpha_1(b, q))/J \quad C(b, q)$ *vs* $x_q \equiv (2 - \alpha)/(2 - \alpha_1(b, q))$ for $q = 3$. All curves with $b > 5$ coincide, within the used scale, with the solid line.

FIG. 5. $b \to \infty$ extrapolations for different values of q. (a) Critical temperature T_c/J *vs* α; (b) critical exponent ν *vs* α. Dashed lines correspond to the exact result ν_2 for $1 < \alpha < 1.5$ and to the Kosterlitz's asymptotic result ν_K for $\alpha \to 2$, both for $q = 2$.

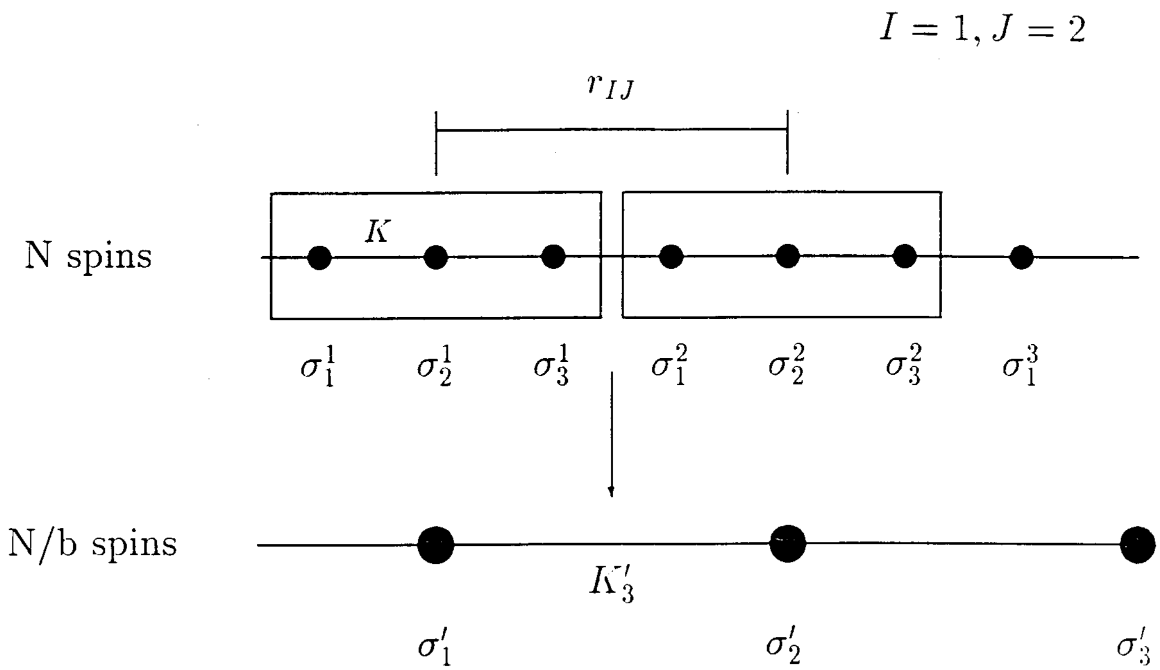
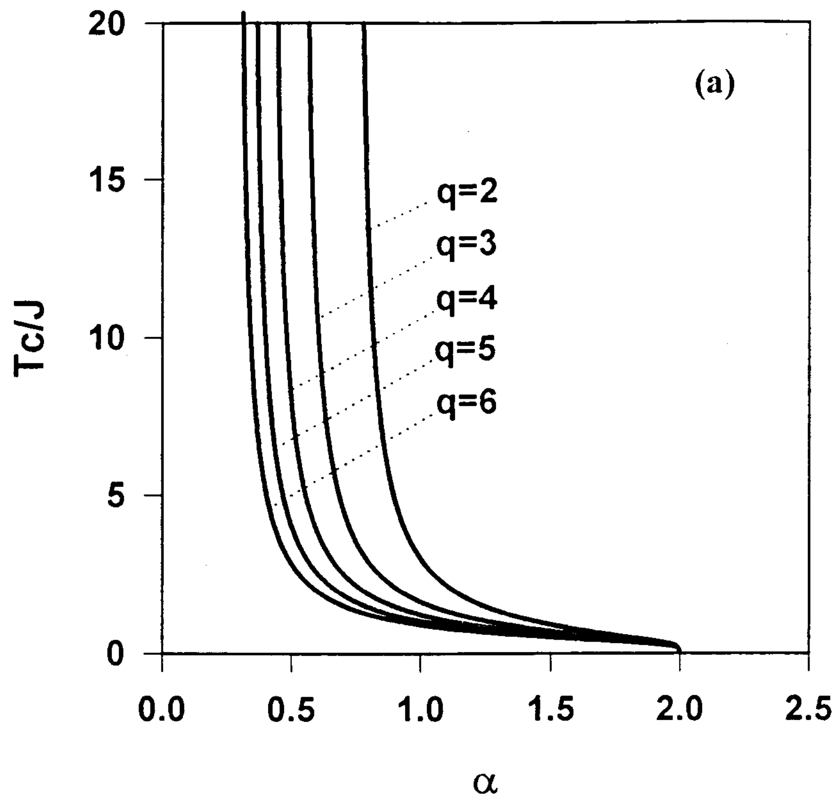
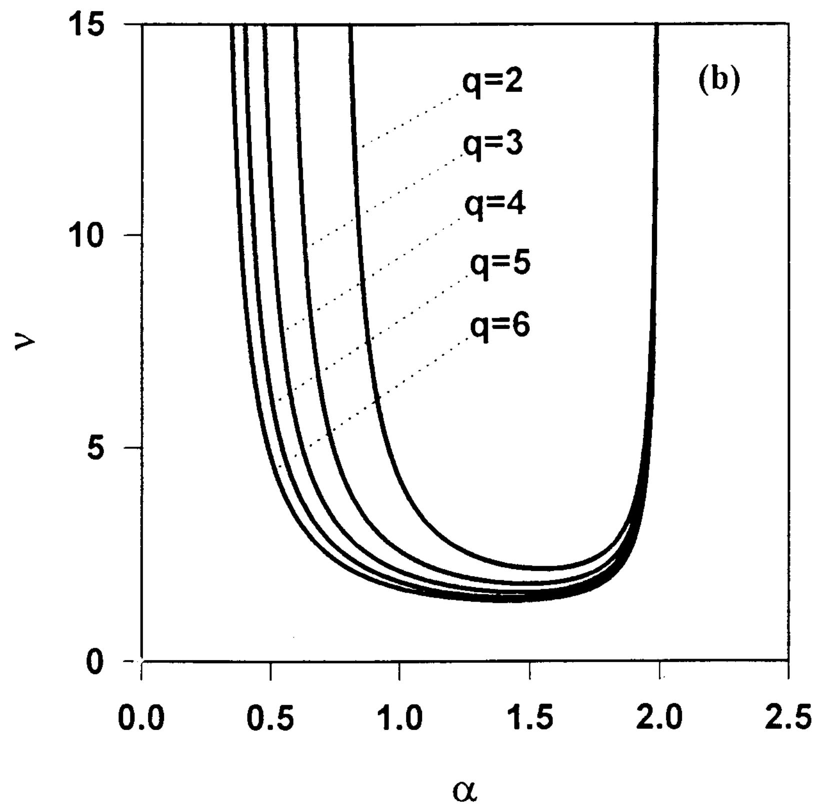

$I = 1, J = 2$
r_{IJ}
N spins
K
σ_1^1 σ_2^1 σ_3^1 σ_1^2 σ_2^2 σ_3^2 σ_1^3
N/b spins
K_3'
σ_1' σ_2' σ_3'

Figure 1



(a)
Tc/J
20
15
10
5
0
0.0
0.5
1.0
1.5
2.0
2.5
q=2
q=3
q=4
q=5
q=6
α



(b)
ν
15
10
5
0
0.0
0.5
1.0
1.5
2.0
2.5
q=2
q=3
q=4
q=5
q=6
α

Figure 2

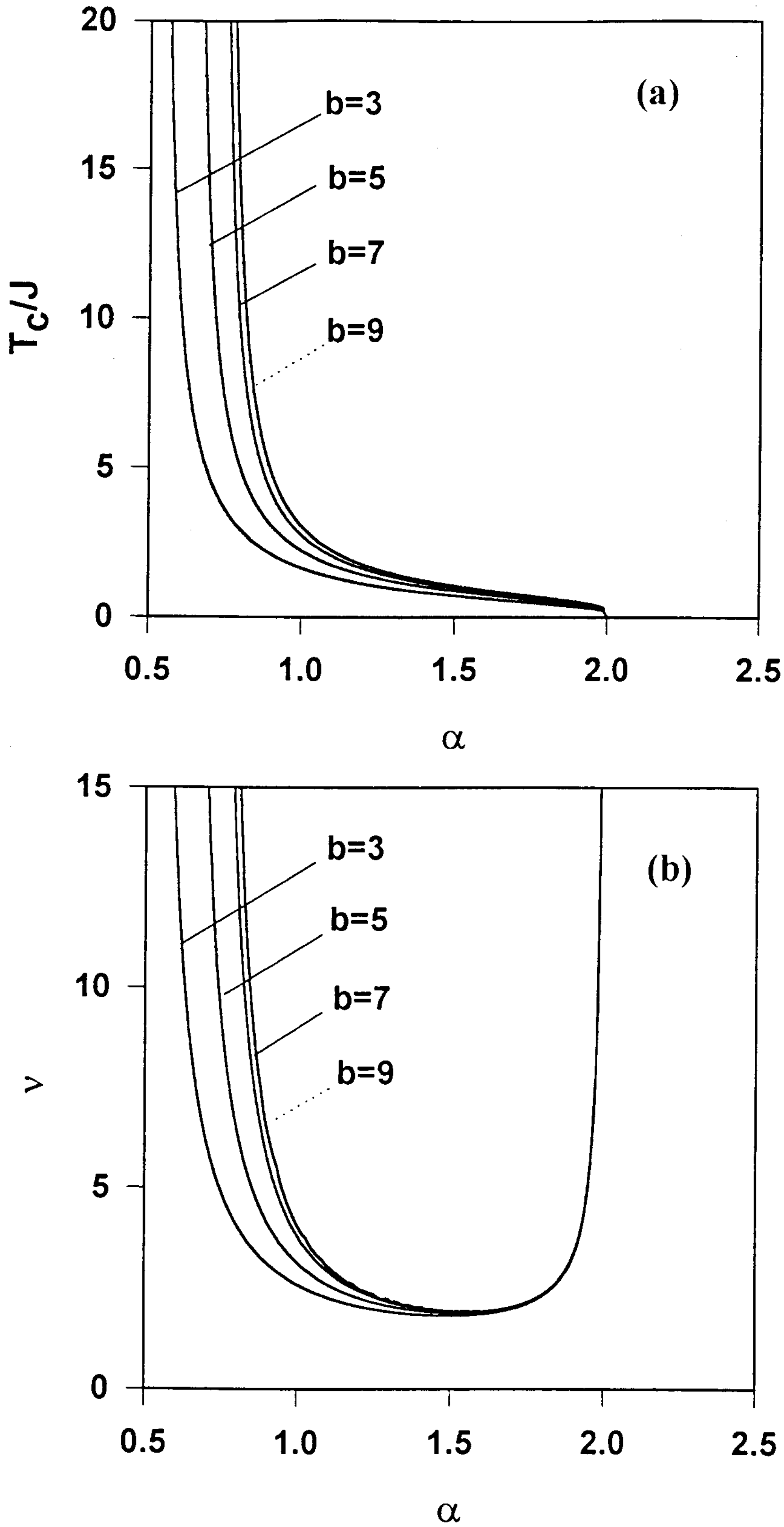

(a)
b=3
b=5
b=7
b=9
T_c/J
20
15
10
5
0
0.5
1.0
1.5
2.0
2.5
α
(b)
b=3
b=5
b=7
b=9
ν
15
10
5
0
0.5
1.0
1.5
2.0
2.5
α

Figure 3

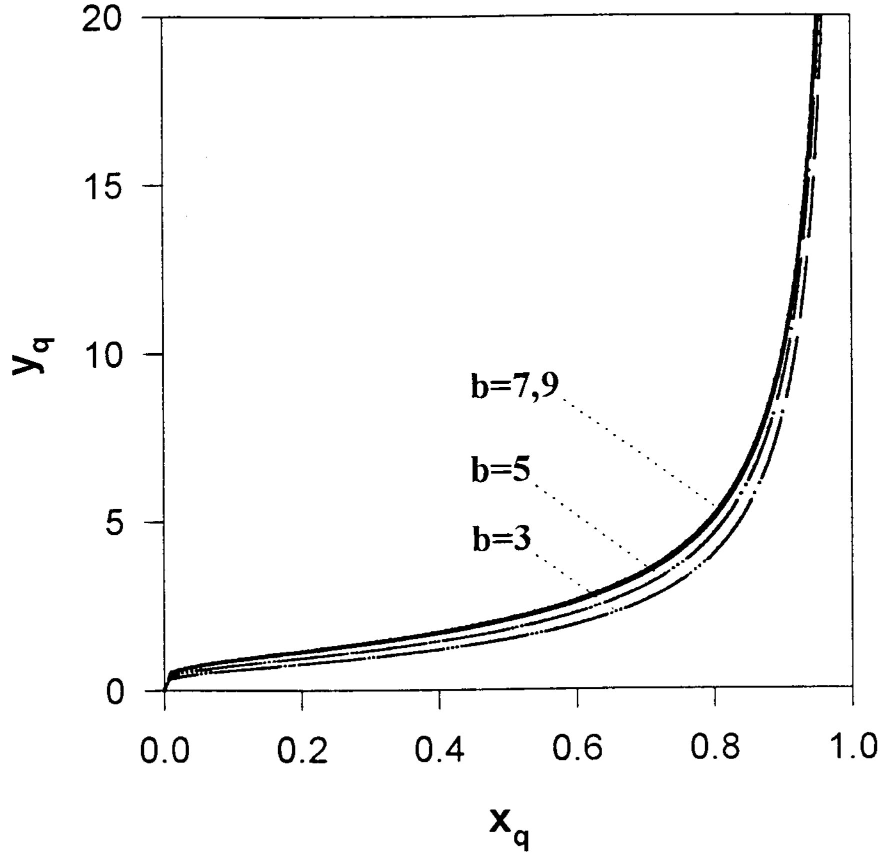


Figure 4

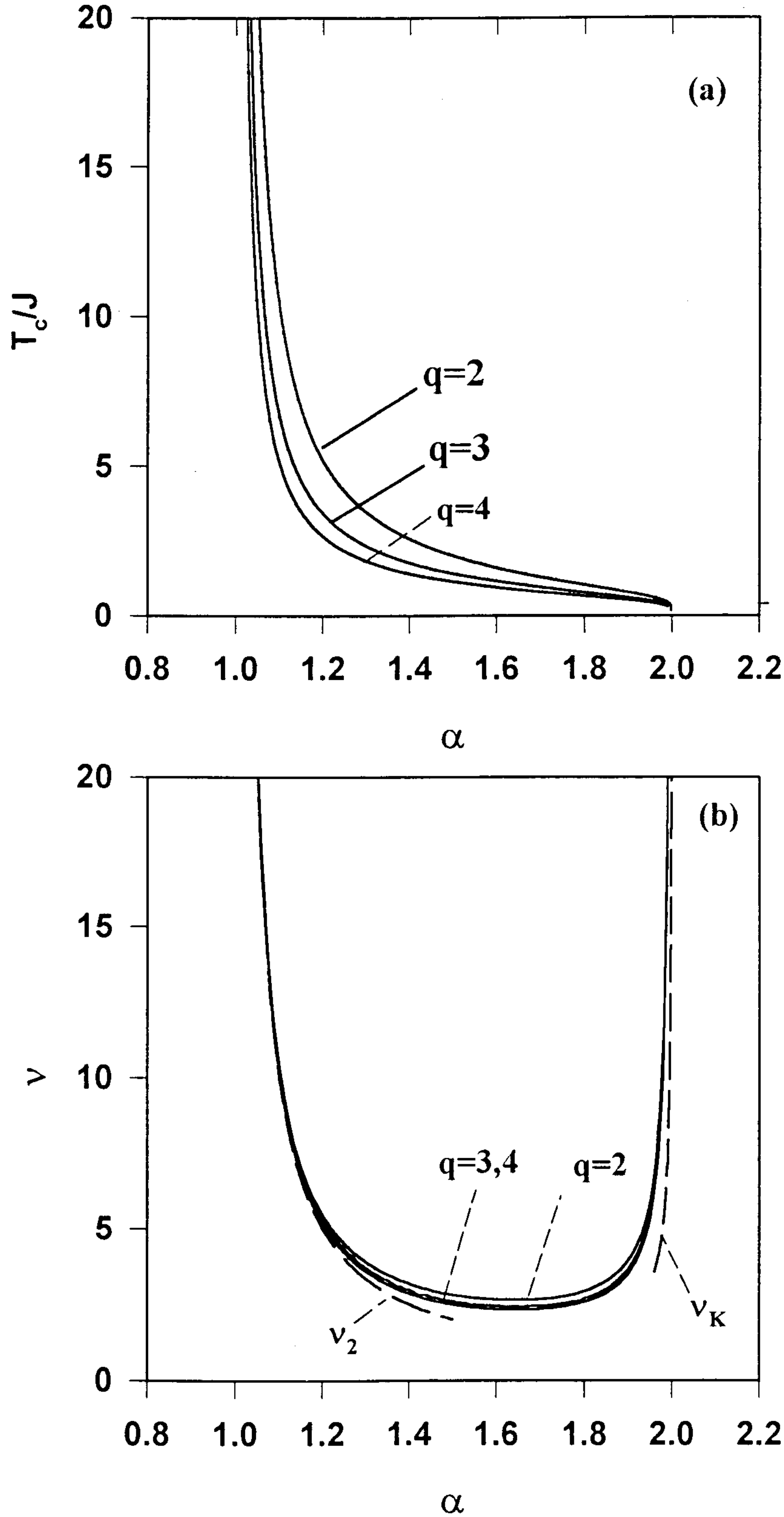


Figure 5

TABLE I. Comparison between our RG ($b \to \infty$) extrapolated values and Glumac and Uzelac FRS calculations of the critical temperature T_c/J for different values of q and α.

	q=2		*q=3*		*q=4*	
α	RG	FRS	RG	FRS	RG	FRS
1.1	10.40	10.787	6.72	7.353	5.16	4.926
1.3	3.48	3.680	2.33	2.589	1.89	2.045
1.5	2.00	2.179	1.41	1.663	1.14	1.402
1.7	1.28	1.463	0.95	1.194	0.78	1.048
1.9	0.77	1.003	0.61	0.874	0.51	0.797

TABLE II. Comparison between our RG $b \to \infty$ extrapolation and Glumac and Uzelac FRS calculations of the critical exponent ν for $q = 2$ and some typical values of α.

α	RG	FRS
1.1	10.48	9.901
1.3	3.90	3.322
1.5	2.81	2.325
1.7	2.66	1.930
1.9	3.90	2.469
2.0	∞ (exact)	3.236

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