

The Static Gravitational Field of a Spherically Symmetric Body

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ABSTRACT

We continue here the exam [1] of a theory of gravity that satisfies the Einstein Equivalence Principle (EEP) for any kind of matter/energy, except for the gravitational energy. This is part of a research program that intends to re-examine the standard Feynman-Deser approach of field theoretical derivation of Einstein's General Relativity. The hypothesis implicit in such precedent derivations [2][3] concerns the universality of gravity interaction. Although there is a strong observational basis supporting the universality of matter to gravity interaction, there is not an equivalent situation that supports the hypothesis that gravity interacts with gravity as any other form of non-gravitational energy. We analyse here a kind of gravity-gravity interaction distinct from GR but, as we shall see, that conforms with the actual status of observation. We exhibit the gravitational field produced by a spherically symmetric static configuration as described in this field theory of gravity. The values that we obtain for the standard PPN parameters ($\alpha = \beta = \gamma = 1$) coincide with those of General Relativity. Thus, as we pointed out in a previous paper, the main different aspect of our theory and GR concerns the velocity of the gravitational waves. Since there is a large expectation that the detection of gravitational waves will occur in the near future, the question of which theory describes Nature better will probably be settled soon.

Key-words: Gravity; Energy; Observational.

1 INTRODUCTION

1.1 A. Introductory Remarks

In a recent paper two of us (LRF and MN) have proposed a critical re-examination of the general treatment concerning Einstein’s Equivalence Principle (EEP). The starting point rests on the well-known fact that EEP is supported by experiment by at least¹ one part in 10^{11} . This statement should be understood in the very limited sense that it concerns observed effects of gravity restrained to the coupling of matter (that is, any non-gravitational form of energy) to the gravitational field. However, many theories of gravity, starting from General Relativity, go far beyond such limited domain of observation and assume a generalized form which states that the EEP comprises absolutely all forms of energy (including gravitational). This, of course, is a speculation that still nowadays remains beyond any experimental test. The recent general excitement among the scientific community concerning the possible detection of gravitational waves led to the belief that the observation of these waves could act as a crucial test on gravity-gravity interaction and to provide for a decision concerning which, among the rivals gravity theories, is the best one.

In [1] a class of alternative models for gravity was presented and a specific one was developed. Our purpose in the present paper is to proceed with such an investigation on the consequences of our theory. Here we will describe a solution of our set of equations of the gravitational field produced by a spherically symmetric and static configuration. We shall show that as far as the post-Newtonian approximation is concerned, the behavior of matter (i.e. any form of non-gravitational energy), in our theory is indistinguishable from General Relativity (GR).

From the experimental standard tests we can state the following results²

- $\alpha = 1$
- $\gamma = 1.000 \pm 0.002$
- $\frac{1}{3}(2\gamma + 2 - \beta) = 1.00 \pm 0.02$
- $\chi < 10^{-3}$

¹See the status of the STEP (Satellite Test of the Equivalence Principle), an international tentative of improving this result to one part in 10^{17} .

²We are following the standard notation for the PPN parameters. The reader not aware of this formulation should consult the Appendix and the references.

- $\alpha_2 < 4x10^{-4}$
- $\frac{2}{3}\alpha_2 - \alpha_1 < 0.02$
- $49\alpha_1 - \alpha_2 - 2.2\chi < 0.1$

We shall prove in this paper that, for the new theory of gravity that we are examining, like in the very same manner as General Relativity, the unique non vanishing parameters are α , β and γ . The extra parameters, present in others theories describe strange non-usual properties of the gravitational interaction³.

This result supports a remark made in our previous paper which states that the crucial distinction of ours gravity theory and GR should be made in evidence from the detection of the gravitational waves.

1.2 B. Synopsis

The plan of the paper is the following. In section 2 we make a very short review of the field theory of gravitation⁴. We present the fundamental variables that describe the gravitational field and its corresponding equation of motion. In section 3 we describe the symmetries of the problem and find the correspondent equations of motion. A solution for the static and spherically symmetric configuration is then presented. In section 3.1 we show the effective geometry that is the observed effect of the previous solution as seen by any non-gravitational form of energy. In section 4 we evaluate the corresponding distribution of gravitational energy within our theory. We end with section 5 in which some comments concerning the present solution and its consequences are shown.

2 Field Theory of Gravity: A Summary

In this section we will provide a short resumé of the theory of gravity that we will analyse. Actually, we will deal here with a field theoretical model of gravitational interaction. In the standard traditional way this field is represented by a symmetric second order tensor

³Like, for instance, possible deviation from Lorentz symmetry, effects of preferred frames, non conservation of energy-momentum at the post-Newtonian limit and an eventual spatial anisotropy of three-body interaction.

⁴The reader that would like to know more about this should look for the references of the quoted papers by Feynman, Deser and our previous paper.

that we denote by $\varphi_{\mu\nu}$. This field is supposed to propagate in an auxiliary background Minkowski geometry⁵.

In order to exhibit the general covariance of the theory we will write the auxiliary metric $\gamma_{\mu\nu}$ in an arbitrary system of coordinates. We define the corresponding covariant derivative by the standard way:

$$V_{\mu};_{\nu} = V_{\mu, \nu} - \Delta_{\mu\nu}^{\alpha} V_{\alpha} \quad (1)$$

in which

$$\Delta_{\mu\nu}^{\alpha} = \frac{1}{2} \gamma^{\alpha\beta} (\gamma_{\beta\mu, \nu} + \gamma_{\beta\nu, \mu} - \gamma_{\mu\nu, \beta}). \quad (2)$$

The associated curvature tensor vanishes identically that is

$$R_{\alpha\beta\mu\nu}(\gamma_{\epsilon\lambda}) = 0. \quad (3)$$

The gravitational field is represented by a three-index tensor $F_{\alpha\beta\mu}$ obtained from the potential $\varphi_{\mu\nu}$ by derivation:

$$F_{\alpha\beta\mu} = \varphi_{\mu[\alpha;\beta]} + \varphi_{, [\alpha} \gamma_{\beta]\mu} + \gamma_{\mu[\alpha} \varphi_{\beta]}^{\lambda}_{;\lambda}. \quad (4)$$

where we are using the anti-symmetrization symbol $[\]$ like

$$[A, B] \equiv AB - BA.$$

We use an analogous form to the symmetrization symbol $()$

$$(A, B) \equiv AB + BA.$$

From the above definition it follows that the field $F_{\alpha\beta\mu}$ is anti-symmetric in the first pair of indices and obeys the cyclic identity, that is

$$F_{\alpha\mu\nu} + F_{\mu\alpha\nu} = 0$$

and

$$F_{\alpha\mu\nu} + F_{\mu\nu\alpha} + F_{\nu\alpha\mu} = 0.$$

The trace of the tensor $F_{\alpha\beta\mu}$ is given by

⁵We shall see that this metric is not observable neither by matter nor by the gravitational field.

$$F_\mu \equiv F_{\mu\alpha\beta}\gamma^{\alpha\beta} = 4(\varphi_\mu{}^\lambda{}_{;\lambda} - \varphi_{,\lambda}). \quad (5)$$

This allow us to re-write the expression of the field in the form

$$F_{\alpha\beta\mu} = \varphi_{\mu[\alpha;\beta]} + \frac{1}{4}F_{[\alpha\gamma\beta]\mu}. \quad (6)$$

The equation of motion of the gravitational field will appear in a more convenient form when written in terms of an associated quantity $M_{\alpha\beta\mu}$ that has the same symmetries as $F_{\alpha\beta\mu}$ and is defined by

$$M_{\alpha\beta\mu} \equiv F_{\alpha\beta\mu} - \frac{1}{2}F_{\alpha\gamma\beta\mu} + \frac{1}{2}F_{\beta\gamma\alpha\mu}. \quad (7)$$

See [1] for other properties of these quantities.

The symbol κ represents Einstein's constant, written in terms of Newton's constant G_N and the velocity of light c by the definition

$$\kappa = \frac{8\pi}{G_N c^4}.$$

We set $c = G_N = 1$.

Any theory of gravity that yields the correct weak field limit must reduces in this limit to the standard Fierz equation of motion. We have shown, in our previous paper, that in order to achieve such correct limit the theory, described in terms of the gravitational field $F_{\alpha\beta\mu}$, must be such that its Lagrangian is a functional of the quantity Z defined in terms of the invariants A and B under the form:

$$Z \equiv A - \frac{3}{4}B. \quad (8)$$

in which

$$A \equiv F_{\alpha\mu\nu}F^{\alpha\mu\nu}$$

and

$$B \equiv F_\alpha F^\alpha.$$

We have argued that a good candidate for this is a theory the Lagrangian of which is given by

$$L = \frac{b^2}{k} \left\{ \sqrt{1 + \frac{1}{b^2}(-A + \frac{3}{4}B)} - 1 \right\}. \quad (9)$$

We will analyse here only this particular form among the whole class of theories that could be constructed with Z .

The equation of motion that follows from this Lagrangian is then given by

$$G^L_{\mu\nu} = \frac{1}{2}L_A^{-1} \{L_{A;\lambda}M^\lambda_{(\mu\nu)} + T^m_{\mu\nu}\} \quad (10)$$

in which we have added the source term represented by the energy-momentum tensor of matter.

In the theory of General Relativity the corresponding equation of motion of the gravitational field in the geometrical representation takes the form:

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = -T^m_{\mu\nu}$$

in which the curvature of the associated Riemannian metric appears explicitly. However there is an equivalent way to describe the equation of motion in GR that seems worth to mention here just in order to exhibit the similarity and the distinction between both theories. Indeed, many authors (see, for instance [3]) have shown that it is possible to re-write Einstein's equation of motion by isolating the linear operator $G^L_{\mu\nu}$ from the non-linear terms, showing explicitly the form of the self-interacting terms. It is given by:

$$G^L_{\mu\nu} = -T^m_{\mu\nu} - t_{\mu\nu} \quad (11)$$

in which $t_{\mu\nu}$ is a complicated non-linear expression constructed with the metric tensor and its derivative⁶ A simple inspection on both theories thus exhibits very clearly the particular characterization of the self-interacting terms in each theory. Let us emphasize that in both cases of equations (10) and (11) they represent the full theory. There is no approximation of any sort in these expressions.

Let us just make one more comment on the properties of these rivals theories of gravity. Since both theories satisfy the restricted EEP, the behavior of matter (or any form of non-gravitational energy) is precisely the same in both theories. Matter follows geodesics in an effective Riemannian geometry. Only the gravity-gravity processes are distinct. Since we are not dealing in the present paper with gravitational waves we will not consider any longer such distinction here.

So much for the review. Let us now look for a solution of our equation (10), in the absence of matter.

⁶The reader not acquainted with this formulation should consult [4] for a very didactic presentation of it.

3 THE SOLUTION

We set for the auxiliary metric of the background the form

$$ds^2 = dt^2 - dr^2 - r^2(d\theta^2 + \sin^2\theta d\varphi^2) \quad (12)$$

This means that all operations of raising and lowering indices are made by this Minkowski metric $\gamma_{\mu\nu}$ ⁷.

We search for the simple solution in such a way that the only non-identically null gravitational potential components $\varphi^{\mu\nu}$ are only

$$\varphi_{00} = \varphi^{00} = \mu(r)$$

and

$$\varphi_{11} = \varphi^{11} = \nu(r).$$

The trace φ is then given by

$$\varphi = \mu - \nu.$$

From this we obtain the gravitational field $F_{\alpha\beta\mu}$. The only non-null terms are

$$F_{100} = -2\left(\mu' + \frac{\nu}{r}\right)$$

$$F_{122} = \mu' r^2 + 3\nu r$$

$$F_{133} = \sin^2\theta F_{122}.$$

in which we have used a prime ' to symbolise the derivative with respect to the radial variable r .

Thus the unique component of the trace that remains is F_1 which is given by:

$$F_1 = -4\left(\mu' + 2\frac{\nu}{r}\right)$$

From these we can evaluate the associated quantities

- The invariant A .

⁷The reader should be attentive to the fact that matter (massive or massless particles –photons, for instance – that is, any form of non-gravitational energy) feels a modified geometry. See the previous paper [1].

- The invariant B .
- The associated tensor $M_{\mu\nu\alpha}$.

We obtain directly the following values:

$$A = -4 \left\{ 3\mu'^2 + 10\frac{\nu\mu'}{r} + 11\frac{\nu^2}{r^2} \right\} \quad (13)$$

$$B = -16 \left\{ \mu'^2 + 4\frac{\nu\mu'}{r} + 4\frac{\nu^2}{r^2} \right\} \quad (14)$$

$$M_{100} = 2\frac{\nu}{r} \quad (15)$$

$$M_{122} = -r^2\left(\mu' + \frac{\nu}{r}\right) \quad (16)$$

$$M_{133} = \sin^2\theta M_{122}. \quad (17)$$

Under the above hypothesis of spherical symmetry of the solution, there remain only two non-trivially satisfied equations, which are given by

$$\left\{ L_A \frac{\nu}{r} \right\}' + 2L_A \frac{\nu}{r} = 0. \quad (18)$$

and

$$\mu' + \frac{\nu}{r} = 0. \quad (19)$$

It seems worth to remark that the above set of equations (18) and (19) are the same for any theory that satisfies the fundamental condition, that is, for functionals only of the quantity Z defined by equation (8).

Let us now specialize this for our theory characterized by equation (9).

We have

$$L_A = -\frac{b^2}{2} (b^2 + L)^{-1} \quad (20)$$

Substituting the above tentative form of a solution in the expressions of the invariants we find for the quantity Z :

$$-A + \frac{3}{4}B = 4\frac{\nu^2}{r^2}. \quad (21)$$

Using this value in the equation (18) we find the solution

$$\nu = -\frac{2M}{r} \left\{ 1 - \left(\frac{r_c}{r}\right)^4 \right\}^{-\frac{1}{2}}. \quad (22)$$

in which the constant r_c is given by

$$r_c^2 \equiv \frac{4M}{b}.$$

The remaining function μ is given in terms of the elliptic function $F(\alpha, \frac{\sqrt{2}}{2})$ by

$$\mu = \frac{1}{2}\sqrt{bM} \left\{ F\left(\alpha, \frac{\sqrt{2}}{2}\right) + \mu_0 \right\} \quad (23)$$

in which the constant μ_0 must be chosen to yield the correct asymptotic limit. The quantity α is given by

$$\alpha \equiv \arcsin \left\{ \frac{\cosh x - 1}{\cosh x} \right\}^{\frac{1}{2}}$$

and x is defined by

$$\cosh x \equiv \left\{ \frac{r}{r_c} \right\}^2.$$

At this point let us make a preliminary comment concerning the value of the constant b . At the value $r = r_c$ the field has apparently a singularity. We shall see in a subsequent section that we deal here with a true singularity once the associated gravitational energy is singular at point r_c . Although the quantity b is a free parameter of the theory, its value must be such that it provides that the domain $r < r_c$ should be hidden inside all known celestial bodies. A typical example associates b to the inverse of Planck length. This provides a good possible value that satisfies the above requirement. We leave the actual value of b , for the time being, as an open parameter, the true value of which is to be decided later on.

3.1 The Effective Geometry

In this section we will turn our attention to a geometric representation of the present theory. Following the standard procedure [2] [3] we define a Riemannian metric tensor in terms of the gravitational potential as⁸

$$g_{\mu\nu} \equiv \gamma_{\mu\nu} + \varphi_{\mu\nu} \quad (24)$$

⁸The reader should note that one can make different forms of geometric representation. For instance one can use the contravariant representation to set $g^{\mu\nu} = \gamma^{\mu\nu} + \varphi^{\mu\nu}$; or use pseudo quantities by means of the determinant γ in the definition of the geometry. Each one of these choices provides non-equivalent representations.

It seems worth to call attention to the reader that this definition has a deep meaning, once for all forms of non-gravitational energy the net effect of the gravitational field is felt precisely as if gravity was responsible of changing the metrical properties of the spacetime from the flatness structure to a curved one that is related to the gravitational field precisely by the above expression⁹. This means that any material body (or photons) follows along the geodesics (null, in the case of photons) as if the metric tensor of spacetime was given by the above expression.

From the previous calculation we find for the effective geometry the form

$$ds^2 = g_{00}dt^2 - g_{11}dr^2 - r^2(d\theta^2 + \sin^2\theta d\varphi^2) \quad (25)$$

with

$$g_{00} = 1 + \mu.$$

and

$$g_{11} = -1 + \nu.$$

Expanding for $r_c \ll r$ we have

$$g_{00} \approx 1 - \frac{2M}{r} - \frac{1}{10} \left\{ \frac{r_c}{r} \right\}^4 + \dots$$

and

$$g_{11} \approx -1 - \frac{2M}{r} - \frac{M}{r} \left\{ \frac{r_c}{r} \right\}^4 + \dots$$

We see that the modification of the flatness of spacetime, as seen by matter, beyond the order $O(\frac{M}{r})$, occurs only at order $O(\frac{M^3}{r^5})$. This is radically different from the results of General Relativity and should provide the basis for a future test.

Before this, however let us look into the corresponding post-Newtonian parameter of our theory. Using the isotropic coordinate system (see the Appendix) we can re-write our effective geometry under the form

$$ds^2 = (1 - 2\alpha \frac{M}{\rho} + 2\beta \frac{M^2}{\rho^2} + \dots) dt^2 - (1 + 2\gamma \frac{M}{\rho}) \{ d\rho^2 + \rho^2 (d\theta^2 + \sin^2\theta d\varphi^2) \}$$

It then follows from a direct inspection on this form of the effective geometry the values of the PPN parameters corresponding to our solution. We can use the associated table in order to compare the values of our model and those obtained from General Relativity.

⁹The reader should consult the previous paper [1] for more details and for a proof of this statement.

Table 1: PPN Parameters

	GR	LN
α	1	1
β	1	1
γ	1	1

A direct inspection on this table led us to conclude that, at the PPN level of observation, both theories provide the same answer for the effects of the gravitational field on matter.

4 Gravitational Energy

The fact that we are dealing with a field theory in the conventional way provide us directly as a by-product, with a well-defined definition of the energy. We can arrive at the expression of the energy-momentum tensor either by a direct variation of the associated background metric or by means of Noether's theorem¹⁰.

Using the definition

$$T^{g\mu\nu} = -\frac{2}{\sqrt{-\gamma}} \frac{\delta L \sqrt{-\gamma}}{\delta \gamma_{\mu\nu}} \quad (26)$$

we find, from the above Lagrangian,

$$T^g_{\mu\nu} = -L\gamma_{\mu\nu} + L_A \left\{ 4F_{\mu\alpha\beta} F_\nu^{\alpha\beta} + 2F_{\alpha\beta\mu} F^{\alpha\beta}_\nu - 3F^\alpha F_{\alpha(\mu\nu)} - \frac{5}{2} F_\mu F_\nu + F^\epsilon F_\epsilon \gamma_{\mu\nu} \right\} \quad (27)$$

Let us specialize this for the case of a static and spherically symmetric configuration. Since in our theory the Lagrangian is given by eq. (9) we have, under the conditions of the above solution,

$$L = \frac{b^2}{k} \left\{ \left(1 - \left(\frac{r_c}{r} \right)^4 \right)^{-\frac{1}{2}} - 1 \right\}. \quad (28)$$

¹⁰The reader should note that many different forms of the energy-momentum tensor of the gravitational field have been described in the literature. Most of them suffer from the disease of being pseudo quantities and not true tensors. Some others, like the one described in [4], are true tensors but have a hidden gauge symmetry. We note that our expression does not seem to suffer of any of these difficulties.

Note that it then follows that

$$L > 0.$$

Finally we obtain, for the density of energy, the expression

$$\frac{k}{b^2} T^g_{00} = 1 - \frac{1}{\sqrt{1 - (\frac{r_c}{r})^4}} + \frac{32M^2 b^2}{r^4} \frac{1}{\sqrt{1 - (\frac{r_c}{r})^4}} \quad (29)$$

We should point out two important conclusions that follow from this expression.

- The density of energy of the gravitational field is positive.
- There exists a true singularity of the field at the point $r = r_c$.

Integrating from $r = R \gg r_c$, the radius of the compact object, until infinity a direct simple calculation gives a finite value for the total energy, provided by

$$E_T = 4\pi \frac{32M^2}{R}.$$

This is the value one should expect on Newtonian grounds.

5 CONCLUSION

In this paper we continue the exam of a new theory of gravity. We have found the exact solution of a compact spherically symmetric and static configuration. We have shown that, although the general system of equations is highly non linear, the system reduces, for this symmetry, to a very simple one that allows a direct integration to be done. An analysis of the properties of our solution shows many points in common with GR. The PPN parameters do not allow a distinction between these theories. In order to exhibit a crucial observable difference between them, we should look for the propagation of gravitational disturbances. Indeed, (see the Appendix) the velocity of the gravitational waves is not the same in these two theories. This should be a clear test to select the good model for gravitational interaction.

6 APPENDIX

6.1 The Velocity of Gravitational Waves

In [1] we have shown that the gravitational waves travel in different cones than the photons. This means that the gravitational waves feel a geometry of the spacetime that is

not the same as the one felt by matter. The propagation of the gravitational disturbances follow along trajectories k^μ that satisfy the equation

$$\left\{ \gamma_{\mu\nu} - \frac{1}{b^2} T^g_{\mu\nu} \right\} k^\mu k^\nu = 0.$$

This is one of the most important distinctions between the present field theory of gravity and General Relativity.

6.2 PPN Formalism

The standard form of the metric in the PPN form is given by

$$g_{00} = 1 - 2\alpha \frac{M}{\rho} + 2\beta \left(\frac{M}{\rho} \right)^2.$$

$$g_{ij} = \left(-1 - 2\gamma \frac{M}{\rho} - \lambda \frac{M^2}{\rho^2} \right) \delta_{ij}.$$

We make a coordinate transformation to pass from this isotropic system $(t, \rho, \theta, \varphi)$ to the spherical one (t, r, θ, φ) :

$$\rho = r \left\{ 1 - \gamma \frac{M}{r} - \frac{1}{2} (\lambda - \gamma^2) \frac{M^2}{r^2} \right\}$$

It then follows that in the new system the metric takes the form

$$ds^2 = \left\{ 1 - 2\alpha \frac{M}{r} - 2(\beta - \alpha\gamma) \frac{M^2}{r^2} \right\} dt^2 - \left\{ 1 + 2\gamma \frac{M}{r} + (2\lambda + \gamma^2) \frac{M^2}{r^2} \right\} dr^2 - r^2 (d\theta^2 + \sin^2\theta d\varphi^2).$$

6.3 Connections of the Background

Just for completeness let us enumerate the non-identically null connections of the background metric 12 defined by

$$\Delta^\alpha_{\mu\nu} \equiv \frac{1}{2} \gamma^{\alpha\beta} \{ \gamma_{\beta\mu,\nu} + \gamma_{\beta\nu,\mu} - \gamma_{\mu\nu,\beta} \}$$

They are:

$$\Delta^1_{22} = -r.$$

$$\Delta^1_{33} = -r \sin^2\theta.$$

$$\Delta^2_{12} = \frac{1}{r}.$$

$$\Delta^2_{33} = -\sin\theta\cos\theta.$$

$$\Delta^3_{13} = \frac{1}{r}.$$

$$\Delta^3_{23} = \cot\theta.$$

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