CBPF-NF-084/95

Anomalous diffusion in the presence of external forces: exact time-dependent solutions and entropy

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ABSTRACT

The optimization of the usual entropy $S_1[p] = -\int du\ p(u)\ ln\ p(u)$ under appropriate constraints is closely related to the Gaussian form of the *exact* time-dependent solution of the Fokker-Planck equation describing an important class of *normal* diffusions. We show here that the optimization of the generalized entropic form $S_q[p] = \{1 - \int du\ [p(u)]^q\}/(q-1)$ (with $q = 1 + \mu - \nu \in \mathcal{R}$) is closely related to the calculation of the *exact* time-dependent solutions of a generalized, nonlinear, Fokker Planck equation, namely $\frac{\partial}{\partial t}p^\mu = -\frac{\partial}{\partial x}[F(x)p^\mu] + D\frac{\partial^2}{\partial x^2}p^\nu$, associated with a physically meaningful *anomalous* diffusion in the presence of the external force $F(x) = k_1 - k_2 x$. Consequently, paradigmatic types of normal ($q = 1$) and anomalous ($q \neq 1$) diffusions occurring in a great variety of physical situations become *unified* in a single picture.

Key-words: Anomalous diffusion; Fokker-Planck equation; Generalized thermostatistics; Generalized entropy.

05.20.-y; 05.40.+j; 05.60.+w; 66.10.Cb

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Anomalous diffusion is intensively studied nowadays, both theoretically and experimentally. It is observed, for instance, in CTAB micelles dissolved in salted water [1], the analysis of heartbeat histograms in a healthy individual [2], financial transactions [3], chaotic transport in laminar fluid flow of a water-glycerol mixture in a rapidly rotating annulus [4], subrecoil laser cooling [5], particle chaotic dynamics along the stochastic web associated with a d=3 Hamiltonian flow with hexagonal symmetry in a plane [6], conservative motion in a d=2 periodic potential [7], transport of fluid in porous media (see [8] and references therein), surface growth [8], NMR relaxometry of liquids in porous glasses [9] and many other interesting physical systems. Its thermostatistical foundation (as it is known for *normal* diffusion) is naturally highly desirable and has, since long, been looked for (see, for instance, [10] and references therein). This goal was recently achieved by Alemany and Zanette [11] and others [12, 13] for *Lévy-like* anomalous diffusion, in the context of a generalized, not necessarily *extensive* (additive), thermostatistics that has been recently proposed [14, 15]. In particular, within this framework, the ubiquity and robustness of Lévy distributions in Nature has been thermostatistically founded on the Lévy-Gnedenko central limit theorem [13]. This thermostatistics (already applied to a considerable variety of physical systems [16] and optimization techniques [17], and experimentally checked for a specific pure-electron-plasma turbulence [18]) is based upon the entropic form

$$S_q[p] \equiv \frac{1 - \int du\, [p(u)]^q}{q-1} \qquad (q \in \mathcal{R}) \tag{1}$$

which reduces, in the $q \to 1$ limit, to the standard Boltzmann-Gibbs entropy

$$S_1[p] \equiv - \int du\; p(u)\; ln\; p(u) \tag{2}$$

We show here that an approach similar to that of [11, 12, 13] makes possible a quite satisfactory discussion of a sensibly different anomalous diffusion, namely of the *correlated* type, characterized by the following generalized, *nonlinear* Fokker-Planck equation:

$$\frac{\partial}{\partial t}[p(x,t)]^{\mu} = -\frac{\partial}{\partial x}\left\{F(x)[p(x,t)]^{\mu}\right\} + D\frac{\partial^2}{\partial x^2}[p(x,t)]^{\nu} \tag{3}$$

where $(\mu, \nu) \in \mathcal{R}^2$, $D > 0$ is a (dimensionless) diffusion constant, $F(x) \equiv -dV(x)/dx$ is a (dimensionless) external force (*drift*) associated with the potential $V(x)$, and (x,t) is a (dimensionless) 1+1 space-time. Let us mention that, in variance with the correlated type we are focusing on here, the Lévy-like anomalous diffusion is associated with a *linear* equation, though in fractional derivatives [6].

We intend to consider here a specific (but very common) drift, namely characterized by $F(x) = k_1 - k_2 x$ ($k_1 \in \mathcal{R}$ and $k_2 \geq 0$; $k_2 = 0$ corresponds to the important case of constant external force, and $k_1 = 0$ corresponds to the so called Uhlenbeck-Ornstein process[19]). The particular case $\mu = \nu = 1$ corresponds to the standard Fokker-Planck equation, i.e., to *normal* diffusion. The particular case $F(x) = 0$ (no drift) has been considered by Spohn [8] for $\mu = 1$ and arbitrary ν (for instance, $\nu = 3$ satisfactorily describes a standard solid-on-solid model for surface growth), and has been extended by Duxbury[20] for arbitrary μ and ν. The case $(\mu, k_1) = (1, 0)$ has been considered by Plastino and Plastino[21]. Our present discussion recovers *all* of these as particular instances.

First, let us illustrate the procedure we intend to follow, by briefly reviewing normal diffusion ($\mu = \nu = 1$). We wish to optimize S_1 (given by Eq. (2)) with the constraints

$$\int du\, p(u) = 1 \ , \tag{4}$$

$$\langle u - u_M \rangle_1 \equiv \int du\, (u - u_M)\, p(u) = 0 \tag{5}$$

and

$$\langle (u - u_M)^2 \rangle_1 \equiv \int du\, (u - u_M)^2\, p(u) = \sigma^2 \ , \tag{6}$$

u_M and σ being fixed *finite* real quantities. The optimization straightforwardly yields the solution

$$p_1(u) = \frac{e^{-\beta(u-u_M)^2}}{Z_1} \tag{7}$$

with

$$Z_1 = \int du\, e^{-\beta(u-u_M)^2} = (\pi/\beta)^{1/2} \tag{8}$$

where $\beta \equiv 1/T$ is the Lagrange parameter associated with the constraint (6) and satisfies $\beta = 1/(2\sigma^2)$.

On the basis of Eq. (7) we propose, for the $\mu = \nu = 1$ particular case of Eq. (3) (i.e., the standard Fokker-Planck equation), the *ansatz*

$$p_1(x,t) = \frac{e^{-\beta(t)[x-x_M(t)]^2}}{Z_1(t)} \tag{9}$$

with

$$\frac{\beta(t)}{\beta(0)} = \left[\frac{Z_1(0)}{Z_1(t)}\right]^\lambda \tag{10}$$

It follows straightforwardly that $\lambda = 2$,

$$\frac{\beta(t)}{\beta(0)} = \left[\frac{Z_1(0)}{Z_1(t)}\right]^2 = \frac{1}{[1 - \frac{2D\beta(0)}{k_2}]e^{-2k_2 t} + \frac{2D\beta(0)}{k_2}} \tag{11}$$

and

$$\frac{dx_M(t)}{dt} = k_1 - k_2 x_M(t) \tag{12}$$

hence

$$x_M(t) = \frac{k_1}{k_2} + \left[x_M(0) - \frac{k_1}{k_2}\right] e^{-k_2 t}. \tag{13}$$

To discuss the $k_2 = 0$ case we can use $e^{-2k_2 t} \sim 1 - 2k_2 t$, which implies $x_M(t) = x_M(0) + k_1 t$ and $1/\beta(t) = [1/\beta(0)] + 4Dt$, which, in the limit $t \to \infty$ (i.e., $t >> 1/[4D\beta(0)]$), yields the familiar result $1/\beta(t) \sim 4Dt$. This result implies, in turn, the celebrated Einstein expression $\langle (x - x_M)^2 \rangle_1 \propto t$ for Brownian motion.

Let us now address the *general* (μ, ν) case. Following along the lines of Alemany and Zanette[11] (and the generic framework of the generalized thermostatistics[14, 15]) we now wish to optimize S_q (given by Eq. (1)). The constraints are Eq. (4),

$$\langle u - u_M \rangle_q \equiv \int du \; (u - u_M) \; [p(u)]^q = 0 \tag{14}$$

(which generalizes Eq. (5)) and

$$\langle (u - u_M)^2 \rangle_q \equiv \int \; (u - u_M)^2 \; [p(u)]^q = \sigma^2 \tag{15}$$

(which generalizes Eq. (6)). This is an appropriate moment for commenting that the reason for using $[p(u)]^q$ (instead of the familiar $p(u)$) in the constraints (14) and (15) is the (very essential) fact that by doing so we *preserve*[15] *the Legendre structure of Thermodynamics* and (through the nonnegativity of C_q/q[22], where C_q denotes the specific heat) *guarantee thermodynamic stability.* Let us consistently stress that the constraint (14) is equivalent to $\langle u \rangle_q = \langle u_M \rangle_q$, but not to $\langle u \rangle_q = u_M$ (since, unless $q = 1$, $\langle u_M \rangle_q \neq u_M$). All these peculiarities are of course originated by the essential *nonextensivity* that the index q introduces in the theory. For example, if we have two *independent* systems A and B (i.e., $p_{A*B}(u_A, u_B) = p_A(u_A)p_B(u_B)$), we immediately verify that $S_q(A*B) = S_q(A) + S_q(B) + (1-q)S_q(A)S_q(B)$.

It is straightforward to see that the above described optimization of S_q yields

$$p_q(u) = \frac{[1 - \beta(1-q)(u - u_M)^2]^{\frac{1}{1-q}}}{Z_q} \tag{16}$$

with

$$Z_q = \int du \; [1 - \beta(1-q)(u - u_M)^2]^{\frac{1}{1-q}} \tag{17}$$

In the limit $q \to 1$, these equations reduce to Eqs. (7) and (8), respectively. The corresponding *ansatz* for solving Eq. (3) now is

$$p_q(x,t) = \frac{\{1 - \beta(t)(1-q)[x - x_M(t)]^2\}^{\frac{1}{1-q}}}{Z_q(t)} \tag{18}$$

with

$$\frac{\beta(t)}{\beta(0)} = \left[\frac{Z_q(0)}{Z_q(t)}\right]^{\lambda} \tag{19}$$

(as before, $\beta(t)$ and $Z_q(t)$ are nothing but the scaling of space with time). A tedious (but straightforward) calculation yields $\lambda = 2\mu$, and $q = 1 + \mu - \nu$. An equation for $Z_q(t)$ is also found, namely

$$2\nu D\beta(0)[Z_q(0)]^{2\mu} - k_2[Z_q(t)]^{\mu+\nu} - \frac{\mu}{\mu+\nu}\frac{d[Z_q(t)]^{\mu+\nu}}{dt} = 0, \tag{20}$$

which can be solved by substituting $\widetilde{Z}(t) = Z_q(t)^{\mu+\nu}$. The resulting solution is (*for all values of* k_1)

$$Z_q(t) = Z_q(0)\left[\left(1 - \frac{1}{K_2}\right)e^{-t/\tau} + \frac{1}{K_2}\right]^{\frac{1}{\mu+\nu}} \tag{21}$$

with

$$K_2 \equiv \frac{k_2}{2\nu D\beta(0)[Z_q(0)]^{\mu-\nu}} \tag{22}$$

and

$$\tau \equiv \frac{\mu}{k_2(\mu+\nu)} \tag{23}$$

(See Fig. 1). The function $x_M(t)$ is the same as in the case of normal diffusion, Eq. (13), since it only describes the motion of the average of the distribution $p_q(x,t)$, and does not depend on the way in which it spreads. $\beta(0)$ and $Z_q(0)$ are determined by the initial condition (i.e., by $p_q(x,0)$).

For $k_2 = 0$, Eq. (21) becomes

$$Z_q(t) = \left\{[Z_q(0)]^{\mu+\nu} + \frac{2\nu(\nu+\mu)D\beta(0)[Z_q(0)]^{2\mu}}{\mu}t\right\}^{\frac{1}{\mu+\nu}} \tag{24}$$

which, for $t \to \infty$, asymptotically recovers Duxbury's solution[20], namely $1/\beta(t) \propto [Z_q(t)]^{2\mu} \propto t^{\frac{2\mu}{\mu+\nu}}$. As we see, $\mu/\nu = 1$, > 1 and < 1 respectively imply that $[x(t) - x_M(t)]^2$ scales like t (*normal diffusion*), faster than t (*superdiffusion*) and slower than t (*subdiffusion*). The limits $\mu/\nu = 0$ and $\mu/\nu = \infty$ correspond to "no diffusion" and ballistic motion, respectively.

For $(\mu, k_1) = (1, 0)$, the present set of equations reduces to that of Plastino and Plastino[21].

Let us mention that the general solution given in Eq. (21) can be derived from that for $\mu = 1$ (and arbitrary k_1) by defining $\widetilde{p}(x,t) = [p(x,t)]^{\mu}$, and $\widetilde{\nu} = \nu/\mu$, as can easily be seen from Eq. (3).

Finally, by using Eq. (19) with $\lambda = 2\mu$, we can verify that

$$\int dx\, p_q(x,t) = [Z_q(t)/Z_q(0)]^{\mu-1} \int dx\, p_q(x,0) \ . \tag{25}$$

Consequently, the norm ("total mass") is generically conserved for all times only if $\mu = 1$ ($\forall\ K_2$) or if $K_2 = 1$ ($\forall \mu$). For $0 \le K_2 < 1$ (a common case), the norm monotonically *increases (decreases)* with time if $\mu > 1$ ($\mu < 1$). If $K_2 > 1$, it is the other way around.

Before ending let us mention that, *also* when t grows to infinity, the solutions we have found must be physically meaningful. This imposes $\mu/\nu > -1$. Indeed, if $k_2 \neq 0$, τ in Eq. (21) must be positive, which implies $\mu/\nu > -1$. Also, if $k_2 = 0$, x must scale with an *increasing* function of t; hence, $\beta(t)$ must *decrease* with t, which implies (through Eqs. (19) and (24)) $2\mu/(\mu+\nu) > 0$, hence, the already mentioned restriction applies once again. The entire picture which emerges is indicated in Fig. 2 (we have not focused the $\mu < 0$ region because that would force us to discuss the stability of the solutions with respect to small departures, and this lies outside of the scope of the present work).

Summarizing, on *general* grounds, we have shown that thermostatistics allowing for nonextensivity constitute a theoretical framework within which a rather nice *unification* of *normal* and *correlated* anomalous diffusions can be achieved. Both types of diffusions have been founded, on equal footing, on primary concepts of (appropriately generalized) Thermodynamics and Information Theory. Moreover, the Lévy-like anomalous diffusion (see [13] and references therein) is described by a *linear* Fokker-Planck-like equation with *fractional* (time and space) derivatives, and the present correlated anomalous diffusion is described by a *nonlinear* Fokker-Planck-like equation with *integer* derivatives (in contrast with normal diffusion, which corresponds to the *linear* Fokker-Planck equation with *integer* derivatives). Since *both* types of anomalous diffusions can be handled within the present generalized thermostatistics, the conjecture is allowed that further unification can possibly be achieved by considering the generic case of a *nonlinear* Fokker-Planck-like equation with *fractional* derivatives.

On *specific* grounds, we have obtained, for a generic linear force $F(x)$, the physically relevant *exact* (space, time)-dependent solutions of a considerably generalized Fokker-Planck equation, namely Eq. (3).

It is with pleasure that one of us (C.T.) acknowledges warm hospitality by B. Widom at the Baker Laboratory. This work was carried out in the research group of B. Widom, and was supported by the National Science Foundation and the Cornell University Materials Science Center.

FIGURES

Fig. 1. The $\mu/\nu = 1/3$ example: (a) Time dependence of $\beta(0)/\beta(t) = [Z_q(t)/Z_q(0)]^{2\mu}$ for $Z_q(0) \neq 0$ and typical values of K_2 (indicated at the right of each curve). The curve for $K_2 = 0$ lies on the vertical axis. For $K_2 = 0.25, 0.5$, and 2 the asymptotic values for $t/\tau \to \infty$ are shown by the dashed lines. (τ is defined in Eq. (23).); (b) Time dependence of $\{\beta(0)[Z_q(0)]^{2\mu}\}/\beta(t) = [Z_q(t)]^{2\mu}$ for $Z_q(0) = 0$, $\beta(0)[Z_q(0)]^{2\mu} \neq 0$, and typical values of $K_2' \equiv k_2/\{2\nu D\beta(0)[Z_q(0)]^{2\mu}\}$ (indicated at the right of each curve). The curve for $K_2 = \infty$ coincides with the horizontal axis. All curves saturate at a finite value as $t \to \infty$, except that for $K_2' = 0$, which is proportional to $t^{2\mu/(\mu+\nu)}$ for all t.

Fig. 2. "Norm conservation" means that $N \equiv \int dx\, p_q(x,t)$ is time-invariant; "Norm creation" means that N monotonically increases (decreases) with time if $K_2 < 1$ ($K_2 > 1$); "Norm dissipation" means that N monotonically decreases (increases) with time if $K_2 < 1$ ($K_2 > 1$). "Normal diffusion", "Superdiffusion" and "Subdiffusion" refer to the fact that, for $k_2 = 0$, $(x - x_M)^2$ scales like t, faster than t and slower than t, respectively. The standard Fokker-Planck equation corresponds to $\mu = \nu = q = 1$. For the precise meaning of "unphysical", see the text. On the $\mu = 1$ line we have $q = 2 - \nu$; consequently, when ν varies from ∞ to -1, q varies from $-\infty$ to 3, which precisely is the interval within which Eq. (4) (and, consistently, $\int dx\, p_q(x,0) = 1$) can be satisfied.

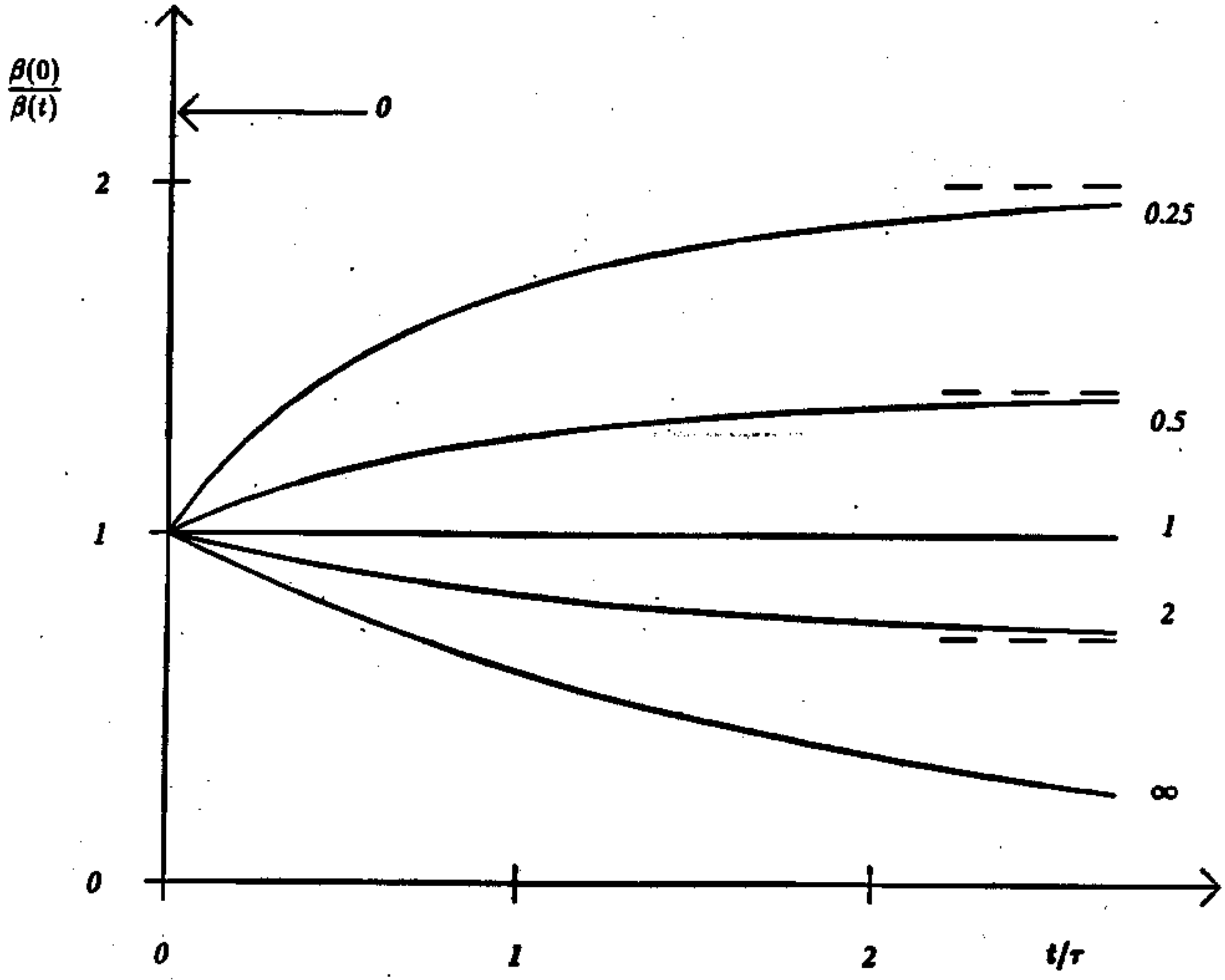

FIG. 1(a)

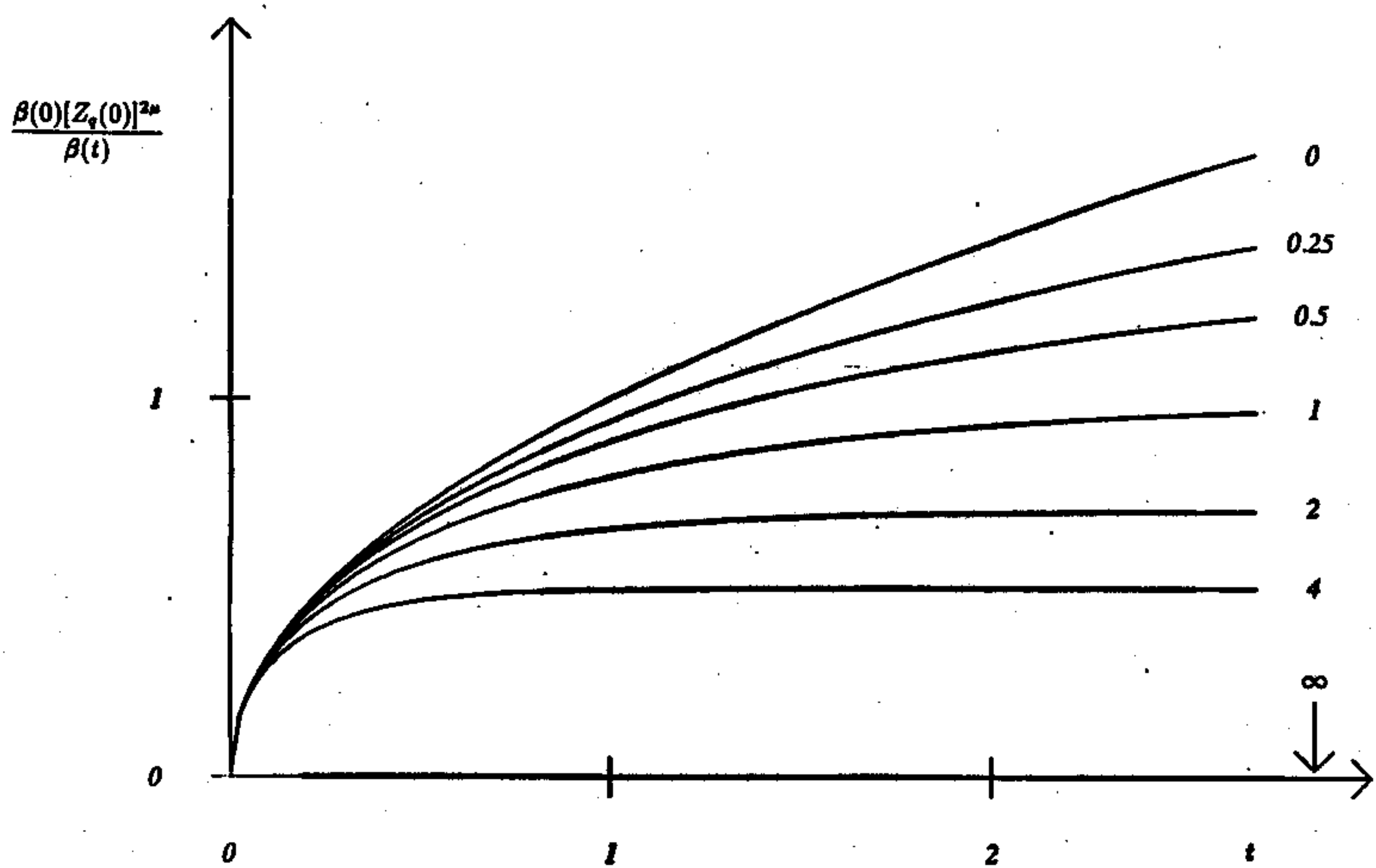

FIG. 1(b)

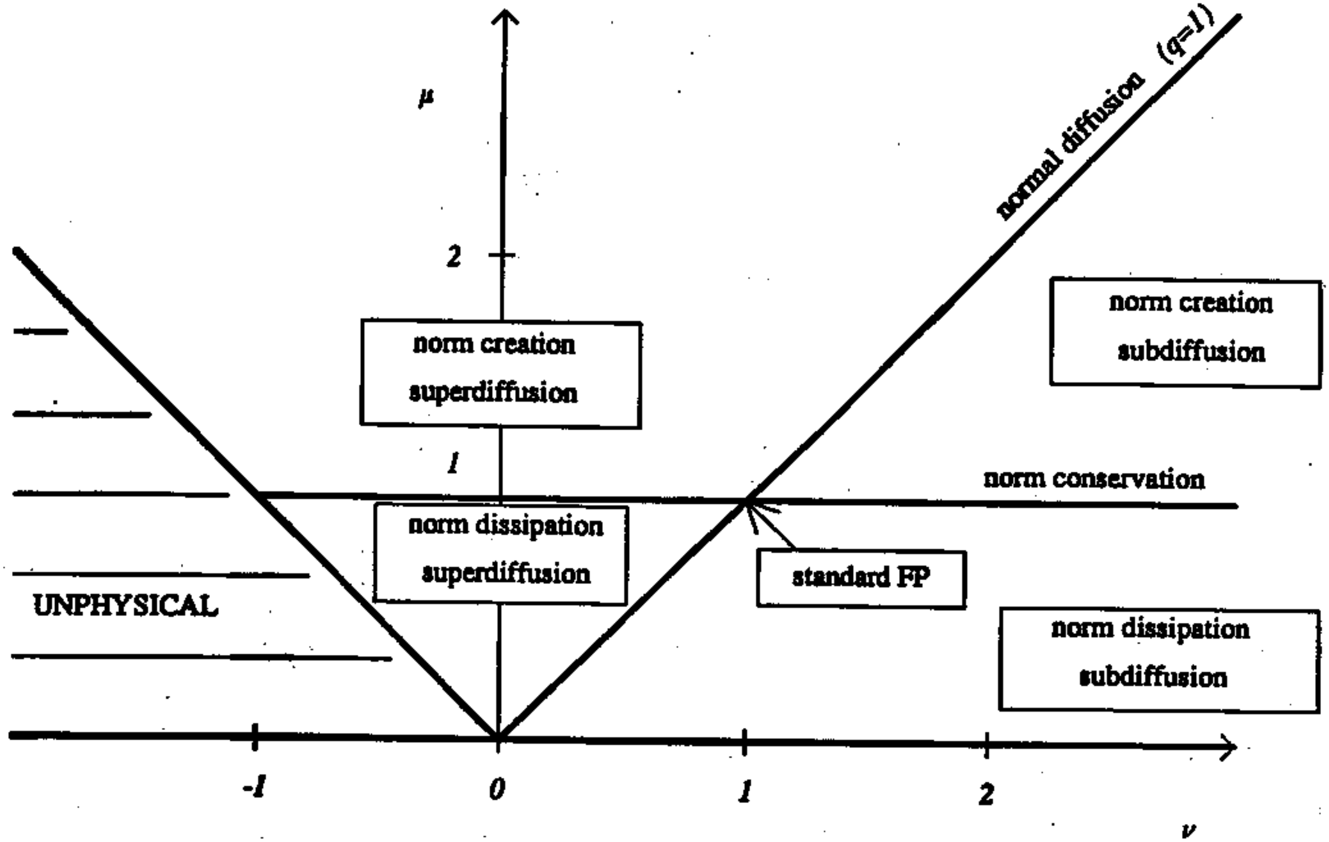
μ
2
1
normal diffusion (q=1)
norm creation
superdiffusion
norm creation
subdiffusion
norm conservation
norm dissipation
superdiffusion
standard FP
UNPHYSICAL
norm dissipation
subdiffusion
-1
0
1
2
ν

FIG. 2

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