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CRITICALITY OF THE ROUTES TO CHAOS OF THE $1-a|x|^z$ MAP

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ABSTRACT

We discuss, within a unified renormalization group (RG) framework, the onset of chaos appearing in the $1-a|x|^z$ map for real $z \geq 1$. In particular we study with some detail the criticality associated with the $k \to \infty$ limit of the p^k bifurcation sequences ($p=2,3,4$, which correspond respectively to the R^{*k}, $(RL)^{*k}$ and $(RL^2)^{*k}$ MSS sequences). The critical points $a_p^*(z)$ monotonically increase, for all values of p, from $a_p^*(1)$ ($1 \leq a_p^*(1) < 2$) to 2 while z grows from 1 to infinity. The z-dependence of $\delta_p(z)$ for $p = 2$ ($\delta_2(z)$ monotonically increases from 2 to infinity while z increases from 1 to infinity) is different from that associated with $p > 2$ ($\delta_p(z)$ diverges in both $z \to 1$ and $z \to \infty$ limits presenting a minimum in the neighborhood of $z = 2$). The present RG recovers, for both a_p and δ_p and all values of p, the exact asymptotic behaviors in the $z \to 1$ limit. It provides, in the $z \to \infty$ limit, the (possibly exact) asymptotic behaviors $a_p^*-2 \propto 1/z$ and $\delta_p \propto z$.

Key-words: Chaos; Bifurcations; One dimensional map; Criticality.

I. INTRODUCTION

Nonlinear dynamics is no doubt an important and very active area of research. More specifically, because of its analogy with critical phenomena as well as because of its strong intrinsic interest, the routes to chaotic behavior are nowadays being intensively studied from both theoretical [1-7] and experimental [8-12] standpoints. In particular the models associated with simple one-dimensional one-parameter maps have received considerable attention. The map

$$x_{t+1} = f_a(x_t) \qquad (1)$$

with

$$f_a(x) \equiv 1-a|x|^z \qquad (2)$$

has been investigated for $z=2$ [2] (quadratic map), for $z \geq 2$ (integer) [4,6,7] as well as for $z \to 1$ [4], and it is by now well established that z determines universality classes (in the sense that more complicate functions presenting the same type of extremum will share critical exponents, in particular the bifurcation ratios δ). The methods that have been used include analytical [4] as well as numerical and renormalization group (RG) [2,4-7] calculations.

Herein we follow along the lines of the latter (more specifically we use the "equality of slopes"-RG [4,7]), and extend the discussion to all real values of $z \geq 1$ (there is no onset of chaos for $z < 1$; see Ref. [5]). We study in detail the criti cality associated with the $k \to \infty$ limit (onset of chaos) of the p^k bifurcation sequences (hereafter referred to as *p-furcation* sequences) for $p = 2$ (bifurcations), $p = 3$ (trifurcations) and $p = 4$ (tetrafurcations), which respectively correspond to the R^{*k}, $(RL)^{*k}$ and $(RL^2)^{*k}$ MSS (Metropolis, Stein and Stein[5]) sequences (for this nomenclature, see Refs. [3,4]). The z-dependences of both critical points a_p^* and p-furcation ratios δ_p are calculated, and the $z \to 1$ and $z \to \infty$ asymptotic behaviors are discussed. We present in Section II the RG formalism, and in Section III the results.

II. RENORMALIZATION GROUP FORMALISM

In order to recall the RG procedure[4,7] let us introduce the notation $f_a^{(1)}(x) \equiv f_a(x)$, $f_a^{(2)}(x) \equiv f_a(f_a(x))$, $f_a^{(3)}(x) \equiv f_a(f_a(f_a(x)))$, and so on. No finite attractive basin exists for the map associated with Eq.(2) if $a > 2$; on the other hand no chaos can appear for $a < 0$. Consequently we restrict our discussion to $0 \leq a \leq 2$, $\forall z$ (as a matter of fact we shall verify later on that no onset of chaos is possible below $a = 1$; therefore all the interesting phenomena are going to occur in the interval $1 \leq a \leq 2$, $\forall z$).

We verify that the equation

$$x = f_a^{(1)}(x) \tag{3}$$

admits two roots, noted $x_1^{(1)}(a) \geq 0$ and $x_2^{(1)}(a) \leq 0$. $x_1^{(1)}(a)$ equals unity for $a=0$ and monotonically decreases for increasing a; it is a stable fixed point for a low enough, and becomes an unstable one above a certain value of a (which equals 3/4 for $z=2$). $x_2^{(1)}(a)$ monotonically increases from minus infinity to zero, while a increases from zero to infinity; it is an unstable fixed point of no particular interest.

We also verify that the equation

$$x = f_a^{(2)}(x) \qquad (4)$$

admits the root $x_1^{(1)}(a)$ (besides $x_2^{(1)}(a)$); but, for a greater than the value mentioned before, a bifurcation occurs and two new roots appear, noted $x_1^{(2)}(a)$ and $x_2^{(2)}(a)$, and satisfying $x_1^{(2)}(a) \geq x_1^{(1)}(a) \geq x_2^{(2)}(a)$, as well as $x_1^{(2)}(1) = 1$ and $x_2^{(2)}(1) = 0$, $\forall z$ (see Fig. 1).

Consider now the equation

$$x = f_a^{(3)}(x) \qquad (5)$$

It admits of course $x_1^{(1)}(a)$ as a root; but for a greater than a certain value (which equals 1.75 for $z=2$), a trifurcation occurs by pairs (see Fig. 2), and six new roots appear, noted $x_1^{(3)}, x_2^{(3)}, \ldots, x_6^{(3)}$.

The next step ($p=4$) is indicated in Fig. 3. In general the equation

$$x = f_a^{(p)}(x) \qquad (p = 1,2,3,\ldots) \qquad (6)$$

will present 2^p roots, some of them having appeared earlier; through one or several p-furcations, new roots appear which will be noted $x_1^{(p)}$, $x_2^{(p)}$, $x_3^{(p)}$,..., starting from the highest among them.

It is well known (at least for $z = 2$) that below $a = 2$ a complex bifurcation scheme occurs. In particular, as a increases from 0 to 2, a succession of attractive fixed cycles determines the recursive evolution of x. The first family of cycles which appears is the 2^k one (i.e., R^{*k}), and is responsible for the first onset of chaos (with critical point a_2^* and bifurcation rate δ_2). An infinite number of other families follow: we are presently interested in the p^k-cycles, with critical points a_p^* and rates δ_p. It is worthy to stress that, for instance, $p = 3$ refers to the (disconnected) family 3^k (i.e., $(RL)^{*k}$), which has to be distinguished from the (connected) family 3×2^k (i.e., $(RL)*R^{*k}$), whose critical point lays between a_2^* and a_3^*, and whose rate is δ_2 (same universality class as 2^k). Let us also recall[1,4] that a single family of primary p-furcations is associated with each value of p for $p = 2,3$ and 4 (see Figs. 1-3); but there are three different families (namely $(RLR^2)^{*k}$, $(RL^2R)^{*k}$ and $(RL^3)^{*k}$) for $p = 5$, and a rapidly increasing number of them for increasing p: each of them determines, for a given value of z, a universality class.

The eigenvalue associated with the p-cycle of $f_a(x)$ is given[4] by

$$\lambda_p(a) \equiv \left.\frac{d\, f_a^{(p)}(x)}{dx}\right|_{x = x_i^{(p)}}$$

$$\equiv \prod_j \left.\frac{d\, f_a^{(1)}(x)}{dx}\right|_{x = x_j^{(p)}} \qquad (p=1,2,3\ldots) \qquad (7)$$

where j runs over the p stable roots (of Eq. (6)) corresponding to the chosen family of sequences and i is anyone among them (e.g., for $p = 3$, it is $j = 1,4,6$; see Fig. 2). The RG's (noted $RG_{b'b}^{(p)}$) describing the onset of chaos corresponding to the chosen family of p-furcations are determined[2,4] by the following recursive relations

$$\lambda_{b'}(a') = \lambda_b(a) \qquad (p=2,3,4,\ldots) \qquad (8)$$

where $b' = 1,p,p^2,p^3,\ldots,$ and $b = pb',p^2b',p^3b',\ldots$ In order to avoid nomenclature confusions let us illustrate this equation through the specific cases we shall discuss: (i) for $RG_{12}^{(2)}$ we calculate $\lambda_1(a')$ and $\lambda_2(a)$ by respectively using the roots $x_1^{(1)}(a')$ and $x_1^{(2)}(a)$ (see Fig. 1); (ii) for $RG_{24}^{(2)}$ we use $x_1^{(2)}(a')$ and $x_3^{(4)}(a)$ (see Fig. 3); (iii) for $RG_{14}^{(2)}$ we use $x_1^{(1)}(a')$ and $x_3^{(4)}(a)$ (see Fig. 3); (iv) for $RG_{13}^{(3)}$ we use $x_1^{(1)}(a')$ and $x_1^{(3)}(a)$ (see Fig. 2); (v) for $RG_{14}^{(4)}$ we use $x_1^{(1)}(a')$ and $x_1^{(4)}(a)$ (see Fig. 3). The approximate critical point $a^*_{p;b',b}$ corresponds to the unstable fixed point of Eq. (8), and therefore satisfies

$$\lambda_{b'}(a^*_{p;b',b}) = \lambda_b(a^*_{p;b',b}) \qquad (p=2,3,4,\ldots) \qquad (9)$$

The approximate p-furcation rate $\delta_{p;b',b}$ is given by

$$\delta_{p;b',b} = \left\{ \left.\frac{da'}{da}\right|_{a^*_{p;b',b}} \right\}^{pb'/b} = \left\{ \left.\frac{d\lambda_b(a)/da}{d\lambda_{b'}(a')/da'}\right|_{a^*_{p;b',b}} \right\}^{pb'/b} \tag{10}$$

The *exact* critical points a^*_p and rates δ_p are, in principle, given by $a^*_p = \lim_{\substack{b\to\infty \\ b'<b}} a^*_{p;b',b}$ and $\delta_p = \lim_{\substack{b\to\infty \\ b'<b}} \delta_{p;b',b}$.

III. RESULTS

Through use of the method described in the previous Section we have obtained the results indicated in Figs. 4,5 and 6 and Table I.

Let us first discuss the $z \to 1$ limit. For $p = 2$, all three $RG^{(2)}_{12}$, $RG^{(2)}_{14}$ and $RG^{(2)}_{24}$ recover the exact[4] asymptotic behavior; indeed they are all numerically consistent with

$$a^*_2(z) \sim 1 + \phi_2(z) \tag{11}$$

and

$$\delta_2(z) \sim 2 + \frac{z-1}{\phi_2(z)} \tag{12}$$

where

$$\phi_2(z) + (z-1)\,\ell n\,\phi_2(z) + z-1 = 0 \tag{13}$$

Notice that $\lim_{z\to 1} a_2^*(z) = 1$, and $\lim_{z\to 1} \delta_2(z) = 2$.

For p=3, the same happens with $RG_{13}^{(3)}$, as it recovers the exact result[4], namely

$$a_3^*(z) \sim \bar{a}_3(z) + \phi_3(z) \tag{14}$$

and

$$\delta_3(z) \sim \frac{(z-1)\bar{a}_3(z)}{\phi_3(z)} \tag{15}$$

where

$$1-\bar{a}_3(z)\left|\bar{a}_3(z)-1\right|^z \sim 0 \tag{16}$$

corresponds to the superstability point of the sequence RL, and

$$\phi_3(z) \sim \left\{ e^2 [2\bar{a}_3(z)-1] [\bar{a}_3(z)-1] [\bar{a}_3(z)]^{\frac{2}{z-1}} \right\}^{-1} \tag{17}$$

Notice that $\lim_{z\to 1} a_3^*(z) = (\sqrt{5}+1)/2 \simeq 1.61803$ (this is the same value into which merges[1,4], in the limit $z\to 1$, the *entire* window 3×2^k), and $\lim_{z\to 1} \delta_3(z) = \infty$.

For p = 4, we have only checked the limit itself, i.e., $RG_{14}^{(4)}$ provides $\lim_{z\to 1} a_{4;1,4}^*(z) = [(19+9\sqrt{11/3})^{1/3} + (19-9\sqrt{11/3})^{1/3}+1]/3 \simeq 1.83929$ (this is the same value into which merges, in the limit $z\to 1$, the *entire* window 4×2^k or $(RL^2)*R^{*k}$), and $\lim_{z\to 1} \delta_{4;1,4}(z) = \infty$, which are the exact results[4]. We recall that the exact value for $a_p^*(1)$ satisfies[4] $f_a^{(p)}(x=0,z=1) = 0$, i.e.

$$\underbrace{1-a\,|1-a\ldots|}_{p-2}\underbrace{1-a\,|\underbrace{1-a\,|}_{1}|}_{2}| = 0 \qquad (p=2,3,4,\ldots); \tag{18}$$

also[4] $\lim_{z\to 1} \delta_p(z) = \infty$, $\forall p > 2$.

Although we have not attempted a general proof, the above set of results strongly suggests that, in the limit $z\to 1$ and for *all p*, the $RG^{(p)}_{b'b}$'s asymptotically recover, for *all* (b',b), the exact results[4].

With respect to finite values of z-1, the overall situation is satisfactory: whenever previous results are available[4-7], they are recovered within the present framework (excepting for a few possible small inadvertences, e.g. Eqs.(4.67) and (4.68) of Ref. [7] are numerically inconsistent among them: the correct value of $\delta_{2;1,2}(2)$ is $[18+2\sqrt{17}]^{1/2} \simeq 5.1231056$, and not 5.1224575 as indicated therein, as well as in Ref. [6]). For $p=2$, the results provided by $RG^{(2)}_{24}$ prove to be numerically quite reliable when compared to the available exact results (see Table I): this is probably true for *all values of* z.

Finally a simple general tendency is numerically verified in the $z \to \infty$ limit, namely

$$a^*_{p;b',b}(z) \sim 2 - \frac{A_{p;b',b}}{z} \tag{19}$$

and

$$\delta_{p;b',b}(z) \sim B_{p;b',b}\, z \tag{20}$$

with $(A_{2;1,2}, B_{2;1,2}) \simeq (6.1\,,\,2.00)$, $(A_{2;1,4}, B_{2;1,4}) \simeq (4.3\,,\,0.51)$, $(A_{2;2,4}, B_{2;2,4}) \simeq (3.4\,,\,0.51)$ (this is probably a good approximation for $p=2$) and $(A_{3;1,3}, B_{3;1,3}) \simeq (0.05, 300)$.

With respect to the p-dependence, notice that, for both $p=3$ and $p=4$, $\delta_p(z)$ becomes minimal in the neighborhood of $z=2$[6] (see Fig.6 and Table I). We verify that roughly $\delta_3^{min}/\delta_2(2) \simeq$

$\simeq \delta_4^{min}/\delta_3^{min} \simeq 15$, which might be an indication that δ_p^{min} (or even $\delta_p(z)$ for all finite values of z-1) grows exponentially with p, in the $p \to \infty$ limit. This would imply that the cycle length critical exponent[4] $\nu_p(z) = \frac{\ell n\, p}{\ell n\, \delta_p(z)}$ vanishes when $p \to \infty$.

IV. CONCLUSION

Within the "equality of slopes"-renormalization group (RG) framework[4,7], we have studied the criticality corresponding to the onsets of chaos associated with the $1-a|x|^z$ map for real $z \geq 1$; we have specifically focused the 2^k family of bifurcations ($p = 2$; doubling-period type), the 3^k family of trifurcations ($p = 3$; tangent type), and finally the 4^k family of tetrafurcations ($p = 4$; tangent type). Our main conclusions can be summarized as follows:

i) We have numerically confirmed that, for $p = 2,3,4$, this RG recovers the *exact*[4] $z \to 1$ asymptotic behaviors; this is possibly true for *all values of p, all the sequences* existing for a given value of p, and *independently of the sizes* (b and b') of the cycles that have been used to construct the RG;

ii) The critical points $a_p^*(z)$ monotonically increase from $a_p^*(1)$ ($1 \leq a_p^*(1) < 2$) to 2 while z increases from 1 to infinity, and satisfy $a_2^*(z) < a_3^*(z) < a_4^*(z)$; along the same variation of z, the bifurcation rate $\delta_2(z)$ monotonically increases from 2 to infinity, while the rates $\delta_3(z)$ and $\delta_4(z)$ (and possibly all $\delta_p(z)$, $\forall p > 2$) present a single minimum in the neighborhood of $z = 2$ ($\delta_2(z) < \delta_3(z) < \delta_4(z)$, $\forall z \geq 1$; $\lim_{z\to 1} \delta_p(z) = \lim_{z\to\infty} \delta_p(z) = \infty$, $\forall\ p > 2$); see Table I for numerical approximations of $\{a_p^*\}$ and $\{\delta_p\}$ as functions of z;

(iii) The present RG results for $z >> 1$ are consistent with the $z \to \infty$ asymptotic behaviors (possibly correct for all $p \geq 2$) $a_p^* \sim 2 - A_p/z$, and $\delta_p \sim B_p z$, with $(A_2, B_2) \simeq (3, 05)$, $(A_3, B_3) \simeq (0.05, 300)$;

(iv) Some preliminary indications exist that, for fixed and finite z-1, $\delta_p(z)$ grows exponentially with p in the $p \to \infty$ limit; if so the cycle length critical exponent $\nu_p(z)$ vanishes in the $p \to \infty$ limit.

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CAPTION FOR FIGURES AND TABLE

Fig. 1 - The roots of the equation $x = f_a^{(2)}(x)$ as functions of a, for typical values of z.

Fig. 2 - The roots of the equation $x = f_a^{(3)}(x)$ as functions of a, for $z = 2$.

Fig. 3 - The roots of the equation $x = f_a^{(4)}(x)$ as functions of a, for $z = 2$.

Fig. 4 - z-dependence of the RG critical points $a^*_{p,b',b}$ associated with bifurcations ($p = 2$), trifurcations ($p=3$) and tetrafurcations ($p = 4$). The available numerically exact results[4] approximately lay between the dashed and dotted lines.

Fig. 5 - z-dependence of the RG bifurcation rates $\delta_{2;b',b}$; the dots denote numerically exact results[4,7].

Fig. 6 - z-dependences of the RG p-furcation rates $\delta_{p;b',b}$; $\delta_{3;1,3}$ ($\delta_{4;1,4}$) attains its minimum at 1.87 (2.3) where it equals 60.9638 (984.335). The dot indicates a numerically exact result[4].

Table I - RG critical points (a^*) and p-furcation rates (δ) for typical values of p and z (the numerically exact values are reproduced from Refs. [4,7]).

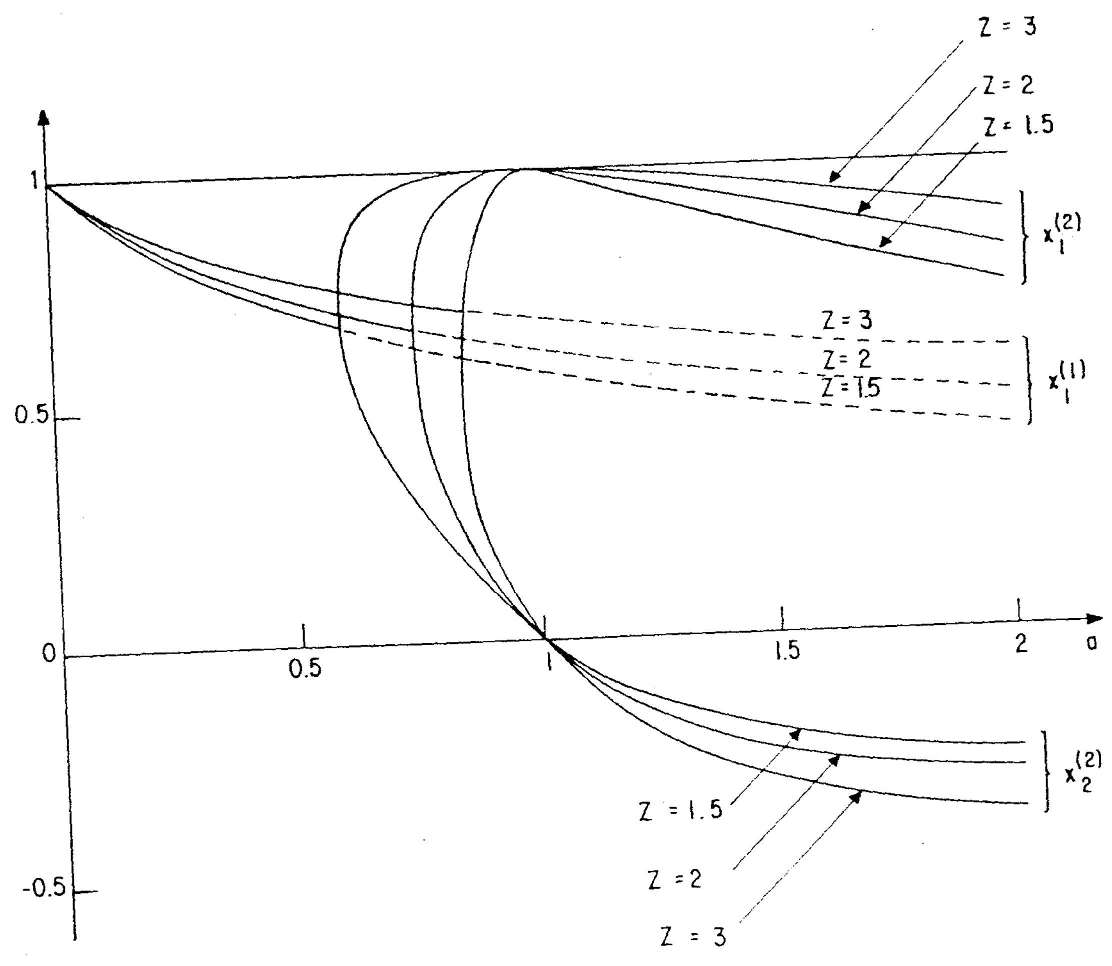

FIG.1

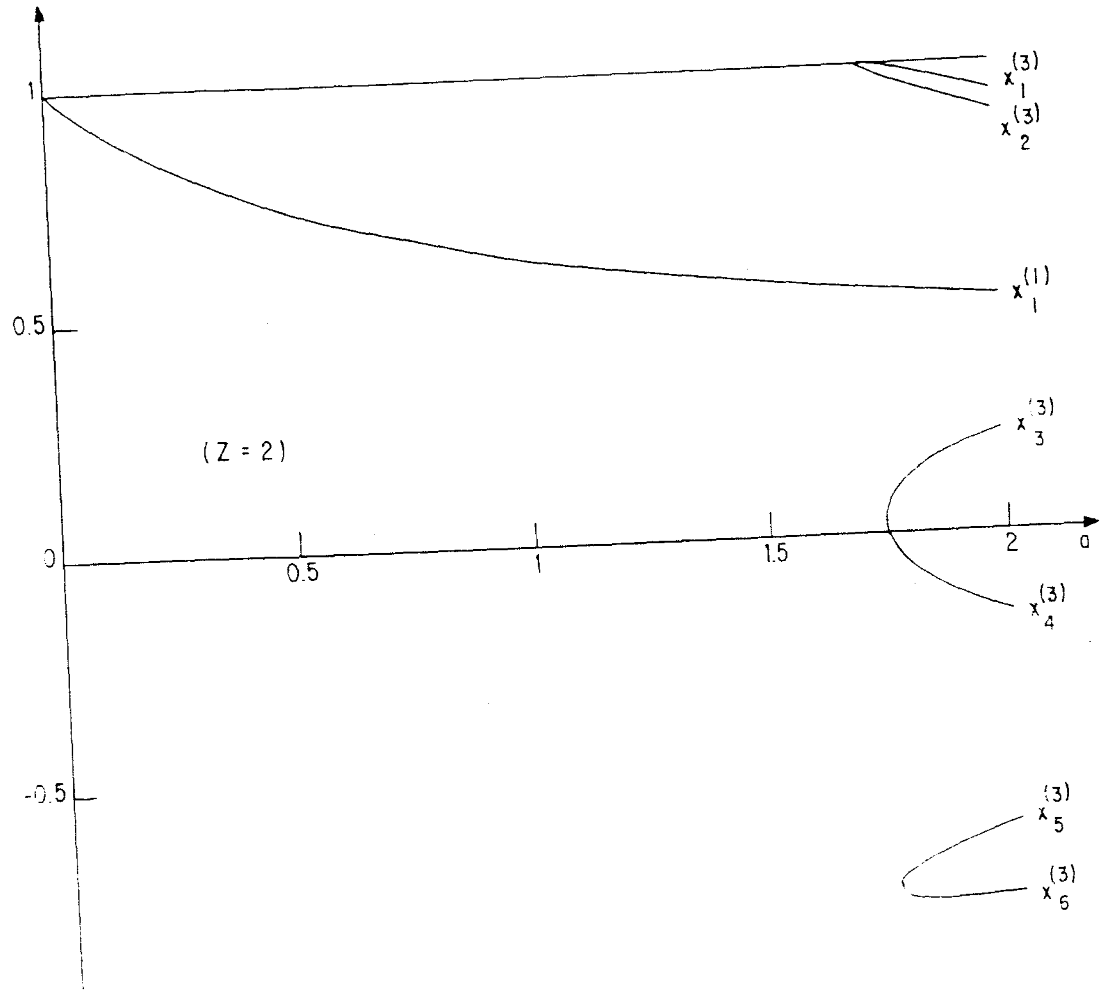

FIG 2

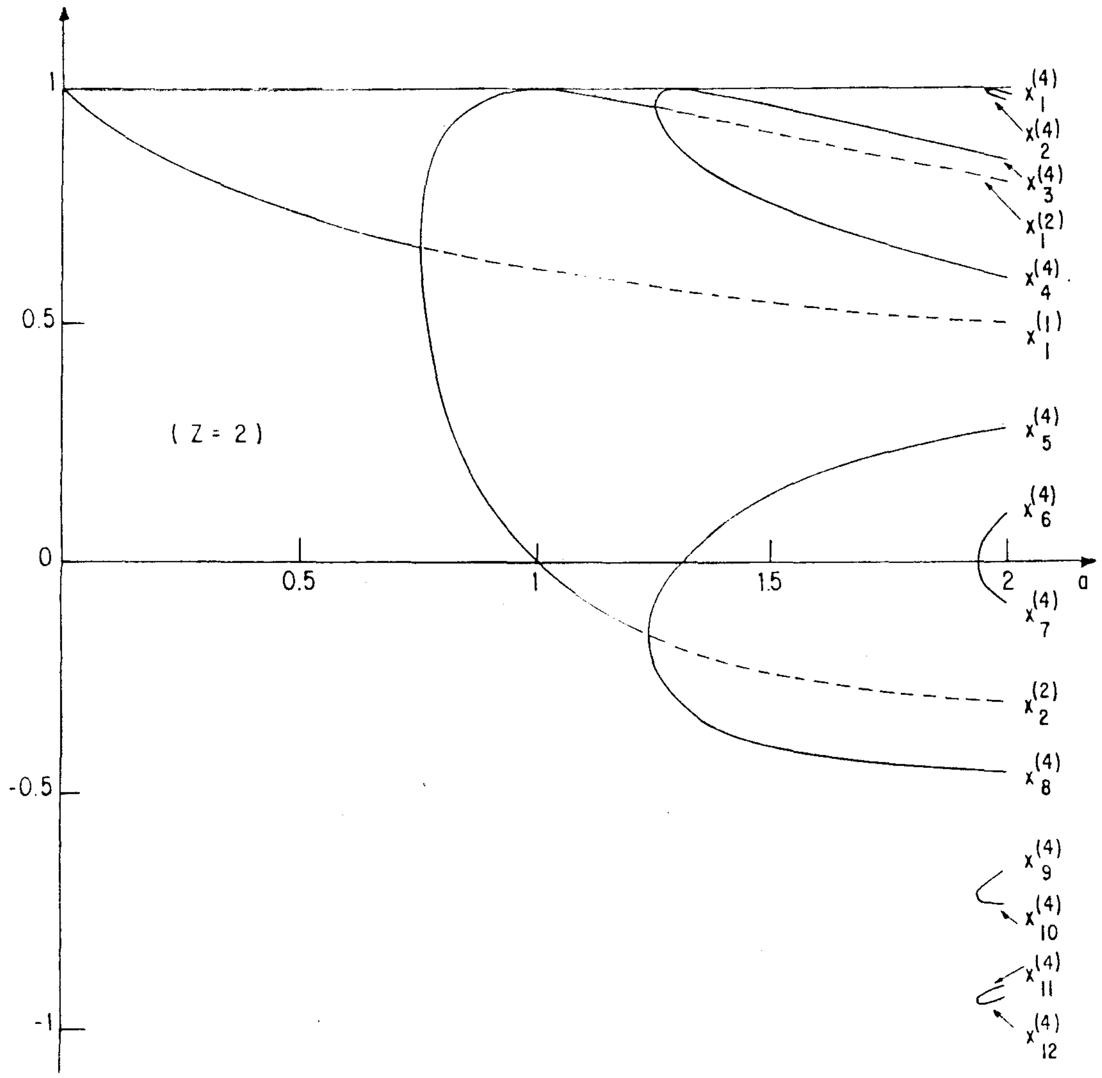
1
0.5
0
-0.5
-1
0.5
1
1.5
2
a
(Z = 2)
$x_1^{(4)}$
$x_2^{(4)}$
$x_3^{(4)}$
$x_1^{(2)}$
$x_4^{(4)}$
$x_1^{(1)}$
$x_5^{(4)}$
$x_6^{(4)}$
$x_7^{(4)}$
$x_2^{(2)}$
$x_8^{(4)}$
$x_9^{(4)}$
$x_{10}^{(4)}$
$x_{11}^{(4)}$
$x_{12}^{(4)}$

FIG.3

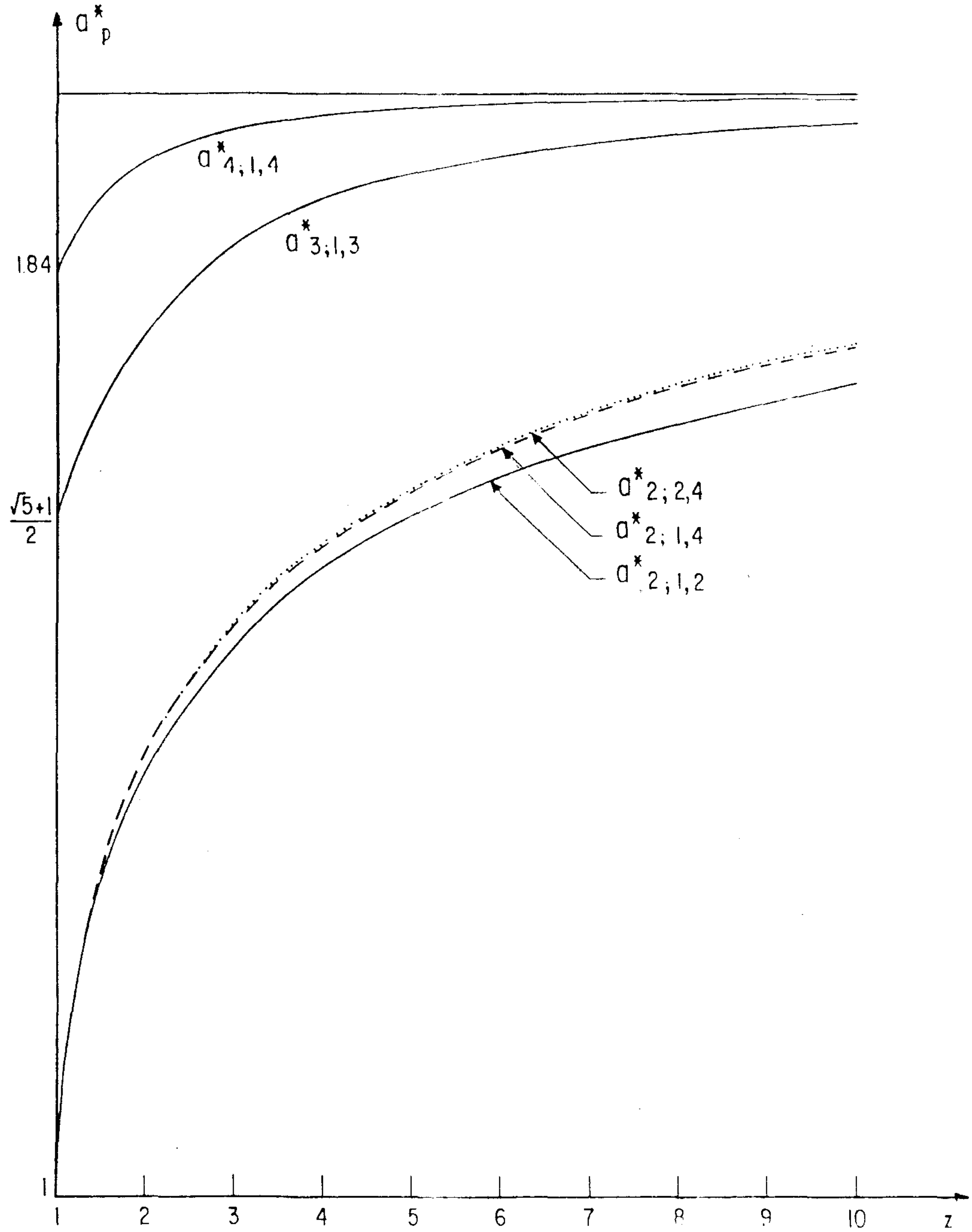

FIG 4

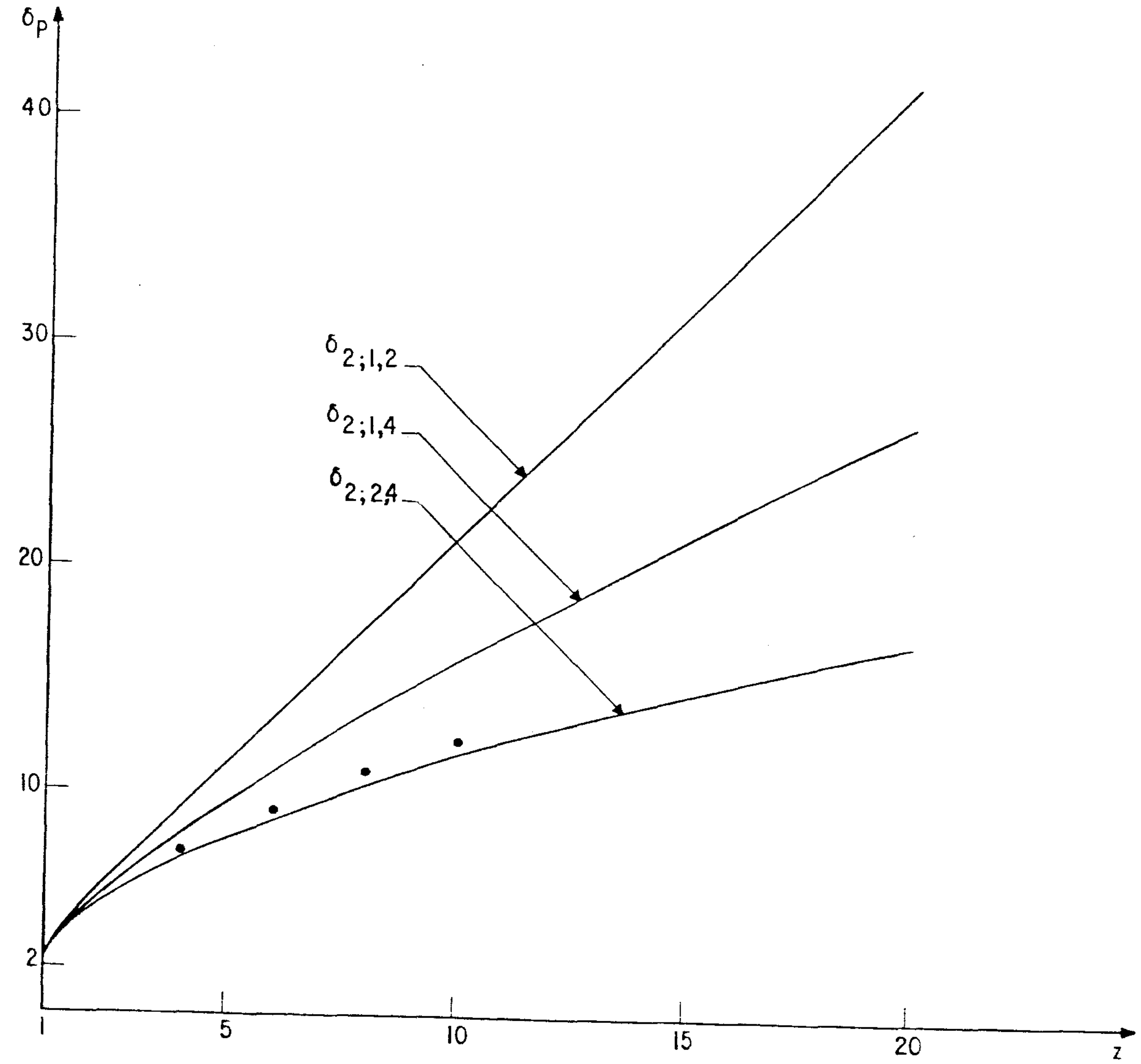

FIG. 5

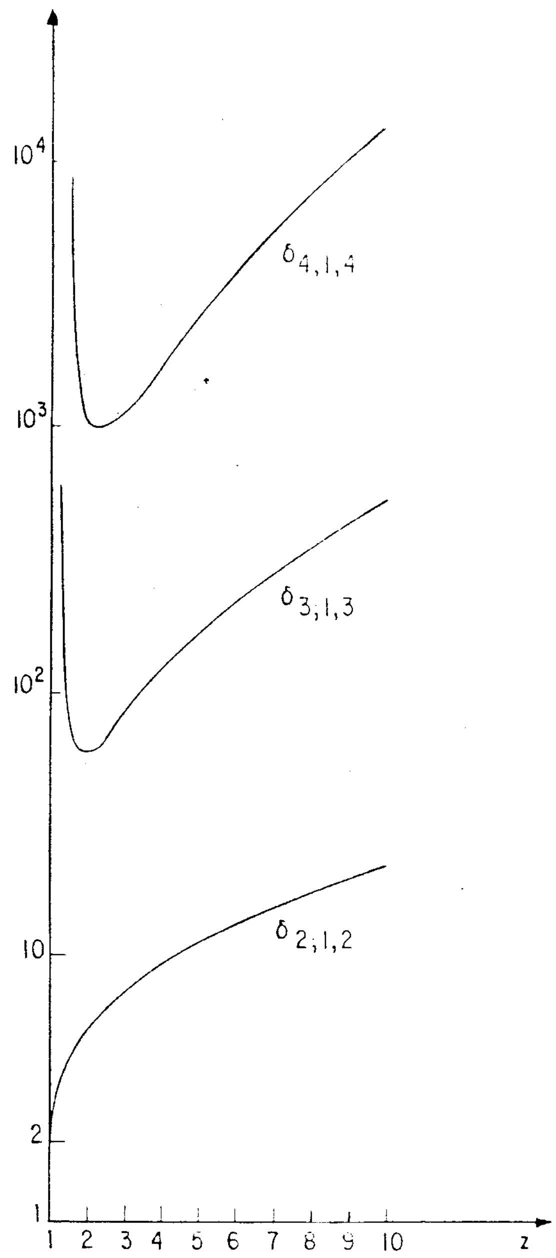

FIG. 6

TABLE I

Z		P = 2				P = 3	P = 4
		$RG_{12}^{(2)}$	$RG_{14}^{(2)}$	$RG_{24}^{(2)}$	EXACT (NUMERICAL)	$RG_{13}^{(3)}$	$RG_{14}^{(4)}$
1	a*	1	1	1	1	$\frac{\sqrt{5}+1}{2}$	1.83929
	δ	2	2	2	2	∞	∞
1.1	a*	1.12519	1.12542	1.12550	1.1236	-	-
	δ	2.84606	2.83818	2.83032	2.857	-	-
1.6	a*	1.31561	1.32036	1.32158	-	1.73047	1.91688
	δ	4.21360	4.09024	3.96930	-	66.7946	1841.21
2	a*	1.39039	1.39941	1.40142	1.401155	1.78597	1.94269
	δ	5.12311	4.86448	4.61483	4.66920	61.6175	1037.93
3	a*	1.50181	1.51940	1.52250	1.51930	1.86656	1.97341
	δ	7.25324	6.55992	5.91871	5.9880	85.6892	1182.48
4	a*	1.56925	1.59230	1.59575	-	1.90765	1.98544
	δ	9.31426	8.08956	6.99948	7.29	123.544	1802.96
5	a*	1.61663	1.64304	1.64652	-	1.93170	1.99111
	δ	11.3499	9.52224	7.94966	-	170.607	2742.07
6	a*	1.65252	1.68093	1.68432	1.68326	1.94713	1.99414
	δ	13.3734	10.88540	8.80831	9.2997	226.182	4023.66
7	a*	1.68100	1.71056	1.71379	-	1.95768	1.99592
	δ	15.3901	12.1939	9.59717	-	290.059	5689.92
8	a*	1.70433	1.73449	1.73755	-	1.96525	1.99704
	δ	17.4027	13.4572	10.3301	10.948	362.150	7787.26
9	a*	1.72391	1.75430	1.75719	-	1.97089	1.99777
	δ	19.4126	14.6820	11.0168	-	442.412	10363.5
10	a*	1.74063	1.77100	1.77373	1.77264	1.97521	1.99828
	δ	21.4206	15.8733	11.6643	12.3762	530.820	13466.9
20	a*	1.83286	1.85915	1.86078	-	1.99177	-
	δ	41.4623	26.5207	16.7790	-	1860.63	-
30	a*	1.87349	1.89565	1.89677	-	1.99585	-
	δ	61.4843	35.7376	20.5271	-	3997.91	-
∞	a*	2	2	2	2	2	2
	δ	∞	∞	∞	∞	∞	∞