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Light-Front Quantized Field Theory: (an introduction) Spontaneous Symmetry Breaking. Phase Transition in ϕ^4 Theory

by

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LIGHT-FRONT QUANTIZED FIELD THEORY: (*an introduction*) *

Spontaneous Symmetry Breaking. Phase Transition in ϕ^4 Theory.

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Abstract

The field theory quantized on the light-front is compared with the conventional equal-time quantized theory. The arguments based on the microcausality principle would imply that the light-front field theory may become nonlocal with respect to the longitudinal coordinate even though the corresponding equal-time formulation is local. This is found to be the case for the scalar theory. The conventional instant form theory is sometimes required to be constrained by invoking external physical considerations; the analogous conditions seem to be already built in the theory on the light-front. In spite of the different mechanisms of the spontaneous symmetry breaking in the two forms of dynamics they result in the same physical content. The phase transition in $(\phi^4)_2$ theory is also discussed. The symmetric vacuum state for vanishingly small couplings is found to turn into an unstable symmetric one when the coupling is increased and may result in a phase transition of the second order in contrast to the first order transition concluded from the usual variational methods.

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1- Introduction:

The possibility of building dynamical theory of a physical system on the three dimensional hypersurface in space-time formed by a plane wave front advancing with the velocity of light was indicated by Dirac [1] in 1949. The initial conditions on the dynamical variables are now specified on the hyperplane (*light-front*), say, $x^0 + x^3 = 0$, which has a light like normal, the *front form*, in contrast to the usual formulation where we employ instead the $x^0 \equiv t = 0$ hyperplane, the *instant form*. He also argued that the light-front formulation should be simpler since seven of the ten generators of the Poincaré group turn out to be kinematical while in the case of the *instant form* there are only six of them. The kinematical generators correspond to the ones which leave the chosen hyperplane invariant. Latter in 1966 the *front form* dynamics was rediscovered by Weinberg [2] in the infinite momentum frame rules in the quantized field theory which were clarified by Kogut and Soper [3] in 1970 to be equivalent to the quantization on the light-front. Even earlier [4] the $p \to \infty$ technique played an important role in the derivation of the current algebra sum rules and it was observed [5] that it amounted to using appropriate light-front current commutators. The *front form* coordinates are also adopted frequently in the string theories [6] in order to be able to work with the physical degrees of freedom and to expose clearly the physical contents.

A remarkable feature of the theory quantized on the light-front is the apparent simplicity of the vacuum state. In many theories the interacting theory vacuum coincides with that of the perturbation (free) theory one. In fact, the four momentum components are now $(k^-, k^+, k^\perp)$ where $k^\pm = (k^0 \pm k^3)/\sqrt{2}$ and $k^\perp \equiv \bar{k} = (k^1, k^2)$ indicate the two components transverse to x^3-direction. For a massive free particle on its mass shell and $k^0 > 0$ we find $k^\pm$ nonvanishing and positive. On the other hand, in the *instant form*, the momentum eigenstates of a particle is specified by the components (k^1, k^2, k^3) which may take positive or negative values. We may construct here eigenstates of zero momentum with arbitrary number of particles (and antiparticles) which may mix with the vacuum state, without any particle, to form the ground state. In contrast in the light-front framework we require $k^+ \to 0$ for each of the particle entering the ground state with vanishing total momentum. Such configurations constitute a point with zero measure in the phase space and may not be of relevance in many cases. It should, however, be remarked that when dealing with momentum space integrals, say, the loop integrals, in some cases a significant contribution may arise precisely from such a (corresponding) configuration in the integrand; the reason being that we have to deal with products of several distributions.

The recent revival of interest [7,8] in the light-front theory has been motivated by the difficulties faced in the nonperturbative QCD- the gauge theory of quarks and gluons- in the usual *instant formulation*. The technique of the regularization on the lattice has been quite successful for some problems but it cannot handle, for example, the light (or chiral fermions) and has not been able yet to demonstrate confinenment of the quarks. We have also the open problem of reconciling the standard constituent quark model and QCD to describe the hadrons [7,8]. In the former we employ few valence quarks while in the latter the QCD vacuum state itself contains an infinite sea of constituent quarks and gluons (partons) with the density of low momentum constituents getting very large in view of infrared slavery. Another problem is that of relativistic bound-state computation in the presence of the complicated vacuum in the *instant form.* Recent studies [7,8] show that the application of Light-front Tamm-Dancoff method [7] may be feasible here. The front-form dynamics may serve as a complementary tool where we have a simple vacuum while the complexity of the problem is now transferred to the light-front Hamiltonian. In the case of the scalar field theory, for example, discussed below the corresponding light-front Hamiltonian is found [9,10] to be nonlocal due to the presence of *constraint equations* in the Hamiltonian formulation. A different description of the spontaneous symmetry breaking is obtained which is, however, equivalent ([9], Sec. 2) in the physical contents to the usual description in the *instant form.* In the latter case we customarily do add to the theory some physical requirements from outside while such conditions seem to be already incorporated in the light-front context through the self-consistency requirements and the constraint equations (Sec. 2). We give arguments that the *nonlocality* mentioned above is not unexpected and it *does not enter into conflict with the microcausality principle.*

A general feature of the *front form* theory is that it describes a constrained dynamical system and the construction of Hamiltonian formulation is not straightforward. The Dirac procedure [11] or its variants must be used to handle such a system. Selfconsistent classical Hamiltonian formulation [1] is very convenient to quantize the theory via the correspondence of the Dirac (Poisson) brackets with the commutators of the corresponding operators and also allows us to unify the principle of (special) relativity in the dynamical theory again by making use of these brackets.

We introduce the following notation. For the coordinate x^μ and for all other vector or tensor quantities we define the $\pm$ components $x^\pm = (x^0 \pm x^3)/\sqrt{2} = x_\mp$. We adopt x^+ to indicate the *light-front time coordinate* and x^- the *spatial longitudinal coordinate.* The *spatial transverse* components will be usually denoted by $\bar{x} \equiv x^\perp = (x^1 = -x_1, x^2 = -x_2)$. The metric tensor for the indices $\mu = (+,-,1,2)$ is given by $g^{++} = g^{--} = g^{12} = g^{21} = 0; g^{+-} = g^{-+} = -g^{11} = -g^{22} = 1$ and it is verified that $g_{\mu\nu}A^\mu B^\nu = A^\mu B_\mu = A^- B^+ + A^+ B^- - A^1 B^1 - A^2 B^2$ is the correct Lorentz invariant expression, for example, $x^\mu x_\mu \equiv x^2 = 2x^+ x^- - \bar{x}^2$. Under the pure Lorentz transformations in $(0,3)$ plane we note that the components $A^\pm$ undergo scale transformations such that both A^+B^- and A^-B^+ are left invariant. Their sum corresponds to the usual invariant $A^0B^0 - A^3B^3$, while the difference to the invariant $A^0B^3 - A^3B^0$ which has a symplectic structure. It is easily verified that the transformation from the usual coordinates $(x^0, x^3, \bar{x})$ to the coordinates $(x^+, x^-, \bar{x})$ is *not* a Lorentz transformation.

It is well known that two distinct points lying on the hyperplane $x^0 = const.$ are separated by space like distance, e.g., $(x-y)^2 = -(\vec{x}-\vec{y})^2 < 0$, and the separation becomes light like when the two points become coincident. The points on the hyperplane $x^+ = const.$ also have space like separation for $x^\perp \neq y^\perp$. It becomes light like when $x^\perp = y^\perp$, however, with the *difference* that the points now need not become coincident, since $(x^- - y^-)$ is not required to be vanishing. This observation when combined with the *microscopic causality* postulate: '*the commutators of two physical observables pertaining to space-time points which are separated by a space like distance be vanishing*', leads to the result that the *front form* (quantized) dynamics may become non-local with respect to the longitudinal (space) coordinate x^-. Consider, for example, the commutator $[A(x^0,\vec{x}), B(0,\vec{0})]$ of two scalar observables $A(x)$ and $B(x)$ where $\vec{x}$ indicates the usual 3-vector (in equal-time formulation). It is function of the invariant x^2 due to Lorentz invariance and vanishes for $x^2 < 0$ if microcausality condition is assumed. Employing the light-front coordinates and evaluating the commutator on the light-front we find that $[A(x^+,x^-,\bar{x}), B(0,0,\bar{0})]_{x^+=0}$ should vanish for $\bar{x} \neq 0$, since $x^2 = -\bar{x}^2 < 0$ when $x^+ = 0$. This commutator hence is non-vanishing only for $\bar{x} = 0$ when also x^2 becomes light-like. We thus expect that its value contains a $\delta^2(\bar{x})$ and its derivatives which would imply locality in $\bar{x}$. No constraint, however, is obtained on the x^- dependence which is arbitrary. In the *instant form* case similar arguments applied to the equal-time commutator $[A(x^0,\vec{x}), B(0,\vec{0})]_{x^0=0}$ lead to the possible presence of $\delta^3(\vec{x})$ and its derivatives implying locality in all the three space coordinates. We remark that in view of the microcausality the knowledge of the equal-x^+ or equal-x^0 commutator is equivalent to finding its value on the light-cone $x^2 = 0$, while approaching it from the space like regions.

In order to obtain some information on the nature of the light-front commutator, say, of the scalar field, we may consider the corresponding *Lehmann spectral representation* [12] for the vacuum expectation value of the commutator given by

$$\langle 0|[\phi(x),\phi(0)]|0\rangle = \int_0^\infty d\sigma^2\, \rho(\sigma^2)\, \Delta(x;\sigma^2). \tag{1.1}$$

Here ϕ is is an Heisenberg operator, $\rho(\sigma^2)$ is Lorentz invariant positive-definite spectral function, and $\Delta(x;\sigma^2)$ is the free field commutator function

$$\Delta(x;\sigma^2) = \frac{1}{(2\pi)^3}\int_{-\infty}^{\infty} d^4k\, \epsilon(k^0)\delta(k^2-\sigma^2)\, e^{-ik.x} \qquad (1.2).$$

where the distribution $\epsilon(y) = -\epsilon(-y) = \theta(y) - \theta(-y) = 1$ for $y > 0$. In case the theory is derived from a local Lagrangian we may also establish, by making use of the canonical equal-time commutation relations, the result

$$\int_0^\infty d\sigma^2\, \rho(\sigma^2) = 1 \qquad (1.3).$$

Let us compute the free field commutator (2) on the light-front $x^+ = 0$. We note that $d^4k = d^2\bar{k}dk^+dk^-$, $k^2 = 2k^+k^- - \bar{k}^2$, $k.x = k^+x^- + k^-x^+ - \bar{k}.\bar{x}$, $and(2|k^+|)\delta(k^2-\sigma^2) = \delta(k^- - [\bar{k}^2+\sigma^2]/(2k^+))$. In view of the mass shell condition implied by the delta function it is easily shown that $(k^-/k^+) > 0$ and from the definition $k^0 = (k^+ + k^-)/\sqrt{2}$ it follows that inside the integral $\epsilon(k^0) = \epsilon(k^+)$. On setting $x^+ = 0$ and integrating over k^- (to remove the delta function) and $\bar{k}$ we obtain

$$\Delta(x^+, x^-, \bar{x}; \sigma^2)|_{x^+=0} = -\frac{i}{4}\delta^2(\bar{x})\epsilon(x^-), \tag{1.4}$$

which does not depend on σ^2. On using (3) we obtain

$$[\phi(x^+, x^-, \bar{x}), \phi(0)]|_{x^+=0} = -\frac{i}{4}\delta^2(\bar{x})\epsilon(x^-), \tag{1.5}$$

as far as the vacuum expectation value is concerned. We will give below an independent derivation of this light-front commutator by quantizing the scalar field theory directly in the *front form* by following the Dirac procedure. From (1.5) we derive

$$[\partial_{x^-}\phi(x^+, x^-, \bar{x}), \phi(0)]|_{x^+=0} = -\frac{i}{2}\delta^2(\bar{x})\delta(x^-). \tag{1.6}$$

Comparing (1.6) with the equal-time commutator $[\pi(x^0, \vec{x}), \phi(0)]|_{x^0=0} = -i\delta^3(\vec{x})$, where $\pi = \partial_t\phi$, it is suggested that the canonical momentum in the light-front quantized theory is $\sim \partial_{x^-}\phi$ and is thus not an independent variable, a result we rederive below.

We note also that in the *front form* the Green's function are ordered with respect to light-front time x^+ rather than $x^0 \equiv t$. However, in view of the microcausality requirement, the retarded commutators $[A(x), B(0)]\theta(x^0)$ and $[A(x), B(0)]\theta(x^+)$ do agree. In the regions $x^0 > 0, x^+ < 0$ and $x^0 < 0, x^+ > 0$ where they seem to disagree x^2 is space like and they are both vanishing. Such retarded commutators appear in the LSZ reduction formulas [13] of the S-matrix elements in terms of the field operators.

We describe briefly the elements of the *Poincaré algebra* on the light-front and some of their properties. In the system of coordinates (x^0, x^1, x^2, x^3) with the metric $g_{\mu\nu} = diag\,(1, -1, -1, -1)$ the generators of the Poincaré algebra satisfy the following commutation relations:

$$[M_{\mu\nu}, P_\sigma] = -i(P_\mu g_{\nu\sigma} - P_\nu g_{\mu\sigma}),$$

$$[M_{\mu\nu}, M_{\rho\sigma}] = i(M_{\mu\rho}g_{\nu\sigma} + M_{\nu\sigma}g_{\mu\rho} - M_{\nu\rho}g_{\mu\sigma} - M_{\mu\sigma}g_{\nu\rho}) \tag{1.7}$$

Introducing the convenient variables $J_i = -(1/2)\epsilon_{ikl}M^{kl}$ and $K_i = M_{0i}$ where $i, j, k, l = 1, 2, 3$ we find

$$[J_i, F_j] = i\epsilon_{ijk}F_k \qquad for \qquad F_l = J_l, P_l \; or \; K_l \tag{1.8}$$

while

$$[K_i, K_j] = -i\epsilon_{ijk}K_k, \quad [K_i, P_l] = -iP_0 g_{ik}, \quad [K_i, P_0] = iP_i, \quad [J_i, P_0] = 0. \qquad (1.9)$$

We note that the six (kinematical) generators P_l, M_{kl} leave the hyperplane $x^0 = 0$ invariant. In the light-front coordinates on the other hand there are seven such generators P_-, P_1, P_2, $M_{12} = -J_3$, $M_{+-} = -K_3$, $M_{1-} = -(K_1 + J_2)/\sqrt{2} \equiv -B_1$, $M_{2-} = -(K_2 - J_1)/\sqrt{2} \equiv -B_2$ which leave the light-front $x^0 + x^3 = 0$ invariant. The remaining three generators related to the dynamic [1] are P_+, $M_{1+} = (K_1 - J_2)/\sqrt{2} \equiv S_1$, $M_{2+} = -(K_2 + J_1)/\sqrt{2} \equiv -S_2$. The generators P^+, P^1, P^2, S_1, S_2, and J_3 commute with P^-. The generators B_1, B_2, J_3, which also commute with P^+, span the E_2 subalgebra of the 'little' group which leaves the light like vector $(n^0 = 1, 0, 0, n^3 = 1)$ invariant. The K_3 boosts along the 3-direction ($[K_3, P^{\pm}] = \mp iP^{\pm}$)

$$e^{-i\eta K_3}\, P^{\pm}\, e^{i\eta K_3} = e^{\mp\eta}\, P^{\pm}, \qquad (1.10)$$

while under Galilean (transverse) boosts

$$e^{-i\bar{u}.\bar{B}} P_j e^{i\bar{u}.\bar{B}} = (P_j + u_j P^+), \qquad j = 1,2 \;\; and \;\; \bar{u}.\bar{B} = [u_1 B_1 + u_2 B_2] \qquad (1.11)$$

$$e^{-i\bar{u}.\bar{B}}\, P^-\, e^{i\bar{u}.\bar{B}} = P^- + \bar{u}.\bar{P} + \frac{1}{2}\bar{u}^2 P^+, \qquad e^{-i\bar{u}.\bar{B}}\, P^+\, e^{i\bar{u}.\bar{B}} = P^+. \qquad (1.12)$$

They are useful for constructing an eigenstates $|p^+, p^1, p^2\rangle$ starting, say, from the state described in the rest frame. We collect also

$$[B_1, P_1] = [B_2, P_2] = iP^+, \quad [B_1, P_2] = [B_2, P_1] = 0, \quad [B_j, P^-] = iP_j, \quad [B_j, P^+] = 0 \qquad (1.12)$$

$$[S_j, P^-] = 0, \qquad [S_j, P^+] = iP_j, \qquad [S_1, S_2] = 0, \qquad [S_1, P_1] = [S_2, P_2] = iP^-, \qquad (1.13)$$

$$[B_1, B_2] = 0, \quad [B_1, J_3] = -iB_2, \qquad [B_2, J_3] = iB_1, \qquad [B_j, K_3] = iB_j, \quad [J_3, K_3] = 0. \qquad (1.14)$$

2- Spontaneous symmetry breaking mechanism in light-front quantized scalar field theory:

The quantization of the scalar field theory in the *instant form* is found in the text books but the quantization on the light-front has been clarified only recently. Working directly in the continuum it was demonstrated [9,10] that corresponding to the local Hamiltonian in the *instant formulation* we in fact now obtain a nonlocal Hamiltonian in the *front form* formulation. The nonlocality arises along the longitudinal direction x^- because of a new ingredient in the form of the nonlocal *constraint equations* in the Hamiltonian formulation. The treatment of the theory in the usually adopted discretized formulation (assuming the finite volume) [14-18,9] introduces spurious finite size contributions which make the physical interpretations difficult. The infinite volume limit may, however, be taken [9] which coincides with the continuum formulation [10]. As argued in Sec. 1 such a nonlocality is not unexpected and we pay the due price for working with a simple vacuum on the light front. The constraint equations allow to describe the tree level spontaneous symmetry breaking [10,14-18] and suggest the modifications that would be introduced by the quantum corrections [9]. The problem had been a a challenge for a long time [8]. The reason seems to be that of not distinguishing clearly between the bosonic condensate associated with the scalar field and the field which describes the (quantum) fluctuations (see also 2*(c)*). We remind that the scalar field theory plays an important role in many branches of physics. It is relevant, say, in connection with the generalized Ising models [19] in the condensed matter theory, is an indispensable ingredient (Higgs sector) of the Standard Model of electro-weak interactions, plays an important role in describing inflationary cosmology, and in the construction, say, of the heterotic string theory [6].

Dirac [1] discussed the problem of fitting together in a dynamical theory the two general requirements: *it should be quantized and also incorporate the special theory of relativity (ignoring gravitation).* He showed that Hamiltonian formulation, where we introduce a new element in the form of Poisson brackets, is a convenient procedure to attain this objective. We would apply the Dirac method [11] to construct this formulation.

Discrete symmetry in two dimensions:

(a)- Continuum formulation:

In order to emphasize the new features which distinguish the *front form* from the *instant form* we consider first the simpler case of a real massive scalar field theory in two dimensions. The Lagrangian density in the *front form* is given by $[\dot{\phi}\phi' - V(\phi)]$, where an overdot and a prime indicate the partial derivatives with respect to the light-front time $\tau \equiv x^+ = (x^0 + x^1)/\sqrt{2}$ and the longitudinal coordinate $x \equiv x^- = (x^0 - x^1)/\sqrt{2}$ respectively. In contrast to the case in the *instant form* dynamics the $\dot{\phi}$ now occurs linearly in the Lagrangian and the eq. of motion is $2\dot{\phi}' = -V'(\phi)$, where a prime on V indicates the variational derivative with respect to ϕ. It shows that the classical solutions, $\phi = const. \equiv \omega$, are allowed and given by solving $V'(\phi) = 0$. On the other hand the vacuum expectation value (vev) $\langle vac|\phi|vac\rangle$ of the quantized field must be a constant in view of the requirement of the translation invariance of the theory. It is suggestive that there is a certain correspondence between the value ω with the tree level *vev* of the quantized field ϕ. We now note that if we integrate the eq. of motion over $-L/2 \leq x \leq L/2$, where $L \to \infty$, we are led to the constraint equation, $\int dx\, V'(\phi) = -2\,\partial_\tau[C(\tau, L)]$, where $C(\tau, L) = [\,\phi(\tau, x = L/2) - \phi(\tau, x = -L/2)]$. The constraint thus seems to depend on the boundary conditions imposed on ϕ and thus needs clarification. We should, however, first formulate the *physical problem* at hand more precisely. Moreover, we need to write the equations in Hamiltonian form as stressed above. Taking into consideration the discussion made here we propose to make the separation $\phi(x,\tau) = \omega(\tau) + \varphi(x,\tau)$ for any fixed value of light-front time τ. The variable ω corresponds to the bosonic condensate or the background field while φ describes the (quantum) fluctuations above the condensate. This separation should be done independently of whether we work in the continuum formulation or in the discretized one where L is taken finite (finite volume), frequently employed. It should be emphasized, however, that a well defined infinite volume limit must exist if the theory is physical and self-consistent [20]. At the classical level the φ field is an *ordinary* function of x such that ($\int dx\, |\varphi| < \infty$) and such that its Fourier transform (or Fourier series) along with its inverse are defined while ϕ is a generalized function of x due to the constant term ω. Since our primary interest is to discuss the vacuum states we will ignore presently the τ dependence of ω and return to the general case latter below.

The Lagrangian then reads as

$$\int_{-L/2}^{L/2} dx\, [\dot{\varphi}\,\varphi' - V(\phi)], \tag{2.1}$$

where for illustration purposes we take, $V(\phi) = -(1/2)m^2\phi^2 + (\lambda/4)\phi^4 + const.$, $\lambda > 0$, with the wrong sign for the mass term and $L \to \infty$. We next construct the classical Hamiltonian formulation for the system which may be used to make transition [1] to a relativistic and quantized theory through the correspondence principle, in analogy with what we do in the construction of the quantum mechanics or alternatively, by employing the functional integral technique. The canonical light-front Hamiltonian is easily obtained to be $\int dx V(\phi)$. However, in the *front form* dynamics there is no physical argument available to minimize the light-front energy in order to obtain the (classical) ground states. In equal-time formulation we *add* new ingredients to the theory invoking physical considerations such as $\partial_t\varphi = 0$ and $\partial_{x^1}\varphi = 0$ which reduce the energy and argue then to minimize the energy functional in order to obtain the ground states. We will demonstrate below that *the light-front dynamics already incorporates such information in the theory through self-consistency requirements and the new ingredients found in the form of the constraint equations. The physical results in the two forms of dynamics* *coincide even though obtained through different mechanisms.* We remind that in the Hamiltonian approach for nonsingular Lagrangians the number of dynamical variables to describe the theory is doubled and it also contains the first-order Hamilton eqs. whose number is twice as compared to the number of the second-order Lagrange eqs. The approach also introduces a new object in the form of a Poisson bracket which allows us to satisfy in the theory both the requirements of the special theory of relativity and of the transition to the quantum theory. It is also richer than the Lagrangian formulation in that it allows for a broader set of general transformations (on the phase space). In a self-consistent Hamiltonian formulation the Lagrangian formulation must be recovered [11]. In contrast to the *instant form* the light-front Lagrangian (2.1) is singular and the Hamiltonian here determines the evolution of the dynamical system with changing τ in place of t. We follow the Dirac method to build the canonical framework at a given τ. Indicating by $\pi(x,\tau)$ the momenta conjugate to $\varphi(x,\tau)$, the primary constraint is found to be $\Phi \equiv \pi - \varphi' \approx 0$ while the canonical Hamiltonian density is $\mathcal{H}_c = V(\phi)$, with the symbol $\approx$ standing for the weak equality [11]. We postulate now the standard Poisson brackets at equal-τ, with the nonvanishing brackets satisfying, $\{\pi(x,\tau), \varphi(y,\tau)\} = -\delta(x-y)$, and assume for the preliminary Hamiltonian the expression

$$H'(\tau) = H_c(\tau) + \int dy\, u(\tau,y)\Phi(\tau,y), \tag{2.2}$$

where u is a Lagrange multiplier function. Using $\dot{f} = \{f, H'\} + \partial f/\partial\tau$ we find

$$\dot{\Phi} \;=\; \{\Phi, H'\} \approx \;-\; V'(\phi) \;-\; 2u'. \tag{2.3}$$

The persistency requirement $\dot{\Phi} \approx 0$ then results in a consistency condition involving the multiplier u and does not generate a new constraint. The only constraint $\Phi(x) \approx 0$ in the theory is second class [11] by itself since

$$\{\Phi(x), \Phi(y)\} = -2\partial_x \delta(x-y) \equiv C(x,y) = -C(y,x) \tag{2.4}$$

is nonvanishing. Its (unique) inverse with the correct symmetry property is $C^{-1}(x,y) = -C^{-1}(y,x) = -\epsilon(x-y)/4$. The equal-$\tau$ *Dirac bracket* $\{,\}_D$ is then constructed as

$$\{f(x), g(y)\}_D = \{f(x), g(y)\} + \frac{1}{4}\int\int du dv \{f(x), \Phi(u)\}\epsilon(u-v)\{\Phi(v), g(y)\}. \tag{2.5}$$

where we suppress τ for convenience of writing. We verify that $\{f, \Phi\}_D = 0$ for any arbitrary functional $f(\varphi, \pi)$ and thereby we are allowed to set $\pi = \varphi'$ even inside the Dirac bracket, e.g., treat it as a strong equality. The eqs. of motion are now given by $\dot{f} = \{f, H_c\}_D + \partial f/\partial\tau$ since $H'(\tau)$ reduces to $H_c(\tau) \equiv P^-$, and whose explicit form is

$$
\begin{aligned}
P^- &\equiv \int_{-L/2}^{L/2} dx\, V(\phi) \\
&= \int_{-L/2}^{L/2} dx \left[\omega(\lambda\omega^2 - m^2)\varphi + \frac{1}{2}(3\lambda\omega^2 - m^2)\varphi^2 + \lambda\omega\varphi^3 + \frac{\lambda}{4}\varphi^4 + const.\right]
\end{aligned}
\tag{2.6}
$$

where $L \to \infty$.

The variable π is not an independent one unlike in the case of the *instant formulation* and from (2.5) we derive

$$\{\varphi(x,\tau), \varphi(y,\tau)\}_D = -(1/4)\epsilon(x-y)\ , \tag{2.7}$$

We make a brief digression on the transition to the quantized theory. To each dynamical variable we associate an operator in the quantized theory and make the correspondence $i\{f,g\}_D \to [f,g]$ where $[f,g]$ indicates a commutator (or anticommutator) between the operators. Such procedure is frequently adopted to obtain the quantum mechanics of a particle in the Heisenberg formulation. For example, from (2.7) we find $[\varphi(x,\tau), \varphi(y,\tau)] = -(i/4)\epsilon(x-y)$ for the φ commutator in two dimensions. This agrees with the one suggested from the considerations on the Lehmann spectral representation in Sec. 1. We note that the antisymmetry of the Dirac bracket imposes that we define $\epsilon(0) = 0$. We also note that unlike in the equal-time case here the commutator does not vanish on the light-cone for non-coincident points with $x \neq y$ and we derive on using $\partial_x \epsilon(x-y) = 2\delta(x-y)$

$$\{\varphi'(x,\tau), \varphi(y,\tau)\}_D = -(1/2)\delta(x-y)\ , \tag{2.8}$$

and

$$\{\varphi'(x,\tau), \varphi'(y,\tau)\}_D = -(1/2)\partial_y\delta(x-y)\ , \tag{2.9}$$

It is easy to show that the translations in the space direction x^- are generated by

$$P^+(\tau) = \int dx\, (\varphi'(x,\tau))^2\ , \qquad \varphi'(x,\tau) = \{\varphi(x,\tau), P^+(\tau)\}_D. \tag{2.10}$$

For example, $\{V(\phi(x,\tau), P^+(\tau)\}_D = \partial_x V(\phi(x,\tau))$ and from which it follows that

$$\{P^-(\tau), P^+(\tau)\}_D \ = \ V(\phi(\infty,\tau)) - V(\phi(-\infty,\tau)). \tag{2.11}$$

The right hand side must vanish in a relativistic invariant theory.

The Hamilton's eq. for φ is found to be

$$\begin{aligned}\dot{\varphi}(x,\tau) &= \{\varphi(x,\tau), P^-(\tau)\}_D \\ &= -\frac{1}{4}\int dy\, \epsilon(x-y)\, \frac{\delta V}{\delta\phi(y,\tau)}\end{aligned} \tag{2.12}$$

and we derive from it the Euler-Lagrange eq.

$$\dot{\varphi}'(x,\tau) = -\frac{1}{2}\,\frac{\delta V}{\delta\phi(x,\tau)} \tag{2.13}$$

The Hamiltonian formulation constructed above is thus self-consistent with the Lagrange formulation. If we substitute the value of $V'(\phi)$ obtained from (2.13) into (2.12) we find after an integration by parts

$$\dot{\varphi}(x,\tau) = \dot{\varphi}(x,\tau) - \frac{1}{2}\Big[\dot{\varphi}(\infty,\tau)\epsilon(\infty - x) - \dot{\varphi}(-\infty,\tau)\epsilon(-\infty - x)\Big]. \tag{2.14}$$

Considering finite values of x we must then require $\dot{\varphi}(\infty,\tau) + \dot{\varphi}(-\infty,\tau) = 0$. On the other hand the equal-τ commutator of φ obtained above may be realized through the momentum space expansion given below in (2.16). Integrating over x the expansion of $\varphi'(x,\tau)$ we easily show that $\varphi(\infty,\tau) - \varphi(-\infty,\tau) = 0$. Combining the two results we are led to $\partial_\tau\varphi(\pm\infty,\tau) = 0$ as a consistency condition. This is similar to $\partial_t\varphi(x^1 = \pm\infty, t) = 0$ which is imposed from the outside in the equal-time formulation based upon the physical considerations.

On integrating (2.13) over the longitudinal space coordinate it follows that we must (also) satisfy the following ***constraint equation*** in the light-front Hamiltonian framework

$$\begin{aligned}\beta(\tau) \equiv\ & lim_{L\to\infty}\frac{1}{L}\int_{L/2}^{L/2} dx\, V'(\phi) \\ &= \omega(\lambda\omega^2 - m^2) + lim_{L\to\infty}\frac{1}{L}\int_{-L/2}^{L/2} dx\Big[(3\lambda\omega^2 - m^2)\varphi + \lambda(3\omega\varphi^2 + \varphi^3)\Big] = 0\end{aligned} \tag{2.15}$$

The Hamiltonian formulation in the present case thus contains a new ingredient in the form of a nonlocal constraint (2.15). This is not unexpected in view of the discussion give in Sec. 1. In ref. 1 we find some illustrations where restrictions on the potential arise due to the necessity of incorporating special relativity in the theory. Because of the constraint eq. the light-front Hamiltonian (2.6) is nonlocal and much involved compared with the local polynomial form for interaction in the corresponding equal-time formulation.

At the classical (or tree) level the integrals appearing in (2.15) are convergent (since $\int dx|\varphi| < \infty$). In the continuum limit, when $L \to \infty$, we find the result $V'(\omega) = 0$ which determines the tree level values for the condensate ω and they are the same as those found in the *instant formulation*, where the condition is added to the theory (imposed) by appealing to physical considerations for minimizing the energy functional. Similar comments hold true as regards $\partial_\tau\varphi(\pm\infty,\tau)=0$. Such considerations in general are *not* available in the *front form* dynamics. In its place constraint equations like (2.15) or consistency conditions like (2.14) arise in the theory itself. The two forms of dynamics should clearly lead to the same physical results even though attained by different paths. This will be explicitly seen in the description of the spontaneous symmetry breaking considered below.

In the *quantized theory* the field φ and its commutation relations are obtained through the correspondence $i\{f,g\}_D \to [f,g]$. It is easy to verify that the equal-τ φ field commutator, $[\varphi(x,\tau),\varphi(0,\tau)] = -(i/4)\epsilon(x)$, can be realized in the momentum space through the following (momentum space) expansion of the field (suppressing τ)

$$\varphi(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}dk\,\frac{\theta(k)}{\sqrt{2k}}\,[a(k)e^{-ikx}+a^\dagger(k)e^{ikx}] \tag{2.16}$$

where $a(k)$ and $a^\dagger(k)$ satisfy the canonical equal-τ commutation relations, $[a(k),a(k')^\dagger]=\delta(k-k')$, $[a(k),a(k')]=0$, and $[a(k)^\dagger,a(k')^\dagger]=0$. We note the presence of $\theta(k)$ in (2.16) originating as a consequence of the integral representation of the sgn function, $\epsilon(x) = (i/\pi)\mathcal{P}\int(dk/k)e^{-ikx}$. Also, in contrast to the case of equal-time formulation, there is no dependence on mass in (2.16).

The vacuum state is defined by $a(k)|vac\rangle=0$, $k>0$. The longitudinal momentum operator and the light-front energy operators are $\int dx\ :\varphi'^2:$ and $P^-=H=\int dx\ :V(\phi):$ respectively. Here we normal order with respect to the creation and destruction operators to drop unphysical infinities. We find $[a(k),P^+]=k\,a(k)$, $[a^\dagger(k),P^+]=-k\,a^\dagger(k)$. The tree level description of the spontaneous (discrete) symmetry breaking may be given as follows. The values of $\omega=\langle|\phi|\rangle_{vac}$ obtained from the tree level condition $V'(\omega)=0$ *characterize* the possible vacua of different type in the theory. Distinct Fock spaces corresponding to different values of ω are built as usual by applying the creation operators on the corresponding vacuum state. The $\omega=0$ corresponds to a *symmetric phase* since the Hamiltonian is then found symmetric under the discrete symmetry transformation $\varphi\to-\varphi$. For $\omega\neq0$ this symmetry is clearly violated and the system is in a *broken or asymmetric phase*. In the case of the right sign for the mass term, $m^2\to-m^2$, and $\lambda=0$ the system is found in the symmetric phase at the tree level. When the interaction is switched on, the constraint eq. (2.15) shows, however, that the symmetric phase may become unstable due to the quantum corrections and the system may undergo a phase transition, as the coupling constant increases, to the broken phase. For the wrong sign for the mass term we have the possibility of both types of phases already at the tree level. It should be stressed that we do *not* have any physical arguments, like for $P^\pm$, in the *front form* theory to normal order the constraint equation (2.15) and consequently the tree level value of the condensate does obtain high order quantum corrections as dictated by the renormalized constraint equation. A factor L may arise in the numerator which cancels the L in the denominator in the integrals involved in (2.15). A self-consistent *front form* Hamiltonian formulation can thus be built in the continuum which also can describe the ssb. The phase transition in two dimensions can be described (Sec. 3) in the renormalized theory based on P^-, $\beta = 0$, and the light-front commutator.

(b)- Discretized formulation in finite volume:

It is instructive to rederive the continuum theory above as the infinite volume limit of the *discretized formulation* in the finite volume [9]. It is some times wrongly affirmed in the literature that the infinite volume limit of the discretized formulation does not exist. We make the Fourier series expansion of the field φ and write

$$\phi(\tau,x) = \omega + \frac{q_0(\tau)}{\sqrt{L}} + \frac{1}{\sqrt{L}} \sum_n{}' \, q_n(\tau)\, e^{-ik_n x} \equiv \omega + \varphi(\tau,x) \tag{2.17}$$

where the periodic boundary conditions are assumed for convenience with $\Delta = (2\pi/L), k_n = n\Delta,\ n = 0, \pm1, \pm2, ..$, and L is now finite. The discretized Lagrangian obtained by integrating the Lagrangian density in (2.1) over the finite interval $-L/2 \leq x \leq L/2$ is given by

$$i\sum_n k_n\, q_{-n}\, \dot{q}_n \, - \int_{-L/2}^{L/2} dx\; V(\phi) \tag{2.18}$$

The momenta conjugate to q_n are then $p_n = ik_n q_{-n}$ and the canonical Hamiltonian is obtained from the expression in (2.6). The primary constraints are thus $p_0 \approx 0$ and $\Phi_n \equiv\ p_n - ik_n q_{-n} \approx 0$ for $n \neq 0$. We postulate initially the standard Poisson brackets at equal τ, viz, $\{p_m, q_n\} = -\delta_{mn}$ and define the preliminary Hamiltonian

$$H' = H_c + \sum_n{}' \, u_n \Phi_n + u_0\, p_0 \,. \tag{2.19}$$

On requiring the persistency in τ of these constraints we find the following weak equality relation

$$\dot{p}_0 \;=\; \{p_0,\, H'\} \;=\! -\frac{\partial H'}{\partial q_0} \;=\; -\frac{1}{\sqrt{L}} \int_{-L/2}^{L/2} dx\, V'(\phi) \;\equiv\; -\frac{1}{\sqrt{L}}\, \beta(\tau) \,\approx 0\,, \tag{2.20}$$

and for $n \neq 0$

$$\dot{\Phi}_n \;=\; \{\Phi_n, H'\} \;=\; -2i\sum_n{}' k_n\, u_{-n} \,-\, \frac{1}{\sqrt{L}} \int_{-L/2}^{L/2} dx\; V'(\phi)\, e^{-ik_n x} \approx 0\,. \tag{2.21}$$

From (2.20) we obtain an interaction dependent secondary constraint $\beta \approx 0$ while (2.21) is a consistency requirement for determining u_n, $n \neq 0$. Next we extend the Hamiltonian to

$$H'' = H' + \nu(\tau)\beta(\tau), \tag{2.22}$$

and check again the persistency of all the constraints encountered above making use of H''. We check that no more secondary constraints are generated if we set $\nu \approx 0$ and we are left only with consistency requirements for determining the multipliers u_n, u_0.

We verify that all the constraints $p_0 \approx 0$, $\beta \approx 0$, and $\Phi_n \approx 0$ for $n \neq 0$ are second class. They may be implemented in the theory by defining Dirac brackets and this may be performed iteratively. We find $(n, m \neq 0)$

$$\{\Phi_n, p_0\} = 0, \qquad \{\Phi_n, \Phi_m\} = -2ik_n\delta_{m+n,0}\,, \tag{2.23}$$

$$\{\Phi_n, \beta\} = \{p_n, \beta\} = -\frac{1}{\sqrt{L}}\int_{-L/2}^{L/2} dx\,[V''(\phi) - V''(\frac{\omega + q_0}{\sqrt{L}})]\,e^{-ik_n x} \equiv -\frac{\alpha_n}{\sqrt{L}}, \tag{2.24}$$

$$\{p_0, \beta\} = -\frac{1}{\sqrt{L}}\int_{-L/2}^{L/2} dx\, V''(\phi) \equiv -\frac{\alpha}{\sqrt{L}}, \tag{2.25}$$

$$\{p_0, p_0\} = \{\beta, \beta\} = 0\,. \tag{2.26}$$

The explicit expressions of α_n and α appear below in the numerator and the denominator of eq. (2.29).

We implement first the pair of constraints $p_0 \approx 0, \beta \approx 0$. The Dirac bracket $\{\}^*$ with respect to them is easily constructed

$$\{f,g\}^* = \{f,g\} - [\{f,p_0\}\,\{\beta,g\} - (p_0 \leftrightarrow \beta)](\frac{\alpha}{\sqrt{L}})^{-1}. \tag{2.27}$$

We may then set $p_0 = 0$ and $\beta = 0$ as strong relations. and the variable p_0 is thus removed from the theory. We conclude easily by inspection that the brackets $\{\}^*$ of the surviving canonical variables coincide with the standard Poisson brackets except for the ones involving q_0 and p_n $(n \neq 0)$

$$\{q_0, p_n\}^* = \{q_0, \Phi_n\}^* = -(\alpha^{-1}\alpha_n) \tag{2.28}$$

For the explicit expression of the potential given above we find $\{q_0, p_n\}^* =$

$$-\frac{3\lambda\,[\,2(\omega + q_0/\sqrt{L})\,\sqrt{L}q_{-n} + \int_{-L/2}^{L/2} dx\,\varphi^2\,e^{-ik_n x}\,]}{[\,3\lambda\,(\omega + q_0/\sqrt{L})^2 - m^2\,]\,L + 6\lambda(\omega + q_0/\sqrt{L})\int_{L/2}^{L/2} dx\varphi + 3\lambda\int_{-L/2}^{L/2} dx\,\varphi^2} \tag{2.29}$$

Next we implement the remaining constraints $\Phi_n \approx 0$ $(n \neq 0)$. We have

$$C_{nm} = \{\Phi_n, \Phi_m\}^* = -2ik_n\delta_{n+m,0} \tag{2.30}$$

and its inverse is given by $C^{-1}{}_{nm} = (1/2ik_n)\delta_{n+m,0}$. The *final* Dirac bracket which takes care of all the constraints of the theory is then given by

$$\{f,g\}_D = \{f,g\}^* - \sum\nolimits_n' \frac{1}{2ik_n}\{f, \Phi_n\}^* \{\Phi_{-n}, g\}^*. \tag{2.31}$$

Inside this final bracket all the constraints may be treated as strong relations and we may now in addition write $p_n = ik_n q_{-n}$. It is straightforward to show that

$$\{q_0, q_0\}_D = 0, \qquad \{q_0, p_n\}_D = \{q_0, ik_n q_{-n}\}_D = \frac{1}{2}\{q_0, p_n\}^*, \qquad \{q_n, p_m\}_D = \frac{1}{2}\delta_{nm}. \tag{2.32}$$

In order to remove the *spurious finite volume effects* in discretized formulation we must take the *continuum limit* $L \to \infty$ [20]. We follow the well known procedure: $\Delta = 2(\pi/L) \to dk$, $k_n = n\Delta \to k$, $\sqrt{L}\,q_{-n} \to lim_{L\to\infty}\int_{-L/2}^{L/2} dx\,\varphi(x)\,e^{ik_n x} \equiv \int_{-\infty}^{\infty} dx\,\varphi(x)\,e^{ikx} = \sqrt{2\pi}\tilde{\varphi}(k)$ for all n, $\sqrt{2\pi}\varphi(x) = \int_{-\infty}^{\infty} dk\,\tilde{\varphi}(k)\,e^{-ikx}$, and $lim_{L\to\infty}(q_0/\sqrt{L}) = 0$. From $\{\sqrt{L}q_m, \sqrt{L}q_{-n}\}_D = L\,\delta_{nm}/(2ik_n)$ following from the Dirac bracket between q_m and p_n for $n,m \neq 0$ in (2.32) we derive, on using $L\delta_{nm} \to lim_{L\to\infty}\int_{-L/2}^{L/2} dx e^{i(k_n-k_m)x} = \int_{-\infty}^{\infty} dx e^{i(k-k')x} = 2\pi\delta(k-k')$, that

$$\{\tilde{\varphi}(k), \tilde{\varphi}(-k')\}_D = \frac{1}{2ik}\delta(k-k') \tag{2.33}$$

where $k, k' \neq 0$. On making use of the integral representation of the sgn function, $\epsilon(x) = (i/\pi)\mathcal{P}\int_{-\infty}^{\infty}(dk/k)e^{-ikx}$ we are led from (2.33) to the light-front Dirac bracket (2.7) for φ obtained in the continuum formulation above.
From (2.29) and (2.32) we derive

$$\{\frac{q_0}{\sqrt{L}}, \varphi'(x)\}_D = -(\frac{3\lambda}{2})\frac{[2\omega\varphi(x) + \varphi^2(x)]}{[(3\lambda\omega^2 - m^2)L + 6\lambda\int dx\,\varphi + 3\lambda\int dx\,\varphi(x)^2]} \tag{2.34}$$

where $L \to \infty$. The eq. (2.34) is consistent in the continuum limit for the values of ω given at the tree level by $V'(\omega) = 0$. The constraint eq. (2.20), $\beta = 0$, in the discretized formulation, goes over to the expression (2.15) of the continuum formulation and the eqs. (2.12) and (2.13) are also recovered in the infinite volume limit.

It is interesting to recount *the history of the constraint equation* in the light-front framework. In the decade of 1970 the light-front commutator was reobtained [21] by other methods not following the Dirac procedure and the constraint escaped the observation. In 1976 the procedure was attempted [14] in two dimensional scalar field theory using the discretized formulation. The constraint $p_0 \approx 0$ was, however, missed. The constraint equation was noted but its implications ignored. Latter in 1989 it was remarked [16], again in the context of the discretized formulation, that the light-front quantization of the scalar field theory would be extremely difficult due to the complexity of the constraint eq. which relates the zero mode q_0 with nonzero modes q_n with $n \neq 0$ (see (2.15)). Moreover, at the quantized level the zero mode (in the finite volume) is an operator which does not commute with the nonzero modes (see (2.34)), making it necessary to order it in the constraint itself. Here also the Dirac procedure was not followed and arguments were based essentially on the equation of motion. In 1991 it was proposed [17] to modify the Dirac method introducing $p_0 \approx 0$ from outside; this may not be regarded as gauge-fixing constraint since we do not have any first class constraint in the theory considered. There was also some confusion caused by not distinguishing between the zero mode of φ and the bosonic condensate ω, clarified only latter. Since 1985, with the proposal [8] of DLCQ-*Discretized light cone quantized theory* in the context of perturbation theory, the zero mode (and the condensate) were ignored until recently. The Dirac quantization of the light-front scalar field theory directly in the continuum while separating the condensate [10,9] was considered only towards the end of 1991. It was strongly believed since 1977, when it was perhaps first mentioned [15], that it was not possible to take the infinite volume limit of the discretized formulation just discussed, ignoring strangely enough the conflict with a basic principle [20]. It was sometimes also affirmed that the light-front quantized field theory could not even be construct directly in the continuum. From our discussion we conclude on the contrary. The theory seems manageable only in the continuum formulation and there are no signs of any inconsistency when properly constructed following the Dirac method without modifications. We loose control over the self-consistency checks if any modifications be introduced in the procedure and even the doubts would arise on the genuinity of the constraint equation obtained. The theory constructed above can also describe the spontaneous symmetry breaking of both the discrete as well as the continuous symmetry (see below) and it contains both the tree (classical) as well as quantized level descriptions. The *suggestion that the Hamiltonian in the light-front context may be nonlocal does not occur in* any *of the earlier works.*

(c)- Continuum formulation with ω dynamical:

In case we maintain [10] the τ dependence in ω the Lagrangian is

$$C\dot{\omega} + \int_{-L/2}^{L/2} dx \, [\dot{\varphi}\,\varphi' - V(\phi)], \tag{2.35}$$

where $C(\tau) = [\,\varphi(\tau, x = L/2) - \varphi(\tau, x = -L/2)]$ and $L \to \infty$. The eqs. of motion are

$$\dot{\varphi}' = (-1/2)\,V'(\phi),$$
$$\dot{\mathcal{C}} = -\int_{-L/2}^{L/2} dx\,V'(\phi) \tag{2.36}$$

Integrating the first on x for $(-L/2 < x < L/2)$ and comparing with the second we find $\dot{\mathcal{C}} = 0$ and consequently the constraint (2.15). It is interesting to note that if we consider $\mathcal{C}$ also as a dynamical variable we obtain $\dot{\omega} = 0$ among the eqs. of motion.

In the case φ is an ordinary function admitting Fourier transformation as well as its inverse the surface term $\mathcal{C}$ drops out since $\varphi(\pm\infty, \tau) = 0$ (from the Fourier transform theory). We note that the Fourier transform of the generalized function ϕ is given by $\tilde{\phi}(k,\tau) = \sqrt{2\pi}\,\omega(\tau)\,\delta(k) + \tilde{\varphi}(k,\tau)$. Indicating the canonical momenta conjugate to ω and φ by p e π respectively, the primary constraints are $p \approx 0$ and $\Phi \equiv \pi - \varphi' \approx 0$ and we start from the preliminary Hamiltonian

$$H'(\tau) = H_c(\tau) + \mu(\tau)p(\tau) + \int dy\; u(\tau, y)\Phi(\tau, y), \tag{2.37}$$

where μ e u are Lagrange multipliers. We find

$$\dot{p} \;=\; \{p, H'\} \approx -\int dx\; V'(\phi) \;\equiv\; -\beta(\tau), \tag{2.38}$$

$$\dot{\Phi} \;=\; \{\Phi, H'\} \approx -\; V'(\phi) \;-\; 2u'. \tag{2.39}$$

The Dirac procedure the leads to the three second class constraints $p \approx 0,\ \beta \approx 0,\ \Phi \approx 0$. In view of $\{\beta(\tau), p(\tau)\} \equiv \alpha(\tau) = \int dx\; V''(\phi)$, $\{\beta, \beta\} = \{p, p\} = 0$, we define the modified bracket $\{,\}^*$

$$\{f(x), g(y)\}^* = \{f(x), g(y)\} - \frac{1}{\alpha}[\{f(x), p\}\{\beta, g(y)\} - (\beta \leftrightarrow p)], \tag{2.40}$$

where

$$\alpha(\tau) = \int dx\, V''(\phi) = L\,(3\lambda\omega^2 - m^2) + 6\lambda\omega \int dx\, \varphi + 3\lambda \int dx\, \varphi^2, \tag{2.41}$$

We verify that for the independent dynamical variables only $\{\omega, \pi\}^* = \{\omega, \Phi\}^* = -\alpha^{-1}\,V''(\phi)$ do not coincide with the corresponding Poisson brackets $\{,\}$. Defining the final Dirac bracket $\{,\}_D$ by

$$\{f(x), g(y)\}_D = \{f(x), g(y)\}^* + \frac{1}{4}\int\!\!\int du dv \{f(x), \Phi(u)\}^* \epsilon(u - v)\{\Phi(v), g(y)\}^*. \tag{2.42}$$

we can also implement $\Phi = 0$. From (2.42) we derive

$$\{\varphi(x), \varphi(y)\}_D = -(1/4)\epsilon(x-y)\,, \tag{2.43}$$

$$\{\omega, \pi(x)\}_D = \{\omega, \varphi'(x)\}_D = \frac{1}{2}\{\omega, \pi(x)\}^*, \qquad \{\omega, \omega\}_D = 0 \tag{2.44}$$

The constraint $\beta = 0$ is interpreted as in the discusssion given above. At the tree level $\alpha \to \infty$ which leads to $\{\omega, \pi\}^* = -\alpha^{-1} V''(\phi) \to 0$. It follows from (2.44) that $\{\omega, \varphi(x)\}_D = 0$, which leads to $\dot{\omega} = 0$, in agreement with the constant values for ω. We also verify that the Lagrange eq. for φ is also recovered. It is thus possible to construct a self-consistent Hamiltonian formulation in the continuum with the proposal of separating first the condensate from the fluctuations represented by φ.

Continuous symmetry in $3+1$ dimensions:

The extension to $3+1$ dimensions and to global continuous symmetry case is straightforward. Consider the multiplet of real scalar fields $\phi_a (a = 1, 2, ..N)$ with nonzero mass which transform as an isovector under the global isospin transformations of the internal symmetry group $O(N)$. We separate the condensate variables and write $\phi_a(x, \bar{x}, \tau) = \omega_a + \varphi_a(x, \bar{x}, \tau)$ while ω_a is assumed independent of τ. The classical Lagrangian density

$$\mathcal{L} = [\, \dot{\varphi}_a \varphi'_a - (1/2)(\partial_i \varphi_a)(\partial_i \varphi_a) - V(\phi)\,] \tag{2.45}$$

is invariant with respect to the global $O(N)$ symmetry group. Here $i = 1, 2$ indicate the transverse space directions. The eqs. of motion are given by $2\dot{\varphi}'_a = [-V'_a(\phi) + \partial_i \partial_i \varphi_a]$. Following the Dirac procedure as above we find

$$[\varphi_a(x, \bar{x}, \tau), \varphi_b(y, \bar{y}, \tau)] = -(i/4)\delta_{ab}\epsilon(x-y)\delta^2(\bar{x} - \bar{y})\,, \tag{2.46}$$

$$P^-(\tau) = \int dx d^2x \Big[V(\phi) + \frac{1}{2}(\partial_i \varphi_a)(\partial_i \varphi_a)\Big], \qquad P^+ = \int dx d^2x \pi_a \varphi_a, \tag{2.47}$$

where $\pi_a(x, \bar{x}, \tau) = \varphi'_a(x, \bar{x}, \tau)$, $\dot{\varphi}_a(\pm\infty, \bar{x}, \tau) = 0$, and the set of coupled constraint equations $\beta_a = 0$ when expanded in Taylor series is given as

$$L\, V'_a(\omega) + V''_{ab}(\omega) \int dx \varphi_b + \frac{1}{2!} V'''_{abc}(\omega) \int dx \varphi_b \varphi_c + ... = 0. \tag{2.48}$$

The momentum space expansion of the fields is now

$$\varphi_b = \frac{1}{(\sqrt{2}\pi)^3} \int dk d^2\bar{k}\, \frac{\theta(k)}{\sqrt{2k}} \left[a_b(k,\bar{k},\tau) e^{-i(kx+\bar{k}.\bar{x})} + a_b^\dagger(k,\bar{k},\tau) e^{i(kx+\bar{k}.\bar{x})}\right] \tag{2.49}$$

where $[a_b(k,\bar{k},\tau), a_c(k',\bar{k}',\tau)^\dagger] = \delta_{bc}\delta(k-k')\delta^2(\bar{k}-\bar{k}')$ etc. At the tree level the values of ω_a are obtained from $V'_a(\omega) = 0$ where V'_a indicates the variational derivative of $V(\phi) = -(1/2)m^2\phi^2 + (\lambda/4)\phi^4 + const.$, with respect to ϕ_a, with $\phi^2 = \phi_a\phi_a$.

The case of discrete symmetry in $3+1$ dimensions is obtained here when there is only one real field.

Consider next the discussion of the field theory symmetry generators and the description of the *spontaneous symmetry breaking of the continuous symmetry.* The classical theory is invariant under global isospin rotations $\delta\varphi_a = -i\epsilon_\alpha(t_\alpha)_{ab}\varphi_b$, $\delta\omega_a = -i\epsilon_\alpha(t_\alpha)_{ab}\omega_b$ where $\alpha, \beta = 1, 2, ..N(N-1)/2$ are the group indices, t_α are hermitian and antisymmetric generators of the group, and $[t_\alpha, t_\beta] = if_{\alpha\beta\gamma}t_\gamma$. The classical conserved Noether currents are given by $J^\mu{}_\alpha = -i\partial^\mu\phi^T t_\alpha\phi = -i\partial^\mu\varphi^T t_\alpha\varphi - i(t_\alpha\omega)^T\partial^\mu\varphi$. In the light-front quantized theory the field theory symmetry generators are

$$\begin{aligned} G_\alpha(\tau) &= \int d^2\bar{x}dx\, J^+ \\ &= -i\int d^2\bar{x}dx[\varphi'_a(t_\alpha)_{ab}\varphi_b - i(t_\alpha\omega)_a\varphi'_a] \\ &= -i\int d^2\bar{x}dx\varphi'_a(t_\alpha)_{ab}\varphi_b = \int d^2\bar{k}\, dk\, \theta(k) a_a(k,\bar{k})^\dagger\, (t_\alpha)_{ab}\, a_b(k,\bar{k}) \end{aligned} \tag{2.50}$$

in view of $\varphi(\pm\infty,\bar{x},\tau) = 0$ and where we used also the (light-front) expansion (2.49) of the field. The continuous symmetry generators come out already normal ordered and as such they annihilate the vacuum state independent of the values assumed by the condensates. We define the system to be in the *symmetric phase* when all the ω_a are vanishing while it is defined to be in the *broken or asymmetric phase* when some of these are nonvanishing.

The situation in the conventional equal-time formulation is different. The symmetry generators here are given by

$$\begin{aligned} Q_\alpha(x^0) &= \int d^3x\, J^0 \\ &= -i\partial_0\varphi_a(t_\alpha)_{ab}\varphi_b - i(t_\alpha\omega)_a \int d^3x \frac{d\varphi_a}{dx_0} \end{aligned} \tag{2.51}$$

In the asymmetric phase the generators do not annihilate now the vacuum state and the symmetry of the vacuum is broken. The generators, however, are conserved even in the quantized theory, since $[Q_\alpha, \phi_a] = -(t_\alpha\phi)_a$ e consequently, $[Q_\alpha, H(t)] = 0$. We call it the spontaneous symmetry breaking because the Hamiltonian remains invariant under the symmetry transformations but the vacuum state does not. In fact, the first term on the last line in (2.51) annihilates the vacuum like in the earlier case but the second one gives a vanishing contribution only for the generators for which $(t_\alpha\omega) = 0$. The set of such linearly independent generators define the group of residual symmetry (of the vacuum state) in the theory.

Returning to the light-front field theory case we verify that $[G_\alpha, \phi_a] = -(t_\alpha)_{ab}\varphi_b - (t_\alpha\omega)/2$, $[G_\alpha, \omega_a] = 0$. They imply that in the asymmetric phase the symmetry of the quantized theory Hamiltonian is broken while the symmetry of the vacuum state is preserved. Only the generators (or the linear combination of original generators) for which $(t_\alpha\omega) = 0$ are left conserved, e.g., the symmetry transformations associated with them leave the Hamiltonian invariant. The set of such generators give rise to the residual symmetry group of the Hamiltonian operator and of the quantized theory.

The spontaneous symmetry breaking in the *front form* is described as follows. At the tree level a particular solution $(\omega_1, \omega_2, \omega_3, ...)$ of $V'_a(\omega) = \omega_a(\lambda\omega^2 - m^2) = 0$ determines a a fixed direction in the isospace which *characterizes* a (non-perturbative) vacuum state, $\langle 0|\phi_a|0\rangle_\omega = \omega_a$. The Fock space of this sector in the quantized theory is built by applying the particle creation operators on the vacuum state. In the symmetric phase, both the vacuum and the Hamiltonian are invariant under the internal symmetry group. In the asymmetric phase the vacuum remains invariant under the initial symmetry group but the Hamiltonian does not remain so under some of the symmetry transformations. The residual symmetry group in both the *front form* and the *instant form* is determined from the condition $(t_\alpha\omega) = 0$. The total number of the corresponding generators does not depend on the particular choice of the isovector as long as $(\lambda\omega^2 - m^2) = 0$. There is an infinite degeneracy corresponding to the continuum of orientations of the (condensate or background field) isovector in the isospin space satisfying this condition. This corresponds to the infinite degeneracy of the vacuum states in the equal-time case. The number of Goldstone bosons may also be counted [22] and is the same as in the usual formulation. Their number (ignoring the case of pseudo-Goldstone bosons) is the difference in the number of generators of the original and the residual symmetry group. The values of the condensates ω_a found at the tree level get altered when the high order quantum corrections are included and we take into account of the set of coupled renormalized constraint equations in the light-front quantized theory.

It is possible, in the above discussion, to allow for an $\bar{x}$ dependence in ω_a. The first term in the constraint equations now gets altered and the tree level configurations are now obtained from $[V'_a(\omega) - \partial_i\partial_i\omega_a] = 0$. As before there is agreement with the Lagrangian formulation. We obtain then the famous *kink* solutions but with an important difference. In the *front form* dynamics the equation for *kinks* depends only on the two transverse directions and not three as in the *instant form* case. In two dimensions theories there are then no kink solutions. This reinforces the affirmation that the vacuum of the light-front quantized theory is simpler than that in the *instant form.* Moreover, the Hamiltonian maintains locality with respect to the transverse directions but it is nonlocal with respect to the longitudinal direction because of the presence of the constraint equations. In the presence of fermionic (or other) fields interacting with the scalar fields the constraint equations would relate the fermionic condensates with the bosonic ones.

We may also consider the still more general case [10] where $\omega_a = \omega_a(\tau, \bar{x})$. In order to implement the constraint $\beta_a(\tau, \bar{x}) \approx 0$ and define the brackets $\{,\}^*$ we need now to invert the matrix

$$C_{ab}(\bar{x},\bar{y}) \equiv \{\beta_a(\bar{x}), p_b(\bar{y})\} = \Big[L[-\delta_{ab}\,\partial_i\partial_i + V''_{ab}(\omega)] + V'''_{abc}(\omega)\int dx\,\varphi_c + ...\Big]\,\delta^2(\bar{x}-\bar{y}). \tag{2.52}$$

When ω_a, given by $\omega_a(\lambda\omega^2 - m^2) = 0$, are zero, the leading term, since $L \to \infty$, of the matrix C is $-L(\partial_i\partial_i + m^2)\,\delta_{ab}\delta^2(\bar{x}-\bar{y})$, while for the case of $(\lambda\omega^2 - m^2) = 0$, the leading term is $L\,[-\delta_{ab}\partial_i\partial_i + 2m^2\,P_{ab}]\delta^2(\bar{x}-\bar{y})$, where $P_{ab} = (\omega_a\omega_b)/\omega^2$ is a projection operator. In both the cases the inverse of the leading term contains a well defined Green's function multiplied by an explicit factor of $1/L$. Consequently, in the continuum, we fall back to the situation similar to that discussed above in connection with the theory in two dimensions. The final conclusions then coincide with those obtained in the beginning of this Section. It is interesting to note that we may now give a new proof in favour of the absence of the Goldstone bosons in two dimensions [23] (*Coleman's theorem*): Since we have no transverse directions in two dimensions, the matrix $C_{ab} \to 2Lm^2P_{ab}$. It cannot be inverted and we are unable to implement the constraints $\beta_a = 0$.

3- Phase transition in $(\phi^4)_2$ theory:

We will discuss now the stability of the vacuum in the ϕ^4 theory when the coupling constant is increased from vanishingly small values to larger values. The light-front framework seems very appropriate to study this problem. On renormalizing the theory we have here at our disposal, in addition to the usual equations like the mass renormalization condition in the equal-time formulation, also the renormalized constraint equations in the theory. For super-renormalizable theories in two and three dimensions these eqs. will be shown to contain all the information needed to study the problem at hand and they can also describe [24] the phase transition. We showed in Sec. 2 that the same physical description of the spontaneous symmetry breaking is obtained whether we use the front form or instant form dynamics in spite of the different mechanisms in the two cases. The same is seen to be true for the problem we consider below.

We recall that there are rigorous proofs [25] on the triviality of ϕ^4 theory in the continuum for more than four space time dimensions and on its interactive nature for dimensions less than four when the theory is also super-renormalizable. In exactly four dimensions the situation is still not clarified [25] and it will be very interesting to study this case on the light-front since it is pertinent to the Higgs sector of the Standard Model of Weinberg and Salam [26]. However, due to the complexity of the renormalization problem in four dimensions we will illustrate our points considering only the two dimensional theory. From the well established results on the generalized Ising models, Simon and Griffiths [19] conjectured some time ago that the two dimensional ϕ^4 theory should show a *second order* phase transition. We do seem to verify this conjecture by studying the theory quantized on the light-front. Variational methods like the Hartree approximation or Gaussian effective potential [27], using *front form* but ignoring the constraint [28], or the one based on a scheme of canonical transformations [29] all lead to a first order phase transition contradicting the conjecture. The post-Gaussian approximation [30] and the non-Gaussian variational method [31] give a second order transition for a particular value of the coupling constant. Our result shows it to be of second order for any coupling above a critical coupling as implied by the mathematical theorem [19]. Our procedure uses the well established Dyson-Wick expansion [13] of perturbation theory and may be improved systematically computing still higher order corrections which is not possible to do in the variational methods. From the considerations on the light-front quantized theory we find that we may not ignore certain contributions in the theory originating from a finite renormalization corrections. If we drop them our results are in complete agreement with those obtained in the variational methods.

Renormalization. Phase transition in two dimensions:

On the light-front we need to renormalize the theory with the Hamiltonian

$$H^{l.f.} = \int d^2x \left[\frac{1}{2}(m_0^2 + 3\lambda\omega^2)\varphi^2 + \lambda\omega\varphi^3 + \frac{\lambda}{4}\varphi^4 + \frac{1}{2}m_0^2\omega^2 + \frac{1}{4}\lambda\omega^4\right], \qquad (3.1)$$

in the presence of the constraint equation

$$\omega\,(\lambda\,\omega^2 + m_0^2) \,+\, \lambda\, lim_{L\to\infty}\, \frac{1}{L}\int_{-L/2}^{L/2} dx\ [3\,\omega\,\varphi^2 + \varphi^3] \,=\, 0\,. \qquad (3.2)$$

and the light-front commutator obtained in Sec. 2. It is clear that it is not convenient to eliminate ω using (3.2) since the resulting Hamiltonian would be quite involved. However, we may renormalize the theory based on (3.1) and obtain thereby a renormalized constraint equation. We have taken here the *correct sign* for the mass term and m_0 indicates the bare mass which is assumed to be nonvanishing. We recall that there is no physical consideration in the light-front framework to normal order the constraint equation. In view of $k \neq 0$ we have $\tilde{\varphi}(k=0) = 0$ (see also ref. [32] which implies $\int dx\varphi = 0$.

We write $M_0^2(\omega) = (m_0^2 + 3\lambda\omega^2)$ and choose $\mathcal{H}_0 = {M_0}^2\varphi^2/2$ so that $\mathcal{H}_{int} = \lambda\omega\varphi^3 + \lambda\varphi^4/4$. The theory being super-renormalizable we need to perform only the mass renormalization. We could follow as is usually done [33] the old fashioned noncovariant perturbation theory. However, we could as well use the covariant propagator (as shown in Appendix A) and use the Dyson-Wick expansion [13] based on the Wick theorem for exponentials ordered with respect to light-front time τ

$$T[e^{i\int d^2x\, j(x)\varphi(x)}] = e^{-\frac{1}{2}\int\int d^2x d^2y\, j(x)G_0(x-y)j(y)} : [e^{i\int d^2x\, j(x)\varphi(x)}] :,$$

where G_0 is the free propagator of the scalar field.

The self-energy correction to *one loop order* is

$$\begin{aligned} -i\Sigma(p) &= -i\Sigma_1 - i\Sigma_2(p) \\ &= (-i6\lambda)\frac{1}{2}D_1({M_0}^2) + (-i6\lambda\omega)^2\frac{1}{2}(-i)D_2(p^2, {M_0}^2), \end{aligned} \tag{3.3}$$

where the divergent contribution D_1 (see (3.5) below) refers to the tadpole graph while the one-loop second term comes from the cubic interaction vertex and gives a finite contribution (Appendix A) with a sign opposite to that of the first. Also the symmetry and other factors from the vertices are explicitly written. We argue below that due to the presence of ω in the second term it is of the same order in λ as the first one and thus cannot be dropped in the one-loop order we will be considering. This term is also quite relevant for determining the nature of the phase transition. We remind that $\langle\varphi(x)\rangle = 0$ and the one particle reducible graphs originating from the cubic term in the interaction are ignored. The divergences will be handled by the dimensional regularization and we adopt the minimal subtraction (MS) prescription [34].

The physical mass $M(\omega)$ is defined [13,34] by

$${M_0}^2(\omega) + \Sigma(p)|_{p^2=-M^2(\omega)} = M^2(\omega) \tag{3.4}$$

where p^μ is the Euclidean space 4-vector and $M(\omega)$ determines the pole of the renormalized propagator. Following the well known procedure of dimensional regularization we have

$$\begin{aligned} D_1(M_0) &= \frac{1}{(2\pi)^n}\int \frac{d^n k}{(k^2 + {M_0}^2)} \\ &= \mu^{(n-2)}\frac{1}{4\pi}(\frac{{M_0}^2}{4\pi\mu^2})^{(\frac{n}{2}-1)}\Gamma(1-\frac{n}{2}) \\ &\rightarrow \frac{\mu^{(n-2)}}{4\pi}[\frac{2}{(2-n)} - \gamma - ln(\frac{M_0^2}{4\pi\mu^2})], \end{aligned} \tag{3.5}$$

where the limit $n \to 2$ is understood. From (3.3-5) we obtain

$$M_0{}^2(\omega) = M^2(\omega) + \frac{3\lambda}{4\pi}\Big[\gamma + ln(\frac{M^2(\omega)}{4\pi\mu^2})\Big] + 18\lambda^2\omega^2 D_2(p, M^2)|_{p^2=-M^2} + \frac{3\lambda}{2\pi}\frac{1}{(n-2)}. \quad (3.6)$$

Here we have taken into account that in view of the tree level result $\omega(\lambda\omega^2 + m_0{}^2) = \omega[M_0{}^2(\omega) - 2\lambda\omega^2] = 0$ the term $\lambda^2\omega^2$ (when $\omega \neq 0$) is, in fact, of the first order in λ and not of the second. We keep the terms only up to first order in λ. We remind, however, that M_0 depends on ω and which in its turn is involved in the constraint equation (3.2). The expression of D_2 is given in the Appendix A; it is finite with the value $D_2(p^2, M^2)|_{p^2=-M^2} = \sqrt{3}/(18M^2)$. To maintain consistency we also replace M_0 by M in the terms which are already multiplied by λ.

From (3.6) we obtain the *mass renormalization condition*

$$M^2 - m^2 = 3\lambda\omega^2 + \frac{3\lambda}{4\pi}ln(\frac{m^2}{M^2}) - \lambda^2\omega^2\frac{\sqrt{3}}{M^2} \quad (3.7)$$

where for convenience of writing we have set $M(\omega) \equiv M$ and $M(\omega = 0) \equiv m$ indicating the physical masses in the asymmetric and symmetric phases respectively. The eq. (3.7) expresses the invariance of the bare mass $m_0{}^2$. For $\omega = 0$ or $\lambda \to 0$ it implies $M^2 = m^2$ (*symmetric phase*).

Consider next the constraint equation (3.2). To the lowest order we find [13,20]

$$\begin{aligned} 3\lambda\omega\langle\varphi(0)^2\rangle &\simeq 3\lambda\omega.iG_0(x,x) = 3\lambda\omega.D_1(M), \\ \lambda\langle\varphi(0)^3\rangle &\simeq \lambda(-i\lambda\omega).6.\int dx\langle T(\varphi(0)^3\varphi(x)^3)\rangle_c^0, \\ &= -6\lambda^2\omega D_3(M) = -6\lambda^2\omega\frac{b}{(4\pi)^2M^2}, \end{aligned} \quad (3.8)$$

where c indicates *connected* diagram and D_3 (Appendix A) is a finite integral with $b \simeq 7/3$. Taking the vacuum expectation value of the constraint equation (3.2) and on making use of (3.5-8) we find that the divergent term cancels and we obtain the *renormalized constraint equation*

$$\beta(\omega) \equiv \omega\Big[M^2 - 2\lambda\omega^2 + \lambda^2\omega^2\frac{\sqrt{3}}{M^2} - \frac{6\lambda^2}{(4\pi)^2}\frac{b}{M^2}\Big] = 0. \quad (3.9)$$

We will verify below that β coincides with the total derivative with respect to ω, in the equal-time formulation, of the (finite) difference $F(\omega)$ (see below) of the renormalized vacuum energy densities in the *asymmetric* ($\omega \neq 0$) and *symmetric* ($\omega = 0$) phases in the theory. The last term in β corresponds to a correction $\simeq \lambda(\lambda\omega^2)$ in this energy difference and thus may not be ignored just like in the case of the self-energy discussed above.

In the equal-time case (3.9) would be required to be *added* to the theory upon physical considerations. It will ensure that the sum of the tadpole diagrams, to the approximation concerned, for the transition $\varphi \to$ *vacuum* vanishes. The physical outcome would then be the same in the two forms of treating the theory here discussed. The variational methods write only the first two ($\approx$ tree level) terms in the expression for β and thus ignore the terms coming from the finite corrections. A similar remark can be made about the last term in (3.7). Both of the eqs. (3.7) and (3.9) and the difference of energy densities above are also found to be independent of the arbitrary mass scale introduced in the dimensional regularization and contain only the finite physical parameters of the theory. The finite renormalization corrections alter the critical coupling and the nature of the phase transition compared to what found in the case of the variational methods.

Consider first the *symmetric phase* with $\omega \approx 0$, which is allowed by (3.9), and for which $M^2 \approx m^2$ as follows from (3.7). The latter also allows us to compute $\partial M^2/\partial\omega = 2\lambda\omega(3 - \sqrt{3}\lambda/M^2)/[1 + 3\lambda/(4\pi M^2) - \sqrt{3}\lambda^2\omega^2/M^4]$ which is needed to find $\beta' \equiv d\beta/d\omega$. Its sign will determine the nature of the stability of the vacuum corresponding to a particular value of ω obtained from (3.9). In the symmetric phase we obtain $\beta'(\omega = 0) = M^2[1 - 0.0886(\lambda/M^2)^2]$. It changes the sign from a positive value for vanishingly weak couplings to a negative value when the coupling increases. In other words the system starts out in a stable symmetric phase for very small coupling but goes over into an unstable symmetric phase for values above the small coupling $g_s \equiv \lambda_s/(2\pi m^2) \simeq 0.5346$.

Consider next the case of *the spontaneously broken symmetry phase* ($\omega \neq 0$). It follows from (3.9) that the nonzero values of ω are found from

$$M^2 - 2\lambda\omega^2 + \frac{\sqrt{3}\lambda}{2} = 0, \qquad (\omega \neq 0), \tag{3.10}$$

where we made use of $2\lambda\omega^2 \simeq M^2$ in the zero order approximation when $\omega \neq 0$. The mass renormalization condition now reads as

$$M^2 - m^2 = 3\lambda\omega^2 + \frac{3\lambda}{4\pi} ln(\frac{m^2}{M^2}) - \lambda\frac{\sqrt{3}}{2}, \tag{3.11}$$

On eliminating ω from (3.10),(3.11) we obtain the *modified duality relation*

$$\frac{1}{2}M^2 + m^2 + \frac{3\lambda}{4\pi} ln(\frac{m^2}{M^2}) + \frac{\sqrt{3}}{4}\lambda = 0. \tag{3.12}$$

which can also be rewritten as $[\lambda\omega^2 + m^2 + (3\lambda/(4\pi))ln(m^2/M^2)] = 0$ and it shows that the real solutions exist only for $M^2 > m^2$. The finite corrections found here are again not considered in the references cited in Sec. 1, for example, they assume (or find) the tree level expression $M^2 - 2\lambda\omega^2 = 0$. In terms of the dimensionless coupling constants $g = \lambda/(2\pi m^2) \geq 0$ and $G = \lambda/(2\pi M^2) \geq 0$ we have $G < g$. The new self-duality eq. (3.12) differs from the old one [27,28] and shifts the critical coupling to a higher value.

We find that: *i)* for $g < g_c = 6.1897$ (*critical coupling*) there is no real solution for G, *ii)* for a fixed $g > g_c$ we have two solutions for G one with the point lying on the upper branch ($G > 1/3$) and the other with that on the lower branch ($G < 1/3$), of the curve describing G as a function of g and which starts at the point ($g = g_c = 6.1897, G = 1/3$), *iii)* the lower branch with $G < 1/3$, approaches to a vanishing value for G as $g \to \infty$, in contrast to the upper one for which $1/3 < G < g$ and G continues to increase. From (3.11) and $\beta = \omega[M^2 - 2\lambda\omega^2 + \sqrt{3}\lambda/2]$ we determine $\beta' \approx (1+0.9405G)$ which is always positive and thus indicates a minimum of the difference of the vacuum energy densities for the nonzero values of ω.

The energetically favored broken symmetry phases become available only after the coupling grows to the critical coupling $g_c = 6.18969$ and beyond this the asymmetric phases would be preferred against the unstable symmetric phase in which the system finds itself when $g > g_s = 0.5346$. The phase transition is thus of the *second order* confirming the conjecture of Simon-Griffiths. If we ignore the additional finite renormalization corrections the well known results following from the variational methods are reproduced exactly in our calculation, e.g., the symmetric phase always remains stable but for $g > 1.4397$ the energetically favored asymmetric phases also do appear, indicating a first order transition.

Vacuum energy density:

The expression for the vacuum energy density in the conventional equal-time formulation is given by

$$\mathcal{E}(\omega) = I_1(M_0) + \frac{1}{2}{m_0}^2\omega^2 + \frac{\lambda}{4}\omega^4 + \frac{\lambda}{4}.3.D_1(M_0)^2 + (-i6\lambda\omega)^2.\frac{1}{2!}.\frac{1}{6}.D_3(M_0), \quad (3.13)$$

where the symmetry and other factors are explicitly written. The first term is the energy density with respect to the free propagator with mass ${M_0}^2$ and is given by [27]

$$\begin{aligned} I_1(M_0) &= \frac{1}{(2\pi)^{(n-1)}}\int d^{(n-1)}k\ \frac{1}{2}\ \sqrt{\vec{k}^2 + {M_0}^2} \\ &= \frac{{M_0}^n}{(4\pi)^{\frac{n}{2}}}\ \frac{1}{n}\ \Gamma(1-\frac{n}{2}) \\ &\to \mu^{(n-2)}\frac{M_0^2}{4\pi}\ \frac{1}{2}\Big[\frac{2}{(2-n)} + 1 - \gamma - ln(\frac{{M_0}^2}{4\pi\mu^2})\Big] \end{aligned} \quad (3.14)$$

The ${D_1}^2$ term represents the two-loop correction of the order λ in the coupling constant and so does the last one in view of the discussion above except for that it gives a finite contribution and carries an opposite sign. We remark that the last term is non-vanishing even in the light-front framework. Here we find in the integrand $\theta(k)\theta(k')\theta(k'')\delta(k+k'+k'')$ multiplied by another distribution. The product distribution, however, may not be considered vanishing. The last term of (3.9) corresponds to the derivative with respect to ω of the last term in (3.13). In the variational methods this term is found ignored.

A finite expression for the difference of the vacuum energy densities in the broken and the symmetric phases, which is also independent of the arbitrary mass μ introduced in the dimensional regularization, is obtained to be [24]

$$\begin{aligned} F(\omega) &= \mathcal{E}(\omega) - \mathcal{E}(\omega=0) \\ &= \frac{(M^2-m^2)}{8\pi} + \frac{1}{8\pi}(m^2+3\lambda\omega^2)\, ln(\frac{m^2}{M^2}) + \frac{3\lambda}{4}\left[\frac{1}{4\pi} ln(\frac{m^2}{M^2})\right]^2 \\ &\quad + \frac{1}{2}m^2\omega^2 + \frac{\lambda}{4}\omega^4 + \frac{1}{2!}.(-i6\lambda\omega)^2.\frac{1}{6}.D_3(M) \end{aligned} \qquad (3.15)$$

We verify that $(dF/d\omega) = \beta$ and $d^2F/d^2\omega = \beta'$ to the one-loop order. Except for the last term it coincides with the results in the earlier works. From numerical computation we verify that at the minima corresponding to the nonvanishing value of ω the value of F is negative and that for a fixed g it is more negative for the point on the lower branch ($G < 1/3$) than for that on the upper branch ($G > 1/3$). To illustrate we find: for $g = 6.366$ and $G = 0.263$ we get $|\omega| = 0.736$, $F = -0.097\lambda$ while for the same g but $G = 0.431$ we find $|\omega| = 0.617$, $F = -0.082\lambda$. For $g = 11.141$ and $G = 0.129$ we get $|\omega| = 1.050$, $F = -0.174\lambda$ while for the same g but $G = 1.331$ we find $|\omega| = 0.493$, $F = -0.111\lambda$. The symbolic manipulation is convenient to handle (3.7) and (3.9).

4- Conclusion:

The present work and the earlier one on the mechanism of spontaneous continuous symmetry breaking [9,10] add to the previous experience [1-8] that the ***front form*** dynamics is a useful complementary method and needs to be studied systematically in the context of QCD and other problems. The physical results following from one or the other form of the theory should come out to be the same though the mechanisms to arrive at them may be different. In the equal-time case we introduce external considerations in order to constrain the theory while the analogous conditions in the light-front formulation seem to be contained in it through the self-consistency equations. In the discussion of the phase transition in ϕ^4 theory the finite renormalization corrections should also be taken in to account. In any case both the light-front and the equal-time (space like) hyperplanes are equally valid for formulating the field theory dynamics.

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Appendix A

Light-front propagator:

The propagator for the free (e.g., in the interaction representation) massive scalar field is defined by $iG_0(x;x') \equiv \langle 0|T\{\varphi(x,\tau)\varphi(x',\tau')\}|0\rangle$, where T indicates the ordering in light-front time τ. On using the momentum space expansion and commutation relations of the operators we arrive at

$$iG_0(x,0) = \frac{1}{2\pi}\int \frac{dk}{2k}\theta(k)\Big[\theta(\tau)\, e^{-i(kx+\epsilon_k\tau)} + \theta(-\tau)\, e^{i(kx+\epsilon_k\tau)}\Big], \qquad (A.1)$$

where $2k\epsilon_k = m^2$. Consider next the usual Feynman propagator, where we change the variables from k^0, k^1 to $k^+ = (k^0+k^1)/\sqrt{2}$, $k^- = (k^0-k^1)/\sqrt{2}$ with $-\infty < k^+, k^- < \infty$,

$$\int\!\!\int \frac{dk^+dk^-}{(2\pi)^2}\frac{i}{(2k^+k^- - m^2 + i\epsilon)}\,[\theta(k^+)+\theta(-k^+)]\; e^{-i(k^+x+k^-\tau)}, \qquad (A.2)$$

and where we have inserted the identity $1 = [\theta(k^+)+\theta(-k^+)]$ valid in the sense of distribution theory. We rewrite this expression as

$$\int\!\!\int \frac{dk^+dk^-}{(2\pi)^2}\, i\, \left[\frac{e^{-i(k^+x+k^-\tau)}\,\theta(k^+)}{2k^+\,(k^- - \epsilon(k^+)+i\epsilon)} + \frac{e^{-i(k^+x+k^-\tau)}\,\theta(-k^+)}{2k^+\,(k^- - \epsilon(k^+)-i\epsilon)}\right]$$

where $\eta(k^+) = m^2/(2k^+)$. Next we make the change $k^+ \to -k^+, k^- \to -k^-$ in the second term to recast it as

$$\int\!\!\int \frac{dk^+dk^-}{(2\pi)^2}\, i\theta(k^+)\Big[\frac{e^{-i(k^+x+k^-\tau)}}{2k^+\,(k^- - \eta(k^+)+i\epsilon)} + \frac{e^{+i(k^+x+k^-\tau)}}{2k^+\,(k^- - \eta(k^+)+i\epsilon)}\Big]$$

If we make the *rule* that the k^- integration has to be performed first we obtain, on using the well known integral representations of $\theta(\tau)$, the light-front propagator $(A.1)$. Inversely we could introduce these representations directly in $(A.1)$ and arrive at $(A.2)$. In the gauge theories with the infrared singularities also present we need to find an adequate procedure to regulate (subtractions) the integrals in momentum space before so as to ensure that the k^- integration may be performed first. We recall that also in the equal-time formulation similar arguments requiring that the k^0 integration be done first are made [35].

Integrals D_2, D_3:

The finite integrals appearing in the text are well known and easily computed after transforming them to the Euclidean space integrals by Wick rotation as usual

$$D_2(p^2, {M_0}^2) = \int \frac{d^2k}{(2\pi)^2}\frac{1}{(k^2+M_0^2)[(p-k)^2+{M_0}^2]} \qquad (A.3)$$

$$D_3(M) = \int\int \frac{d^2k}{(2\pi)^2}\frac{d^2q}{(2\pi)^2}, \frac{1}{(k^2+M^2)(q^2+M^2)[(q+k)^2+M^2]},$$
$$= \frac{1}{(4\pi)^2M^2}\int_0^1\int_0^1 dxdy \frac{1}{[1-y+xy(1-x)]} \equiv \frac{b}{(4\pi)^2M^2}, \qquad (A.4)$$

We find $D_2(p^2 = -{M_0}^2, {M_0}^2) = \sqrt{3}/(18{M_0}^2)$. We could alternatively perform the corresponding computation using the propagators in (A.1) and follow the old fashioned perturbation theory [36] obtaining the same results.

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