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*Chiral Symmetry, Renormalization Group and  
Fixed Points for Lattice Fermions*

*by*

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### Abstract

We discuss fixed point actions for various types of free lattice fermions. The iterated block spin renormalization group transformation yields lines of local but chiral symmetry breaking fixed points. For staggered fermions at least the  $U(1) \otimes U(1)$  symmetry can be preserved. This provides a basis for approximating perfect actions for asymptotically free theories far from the critical surface. For a class of lattice fermions that includes Wilson fermions we find in addition one non local but chirally invariant fixed point. Its vicinity is studied in the framework of the Gross Neveu model with weak four Fermi interaction.

In the following I report on a work done in collaboration with U.-J. Wiese from HLRZ Jülich, Germany.

The naive action of free fermions on a hypercubic lattice of unit spacing in  $d$  dimensional Euclidean space reads:

$$S_{naive} = \frac{1}{2} \sum_x \sum_{\mu=1}^d [\bar{\psi}_x \gamma_\mu \psi_{x+\hat{\mu}} - \bar{\psi}_{x+\hat{\mu}} \gamma_\mu \psi_x] + m \sum_x \bar{\psi}_x \psi_x \quad (1)$$

The spinors are defined on the lattice sites  $x$ ,  $|\hat{\mu}| = 1$  and  $\{\gamma_\mu, \gamma_\nu\} = 2\delta_{\mu\nu}$ .

The corresponding propagator in momentum space,

$$G(p) = [i \sum_{\mu} \gamma_{\mu} \sin p_{\mu} + m]^{-1} \quad (2)$$

displays the notorious "fermion doubling": for  $m = 0$  there are poles in the Brillouin zone  $B = ]-\pi, \pi]^d$  for  $p_{\mu} = 0$  or  $p_{\mu} = \pi$ , i.e. there appear  $2^d$  fermions instead of one. The additional fermions are an artifact of the lattice; it is an outstanding problem how to get rid of them. Hopes to realize this in a simple way were destroyed by the No Go theorem of Nielsen and Ninomyia [1]. The exact minimal assumptions for its proof are complicated; we simplify them to:

1) Chiral invariance 2) Locality 3) Technicality ( $G(p)$  has only a finite number of poles in  $B$ ) 4) Unitarity 5) Translational invariance of  $S$  with respect to any lattice vector.

Then there are as many left- as righthanded fermions with the same internal quantum numbers. In particular we can not put a single fermion on the lattice.

There are many attempts to circumvent this theorem by violating one of its assumptions and hoping that this violation disappears in the continuum limit. We give an (incomplete) list of them <sup>2</sup> referring to the list of assumptions:

1) Wilson fermions [2]:

$$S_{Wilson} = S_{naive} + \frac{r}{2} \sum_{x,\mu} (2\bar{\psi}_x \psi_x - \bar{\psi}_x \psi_{x+\hat{\mu}} - \bar{\psi}_x \psi_{x-\hat{\mu}}) \quad (3)$$

<sup>2</sup>We ignore e.g. the Kaplan fermions, which are - like the staggered fermions - not described by the ansatz (7), and the Zaragoza fermions, one type of which is similar to the Stamatescu-Wu fermions, see below. Moreover we don't discuss the attempts to solve the problem by using random lattices.

If  $p$  has  $n$  components equal to  $\pi$ , then the mass increases as  $m \rightarrow m + 2nr$ . Hence the unphysical poles disappear, but Wilson's term breaks explicitly the chiral symmetry. The parameter  $r$  is supposed to disappear in the continuum, restoring the chiral symmetry.

2) SLAC fermions [3]: here a non locality is introduced by hand. The sinus in (2) is linearized such that  $G^{-1}(p)$  performs a finite gap at the boundary of  $B$ . Also Rebbi fermions [4] contain an artificial non locality. Here  $G^{-1}(p)$  has poles.

3) Smeared fermions [5] include couplings to next to nearest neighbors. They are chirally symmetric, but violate technicality:  $G^{-1}(p)$  vanishes on the whole boundary of  $B$ .

4) Stamatescu-Wu fermions [6]: take a one-sided difference for the lattice derivative instead of the symmetrized expression in (1). The extra poles disappear, but unitarity is lost.

5) Staggered fermions [7]: we decouple the flavors, most easily by the substitution [8]

$$\bar{\psi}_x \rightarrow \bar{\psi}_x \gamma_1^{x_1} \gamma_2^{x_2} \dots \gamma_d^{x_d}; \quad \psi_x \rightarrow \gamma_d^{x_d} \dots \gamma_1^{x_1} \psi_x,$$

and then consider only one of them:

$$S_{stag} = \sum_x \left[ \frac{1}{2} \sum_{\mu} \sigma_{\mu}(x) (\bar{\chi}_x \chi_{x+\hat{\mu}} - \bar{\chi}_{x+\hat{\mu}} \chi_x) + m \bar{\chi}_x \chi_x \right] \quad (4)$$

$\bar{\chi}$ ,  $\chi$  are one component Grassmann fields and the sign factor  $\sigma_{\mu}(x) \doteq (-1)^{x_1 + \dots + x_{\mu-1}}$  makes  $S_{stag}$  explicitly  $x$  dependent; it is only translational invariant under an even number of lattice spacings. Here we reduce the fermion multiplication to the factor  $2^{d/2}$ .

For free fermions, the critical surface corresponds to the chiral limit  $m = 0$ . There we perform renormalization group transformations (RGT) and hope to find a fixed point action (FPA). The FPA is free of cutoff effects.

Kadanoff's block spin RGT acts like this:

$$e^{-S'[\bar{\psi}', \psi']} = \int D\bar{\psi} D\psi K[\bar{\psi}', \psi', \bar{\psi}, \psi] e^{-S[\bar{\psi}, \psi]} \quad (5)$$

$\bar{\psi}', \psi'$  are new spinors defined on the centers of hypercubic blocks, i.e. on a coarser lattice, and  $S'$  is the transformed action. This transformation must

not change the partition function. Thus we reduce systematically the number of degrees of freedom by integrating out the short range fluctuations inside a block (i.e. the high momentum modes). We obtain a sequence  $S, S', S''$  etc. which might converge to an FPA  $S^*$ . In the fermionic case, a sensible choice for the transformation term is [9, 10]

$$K = \exp\left\{-a(\bar{\psi}'_{x'} - \frac{b}{n^d} \sum_{x \in x'} \bar{\psi}_x)(\psi'_{x'} - \frac{b}{n^d} \sum_{x \in x'} \psi_x)\right\} \quad (6)$$

$$\propto \int D\bar{\eta} D\eta \exp\left\{\bar{\eta}_{x'}(\psi'_{x'} - \frac{b}{n^d} \sum_{x \in x'} \psi_x) + (\bar{\psi}'_{x'} - \frac{b}{n^d} \sum_{x \in x'} \bar{\psi}_x)\eta_{x'} + \frac{1}{a}\bar{\eta}_{x'}\eta_{x'}\right\}$$

where  $\bar{\eta}, \eta$  are auxiliary Grassmann fields on the sites  $x'$  of a coarse lattice and  $x \in x'$  extends over  $n^d$  blocks.  $b$  is a renormalization parameter. The second representation displays that for the case of an exact  $\delta$  function RGT,  $a \rightarrow \infty$ , the transformation term is chirally invariant, whereas for finite  $a$  (smeared  $\delta$  function) it breaks chiral symmetry explicitly.

U.-J. Wiese [10, 15] has applied this transformation to a general bilinear ansatz for the action, namely:

$$S[\bar{\psi}, \psi] = \sum_{x,y} \left[ i \sum_{\mu} \rho_{\mu}(x-y) \bar{\psi}_x \gamma_{\mu} \psi_y + \lambda(x-y) \bar{\psi}_x \psi_y \right] \quad (7)$$

where  $\rho_{\mu}, \lambda$  are arbitrary functions; the only assumption here is lattice translational invariance. The recursion relations for these functions in an RGT can be calculated analytically and yield the same structure as in the bosonic case, which had been analyzed by Bell and Wilson a long time ago [11]. If we insert Wilson fermions as initial action, it turns out that only  $b = n^{(d-1)/2}$  leads to a non trivial fixed point, in accordance with a dimensional consideration. In the fixed point, the quantities:  $\alpha_{\mu}(p) \doteq \rho_{\mu}(p)/[\rho^2(p) + \lambda^2(p)]$ ;  $\beta(p) \doteq \lambda(p)/[\rho^2(p) + \lambda^2(p)]$  take the form:

$$\alpha_{\mu}^*(p) = \sum_{\ell \in Z^d} \frac{p_{\mu} + 2\pi\ell_{\mu}}{(p + 2\pi\ell)^2} \prod_{\nu} \left( \frac{\sin(p_{\nu}/2)}{p_{\nu}/2 + \pi\ell_{\nu}} \right)^2 ; \quad \beta^*(p) = \frac{n}{(n-1)a} \quad (8)$$

Wilson's parameter  $r$ , i.e. the initial chiral symmetry breaking, disappears and the doublers do not return. Hence for  $a \rightarrow \infty$  the FPA is chirally invariant. However, numerical studies show that in this case it is non local, therefore there is no contradiction with the Nielsen Ninomyia theorem. For

any finite  $a$  the FPA becomes local, but chiral symmetry is broken. It turns out that for  $a \simeq 4$  the locality is optimal. <sup>3</sup>

The disappearance of Wilson's term can be easily understood from the fact that it represents a (discretized) second derivative, i.e. a quadratic momentum. In the FPA only the leading order of  $\rho_\mu(p)$  survives and for Wilson fermions this order is linear. For dimensional reasons, the latter is true for all reasonable fermionic actions. <sup>4</sup> Instead of only Wilson's term we might add discretized derivatives of any higher orders with arbitrary coefficients. They all disappear in the FPA and for all even orders the effect is the same as for Wilson's term (elimination of the doublers but explicit breaking of the chiral symmetry in the initial action), whereas for all odd orders ( $> 1$ ) chiral symmetry persists, but the doublers too.

We arrive at exactly the same FPA if we insert SLAC or Rebbi fermions for the initial action. This holds for all lattice fermions which coincide in the leading order of  $G^{-1}(p)$  with the naive lattice fermions and which avoid doubling by non locality.

We arrive again at the same FPA if we insert the ansatz of Stamatescu and Wu as initial action. We may even generalize also this approach by using an arbitrary real parameter  $q$  for the unitarity violation:

$$S_{SW} = \sum_x \left\{ \sum_\mu [q \bar{\psi}_x \gamma_\mu \psi_{x+\hat{\mu}} - (1-q) \bar{\psi}_{x+\hat{\mu}} \gamma_\mu \psi_x + (1-2q) \bar{\psi}_x \gamma_\mu \psi_x] + m \bar{\psi}_x \psi_x \right\} \quad (9)$$

For  $q = \frac{1}{2}$  we obtain naive fermions and  $q = 0, 1$  are the one sided derivatives discussed by Stamatescu and Wu. For every  $q \neq \frac{1}{2}$  doubling is avoided, but unitarity violated. But any  $q \neq \frac{1}{2}$  leads to the same fixed point as Wilson fermions, in particular unitarity is recovered there [19].

If we start from naive or smeared fermions, the doublers are still present in the FPA, i.e. here we don't arrive at a useful result by iterating the RGT (6).

<sup>3</sup>This has been shown analytically for  $d = 1$  and numerically for  $d = 2$ . It coincides with the optimal value that Bell/Wilson and Hasenfratz/Niedermayer found in their treatment of the free scalar fields and of the non linear  $\sigma$  model, respectively [11, 12].

Note also that the Kadanoff transformations have properties of an Abelian semigroup only for  $a \rightarrow \infty$ . Therefore it is natural that for finite  $a$  the FPA depends on the blocking factor  $n$ .

<sup>4</sup>Analogously to Bell/Wilson we could generalize the power of the leading momentum order to  $1 + \varepsilon$ . Then we arrive at a non trivial fixed point for  $b = n^{(d-1-\varepsilon)/2}$ . For  $\varepsilon \neq 0$  these fixed points are non local, like the initial action. But this case is quite artificial.

Not described by the ansatz (7) are e.g. staggered fermions, because their action has not the full lattice translational invariance. However, they can be treated similarly if the RGT is modified [14, 15]. At the end, corners of hypercubic blocks (pseudoflavors) are to reconstruct spinors, therefore it is important not to mix them in the RGT. Kalkreuter, Mack and Speh have proposed a suitable transformation, where each pseudoflavor builds its own block spin with blocking factor 3 [13]. We have first applied the exact  $\delta$  RGT to this blocking scheme and observed that the fixed point – which is reached for the suitable value of the renormalization parameter – is local. In analogy to the chiral invariance of (6) for  $a \rightarrow \infty$ , here the  $U(1) \otimes U(1)$  symmetry for  $m = 0$  – the remainder of the chiral symmetry – still holds in the FPA. There is no contradiction with the Nielsen Ninomyia theorem, since this symmetry does not imply full chiral invariance of the spinors, which can be reconstructed. We can even smear the  $\delta$  RGT without loss of any symmetry by adding a “kinetic term of the auxiliary fields” instead of the mass term in (6) (the latter would break the  $U(1) \otimes U(1)$  symmetry) and obtain in this way an extremely local FPA.

If we denote the pseudoflavors as  $\bar{\chi}^{(i)}, \chi^{(i)}$ ,  $i = 1 \dots 2^d$ , then the general ansatz for the staggered fermion action – analogous to (7) – reads:

$$S[\bar{\chi}, \chi] = i \sum_{x,y} \sum_{i,j} \bar{\chi}_x^{(i)} \rho_{ij}(x-y) \chi_y^{(j)} \quad (10)$$

where  $x, y$  refer now to the block centers (spacing 2). After exploiting the symmetry properties, we are left with  $d$  independent elements of  $\rho$ .

In particular for  $d = 2$  and  $\bar{\chi}_x^{(i)}, \chi_x^{(i)} = \bar{\chi}_{x+a_i}, \chi_{x+a_i}$ ,  $a_i = (n_1 - \frac{1}{2})\hat{1} + (n_2 - \frac{1}{2})\hat{2}$  with  $i = 1 + n_1 + 2n_2$ ,  $\rho$  can be written as:

$$\begin{aligned} \rho(p) &= \frac{1}{\alpha_1(p)^2 + \alpha_2(p)^2} \begin{pmatrix} 0 & \alpha_1(p) & \alpha_2(p) & 0 \\ \alpha_1(p) & 0 & 0 & -\alpha_2(p) \\ \alpha_2(p) & 0 & 0 & \alpha_1(p) \\ 0 & -\alpha_2(p) & \alpha_1(p) & 0 \end{pmatrix} \\ &= \frac{1}{\alpha_1(p)^2 + \alpha_2(p)^2} \rho(p)^{-1} \end{aligned} \quad (11)$$

and in the fixed point we obtain:

$$\alpha_\mu^*(p) = 2 \sum_{\ell \in \mathbb{Z}^2} \frac{p_\mu + 2\pi\ell_\mu}{(p + 2\pi\ell)^2} (-1)^{\ell_\mu} \prod_\nu \left( \frac{\sin(p_\nu/2)}{p_\nu/2 + \pi\ell_\nu} \right)^2 + \frac{9}{8a} \sin(p_\mu/2) \quad (12)$$



where the auxiliary kinetic term is suppressed by a factor  $1/a$ . Optimal locality is reached for  $a \simeq 9/4$ , as we see again analytically for  $d = 1$  and numerically for  $d = 2$ .

We also treated Wilson fermions with a weak four fermion interaction in the framework of the Gross Neveu model [15]. There we studied for  $d = 2$  the vicinity of the non local fixed point mentioned above. The interaction was expressed by an auxiliary scalar field  $\phi$  with a Yukawa coupling  $y$ . The general ansatz for the Yukawa coupling after a number of RGT reads:

$$\frac{1}{(2\pi)^4} \int dp dk \bar{\psi}(p) \sigma(p, k) \psi(k) \phi(-p - k)$$

The RGT also includes a blocking of  $\phi$  to coarser lattices. At the fixed point, the matrix  $\sigma$  takes to the first order of  $y$  the form:

$$\begin{aligned} \sigma^*(p, k) = & y \rho_\mu^*(p) \gamma_\mu \sum_{\ell, l \in \mathbb{Z}^2} \frac{p_\nu + 2\pi\ell_\nu}{(p + 2\pi\ell)^2} \gamma_\nu \frac{k_\rho + 2\pi l_\rho}{(k + 2\pi l)^2} \gamma_\rho \rho_\sigma^*(k) \gamma_\sigma \\ & \prod_\lambda \frac{2 \sin((p_\lambda + k_\lambda)/2)}{p_\lambda + 2\pi\ell_\lambda + k_\lambda + 2\pi l_\lambda} \frac{2 \sin(p_\lambda/2)}{p_\lambda + 2\pi\ell_\lambda} \frac{2 \sin(k_\lambda/2)}{k_\lambda + 2\pi l_\lambda} \end{aligned} \quad (13)$$

For interacting fermions there is a generalization of the No Go theorem due to Pelissetto [16], which applies to certain non local theories involving a singularity of  $G^{-1}(p)$ .<sup>5</sup> For Rebbi fermions, which have some similarity with our non local fixed point, it turned out that there are still doublers beyond the tree level [18]. We refer to spurious ghost states which are dynamically generated. It remains to be checked if our FPA suffers from this problem too. There is hope that this is not the case since in our case the poles in  $G^{-1}(p)$  appear naturally from the RGT, unlike those which Rebbi has created by hand.

The analogous consideration for staggered fermions is in progress. Since the Gross Neveu model is asymptotically free, there is one (weakly) relevant direction (to lowest order it is marginal), which can be determined perturbatively. This is the tangent to the curve of perfect actions – actions free of cutoff effects – that emanates from the FPA, away from the critical surface.

<sup>5</sup>The type of non local lattice fermions with a finite gap of  $G^{-1}(p)$  – such as SLAC fermions – is also troublesome on the loop levels; for this case there are problems to recover Lorentz invariance in the continuum [17].

The concept of Hasenfratz and Niedermayer [12] consists now of following this tangent to a point where the correlation length is short enough for simulations and hoping to be still close to a perfect action. Then continuum physics can be described to a very good approximation with only little numerical effort.

The final goal of this concept is to improve the scaling of QCD. As a preparatory work for the combination with gauge theory, we considered the pure Schwinger model with a simple  $\delta$  RGT – that just sums over the frames of hypercubic blocks – and found also there analytically a similar recursion relation and a fixed point [14]. To smear the  $\delta$  RGT, however, a more sophisticated blocking scheme is required. Probably it will be necessary to include all the gauge fields on the fine lattice in the RGT. We note that if we switch on a gauge interaction between fermions, the “kinetic smearing” of the  $\delta$  RGT is not permitted any more.

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G. Bodwin, E. Kovacs, Phys. Lett. B193 (1987) 273; Phys. Rev. D35 (1987) 3198
- [19] Stamatescu and Wu motivate their new animal in the lattice fermion zoo with a postulate that forbids any continuous parameters which disappear in the continuum limit – except for the lattice spacing – such as Wilson's parameter  $r$ , because "irrelevant" parameters can play quite a relevant role sometimes. From our point of view, their fermion type deserves attention, since it leads to a unitary fixed point, but we would recommend to drop the justification from this postulate. Firstly the postulate itself is rather dubious and secondly, as we illustrate above, their own fermions do not really respect it. The authors have just hidden the possibility of such a parameter  $q$  by focusing only on particular values of it. Similarly we might set Wilson's  $r = 1$  and claim that there is no continuous parameter involved. In their last appendix, Stamatescu and

We mention the possibility of a continuous unitarity breaking parameter, but reject it arguing with a consideration on stationary points of the dispersion law. We are not convinced by this argument.