

The Antighost Equation in $N = 1$ Super-Yang-Mills Theories

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ABSTRACT

The antighost equation valid for usual gauge theories in the Landau gauge, is generalized to the case of $N = 1$ supersymmetric gauge theories in a supersymmetric version of the Landau gauge. This equation, which expresses the nonrenormalization of the Faddeev-Popov ghost field, plays an important role in the proof of the nonrenormalization theorems for the chiral anomalies.

Key-words: BRS; Supersymmetry; Anomalies.

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1 Introduction and Conclusions

The nonrenormalization theorems for the chiral anomalies [1, 2, 3, 4] in ordinary gauge theories heavily rely on the (ultraviolet) finiteness of the anomaly operator, more precisely on the vanishing of its anomalous dimension. It has been shown that the latter property is linked to a so-called *antighost equation* [5, 4], valid in the Landau gauge, expressing the absence of anomalous dimension for the ghost field c , and more generally, for any invariant polynomial of c without derivatives. This has allowed for a simple, general and renormalization scheme independent proof of the nonrenormalization theorems [3, 4]. Although the antighost equation holds only in the Landau gauge, the ensuing nonrenormalization theorem is valid in any gauge due to the gauge independence of the anomaly.

There is an analogous nonrenormalization theorem in supersymmetric gauge theories (see Appendix A of [6]). As in the usual theories, one of the ingredients of the proof is the zero anomalous dimension of the ghost fields and of their polynomials, which is equivalent to the finiteness of the diagrams involving ghost c external lines and insertions which are polynomial in c . However, in order to prove this finiteness, use has been made of the "supersymmetric nonrenormalization theorem" stating the finiteness of the chiral vertices (see [7] and p.126 of [8]), which may be applied here since c is a chiral superfield. Since the supersymmetric nonrenormalization theorem assumes the superspace Feynman rules of exact supersymmetry, it follows that this proof of the nonrenormalization theorem for the chiral anomalies is valid only for theories where the supersymmetry is unbroken. But it is a fact that all physically interesting theories have explicitly and softly broken supersymmetry [9], and also that, due to infrared effects, such a breakdown naturally appears in the renormalization of supersymmetric theories with a supersymmetric gauge fixing [8]. Hence it is desirable to extend the proof to these cases.

The aim of the present paper is to derive the generalization of the antighost equation to the case of supersymmetric gauge theories quantized in a supersymmetric extension of the Landau gauge, and to indicate how it leads to a nonrenormalization theorem for the chiral anomalies which is valid in the broken symmetry case.

2 BRS Transformations and Classical Action

The notations and conventions are those of [8]. The gauge group is a compact simple Lie group G , with generators represented by matrices τ_a obeying the algebra $[\tau_a, \tau_b] = if_{abc}\tau_c$, with a normalized trace $\text{Tr } \tau_a \tau_b = \delta_{ab}$. The gauge real superfield field ϕ – the prepotential – is written in matrix form: $\phi = \tau_a \phi^a$, as well as the Lagrange multiplier, ghost and antighost chiral superfields B , c_+ and c_- , as well as their conjugates \bar{B} , \bar{c}_+ and \bar{c}_- . Matter is described by a set of chiral superfields A^i belonging to some – may be reducible – representation of the gauge group G , the generators being represented by Hermitean matrices $T_a^i{}_j$. The BRS

transformations read

$$\begin{aligned}
 s\phi &= \frac{1}{2}[\phi, c_+ + \bar{c}_+] + M(\phi)(c_+ - \bar{c}_+) \\
 &= c_+ - \bar{c}_+ + \frac{1}{2}[\phi, c_+ + \bar{c}_+] + \frac{1}{12}[\phi, [\phi, c_+ - \bar{c}_+]] + \dots, \\
 sc_+ &= -\frac{1}{2}\{c_+, c_+\}, & s\bar{c}_+ &= -\frac{1}{2}\{\bar{c}_+, \bar{c}_+\}, \\
 sA &= -c_+^a T_a A, & s\bar{A} &= \bar{c}_+^a T_a \bar{A}, \\
 sc_- &= B, & s\bar{c}_- &= \bar{B}, \\
 sB &= 0, & s\bar{B} &= 0.
 \end{aligned} \tag{2.1}$$

A concise notation has been used. For example, the first line means

$$s\phi^a = \frac{i}{2}f_{abc}\phi^b(c_+ + \bar{c}_+)^c + M_{ab}(\phi)(c_+ - \bar{c}_+)^b, \tag{2.2}$$

where the matrix M_{ab} may be computed as a power series in ϕ from the usual definition of the BRS transformation of the prepotential ϕ (written in matrix notation):

$$se^\phi = e^\phi c_+ - \bar{c}_+ e^\phi. \tag{2.3}$$

It is a remarkable fact that, except the term linear in ϕ , all the contributions to the BRS transformation of ϕ depend only on the difference³ $(c_+ - \bar{c}_+)$. In the supersymmetric Landau gauge, defined by

$$\frac{\delta\Sigma}{\delta B} = \frac{1}{8}\bar{D}^2 D^2 \phi, \quad \frac{\delta\Sigma}{\delta\bar{B}} = \frac{1}{8}D^2 \bar{D}^2 \phi, \tag{2.4}$$

the complete invariant classical action Σ reads

$$\Sigma = \Sigma_{\text{gauge inv.}} + \Sigma_{\text{gauge fixing}} + \Sigma_{\text{ext. fields}}, \tag{2.5}$$

where

$$\begin{aligned}
 \Sigma_{\text{gauge inv.}} &= -\frac{1}{128g^2}\text{Tr} \int dS F^\alpha F_\alpha + \frac{1}{16} \int dV \bar{A} e^{\phi^a T_a} A + \int dS U(A) + \int d\bar{S} \bar{U}(A), \\
 \Sigma_{\text{gauge fixing}} &= s \left[\frac{1}{8} \text{Tr} \int dV (c_- D^2 \phi + \bar{c}_- \bar{D}^2 \phi) \right] \\
 &= \frac{1}{8} \text{Tr} \int dV (B D^2 \phi + \bar{B} \bar{D}^2 \phi) - c_- D^2 s\phi - \bar{c}_- \bar{D}^2 s\phi, \\
 \Sigma_{\text{ext. fields}} &= \int dV \text{Tr} \phi^* s\phi + \left[\int dS (A^{*i} sA_i + \text{Tr} c_+^* s c_+) + \text{c.c.} \right].
 \end{aligned}$$

F_α denotes the supersymmetric field strength $\bar{D}^2(e^{-\phi} D_\alpha e^\phi)$, and $U(A)$ the superpotential – an invariant polynomial of degree 3 in A . Moreover, ϕ^* , A^* and c_+^* are the external fields coupled to the BRS transformations of ϕ , A and c_+ , respectively.

³This feature, i.e. the structure of (2.2) is still true if one replaces ϕ by any Lie algebra valued function $\mathcal{F}(\phi)$ in (2.3), this leading to a most general [8] transformation law for ϕ .

Denoting by $\Gamma(\phi, A, c_+, B, \phi^*, A^*, c_+^*)$ the generating functional of the 1PI Green functions, which coincide with the classical action (2.5) in the tree graph approximation, the BRS invariance of the theory is expressed by the Slavnov-Taylor identity⁴

$$S(\Gamma) := \text{Tr} \int dV \frac{\delta\Gamma}{\delta\phi^*} \frac{\delta\Gamma}{\delta\phi} + \left(\int dS \left\{ \frac{\delta\Gamma}{\delta A_i^*} \frac{\delta\Gamma}{\delta A_i} + \text{Tr} \frac{\delta\Gamma}{\delta c_+^*} \frac{\delta\Gamma}{\delta c_+} + \text{Tr} B \frac{\delta\Gamma}{\delta c_-} \right\} + \text{c.c.} \right) = 0. \quad (2.6)$$

The nilpotent linearized operator associated to the Slavnov-Taylor operator at Γ reads

$$S_\Gamma = \text{Tr} \int dV \left(\frac{\delta\Gamma}{\delta\phi^*} \frac{\delta}{\delta\phi} + \frac{\delta\Gamma}{\delta\phi} \frac{\delta}{\delta\phi^*} \right) + \left(\int dS \left(\frac{\delta\Gamma}{\delta A^*} \frac{\delta}{\delta A} + \frac{\delta\Gamma}{\delta A} \frac{\delta}{\delta A^*} + \text{Tr} \frac{\delta\Gamma}{\delta c_+^*} \frac{\delta}{\delta c_+} + \text{Tr} \frac{\delta\Gamma}{\delta c_+} \frac{\delta}{\delta c_+^*} + \text{Tr} B \frac{\delta}{\delta c_-} \right) + \text{c.c.} \right). \quad (2.7)$$

3 The Supersymmetric Antighost Equation

In order to derive the classical form of the supersymmetric antighost equation, let us differentiate the classical action (2.5) with respect to the ghost field c_+ :

$$\begin{aligned} \frac{\delta\Sigma}{\delta c_+} &= \frac{1}{16} \bar{D}^2 [D^2 c_-, \phi] + \frac{1}{16} \bar{D}^2 [\bar{D}^2 \bar{c}_-, \phi] \\ &\quad - \frac{1}{2} \bar{D}^2 [\phi^*, \phi] - \bar{D}^2 (\hat{\phi}^* M(\phi)) + [c_+^*, c_+] + A^* T_a A \tau_a. \end{aligned} \quad (3.1)$$

At this point one should observe that the right-hand side of (3.1), besides terms which are linear in the quantum fields, also contains nonlinear terms due to the presence of the formal power series $M(\phi)$ entering the BRS transformation (2.1) of the gauge superfield. These composite terms, being subject to renormalization, spoil the usefulness of this equation. However, considering the corresponding equation for \bar{c}_+ :

$$\begin{aligned} \frac{\delta\Sigma}{\delta \bar{c}_+} &= \frac{1}{16} D^2 [\bar{D}^2 \bar{c}_-, \phi] + \frac{1}{16} D^2 [D^2 c_-, \phi] \\ &\quad - \frac{1}{2} D^2 [\phi^*, \phi] + D^2 (\hat{\phi}^* M(\phi)) + [\bar{c}_+^*, \bar{c}_+] - \bar{A}^* T_a \bar{A} \tau_a, \end{aligned} \quad (3.2)$$

adding together the superspace integrals of the equations (3.1), (3.2) and using⁵ the Landau gauge conditions (2.4), one obtains the antighost equation we are looking for:

$$\mathcal{G}_- \Sigma = \Delta_{\text{class}}, \quad (3.3)$$

⁴We assume the absence of gauge anomaly.

⁵Use has been made of the identity

$$\int dS \bar{D}^2 [D^2 c_-, \phi] = \int dS [c_-, \bar{D}^2 D^2 \phi],$$

and of its complex conjugate.

with

$$\mathcal{G}_- := \int dS \left(\frac{\delta}{\delta c_+} - \left[c_-, \frac{\delta}{\delta B} \right] \right) + \int d\bar{S} \left(\frac{\delta}{\delta \bar{c}_+} - \left[\bar{c}_-, \frac{\delta}{\delta \bar{B}} \right] \right) \quad (3.4)$$

and

$$\Delta_{\text{class}} := - \int dV [\phi^*, \phi] + \int dS \cdot ([c_+^*, c_+] + (A^* T_a A) \tau_a) + \int d\bar{S} \cdot ([\bar{c}_+^*, \bar{c}_+] - (\bar{A} T_a \bar{A}^*) \tau_a) . \quad (3.5)$$

We remark that the undesired nonlinear terms present in each of the equations (3.1) and (3.2) have been cancelled. We are thus left with the breaking (3.5) which, being now linear in the quantum fields, will not be renormalized, i.e., it will remain a classical breaking.

Equation (3.3) has now a form which allows one to consider its validity to all orders of perturbation theory. That it indeed holds as it stands at the quantum level:

$$\mathcal{G}_- \Gamma = \Delta_{\text{class}} , \quad (3.6)$$

may be shown without any difficulty by repeating exactly the argument given in [5, 4] for the nonsupersymmetric case.

Let us finally remark that the sum of the superspace-integrated functional derivatives with respect to c_+ and \bar{c}_+ in (3.4) is in fact the space-time integral of the functional derivative with respect to the real part of the $\theta = 0$ component of c_+ . It coincides with the functional operator appearing in the nonsupersymmetric version of the antighost equation.

Rigid Invariance:

Using the “anticommutation relation”

$$\mathcal{G}_- \mathcal{S}(\Gamma) + \mathcal{S}_\Gamma \Delta_{\text{class}} = - \int dS \left(\left\{ c_-, \frac{\delta \Gamma}{\delta c_-} \right\} + \left[B, \frac{\delta \Gamma}{\delta B} \right] \right) - \int d\bar{S} \left(\left\{ \bar{c}_-, \frac{\delta \Gamma}{\delta \bar{c}_-} \right\} + \left[\bar{B}, \frac{\delta \Gamma}{\delta \bar{B}} \right] \right) ,$$

one easily checks that the following identity holds:

$$\begin{aligned} W_{\text{rigid}} \Gamma &:= \int dV \left(\left[\phi, \frac{\delta \Gamma}{\delta \phi} \right] + \left\{ \phi^*, \frac{\delta \Gamma}{\delta \phi^*} \right\} \right) \\ &+ \int dS \left(\left\{ c_+, \frac{\delta \Gamma}{\delta c_+} \right\} + \left[c_+^*, \frac{\delta \Gamma}{\delta c_+^*} \right] + \left[B, \frac{\delta \Gamma}{\delta B} \right] + \left\{ c_-, \frac{\delta \Gamma}{\delta c_-} \right\} + \left(\frac{\delta \Gamma}{\delta A} T_a A \right) \tau_a - \left(A^* T_a \frac{\delta \Gamma}{\delta A^*} \right) \tau_a \right) \\ &+ \int d\bar{S} \left(\left\{ \bar{c}_+, \frac{\delta \Gamma}{\delta \bar{c}_+} \right\} + \left[\bar{c}_+^*, \frac{\delta \Gamma}{\delta \bar{c}_+^*} \right] + \left[\bar{B}, \frac{\delta \Gamma}{\delta \bar{B}} \right] + \left\{ \bar{c}_-, \frac{\delta \Gamma}{\delta \bar{c}_-} \right\} - \left(\bar{A} T_a \frac{\delta \Gamma}{\delta \bar{A}} \right) \tau_a - \left(\frac{\delta \Gamma}{\delta \bar{A}^*} T_a \bar{A}^* \right) \tau_a \right) \\ &= 0 . \end{aligned} \quad (3.7)$$

This is the Ward identity expressing the invariance of the theory under the rigid transformations, corresponding to the transformations of the gauge group, but with constant (superspace independent) parameters.

4 Nonrenormalization Theorems for the Chiral Anomalies

This theorem is stated in general terms and proven in [6] as Theorem A.1. Its main consequence is Corollary A.2 of [6], which may be paraphrased as follows.

Let the infinitesimal linear transformation⁶

$$\delta A^i = ie^i_j A^j, \quad \delta \bar{A}_i = -i\bar{A}_i e^j_i, \quad (4.1)$$

acting on the matter chiral superfields – the numbers e^i_j being the elements of an hermitian matrix which commutes with the gauge group generators T_a – be a symmetry of the theory up to a possible soft breaking due to the mass terms. Such a symmetry is renormalizable [4] and may be expressed by a Ward identity

$$W\Gamma := e^i_j \left[\int dS A^j \frac{\delta \Gamma}{\delta A^i} - \int d\bar{S} \bar{A}_i \frac{\delta \Gamma}{\delta \bar{A}_j} \right] \Gamma \sim 0, \quad (4.2)$$

where \sim means equality up to soft breakings.

Then there exists a BRS invariant “current” J_{inv} – a real scalar superfield insertion – such that

$$\bar{D}^2 J_{\text{inv}} \cdot \Gamma = e^i_j A^j \frac{\delta \Gamma}{\delta A^i} + \tau \bar{D}^2 K^0 \cdot \Gamma. \quad (4.3)$$

Without the last term, which will be explained below, this would be just the Ward identity expressing the conservation of the current represented by the vector component of the superfield J_{inv} , i.e. of the Noether current associated to the invariance under the transformation (4.1). The last term is an anomaly since it cannot be reabsorbed in a BRS invariant way as a counterterm to J_{inv} . Indeed K^0 is a solution – unique up to trivial terms – of the following supersymmetric version of the descent equations:

$$\begin{aligned} \mathcal{S}_\Gamma[K^0 \cdot \Gamma] &= \bar{D}_{\dot{\alpha}}[K^{1\dot{\alpha}} \cdot \Gamma], \\ \mathcal{S}_\Gamma[K^{1\dot{\alpha}} \cdot \Gamma] &= (\bar{D}^{\dot{\alpha}} D^\alpha + 2D^\alpha \bar{D}^{\dot{\alpha}})[K^2_\alpha \cdot \Gamma], \\ \mathcal{S}_\Gamma[K^2_\alpha \cdot \Gamma] &= D_\alpha[K^3 \cdot \Gamma], \\ \mathcal{S}_\Gamma[K^3 \cdot \Gamma] &= 0, \end{aligned} \quad (4.4)$$

whose solution is a quantum extension of the following superfield polynomials:

$$\begin{aligned} K^0 &= \text{Tr}(\varphi^\alpha \bar{D}^2 \varphi_\alpha), \\ K^{1\dot{\alpha}} &= -\text{Tr}(D^\alpha c_+ \bar{D}^{\dot{\alpha}} \varphi_\alpha + \bar{D}^{\dot{\alpha}} D^\alpha c_+ \varphi_\alpha), \\ K^2_\alpha &= \text{Tr}(c_+ D_\alpha c_+), \\ K^3 &= \frac{1}{3} \text{Tr} c_+^3, \end{aligned} \quad (4.5)$$

⁶These transformations commute with supersymmetry. The nonrenormalization theorem in fact also holds for the Fayet R symmetry [6].

where we have introduced the chiral superconnection

$$\varphi_\alpha = e^{-\phi} D_\alpha e^\phi .$$

The main statement of the nonrenormalization theorem is that the anomaly coefficient r in (4.3) is exactly given by its one-loop approximation.

The proof, which we shall not repeat, is based on the finiteness of the insertion K^3 – which is equivalent to the vanishing of its anomalous dimension. The latter property was shown in [6] as a consequence of the nonrenormalization theorem for the chiral vertices [7]. But, since the latter theorem requires exact supersymmetry and exact superspace Feynman rules, and since supersymmetry is broken in some way in most cases of interest, an alternative for the finiteness of K^3 , generalizing the one given in [5] for the nonsupersymmetric case, will be given now using the antighost equation (3.6). This proof is purely algebraic [4] and thus covers all the situations, in particular those with a supersymmetry breaking described in terms of superfields shifted by θ -dependent, x -independent quantities [8, 10].

The proof of the finiteness of the ghost monomial $\text{Tr } c_+^3$ follows exactly the one given in [5, 4] for the nonsupersymmetric theories. We couple it to an external chiral superfield η , of dimension 3 and ghost number -3 , i.e. we add to the action the term

$$\frac{1}{3} \int dS \eta \text{Tr } c_+^3 - \frac{1}{3} \int d\bar{S} \bar{\eta} \text{Tr } \bar{c}_+^3 . \quad (4.6)$$

The total action is still BRS invariant, the Slavnov-Taylor identity, the gauge condition and the ghost equation remain unchanged. The antighost equation stays as in (3.6), with the same classical breaking, but with the modified differential operator

$$\mathcal{G}_- = \int dS \left(\frac{\delta}{\delta c_+} - \left[c_-, \frac{\delta}{\delta B} \right] - \eta \frac{\delta}{\delta c_+^*} \right) + \int d\bar{S} \left(\frac{\delta}{\delta \bar{c}_+} - \left[\bar{c}_-, \frac{\delta}{\delta \bar{B}} \right] + \bar{\eta} \frac{\delta}{\delta \bar{c}_+^*} \right) . \quad (4.7)$$

Let us now look at the possible η -dependent invariant counterterms Δ one may add to the action – i.e. the counterterms corresponding to a renormalization of $\text{Tr } c_+^3$. The only possible one is (4.6) itself, which however is ruled out by the condition

$$\mathcal{G}_- \Delta = 0 . \quad (4.8)$$

This concludes the algebraic proof of the nonrenormalization theorem of the anomaly of the type corresponding to the descent equations (4.4).

The finiteness of the higher invariant ghost monomials $\text{Tr } c_+^{2p+1}$, $p \geq 2$, as well as of the ghost field itself, goes along the same lines, as shown e.g. for $\text{Tr } c_+^5$ in usual gauge theories [11] in the proof of the nonrenormalization of the gauge anomaly.

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