CBPF-NF-065/96

Motions of Charged Particles in Gödel-type Spacetimes

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ABSTRACT

Gödel-type spacetimes in Hehl's nonpropagating torsion theory are reconsidered by supposing that the curvature source is a Weyssenhoff-Raabe fluid and an electromagnetic field. The electromagnetic field implies spacetime homogeneity and admits a dual interpretation. From the trajectories of test particles, it is shown that there is a class of such spacetimes for which charged particles can reach regions inaccessible to neutral particles or even photons.

PACS numbers: 04.20, 98.80

Key-words: Einstein-Cartan-Maxwell theory; Gödel-type spacetimes; Accelerated particle trajectories.

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1 Introduction

Homogeneous Gödel-type spacetimes in the framework of Hehl's nonpropagating torsion theory [1] were discussed by Oliveira *et al.* [2]. To solve the Einstein-Cartan (EC) equations, they assumed a Weyssenhoff-Raabe (WR) fluid [3] as curvature source and vanishing cosmological constant. Imposing spacetime homogeneity, they found hyperbolic and circular metrics, but the intermediary case known as Som-Raychaudhuri's metric in fact reduces to a Minkowskian metric endowed with torsion. Amongst the hyperbolic metrics, they found a subclass which is stably causal [4]. Here we intend to generalize the above work and discuss what the behaviour of charged test particles in Gödel-type spacetimes would be.

In Section 2, we modify the hypotheses of Oliveira *et al.* by adding to the Einstein-Cartan equations the cosmological constant and superposing to the energy-momentum tensor of the WR fluid the contribution of a sourceless electromagnetic field. Then it turns out that spacetime homogeneity is a consequence of the EC-Maxwell equations and not an *a priori* requirement. As a particular solution, we find a diagonal circular metric. Although this be just a mathematical possibility, since it demands a very high spin density, it survives in the Riemannian limit as an electrovac solution with cosmological constant. Furthermore, the electromagnetic field may as well eliminate causal hyperbolic metrics. In Section 3, we perform a duality transformation in the electromagnetic field and show that those spacetimes, except the ones with diagonal metric, can also be interpreted as resulting from a charged WR fluid. In Section 4, we discuss the qualitative features of charged test particle trajectories in hyperbolic spacetimes. It is found that, contrary to the case of geodesics, trajectories unbounded in the r coordinate are possible for noncausal metrics. A summary of the main results and some comments on the quantum dynamics of charged test fields in such backgrounds are presented in Section 5.

2 Solutions to Einstein-Cartan-Maxwell Equations

A Gödel-type spacetime, whose line element is (c=1)

$$ds^2 = [dt + H(r)d\varphi]^2 - dr^2 - [D(r)d\varphi]^2 - dz^2, \tag{1}$$

is homogeneous if, and only if, the functions D and H satisfy the conditions [5, 6]

$$\frac{d^2D}{dr^2} = 4l^2D \quad , \quad \frac{dH}{dr} = 2\Omega D, \tag{2}$$

in which Ω and l^2 are real constants. Moreover, Maitra's regularity conditions — $\lim_{r\to 0} D = r$ and $\lim_{r\to 0} H \propto r^2$ — permit to interpret r, φ and z as cylindrical coordinates and integrate Eqs. (2) to obtain

$$D = \frac{\sinh(2lr)}{2l}, \quad H = \frac{\Omega \sinh^2(lr)}{l^2}. \tag{3}$$

The metrics are called hyperbolic if $l^2 > 0$, circular if $l^2 < 0$ and, for $\Omega \neq 0$, the limit $l \to 0$ gives the Som-Raychaudhuri metric [7]: $D = r, H = \Omega r^2$.

The parameters Ω and l will be determined from the EC equations of Hehl's nonpropagating torsion theory:

$$G_{\mu\nu} + \Lambda g_{\mu\nu} \;=\; kT_{\mu\nu} \quad \text{and} \quad T_{\mu\nu\sigma} \;=\; kS_{\mu\nu\sigma}. \tag{4}$$

In the first set of equations $G_{\mu\nu}$ is the Einstein tensor built from the Riemann-Cartan connection $\Gamma^{\mu}{}_{\nu\sigma}$, $T_{\mu\nu}$ is the canonical energy-momentum tensor and k is the gravitational constant. In the second set $S_{\mu\nu\sigma}$ is the spin tensor and $T_{\mu\nu\sigma}$ is the modified torsion. The latter is related with the torsion tensor $\tau_{\mu\nu\sigma}$, here defined by

$$\tau_{\mu\nu\sigma} := \frac{1}{2} g_{\mu\rho}(\Gamma^{\rho}{}_{\nu\sigma} - \Gamma^{\rho}{}_{\sigma\nu}),$$

through

$$T_{\mu\nu\sigma} = \tau_{\mu\nu\sigma} - \tau^{\lambda}{}_{\mu\sigma} g_{\nu\sigma} - \tau^{\lambda}{}_{\nu\sigma} g_{\mu\sigma}. \tag{5}$$

(a colon followed by the equal sign means "equal by definition"). We assume that the torsion is generated by the spin $S^{\mu}{}_{\nu\sigma}$of a WR fluid, that is,

$$S^{\lambda}{}_{\nu\sigma} := u^{\lambda} S_{\nu\sigma}, \quad S_{\nu\sigma} u^{\sigma} := 0, \tag{6}$$

u^{μ} being the fluid four-velocity and $S_{\mu\nu} = -S_{\nu\mu}$ the spin density. Then, Eqs. (4- 6) and the metric postulate imply that the full connection is given by

$$\Gamma^{\mu}{}_{\nu\sigma} = \widetilde{\Gamma}^{\mu}{}_{\nu\sigma} + k\left(u^{\mu} S_{\nu\sigma} + u_{\nu} S_{\sigma}{}^{\mu} - u_{\sigma} S^{\mu}{}_{\nu}\right), \tag{7}$$

where $\widetilde{\Gamma}^{\mu}{}_{\nu\sigma}$ is the Christoffel symbol of second kind. On the other hand, the WR fluid energy-momentum tensor, $T^{w}_{\mu\nu}$, is given by

$$T^{w}_{\mu\nu} = (p+\rho) u_{\mu} u_{\nu} - p g_{\mu\nu} + 2u^{\sigma} u^{\rho} \nabla_{\rho}(u_{\sigma} S_{\mu\nu}), \tag{8}$$

p denoting isotropic pressure, ρ energy density and ∇_{ρ} the Riemann-Cartan covariant derivative operator. Thus, taking $T_{\mu\nu} = T^{w}_{\mu\nu} + t_{\mu\nu}$, Eqs. (4) can be written in the "quasi-Riemannian" form [8],

$$\widetilde{G}_{\mu\nu} + \Lambda g_{\mu\nu} = k\Sigma_{\mu\nu} + kt_{\mu\nu}. \tag{9}$$

$\widetilde{G}_{\mu\nu}$ is the conventional Einstein tensor and

$$\Sigma_{\mu\nu} := (p+\rho-2kS^2)u_\mu u_\nu - (p-kS^2)g_{\mu\nu} + 2(u^\sigma u^\rho - g^{\sigma\rho})\widetilde{\nabla}_\sigma\left[u_{(\nu}S_{\mu)\rho}\right], \qquad (10)$$

where $S^2 := S_{\sigma\rho}S^{\sigma\rho}/2$, $\widetilde{\nabla}_\rho$ is the Riemannian derivative operator and the parentheses enclosing indices mean symmetrization in these indices.

As usual we choose the inertial frame of reference defined by the differential 1-forms Θ^a, $\Theta^a := e^a_\mu dx^\mu$,

$$\Theta^{\hat{0}} = dt + Hd\varphi,\ \Theta^{\hat{1}} = dr,\ \Theta^{\hat{2}} = Dd\varphi,\ \Theta^{\hat{3}} = dz, \qquad (11)$$

or, equivalently, by the tetrad basis

$$e^{\hat{0}}_0 = e^{\hat{1}}_1 = e^{\hat{3}}_3 = 1, \quad e^{\hat{0}}_2 = H, \quad e^{\hat{2}}_2 = D, \qquad (12)$$

where t, r, φ and z were numbered as 0, 1, 2 and 3, respectively, and the hats denote tetrad indices. For an observer comoving with the fluid, *viz.*, $u^\mu = \delta^\mu_0$, the only nonvanishing kinematical parameter, computed with the connection given in Eq. (7), is the rotation vector ω^μ whose tetrad components are

$$\omega^a = (0, kS_{\hat{2}\hat{3}}, kS_{\hat{3}\hat{1}}, \Omega + kS_{\hat{1}\hat{2}}).$$

We assume that Ω and S_{ab} depend solely on the coordinate r and, in order to have the same symmetry axis as in General Relativity (GR), we take $S_{\hat{1}\hat{3}} = S_{\hat{2}\hat{3}} = 0$. Then, putting $S(r) := S_{\hat{1}\hat{2}}$, it follows that

$$\omega^a = [\Omega(r) + kS(r)]\,\delta^a_3. \qquad (13)$$

Now we shall suppose that $t_{\mu\nu}$ in Eq. (9) is due to an electromagnetic field. Considering that in Hehl's theory the electromagnetic field does not couple to torsion, in the sense that $f_{\mu\nu} := \partial_\mu A_\nu - \partial_\nu A_\mu$, the Maxwell equations are formally the same as in GR, i. e.,

$$\widetilde{\nabla}_\nu f^{\mu\nu} = j^\mu, \quad \widetilde{\nabla}_{[\sigma}f_{\mu\nu]} = 0, \qquad (14)$$

the square brackets and j^μ standing for antisymmetrization and four-current, respectively. So, taking for granted that $t_{\mu\nu} = E_{\mu\nu}$ in Eq. (9), where

$$E_{\mu\nu} := f_{\mu\sigma}f^\sigma{}_\nu + \frac{1}{4}g_{\mu\nu}f_{\sigma\rho}f^{\sigma\rho}, \qquad (15)$$

we find that the only nonvanishing components of f_{ab} compatible with the EC equations are $f_{\hat{1}\hat{2}}$ and $f_{\hat{0}\hat{3}}$. The sourceless Maxwell equations then yield

$$\partial_\mu f_{\hat{1}\hat{2}} = \partial_\mu f_{\hat{0}\hat{3}} = 0 \ \ (\mu = 0,1,2), \quad \frac{df_{\hat{0}\hat{3}}}{dz} - 2\Omega f_{\hat{1}\hat{2}} = 0, \quad \frac{df_{\hat{1}\hat{2}}}{dz} + 2\Omega f_{\hat{0}\hat{3}} = 0. \qquad (16)$$

and these equations require that Ω be constant. Hence we obtain

$$f_{\hat{0}\hat{3}} = b_0 \sin[2\Omega(z - z_0)], \quad f_{\hat{1}\hat{2}} = b_0 \cos[2\Omega(z - z_0)], \tag{17}$$

in which b_0 and z_0 are integration constants. Accordingly E_{ab} is given by

$$E_{ab} = \mathrm{diag}(\rho_{em}, \rho_{em}, \rho_{em}, -\rho_{em}), \tag{18}$$

where $\rho_{em} := b_0^2/2$ is the electromagnetic energy density. The expressions (16-18) have the same form as in GR, but now Ω takes into account the torsion since the $\hat{0}\hat{2}$ component of the EC equations gives

$$\frac{d\Omega}{dr} = -k\frac{dS}{dr} \quad \text{or} \quad \Omega = \Omega_0 - kS. \tag{19}$$

Ω_0 is an integration constant to be determined, but from Eq. (13) we get

$$\omega^a = \Omega_0 \delta^a_3 \tag{20}$$

and, by this reason, Ω_0 must be interpreted as the magnitude of the rotation vector. In view of Eqs. (18, 19), the remaining EC equations lead to

$$k\rho = \Omega_0^2 + k\rho_{em} + \Lambda, \quad kp = \Omega_0^2 - k\rho_{em} - \Lambda, \quad 2l^2 = \Omega_0^2 - k\rho_{em} - kS\Omega_0. \tag{21}$$

From the last equation it comes out that not only Ω but also l^2 is constant. Therefore, the spacetime homogeneity is consequence of the EC-Maxwell equations by themselves. In the absence of electromagnetic fields, inhomogeneous solutions are also possible (if $S \neq 0$) because we do not have Eq. (16) to assure that Ω and S are constants in Eqs. (19) and (21). It follows as well that the rotation vector (covariantly defined through the EC affinity) is not affected by the the torsion and/or electromagnetic field since Eqs. (21) imply the same expression for Ω_0 as in GR, that is,

$$\Omega_0^2 = \frac{k}{2}(\rho + p) \tag{22}$$

Besides this, the positivity of ρ and p together with the state equation $0 \leq p \leq \rho$ imposes limits on the values of the cosmological constant,

$$-k\rho_{em} \leq \Lambda \leq \Omega_0^2 - k\rho_{em}, \tag{23}$$

and then, again like in GR [9, 10, 11], the minimum value of Λ is associated with a stiff fluid and the maximum value corresponds to zero pressure. However, although the torsion does not modify the rotation vector and the interval for Λ, it alters the expressions for the metrical parameters Ω and l (and consequently for the electromagnetic field) as we see from Eqs. (19,21).

For circular metrics ($l^2 < 0$), the last equation in (21) implies that we must have

$$kS\Omega_0 > \Omega_0^2 - k\rho_{em} \quad \text{or} \quad kS > \frac{\sqrt{k}(p+\rho-2\rho_{em})}{\sqrt{2(p+\rho)}},$$

where the second inequality was obtained by supposing $\Omega_0 > 0$ and by using Eq. (22). Hence circular metrics without electromagnetic field would require a very high spin density in comparison with the matter density, namely: $kS^2 > (p+\rho)/2$. The same would happen in the case of a diagonal circular metric ($\Omega = 0$) in the presence of electromagnetic field since we would have $2l^2 = -k\rho_{em}$ and $kS^2 = (p+\rho)/2$. However its Riemannian limit gives a diagonal electrovac solution with $\Lambda = -k\rho_{em}$.

Predominance of spin effects would also be required in the case of Riemannian flatness and hyperbolic metrics with $l^2 > \Omega^2$. Indeed, rewriting the last of the Eqs. (21) as $2l^2 = \Omega_0\Omega - k\rho_{em}$, we see that the condition to Riemannian flatness ($\Omega = l = 0$) is equivalent to $\rho_{em} = 0$ and $kS^2 = (p+\rho)/2$. Moreover for $\rho_{em} = l = 0$ we also obtain, as it was stated in the Introduction, the previous flat metric and not the Som-Raychaudhuri one, unless we had admitted a spin density in absence of matter ($\Omega = -kS, p = \rho = 0$) or negative pressures ($p = -\rho, \Omega = -kS$). On the other hand, Oliveira *et al.* [2] have pointed out that the torsion allows Gödel-type spacetimes to cover the region of parametrization $l^2 > \Omega^2$, in which there is no causality problem. However, for this class of hyperbolic metrics, Eqs. (19,21) require that

$$kS \;\epsilon \left(\frac{3\Omega_0}{4} - \frac{1}{4}\sqrt{\Omega_0^2 - 8k\rho_{em}}\;, \frac{3\Omega_0}{4} + \frac{1}{4}\sqrt{\Omega_0^2 - 8k\rho_{em}}\;\right). \tag{24}$$

When $8k\rho_{em}$ increases from zero to Ω_0^2, the lower bound of the above interval increases from $\Omega_0/2$ to $3\Omega_0/4$ and, therefore, we always must have $kS^2 > (p+\rho)/8$ and this means a very high spin density as before. The interval becomes meaningless if $8k\rho_{em} \geq \Omega_0^2$, that is, on this condition the electromagnetic field eliminates the mathematical possibility of hyperbolic metrics with $l^2 > \Omega^2$.

Now we derive some relations to be used afterwards. From Eqs. (1,12,17) we find that the components of the electromagnetic field, $f_{\mu\nu}$, and of its dual,

$${}^*f^{\mu\nu} := -\frac{1}{2\sqrt{-g}}\varepsilon^{\mu\nu\sigma\rho}f_{\sigma\rho}, \quad {}^*f_{\mu\nu} := \frac{1}{2}\sqrt{-g}\varepsilon_{\mu\nu\sigma\rho}f^{\sigma\rho},$$

are given by

$$\begin{gathered} f_{03}=b_0\sin[2\Omega(z-z_0)],\, f_{12}=b_0D(r)\cos[2\Omega(z-z_0)],\, f_{23}=b_0H(r)\sin[2\Omega(z-z_0)], \\ {}^*f_{03}=f_{12}/D, \quad {}^*f_{12}=-Df_{03}, \quad {}^*f_{23}=Hf_{12}/D, \\ f^{01}=Hf_{12}/D^2, \quad f^{03}=-f_{03}, \quad f^{12}=f_{12}/D^2, \\ {}^*f^{01}=-f_{23}/D, \quad {}^*f^{03}=-f_{12}/D, \quad {}^*f^{12}=-f_{03}/D. \end{gathered} \tag{25}$$

A suitable choice for the vector potential A_μ is

$$A_0=-\frac{b_0}{\Omega}\sin^2[\Omega(z-z_0)], \quad A_2=\frac{b_0}{2\Omega}H(r)\cos[2\Omega(z-z_0)], \tag{26}$$

where the integration constants were chosen so as to have a finite limit for the diagonal metric, namely,

$$f_{12}=b_0D(r), \quad A_2=\frac{b_0}{2l^2}\sinh^2(lr). \tag{27}$$

Finaly we find that the two invariants of the electromagnetic field are given by

$$f_1:=\frac{1}{2}f_{\mu\nu}f^{\mu\nu}=b_0^2\cos[4\Omega(z-z_0)], \quad f_2:=\frac{1}{2}f_{\mu\nu}{}^*f^{\mu\nu}=-b_0^2\sin[4\Omega(z-z_0)]. \tag{28}$$

3 Duality Rotation in The Electromagnetic Field

Now we can show that the solutions we have presented may also be interpreted as resulting from a WR fluid with a charge distribution $J^\mu=\sigma\delta^\mu_0$. In effect, the Som-Raychaudhuri metric was firstly found as a limit of cylindrically symmetric solutions to Einstein's equations having a charged perfect fluid as curvature source and, later on, it appeared as a particular case of Gödel -type spacetimes with a perfect fluid and a sourceless electromagnetic field. Based on this fact, Raychaudhuri and Thakurta suggested that this duality would be true for the whole class of Gödel-type metrics. The validity of such supposition may be demonstrated by starting from the Maxwell -Einstein equations for a charged perfect fluid, but in place of this we prefer to perform a duality rotation in the electromagnetic field following a procedure due to Gopala Rao [12]. Thus, if we define the complex field $w_{\mu\nu}$ by $w_{\mu\nu}:=f_{\mu\nu}+i\,{}^*f_{\mu\nu}$, the Maxwell equations and energy-momentum tensor, Eqs. (14,15), are written respectively as

$$\widetilde{\nabla}_\nu w^{\mu\nu}=j^\mu, \quad E_{\mu\nu}=w_\mu{}^\sigma\overline{w}_{\nu\sigma}/2,$$

in which $\overline{w}_{\nu\mu}$ stands for the complex conjugate of $w_{\nu\mu}$. The dual rotated field $W_{\mu\nu}$, defined by

$$W_{\mu\nu}:=e^{i\theta}w_{\mu\nu}, \tag{29}$$

where θ is a real scalar function depending on the coordinates, does not change the value of $E_{\mu\nu}$ and therefore does not modify the solutions to the EC equations. The requirement that $J^\mu := \widetilde{\nabla}_\nu W^{\mu\nu}$ be real and divergence-free provides a set of differential equations, namely,

$$\begin{aligned} \cos\theta f^{\mu\nu}\partial_\nu\theta + \sin\theta(j^\mu - {}^*f^{\mu\nu}\partial_\nu\theta) = 0, \\ J^\mu = -\sin(\theta)f^{\mu\nu}\partial_\nu\theta + \cos\theta(j^\mu - {}^*f^{\mu\nu}\partial_\nu\theta). \end{aligned} \tag{30}$$

The first set of these equations gives the condition under which J^μ is real and may be used to determine θ. However, here we are dealing with non-null fields with vanishing Lorentz force and $f_2 \neq 0$. In such a case θ can be determined from [12]

$$\tan(2\theta) = -f_2/f_1. \tag{31}$$

Then, by means of Eq. (28), we obtain

$$\theta = 2\Omega(z - z_0) + n\pi \qquad (n = \text{integer}) \tag{32}$$

and, on account of Eqs. (25,30) with $j^\mu = 0$, we get

$$J^\mu = (-1)^n 2\Omega b_0 \delta^\mu_0, \tag{33}$$

Thus the charge density is given by $\sigma = (-1)^n 2\Omega b_0$. The expressions for the transformed fields $F_{\mu\nu}$ and ${}^*F_{\mu\nu}$,

$$F_{\mu\nu} = f_{\mu\nu}\cos\theta - {}^*f_{\mu\nu}\sin\theta, \quad {}^*F_{\mu\nu} = {}^*f_{\mu\nu}\cos\theta + f_{\mu\nu}\sin\theta, \tag{34}$$

are obtained by equating the right hand side of Eq. (29) to $F_{\mu\nu} + i\ {}^*F_{\mu\nu}$. Then, using Eqs. (25), we get $[B := (-1)^n b_0]$

$$F_{12} = BD, \ {}^*F_{03} = B, \ {}^*F_{23} = BH, \tag{35}$$

and, putting $F_{\mu\nu} := \partial_\mu\overline{A}_\nu - \partial_\nu\overline{A}_\mu$, the only nonvanishing component for $\overline{A}_\mu$ is

$$\overline{A}_2 = \frac{B}{2\Omega}H(r). \tag{36}$$

From Eq. (34) we find

$$F_1 = f_1\cos(2\theta) - f_2\sin(2\theta), \quad F_2 = f_1\sin(2\theta) + f_2\cos(2\theta),$$

and consequently the new electromagnetic invariants now are

$$F_1 = B^2, \quad F_2 = 0. \tag{37}$$

Eq. (31) does not hold for $\Omega = 0$ ($f_2 = 0$) but for this particular case we can use Eqs. (30) to find θ =constant and $J^\mu = 0$. This trivial transformation is precisely what we would expect on the basis of Gopala Rao's formalism. It reduces to the identity transformation if we take $\theta = 0$ and, in this case, it is included in the preceding equations. In this manner, we have found the expressions for the dual-rotated electromagnetic field in Gödel-type spacetimes. We repeat that the results of this section may be obtained by starting from the EC-Maxwell equations with a charge current given by $J^\mu = \sigma\delta^\mu_0$. Note, moreover, that we can take $n = 0$ in Eq. (32) since for an even n the results are the same as for $n = 0$ and for an odd n the only modification is the change of the sign of B. With this proviso, we can also put $B = b_0$.

4 Charged Test Particles in Hyperbolic Spacetimes

The ambiguity in the electromagnetic field concerns only the spacetime geometry, inasmuch as the behaviour of charged test particles and fields will be different in both cases. In this section we discuss the dynamics of charged test particles in spacetimes with hyperbolic metrics. We shall suppose the electromagnetic field results from a charged fluid.

The equations of motion for a charged test particle, with mass M and charge q, interacting with a gravitational field $g_{\mu\nu}$ and an electromagnetic field A_μ, can be obtained from the Lagrangian function L [13],

$$L = \frac{1}{2} g_{\mu\nu} \frac{dx^\mu}{d\sigma}\frac{dx^\nu}{d\sigma} + A_\mu \frac{dx^\mu}{d\sigma}.$$

As $g_{\mu\nu}$ and A_μ do not depend explicitly on the parameter σ, the problem presents a constant of motion which we identify with the mass according to

$$g_{\mu\nu}\frac{dx^\mu}{d\sigma}\frac{dx^\nu}{d\sigma} = M^2,$$

what amounts to taking $\tau = M\sigma$, where τ is the proper time. The Hamiltonian function $\mathcal{H}$ corresponding to L is

$$\mathcal{H} = \frac{1}{2} g^{\mu\nu}(p_\mu - qA_\mu)(p_\nu - qA_\nu), \tag{38a}$$

where the p_μ are the momenta conjugate to x^μ and the constant of motion now reads

$$\mathcal{H} = \frac{M^2}{2}. \tag{38b}$$

For Gödel-type spacetimes with $A_\mu = (A_0, 0, A_2, 0)$ we get

$$\mathcal{H} = \frac{1}{2}\left\{(p_t - qA_0)^2 - p_r^2 - p_z^2 - \frac{1}{D^2}[(p_t - qA_0)H - (p_\varphi - qA_2)]^2\right\}. \tag{39}$$

Restricting to the potential given by Eq. (36), where $A_0 = 0$ and $A_2 := \overline{A}_2$ is function of r uniquely, $\mathcal{H}$ does not depend on the coordinates t, φ and z. Thus the momenta p_t, p_φ and p_z are constants of motion and the the Hamilton equations along with Eq. (38b) lead to

$$\begin{cases} M^2\left(\frac{dr}{d\tau}\right)^2 = p_t^2 - p_z^2 - M^2 - \frac{1}{D^2}\left[\left(p_t + \frac{qB}{2\Omega}\right)H - p_\varphi\right]^2, \\ M\frac{dz}{d\tau} = -p_z, \\ M\frac{dt}{d\tau} = p_t - \frac{H}{D^2}\left[\left(p_t + \frac{qB}{2\Omega}\right)H - p_\varphi\right], \\ M\frac{d\varphi}{d\tau} = \frac{1}{D^2}\left[\left(p_t + \frac{qB}{2\Omega}\right)H - p_\varphi\right]. \end{cases} \tag{40}$$

Replacing D and H in the first equation by their expressions given in Eq. (3), we have

$$\frac{M^2}{4l^2\rho(\rho+1)}\left(\frac{d\rho}{d\tau}\right)^2 = p_t^2 - V(\rho), \qquad V(\rho) := \beta^2 + \frac{W^2}{l^2}\frac{(\rho-\alpha)^2}{\rho(\rho+1)}, \tag{41}$$

where we have also used the definitions

$$\beta^2 := M^2 + p_z^2, \quad \alpha := \frac{l^2 p_\varphi}{W}, \quad W := \Omega p_t + \frac{qB}{2}, \quad \rho := \sinh^2(lr). \tag{42}$$

A necessary condition for the motion is that the right hand side of Eq. (41) be zero or positive. Particularly, if the function $V(\rho)$ has a minimum value V_m with respect to ρ, we must have

$$p_t^2 \geq V_m. \tag{43}$$

Effectively there is a minimum for V only if $-1/2 < \alpha < \infty$ (for $\alpha = 0$, V increases monotonously) and then we find

$$V_m = \begin{cases} \beta^2, & \text{in } \rho = \alpha, \qquad \text{if } \alpha \geq 0 \\ \beta^2 - \frac{4W^2}{l^2}(\alpha^2 + \alpha), & \text{in } \rho = -\frac{\alpha}{1+2\alpha} \quad \text{if } -\frac{1}{2} < \alpha < 0 \end{cases} \tag{44}$$

Note that when α decreases in the interval $-\frac{1}{2} < \alpha < 0$, V_m increases spanning the interval

$$\beta^2 < V_m < \beta^2 + \frac{W^2}{l^2}. \tag{45}$$

For $\alpha \leq -\frac{1}{2}$, V is a dereasing function of ρ. Furthermore, for any α, $V(\rho)$ has the same expression for its asymptotic value, V_a, that is,

$$V_a := \lim_{\rho\to\infty} V(\rho) = \beta^2 + \frac{W^2}{l^2}, \quad (\forall\alpha). \tag{46}$$

We find as well that

$$\lim_{\rho\to 0} V(\rho) = \begin{cases} \beta^2, & \text{if } \alpha = 0 \\ \infty, & \text{if } \alpha \neq 0 \end{cases} \tag{47}$$

From the previous results, we can sketch the graphs below where in the ordinate axis, instead of $V(\rho)$, we have plotted the the function $U(\rho)$,

$$U(\rho) := \frac{(\rho-\alpha)^2}{\rho(\rho+1)}. \tag{48}$$

From these graphs it becomes clear that, for $\alpha > -1/2$, V_a works as a potential barrier and we shall have motions bounded in the r coordinate whenever

$$V_m \leq p_t^2 < V_a := \beta^2 + \frac{W^2}{l^2} \quad \text{(bounded motion).} \tag{49a}$$

As a matter of fact the lower limit for α depends on p_t since, using Eqs. (43-44), we find

$$-\frac{1}{2} + \frac{1}{2}\sqrt{1 - \frac{l^2}{W^2}(p_t^2 - \beta^2)} \leq \alpha < \infty \quad \text{(bounded motion).} \tag{49b}$$

Furthermore, writing Eq. (41) as

$$\frac{M^2}{4l^2}\left(\frac{d\rho}{d\tau}\right)^2 = \left(p_t^2 - \beta^2 - \frac{W^2}{l^2}\right)\rho^2 + \left(p_t^2 - \beta^2 + \frac{2\alpha W^2}{l^2}\right)\rho - \frac{\alpha^2 W^2}{l^2} \tag{50}$$

and equating its right hand side to zero we obtain the turning points $\rho_{(\pm)}$ for confined motions,

$$\rho_{(\pm)} = \frac{p_t^2 - \beta^2 + \frac{2\alpha W^2}{l^2} \pm \sqrt{(p_t^2-\beta^2)\left[p_t^2 - \beta^2 + \frac{4W^2}{l^2}(\alpha^2+\alpha)\right]}}{2(W^2/l^2 + \beta^2 - p_t^2)}. \tag{51}$$

As expected, this gives $\rho_{(-)} = 0$ when $\alpha = 0$ while for $p_t^2 = V_m$ we have

$$\rho_{(-)} = \rho_{(+)} = \begin{cases} \alpha, & \text{if } \alpha \geq 0 \\ -\frac{\alpha}{1+2\alpha}, & \text{if } -\frac{1}{2} < \alpha < 0 \end{cases}$$

that is, $\rho_{(-)} = \rho_{(+)}$ and then there is no radial motion.

On the other hand, it is also clear that unbounded motions may occur if

$$p_t^2 > \beta^2 + \frac{1}{l^2}W^2, \text{ (unbonded motion)} \tag{52}$$

or, equivalently, if

$$g := (\Omega^2 - l^2)p_t^2 + qB\Omega p_t + \frac{q^2B^2}{4} + l^2\beta^2 < 0. \tag{53}$$

and in this case there are no restrictions on α. The motion, if any, is unbounded in the sense that there is a lower bound ρ_0 for ρ,

$$\rho_0 := \frac{\beta^2 - p_t^2 - \frac{2\alpha W^2}{l^2} + \sqrt{(p_t^2 - \beta^2)\left[p_t^2 - \beta^2 + \frac{4W^2}{l^2}(\alpha^2 + \alpha)\right]}}{2(p_t^2 - W^2/l^2 - \beta^2)}, \tag{54}$$

but there is no upper bound. For $\alpha = 0$, $\rho_0 = 0$ and unbounded motions would cover all the possible values for ρ. Moreover, from the plots it follows that we may as well have unbounded motions if $p_t^2 = V_a$, provided that $\alpha > -1/2$. The radial velocities of these motions, $dr/d\tau$, approach to zero when $\rho \to \infty$. Mathematically we can express that by writing the Eq. (50) as

$$\frac{M^2}{4}\left(\frac{d\rho}{d\tau}\right)^2 = W^2\left[(1 + 2\alpha)\rho - \alpha^2\right].$$

As the motion is possible only when the right hand side of this equation is zero or positive, we must have

$$\rho \geq \frac{\alpha^2}{1 + 2\alpha} \text{ and } \alpha > -\frac{1}{2} \quad (\text{if } p_t^2 = V_a). \tag{55}$$

Now we are ready to analize the possibility of unbounded radial motions in hyperbolic spacetimes. For the sake of comparison, note that for $q = 0$ or/and $B = 0$ we have the following situation:

$$\begin{array}{ll} \Omega^2 < l^2: & \left\{ \begin{array}{ll} \text{bounded motions,} & \text{if } V_m \leq p_t^2 < \frac{\beta^2 l^2}{l^2 - \Omega^2}, \\ \text{unbounded motions,} & \text{if } p_t^2 \geq \frac{\beta^2 l^2}{l^2 - \Omega^2}, \end{array} \right. \\ \Omega^2 \geq l^2: & \text{bounded motions only,} \quad p_t^2 \geq V_m. \end{array} \tag{56}$$

Therefore, there are no unbounded geodesic motions when $\Omega^2 \geq l^2$.

In any consideration we have to remember that for both kinds of motions there is a prohibited range for the energies:

$$\Omega p_t \ni (- \mid \Omega\sqrt{V_m} \mid, \mid \Omega\sqrt{V_m} \mid). \tag{57}$$

Furthermore, from Eq. (53) it is also clear that in the case of interaction: i) if $\Omega^2 < l^2$ unbounded motions are possible regardless the relative signs of Ωp_t and qB; ii) if $\Omega^2 \geq l^2$ we may have unbounded motion only when Ωp_t and qB present opposite signs. Here we are concerned mainly with the occurrence of unbounded motions in the second case; they will be possible because, for opposite signs of Ωp_t and qB, V_a can be smaller than the corresponding quantity for the case without electromagnetic interaction. In others words, the electromagnetic interaction can diminishes the height of the "potential barrier" which is responsible by the confinement. Hence, a necessary condition for unbounded motion, when $\Omega^2 \geq l^2$, is that $V_a < \Omega^2 p_t^2/l^2 + \beta^2$ or, equivalently,

$$\Omega p_t qB < -\frac{q^2B^2}{4}. \tag{58}$$

We shall discuss separately the cases $l^2 > \Omega^2$, $\Omega^2 = l^2$ and $l^2 < \Omega^2$. We fix the sign of the charge by supposing $qB > 0$.

Case I: $l^2 > \Omega^2$. In this case the function g presents a maximum value with respect to Ωp_t, g_{max},

$$g_{max} = l^2\left[\beta^2 + \frac{q^2B^2}{4(l^2-\Omega^2)}\right], \quad \text{in } \Omega p_t = \frac{qB\Omega^2}{2(l^2-\Omega^2)},$$

and $g \leq 0$ (unbounded motion) if

$$\begin{array}{l} -\infty < \Omega p_t \leq -\frac{\Omega^2}{2(l^2-\Omega^2)}\left(-qB+\sqrt{\Delta}\right), \\ \frac{\Omega^2}{2(l^2-\Omega^2)}\left(qB+\sqrt{\Delta}\right) \leq \Omega p_t < \infty, \end{array} \tag{59}$$

where the equal sign holds only if $\alpha > -1/2$ and

$$\Delta := q^2B^2 + \frac{4(l^2-\Omega^2)}{\Omega^2}\left(l^2\beta^2 + \frac{q^2B^2}{4}\right). \tag{60}$$

On the other hand, $g > 0$ if

$$-\frac{\Omega^2}{2(l^2-\Omega^2)}\left(-qB+\sqrt{\Delta}\right) < \Omega p_t < \frac{\Omega^2}{2(l^2-\Omega^2)}\left(qB+\sqrt{\Delta}\right)$$

This together with the condition (57) imply that for bounded motion we must have

$$\begin{array}{l} -\frac{\Omega^2}{2(l^2-\Omega^2)}\left(-qB+\sqrt{\Delta}\right) < \Omega p_t \leq -\mid \Omega\sqrt{V_m}\mid, \\ \mid \Omega\sqrt{V_m}\mid \leq \Omega p_t < \frac{\Omega^2}{2(l^2-\Omega^2)}\left(qB+\sqrt{\Delta}\right). \end{array} \tag{61}$$

For $qB \to 0$ the inequalities (59) and (61) reduce to those in (56).

Case II: $l^2 = \Omega^2$. When $l^2 \to \Omega^2$ and $qB > 0$ we have

$$\sqrt{\Delta} \to qB + \frac{2(l^2-\Omega^2)}{qB}\left(\Omega^2\beta^2 + \frac{q^2B^2}{4}\right). \tag{62}$$

Thus the limit for unbounded motion, obtained from (59), gives

$$-\infty < \Omega p_t \leq -\frac{1}{qB}\left(\Omega^2\beta^2 + \frac{q^2B^2}{4}\right), \tag{63}$$

since the second inequality affords an empty set. This latter fact is what we would expect since for $\Omega^2 \geq l^2$ unbounded motions require opposite signs for Ωp_t and qB. Note as well that the right hand side of the above inequality satisfies the condition given in (58). For bounded motions the limits of (61) are

$$\begin{array}{c} -\frac{1}{qB}\left(\Omega^2\beta^2 + \frac{q^2B^2}{4}\right) < \Omega p_t \leq - \mid \Omega\sqrt{V_m} \mid, \\ \mid \Omega\sqrt{V_m} \mid \leq \Omega p_t < \infty. \end{array} \tag{64}$$

The limit of the interval (63) when $qB \to 0$ is an empty interval.

Case III: $l^2 < \Omega^2$. The function g presents a minimum g_{min} with respect to Ωp_t,

$$g_{min} = l^2\left[\beta^2 - \frac{q^2B^2}{4(\Omega^2 - l^2)}\right], \quad \text{in } \Omega p_t = -\frac{qB\Omega^2}{2(\Omega^2 - l^2)}, \tag{65}$$

and consequently a condition for unbounded motions is

$$\beta^2 \leq \frac{q^2B^2}{4(\Omega^2 - l^2)}. \tag{66}$$

In this case the energies are comprised in the interval

$$-\frac{\Omega^2}{2(\Omega^2 - l^2)}(qB + \sqrt{\Delta}) \leq \Omega p_t \leq -\frac{\Omega^2}{2(\Omega^2 - l^2)}(qB - \sqrt{\Delta}) \quad \text{(unbounded)}, \tag{67}$$

where Δ is the same as in Eq. (60) and again the equal sign holds only if $\alpha > -1/2$. Ωp_t and qB have opposite signs as expected, on the contrary of bounded motions for which we find

$$\begin{array}{c} -\infty < \Omega p_t < -max\left\{\mid \Omega\sqrt{V_m} \mid, \frac{\Omega^2}{2(\Omega^2 - l^2)}\left(qB + \sqrt{\Delta}\right)\right\} \\ -\frac{\Omega^2}{2(\Omega^2 - l^2)}\left(qB - \sqrt{\Delta}\right) < \Omega p_t \leq - \mid \Omega\sqrt{V_m} \mid \\ \mid \Omega\sqrt{V_m} \mid \leq \Omega p_t < \infty. \end{array} \tag{68}$$

Observe that Ωp_t in (65) and (67) satisfies the condition (58). On the other hand, if

$$\beta^2 > \frac{q^2B^2}{4(\Omega^2 - l^2)} \text{ and } p_t^2 \geq V_m, \tag{69}$$

the only possibility is the occurrence of bounded motions. The results for $\Omega^2 = l^2$ may be reobtained by taking the limits of (66-69) when $\Omega^2 \to l^2$. The limit of the first interval in (68) gives a empty set; the same is true for the limit of (69) since it would require infinite values for β^2 and p_t^2.

Thus, through a qualitative analysis of the motion, we have seen that it is possible to have unbounded motions in the r coordinate for hyperbolic metrics $\Omega^2 \geq l^2 > 0$, that is, for noncausal hyperbolic spacetimes. When $q = 0$ or/and $A_\mu = 0$ that is not possible [14], but in our case we have a Lorentz force acting upon the particles and that could favour or inhibit this kind of motion, depending on the sign of q. If $A_0 = 0$ and A_2 depends only on r, as we supposed in this section, then p_t, p_φ and p_z are constants of motion and we may readily integrate the Hamilton equations. However, when the potential A_μ depends on z, as in Eq. (26), p_z is not constant, but the Hamilton-Jacobi equation corresponding to Eq. (39) is separable and a constant of separation take the place of p_z. Solutions to the equations of motion would lead to elliptical integrals but a qualitative analysis of the motion can be carried out once more and preliminary computations suggested us that the Lorentz force can give rise to motions bounded (periodic) in the z coordinate, in addition to motions unbounded in r when $\Omega^2 \geq l^2 > 0$.

5 Final Comments

In Section 2 we have found that, in the absence of electromagnetic fields, circular metrics and hyperbolic ones with $l^2 > \Omega^2$, in Hehl's theory, are just mathematical possibilities since physically they require a very high spin density in comparison with the matter density. Furthermore, the spacetime homogeneity must be imposed *a priori* and the limit $l \to 0$ gives a flat metric endowed with torsion instead of the Som-Raychaudhuri metric. As expected, the presence of the electromagnetic field eliminates these problems, except for the fact that we again would have high spin density as a condition for hyperbolic metrics with $l^2 > \Omega^2$; by continuity this must also be true for $l^2 \leq \Omega^2$ when Ω^2 tends to l^2. However, if $8k\rho_{em} \geq \Omega_0^2$, that class of metrics is ruled out. The presence of an electromagnetic field allows as well a circular diagonal metric which, in the limit of vanishing torsion, survives as an electrovac solution to the Einstein equations with cosmological constant.

In Section 3, we have shown that the electromagnetic field admits a dual interpretation: it can be envisaged as a free electromagnetic field or as resulting from a charged rotating fluid. An exception is the diagonal metric for which only the first viewpoint is admitted. In both cases we have found the expressions to the electromagnetic fields aiming future applications. In effect, the dynamics of test particles and fields will be different in both situations due to the different dependence of A_μ on the spacetime coordinates.

In section 4, we have dealt with trajectories of charged test particles. Our analysis was restricted to hyperbolic metrics and to electromagnetic field not depending on the z coordinate. We found that motions unbounded in the r coordinate are possible when

$\Omega^2 \geq l^2 > 0$. Therefore, for this class of metrics, charged particles can reach regions inaccessible to neutral particles as light rays, for example. That is not surprising, since it is known that null and timelike geodesics can never go beyond a critical radius in a noncausal hyperbolic spacetime, but timelike trajectories of accelerated particles can do [15]. Thus what we have done was to show that the interaction of a charged particle with a cosmological electromagnetic field is a mechanism responsible for that acceleration. We have also mentioned that, for the case in which the electromagnetic potential depends on the z coordinate, a qualitative analysis of the motion, based on the Hamilton-Jacobi formalism, suggest that motions periodic in the z coordinate can appear, in addition to motions unbounded in r.

We note that in Hehl's theory the spin is directly related with the torsion, vanishing in the Riemannian limit. Thus the solutions of Section 2 may not be confused with the solutions found by Vaidya *et al.* [16] to Einstein's equations in the presence of a spinning fluid and a sourceless electromagnetic field. Nevertheless, the results of Sections 3 and 4 stand even for this case.

Now, some comments on the behaviour of test scalar fields. It is known that in Gödel-type spacetimes there is the following analogy between geodesic motions and the dynamics of neutral scalar test fields minimally coupled to gravity [17]: to geodesics bounded in r correspond discrete energy levels for the scalar field and to geodesics unbounded in r corresponds a continuous energy spectrum. Now, extrapolating this correspondence to charged test particles and scalar fields, we have beforehand an idea about the properties of the solutions to Klein-Gordon equation: we must for example expect to find a continuous energy spectrum in the case of hyperbolic metrics with $\Omega^2 \geq l^2 > 0$. This analogy also suggests asking what would be the quantum analogue to motions peridic in z, if any, when the vector potential A_μ depends on z. For this case, we may show that, parallelly to the separability of the Hamilton-Jacobi equation, the Klein-Gordon equation is also separable and its z dependence is governed by a generalized spheroidal wave equation containing a separation constant. Spheroidal wave equations admit formal analytical solutions [18, 19, 20] but we have also to solve a *characteristic* equation which in principle can affords discrete values to that separation constant. Thus the quantum analogue to motions periodic in z would be the quantization of a constant of separation; inversely, this fact lends further support to think that it is possible to have motions bounded in the z coordinate. Concening the Dirac equation we just remark that, if A_μ depends on z, a constant of motion found by Soares and Tiomno [17, 21], used to decouple the spinor components, is no longer valid. These are some problems to be examined elsewhere.

Acknowledgements

We are grateful to Dr. Maurício Ortiz Calvão for useful discussions and suggestions and to Dr. Nelson Pinto Neto for a critical reading of an early draft of this article.

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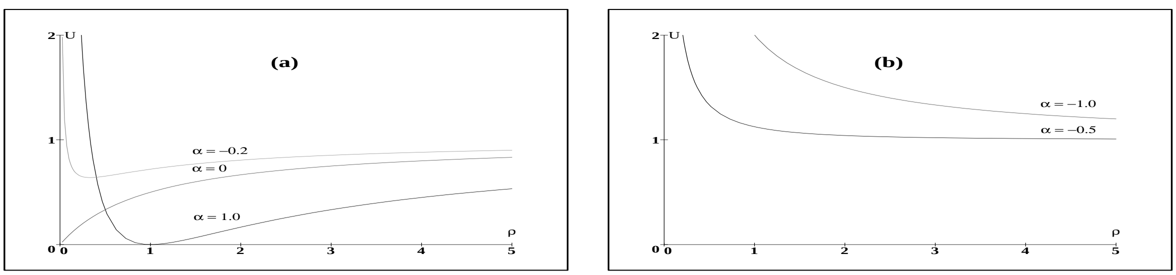

Figure 1: (a) For $\alpha \geq -1/2$ there is a minimum value for $U(\rho)$ and both kinds of motions may be possible; (b) for $\alpha \leq -1/2$ there is no minimum value for $U(\rho)$ and only unbounded motions may occur.