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Abstract

We consider the large q limit of systems made of deformed Heisenberg operators. When the deformation parameter is infinite the Fock space and the statistical properties have a fermionic behaviour. We also investigate the ideal q-gas and find the virial expansion of its equation of state.

Key-words: Quantum algebras; Statistical mechanics.

1 Introduction

There has been a great interest in Quantum Groups [1–4] in the last years, both from physicists and mathematicians. This mathematical structure, also called Quasitriangular Hopf algebras, has emerged as an appealing non-trivial generalization of Lie algebras and groups which are recovered when the deformation parameter (or a set of parameters) goes to one.

Quantum Groups have left their trace in several areas of physics [5–8] and deformed Heisenberg algebras [9] have been attracting increasing interest mainly due to the role played by Heisenberg algebras in a wide range of problems. Recently, the connection of q-oscillators with quantum algebras was investigated [10] thus permitting the discussion of the thermal properties of q-oscillators systems [11–15] and the analysis of possible applications of Quantum Groups to physical phenomena [10–12].

Due to the nature of deformed Hamiltonians, made of bosonic q-oscillators, it is quite difficult to obtain exact expressions when studying their statistical properties. Most of the works done in this area have considered approximations around q (the deformation parameter) equal to one. In this paper we are going to analyse the large q limit of deformed systems. In section 2 we consider the canonical ensemble for the bosonic q-oscillators and we find that for infinite deformation the statistical properties are those of fermions. In addition, it is shown that for large q the system behaves like a deformation of fermions. Section 3 is devoted to the study of a deformed ideal system with large q, and we find the virial expansion for its equation of state. Final remarks are given in section 4.

2 Bosonic q-Oscillators in the large $\hat{q}$ limit

One calls bosonic q-oscillators the associative algebra generated by the elements α, α^+ and N satisfying the relations [10–16]

$$\begin{aligned} [N,\alpha^+] &= \alpha^+ \quad , \; [N,\alpha] = -\alpha \\ [\alpha,\alpha^+]_\alpha &= f_\alpha(N). \end{aligned} \tag{2.1}$$

We are going to consider here the following forms of the above algebra (2.1):

$$[a,a^+]_a \equiv aa^+ - qa^+a = q^{-N} \tag{2.2.a}$$

$$[A,A^+]_A \equiv AA^+ - q^2A^+A = 1. \tag{2.2.b}$$

The above two algebras can be related to each other via

$$A = q^{N/2}a \quad , \quad A^+ = a^+q^{N/2}$$

with q a real parameter.

It is possible to construct representations of the relations (2.2) in the Fock space $\mathcal{F}$ spanned by the normalized eigenstates $|n>$ of the number operator N as

$$\begin{aligned} \alpha|0> &= 0 \quad , \quad N|n>= n|n> \qquad n = 0,1,2,... \\ |n> &= \frac{1}{\sqrt{[n]_\alpha!}}\,(\alpha^+)^n|0> \end{aligned} \tag{2.3}$$

where $[n]_\alpha! \equiv [n]_\alpha \cdots [1]_\alpha, [n]_a = (q^n - q^{-n})/(q - q^{-1})$ and $[n]_A = (q^{2n} - 1)/(q^2 - 1)$.

In the Fock space $\mathcal{F}$ it is possible to express the deformed oscillators in terms of the standard bosonic ones b, b^+ as [16–17]

$$\alpha = \left(\frac{[N+1]_\alpha}{N+1}\right)^{1/2} b \ , \ \alpha^+ = b^+ \left(\frac{[N+1]_\alpha}{N+1}\right)^{1/2} ; \tag{2.4}$$

it can easily be shown in $\mathcal{F}$ that

$$\alpha\alpha^+ = [N+1]_\alpha \quad , \quad \alpha^+\alpha = [N]_\alpha \ , \tag{2.5}$$

and as expected the standard bosonic algebra is obtained in the $q \to 1$ limit.

We are now going to investigate highly deformed q-bosons ($q \to \infty$ limit). In this limit for $n \geq 2$, $[n]_\alpha \to \infty$ and as a result when $q = \infty$ Fock space (2.3) is reduced to a fermionic one since the eigenstates $|n>$ vanish for $n \geq 2$. Consequently, the statistical properties of q-deformed oscillators (2.2) become those of fermions.

In order to exhibit the statistical properties close to this fermionic limit let us consider the canonical partition function for the Hamiltonian

$$H = \omega A^+ A = \omega [N]_A, \tag{2.6}$$

which is given by

$$Z = 1 + e^{-\beta\omega} + e^{-\beta\omega(1+q^2)} + \cdots + e^{-\beta\omega[1+q^2+\cdots q^{2(n-1)}]} + \cdots . \tag{2.7}$$

where $\beta = (k_B T)^{-1}$, with k_B the Boltzmann constant. The above expression has often been taken as the starting point in the analysis of q-bosons at finite temperature [11–15], which is indeed the case when q is close to one. On the other hand for infinite deformations, (2.7) is clearly the partition function of fermions. Therefore, in the canonical ensemble, when $q >> 1$ expression (2.7) can be understood as a deformation around fermions.

The average of N to first order in the large q limit is given by

$$< N > \cong \frac{1 + 2e^{-\beta\omega q^2}}{1 + e^{\beta\omega} + e^{-\beta\omega q^2}} . \tag{2.8}$$

As expected in the $q = \infty$ limit the Fermi-Dirac distribution is recovered.

3 Highly Deformed Ideal q-Gas

Following the standard lore [14] we define the Hamiltonian of an ideal deformed system as:

$$H = \sum_i \omega_i A_i^+ A_i = \sum_i \omega_i [N_i]_A \ , \tag{3.1}$$

where A_i, A_i^+ and N_i are interpreted respectively as annihilation, creation and occupation number operators of particles in level i, with energy ω_i. These operators satisfy the algebra (2.2b) and commute for different levels.

The grand canonical partition function is given by:

$$Z = Tr \ exp[-\beta(H - \mu N)] = exp(-\beta\Omega) \tag{3.2}$$

where N is the total number operator

$$N = \sum_i N_i \,, \tag{3.3}$$

μ is the chemical potential and Ω is the grand canonical potential. For the above system Z factorizes and the grand canonical potential is given by a sum over single level partition functions

$$\Omega = -\frac{1}{\beta}\sum_i log Z_i^0(\omega_i,\beta,\mu) \,, \tag{3.4}$$

where

$$Z_i^0(\omega_i,\beta,\mu) = \sum_{n=0}^{\infty} e^{-\beta(\omega_i[n]_A - \mu n)} \,. \tag{3.5}$$

The energy of the non-relativistic q-boson is

$$\omega_i = \vec{p}^{\,2}/2m \,, \tag{3.6}$$

and the usual approach [18] is to enclose the system in a large volume V, which allows the sum over levels to be replaced by an integral over the p space:

$$\sum_i \to \frac{V}{(2\pi h)^3}\int d^3p \,. \tag{3.7}$$

If we keep only the first correction in the large q limit, the grand canonical potential is found to be

$$-\beta\Omega \cong \frac{1}{2\pi^2} V h^{-3} \int_0^\infty dp\, p^2 \,\ln[1 + z\, e^{-\beta p^2/2m} + z^2\, e^{-\beta(q^2+1)p^2/2m}] \,, \tag{3.8}$$

where z is the fugacity, $z = e^{\beta\mu}$; later we shall discuss the region of validity of the above approximation. After integrating by parts and defining the new variable $\eta = \beta p^2/2m$ (3.8) reduces to

$$-\beta\Omega \cong \frac{V}{6\pi^{7/2}}\Lambda^{-3}\int_0^\infty d\eta\, \eta^{3/2}\, \frac{z\, e^{-\eta} + z^2(q^2+1)e^{-(q^2+1)\eta}}{1 + z\, e^{-\eta} + z^2 e^{-(q^2+1)\eta}} \,, \tag{3.9}$$

where $\Lambda = (h^2\beta/2\pi m)^{1/2}$, also called thermal wavelength, is the relevant expansion parameter in the thermodynamic functions.

Finally, assuming that the fugacity z is small compared to one, expanding Ω and keeping terms up to third order in z the pressure $P = -\Omega/V$ is given by

$$P = \frac{\beta^{-1}\Lambda^{-3}}{(2\pi)^3}\, z\left\{1 + z[-2^{-5/2} + (q^2+1)^{-3/2}] + z^2[3^{-5/2} - (q^2+2)^{-3/2}] + 0(z^3)\right\}. \tag{3.10}$$

The q-boson density $n = \frac{\partial P}{\partial \mu}|_{T,V}$ is easily found to be

$$n = \frac{\Lambda^{-3}}{(2\pi)^3}\, z\left\{1 + 2z[-2^{-5/2} + (q^2+1)^{-3/2}] + 3z^2[3^{-5/2} - (q^2+2)^{-3/2}] + 0(z^3)\right\} \,. \tag{3.11}$$

Inverting the power series above we obtain

$$z = n\Lambda^3 - 2Q_1(n\Lambda^3)^2 + (8Q_1^2 - 3Q_2)(n\Lambda^3)^3 + \cdots, \tag{3.12}$$

where

$$Q_1 = -2^{-5/2} + (q^2+1)^{-3/2} \quad , \quad Q_2 = 3^{-5/2} - (q^2+2)^{-3/2} . \tag{3.13}$$

Substituting (3.12-13) in (3.10) and expanding in powers of n, we obtain the virial expansion of the equation of state

$$P = \frac{n}{(2\pi)^3\beta}\left\{1 + \left[\frac{1}{2^{5/2}} - \frac{1}{(q^2+1)^{3/2}}\right] n\Lambda^3 + \right.$$
$$\left. + \left[\frac{1}{8} - \frac{2}{3^{5/2}} + \frac{4}{(q^2+1)^3} - \frac{4}{(2q^2+2)^{3/2}} + \frac{2}{(q^2+2)^{3/2}}\right] n^2\Lambda^6 + \cdots\right\}. \tag{3.14}$$

Looking at eq. (3.14) one immediately sees that in the $q=\infty$ limit our q-gas behaves exactly like a non-relativistic Fermi-gas [18]. For finite large values of q the pressure is reduced with respect to the Fermi-gas. We would like to stress that analogously to non-deformed Bose or Fermi gases, the approximations done here are valid for large V,$z<<1$ and $n\Lambda^3<<1$, implying that for a given density n, we have a high-temperature approximation or, for a given temperature, a low-density approximation.

A similar procedure is employed in the case of ultrarelativistic q-bosons whose energy is given by $\omega_i = c\vec{p}$. The virial expansion of the equation of state in this regime is

$$P = \frac{n}{(2\pi)^3\beta}\left\{1 + \left[\frac{1}{2^4} - \frac{1}{(q^2+1)^3}\right] \Delta^3 n + \right.$$
$$\left. \left[\frac{1}{2^6} - \frac{2}{3^4} - \frac{1}{2(q^2+1)^3} - \frac{2}{(q^2+2)^3} + \frac{4}{(q^2+1)^6}\right] \Delta^6 n^2 + \cdots\right\}, \tag{3.15}$$

where now the relevant parameter of the expansion is $\Delta^3 n$ with $\Delta = ch\beta/2\pi^{1/3}$, the so called optical wavelength. The comments in the last paragraph about the approximations performed remain valid for (3.15). In the $q=\infty$ limit (3.15) is the ultra-relativistic equation of state and also here the effect of finite q is to reduce the pressure as compared with the infinitely deformed gas.

4 Final Comments

In this paper we have analysed some statistical properties of q-oscillators in the large q limit. It is quite remarkable that infinitely deformed q-bosons acquire a fermionic behaviour. This point deserves further investigation at the level of quantum algebras.

Due to the relevance of Fermi-gases in Condensed Matter and Nuclear Physics we expect that our results can find an application in these fields.

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