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CRITICALITY OF THE $D = 2$ BOND-DILUTE ANISOTROPIC
HEISENBERG FERROMAGNET

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ABSTRACT

We discuss the critical frontier and critical exponents associated with the quenched bond-dilute quantum anisotropic spin $\frac{1}{2}$ Heisenberg ferromagnet in square lattice. To perform the calculations, we extend (and analyse with some detail) an approximate real-space renormalisation-group framework recently developed by some of us for the pure model. Whenever comparison with available exact results is possible, the agreement is either perfect or quite satisfactory. Some effort has been dedicated to extract the main asymptotic behaviours of the critical frontier. Also several interesting quantum effects appearing in the composition laws of (Heisenberg) bond arrays are exhibited.

Key-words: Heisenberg ferromagnet; Bond-dilution; Quantum effects; Criticality.

I INTRODUCTION

During recent years several works have been dedicated to the study, within real space renormalisation group (RG) frameworks, of phase transitions in quantum systems (see Pfeuty et al 1982 for a review). The noncommutative nature of the problem makes it non trivial. One of the main models that have been focused is the Heisenberg ferromagnet (Suzuki and Takano 1979, Stinchcombe 1979 (a,b), 1981, Takano and Suzuki 1981, Castellani et al 1982, Tsallis et al 1984, Plascak 1984). In particular, some of us (Caride et al 1983 (a,b)) have recently discussed, within a RG scheme, the critical line and thermal critical exponents associated with the spin - $\frac{1}{2}$ anisotropic Heisenberg model on square lattice. The procedure is a cluster based one, and uses self-dual (Wheatstone bridge like) two-rooted graphs to simulate the square lattice; in other words, the square lattice is approached through convenient self-dual hierarchical lattices. This formalism provides results which, whenever comparison is possible, precisely (or nearly so) reproduce well established ones (such as the critical points and thermal critical exponents corresponding to the square lattice pure Ising and isotropic Heisenberg models).

In the present work we generalize the treatment of Caride et al 1983 in order to cover the quenched bond-diluted case of the same model. The critical surface in the T (temperature)- Δ (anisotropy)- p (concentration) space is calculated, and the universality classes are established. Some effort is dedicated to the discussion of the main asymptotic behaviours.

In Section II we present the pure model and introduce the

formalism; in Section III we discuss several interesting quantum effects; in Section IV we present the bond-diluted model and its RG treatment; in Section V we present the main results; finally we conclude in Section VI.

II PURE MODEL AND FORMALISM

II.1 Model

We consider a system (anisotropic Heisenberg model) whose dimensionless Hamiltonian is given by

$$\mathcal{H} = \sum_{\langle i,j \rangle} K_{ij} \left[(1 - \Delta_{ij}) (\sigma_i^x \sigma_j^x + \sigma_i^y \sigma_j^y) + \sigma_i^z \sigma_j^z \right] \quad (1)$$

where $K_{ij} \equiv J_{ij}/k_B T$ (J_{ij} is the exchange coupling constant; the factor $(-1/k_B T)$ has been incorporated in the Hamiltonian), Δ_{ij} is the anisotropy in the spin space, the σ 's are the standard Pauli matrix associated with spins $1/2$, and the sum runs over all pairs of nearest-neighbouring sites of a square lattice. The particular cases $\Delta_{ij} = 1$, $\Delta_{ij} = 0$ and $\Delta_{ij} = -\infty$ respectively correspond to the Ising, isotropic Heisenberg and XY models. Because of noncommutativity, the system will exhibit quantum effects whenever $\Delta_{ij} \neq 1$. The standard pure model corresponds to $K_{ij} = K$ and $\Delta_{ij} = \Delta$, $\forall (i,j)$. However before focusing this model and its diluted version, we shall consider simpler systems, namely two-terminal arrays of spins, i.e. we shall consider two-rooted graphs, each site of which

carries a set of Pauli matrix, and each bond of which denotes an anisotropic Heisenberg interaction.

II.2 Parallel array

We consider first a *parallel array of two bonds* characterized respectively by (K_1, Δ_1) and (K_2, Δ_2) (see Fig. 1a). It is "equivalent" to a *single bond* characterized by (K_p, Δ_p) given by

$$K_p = K_1 + K_2 \quad (2)$$

$$K_p(1 - \Delta_p) = K_1(1 - \Delta_1) + K_2(1 - \Delta_2) \quad (3)$$

hence

$$K_p \Delta_p = K_1 \Delta_1 + K_2 \Delta_2 \quad (3')$$

By equivalent we mean that the Hamiltonian (and consequently the associated partition function) remains invariant. If we are dealing with n parallel bonds, then Eqs. (2) and (3') are obviously generalized into

$$K_p = \sum_{i=1}^n K_i \quad (4)$$

$$K_p \Delta_p = \sum_{i=1}^n K_i \Delta_i \quad (5)$$

If this *parallel array* is a subarray of a more complex array, it can *always* be replaced by a *single bond* (with (K_p, Δ_p) given by Eqs. (4) and (5)) without any modification of the whole Hamiltonian (or its corresponding partition function). All this

simplicity will disappear for series arrays because of noncommutativity effects.

II.3 Series array

Let us consider a *series array of two bonds* characterized by (K_1, Δ_1) and (K_2, Δ_2) (see Fig. 1.b). The corresponding Hamiltonian is given by

$$\begin{aligned} \mathcal{H}_{123} = & K_1 \left[(1 - \Delta_1) (\sigma_1^x \sigma_3^x + \sigma_1^y \sigma_3^y) + \sigma_1^z \sigma_3^z \right] \\ & + K_2 \left[(1 - \Delta_2) (\sigma_3^x \sigma_2^x + \sigma_3^y \sigma_2^y) + \sigma_3^z \sigma_2^z \right] \end{aligned} \quad (6)$$

The entire equilibrium thermal statistics of this system as determined by its two terminals (sites 1 and 2 in Fig. 1b) is identical to that of a *single bond* (characterized by (K_s, Δ_s)) if we preserve the partition function, i.e.

$$e^{\mathcal{H}'_{12}} = \text{Tr}_3 e^{\mathcal{H}_{123}} \quad (7)$$

where

$$\mathcal{H}'_{12} = K_s \left[(1 - \Delta_s) (\sigma_1^x \sigma_2^x + \sigma_1^y \sigma_2^y) + \sigma_1^z \sigma_2^z \right] + K'_0 \quad (8)$$

Tr denotes the tracing operation over the states of the spin 3 at site 3 (see Fig. 1b); K'_0 is an additive constant that has to be included in order to Eq. (7) be possible. K_s , Δ_s and K'_0 are functions of K_1 , Δ_1 , K_2 and Δ_2 to be found. Let us now determine these functions.

The expansion of $\exp(\mathcal{H}'_{12})$ provides

$$\exp(\mathcal{H}'_{12}) = a' + b'_{12}(\sigma_1^x \sigma_2^x + \sigma_1^y \sigma_2^y) + c'_{12} \sigma_1^z \sigma_2^z \quad (9)$$

where a' , b'_{12} and c'_{12} depend on K_s , Δ_s and K'_0 . Analogously the expansion of $\exp(\mathcal{H}'_{123})$ provides

$$\begin{aligned} \exp(\mathcal{H}'_{123}) = & a + b_{13}(\sigma_1^x \sigma_3^x + \sigma_1^y \sigma_3^y) + b_{23}(\sigma_2^x \sigma_3^x + \sigma_2^y \sigma_3^y) \\ & + b_{12}(\sigma_1^x \sigma_2^x + \sigma_1^y \sigma_2^y) + c_{13} \sigma_1^z \sigma_3^z + c_{23} \sigma_2^z \sigma_3^z + c_{12} \sigma_1^z \sigma_2^z \quad (10) \end{aligned}$$

where a , $\{b_{ij}\}$ and $\{c_{ij}\}$ depend on K_1, Δ_1, K_2 and Δ_2 . Eq. (10) implies

$$\text{Tr}_3 \exp(\mathcal{H}'_{123}) = 2a + 2b_{12}(\sigma_1^x \sigma_2^x + \sigma_1^y \sigma_2^y) + 2c_{12} \sigma_1^z \sigma_2^z \quad (11)$$

From Eqs. (7), (9) and (11) it follows immediately that

$$a' = 2a \quad (12.a)$$

$$b'_{12} = 2b_{12} \quad (12.b)$$

and

$$c'_{12} = 2c_{12} \quad (12.c)$$

which implicitly determine K_s , Δ_s and K'_0 as functions of K_1 , Δ_1 , K_2 and Δ_2 . Let us now proceed to the last steps of the calculation.

We first take care of \mathcal{H}'_{12} , and express it in the basis which simultaneously diagonalizes σ_1^z and σ_2^z , and which we note

$|++\rangle$, $|+-\rangle$, $| -+\rangle$ and $|--\rangle$. In this basis U'_{12} becomes

$$\begin{pmatrix} K_s + K'_0 & 0 & 0 & 0 \\ 0 & -K_s + K'_0 & 2W_s & 0 \\ 0 & 2W_s & -K_s + K'_0 & 0 \\ 0 & 0 & 0 & K_s + K'_0 \end{pmatrix} \quad (13)$$

where

$$W_s \equiv K_s (1 - \Delta_s) \quad (14)$$

The corresponding eigenvalues are

$$\epsilon'_1 = K_s + K'_0 \quad (15.a)$$

$$\epsilon'_2 = -K_s + K'_0 + 2W_s \quad (15.b)$$

$$\epsilon'_3 = -K_s + K'_0 - 2W_s \quad (15.c)$$

$$\epsilon'_4 = \epsilon'_1 \quad (15.d)$$

The corresponding normed eigenvectors, arranged in columns, provide the following unitary matrix

$$U'_{12} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad (16)$$

If we note $e^{\mathcal{H}'_{12}D}$ the diagonal form of $e^{\mathcal{H}'_{12}}$, the following relation is satisfied:

$$e^{\mathcal{H}'_{12}} = U'_{12} e^{\mathcal{H}'_{12}D} U'^{\dagger}_{12} \quad (17)$$

hence

$$e^{\mathcal{H}'_{12}} = \begin{pmatrix} e^{\epsilon'_1} & 0 & 0 & 0 \\ 0 & \frac{e^{\epsilon'_2} + e^{\epsilon'_3}}{2} & \frac{e^{\epsilon'_2} - e^{\epsilon'_3}}{2} & 0 \\ 0 & \frac{e^{\epsilon'_2} - e^{\epsilon'_3}}{2} & \frac{e^{\epsilon'_2} + e^{\epsilon'_3}}{2} & 0 \\ 0 & 0 & 0 & e^{\epsilon'_1} \end{pmatrix} \quad (18)$$

On the other hand, the right side of Eq. (9), expressed in the $|++\rangle, |+-\rangle, |-+\rangle, |--\rangle$ basis becomes

$$\begin{pmatrix} a' + c'_{12} & 0 & 0 & 0 \\ 0 & a' - c'_{12} & 2b'_{12} & 0 \\ 0 & 2b'_{12} & a' - c'_{12} & 0 \\ 0 & 0 & 0 & a' + c'_{12} \end{pmatrix} \quad (19)$$

This expression, together with Eqs. (9) and (18), yields

$$e^{4K_s} = \frac{(a' + c'_{12})^2}{(a' - c'_{12})^2 - 4(b'_{12})^2} \quad (20.a)$$

$$e^{4W_s} = \frac{a' - c'_{12} + 2b'_{12}}{a' - c'_{12} - 2b'_{12}} \quad (20.b)$$

$$e^{K'_0} = \frac{a' + c'_{12}}{e^{K_s}} \quad (20.c)$$

where we have used Eqs. (15).

Let us now take care of \mathcal{H}_{123} , and express it in the basis $|+++ \rangle, |++-\rangle, |+-+\rangle, | -++ \rangle, |--+\rangle, | -+- \rangle, |+-- \rangle, |--- \rangle$. We obtain

$$\mathcal{H}_{123} = \begin{pmatrix} A_1 & 0 & 0 & 0 \\ 0 & A_3 & 0 & 0 \\ 0 & 0 & A_3 & 0 \\ 0 & 0 & 0 & A_1 \end{pmatrix} \quad (21)$$

where the 1×1 A_1 and 3×3 A_3 matrices are given by

$$A_1 = K_1 + K_2 \quad (22.a)$$

and

$$A_3 = \begin{pmatrix} -(K_1 + K_2) & 2W_2 & 2W_1 \\ 2W_2 & K_1 - K_2 & 0 \\ 2W_1 & 0 & K_2 - K_1 \end{pmatrix} \quad (22.b)$$

where

$$W_i \equiv K_i (1 - \Delta_i) \quad (i = 1, 2) \quad (23)$$

The eigenvalues of A_3 (ϵ_i , $i = 1, 2, 3$) are the roots of the following cubic equation:

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$$\epsilon^3 + \alpha\epsilon^2 - \left[\beta^2 + 4(W_2^2 + W_1^2) \right] \epsilon - \beta \left[\alpha\beta + 4(W_2^2 - W_1^2) \right] = 0 \quad (24.a)$$

where

$$\alpha = K_1 + K_2 \quad (24.b)$$

and

$$\beta = K_1 - K_2 \quad (24.c)$$

The 8 x 8 unitary matrix diagonalizing Δ_{123} into Δ_{123}^D is given by

$$U_{123} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & U_3 & 0 & 0 \\ 0 & 0 & U_3 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad (25)$$

where U_3 is a 3 x 3 matrix given by

$$U_3(i,j) = \lambda_j^i \quad (i,j = 1,2,3) \quad (26.a)$$

with

$$\begin{aligned} \lambda_j^1 &= \left(\frac{\epsilon_j + \beta}{2W_1} \right) \lambda_j^3 (1 - \delta_{\Delta_1}^1) (1 - \delta_{\Delta_2}^1) + \sqrt{\frac{4W_2^2}{4W_2^2 + (\epsilon_j + \alpha)^2}} (1 - \delta_{\Delta_2}^1) \delta_{\Delta_1}^1 (1 - \delta_j^3) \\ &+ \sqrt{\frac{4W_1^2}{4W_1^2 + (\epsilon_j + \alpha)^2}} (1 - \delta_{\Delta_1}^1) \delta_{\Delta_2}^1 (1 - \delta_j^2) + \delta_{\Delta_1}^1 \delta_{\Delta_2}^1 \delta_j^1 \end{aligned} \quad (26.b)$$

$$\lambda_j^2 = \left[\frac{\epsilon_j^2 + \epsilon_j \beta - 4W_1^2}{\alpha(\beta - \epsilon_j) + 4W_2^2} \right] \left(\frac{W_2}{W_1} \right) \lambda_j^3 (1 - \delta_{\Delta_1}^1) (1 - \delta_{\Delta_2}^1) + \sqrt{\frac{4W_2^2}{4W_2^2 + (\epsilon_j - \beta)^2}} (1 - \delta_{\Delta_2}^1)$$

$$\delta_{\Delta_1}^1 (1 - \delta_j^3) + (1 - \delta_{\Delta_1}^1) \delta_{\Delta_2}^1 \delta_j^2 + \delta_{\Delta_1}^1 \delta_{\Delta_2}^1 \delta_j^2 \quad (26.c)$$

$$(\lambda_j^3)^2 = 1 - (\lambda_j^1)^2 - (\lambda_j^2)^2 \quad (26.d)$$

(the δ 's are Kroenecker delta functions)

By following through the same steps as before (namely Eqs. (17) and (18) for \mathcal{D}'_{12}) we obtain the 8×8 matrix $\exp(\mathcal{D}_{123})$ as an explicit function of K_1 , W_1 , K_2 and W_2 . On the other hand, the right side of Eq. (10), expressed in the 8-dimensional basis mentioned before, becomes

$$\begin{pmatrix} B_1 & 0 & 0 & 0 \\ 0 & B_3 & 0 & 0 \\ 0 & 0 & B_3 & 0 \\ 0 & 0 & 0 & B_1 \end{pmatrix} \quad (27)$$

where

$$B_1 = a + c_{12} + c_{13} + c_{23} \quad (28, a)$$

and

$$B_3 = \begin{pmatrix} a+c_{12}-c_{13}-c_{23} & 2b_{23} & 2b_{13} \\ 2b_{23} & a-c_{12}+c_{13}-c_{23} & 2b_{12} \\ 2b_{13} & 2b_{12} & a-c_{12}-c_{13}+c_{23} \end{pmatrix} \quad (28. b)$$

Expression (27) (with Eqs.(28)), together with Eq. (10) and the explicit expression of $\exp(\mathcal{D}_{123})$ mentioned above, yields

$$a = \frac{1}{4} \left[e^{(K_1+K_2)} + \sum_{i=1}^3 \sum_{j=1}^3 e^{\varepsilon_j} (\lambda_j^i)^2 \right] \quad (29. a)$$

$$b_{12} = \frac{1}{2} \sum_{j=1}^3 e^{\varepsilon_j} (\lambda_j^2 \lambda_j^3) \quad (29. b)$$

and

$$c_{12} = \frac{1}{2} \left[e^{(K_1+K_2)} + \sum_{j=1}^3 e^{\varepsilon_j} (\lambda_j^1)^2 \right] - a \quad (29. c)$$

Finally Eqs. (29) substituted into Eqs. (12), and the results substituted into Eqs. (20) provide the expressions we were looking for, i.e. K_s , W_s and K_0' as explicit functions of K_1 , W_1 , K_2 and W_2 .

For the pure Ising particular case ($\Delta_1 = \Delta_2 = 1$) we reobtain the well known results (see, for instance, Tsallis and Levy 1981 and references therein)

$$\tanh(K_s) = \tanh(K_1) \tanh(K_2) \quad (30.a)$$

and

$$\Delta_s = 1 \quad (30.b)$$

as well as

$$K_0' = \frac{1}{2} \ln \left\{ 2 \left[\cosh(2K_1) + \cosh(2K_2) \right] \right\} \quad (30.c)$$

For the isotropic Heisenberg particular case ($\Delta_1 = \Delta_2 = 0$) we obtain $\Delta_s = 0$, and recover results of Tsallis et al 1984, namely, Eq. (4) therein for K_s , and Eq. (A 26a) (with $K_{13} = 0$) therein for K_0' . Finally, for the particular case $K_1 = K_2 \equiv K$ and $W_1 = W_2 \equiv W$, we obtain

$$e^{4W_s} = \frac{[(\lambda + K)e^\lambda + (\lambda - K)e^{-\lambda}]e^{-K}}{2\lambda} \quad (31.a)$$

$$e^{2K_s} = \frac{e^{2K} + [(\lambda - K)e^\lambda + (\lambda + K)e^{-\lambda}]e^{-K}/2\lambda}{2e^{2W_s}} \quad (31.b)$$

and

$$K_0' = \ln 2 + K_s + 2W_s \quad (31.c)$$

where

$$\lambda \equiv \sqrt{K^2 + 8W^2} \quad (32)$$

As anticipated in Section II.2, the series case is, because of noncommutativity, considerably more complex than the parallel one; in particular the search of additive properties (such as those Eqs. (4) and (5)) or factorizing ones (such as that of Eq. (30.a)) becomes practically untractable. Furthermore, a *series array* (e.g., two bonds characterised by (K_1, Δ_1) and (K_2, Δ_2)) which is a subarray of a more complex one, *cannot in general be replaced by a single bond* (associated, in our example, with $K_s = K_s(K_1, \Delta_1; K_2, \Delta_2)$ and $\Delta_s = \Delta_s(K_1, \Delta_1; K_2, \Delta_2)$). This fact (which will be illustrated in Section III) generates interesting quantum effects, but makes hopeless (at least for the present type of quantum systems) the search of operationally simple procedures similar to the Break-collapse Method (initially devised by Tsallis and Levy 1981 for the q-state Potts model, and afterwards extended for other classical models; see Tsallis et al 1983, and Tsallis and Redner 1983) which, through elementary *local* topological operations, enables the calculation of *arbitrary* arrays.

The following properties are worthy to be noted:

$$i) \quad K_s(K_1, \Delta_1; K_2, \Delta_2) = K_s(K_2, \Delta_2; K_1, \Delta_1) \quad (33.a)$$

and

$$\Delta_s(K_1, \Delta_1; K_2, \Delta_2) = \Delta_s(K_2, \Delta_2; K_1, \Delta_1) \quad (33.b)$$

$$\text{ii) } K_s(0, \Delta_1; K_2, \Delta_2) = 0 \quad , \quad \forall (\Delta_1, K_2, \Delta_2) \quad ; \quad (34)$$

$$\text{iii) in the high-temperature limit } (|K_1|, |K_2| \ll 1), \\ K_s(K_1, \Delta_1; K_2, \Delta_2) \sim K_1 K_2 \quad \forall (\Delta_1, \Delta_2); \quad (35)$$

$$\text{iv) } K_s(\pm \infty, \Delta_1; K_2, \Delta_2) = \pm \eta(\Delta_1, \Delta_2; K_2) K_2 \quad (36)$$

where $\eta(\Delta_1, \Delta_2, K_2)$ is a finite quantity which satisfies $\eta(1, 1, K_2) \equiv 1, \forall(K_2)$, and which, for $\Delta_1 = \Delta_2 = 0$, η monotonously increases from $\frac{2}{3}$ to $\frac{3}{4}$ when $|K_2|$ increases from zero to infinity.

$$\text{v) in the low-temperature limit } (|K_1|, |K_2| \gg 1 \text{ and } |K_1| \leq |K_2|);$$

$$|K_s| \sim |K_1| \quad \text{if } \Delta_1 = \Delta_2 = 1 \quad (37.a)$$

$$|K_s| \leq |K_1| \quad \text{if } 0 < \Delta_1, \Delta_2 < 1 \quad (37.b)$$

$$|K_s| \sim \frac{1}{2} \left[K_1 + K_2 - \sqrt{K_1^2 + K_2^2 - K_1 K_2} \right] \quad \text{if } \Delta_1 = \Delta_2 = 0 \quad (37.c)$$

II.4 Arbitrary two-terminal array

We consider here an arbitrary connected two-terminal array (see examples in Fig. 2) containing N sites (2 terminals and $(N - 2)$ internal nodes), and $N(N - 1)/2$ bonds joining them (all sets of parallel bonds are assumed to be replaced by the corresponding equivalent bonds). The sites are noted i, j, k, ℓ , etc; the bonds are noted according to the sites they

join (ij, ik, jk, etc); the two terminals are noted 1 and 2. The ij-bond (and analogously all the others) is characterized by (K_{ij}, Δ_{ij}) ; an absent bond is equivalent to the corresponding K equal to zero. The Hamiltonian of this system is given by

$$\mathcal{H}_{12\dots N} = \frac{N(N-1)}{2} \sum_{\text{bonds}} K_{ij} \left[(1 - \Delta_{ij})(\sigma_i^x \sigma_j^x + \sigma_i^y \sigma_j^y) + \sigma_i^z \sigma_j^z \right] \quad (38)$$

We intend to replace this array by a single bond whose Hamiltonian \mathcal{H}'_{12} is that of Eq. (8) (in this Section we shall use the notation (K'_{12}, Δ'_{12}) rather than (K_s, Δ_s)). We impose the preservation of the partition function, i.e.

$$e^{\mathcal{H}'_{12}} = \text{Tr}_{3,4,\dots,N} e^{\mathcal{H}_{12\dots N}} \quad (39)$$

Our purpose in the present Section is to outline the general procedure for finding K'_{12} , Δ'_{12} and K'_0 as functions (explicit whenever possible) of $\{K_{ij}\}$ and $\{\Delta_{ij}\}$ (the convenient notation $W_{ij} \equiv K_{ij}(1 - \Delta_{ij})$ will sometimes be used).

Let us introduce the total angular momentum $\vec{S} \equiv \left(\sum_{i=1}^N \vec{\sigma}_i \right) / 2$, and note S^z its z-component. The Hamiltonian $\mathcal{H}_{12\dots N}$ always commutes with S^z , but not necessarily with S^2 (excepting for $\Delta_{ij} = 0, \forall(ij)$; for this case, see Tsallis et al 1984, where a misprint should be noted: a few lines below Eq. (A4), it should be "It is convenient to work ...", instead of "It is difficult to work..."). Consequently the $2^N \times 2^N$ matrix $\mathcal{H}_{12\dots N}$ is conveniently represented in the basis $|\pm\rangle_1 |\pm\rangle_2 |\pm\rangle_3 \dots |\pm\rangle_N \equiv |\pm \pm \pm \dots \pm\rangle$, where it exhibits the following block form:

$$\mathcal{H}_{12\dots N} = \begin{pmatrix} A_1 & 0 & \dots 0 & \dots 0 & 0 \\ 0 & A_N & \dots 0 & \dots 0 & 0 \\ \vdots & \vdots & \dots & \vdots & \vdots \\ 0 & 0 & \dots A_{\binom{N}{r}} & \dots 0 & 0 \\ \vdots & \vdots & \dots & \vdots & \vdots \\ 0 & 0 & \dots 0 & \dots A_N & 0 \\ 0 & 0 & \dots 0 & \dots 0 & A_1 \end{pmatrix} \quad (40)$$

where $A_{\binom{N}{r}}$ is a $\binom{N}{r} \times \binom{N}{r}$ matrix ($r = 0, 1, \dots, N$). A_1 corresponds to $|++\dots+\rangle$ (and also to $|- - \dots - \rangle$); A_N corresponds to the set of N vectors $|++\dots-\rangle, \dots, |+-\dots+\rangle$ and $|-+\dots+\rangle$ (and also to the corresponding N sign-inverted vectors); analogously for $A_{\binom{N(N-1)}{2}}$ etc. The eigenvectors of $A_{\binom{N}{r}}$ will be noted $X_1^{(\binom{N}{r})}, X_2^{(\binom{N}{r})}, \dots, X_{\binom{N}{r}}^{(\binom{N}{r})}$ and the corresponding unitary matrix (whose columns are the normed eigenvectors) will be noted $U_{\binom{N}{r}}$. If we note $\mathcal{H}_{12\dots N}^D$ the diagonal form of $\mathcal{H}_{12\dots N}$, the following relation is satisfied:

$$e^{\mathcal{H}_{12\dots N}} = U_{12\dots N} e^{\mathcal{H}_{12\dots N}^D} U_{12\dots N}^+ \quad (41)$$

where the $2^N \times 2^N$ unitary matrix $U_{12\dots N}$ has the same form of Eq. (40), $U_{\binom{N}{r}}$ playing the role of $A_{\binom{N}{r}}$. By using (41) we can express $\exp(\mathcal{H}_{12\dots N})$ as function of $\{K_{ij}, \Delta_{ij}\}$.

On the other hand $\exp(\mathcal{H}_{12\dots N})$ can be expanded as follows:

$$\begin{aligned}
e^{i\phi_{12\dots N}} = & a + \sum_{\frac{N(N-1)}{2} \text{ bonds}} \left[b_{ij} (\sigma_i^x \sigma_j^x + \sigma_i^y \sigma_j^y) + c_{ij} \sigma_i^z \sigma_j^z \right] \\
& + \sum_{(ij) \neq (kl)} \left[d_{ij,kl} (\sigma_i^x \sigma_j^x + \sigma_i^y \sigma_j^y) \sigma_k^z \sigma_l^z \right. \\
& + e_{ij,kl} (\sigma_i^x \sigma_j^x + \sigma_i^y \sigma_j^y) (\sigma_k^x \sigma_l^x + \sigma_k^y \sigma_l^y) + f_{ij,kl} \sigma_i^z \sigma_j^z \sigma_k^z \sigma_l^z \left. \right] \\
& + \sum_{(ij) \neq (kl) \neq (mn)} \left[g_{ij,kl,mn} (\sigma_i^x \sigma_j^x + \sigma_i^y \sigma_j^y) \sigma_k^z \sigma_l^z \sigma_m^z \sigma_n^z \right. \\
& + \dots \left. \right] + \dots \tag{42}
\end{aligned}$$

This expansion can be represented in the $|\pm \pm \dots \pm\rangle$ basis, and then compared to that obtained in Eq. (41), thus obtaining a, b_{12} and c_{12} (as well as any other coefficient we might be interested in, such as $d_{ij,kl}, e_{ij,kl}$, etc) as explicit functions of $\{K_{ij}, \Delta_{ij}\}$. Let us now proceed through the final step of our calculation. Eq. (39), together with expansions (9) and (42), implies

$$a' = 2^{N-2} a \tag{43.a}$$

$$b'_{12} = 2^{N-2} b_{12} \tag{43.b}$$

$$c'_{12} = 2^{N-2} c_{12} \tag{43.c}$$

Substituting these expressions into Eqs. (20) (where we have to remember that $K'_{12} \equiv K_s$ and $W'_{12} \equiv W_s$) we close the procedure, thus obtaining K'_{12}, W'_{12} and K'_0 as functions (explicit whenever

N is low enough, or the symmetry of the array high enough) of $\{K_{ij}, W_{ij}\}$ as desired.

III SOME QUANTUM EFFECTS

The present section is dedicated to the analysis of some interesting quantum effects.

III.1 Linear chain arrays

If we are dealing with a classical system (i.e., all relevant commutators vanish), a linear chain can be solved "by pieces", i.e., the exact solution corresponding to the entire chain can be straightforwardly achieved by exactly solving finite pieces (e.g, linear arrays of 3 or 4 spins) of the same chain. This simplifying property is lost whenever the system presents non-commutative quantum aspects. This effect has already been examined by Takano and Suzuki 1981, who have analyzed (in detail, for high temperatures) the discrepancy existing between the exact procedure (consideration of the *entire* array) and the approximative one (decomposition into subarrays). In the present Section we calculate, for all temperatures, an illustrative and important example of this discrepancy, namely that corresponding to a linear array of 4 spins (see Fig. 2 (g)). All 3 bonds are assumed of the same type, and are associated with (K, Δ) .

We note $K_s^{(n)}(K_1, \Delta_1; K_2, \Delta_2; \dots; K_n, \Delta_n)$ and $\Delta_s^{(n)}(K_1, \Delta_1; K_2, \Delta_2; \dots;$

K_n, Δ_n) the values K'_{12} and Δ'_{12} (see Section II.4) corresponding to a linear array of n bonds and $(n+1)$ spins. In particular $K_s^{(2)}(K_1, \Delta_1; K_2, \Delta_2)$ and $\Delta_s^{(2)}(K_1, \Delta_1; K_2, \Delta_2)$ are respectively identical to K_s and Δ_s calculated in Section II.3 (Eqs. (31.a) and (31.b) provide explicit expressions for the particular case $(K_1, \Delta_1) = (K_2, \Delta_2) \equiv (K, \Delta)$).

The exact calculation of the array of Fig. 2(g) provides (through the general procedure indicated in Section II.4) $K_s^{(3)}(K, \Delta; K, \Delta; K, \Delta)$. The approximate calculation ("by pieces") of the same array provides $K_s^{(2)}(K_s^{(2)}(K, \Delta; K, \Delta), \Delta_s^{(2)}(K, \Delta; K, \Delta); K, \Delta)$. We finally introduce the convenient ratio

$$R_s^K \equiv \frac{K_s^{(2)}(K_s^{(2)}(K, \Delta; K, \Delta), \Delta_s^{(2)}(K, \Delta; K, \Delta); K, \Delta)}{K_s^{(3)}(K, \Delta; K, \Delta; K, \Delta)} \quad (44)$$

which depends on (K, Δ) . If $\Delta = 1$ (classical system), R_s^K equals 1 for all values of K . If $\Delta \neq 1$ (quantum system), R_s^K presents oscillations as a function of K . The effect has been illustrated for $\Delta \in [0, 1]$ in Fig. 3. Note that, for high temperatures ($k_B T/J \gg 1$), $R_s^K \simeq 1$ for all values of Δ . Accordingly to these results the approximate procedure is reliable for high temperatures, but might be misleading at low temperatures. This is precisely what is observed in the calculation, by Takano and Suzuki 1981, of the internal energy associated with the quantum linear chain.

Let us finally add that the analysis of the ratio $R_s^\Delta \equiv \Delta_s^{(2)}(K_s^{(2)}(K, \Delta; K, \Delta), \Delta_s^{(2)}(K, \Delta; K, \Delta); K, \Delta) / \Delta_s^{(3)}(K, \Delta; K, \Delta; K, \Delta)$ shows the same type of behaviour as the ratio R_s^K . This is to say, for $\Delta = 1$, $R_s^\Delta = 1$ for all values of K , and for $\Delta \neq 1$, R_s^Δ os-

cillates as a function of K . In the limit of high temperatures, R_s^Δ tends to unity for all values of Δ , thus exhibiting the common tendency of most quantum systems to behave classically at high temperatures.

III.2 Dangling bonds

A subgraph (e.g., a bond) of a two-terminal graph is said to be *dangling* if it is connected to the terminals through a *single* site (which might be internal or terminal). If the graph was a wire-made one, and a voltage was applied between the two terminal sites, the dangling subgraphs are those along which no current circulates. For example, the graph of Fig. 2(e) can be seen as that of Fig. 2(h) with a dangling bond; the same holds for the graph of Fig. 2(f). In the present Section, we analyze the role played by dangling subgraphs. For the sake of simplicity we assume all the bonds to be equal and associated with (K, Δ) .

If $\Delta = 1$ (classical system), then $\Delta_{12}^i = 1$ and K_{12}^i is not sensitive to dangling bonds (or subgraphs). This is no more true if $\Delta \neq 1$ (quantum system). We shall illustrate the situation by comparing the graph of Fig. 2(h) (associated with $[K_{12}^{i(h)}(K, \Delta), \Delta_{12}^{i(h)}(K, \Delta)] \equiv [K_s^{(2)}(K, \Delta; K, \Delta), \Delta_s^{(2)}(K, \Delta; K, \Delta)]$) with that of Fig. 2(e) (associated with $[K_{12}^{i(e)}(K, \Delta), \Delta_{12}^{i(e)}(K, \Delta)]$) and that of Fig. 2(f) (associated with $[K_{12}^{i(f)}(K, \Delta), \Delta_{12}^{i(f)}(K, \Delta)]$). $K_s^{(2)}(K, 1; K, 1) = K_{12}^{i(h)}(K, 1) = K_{12}^{i(e)}(K, 1) = K_{12}^{i(f)}(K, 1)$ presents a smooth monotonous (and relatively uninteresting) K -dependence. In order to be free of such type of trivial dependence, let us introduce the following convenient ratios:

$$R^{(i)} \equiv \frac{K_{12}^{(i)}(K, \Delta)}{K_s^{(2)}(K, 1; K, 1)} \quad (i = e, f, h) \quad (45)$$

The results for typical values of $\Delta \in [0, 1]$ are indicated in Fig. 4. In the high temperature limit all three $R^{(e)}$, $R^{(f)}$ and $R^{(h)}$ tend (through $1/T$ corrections) to unity for all values of Δ . In the $T \rightarrow 0$ limit we obtain the following results (for $\Delta \in [0, 1]$):

$$R^{(e)} \rightarrow \frac{1}{2}(3 - \sqrt{3\Delta^2 - 6\Delta + 4}) \quad \in [1/2, 1] \quad (46)$$

$$R^{(f)} \rightarrow \begin{cases} \frac{1}{2}(2 + \Delta - \sqrt{5\Delta^2 - 8\Delta + 4}) & \in [0, 1], \quad \text{if } \Delta > 0 \\ 1 - \sqrt{2}/2 & , \quad \text{if } \Delta = 0 \end{cases} \quad (47.a)$$

$$, \quad \text{if } \Delta = 0 \quad (47.b)$$

$$R^{(h)} \rightarrow \frac{1}{4}(5 - \sqrt{8\Delta^2 - 16\Delta + 9}) \quad \in [1/2, 1] \quad (48)$$

An interesting non uniform convergence is observed for $R^{(f)}$ in the $0 < \Delta \ll 1$ region: at $T = 0$, $R^{(f)} \sim (3/2)\Delta$, then at a small *but finite* temperature (which tends to zero when Δ tends to zero) rapidly grows up to about $(1 - \sqrt{2}/2)$, then very smoothly grows from this value up to unity, when T keeps increasing up to infinity.

The fact that all three $R^{(e)}$, $R^{(f)}$ and $R^{(h)}$ are smaller than unity for finite temperatures, and monotonously decrease for decreasing temperatures, shows (as intuitively expected) that the quantum nature of the problem tends to make more difficult the transmission of thermal information between the spins of the system (i.e., the two-body correlation decreases).

A similar discussion can be done for $\Delta_{12}^{(i)}(K, \Delta)$ ($i = e, f, h$). For $\Delta = 0$ and $\Delta = 1$, $\Delta_{12}^{(i)}(K, \Delta)$ respectively equals 0 and 1 for $i = e, f, h$ and all finite values of T . For $0 < \Delta < 1$, the following inequality is verified for all $T \geq 0$:

$$\Delta_{12}^{(f)}(K, \Delta) > \Delta_{12}^{(e)}(K, \Delta) > \Delta_{12}^{(h)}(K, \Delta) \quad (49)$$

At $T = 0$, all three values belong to the interval $[0, 1]$. When T increases, all three monotonously decrease, and smoothly join together down to a common asymptotic value which is achieved in the $T \rightarrow \infty$ limit. This asymptotic value is given by

$$\Delta_{12}^{(i)} \rightarrow \Delta(2 - \Delta) \in [0, 1] \quad (i = e, f, h) \quad (50)$$

The fact that decreasing temperature enhances $\Delta_{12}^{(i)}$ (the problem becoming therefore closer to the Ising limit $\Delta = 1$), corresponds to the somehow intuitive notion that the lower the temperature is, the stronger the symmetry trends are (see also Castellani et al 1982 for interesting considerations concerning the fundamental state).

III. 3 Diamond array

Our last simple array will be the diamond-like (see Fig. 2(c)) commonly used in approximations of Migdal-Kadanoff type. The exact calculation of the diamond array will be noted $[\Delta_{12}^{(c)}(K, \Delta), \Delta_{12}^{(c)}(K, \Delta)]$. The approximate calculation ("by pieces") provides $[\Delta_s^{(2)}(K, \Delta; K, \Delta), \Delta_s^{(2)}(K, \Delta; K, \Delta)]$, where we

have used Eqs. (2) and (3'). Let us introduce the convenient ratios:

$$R_d^K \equiv \frac{2K_s^{(2)}(K, \Delta; K, \Delta)}{K_{12}'^{(c)}(K, \Delta)} \quad (51)$$

and

$$R_d^\Delta \equiv \frac{\Delta_s^{(2)}(K, \Delta; K, \Delta)}{\Delta_{12}'^{(c)}(K, \Delta)} \quad (52)$$

The K-dependences of R_d^K and R_d^Δ for typical values of $\Delta \in [0, 1]$ are indicated in Fig. 5 (a,b). In the $T \rightarrow \infty$ limit, both R_d^K and R_d^Δ tend (through $1/T^2$ corrections) to unity, as expected. In the $T \rightarrow 0$ limit, they tend to the following values:

$$R_d^K \rightarrow \frac{5 - \sqrt{8\Delta^2 - 16\Delta + 9}}{2(1 + \Delta)} \in [1, 1, 10] \quad (53)$$

$$R_d^\Delta \rightarrow \frac{(1 + \Delta)(6 - 2\sqrt{8\Delta^2 - 16\Delta + 9})}{2\Delta(5 - \sqrt{8\Delta^2 - 16\Delta + 9})} \in [1, 4/3] \quad (54)$$

For intermediate temperatures, both R_d^K and R_d^Δ remain roughly in the same intervals as for $T = 0$. In other words, the procedure "by pieces", strictly correct for classical systems ($\Delta = 1$), is an excellent approximation for all temperatures and all values of $\Delta \in [0, 1]$. This fact justifies "a posteriori" the use of this procedure, commonly adopted in Migdal-Kadanoff-like RG treatments of quantum systems (Suzuki and Takano 1979, Stinchcombe 1979 (b), 1981, Takano and Suzuki 1981, Tsallis et al 1984, and many others).

IV BOND-DILUTED MODEL AND RG APPROACH

We shall now use the concepts and mathematical tools introduced in previous sections to study the quenched bond-diluted Heisenberg ferromagnet in square lattice. (this problem has already been studied, within a Migdal-Kadanoff renormalization group approach, in one, two and three dimensions by Stinchcombe 1981). We consider the following random form of Hamiltonian (1):

$$\mathcal{H} = \sum_{\langle i,j \rangle} K_{ij} \left[(1-\Delta) (\sigma_i^x \sigma_j^x + \sigma_i^y \sigma_j^y) + \sigma_i^z \sigma_j^z \right] \quad (55)$$

with $0 \leq \Delta \leq 1$, and where K_{ij} is a random variable which vanishes with probability $(1-p)$, and takes the value K with probability p ($K > 0$; $0 \leq p \leq 1$). This model is properly represented by the following probability law:

$$P(K_{ij}, K_{ij} \Delta_{ij}) = (1-p) \delta(K_{ij}) \delta(K_{ij} \Delta_{ij}) + p \delta(K_{ij} - K) \delta(K_{ij} \Delta_{ij} - K\Delta) \quad (56)$$

where we have introduced, for future convenience, the random variable Δ_{ij} . The variable $K_{ij} \Delta_{ij}$ has been used (instead of Δ_{ij} alone) in order to avoid the indeterminacy of Δ_{ij} when K_{ij} vanishes (absent bond). In fact, it is the variable $K_{ij} \Delta_{ij}$ (and not Δ_{ij}) which naturally appears in the Hamiltonian of the system. The pure model ($p=1$) is the one treated by Caride et al 1983 (a,b). Following along the same lines, we choose, for constructing the RG recursive relations, the self-dual cluster appearing in Fig. 2(b). The corresponding probability

law P_H is given by

$$\begin{aligned}
P_H(K_{ij}, K_{ij}\Delta_{ij}) &= p^5 \delta(K_{ij} - K_{12}'^{(b)}) \delta(K_{ij}\Delta_{ij} - K_{12}'^{(b)} \Delta_{12}'^{(b)}) \\
&+ p^4(1-p) \left[\delta(K_{ij} - K_{12}'^{(c)}) \delta(K_{ij}\Delta_{ij} - K_{12}'^{(c)} \Delta_{12}'^{(c)}) \right. \\
&\quad \left. + 4\delta(K_{ij} - K_{12}'^{(d)}) \delta(K_{ij}\Delta_{ij} - K_{12}'^{(d)} \Delta_{12}'^{(d)}) \right] \\
&+ 2p^3(1-p)^2 \left[\delta(K_{ij} - K_{12}'^{(e)}) \delta(K_{ij}\Delta_{ij} - K_{12}'^{(e)} \Delta_{12}'^{(e)}) \right. \\
&\quad \left. + 2\delta(K_{ij} - K_{12}'^{(f)}) \delta(K_{ij}\Delta_{ij} - K_{12}'^{(f)} \Delta_{12}'^{(f)}) \right. \\
&\quad \left. + \delta(K_{ij} - K_{12}'^{(g)}) \delta(K_{ij}\Delta_{ij} - K_{12}'^{(g)} \Delta_{12}'^{(g)}) \right] \\
&+ 2p^2(1-p)^3 \delta(K_{ij} - K_{12}'^{(h)}) \delta(K_{ij}\Delta_{ij} - K_{12}'^{(h)} \Delta_{12}'^{(h)}) \\
&+ \left[2p^3(1-p)^2 + 8p^2(1-p)^3 + 5p(1-p)^4 + (1-p)^5 \right] \delta(K_{ij}) \delta(K_{ij}\Delta_{ij})
\end{aligned} \tag{57}$$

where $(K_{12}'^{(i)}, \Delta_{12}'^{(i)})$ ($i = b, c, d, e, f, g, h$) respectively correspond to the clusters appearing in Fig. 2, all the bonds of which are assumed equal and associated with (K, Δ) . $(K_{12}'^{(i)}, \Delta_{12}'^{(i)})$ have been analytically calculated as explicit functions of (K, Δ) ; however their expressions are too lengthy to be reproduced herein (we recall nevertheless that the results corresponding to $i = h$ are indicated in Eqs. (31)). The law given in Eq. (57) clearly is not anymore binary (in contrast with the original one given by Eq. (56)). Following satisfactory previous attempts for the Ising and Potts models (Yeomans and Stinchcombe, 1979; Levy et

al 1980, and many others), we shall approximate P_H by the binary law P' (associated with a renormalised single bond) given by

$$P'(K_{ij}, K_{ij}\Delta_{ij}) = (1 - p')\delta(K_{ij})\delta(K_{ij}\Delta_{ij}) + p'\delta(K_{ij} - K')\delta(K_{ij}\Delta_{ij} - K'\Delta') \quad (58)$$

where p' , K' and Δ' are quantities to be determined as functions of p , K and Δ . For the first of these equations we adopt the bond percolation one corresponding to the chosen self-dual cluster, i.e. (Reynolds et al 1977)

$$p' = p^5 + 5p^4(1 - p) + 8p^3(1 - p)^2 + 2p^2(1 - p)^3 \quad (59)$$

The other two equations can be obtained by matching first moments of K_{ij} and $K_{ij}\Delta_{ij}$, or of any convenient functions of them. A particularly performant choice for the Ising model ($\Delta = 1$) has proven to be (Tsallis and dos Santos 1983 and references therein)

$$S(K_{ij}) \equiv \frac{\ln(1 + \tanh K_{ij})}{\ln 2} \in [0, 1] \quad (60)$$

The matching we are looking for is going to be

$$\langle S(K_{ij}) \rangle_{P'} = \langle S(K_{ij}) \rangle_{P_H} \quad (61)$$

and

$$\langle S(K_{ij}\Delta_{ij}) \rangle_{P'} = \langle S(K_{ij}\Delta_{ij}) \rangle_{P_H} \quad (62)$$

hence

$$\begin{aligned} p'S(K') &= p^5 S(K_{12}'^{(b)}) + p^4(1-p) \left[S(K_{12}'^{(c)}) + 4S(K_{12}'^{(d)}) \right] \\ &+ \dots + 2p^2(1-p)^3 S(K_{12}'^{(h)}) \equiv F(p,K,\Delta) \end{aligned} \quad (63)$$

and

$$\begin{aligned} p'S(K'\Delta') &= p^5 S(K_{12}'^{(b)}\Delta_{12}'^{(b)}) + p^4(1-p) \left[S(K_{12}'^{(c)}\Delta_{12}'^{(c)}) + 4S(K_{12}'^{(d)}\Delta_{12}'^{(d)}) \right] \\ &+ \dots + 2p^2(1-p)^3 S(K_{12}'^{(h)}\Delta_{12}'^{(h)}) \equiv G(p,K,\Delta) \end{aligned} \quad (64)$$

Eqs. (59), (63) and (64) provide explicit RG recursive relations in the (p,K,Δ) space, the problem being thus formally solved. The critical properties associated with the present RG recurrence are described and analyzed in the next Section.

V RESULTS

The numerical iteration of the explicit RG recurrence (from Eqs. (59), (63) and (64)) provides the critical surface in the (p,K,Δ) space, or equivalently in the $(p,k_B T/J,\Delta)$ space: see Fig. 6. The most important fixed points are located at the following positions: (i) two trivial (fully stable) fixed points at $(p,k_B T/J, \Delta) = (1,0,1)$ and at $(1,\infty,1)$, respectively cor-

responding to the ferro-and paramagnetic phases; (ii) three semi-stable critical fixed points, at $(1, 2.269\dots, 1)$ (pure Ising), at $(1, 0, 0)$ (pure Heisenberg) and at $(1/2, 0, 1)$, (Ising bond-percolation), the first of them being the attractor of all $T \neq 0$ points of the critical surface (note that the present RG recovers, for all these three fixed points, well known *exact* numerical values); (iii) one fully unstable critical fixed point at $(1/2, 0, 0)$ (Heisenberg bond-percolation), this location very probably being the *exact* one as well.

Let us now focus the relevant critical exponents. We consider the Jacobian $\partial(p', 1/K', \Delta')/\partial(p, 1/K, \Delta)$ (calculated at the four critical fixed points) and its eigenvalues, noted λ_p, λ_T and λ_Δ (for the Ising bond-percolation point we use the natural variable $\tanh K$, instead of $1/K$; see Stauffer 1975, Lubensky 1977 and Stinchcombe 1979 (b)). The corresponding correlation length critical exponents are given by

$$\nu_r = \ln b / \ln \lambda_r \quad (r = p, T, \Delta) \quad (65)$$

where b is the RG linear scale (in our case $b = 2$), and where only λ_r satisfying $\lambda_r \geq 1$ (unstable directions) are to be considered. It is also convenient to introduce also the cross over exponents, through

$$\phi_s = \nu_s / \nu_T \quad (s = p, \Delta) \quad (66)$$

The results are indicated in Table I. The numerical values are, on the whole, quite satisfactory when compared to exact

available results (in particular, they improve on those obtained, within a Migdal-Kadanoff-like RG, by Stinchcombe 1979 (b), 1981).

We intend now to discuss the main asymptotic behaviours of the critical surface in the $(p, k_B T/J, \Delta)$ space. We start by considering the fixed Δ cuts of the critical surface (see Fig. 6(b)). In the $p \rightarrow 1$ limit, we obtain

$$\frac{T_c(p=1, \Delta) - T_c(p, \Delta)}{T_c(p=1, \Delta)} \sim A(\Delta) (1-p) \quad (67)$$

with $A(\Delta)$ shown in Fig. 7. We obtain $A(1) \simeq 1.3314$, to be compared with the exact value $2\sqrt{2}/(1+\sqrt{2})\ln(1+\sqrt{2}) \simeq 1.3293$ (Harris 1974). In the $p \rightarrow 1/2$ limit, we obtain

$$e^{-B(\Delta)J/k_B T_c(p, \Delta)} \sim C(\Delta)(p - p_c) \quad (68)$$

with $B(\Delta)$ shown in Fig. 8, and $C(\Delta)$ practically independent from Δ , at least for Δ not too small, let us say $0.3 \leq \Delta \leq 1$ (for Δ below 0.3 numerical precision difficulties appear). We obtain $B(1) = 2$ (which is the exact result) and $c(1) \simeq 1.395$, to be compared with the exact value $2 \ln 2 \simeq 1.386$ (Domany 1978).

We shall now consider the fixed p cuts of the critical surface (see Fig. 6(c)). In the $\Delta \rightarrow 1$ limit we obtain

$$\frac{T_c(p, \Delta=1) - T_c(p, \Delta)}{T_c(p, \Delta=1)} \sim D(p)(1-\Delta)^2 \quad (69)$$

with $D(p)$ shown in Fig. 9; $D(1) \simeq 0.295$ reproduces the value obtained by Caride et al 1983 (a,b). In the $\Delta \rightarrow 0$ limit the numerical precision difficulties become particularly hard.

For $p = 1$ we recover Caride et al 1983 (a,b) asymptotic law, namely

$$\frac{J}{k_B T_c(p, \Delta)} e^{-E(p) J/k_B T_c(p, \Delta)} \sim F(p) \Delta \quad (70)$$

with $E(1) = 4$ and $F(1) \simeq 4$. For p decreasing below 1, Eq. (70) is roughly satisfied with $E(p)$ smoothly decreasing and $F(p)$ increasing in such a way that $E(p)F(p) \simeq 16$. However these $p < 1$ results should be considered only as indicative trends. Within a Migdal-Kadanoff RG approach, Stinchcombe 1981 obtained for the pure case ($p = 1$), the following asymptotical behaviours: (a) in the $\Delta \rightarrow 1$ limit, Eq. (69) with $D(1) \simeq 0.32$; (b) in the $\Delta \rightarrow 0$ limit, $\exp(-J/k_B T_c) \sim \Delta^a$, where $a \simeq 0.21$.

To make a final comment, let us cut the critical surface (see Fig. 6(a)) by a plane which contains the straight line $p = \Delta = 1$ and turns around it, and consider the variable $\ell \equiv [(1 - \Delta)^2 + (1 - p)^2]^{1/2}$. We verify that the departure of $T_c(p, \Delta)$ from $T_c(1, 1)$ is *linear* in ℓ for all angular positions excepting that corresponding to the $p = 1$ plane, where the dependence is *quadratic* in ℓ (see Eq. (69)). The proportionality factor associated with the linearity (in ℓ) just mentioned, smoothly decreases from $1.33 T_c(1, 1)$ (see Eq. (67)), in the $\Delta = 1$ plane, down to zero, in the $p = 1$ plane.

VI CONCLUSION

We have fully developed and illustrated the method outlined by Caride et al 1983 (a,b) for *exactly* treating the thermal statistics of two-terminal graphs whose sites are occupied by $s = 1/2$ spins, and whose bonds represent anisotropic Heisenberg interactions admitting at least one axis of rotational symmetry in spin space (the Ising, XY and isotropic Heisenberg interactions satisfy this requirement). As a matter of fact, the procedure can be straightforwardly extended to cover the general Heisenberg interaction ($J_x \neq J_y \neq J_z$) or even more complex ones, as well as $s > 1/2$ systems. This method can in principle easily be adapted to treat a quite large class of quantum systems, including fermionic ones. We are presently working along these lines.

At the graph level several interesting quantum effects have been exhibited and analyzed. In particular, the non trivial consequences (on the thermal statistics) of dangling bonds have been discussed. Depending on their location in the graph, and on temperature, they can increase or decrease (the more intensively the more quantic the system is) the correlation between spins; their importance gradually disappears for increasing temperatures. We have also compared the *exact* solution of typical graphs with that *approximate* obtained by solving the graph "by pieces" (i.e., conveniently inserting the results associated with subgraphs in order to achieve a solution for the entire graph). In particular, the discussion of the standard diamond graph, commonly used in Migdal-Kadanoff-like

renormalisation groups, provides an "a posteriori" justification of some approximate procedures (i.e., solution "by pieces") which are frequently found in the literature.

The exact procedures developed here for two-terminal graphs, provide the mathematical tool for constructing renormalisation groups approaching Bravais lattices through hierarchical ones. Were it not for the non-commutation present in quantum systems, the treatment would be the *exact* solution of the hierarchical lattice under consideration. It is believed however (see also Suzuki and Takano 1979 and Takano and Suzuki 1981) that the solution thus achieved is, by means of certain compensations, an excellent approximation *even* if the nature of the problem is not classical.

We use the framework developed here, to study the criticality associated with the quenched bond-diluted quantum spin 1/2 anisotropic Heisenberg ferromagnet in square lattice, for anisotropy Δ varying from $\Delta = 0$ (isotropic Heisenberg model) to $\Delta = 1$ (Ising model). The cluster we use is the self-dual Wheatstone-bridge one, which has been shown to be particularly performant for the square lattice. The para-ferromagnetic critical surface we obtain recovers (precisely or almost) all the already known exact results. Furthermore, the asymptotic behaviours corresponding to $\Delta \rightarrow 0$ (almost isotropic), $\Delta \rightarrow 1$ (almost Ising), $p \rightarrow 1$ (almost pure) and $p \rightarrow 1/2$ (just percolating) have been (either precisely or roughly) established for arbitrary values of (p, Δ) . Finally the various thermal (ν_T), percolation (ν_p) and anisotropy (ν_Δ) correlation length critical exponents we obtain satisfactorily agree

with the exact whenever available .

With respect to universality classes, the common expectations (i.e., pure Ising for $T > 0$, $\forall p$, $\forall \Delta$, pure Heisenberg for $T = 0$, $\Delta = 0$ and $1/2 < p \leq 1$, Ising percolation for $T = 0$, $p = 1/2$ and $0 < \Delta \leq 1$, and Heisenberg percolation for $T = 0$, $p = 1/2$ and $\Delta = 0$) are verified.

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REFERENCES

- Caride AO, Tsallis C and Zanette SI 1983, (a) Phys. Rev. Lett. 51, 145; (b) Phys. Rev. Lett. 51, 616
- Castellani C, Di Castro C and Ranninger J 1982, Nuclear Physics B 200, 45
- den Nijs MPM 1979, Physica A 95, 449
- Domany E 1978, J. Phys. C. 11, 337
- Harris AB 1974, J. Phys. C. 7, 1671
- Kramers HA and Wannier GH 1941, Phys. Rev. 60, 252
- Levy SVF, Tsallis C and Curado EMF 1980, Phys. Rev. B 21, 2991
- Lubensky TC 1977, Phys. Rev. B 15, 311
- Mermin ND and Wagner H 1966, Phys. Rev. Lett. 17, 1133
- Pfeuty P, Jullien R and Penson KA 1982, "Renormalization for Quantum Systems" Chap. 5 in "Real-Space Renormalization", ed. Burkhardt TW and van Leeuwen JMJ (Springer-Verlag)
- Plascak JA 1984, J. Phys. A 17, L597
- Polyakov AM 1975, Phys. Lett. B, 59, 79
- Reynolds PJ, Klein W and Stanley HE 1977, J. Phys. C, 10, L167
- Stauffer D 1975, Z. Physik B 22, 161
- Stinchcombe RB 1979, (a) J. Phys. C 12, 2625; (b) J. Phys. C 12, 4533
- Stinchcombe RB 1981, J. Phys. C 14, 397
- Suzuki M and Takano H 1979, Phys. Lett. 69A, 426
- Sykes MF and Essam JW 1963, Phys. Rev. Lett. 10, 3
- Sykes MF and Essam JW 1964, J. Math. Phys. 5, 1117
- Takano H and Suzuki M 1981, J. Stat. Phys. 26, 635
- Tsallis C and Levy SVF 1981, Phys. Rev. Lett. 47, 950
- Tsallis C and dos Santos RJV 1983, J. Phys. A 16, 360

- Tsallis C, Coniglio A and Redner S 1983, J. Phys. C. 16, 4339
Tsallis C and Redner S 1983, Phys. Rev. B 28, 6603
Tsallis C, Stinchcombe RB and Buck B 1984, J. Phys. C 17, 301
Wallace DJ and Young AP 1978, Phys. Rev. B 17, 2384
Wu TT 1966, Phys. Rev. 149, 380
Yeomans JM and Stinchcombe RB 1979, J. Phys. C 12, 347

CAPTIONS FOR TABLE AND FIGURES

Table 1: Present RG critical points and exponents as compared with Migdal-Kadanoff-like and exact results (whenever available).

Fig. 1 : Two-terminal graphs used for renormalisation purposes: $\circ(\bullet)$ denotes terminal (internal) nodes: (a) parallel array; (b) series array.

Fig. 2 : (a) and (b): self-dual two-terminal graphs on which the present RG is constructed. (c)-(h): all the non equivalent percolating sub-graphs of graph (b) (generated by bond-dilution).

Fig. 3 : Thermal behaviour of R_s^K (Eq.(44)), for selected values of the anisotropy parameter Δ . The oscillations around the classical value $R_s^K = 1$ are more pronounced for $0.3 \lesssim \Delta \lesssim 0.6$.

Fig. 4 : Thermal dependences of the ratios $R^{(e)}$ (dashed), $R^{(f)}$ (dot-dashed) and $R^{(h)}$ (full) defined in Eq. (45), for typical values of Δ . $R^{(i)} = 1$ ($i = e, f, h$) is the classical limit ($\Delta = 1$).

Fig. 5 : Thermal dependences of the ratios R_d^K (a) and R_d^Δ (b) respectively defined by Eqs. (51) and (52), for typical values of Δ . We note that the Migdal-Kadanoff approach overestimates the coupling constant K and the anisotropy parameter Δ . In the less favourable region (low temperatures), the errors are less than 10% for R_d^K and 45% for R_d^Δ ; $R_d^K = R_d^\Delta = 1$ are

the classical ($\Delta=1$) limits.

Fig. 6 : (a) The ferromagnetic critical frontier (full line) in the $(\Delta, k_B T/J, 1-p)$ space; the most important fixed points and flux lines are indicated (o, ● and ■ respectively denote the fully unstable, semi-stable and fully stable fixed points); all $T > 0$ critical points flow towards the pure Ising fixed point (I). (b) Fixed Δ cuts of the critical frontier for selected values of Δ ; the $p \rightarrow 1$ ($p \rightarrow p_c = 0.5$) asymptotical behaviour of the cuts is given by Eq. (67) (Eq. (68)). (c) Fixed p cuts of the critical frontier for typical values of p . The $\Delta \rightarrow 1$ ($\Delta \rightarrow 0$) asymptotical behaviour is given by Eq. (69) (Eq. (70)).

Fig. 7 : Δ -dependence of the coefficient $A(\Delta)$ (defined through Eq. (67)).

$$A(1) \simeq 1.33 \text{ and } A(0) \simeq 3 .$$

Fig. 8 : Δ -dependence of the coefficient $B(\Delta)$ (defined through Eq. (68)).

The dashed line is indicative (for $\Delta \lesssim 0.3$ numerical difficulties appear).

Fig. 9 : p -dependence of the asymptotical coefficient $D(p)$ (defined through Eq. (69)). $D(1) \simeq 0.295$ and $D(0.5) \simeq 0.65$.

TABLE 1

		Exact	Migdal-Kadanoff	Present	
Ising ($\Delta = 1$)	pure ($p = 1$)	$k_B T/J$	2.269 ⁽¹⁾	1.64 ⁽²⁾	2.269...
		ν_T	1 ⁽³⁾	1.34 ⁽²⁾	1.15
	percolation ($T = 0$)	p_c	0.5 ⁽⁴⁾	0.62 ⁽²⁾	0.5
		ν_P	1.33 ⁽⁵⁾	1.64 ⁽²⁾	1.43
		$\phi_P = \nu_P / \nu_T$	1 ⁽⁶⁾	1 ⁽²⁾	1
	Heisenberg ($\Delta = 0$)	pure ($p = 1$)	$k_B T/J$	0 ⁽⁷⁾	0 ⁽²⁾
ν_Δ			-	-	1.22
ν_T			∞ ⁽⁸⁾	∞ ⁽²⁾	∞
percolation ($T = 0$)		p_c	0.5 ⁽⁴⁾	0.62 ⁽²⁾	0.5
		ν_P	1.33 ⁽⁵⁾	1.64 ⁽²⁾	1.43
		ν_Δ	-	0.71 ⁽⁹⁾	0.69
$\phi_P = \nu_P / \nu_T$	-	1.13 ⁽²⁾	1.81		
$\phi_\Delta = \nu_\Delta / \nu_T$	-	0.49 ⁽⁹⁾	0.87		

- (1) Kramers and Wannier 1941
(2) Stinchcombe 1979 (b)
(3) Wu 1966
(4) Sykes and Essam 1963, 1964
(5) den Nijs 1979
(6) Wallace and Young 1978
(7) Mermin and Wagner 1966
(8) Polyakov 1975
(9) Stinchcombe 1981

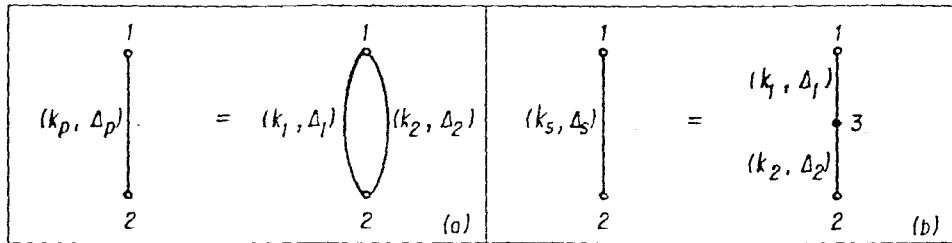


FIG. 1

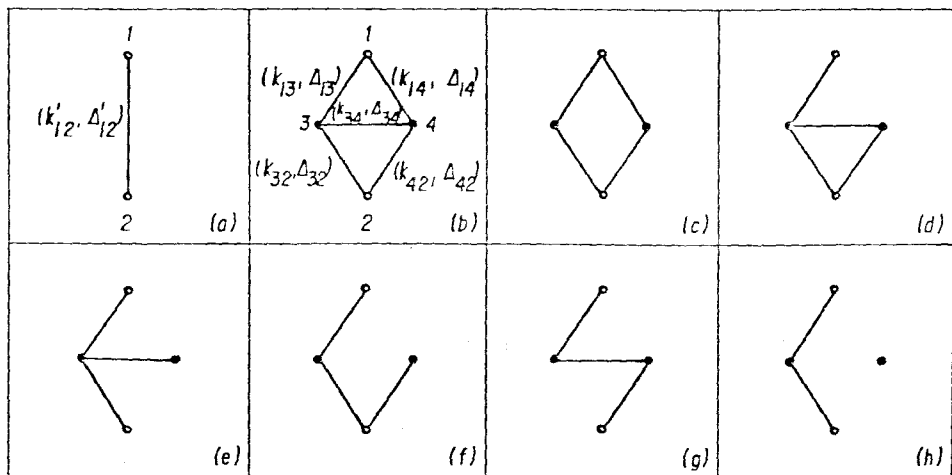


FIG. 2

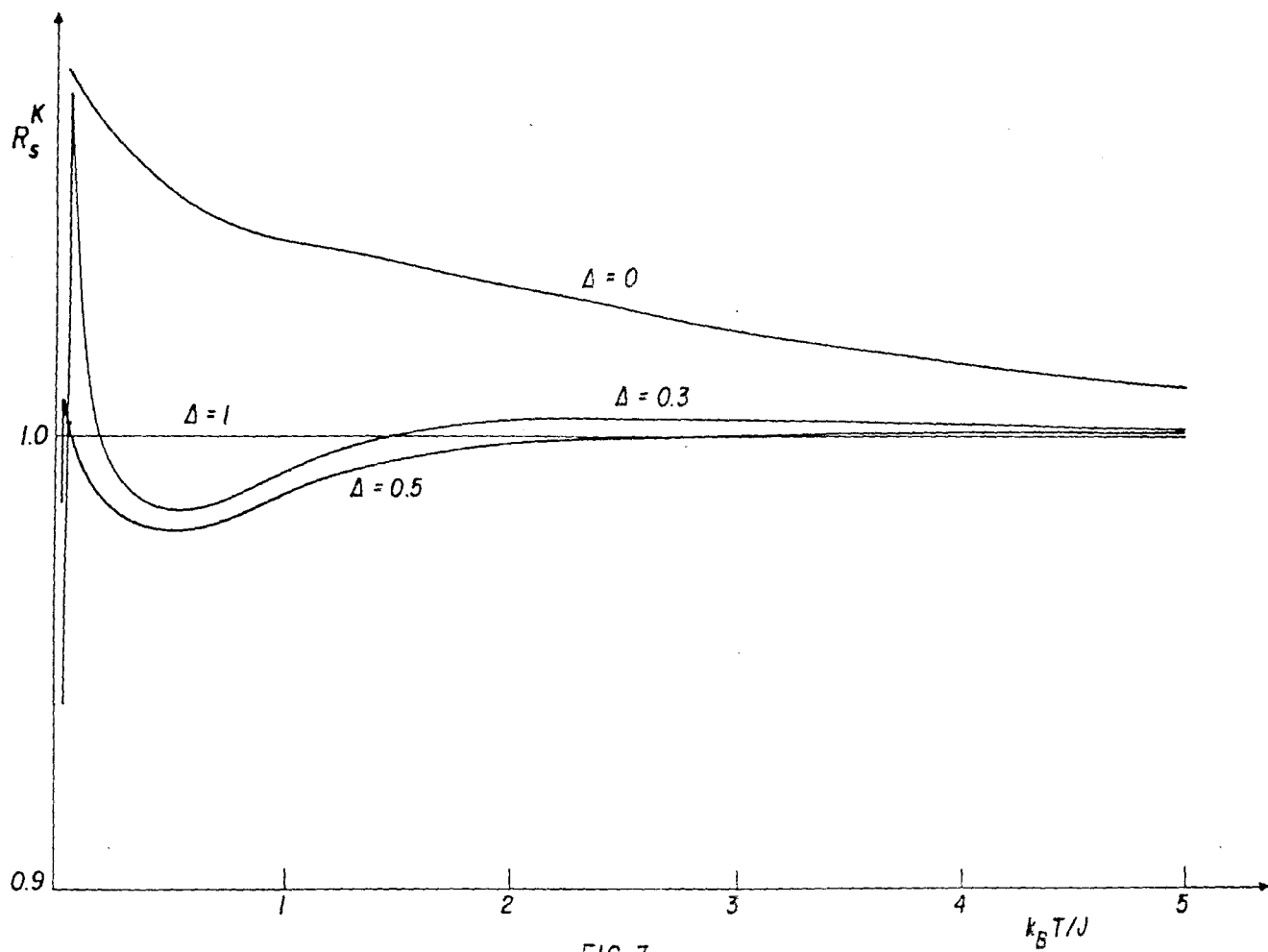


FIG. 3

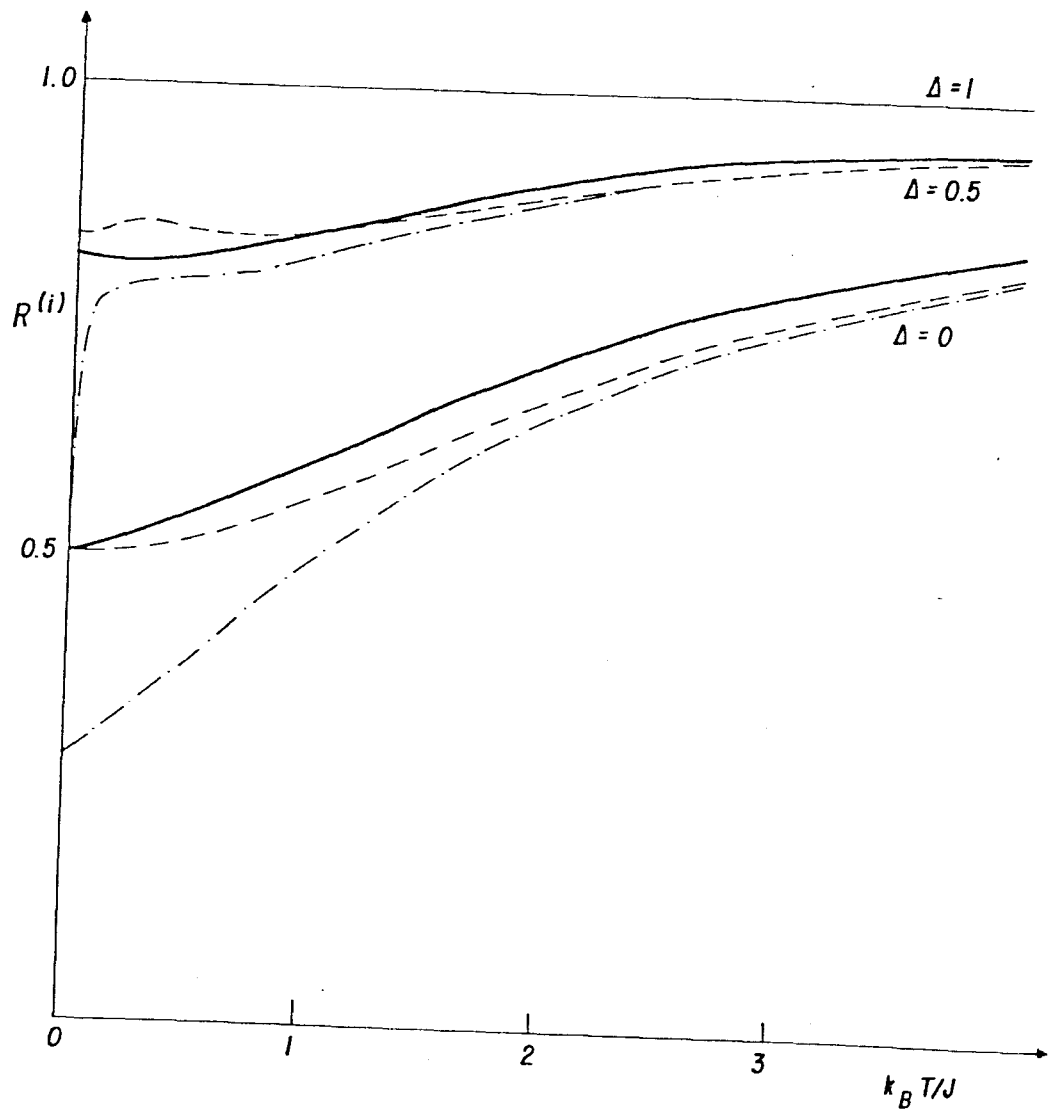


FIG. 4

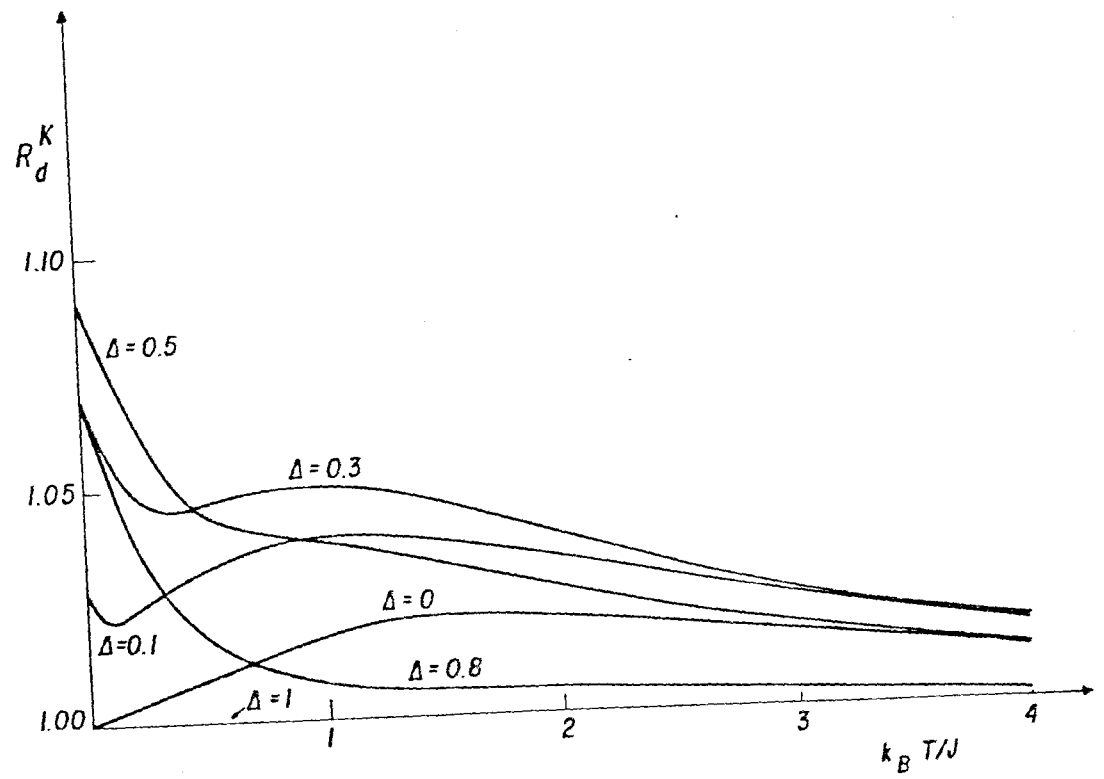


FIG. 5-a

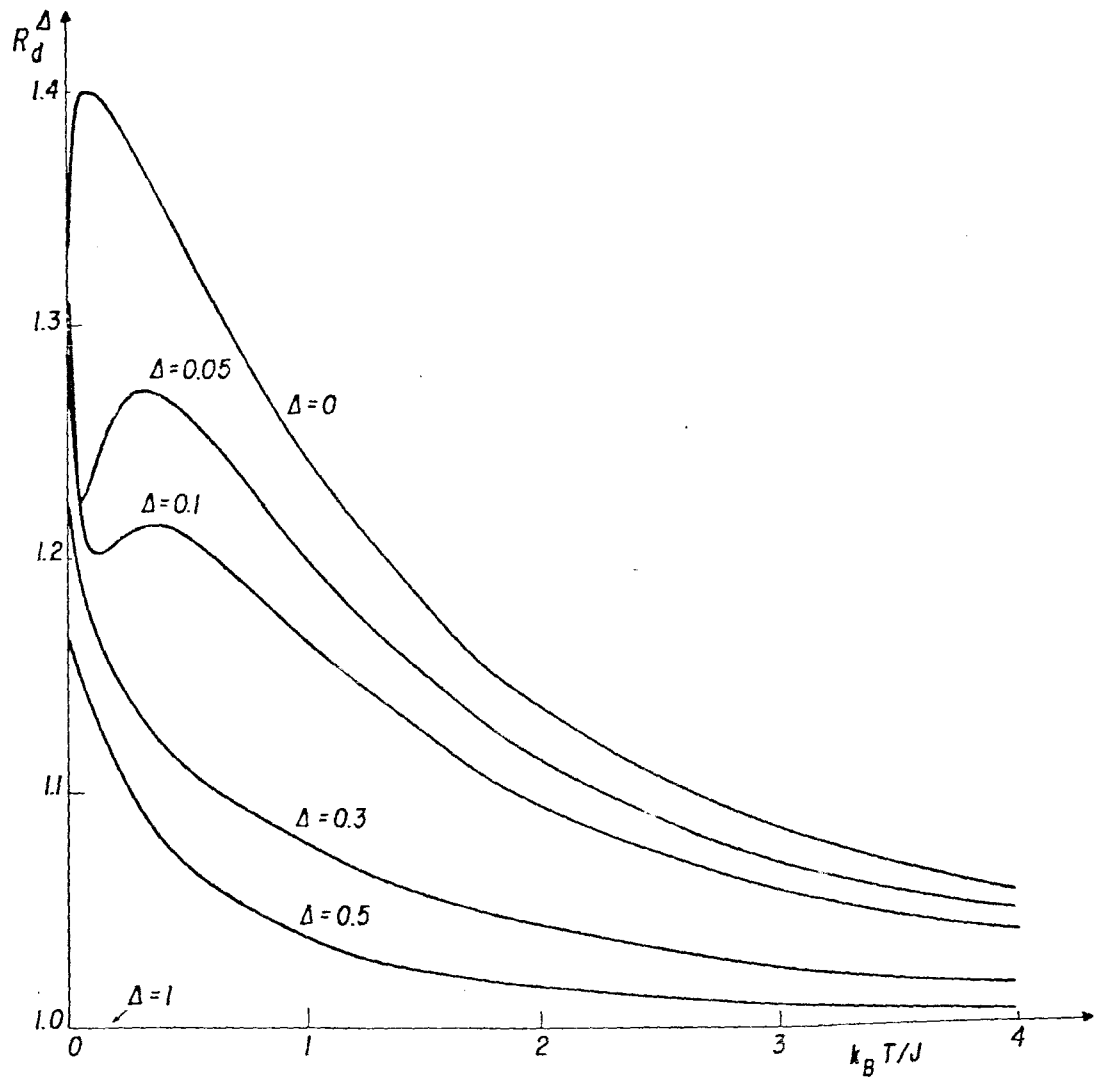


FIG. 5-b

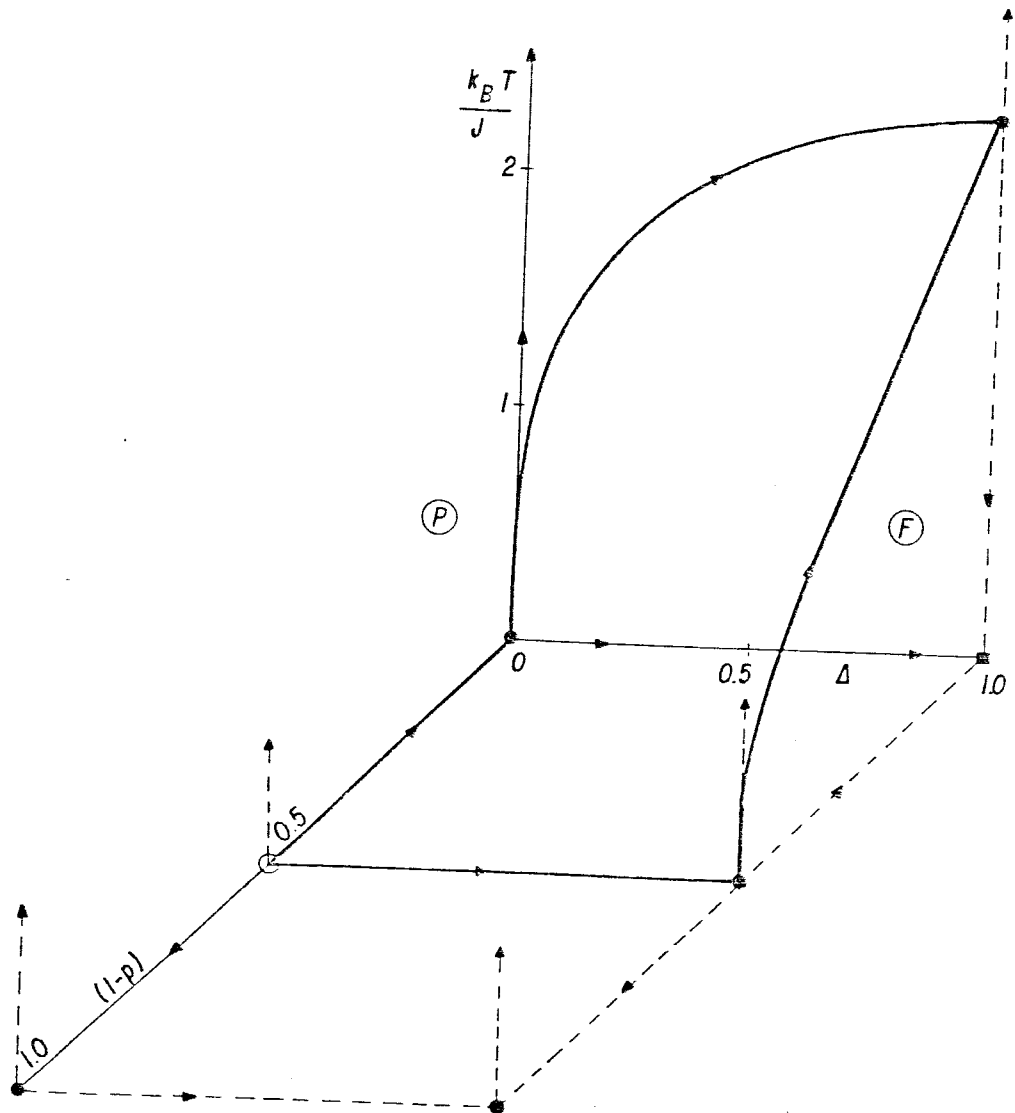


FIG. 6-a

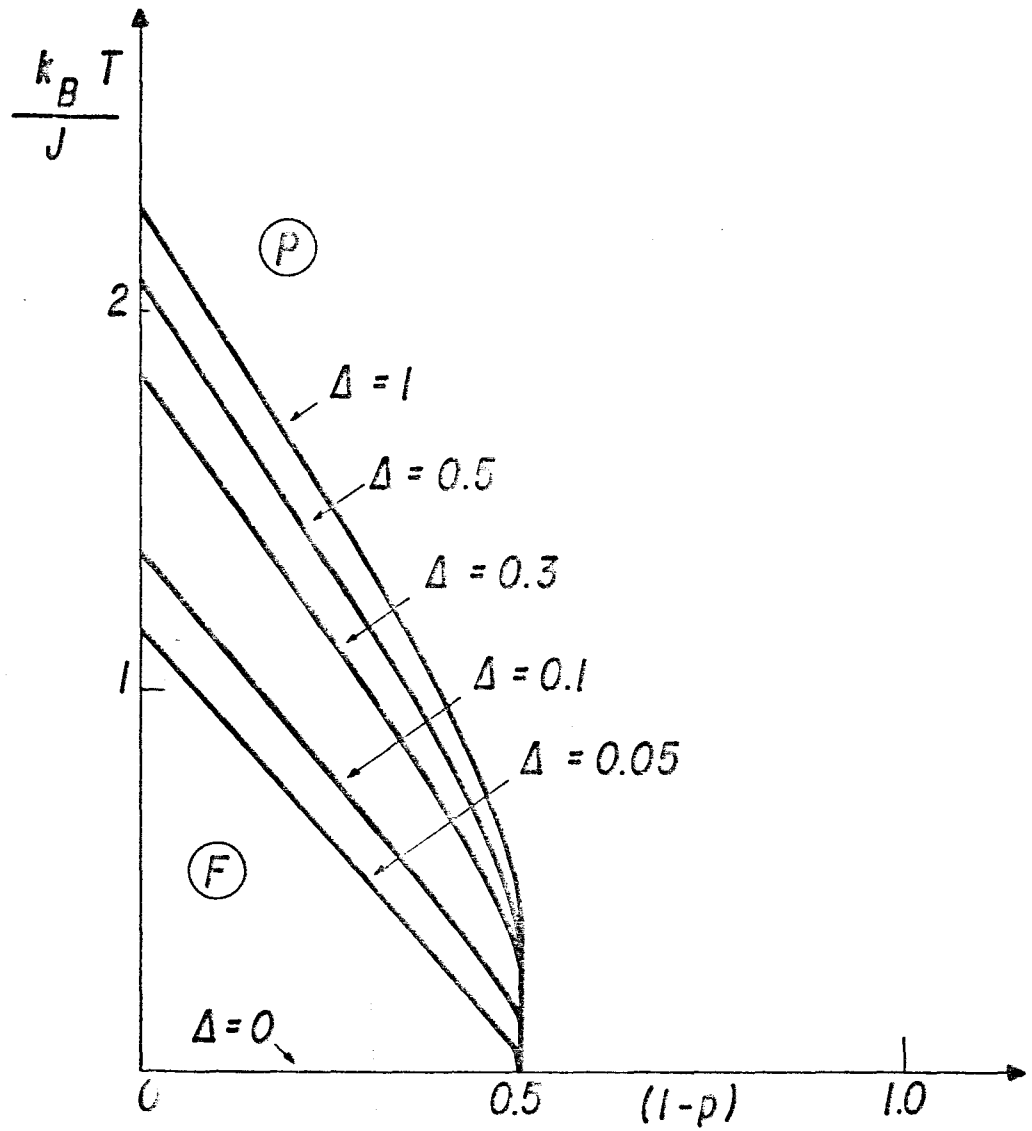


FIG. 6-b

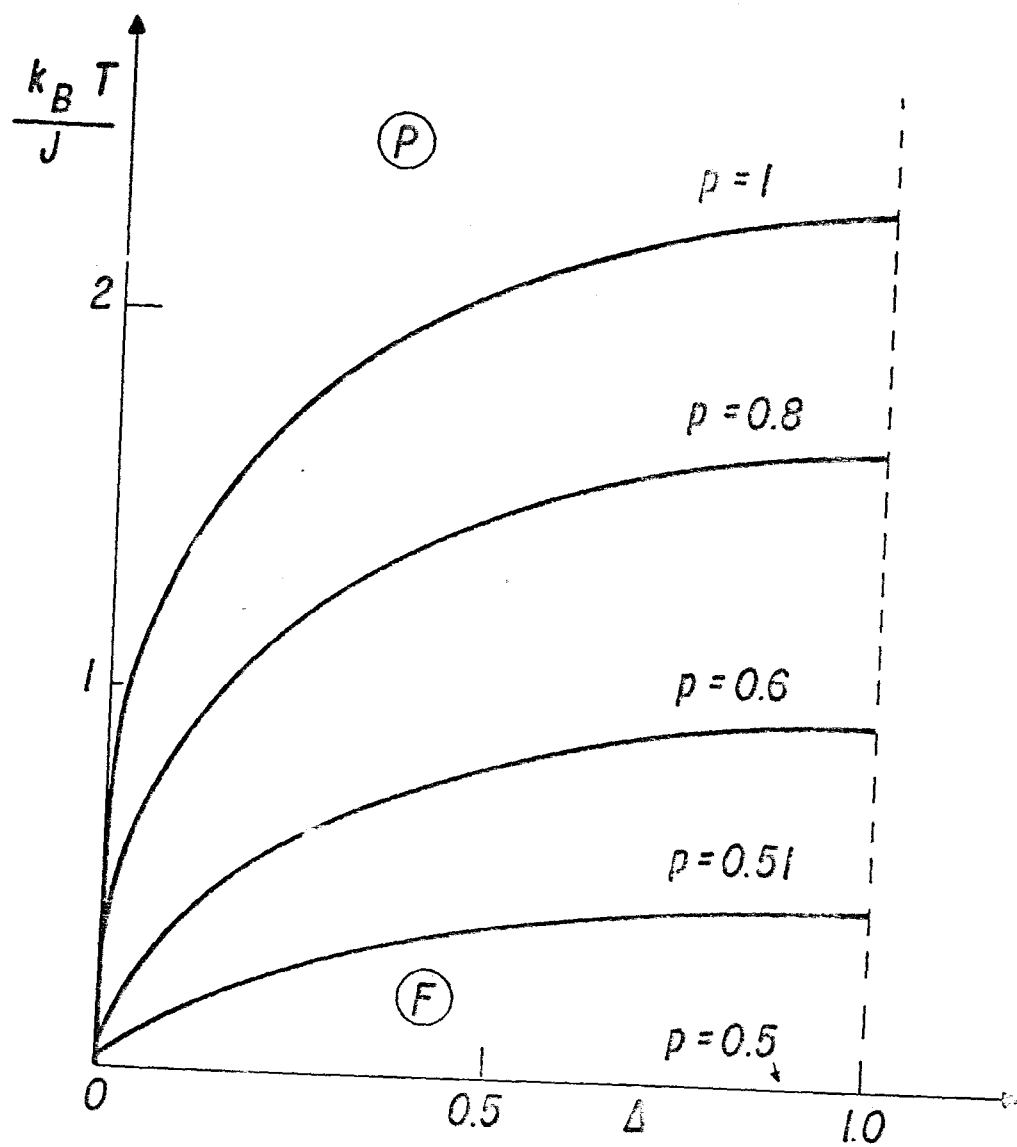


FIG.6 - c

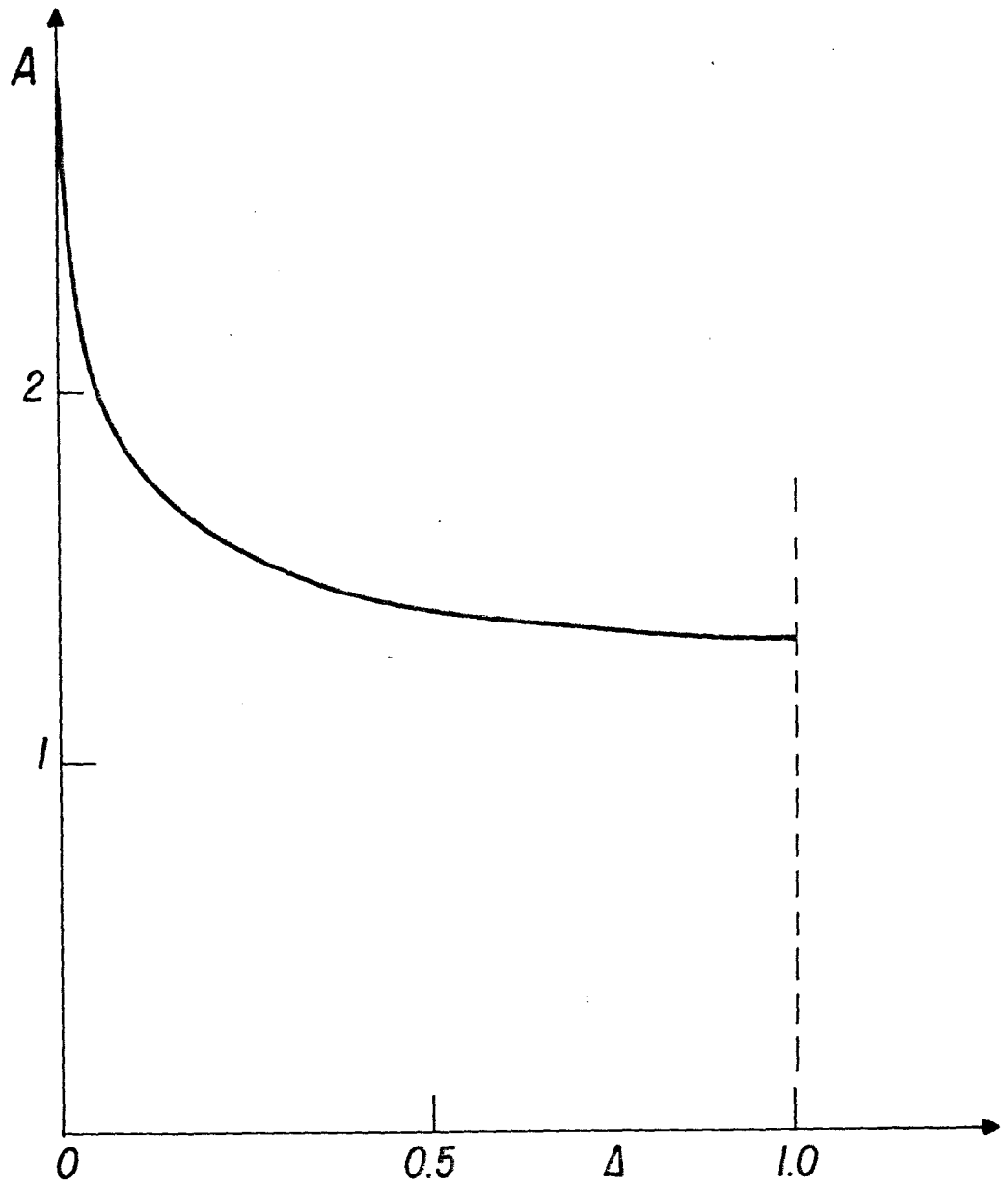


FIG. 7

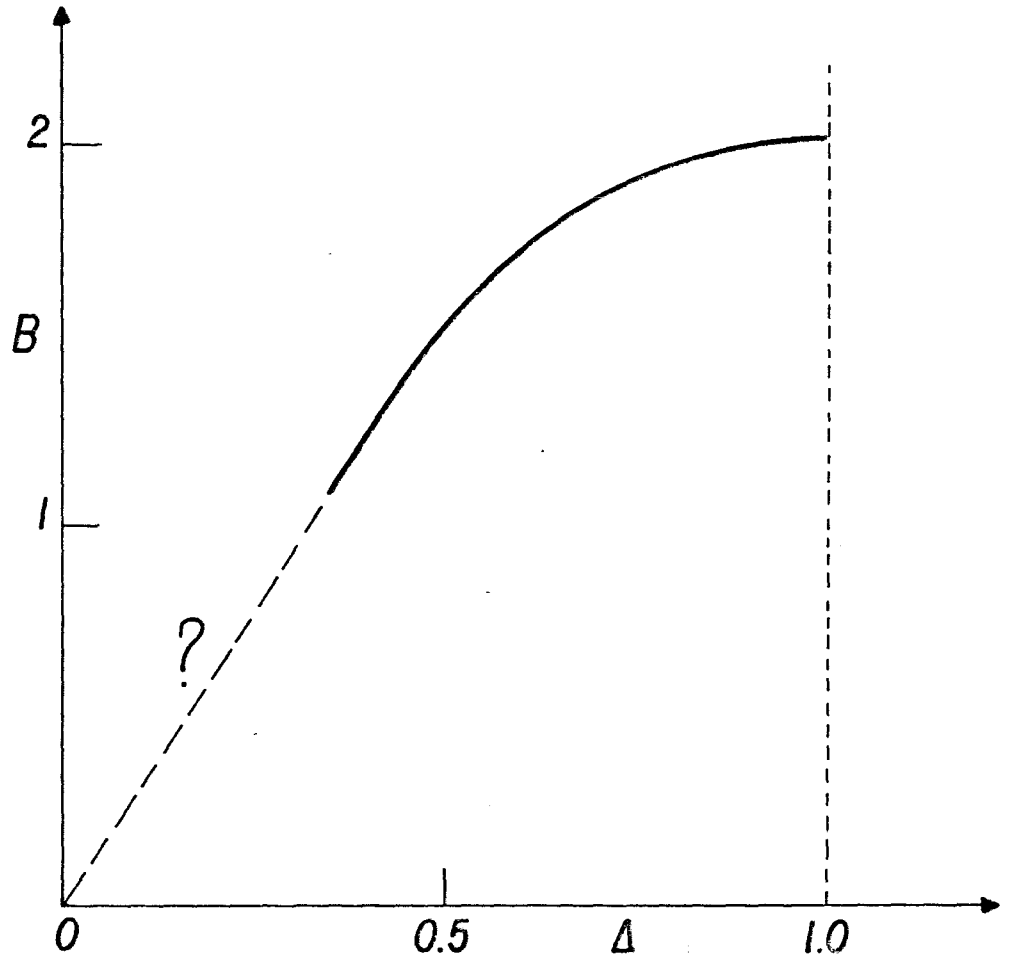


FIG. 8

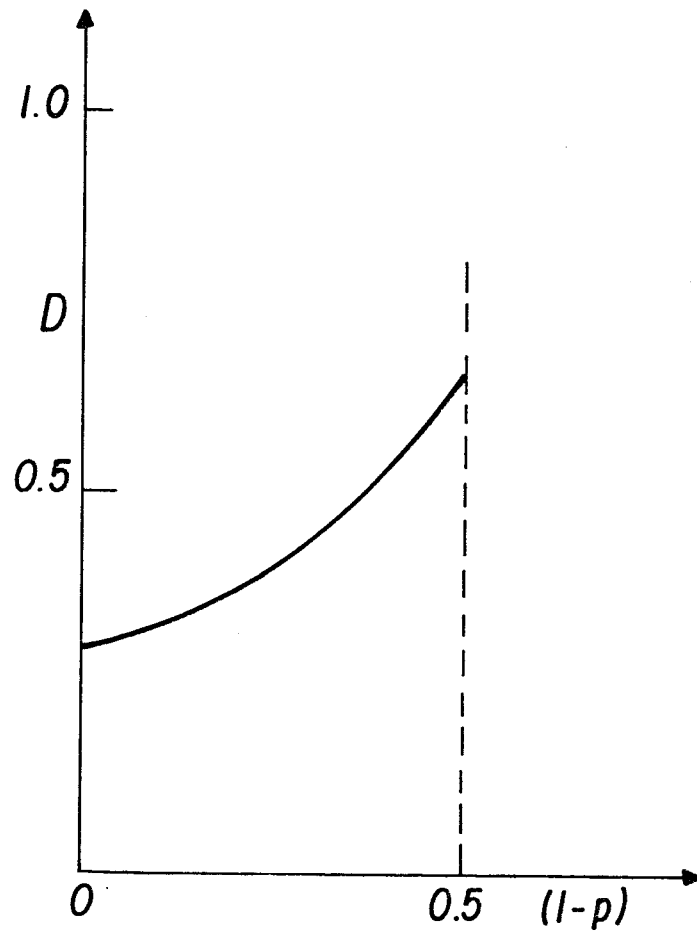


FIG. 9