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TORSION AND CURVATURE IN HIGHER DIMENSIONAL SUPERGRAVITY THEORIES

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Abstract

This work is an extension of Dragon's theorems to higher dimensional space-time. We show that the first set of Bianchi identities allow us to express the curvature components in terms of torsion components and its covariant derivatives. It is also shown that the second set of Bianchi identities does not give any new information which is not already contained in the first one.

Key-words: Supergravity;Torsion and Curvature.

I. Introduction

It has already been emphasized the importance in supergravity (supersymmetry) of a) solving the Bianchi identities [1-4] and b) working in higher space-time dimensions [5-7]. For this very reason we have extended Dragon's theorems [8] to higher dimensional space-time formulations of supergravity theories.

The first theorem tell us that the first set of Bianchi identities allows us to express the curvature components in terms of torsion components and its covariant derivatives. The second theorem tell us that the second set of Bianchi identities does not give any new information which is not already contained in the first one. In the next section we consider the general formalism of superspace [9]. In section III we demonstrate the theorems stated above and make the conclusions.

II. General Formalism

We are going to formulate this work in superspace [9] which is a generalization of space time with D bosonic-coordinates x^m and $2^{[\frac{D}{2}]}$ anticommuting (fermionic) coordinates θ^μ. The superspace coordinates are going to be labeled by $z^M \sim (x^m, \theta^\mu)$. The (anti) commuting properties are usually stated as $n^A n^B = (-1)(-1)^{d(A)d(B)} n^B n^A$ where d(A) is (one) zero for (anti) commuting objects. Following [8] the same equation can be written as $n^A n^B = n^B n^A$ because if all superindices are filled into the equation from left in the same order, commuting the indices into their final position gives minus signus, depending on the grading d(A) of the indices.

The basic dinamic variables are the vielbein

$$E^A = dz^M E_M{}^A(z) \qquad (2.1)$$

and the connection

$$\phi_A{}^B = dz^M \phi_{M\ A}{}^B(z) \qquad (2.2)$$

Where $M = (m,\mu)$ is a superspace index, called world (or Einstein) index and $A = (a,\alpha)$ is an index reserved to the tangent space group.

The connexion is supposed to be Lie-algebra valued.

We are going to choose the Lorentz group SO(1,D-1) as the structure group. The Lorentz generators are given by

$$(L_{ab})_{CD} = \begin{vmatrix} (L_{ab})_{cd} & 0 \\ 0 & \frac{1}{4}(\Gamma_{ab})_{\alpha\beta} \end{vmatrix} \qquad (2.3)$$

where

$$(L_{ab})_{cd} = -(\eta_{ac}\eta_{bd}-\eta_{bc}\eta_{ad}) \qquad (2.4)$$

with

$$\eta_{ac} = \text{diag}(-,+,+,\ldots,+) \qquad (2.5)$$

and

$$(\Gamma^{ab})_{\alpha\beta} = \frac{1}{4}(\Gamma^{a}_{\alpha\lambda}\Gamma^{b}_{\lambda\beta}-\Gamma^{b}_{\alpha\lambda}\Gamma^{a}_{\lambda\beta}) \qquad (2.6)$$

(a summation over doubled indices is always implied).

$\Gamma^{a}_{\alpha\lambda}$ are Dirac matrices obeying the Clifford algebra in D dimensions [10]

$$\{\Gamma^{a},\Gamma^{b}\} = 2\eta^{ab} \qquad (2.7)$$

$$a = 1,\ldots,D$$

and

$$\alpha = 1,\ldots,2^{[D/2]}$$

The torsion and curvature are defined as

$$T^{A} = dE^{A} + E^{B}\phi_{B}{}^{A} \qquad (2.8)$$

$$R_{A}{}^{B} = d\phi_{A}{}^{B} + \phi_{A}{}^{C}\phi_{C}{}^{B} \qquad (2.9)$$

respectively

Using the Poincaré lemma $(ddw = 0)$ [11] we obtain the two set of Bianchi identities:

$$\mathcal{D}_{\{A}T_{BC\}}{}^{D} + T_{\{AB}{}^{E}T_{EC\}}{}^{D} + R_{\{ABC\}}{}^{D} = 0 \qquad (2.10)$$

$$\mathcal{D}_{\{A}R_{BC\}D}{}^{E} + T_{\{AB}{}^{F}R_{FC\}D}{}^{E} = 0 \qquad (2.11)$$

The curly bracket denotes the cyclic sum over the three enclosed indices.

The identities coming from (2.10) are:

$$\mathcal{D}_{\{a} T_{bc\}}{}^{d} + T_{\{ab}{}^{E} T_{Ec\}}{}^{d} + R_{\{abc\}}{}^{d} = 0 \tag{2.10a}$$

$$\mathcal{D}_{\{a} T_{\beta c\}}{}^{d} + T_{\{a\beta}{}^{F} T_{Fc\}}{}^{d} + R_{\{a\beta c\}}{}^{d} = 0 \tag{2.10b}$$

$$\mathcal{D}_{\{a} T_{\beta\gamma\}}{}^{d} + T_{\{a\beta}{}^{F} T_{F\gamma\}}{}^{d} + R_{\{a\beta\gamma\}}{}^{d} = 0 \tag{2.10c}$$

$$\mathcal{D}_{\{a} T_{bc\}}{}^{\alpha} + T_{\{ab}{}^{F} T_{Fc\}}{}^{\alpha} = 0 \tag{2.10d}$$

$$\mathcal{D}_{\{a} T_{\beta\gamma\}}{}^{\alpha} + T_{\{a\beta}{}^{F} T_{F\gamma\}}{}^{\alpha} + R_{\{a\beta\gamma\}}{}^{\alpha} = 0 \tag{2.10e}$$

$$\mathcal{D}_{\{\alpha} T_{\beta\gamma\}}{}^{d} + T_{\{\alpha\beta}{}^{F} T_{F\gamma\}}{}^{d} = 0 \tag{2.10f}$$

$$\mathcal{D}_{\{\alpha} T_{\beta\gamma\}}{}^{\delta} + T_{\{\alpha\beta}{}^{F} T_{F\gamma\}}{}^{\delta} + R_{\{\alpha\beta\gamma\}}{}^{\delta} = 0 \tag{2.10g}$$

$$\mathcal{D}_{\{a} T_{\beta c\}}{}^{\alpha} + T_{\{a\beta}{}^{F} T_{Fc\}}{}^{\alpha} + R_{\{a\beta c\}}{}^{\alpha} = 0 \tag{2.10h}$$

III. The Curvature Tensor and the Second Set of Bianchi Identities

First we are going to show that we can express the curvature in terms of the torsion.

Equation (2.10h) gives us:

$$R_{ca\beta}{}^{\alpha} = \mathcal{D}_{\{a} T_{\beta c\}}{}^{\alpha} + T_{\{a\beta}{}^{F} T_{Fc\}}{}^{\alpha} \tag{3.1}$$

After some algebra equation (2.10b) gives us:

$$R_{\beta cad} = \frac{1}{2}[\mathcal{D}_{\{a} T_{\beta c\}d} + T_{\{a\beta}{}^{F} T_{Fc\}d} + \mathcal{D}_{\{c} T_{\beta d\}a} + T_{\{c\beta}{}^{F} T_{Fd\}a} -$$

$$- \mathcal{D}_{\{d} T_{\beta a\}c} - T_{\{d\beta}{}^{F} T_{Fa\}c}] \tag{3.2}$$

Equation (2.10c) gives us

$$R_{\beta\gamma a}{}^{d} = \mathcal{D}_{\{a} T_{\beta\gamma\}}{}^{d} + T_{\{a\beta}{}^{F} T_{F\gamma\}}{}^{d} \tag{3.3}$$

Now due to the Lie-algebra valuedness of $R_{ABC}{}^{D}$ we can obtain

$$R_{abcd} \sim (\Gamma_{cd})_{\alpha}{}^{\beta} R_{ab\beta}{}^{\alpha} \tag{3.4}$$

$$R_{a\beta\gamma}{}^{\alpha} \sim (\Gamma^{cd})_{\gamma}{}^{\alpha} R_{a\beta cd} \tag{3.5}$$

$$R_{\alpha\beta\gamma}{}^{\delta} \sim (\Gamma^{cd})_{\gamma}{}^{\delta} R_{\alpha\beta cd} \tag{3.6}$$

where $R_{ab\beta}{}^{\alpha}$, $R_{a\beta cd}$ and $R_{\alpha\beta cd}$ are given by (3.1), (3.2) and (3. 3) respectively. So we have expressed the curvature in terms of torsion components and its covariant derivatives.

We are going to show now that without any restriction on the torsion the second set of Bianchi identities follows from the equations (2.10) and from the equations:

$$[\mathcal{D}_A \mathcal{D}_B] T_{CD}{}^{E} = R_{ABC}{}^{F} T_{FD}{}^{E} + R_{ABD}{}^{F} T_{CF}{}^{E} - R_{ABF}{}^{E} T_{CD}{}^{F} - T_{AB}{}^{F} \mathcal{D}_F T_{CD}{}^{E} \tag{3.7}$$

Following [8] we introduce the abbreviation

$$M_{ABCD}{}^{E} \equiv \mathcal{D}_{\{A} R_{BC\}D}{}^{E} + T_{\{AB}{}^{F} R_{FC\}D}{}^{E} \qquad (3.8)$$

and state the lemma: the equation

$$M_{ABCD}{}^{E} - M_{BCDA}{}^{E} + M_{CDAB}{}^{E} - M_{DABC}{}^{E} = 0 \qquad (3.9)$$

follows from (2.10) and (3.7).

The proof of this lemma is the same of that found in reference[8]. But what we want to prove is that equation (3.9) implies $M_{ABCD}{}^{E} = 0$.

The Lie-algebra valuedness of $M_{ABCD}{}^{E}$ in the indices DE implies the vanishing of $M_{ABCa}{}^{\alpha}$ and $M_{ABC\alpha}{}^{a}$. The remaining equations coming from (3.9) are:

$$M_{ab\gamma\delta\eta} + M_{ab\delta\gamma\eta} = 0 \qquad (3.9a)$$

$$M_{ab\gamma c}{}^{d} - M_{b\gamma ca}{}^{d} + M_{\gamma cab}{}^{d} = 0 \qquad (3.9b)$$

$$-M_{b\gamma\delta a}{}^{c} + M_{a\gamma\delta b}{}^{c} = 0 \qquad (3.9c)$$

Equation (3.9c) implies that $M_{a\gamma\delta b}{}^{c}$ is symmetric in its indices ab and antisymmetric in its indices bc. So it has to vanish.

From (3.9a) we obtain

$$(\Gamma^{cd})_{\delta\eta} M_{ab\gamma cd} + (\Gamma^{cd})_{\gamma\eta} M_{ab\delta cd} = 0 \qquad (3.10)$$

Antisymmetrization of eq. (3.10) in $\gamma\eta$ gives us

$$(\Gamma^{cd})_{\delta\eta} M_{ab\gamma cd} - (\Gamma^{cd})_{\delta\gamma} M_{ab\eta cd} = 0 \qquad (3.11)$$

So eq. (3.10) tell us that

$$(\Gamma^{cd})_{\delta}{}^{\eta} M_{ab\gamma cd} = 0 \tag{3.12}$$

so

$$M_{ab\gamma\delta}{}^{\eta} = 0 \quad \text{and} \quad M_{ab\gamma cd} = 0 \tag{3.13}$$

Thus we have seen that the second set of Bianchi identities does not give any new information which is not already given by the first one, once this result comes from eqs. (2.10) and (3.7).

One usually imposes constraints on the torsion components in order to eliminate some degrees of freedom. In this case the equations (2.10a) and (2.10d-2.10g) which will involve only the torsion, after using equations (3.4), (3.5) and (3.6), can tell us if these constraints are too restrictive or not, i.e., if they imply or not equations of motion.

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