

Sp(2) Covariant Quantisation of General Gauge Theories

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ABSTRACT

This letter studies the $Sp(2)$ covariant quantisation of gauge theories. The geometrical interpretation of gauge theories in terms of quasi principal fibre bundles $Q(M_S, G_S)$ is reviewed. It is then described the $Sp(2)$ algebra of ordinary Yang-Mills theory. A consistent formulation of covariant lagrangian quantisation for general gauge theories based on $Sp(2)$ BRST symmetry is established. The original $N = 1$, ten dimensional superparticle is considered as an example of infinitely reducible gauge algebras, and given explicitly its $Sp(2)$ BRST invariant action.

Key-words: Gauge theories; BRST symmetry; Fibre bundles; Covariant quantisation

1. Gauge theories in terms of quasi-principal fibre bundles.

Gauge theories have a nice geometrical interpretation in terms of connections on a principal fibre bundle (pfb) $P(M, G)$, where M is the base space-time manifold and G is the gauge group [1,2, 3, 4]. However, quantisation of gauge theories requires the introduction of fields (c_m^n, π_m^n) . It would be then desirable to have a formalism where those extra fields fit into some representation of a larger group and all the fields are components of a superfield. This is a step in the direction of recovering a geometrical interpretation of quantum gauge theories. The main ingredients in the construction of geometrical quasi-principal fibre bundles (qpfb) are a space-time base manifold M , a gauge group G , an extended superspace manifold M_S which is obtained by adding two extra Grassmann variables θ^a ($a = 1, 2$) to M , in the case of $Sp(2)$ symmetry, and a supergroup G_S . The construction is performed basically in three steps [1,2, 3]. It starts with a pfb $P(M, G)$ and extend the gauge group G to a supergroup G_S . The composition of G with a Grassmann algebra B prolongs $P(M, G)$ to a pfb $P'(M, G_S)$. The most general supergroup G_S can be represented in matrix form. In particular, $OSp(N/M)$ groups are represented by block matrices of the form

$$\begin{pmatrix} A & E \\ C & D \end{pmatrix} \quad (1.1)$$

where A, D are $(N \times N)$ and $(M \times M)$ matrices whose elements are taken from the even part of the Grassmann algebra B constructed over a complex vector space W , whilst E, C are $(N \times M)$ and $(N \times M)$ rectangular matrices whose elements belong to the odd part of B . Next, it is enlarged the base space manifold M to a superspace M_S in $P'(M, G_S)$ by adding Grassmann variables. At this stage, a pfb $P''(M_S, G_S)$ is obtained. Finally, the pfb $P''(M_S, G_S)$ is transform into a quasi-principal fibre bundle $Q(M_S, G_S)$. For instance, given a one-form valued function $\alpha(x) = A_\mu dx^\mu$ on M this induces a connection ω on the pfb $P(M, G)$. Then a one-form valued function α' on M_S is found by

$$\alpha'(x, \theta^a) = g^{-1} A_\mu dx^\mu g + g^{-1} dg \quad (1.2)$$

where $g = g(x^\mu, \theta^a)$, ($a = 1, 2, \dots$) which induces a connection ω' on the qpfb $Q(M_S, G_S)$ [1,2,3,4].

2. The $Sp(2)$ BRST Algebra of Yang-Mills Theory.

It has been realized for some time [5,6,7,8] that a geometrical construction can be useful for the discussion of BRST and anti-BRST symmetry. The idea is to use a superspace with coordinates $Z^M = (x^\mu, \theta^a)$, where ($a = 1, 2$) and θ^a is an anti-commuting scalar coordinate and the BRST generators s^a are realized as differential operators on superspace, $s^a = \frac{\partial}{\partial \theta^a}$, so that $s^a s^b + s^b s^a = 0$ holds automatically¹. For example, in Yang-Mills theory the gauge potential A_μ^i and the Faddeev-Popov ghost $(c^a)^i$ (where i is an adjoint group

¹ It is usually defined a bosonic operator $\sigma = \frac{1}{2} \epsilon_{ab} s^a s^b$ where ϵ_{ab} is the symplectic invariant form of $Sp(2)$, so that $\epsilon_{ab} = -\epsilon_{ba}$, $\epsilon^{ab} \epsilon_{bc} = \delta_{ac}$ and $\epsilon_{12} = 1$. The generator σ is invariant

index) can be combined into a super-gauge field $\mathcal{A}_M^i(Z)$ whose lowest order components are $\mathcal{A}_M^i(Z)|_{\theta^a=0} = (\mathcal{A}_\mu^i, \mathcal{A}_{\theta^a}^i)|_{\theta^a=0} = (A_\mu^i, c^a)$. Then the standard Yang-Mills BRST transformations arise from imposing the constraints $\mathcal{F}_{\mu\theta^a}^i = 0$, $\mathcal{F}_{\theta^a\theta^b}^i = 0$ on the superfield strength \mathcal{F}_{MN}^i [5,6]. This gives an elegant geometrical description of BRST and anti-BRST symmetry.

Let us review the construction of gauge theories in the superspace with coordinates $Z^M = (x^\mu, \theta^a)$, which gives a geometric formulation of $Sp(2)$ BRST symmetry. We consider matter fields $\Phi^i(x, \theta^a)$ and a gauge potential $\mathcal{A}_M^i(x, \theta^a) = (\mathcal{A}_\mu^i(x, \theta^a), \mathcal{A}_{\theta^a}^i(x, \theta^a))$. These can be used to define a covariant derivative

$$\mathcal{D}_M \Phi^i = \partial_M \Phi^i - (T^k)^i_j \mathcal{A}_M^k \Phi^j \quad (2.1)$$

and the field strength

$$\mathcal{F}_{MN}^i = \partial_M \mathcal{A}_N^i - (-1)^{MN} \partial_N \mathcal{A}_M^i + f^i_{jk} \mathcal{A}_M^j \mathcal{A}_N^k \quad (2.2)$$

where $(-1)^{MN}$ is 1 unless both M and N are indices referring to anti-commuting coordinates, in which case it is -1 . The gauge potential \mathcal{A}_M contains more component fields than the physical gauge and ghost fields and so, as in supersymmetric theories, constraints should be imposed on the field strength \mathcal{F} . Appropriate constraints are [5,6]

$$\mathcal{F}_{\theta^a\theta^b} = 0, \quad \mathcal{F}_{\mu\theta^a} = 0. \quad (2.3)$$

These can be written more explicitly as

$$\partial_\mu \mathcal{A}_{\theta^a} - \partial_{\theta^a} \mathcal{A}_\mu + [\mathcal{A}_\mu, \mathcal{A}_{\theta^a}] = 0 \quad (2.4)$$

$$\partial_{\theta^a} \mathcal{A}_{\theta^a} + \frac{1}{2} [\mathcal{A}_{\theta^a}, \mathcal{A}_{\theta^a}] = 0 \quad (2.5)$$

$$\epsilon^{ab} \left(\partial_{\theta^a} \mathcal{A}_{\theta^b} + [\mathcal{A}_{\theta^a}, \mathcal{A}_{\theta^b}] \right) = 0. \quad (2.6)$$

Defining the component expansions

$$\mathcal{A}_\mu(x, \theta^a) = A_\mu(x) + \theta^a \Lambda_{a\mu}(x) + \theta^a \theta^b \Omega_{ab\mu}(x) \quad (2.7)$$

$$\mathcal{A}_{\theta^a}^b(x, \theta^a) = c^b(x) + \theta^a \Upsilon_a^b(x) + \theta^a \theta^c \omega_{ac}^b(x), \quad (2.8)$$

however, it was found that if A_μ , c^a , π are identified with the gauge, ghost (anti-ghost) and auxiliary fields respectively then the supergauge fields have the expansions *

$$\mathcal{A}_\mu = A_\mu + \theta^a (s^a A_\mu) + \theta^a \theta^b (s^a s^b A_\mu) \quad (2.9)$$

under $Sp(2)$ and satisfies $s^a \sigma = 0$. The $Sp(2)$ generators σ^i ($i = \pm, 0$) and the fermionic charges s^a together form an algebra which is a contraction of $OSp(1,1/2)$ and denoted as $ISp(2)$ [9].

* In Ref [6], it was obtained explicitly a geometrical formulation of BRST and anti-BRST symmetries and given the field content of Λ , Υ , Ω and ω . The components in the expansion can also be read as conditions on the mapping of the coordinates ϕ^i of the fibres over $\{\mathcal{U}_i\}$ (covering set of M_G) and expressed as cocycle conditions.

$$A_{\theta^a}^b = c^b + \theta^a (s^a c^b) + \theta^a \theta^c (s^a s^c c^b). \quad (2.10)$$

The BRST and anti-BRST generators s^a are then identified with the superspace differential operators ∂_{θ^a} [5,6], and the complete set of BRST and anti-BRST transformations are given by [10,11,12]

$$\begin{aligned} s^a \phi^i &= R_\alpha^i c^{\alpha a}, & s^a c^{\alpha b} &= \epsilon^{ab} \pi^\alpha - \frac{1}{2} f_{\beta\gamma}^\alpha c^{\beta a} c^{\gamma b}, \\ s^a \pi^\alpha &= \frac{1}{2} f_{\beta\gamma}^\alpha \pi^\beta c^{\gamma a} - \frac{1}{12} (f_{\beta\gamma}^\alpha f_{\delta\tau}^\beta + f_{\delta\tau, i}^\alpha R_\gamma^i) c^{\delta a} c^{\tau e} \epsilon_{eb} c^{\gamma b}, \end{aligned} \quad (2.11)$$

where the generators R_α^i for the gauge field A_μ^i read off from $s^a A_\mu^i = (D_\mu c^a)^i$, and π^α is an auxiliary field which connects ghost and antighost sectors.

For the matter fields $\Phi^i(x, \theta^a)$, we impose the constraint

$$D_{\theta^a} \Phi^i = \partial_{\theta^a} \Phi^i - (T^k)^i_j A_{\theta^a}^k \Phi^j = 0 \quad (2.12)$$

which implies

$$\Phi^i = \psi^i + \theta^a (s^a \psi^i), \quad (2.13)$$

and the BRST and anti-BRST transformations again corresponds to translations in the θ^a direction with $\frac{\partial}{\partial \theta^a}$ realized as differential operators on the extended space manifold M_S .

3. $Sp(2)$ formalism for General gauge theories.

Consider a general gauge theory with classical fields $A^i(x^\mu)$ ($i = 1, 2, \dots, n$) and classical action $S_0(A^i)$. The action is invariant under gauge transformations

$$\delta A^i = R_\alpha^i \xi^\alpha, \quad (3.1)$$

where ξ^α is the local gauge parameter. The Noether equations are given by

$$S_{0, i} R_\alpha^i = 0, \quad \alpha = 1, 2, \dots, m \quad (0 \leq m \leq n), \quad (3.2)$$

and the generators of the gauge transformations satisfy

$$R_{\alpha, j}^i R_\beta^j - (-)^{\alpha\beta} R_{\beta, j}^i R_\alpha^j = -R_\gamma^i f_{\alpha\beta}^{\gamma} - S_{0, j} M_{\alpha\beta}^{ij}. \quad (3.3)$$

To construct a covariant lagrangian formalism for general gauge theories either with open ($M_{\alpha\beta}^{ij} \neq 0$) or closed algebras, and based on $Sp(2)$ BRST symmetry, it is needed to enlarge the base manifold M to M_S . It is then defined a superspace M_S to include classical fields Φ^A and $Sp(2)$ doublets of anti-fields Φ_A^* , Φ_A^{**} [13,14,15]. The properties of these fields and anti-fields are $\epsilon(\Phi^A) = \epsilon_A$, $\epsilon(\Phi_A^*) = \epsilon_A + 1$, $\epsilon(\Phi_A^{**}) = \epsilon_A$, $gh(\Phi_A^*) = (-)^a - gh(\Phi^A)$ and $gh(\Phi_A^{**}) = -gh(\Phi^A)$. An extended Poisson superbracket is defined by

$$(F, G) = \frac{\delta_r}{\delta \Phi^A} F \frac{\delta_l}{\delta \Phi_A^*} G - (-)^{\epsilon_F \epsilon_G} \frac{\delta_r}{\delta \Phi^A} G \frac{\delta_l}{\delta \Phi_A^*} F, \quad (3.4)$$

where ϵ_F, ϵ_G denotes the Grassmann parity of the F, G functions on M_S , and left (right) derivatives are understood with respect to anti-fields (fields) unless otherwise stated. The extended anti-bracket (3.4) satisfies

$$\epsilon((F, G)) = \epsilon(F) + \epsilon(G) + 1 \quad (3.5)$$

$$gh((F, G)^a) = -(-)^a + gh(F) + gh(G), \quad a = 1, 2 \quad (3.6)$$

$$(F, G) = -(-)^{\epsilon_F \epsilon_G} (G, F) \quad (3.7)$$

and

$$(-)^{\epsilon_F} \epsilon_G((F, G), H) + [\text{cycl. perm } (F, G, H)] = 0. \quad (3.8)$$

A bosonic action functional $S = S(\Phi^A, \Phi_A^*, \Phi_A^{**})$ is constructed on M_S . This action satisfy the following *generating equation*

$$\bar{\Delta}^a \exp \frac{i}{\hbar} S(\Phi^A, \Phi_A^*, \Phi_A^{**}) = 0, \quad (a = 1, 2), \quad (3.9)$$

together with the boundary condition

$$S(\Phi^A, \Phi_A^*, \Phi_A^{**}) \Big|_{\Phi_A^* = \Phi_A^{**} = 0} = S(\Phi_A). \quad (3.10)$$

The operator $\bar{\Delta}^a$ is defined by

$$\bar{\Delta}^a = \Delta^a + (i/\hbar)V^a, \quad \Delta^a = (-)^{\epsilon_A} \frac{\delta}{\delta \Phi^A} \frac{\delta_l}{\delta \Phi_{Aa}^*} \quad V^a = \epsilon^{ab} \Phi_{Ab}^* \frac{\delta_r}{\delta \Phi_A^{**}}. \quad (3.11)$$

The algebra of operators (3.11) satisfy the important property ²

$$\bar{\Delta}^a \bar{\Delta}^b = 0. \quad (3.12)$$

The solution to the generating equation (3.9) is given as a power series of the Planck constant

$$S(\Phi^A, \Phi_A^*, \Phi_A^{**}) = \sum_{n=0}^{+\infty} \hbar^n S_{(n)}, \quad (3.13)$$

where the classical approximation $S_{(0)}$ satisfies

$$\frac{1}{2}(S_{(0)}, S_{(0)})^a + V^a S_{(0)} = 0. \quad (3.14)$$

For a theory in which the $Sp(2)$ algebra closes off-shell the classical solution $S_{(0)}$ takes the form

$$S_{(0)} = S_0 + \Phi_{Aa}^* s^a \Phi^A + \frac{1}{2} \Phi_A^{**} \epsilon_{ab} s^a s^b \Phi^A + F_{AB} \epsilon_{ab} s^a \Phi^A s^b \Phi^B. \quad (3.15)$$

For more complicated theories like superparticles or superstrings, $S_{(0)}$ has terms of higher order in the fields Φ_A^*, Φ_A^{**} to compensate those terms which makes the $Sp(2)$ algebra to close on-shell. The classical solution to the generating equation is $Sp(2)$ BRST invariant under modified BRST generators \tilde{s}^a which satisfy $\tilde{s}^a \tilde{s}^b + \tilde{s}^b \tilde{s}^a = 0$.

² A supercommutative, associative algebra \mathcal{A} equipped with an extended Poisson anti-bracket structure plus a nilpotent property it is known as a BV-algebra, or coboundary Gersterhaber algebra (CGA) [16].

4. Orthosymplectic Structure of the Original BSC Superparticle.

The original BSC superparticle S_{BSC} and further models are known to yield the same spectrum as that of $D = 10$, $N = 1$ super-Yang-Mills theory [17]. It is used here as an illustrative example to construct its $Sp(2)$ covariant lagrangian, since the model has an infinitely reducible algebra. The BSC superparticle action is given by [18]

$$S_0 = \int d\tau [p_\mu \dot{x}^\mu - i\theta \dot{\theta} - \frac{1}{2} e p^2]. \quad (4.1)$$

This action describes a particle with world-line parametrized by τ moving through a ten-dimensional $N = 1$ superspace with coordinates (x^μ, θ_A) . The superparticle action S_{BSC} is invariant under a 10 dimensional super-Poincaré symmetry

$$\delta\theta = \epsilon, \quad \delta x^\mu = i\epsilon \Gamma^\mu \theta, \quad (4.2)$$

together with world-line reparametrisations and a local fermionic symmetry

$$\begin{aligned} \delta\theta &= \not{p}\kappa, & \delta e &= 4i\kappa\dot{\theta} + \dot{\xi}, \\ \delta x^\mu &= i\theta \Gamma^\mu \not{p}\kappa + \xi p_\mu. \end{aligned} \quad (4.3)$$

The Grassmann spinor κ_A parametrizes the local symmetry while ξ parametrizes a linear combination of world-line diffeomorphisms and a local *trivial* local symmetry. To construct a covariant $Sp(2)$ orthosymplectic structure for this model, it is required the formalism of the previous section since the classical infinitely reducible gauge algebra \mathcal{A} closes on-shell. It is then defined a superspace M_S to include the classical fields $\Phi^A = (x^\mu, p_\mu, e, \theta_A)$ and $Sp(2)$ doublets of anti-fields Φ_A^* , Φ_A^{**} . The classical approximation $S_{(0)}$ which satisfies (3.14) is given by

$$S_{(0)} = S_{BSC} + S_1 + S_2 + S_3, \quad (4.4)$$

where S_{BSC} is the classical action of the original superparticle and S_1 , S_2 , and S_3 are

$$\begin{aligned} S_1 &= \int d\tau [\theta_a^* \not{p}\kappa_1^a + e_a^* (4i\kappa_1^a \dot{\theta} + \dot{c}^a) + \kappa_{nab}^* ((-)^n \not{p})(f_c^{ab} \kappa_{n+1}^c + \epsilon^{ab} \pi_n) \\ &\quad + x_{\mu a}^* (i\theta \gamma^\mu \not{p}\kappa_1^a + p_\mu c) + c_{da}^* (-2i f_{rs}^{ad} \kappa_1^r \not{p}\kappa_1^s + \epsilon^{ad} \pi)], \end{aligned} \quad (4.5)$$

$$\begin{aligned} S_2 &= \int d\tau [\theta^{**} (-p^2 \epsilon_{ab} f_c^{ab} \kappa_2^c) + e^{**} (-4i \epsilon_{ab} f_c^{ab} \kappa_2^c \not{p}\dot{\theta} + 2i \epsilon_{ab} \kappa_1^a \not{p}\kappa_1^b) \\ &\quad + \kappa_{nr}^{**} (-p^2) (\epsilon_{ab} f_c^{br} f_a^{ac} \kappa_{n+2}^s + f_b^{br} \pi_{n+1}) \\ &\quad + x_\mu^{**} (ip^2 \epsilon_{ab}) (f_c^{ab} \theta \gamma^\mu \kappa_2^c - \kappa_1^a \gamma^\mu \kappa_1^b) \\ &\quad + c_e^{**} (-4ip^2) (\epsilon_{ab} f_{rs}^{be} f_c^{as} \kappa_1^r \kappa_2^c + f_{rb}^{be} \kappa_1^r \pi_1)], \end{aligned} \quad (4.6)$$

and

$$\begin{aligned} S_3 &= \int d\tau \frac{1}{2} e_a^* [\theta_b^* (-\kappa_2^c) (2f_c^{ab} + \epsilon^{ab} \epsilon_{rs} f_{rs}^c) \\ &\quad + x_{\mu b}^* (2i\kappa_1^a \gamma^\mu \kappa_1^b + i\epsilon^{ab} \epsilon_{rs} \kappa_1^r \gamma^\mu \kappa_1^s - i\theta \gamma^\mu \kappa_2^c) (2f_c^{ab} + \epsilon^{ab} \epsilon_{rs} f_{rs}^c) \\ &\quad + \kappa_{nAb}^* (-\kappa_{n+2}^c (\epsilon^{ab} \epsilon_{ps} f_q^{sA} f_p^{pq} + 2f_s^{Ab} f_c^{as}) - \pi_{n+1} (2\epsilon^{as} f_s^{Ab} + \epsilon^{ab} f_p^{pA})) \\ &\quad - 4i c_{Ab}^* (\kappa_1^r \kappa_2^c (\epsilon^{ab} \epsilon_{ps} f_{rq}^{sA} f_p^{pq} + 2f_{rs}^{bA} f_c^{as}) + \kappa_1^r \pi_1 (\epsilon^{ab} f_{rs}^{bA} + \epsilon^{ab} f_{qp}^{pA}))]. \end{aligned} \quad (4.7)$$

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