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by

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Abstract

Using dimensional regularization, the one-loop approximation for the effective potential (finite temperature) is computed as an analytic function of the number of dimensions. It is shown that a simple relation exists between potentials for different dimensions. This relation reduces to a simple derivative when these numbers differ by two units. The limit of zero temperature is calculated and also the finite temperature corrections are given.

Key-word: Effective potentials.

By using dimensional regularization we express the one-loop contribution to the effective potential [1], as an analytic function of the dimension ν.

We will then be able to find some of its general properties and also interesting relations between different dimensions. In what follows we will consider a $\lambda\phi^4$ theory.

The one loop approximation in ν spacial dimensions for finite temperature $T = \beta^{-1}$ (β = period in time of the field) is given by [2]:

$$U_\nu(\beta,\psi) = \frac{1}{\beta} \sum_{n=-\infty}^{+\infty} \int \frac{d^\nu p}{(2\pi)^\nu} \ln\left[p^2 + 4\pi^2 n^2 T^2 + \psi^2\right] \tag{1}$$

where $\psi^2 = \frac{\lambda}{2}\phi^2 - \mu^2$

For each n we have ([3] p. 563)

$$I_n = \int d^\nu p \ \ln(p^2 + \alpha_n^2) = \frac{2\Pi^{\frac{\nu}{2}}\alpha_n^\nu}{\Gamma(\frac{\nu}{2})} \int_0^\infty dp \ p^{\nu-1} \ln(p^2+1) = -\alpha_n^\nu \Pi^{\frac{\nu}{2}} \Gamma(-\tfrac{\nu}{2}) \tag{2}$$

Using (2) we can write (1) as:

$$U_\nu(\beta,\phi) = -\frac{\Gamma(-\frac{\nu}{2})}{(4\pi)^{\nu/2}\beta} \sum_n (4\pi^2 n^2 T^2 + \psi^2)^{\frac{\nu}{2}} \tag{3}$$

which by term by term derivation is easily seen to obbey

$$\frac{dU_\nu}{d\psi^2} = -\frac{1}{4\pi} U_{\nu-2} = -\frac{dU_\nu}{d\mu^2} \tag{4}$$

We will now write $U_\nu(\beta,\phi)$ in a way more suitable for actual calculations. We shall use (Ref. [3] p. 713)

$$\sum_{n=1}^{\infty} (n^2\pi^2 + A^2)^{-\mu-\frac{1}{2}} = \frac{\sqrt{\pi}}{(2A)^{\mu}\Gamma(\mu+\frac{1}{2})} \int_0^{\infty} \frac{J_{\mu}(Ax)}{e^{\pi x}-1}\, dx \qquad (5)$$

Using (5) in (3) we get, after separation of the $n = 0$ term:

$$U_{\nu} = - \frac{\Gamma(-\frac{\nu}{2})}{(4\pi)^{\frac{\nu}{2}}\beta} \psi^{\nu} - 2\Pi^{\frac{1-\nu}{2}} (\frac{\psi}{\beta})^{\frac{1+\nu}{2}} \int_0^{\infty} \frac{x^{\frac{\nu+1}{2}}}{e^{\pi x}-1} J_{-\frac{\nu+1}{2}} (\frac{\beta\psi x}{2})\, dx \qquad (6)$$

From here, it is easy to prove eq. (4), by using a Bessel function recurrence relation. (Ref. [3] form 2.p. 968).

Now we use the integral representation (Ref. [3] p. 953 form 9)

$$J_{-\alpha}(\chi\mu) = (\frac{\chi}{2\mu})^{\alpha} \frac{1}{\Gamma(\alpha+\frac{1}{2})} \frac{2}{\sqrt{\pi}} \int_0^{\infty} (t^2 - \mu^2)^{\alpha-\frac{1}{2}} \sin \chi t\, dt \qquad (7)$$

in eq. (6). Integrating first over the χ-variable and using (Ref. [3] p. 494 form. 12)

$$\int_0^{\infty} \frac{dx \,\mathit{sen}\, x t}{e^{\pi x}-1} = \frac{1}{2} \mathit{cotgh}\, t - \frac{1}{2t} \qquad (8)$$

eq. (6) takes the form:

$$U_{\nu} = - \frac{\Gamma(-\frac{\nu}{2})}{(4\pi)^{\frac{\nu}{2}}\beta} \psi^2 - \frac{2}{\beta^{1+\nu}\Pi^{\frac{\nu}{2}}\Gamma(1+\frac{\nu}{2})} \int_u^{\infty} dt\,(t^2 - u^2)^{\frac{\nu}{2}} (\mathit{cotgh}\, t - \frac{1}{t}) \qquad (9)$$

with $u = \frac{\beta\psi}{2}$

The integral involving the last term t^{-1} is seen to compensate exactly the term in ψ^{ν} (Ref. [3] p. 295, form. 3)

We are then left with:

$$U_\nu(\beta,\psi) = - \frac{2}{\beta^{1+\nu}\Pi^{\frac{\nu}{2}}\Gamma(1+\frac{\nu}{2})} \int_{u=\frac{\beta\psi}{2}}^{\infty} dt\,(t^2 - u^2)^{\frac{\nu}{2}} cotgh\, t \qquad (10)$$

Eq. (10) can be written in a more compact form by using Liouville's concept of "fractional derivative" [4]

$$U_\nu(\beta,\psi) = \frac{1}{\beta^{1+\nu}(\pi e^{i\pi})^{\frac{\nu}{2}}} \frac{d^{-\frac{\nu}{2}-1}}{(du^2)^{-\frac{\nu}{2}-1}} \left[\frac{cthu}{u}\right]_{(u = \frac{\beta\psi}{2})} \qquad (11)$$

Or also

$$U_\nu = \frac{1}{\beta^{\nu}(\pi e^{i\pi})^{\frac{\nu}{2}}} \frac{d^{-\frac{\nu}{2}}}{(du^2)^{-\frac{\nu}{2}}} U_o ;$$

$$U_o = \frac{1}{\beta} \frac{d^{-1}}{(du^2)^{-1}} \left[\frac{cthu}{u}\right]; U_o = \frac{1}{\beta}\, \ell n\, sh\, \frac{\beta\psi}{2} \qquad (12)$$

Equivalently, we may take derivatives with respect to ψ^2, obtaining again

$$\frac{dU_\nu}{d\psi^2} = - \frac{1}{4\pi} U_{\nu-2} \qquad (13)$$

Or, more generally:

$$\frac{d^\alpha}{(d\mu^2)^\alpha} U_\nu = \frac{1}{(4\pi)^\alpha} U_{\nu-2\alpha} \qquad (14)$$

Going back to eq. (10), we note that it is easy to compute the limit $\beta \to \infty$ (zero temperature), as only $t > \frac{\beta\psi}{2}$ contribute to the integral. So that we can take $\coth t = 1$, and use formula 3, p. 295, of Reference [3]:

$$U_\nu \xrightarrow[\beta\to\infty]{} - \left(\frac{\psi}{2\sqrt{\pi}}\right)^{\nu+1} \Gamma\left(-\frac{\nu+1}{2}\right) \tag{15}$$

It is obvious that (15) satisfies (13).

When the number of dimension $(n = \nu + 1)$ is even, there is a pole in the Γ-function of (15). We then use the well known procedure to take the finite part of an analytic function at a pole $\bar{\nu}$; i.e.:

$$\mathrm{Pf.}f(\nu)\Big|_{\nu=\bar{\nu}} = \frac{d}{d\nu}\left[(\nu-\bar{\nu})f(\nu)\right]\Big|_{\nu=\bar{\nu}} \tag{16}$$

The finite part of (15) is defined by (16) plus a term proportional to the residue which is taken care of by the renormalization procedure.

With (16) we get, for n even:

$$U_\nu \xrightarrow[\nu+1=n]{} \psi^n \ell n\, \psi \qquad (n \text{ even}) \tag{17}$$

For $n = 0$, in particular, ν goes to -1. It is amusing that this result corresponds, according to form (1), to a space of dimension -1.

Back to (10). We can use

$$\coth x = 1 + 2 \sum_{p=1}^{\infty} e^{-px}$$

and remembering that: Ref. [3] p. 322 for 6.

$$\int_\mu^\infty \frac{(x^2-\mu^2)^{\frac{\nu}{2}}}{\Gamma(1+\frac{\nu}{2})} e^{-px}\, dx = \frac{1}{\sqrt{\pi}} \left(\frac{2\mu}{p}\right)^{\frac{\nu+1}{2}} K_{\frac{\nu+1}{2}}(p\mu) \tag{18}$$

We get finally

$$U_\nu = -\frac{4\pi^{-\frac{\nu+1}{2}}}{\beta^{1+\nu}} \sum_{p=1}^{\infty} \left(\frac{\beta\psi}{p}\right)^{\frac{\nu+1}{2}} K_{\frac{\nu+1}{2}}\left(\frac{p\beta\psi}{2}\right) - \left(\frac{\psi}{2\sqrt{\pi}}\right)^{\nu+1} \Gamma\left(-\frac{\nu+1}{2}\right) \qquad (19)$$

(remember (17) when $(\nu+1)$ is even)

(19) can also be looked upon as a low temperature expansion [6]

We want to point out that a similar relation to (4) exists also for the Green function $\Delta_\nu(\mu^2)$ of the Klein Gordon equation; i.e:

$$\frac{d}{d\mu^2} \Delta_\nu(\mu^2,x) = \pi^2 \Delta_{\nu-2}(\mu^2,x) \qquad (20)$$

As can be verified by direct calculation. (See [4] p. 362).

The expression (10) can be used for the calculation of the transition temperature as a function of the number of dimensions; i.e. the value of β at which the relative minimum disappears [5].

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