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Non-Local Theory of Gravity

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ABSTRACT

We use a three-index tensor $A_{\mu\nu\alpha}$ to describe spin-two fields. This quantity acts as the potential of the standard variable $h_{\mu\nu}$. We apply the new variables to examine the coherence of a non-local non-linear theory of gravity.

1 Introduction

In the last few years the task of formulating a complete linear theory of spin-two field using a three-index tensor $A_{\mu\nu\alpha}$ [1] has been undertaken. In a series of papers its dynamics and the corresponding quantum formulation was presented.The study of exceptional Lagrangeans (in the sense of Lax [3]) has shown how to construct a particular example of a non-linear theory of the spin-two field $A_{\mu\nu\alpha}$ resulting in a model very similar to the one introduced by Born and Infeld for spin one field. All these previous works considered the free field. The purpose of the present paper is to examine the interaction with matter and to consider some models of non-linearity. We show that the use of the variable $A_{\mu\nu\alpha}$ sets a natural framework to deal with a non-local theory of gravity. We contemplate an example of a non-local theory and show how it modifies the background geometry.

Before going into this let us make a brief comment concerning the insertion of the present paper among some other related ones. In the last years a certain number of proposals were made concerning the alternative description of gravity through the use of connections (usually presented in its spinorial version [4] [5]) instead of the traditional Einstein's approach that deals with two index symmetric tensor (interpreted as the metric tensor of the spacetime) as the basic variable. This connection appears as a modification of the derivative in order to restore a lost symmetry induced by the passage from a global to a local group invariance through a typical gauge-like procedure. We will not follow this road here. Nevertheless we shall use the new variable as a really fundamental one. This means that the associated two-tensor $h^{\mu\nu}$ (which will give origin to a modification of the metric tensor) is obtained as a function of derivatives of the basic field $A_{\mu\nu\alpha}$.

2 Synopsis

Although one may regard the curvature of spacetime as an evidence, in Einstein's model, of a non-local gravitational contribution to the total energy-momentum tensor [6],we can take General Relativity (GR) as an example of a non-linear local field theory. A dramatic consequence of this is that the gravitational field can exist without any source. This means that it is conceivable the existence of a Universe (i.e.,a global structure) without any matter whatsoever (see, for instance, the solution found by Kasner [7]). As such structure is an unobservable, one should wonder if it is possible to keep most of the GR without allowing the existence of such kind of completely empty self-gravitating Universe. We shall see that the use of the new variable as the fundamental one to describe the gravitational field will lead us naturally to a class of non-local non-linear theories of gravity where the presence of matter is essential to induce the non-linearity and to the consideration of more general types of non-linearity having a local origin. Our results will then be re-written in terms of the standard two-index tensor $h_{\mu\nu}$.

A very particular example of the theory which we consider is equivalent to a model presented some years ago [9] as will be shown.

The presentation of the paper is the following. In the next section we give a short

[1]This variable is not a new one. It has been introduced by Fierz many years ago but only recently a complete study of its dynamics was considered [1] [2].

resumé of the main properties of the variable $A_{\mu\nu\lambda}$, its relationship with the standard one and its use in previous works. From $A_{\mu\nu\lambda}$ we define the tensor field $F_{\mu\nu\alpha\beta}$ following the spin-one analogy. Its algebraic properties and the gauge symmetry under an internal Poincaré mapping are then exhibited. Some possible candidates for the dynamics are presented.

In section 4 we consider the interaction with matter represented here for the sake of simplicity by a scalar field. The generalization to other types of substance does not present further difficulty. (This will be described in a future publication). Here, the main lesson we learn is that for the kind of interaction of the field $A_{\mu\nu\alpha}$ with matter that we consider, the gravitational field appears only in the particular form

$$\sqrt{-g}g^{\mu\nu} = \sqrt{-\gamma}(\gamma^{\mu\nu} + h^{\mu\nu}).$$

As it will be shown later on, $h_{\mu\nu}$ is a given linear combination of derivatives of the fundamental field $A_{\mu\nu\alpha}$. Thus the net effect of the interaction can be described by means of a modification of the background geometry $\gamma_{\mu\nu}$ into an effective one $g_{\mu\nu}$ generated by a particular combination of the derivatives of $A_{\mu\nu\alpha}$. One is then compelled to examine the equation of motion that this particular combination of the field $A_{\mu\nu\alpha}$, which we represent by $h_{\mu\nu}$, satisfies. This is the subject of section 5. We compare the present theory with some others and discuss the non-observability of $\gamma_{\mu\nu}$. We end in section 6 with some conclusions and future proposals.

2.1 Notation

The background metric, $\gamma_{\mu\nu}$, is taken to be the Minkowski one. We shall see that this is just a simplification devoid of any further observational consequence. Thus all indices are lowered and raised by means of $\gamma_{\mu\nu}$. We set $\gamma \equiv det\gamma_{\mu\nu}$. A semi-colon (;) denotes the covariant derivative with respect to the background metric.

The symmetrization and antisymmetrisation symbols will be denoted, respectively as:

$$T_{(\mu\nu)} \equiv T_{\mu\nu} + T_{\nu\mu}$$

$$T_{[\mu\nu]} \equiv T_{\mu\nu} - T_{\nu\mu}.$$

The symbol $*$ means the dual. Let $F_{\mu\nu}$ be an arbitrary antisymmetric tensor. We define

$$F^*_{\mu\nu} \equiv \frac{1}{2}\eta_{\mu\nu}^{\alpha\beta}F_{\alpha\beta},$$

in which $\eta_{\mu\nu}^{\alpha\beta}$ represents the completely antisymmetric Levi-Civita symbol. We work in the natural system of unities ($\hbar = c = 1$) where the dimensionality of $A_{\mu\nu\lambda}$ is $(length)^{-1}$.

3 The Fundamental Variable $A_{\mu\nu\alpha}$

Let $A_{\mu\nu\lambda}$ be a three-index tensor that satisfy the following algebraic identities :

$$A_{\alpha\beta\mu} + A_{\beta\alpha\mu} = 0 \tag{1}$$

$$A_{\alpha\beta\mu} + A_{\beta\mu\alpha} + A_{\mu\alpha\beta} = 0. \tag{2}$$

Thus defined this quantity has twenty independent components. In [2] it was shown that it describes two spin-two fields, one of which is a pseudo-tensor. Throughout this paper we will take $A_{\alpha\beta\mu}$ as the fundamental quantity which will be used to represent spin-two fields. However, in order to make contact with the standard formulation we define two derived quantities $\Phi_{\mu\nu}$ and $\Psi_{\mu\nu}$ by setting

$$\Phi_{\mu\nu} = A_{(\mu}{}^{\epsilon}{}_{\nu);\epsilon} + \eta A_{(\mu;\nu)} + \xi A^{\lambda}{}_{;\lambda}\gamma_{\mu\nu} \tag{3}$$

$$\Psi_{\mu\nu} = A^{*}_{(\mu\epsilon\nu)}{}^{;\epsilon}, \tag{4}$$

in which we used $A_{\mu} = A_{\mu\nu}{}^{\nu}$. It was shown [1] that $\Phi_{\mu\nu}$ and $\Psi_{\mu\nu}$ constitute respectively, the two independent spin-two tensor and pseudo-tensor constructed from $A_{\mu\nu\alpha}$. We take the trace of (3) to obtain

$$\Phi = 2(2\xi + \eta - 1)A^{\mu}{}_{;\mu}. \tag{5}$$

For reasons which will become clear later on in section 5 it is useful to define the quantity $h_{\mu\nu}$:

$$h_{\mu\nu} \equiv \Phi_{\mu\nu} - \frac{1}{2}\Phi\gamma_{\mu\nu}. \tag{6}$$

Then, it follows that

$$h_{\mu\nu} = A_{(\mu}{}^{\epsilon}{}_{\nu);\epsilon} + \eta A_{(\mu;\nu)} + \frac{(1-\eta-\xi)}{2(1-\eta-2\xi)} h\gamma_{\mu\nu}. \tag{7}$$

in which the trace of $h_{\mu\nu}$ is given by

$$h = 2(1-\eta-2\xi)A^{\alpha}_{;\alpha}. \tag{8}$$

3.1 The Standard Representation-I

In order to eliminate one of the two spin-two fields contained in $A_{\mu\nu\beta}$ we can use the definitions given in the last section for $\Phi_{\mu\nu}$ and/or $\Psi_{\mu\nu}$. In the case we eliminate the pseudo tensor we must set:

$$\Psi_{\mu\nu} = 0, \tag{9}$$

which by the definition eq (4) is equivalent to

$$A^{*}_{\mu\epsilon\nu}{}^{;\epsilon} + A^{*}_{\nu\epsilon\mu}{}^{;\epsilon} = 0. \tag{10}$$

If we multiply eq(10) by the Levi-Civita symbol we get the expression:

$$2[A_{\lambda\alpha}{}^{\nu}{}_{;\beta} + A_{\alpha\beta}{}^{\nu}{}_{;\lambda} + A_{\beta\lambda}{}^{\nu}{}_{;\alpha}] = \delta^{\nu}_{\alpha}A_{[\beta\lambda]} + \delta^{\nu}_{\beta}A_{[\lambda\alpha]} + \delta^{\nu}_{\lambda}A_{[\alpha\beta]}. \tag{11}$$

We have used the definition

$$A_{\mu\nu} \equiv A_{\mu}{}^{\epsilon}{}_{\nu;\epsilon} - A_{\mu;\nu}. \tag{12}$$

It is now easy to show that eq (10) implies a reduction of $A_{\mu\nu\epsilon}$ to the standard representation (two-index tensor) once from it follows the existence of a tensor $\Theta_{\mu\nu}$ in terms of which $A_{\mu\nu\epsilon}$ can be expressed:

$$A_{\mu\epsilon\nu} = \Theta_{\nu[\mu;\epsilon]} + p\Theta_{[\mu}{}^{\lambda}{}_{;\lambda}\gamma_{\epsilon]\nu} + q\Theta_{,[\mu}\gamma_{\epsilon]\nu}. \tag{13}$$

Here, p and q are constants and $\Theta = \Theta_{\mu}{}^{\mu}$. We note that it is possible then to relate the two second-order tensors $\Phi_{\mu\nu}$ and $\Theta_{\mu\nu}$ by a simple expression:

$$\begin{aligned}
\Phi_{\mu\nu} &= 2\Box\Theta_{\mu\nu} + [p - 1 + \eta(1+3p)]\Theta_{(\mu}{}^{\epsilon}{}_{;\nu);\epsilon} \\
&+ 2[q - \eta(1-3q)]\Theta_{,\mu;\nu} + [-2p + \xi(1+3p)]\Theta^{\epsilon\lambda}{}_{;\epsilon;\lambda} \\
&- [2q + \xi(1-3q)]\Box\Theta\gamma_{\mu\nu}
\end{aligned} \tag{14}$$

Although the constants p and q can take any value, there is a particular set which is worth to consider. Indeed, let us choose

$$p = -q = -\frac{(1+\eta)}{(1+3\eta)} = \frac{(\xi - 2)}{(2 - 3\xi)}.$$

Inserting these values in the expression eq (14) we get

$$\Phi_{\mu\nu} = 2\mathbf{D}_{\mu\nu}{}^{\alpha\beta}\Theta_{\alpha\beta}, \tag{15}$$

in which the operator $\mathbf{D}$ is given by

$$\begin{aligned}
\mathbf{D}^{\mu\nu}_{\alpha\beta} &= (\delta^{\mu}_{\alpha}\delta^{\nu}_{\beta} - \gamma^{\mu\nu}\gamma_{\alpha\beta})\Box + \gamma_{\alpha\beta}\nabla^{\mu}\nabla^{\nu} \\
&+ \gamma^{\mu\nu}\nabla_{\alpha}\nabla_{\beta} - \delta^{\nu}_{\alpha}\nabla^{\mu}\nabla_{\beta} - \delta^{\mu}_{\beta}\nabla^{\nu}\nabla_{\alpha}.
\end{aligned} \tag{16}$$

Let us note that the operator $\mathbf{D}_{\alpha\beta}{}^{\mu\nu}$ is the standard divergence-free Fierz-Pauli operator of spin-two field equation.

In this paper we will make use of eq. (9) to get rid of the pseudo-tensorial spin-two field present in our theory. As we shall see, this condition does not influence the conclusions we will reach in this work. It is considered just to make easier the task of comparing our results with stablished theories. The presence of this extra spin-two field will be the subject of a future study.

3.2 The Standard Representation-II

Instead of eq (10) we could have imposed on the variable $A_{\alpha\beta\mu}$ a more restrictive condition to eliminate one spin-two field:

$$A^{*}_{\mu\nu\alpha}{}^{;\nu} = 0. \tag{17}$$

This condition admits the following equivalent forms

$$A_{[\mu\nu]} = 0 \tag{18}$$

and

$$A^{\nu}_{\lambda\alpha;\beta} + A^{\nu}_{\alpha\beta;\lambda} + A^{\nu}_{\beta\lambda;\alpha} = 0, \tag{19}$$

the last one being adequate for use in algebraic manipulations to be performed in the next section.

3.3 The Field $F_{\alpha\beta\mu\nu}$

From the tensor $A_{\alpha\beta\mu}$ we define the field $F_{\alpha\beta\mu\nu}$ in analogy to the electromagnetic field:

$$F_{\alpha\beta\mu\nu} = A_{\alpha\beta[\mu;\nu]} + A_{\mu\nu[\alpha;\beta]}. \tag{20}$$

3.3.1 Algebraic Properties

From the above definition follow the symmetry properties:

$$F_{\alpha\beta\mu\nu} = -F_{\beta\alpha\mu\nu} = -F_{\alpha\beta\nu\mu} = F_{\mu\nu\alpha\beta}. \tag{21}$$

Besides these trivial identities the tensor $F_{\alpha\beta\mu\nu}$ satisfies the cyclic property:

$$F^{\alpha}{}_{\beta\mu\nu} + F^{\alpha}{}_{\mu\nu\beta} + F^{\alpha}{}_{\nu\beta\mu} = 0. \tag{22}$$

Let us now evaluate the cyclic derivative of $F_{\alpha\beta\mu\nu}$ in order to see under what conditions this tensor satisfies a Bianchi-type identity. We have:

$$\begin{aligned} F^{\alpha\beta}{}_{\mu\nu;\lambda} &+ F^{\alpha\beta}{}_{\nu\lambda;\mu} + F^{\alpha\beta}{}_{\lambda\mu;\nu} \\ &= (A_{\mu\nu}{}^{\alpha}{}_{;\lambda} + A_{\nu\lambda}{}^{\alpha}{}_{;\mu} + A_{\lambda\mu}{}^{\alpha}{}_{;\nu})^{;\beta} \\ &- (A_{\mu\nu}{}^{\beta}{}_{;\lambda} + A_{\nu\lambda}{}^{\beta}{}_{;\mu} + A_{\lambda\mu}{}^{\beta}{}_{;\nu})^{;\alpha}. \end{aligned} \tag{23}$$

We thus see that if the condition (19) is satisfied then the tensor $F_{\alpha\beta\mu\nu}$ obeys the cyclic derivative identity.

We note that the trace of this equation yields:

$$F^{\mu\nu\alpha\beta}{}_{;\beta} = F^{\mu\alpha;\nu} - F^{\nu\alpha;\mu}, \tag{24}$$

in which we have used the definition

$$F_{\mu\nu} = A_{(\mu\epsilon\nu)}{}^{;\epsilon} - A_{(\mu;\nu)}.$$

3.3.2 Gauge

The theory we are examining is covariant once it is written in arbitrary coordinate system. The auxiliar background metric $\gamma_{\mu\nu}$ is taken, just for simplicity, to be Minkowskian. We shall see in a forthcoming section that it is an unobservable for the matter. However, as far as the internal symmetry of the field $F_{\alpha\beta\mu\nu}$ is concerned, the structure of the invariance of the field depends on the background geometry. This property can be understood if we note that the three-index field contains more information than the restricted quantity $h_{\mu\nu}$. Indeed, let us consider the map

$$\tilde{A}_{\mu\epsilon\nu}(x) = A_{\mu\epsilon\nu}(x) + Z_{\mu}(x)\gamma_{\epsilon\nu} - Z_{\epsilon}(x)\gamma_{\mu\nu}. \tag{25}$$

It then follows that the associated fields $\Phi_{\mu\nu}$ and $\Psi_{\mu\nu}$ changes, respectively, as

$$\tilde{\Phi}_{\mu\nu}(x) = \Phi_{\mu\nu}(x) + (1+3\eta)Z_{(\mu;\nu)} - (2-3\xi)Z^{\alpha}{}_{;\alpha}\gamma_{\mu\nu} \tag{26}$$

and

$$\tilde{\Psi}_{\mu\nu}(x) = \Psi_{\mu\nu}(x). \tag{27}$$

For the tensor $F_{\alpha\beta\mu\nu}$ we obtain:

$$\tilde{F}_{\alpha\beta\mu\nu} = F_{\alpha\beta\mu\nu} - \gamma_{\alpha\mu}Z_{(\beta;\nu)} + \gamma_{\beta\mu}Z_{(\alpha;\nu)} + \gamma_{\alpha\nu}Z_{(\beta;\mu)} - \gamma_{\beta\nu}Z_{(\mu;\alpha)}. \tag{28}$$

In order that the above map be a symmetry of the field $F_{\alpha\beta\mu\nu}$ the vector Z_{μ} must satisfy the equation

$$Z_{\mu;\nu} + Z_{\nu;\mu} = 0. \tag{29}$$

We note that for the spin-1 Maxwell field W_{μ} there is an analogous map given by:

$$\tilde{W}_{\mu} = W_{\mu} + \Lambda_{\mu}.$$

The field $F_{\mu\nu} = W_{\mu;\nu} - W_{\nu;\mu}$ remains invariant under the above map if the vector Λ_{μ} satisfies the equation $\Lambda_{\mu;\nu} - \Lambda_{\nu;\mu} = 0$, which differs from (29) by the sign between its terms. Once the background geometry is supposed to be torsion-free (that is, the corresponding Christoffel symbols are symmetric) this condition is satisfied if the vector Λ_{μ} is a gradient. However in the case of the field $F_{\alpha\beta\mu\nu}$ the appearance of the plus sign yields for Z_{μ} a more elaborate property. Indeed, we recognize that (29) is nothing but the condition that makes Z_{μ} a Killing vector of the background geometry. Thus, the invariance of the field $F_{\alpha\beta\mu\nu}$ under the above internal map (25) is nothing but the Poincaré gauge symmetry. One should be tempted at this point to interpret the field $F_{\alpha\beta\mu\nu}$ as a candidate to be used in a Poincaré gauge invariant gravity theory. However, we note that the invariance of $F_{\alpha\beta\mu\nu}$ implies the invariance of the field $h_{\mu\nu}$. Once we will examine in this paper only the restricted case in which the matter interacts just with a functional of the quantity $h_{\mu\nu}$ the above symmetry will be a hidden one.

3.4 The Irreducible Parts of $F_{\alpha\beta\mu\nu}$

In order to decompose the tensor $F_{\alpha\beta\mu\nu}$ into its irreducible components, we define the quantity $M_{\alpha\beta\mu\nu}$ by the expression

$$M_{\alpha\beta\mu\nu} = \frac{1}{2}[F_{\alpha\mu}\gamma_{\beta\nu} + F_{\beta\nu}\gamma_{\alpha\mu} - F_{\alpha\nu}\gamma_{\beta\mu} - F_{\beta\mu}\gamma_{\alpha\nu}], \tag{30}$$

in which $F_{\alpha\mu} = F_{\alpha\epsilon\mu}{}^{\epsilon}$. The trace of $F_{\alpha\mu}$, represented by F, is given by

$$F = -4A^{\lambda}_{;\lambda}. \tag{31}$$

We can then write

$$F_{\alpha\beta\mu\nu} = C_{\alpha\beta\mu\nu} + M_{\alpha\beta\mu\nu} - \frac{1}{6}F\gamma_{\alpha\beta\mu\nu}, \tag{32}$$

in which $\gamma_{\alpha\beta\mu\nu} \equiv \gamma_{\alpha\mu}\gamma_{\beta\nu} - \gamma_{\alpha\nu}\gamma_{\beta\mu}$. The tensor $C_{\alpha\beta\mu\nu}$ is the traceless part of $F_{\alpha\beta\mu\nu}$.

3.4.1 Invariance Properties of $C_{\alpha\beta\mu\nu}$

The tensor $C_{\alpha\beta\mu\nu}$ was used in previous papers as the fundamental field in terms of which a Lagrangean can be constructed. Thus it seems worthwhile to examine its invariant properties. Let us come back to the map (25). Using (28) as the corresponding map for the field $F_{\alpha\beta\mu\nu}$, it follows by contraction

$$\tilde{F}_{\alpha\mu} = F_{\alpha\mu} - 2[Z_{(\alpha;\mu)} + Z^{\lambda}_{;\lambda}\gamma_{\alpha\mu}], \tag{33}$$

and for the trace

$$\tilde{F} = F - 12Z^{\alpha}_{;\alpha}. \tag{34}$$

Using these results into the expression (32) we find that $C_{\alpha\beta\mu\nu}$ does not change, independently of the value of vector Z_{μ}. Besides this invariance, the traceless part of the tensor $F_{\alpha\beta\mu\nu}$ has another invariance. Indeed, let us consider the map generated by an anti-symmetric tensor $\omega_{\mu\nu}$ given by

$$\tilde{A}_{\alpha\beta\mu} = A_{\alpha\beta\mu} + \omega_{\alpha\beta;\mu} - \frac{1}{2}\omega_{\mu[\alpha;\beta]} + \frac{1}{2}\omega_{[\beta\lambda}{}^{;\lambda}\gamma_{\mu\alpha]}. \tag{35}$$

Then, a straightforward calculation shows that under the above map the tensor $C_{\alpha\beta\mu\nu}$ does not change.

4 Dynamics-I

In previous papers dealing with the variable $A_{\mu\nu\alpha}$ [1, 2] the dynamics of the field was constructed based on a functional of the trace-free tensor $C_{\mu\nu\alpha\beta}$. A typical action is

$$S = -\frac{1}{8}\int \sqrt{-\gamma}C_{\alpha\beta\mu\nu}C^{\alpha\beta\mu\nu}d^4x. \tag{36}$$

The equations of motion that follow from this action are

$$\Box A_{\mu\nu\epsilon} = 0, \tag{37}$$

provided that we use as a choice of gauge (allowed by the above invariances),

$$A_{\mu} = 0 \tag{38}$$

for the trace part of the field and

$$A_{\alpha\mu\nu}{}^{;\nu} = 0 \tag{39}$$

for its divergence. The spin content of this theory becomes more transparent if we decompose the field in a $3+1$ split of space-time. For the inertial observer characterized by its four-velocity $V_\mu = \delta_\mu{}^0$, we can decompose $A_{\mu\nu\epsilon}$ into the quantities $\xi, \beta_{ij}, \alpha_{ij}, \Delta_{ij}$, defined by:

$$\xi_i = A_{0i0} \quad (40)$$
$$\beta_{ij} = A_{ij0} \quad (41)$$
$$\alpha_{ij} = -A^0{}_{(ij)} \quad (42)$$
$$\Delta_{ij} = \frac{1}{2}\epsilon^{ab}{}_{(i}\Delta_{abj)} \quad (43)$$

in which $\Delta_{ijk} = A_{ijk} + \frac{1}{2}\gamma_{k[i}\xi_{j]}$, and ϵ_{ijk} is the 3-dim Levi-Civita pseudo-tensor. The gauge's choice yields:

$$\xi_i = 0 \quad (44)$$
$$\beta_{ij} = 0 \quad (45)$$
$$\alpha^{ij}{}_{,j} = 0 \quad (46)$$
$$\Delta^{ij}{}_{,j} = 0 \quad (47)$$

After inserting the above additional information in the equations of motion, a direct calculation gives for the independent 3-tensors α_{ij} and Δ_{ij} the equations:

$$\Box\alpha_{ij} = 0 \quad (48)$$

$$\Box\Delta_{ij} = 0. \quad (49)$$

Let us make here two additional comments which follow directly from the above projection, through the exam of the Hamiltonian and of the plane wave solutions of equations (48) and (49) (see [2] for the proof of these statements):

- The energy of the independent fields α_{ij} and Δ_{ij} have opposite signs.
- The waves have helicity ± 2.

This accomplishes the task of showing that we are indeed dealing with two spin-2 fields when we consider $A_{\mu\nu\alpha}$ as the basic fundamental quantity.

5 Dynamics II

We shall not consider further in this paper, the theory provided by the action (36). Instead of this our Lagrangean will be constructed with the field $F_{\mu\nu\alpha\beta}$. However, before considering this new action let us turn our attention to the interaction term, once as we shall see it will guide us in the choice of the Lagrangean for the free part of the field.

The source of the gravitational field $A_{\mu\nu\alpha}$ must be a current $J_{\mu\nu\alpha}$ which has the same symmetries as the field itself. The interaction term is given by:

$$S_{int} = -\int \sqrt{-\gamma} A_{\mu\nu\alpha} J^{\mu\nu\alpha} d^4x. \quad (50)$$

The current $J^{\mu\nu\alpha}$ must be a functional of the stress-energy tensor $t_{\mu\nu}$ which, for the sake of compatibility with the equations of motion, must obey the conservation identity

$$J^{\mu\nu\alpha}{}_{;\alpha} = 0.$$

The simplest choice satisfying identically such condition is given by the expression

$$2J_{\mu\epsilon\nu} = t_{\nu[\mu;\epsilon]} - t^{\lambda}{}_{[\mu;\lambda}\gamma_{\epsilon]\nu} + \frac{1}{2}\xi t_{;[\mu}\gamma_{\epsilon]\nu}, \tag{51}$$

in which $t = t^{\mu}_{\mu}$ and the energy tensor is given by the standard expression $t_{\mu\nu} = \frac{2}{\sqrt{-\gamma}}\frac{\delta L^m}{\delta\gamma_{\mu\nu}}$ where L^m is the matter Lagrangean. Using the definition of the field $h_{\mu\nu}$ given in equation (7) we can re-write this Action up to surface terms in the form

$$S_{int} = \frac{1}{2}\int\sqrt{-\gamma}t^{\mu\nu}(h_{\mu\nu} - \frac{1}{2}h\gamma_{\mu\nu})d^4x, \tag{52}$$

in which (cf equation (7))

$$h_{\mu\nu} - \frac{1}{2}h\gamma_{\mu\nu} = A_{(\mu}{}^{\epsilon}{}_{\nu);\epsilon} - A_{(\mu;\nu)} + \xi A^{\lambda}_{;\lambda}\gamma_{\mu\nu} \tag{53}$$

We thus see that the above interaction is nothing but the conventional coupling of matter with gravity in the standard variable. Let us just give a simple example.

5.1 The Case of the Scalar Field

Let φ be a scalar matter field. We take for the free Lagrangean the expression

$$S_{\varphi} = \frac{1}{2}\int\sqrt{-\gamma}\varphi_{,\mu}\varphi_{,\nu}\gamma^{\mu\nu}d^4x. \tag{54}$$

It then follows that the stress-energy tensor is given by

$$t_{\mu\nu} = \varphi_{,\mu}\varphi_{,\nu} - \frac{1}{2}\varphi_{,\lambda}\varphi_{,\epsilon}\gamma^{\lambda\epsilon}\gamma_{\mu\nu}.$$

As we saw in the last section the interaction term is given by eq (52). This expression can be rewritten in the following form:

$$S_{int} = \frac{1}{2}\int\sqrt{-\gamma}h^{\mu\nu}(t_{\mu\nu} - \frac{1}{2}t\gamma_{\mu\nu})d^4x. \tag{55}$$

For the scalar field we are considering we have:

$$S_{int} = \frac{1}{2}\int\sqrt{-\gamma}h^{\mu\nu}\varphi_{,\mu}\varphi_{\nu}d^4x. \tag{56}$$

The sum of the free and interaction actions for the scalar field can thus be written as:

$$S_{\varphi} + S_{int} = \frac{1}{2}\int\sqrt{-\gamma}(\gamma^{\mu\nu} + h^{\mu\nu})\varphi_{,\mu}\varphi_{,\nu}d^4x. \tag{57}$$

Collecting all these informations we see that the net effect of the interaction of the scalar field with gravity in the present framework, where $A_{\mu\nu\alpha}$ is considered the fundamental variable, can be described as a change of the metric of the background. That is, we can say that the matter is now subjected to an effective metric $g^{\mu\nu}$ characterized completely by the field through the formula

$$\sqrt{-g}g^{\mu\nu} = \sqrt{-\gamma}[\gamma^{\mu\nu} + h^{\mu\nu}],$$

in which the quantity $h^{\mu\nu}$ is a combination of derivatives of the basic field $A_{\mu\nu\alpha}$.

5.2 Non-Locality

Let us pause for a while and see what we achieved. We have an action for the matter represented by the scalar field φ,both for the free case (54) and for its interaction with the gravitational field . One could then be tempted to take (36) to provide the free part of gravitation in the total action. However if we assume this expression for free gravity it follows that the matter sees itself imbedded in a geometry in which the metric tensor is a non-local function of the gravity field $A_{\mu\nu\alpha}$ (see the Appendix). One could proceed with this theory once, in principle, it constitutes a coherent model. We decided however, in this paper to follow another road which we will now present.

The motivation of taking Lagrangean (36) was provided by an analogy with Maxwell's electrodynamics. This led, due to the choice of the term describing the interaction with matter to a non-local geometry. Thus it seems natural to try to remove this non-locality away by a convenient choice of the gravity Lagrangean. One can easily be convinced however, that the only possible modification, due to the choice of the fundamental variable $A_{\mu\nu\alpha}$ and its interaction with matter, is the consideration of a Lagrangean containing higher order derivatives. In this way we are able to move the non-locality from the induced geometry to the equations of motion satisfied by gravity, an indication that in the present context the non-locality is an irremovable basic property. In this vein, the above analogy with Maxwell's electrodynamics should be abandoned. This led us naturally to a non-local theory of gravity. Curiously, the new theory has a counterpart in a theory of electrodynamics formulated by Podolsky [8].

The action S of the model is:

$$\begin{aligned} S_{grav} &= \frac{1}{2}\int\sqrt{-\gamma}[-\frac{1}{8}F^{\alpha\beta\mu\nu;\lambda}F_{\alpha\beta\mu\nu;\lambda} \\ &+ \lambda F^{\mu\nu;\epsilon}F_{\mu\nu;\epsilon} + \sigma F_{,\epsilon}F^{,\epsilon}]d_4x, \end{aligned} \tag{58}$$

in which λ and σ are nondimensional constants. Then the variational principle yields:

$$\delta S_{grav} = \frac{1}{2}\int \Box\Omega_{\alpha\beta\mu}\delta A^{\alpha\beta\mu}d^4x, \tag{59}$$

with $\Omega_{\alpha\beta\mu}$ given by:

$$\begin{aligned} \Omega_{\alpha\beta\mu} &= -F_{\alpha\beta\mu\nu}{}^{;\nu} + 2\lambda[F_{\alpha\mu;\beta} - F_{\beta\mu;\alpha} + F_{\beta\epsilon}{}^{;\epsilon}\gamma_{\alpha\mu} - F_{\alpha\epsilon}{}^{;\epsilon}\gamma_{\beta\mu}] \\ &- 4\sigma[F_{,\alpha}\gamma_{\beta\mu} - F_{,\beta}\gamma_{\alpha\mu}]. \end{aligned} \tag{60}$$

We note that the identity $\Omega^{\alpha\beta\mu}{}_{;\mu} = 0$ is valid for this expression.

Taking together the two Lagrangeans we set:

$$S = S_{grav} + S_{int}.$$

This yields the following equation of motion:

$$\frac{1}{2}\Box\Omega_{\alpha\beta\mu} = J_{\alpha\beta\mu}. \tag{61}$$

At this point it seems worth to stop for a while the use of the variable $A_{\alpha\mu\nu}$ and see how this theory looks in terms of the variables $h_{\mu\nu}$. The procedure for this is simple: one has to take the divergence of the equation (61) and use the definition of $h_{\mu\nu}$ in terms of the fundamental variables $A_{\alpha\mu\nu}$. The calculation is straightforward and we present only some intermediate steps for future reference. We have:

$$\begin{aligned} F^{\alpha\beta\mu\nu}{}_{;\beta;\nu} &= \Box h^{\alpha\mu} - h^{(\alpha\epsilon;\mu)}{}_{;\epsilon} + \frac{1-\xi}{1-\eta-2\xi} h^{;\alpha\mu} \\ &- \frac{1-\eta-\xi}{2(1-\eta-2\xi)} \Box h \gamma^{\alpha\mu}, \end{aligned} \tag{62}$$

.

$$\begin{aligned} F_{\mu[\alpha;\beta]} &= h_{[\alpha\mu;\beta]} - (1+\eta) A_{(\alpha;\mu);\beta} - (1+\eta) A_{(\beta;\mu);\alpha} \\ &- \frac{1-\eta-\xi}{2(1-\eta-2\xi)} h_{;[\beta}\gamma_{\alpha]\mu}. \end{aligned} \tag{63}$$

We have also,

$$F = \frac{-2}{1-\eta-2\xi} h. \tag{64}$$

Thus collecting all these informations it follows that:

$$\begin{aligned} \Omega^{\alpha\beta\mu}{}_{;\beta} &= -\Box h^{\alpha\mu} + h^{(\alpha\epsilon;\mu)}{}_{;\epsilon} \\ &- \frac{1-\xi}{1-\eta-2\xi} h^{;\alpha\mu} + \frac{1-\eta-\xi}{2(1-\eta-2\xi)} \Box h \gamma^{\alpha\mu} \\ &+ 2\lambda[\Box h^{\alpha\mu} - h^{(\alpha\epsilon;\mu)}{}_{;\epsilon}] \\ &+ 2\lambda[\frac{-3+\eta+\xi}{2(1-\eta-2\xi)} \Box h \gamma^{\alpha\mu} + \frac{2-\xi}{1-\eta-2\xi} h^{;\alpha\mu}] \\ &+ \frac{8\sigma}{1-\eta-2\xi} [h^{;\alpha\mu} - \Box h \gamma^{\alpha\mu}] \end{aligned} \tag{65}$$

Let us remark that, from the definition of $h_{\alpha\mu}$ it follows identically:

$$h^{\epsilon\mu}{}_{;\epsilon;\mu} = \frac{1+\eta-\xi}{2(1-\eta-2\xi)} \Box h. \tag{66}$$

Using these results into (61) yields the required equation of motion for the derived quantity $h_{\mu\nu}$,

$$\Box h_{\mu\nu} - h_{(\mu\epsilon;\nu)}{}^{;\epsilon} + m h_{;\mu\nu} + n \Box h \gamma_{\mu\nu} = J_{\mu\nu}, \tag{67}$$

in which we have used the definition $J_{\mu\nu} \equiv \frac{-1}{1-2\lambda} \Box^{-1} J_{\mu\beta\nu}{}^{;\beta}$. In the above equation m and n are constants that depends on the parameters η and ξ of the definition of $h_{\mu\nu}$ and on the pair σ, λ. They are:

$$m = \frac{-2\lambda(2-\xi) - \xi + 1 - 8\sigma}{(1-2\lambda)(1-\eta-2\xi)}$$

and

$$n = \frac{-1 - 2\lambda(\eta+\xi-3) + \eta + \xi + 16\sigma}{2(1-2\lambda)(1-\eta-2\xi)}.$$

In the absence of matter, compatibility of the trace of eq. (67) with our choice eq. (7) requires the following relation between the constants:

$$1 - 24\sigma - 8\lambda = 0. \tag{68}$$

It is convenient then to re-write equation (61) under the equivalent integral form:

$$\Omega_{\alpha\mu\nu} = \frac{2}{\Box} J_{\alpha\mu\nu}. \tag{69}$$

The Green's function $\Box^{-1}$ is to be taken in the Minkowski background, in which it is a well defined operator controlled by causal requirements. In order to make contact with other theories let us consider the particular case in which we choose the free parameters of the present theory to be given by: $\eta = -1$, $\xi = 0$ and $\lambda = 1$. It then follows that the current $J_{\mu\nu}$ given by (51) reduces to the form:

$$J_{\mu\nu} = t_{\mu\nu} - \Box^{-1} t_{\epsilon(\nu;\mu)}{}^{;\epsilon} + \gamma_{\mu\nu} \Box^{-1} t^{\epsilon\lambda}{}_{;\epsilon\lambda}. \tag{70}$$

This expression was used in [9] to provide a non-local theory of gravity which although being non-linear does not provide a direct graviton-graviton interaction. The source of the gravitational field consists of two parts:

- A local contribution, which represents the influence at an arbitrary point P of the matter/energy present at P.
- Non-local contributions, due to the existence of matter at other places than P (to be selected by the Causal Principle).

Let us come back to our previous example and consider the gravitational interaction with a scalar field. The total action is:

$$S = S_{\varphi} + S_{grav} + S_{int}.$$

The corresponding equation of motion for the field φ is given by

$$[\sqrt{-g} g^{\mu\nu} \varphi_{,\mu}]_{,\nu} = 0, \tag{71}$$

in which the effective metric $g^{\mu\nu}$ is a function of derivatives of $A_{\mu\nu\alpha}$ through the expression

$$\sqrt{-g} g^{\mu\nu} = \sqrt{-\gamma} [\gamma^{\mu\nu} + h^{\mu\nu}]. \tag{72}$$

and $h_{\mu\nu}$ is given by

$$h_{\mu\nu} - \frac{1}{2} h \gamma_{\mu\nu} \equiv A_{(\mu\epsilon\nu)}{}^{;\epsilon} - A_{(\mu;\nu)}. \tag{73}$$

Note that g is the determinant of the inverse matrix of $g^{\mu\nu}$ which we will represent by $g_{\mu\nu}$, that is

$$(g^{\mu\nu})^{-1} \equiv g_{\mu\nu},$$

satisfying $g_{\mu\nu}g^{\nu\alpha} = \delta^{\alpha}_{\mu}$. It seems worth to point out that except in this particular case of the induced geometry $g_{\mu\nu}$[2] all operations of changing from contravariant to covariant representations are made by means of the background metric $\gamma_{\mu\nu}$.

The evolution of $A_{\mu\nu\alpha}$ is provided by equation (69) or the equivalent one (67) in terms of the derived variable $h_{\mu\nu}$. The present theory has two important consequences:

- The background geometry $\gamma_{\mu\nu}$ is not observable: a soft version of Mach's principle.
- The non-linearity of the gravitational field is induced by the existence of matter somewhere in the Universe.

Indeed, the background metric $\gamma_{\mu\nu}$ is not an observable entity, once matter, represented here by the scalar field is sensitive only to the effective geometry $g_{\mu\nu}$. Besides, from the form of the current (70) it follows that if there is no matter/energy of any kind in the Universe then there is no possibility of induced self gravitating energy. In the present theory the non-linearity is not a fundamental property of gravity but it becomes a consequence of the existence of a source elsewhere. The situation becomes somehow equivalent to the case of the interaction of source and field in electrodynamics. The electromagnetic field A_μ satisfies the linear Maxwell theory; the electron Ψ satisfies Dirac's linear equation. The coupled system A_μ and Ψ taken together, however, satisfy a non-linear system of equations. In the present gravitational theory one cannot eliminate the non-local influence of distant matter: the coupled system matter-gravity constitutes an indivisible reality.

5.3 Non-Machian Non-Linearity

In the precedent section we examined what we will call from now on the *Machian* non-linearity of gravity, which is induced by the presence of matter somewhere in the universe. Besides this sort of *induced* non-linearity one should consider other forms which have, as in the framework of General Relativity (GR), a local origin. In this case, one must modify the action by introducing explicitly in the Lagrangean the desired extra non-linear terms. From what we learned above in the interaction of gravity with a scalar field one could try to use a functional of the derived quantity g as the additional term. Just to exemplify this let us take the simplest case in which the Lagrangean is a functional of $\tilde{g^{\mu\nu}} \equiv \tilde{\gamma^{\mu\nu}} + \tilde{h^{\mu\nu}}$ in which $\tilde{\gamma}^{\mu\nu} \equiv \sqrt{-\gamma}\gamma^{\mu\nu}$ and $\tilde{g}^{\mu\nu} \equiv \sqrt{-g}g^{\mu\nu}$. We set for the 4-dimensional action[3]:

$$S = \int f(\tilde{g}^{\mu\nu})$$

The fundamental variable to be varied is the quantity $A_{\mu\nu\alpha}$. We then have:

$$\delta S \;=\; \int \frac{\delta f}{\delta \tilde{g}^{\mu\nu}} \delta \tilde{g}^{\mu\nu} = \int \frac{\delta f}{\delta \tilde{g}^{\mu\nu}} \delta \tilde{h}^{\mu\nu}$$

[2]We note that this is just a matter of notation. The quantity $g_{\mu\nu}$ is taken to represent the inverse metric and not the quantity $g^{\alpha\beta}\gamma_{\alpha\mu}\gamma_{\beta\nu}$.

[3]We represent the 4-dimensional integration as $\int F = \int F d^4x$

$$
\begin{aligned}
&= \int [-\left\{\frac{\delta f}{\delta \tilde{g}^{\mu\nu}}\right\}_{;\epsilon} + \left\{\frac{\delta f}{\delta \tilde{g}^{\epsilon\nu}}\right\}_{;\mu}]\delta \tilde{A}^{\mu\epsilon\nu} \\
&+ 2\int \left\{\frac{\delta f}{\delta \tilde{g}^{\mu\lambda}}\right\}_{;\lambda} \gamma_{\epsilon\nu}\delta \tilde{A}^{\mu\epsilon\nu} \\
&- 2\int \left\{\frac{\delta f}{\delta \tilde{g}^{\alpha\beta}}\right\}_{;\mu} \gamma^{\alpha\beta}\delta \tilde{A}^{\mu\epsilon\nu}\gamma_{\epsilon\nu}.
\end{aligned} \tag{74}
$$

In order to present a specific simple example let us choose the extra Lagrangean to be given by $f = b\sqrt{-g}$, in which b is a nondimensional constant. We then have

$$\int \delta\sqrt{-g} = \int \sqrt{-\gamma} I_{\mu\epsilon\nu}\delta A^{\mu\epsilon\nu}. \tag{75}$$

Now, using the definition of g in terms of the fundamental variable $A_{\mu\nu\alpha}$ we have:

$$\delta\sqrt{-g} = \sqrt{-\gamma} g_{\mu\nu}[\delta A^{\mu\epsilon\nu}{}_{;\epsilon} - \delta A^{\mu;\nu} + \delta A^{\lambda}{}_{;\lambda}\gamma^{\mu\nu}]. \tag{76}$$

The quantity $g_{\mu\nu}$ is the inverse of $g^{\mu\nu}$. From the decomposition (72) it follows that the inverse metric tensor is an infinite series

$$\frac{1}{\sqrt{-g}} g_{\mu\nu} = \frac{1}{\sqrt{-\gamma}}[\gamma_{\mu\nu} - h_{\mu\nu} + h_{\mu\alpha}h^{\alpha}_{\nu} - \cdots].$$

We recover here the same ideas as in [10] in which the infinite series (contained in the passage from the quantity $g^{\mu\nu}$ to its inverse $g_{\mu\nu}$) appears in the gravitational interaction as a consequence of the sequence of self-couplings. It then follows:

$$
\begin{aligned}
I_{\mu\epsilon\nu} &= -\frac{1}{2}(g_{\mu\nu;\epsilon} - g_{\epsilon\nu;\mu}) + \frac{1}{2}(g_{\mu\lambda}{}^{;\lambda}\gamma_{\epsilon\nu} - g_{\epsilon\lambda}{}^{;\lambda}\gamma_{\mu\nu}) \\
&- \frac{1}{2}(g_{\alpha\beta;\mu}\gamma^{\alpha\beta}\gamma_{\epsilon\nu} - g_{\alpha\beta;\epsilon}\gamma^{\alpha\beta}\gamma_{\mu\nu}),
\end{aligned} \tag{77}
$$

Adding this extra term to the dynamics of $A_{\mu\nu\alpha}$ yields:

$$\frac{1}{2}\Box\Omega_{\alpha\beta\mu} = J_{\alpha\beta\mu} + bI_{\alpha\beta\mu}. \tag{78}$$

We can now proceed as previously in order to pass from the new variable to the standard one $h_{\mu\nu}$. Taking the divergence of the above expression we obtain

$$G^L_{\mu\nu} = J_{\mu\nu} + bI_{\mu\nu}, \tag{79}$$

in which $G^L_{\mu\nu}$ represents the linear part of the theory, that is, the left-hand side of (8) and $I_{\mu\nu} \equiv \Box^{-1} I_{\mu\epsilon\nu}{}^{;\epsilon}$. From the above construction it follows identically that $I_{\mu\nu}$ is divergence-free. Let us point out that the construction of the additional term $I_{\mu\nu}$ from variation of $g^{\mu\nu}$ (and, consequently, the introduction in the dynamics of the gravitational field $A_{\mu\nu\alpha}$ of nonlinearities contained in the infinite series defined by the inverse quantity $g_{\mu\nu}$) was made possible due to the fact that $g^{\mu\nu}$ is not the fundamental quantity. In other words, the effective geometry felt by matter (the scalar field φ in the above example) is a subsidiary functional of $A_{\mu\nu\alpha}$. At this point one should wonder on the possibility of detecting the auxiliary metric $\gamma_{\mu\nu}$ by using not matter but gravitational waves. That this is not the case will be shown next.

5.4 Gravitational Disturbances

The introduction of a term proportional to $\sqrt{-g}$ in the Einstein's framework has the function of representing the whole effect of vacuum of all others fields in Nature. The constant b in that case is called the *cosmological constant.* Here, however it has a different role: it provides the condition of unobservability of the background metric $\gamma_{\mu\nu}$. Indeed, let Σ be a surface of discontinuity for the field $A_{\mu\nu\alpha}$ and let us represent the discontinuity of any quantity F through Σ by the symbol $[F]_\Sigma$. We set in accordance with Hadamard's treatment the sequence of expressions:

$$[h_{\mu\nu;\lambda}]_\Sigma = k_\lambda \varphi_{\mu\nu} \tag{80}$$
$$[h_{\mu\nu;\lambda\epsilon}]_\Sigma = k_\lambda k_\epsilon \varphi_{\mu\nu} + \ldots \tag{81}$$
$$[h_{\mu\nu;\lambda\epsilon\rho}]_\Sigma = k_\lambda k_\epsilon k_\rho \varphi_{\mu\nu} + \ldots \tag{82}$$

in which the vector k_μ is orthonormal to the hypersurface Σ. For our purpose here it is sufficient to consider only the main terms of the discontinuities. The presence of the non Machian nonlinear terms induced by the action involving $\sqrt{-g}$ has an important consequence: the matter and the gravitational field itself are not able to detect the background metric since the gravitational disturbances travels in a modified effective geometry. The reason for this is the presence on $I_{\mu\nu}$ of second order derivatives of the infinite series represented in a condensed manner by the inverse metric $g_{\mu\nu}$.

Note however that the present theory is not identical to General Relativity. Indeed, the sources $J_{\mu\nu}$ and $I_{\mu\nu}$ are independently conserved (i.e. divergence-free). One could modify the present theory, adding some other Lagrangean in the same lines as in Feynman 's work on the road to Einstein's General Relativity, in a by now well known procedure. We do not follow this procedure here.

6 Conclusion

In this paper we examined some consequences of considering the three-index tensor $A_{\mu\nu\alpha}$ as the fundamental variable to describe gravity. In order to make contact with the standard two-index tensor description we exhibit the bridge formula that allows the passage from one representation to the other. In the linear case we consider a model obtained by a direct analogy with Maxwell's electrodynamics. The theory contains a modification of the background structure of the space-time by means of the appearance of an effective non-local geometry. This non-locality in the induced geometry can be removed to the equations of motion. The simplest way to do this can be represented by a change from Maxwell's type of Lagrangean to Podolsky's.

The gravitational theory thus constructed has a source which is a combination of local and non-local matter contributions. The non-local part of the source can be interpreted as representing the non-linearity of the gravitational field at any point P of the space-time induced by the presence of the matter at other places in the region causally connected to P. This result is in accordance with Mach's principle. Thus Lagrangean given in eq.(75) is responsible by what we could call the Machian non-linearity of gravity. Besides, we examined the consequences of introducing explicit nonlinear terms of local origin.

7 Appendix

In section 5.2 we have argued against the use of the linear Maxwell-type Lagrangean (36). The main argument comes from the analysis of the interaction of the gravitational field with matter resulting in a nonlocal modification of the effective geometry. Let us present here briefly the proof of this statement. Combining equations (36) and (50) we obtain

$$\Box A_{\mu\nu\epsilon} + H_{\mu\nu\epsilon} = J_{\mu\nu\epsilon}, \tag{83}$$

where $H_{\mu\nu\epsilon}$ contains terms involving divergence and traces of the field $A_{\mu\nu\epsilon}$. Let us take (see the text) the following expression to represent the source:

$$J_{\mu\nu\epsilon} = C_{\epsilon[\mu;\nu]} + \eta C_{[\mu\lambda}{}^{;\lambda}\gamma_{\nu]\epsilon} + \xi C_{,[\mu}\gamma_{\nu]\epsilon}. \tag{84}$$

As we already know eq.(83) can be expressed by means of the two-index variable $h_{\mu\nu}$ (see the text) yielding

$$\Box h_{\alpha\mu} - h_{(\alpha\lambda;\mu)}{}^{;\lambda} + a_1 h_{,\alpha;\mu} + a_2\gamma_{\alpha\mu}\Box h = J_{(\alpha\lambda\mu)}{}^{;\lambda}, \tag{85}$$

where a_1 and a_2 are constants (which values depend on the choice of the gauge representation).

This is the standard linear Pauli-Fierz equation for a spin-two field if we identify the right-hand side to the energy-momentum tensor of matter $T_{\mu\nu}$. Then, the symmetric tensors $T_{\mu\nu}$ and $C_{\mu\nu}$ are related through the expression

$$T^{\alpha\lambda} = \mathcal{D}^{\alpha\lambda}_{\rho\sigma} C^{\rho\sigma}, \tag{86}$$

where

$$\begin{aligned} \mathcal{D}^{\alpha\lambda}_{\rho\sigma} &\equiv \frac{1}{2}\delta^{(\alpha}_{\rho}\delta^{\lambda)}_{\sigma}\Box + (\eta - 1)\delta^{(\lambda}_{\sigma}\nabla^{\alpha)}\nabla_{\rho} - \eta\gamma^{\alpha\lambda}\nabla_{\sigma}\nabla_{\rho} \\ &+ \xi\gamma_{\rho\sigma}(\nabla^{\alpha}\nabla^{\lambda} - \gamma^{\alpha\lambda}\Box). \end{aligned} \tag{87}$$

For the interaction with matter we have:

$$S_{int} = \int \sqrt{-\gamma} A_{\mu\nu\epsilon} J^{\mu\nu\epsilon}. \tag{88}$$

Inserting here the expression eq.(84) and using eq.(86) gives

$$S_{int} = \int \sqrt{-\gamma} C^{\alpha\beta} h_{\alpha\beta} \tag{89}$$

or

$$S_{int} = \int \sqrt{-\gamma} \frac{1}{\mathcal{D}^{\alpha\lambda}_{\rho\sigma}} T^{\alpha\lambda} h_{\rho\sigma}. \tag{90}$$

The symbol $\frac{1}{\mathcal{D}^{\alpha\lambda}_{\rho\sigma}}$ is to be understood as the Green function associated to the differential operator $\mathcal{D}$.

We now define

$$m_{\alpha\lambda} = \frac{1}{\mathcal{D}^{\alpha\lambda}_{\rho\sigma}} h^{\rho\sigma}.$$

We can then write the interaction as :

$$S_{int} = \int \sqrt{-\gamma} T^{\alpha\beta} m_{\alpha\beta}.$$

In the case of the scalar field φ we have

$$S_{\varphi} + S_{int} = \frac{1}{2} \int \sqrt{-\gamma} (\gamma^{\mu\nu} + m^{\mu\nu}) \varphi_{,\mu} \varphi_{,\nu}. \tag{91}$$

The identification $\sqrt{-g} g^{\mu\nu} = \sqrt{-\gamma}(\gamma^{\mu\nu} + m^{\mu\nu})$, where $m^{\mu\nu}$ is given above, shows clearly that indeed, as stated in the text, we got a non-local expression for the effective geometry in the *Maxwellian* scheme.

Instead of using such complicated expression for the effective geometry we decided to modify the Lagrangean of gravity by the lines presented in the text. The result is a non-local theory of gravity.

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