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On the Geometry of Two-Dimensional Anisotropic Non-Linear Sigma-Models

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Abstract

One discusses here the connection between σ-model gauge anomalies and the existence of a connection with torsion that does not flattten the Ricci tensor of the target manifold. The influence of an eventual anisotropy along a certain internal direction is also contemplated.

Key-words: Sigma-model; Anisotropy; Ricci-flat.

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The geometrical setting, the classical dynamics and quantum-mechanical properties of non-linear σ-models have been extensively discussed by field-theorists over the past decade. Phenomenological motivations connected to low-energy effective physics [1] and considerations of a more formal nature, regarding ultraviolet behaviour, renormalisability [2] and all-order finiteness of some classes of σ-models [3] have been the leitmotif for pursuing a thorough investigation of these non-linear theories.

Introducing supersymmetry, a number of issues concerning the formulation of σ-models are brought about [4] [5]. One of the outstanding points that come out has to do with the fermions that, by means of supersymmetry, naturally couple to the bosonic coordinates of the target space.

The dynamics of fermions coupled to a σ-model requires a vector bundle, B, with a connection defined over the target space, M, on which the bosonic fields, φ^i, map the space-time manifold. The coupling amongst eventual chiral fermions, as dictated by supersymmetry, and the connection naturally defined on B is a potential source of anomalies, as clearly discussed in the works of ref. [6].

Our paper sets out to analyse how it might happen that possible geometrical constraints can be imposed on the target space geometry, so as to prevent the appearance of anomalies connected to the gauge fields associated to the connection introduced in B, or to the isometry gauge fields that may come in whenever one performs the gauging of isometries or isotropies of σ-models defined on homogeneous spaces of the type isometry group G/H.

If $dim\ G/H = D$, the anomaly alluded to in ref. [6] is the one associated to the $SO(D)$ – pull-back connection of B. In our case, we shall take into account the situation where one introduces extra gauge fields that come into play with the purpose of gauging the subgroup H (suitably embedded in $SO(D)$) or even the whole group G.

This problem has been previously tackled in a series of works [7], and the connection between the geometrical structure of G/H, but the mechanism for cancellation of the isometry group gauge anomaly has been worked out in a paper by Alvarez- Gaumé and Ginsparg [11], where the authors succeed in fixing conditions on the H-subgroup content of the fermion fields so as to eliminate the isometry group anomaly. Our purpose here is to understand if there is a relationship between the H-subgroup attributes of the fermions and the torsion on G/H, in such a way to characterize the anomaly suppression mechanism more directly in terms of the geometrical structure of the target space. The main motivation behind our attempt is the coupling between the torsion of the σ-model manifold and the fermion bilinears. We choose to consider here non-symmetric coset spaces with non-vanishing torsion, having in mind that interesting geometrical properties may arise at the expenses of working with a metric connection with torsion. We also try to exploit the geometric nature of anisotropic non-linear σ-models. The topological nature of the latter has been considered by Watanabe and Otsu [8]. These authors have shown that anisotropy may lad to non-trivial metastable states that generate local minima of the energy (instantons).

Following the work of ref. [10], we shall specify the geometry of G/H in terms of group-theoretical properties of G and H. First of all, we assume the splitting dictated by the decomposition

$$\begin{aligned} adj\ G &= adj\ H \bigoplus V : \\ [Q_i, Q_j] &= f_{ij}{}^k Q_k, \\ [Q_i, Q_a] &= f_{ia}{}^b Q_b, \\ [Q_a, Q_b] &= f_{ab}{}^i Q_i + f_{ab}{}^c Q_c, \end{aligned} \tag{1}$$

where the Q_i's ($i = 1, ..., dim\ H$) denote the generators of H and V refers to the Q_a's ($a = 1, ..., D$) the generators of G outside H.

Denoting by φ^α the coordinates of a point in the coset G/H (α is the world index and runs from 1 to D), the *vielbein* and *connection* are obtained by means of the following G-Lie-algebra-valued one-form:

$$e(\varphi) \equiv L_\varphi^{-1} dL_\varphi = [e^i_\alpha(\varphi) Q_i + e^a_\alpha(\varphi) Q_a] d\varphi^\alpha, \tag{2}$$

where L_φ is a coset representative and $a = 1, ..., D$ is to be identified with the tangent space index (local frame index).

With the help of the Cartan-Maurer equation for the 2-form $de(\varphi)$, namely,

$$de(\varphi) = -e(\varphi) \wedge e(\varphi), \tag{3}$$

with $e \equiv e^i Q_i + e^a Q_a$, the geometrical objects (torsion 2-form, T^a, and curvature 2-form, $R^a{}_b$), that obey the equations

$$de^a + w^a{}_b \wedge e^b = T^a \tag{4}$$

and

$$R^a{}_b = dw^a{}_b + w^a{}_c \wedge w^c{}_b \tag{5}$$

turn out to read as follows:

$$w^a{}_b = -f^a{}_{bi}e^i - \frac{1}{i}f^a{}_{bc}e^c - \frac{1}{i}T^a{}_{bc}e^c. \tag{6}$$

If one chooses the torsion 2-form to be of the form

$$T^a = \frac{1}{2}kf^a{}_{bc}e^b \wedge e^c, \tag{7}$$

k being an arbitrary coefficient, the the connection becomes

$$w^a{}_b = -f^a{}_{bi}e^i - (1+k)f^a{}_{bc}e^c; \tag{8}$$

this, in turn, yields

$$\begin{aligned} R^a{}_{bcd} &= f^a{}_{bi}f^i{}_{cd} + \frac{1}{c}(1+k)f^a{}_{be}f^e{}_{cd} + \\ &+ \frac{1}{4}(1+k^2)(f^a{}_{ce}f^e{}_{db} - f^a{}_{de}f^e{}_{cb}). \end{aligned} \tag{9}$$

Torsion shall be absent if $k = 0$, or in the case the embedding of H into G is a symmetric one.

The anisotropic non-linear σ-models are theories of maps between manifolds. More precisely, the scalar fields ϕ^i of the theory map a given space-time, X, to a given target space, M. The action of model is obtainable from pure kinetic term $\frac{1}{2}[\sum_i \partial_\mu\varphi^i\partial^\mu\varphi^i + \lambda\partial_\mu\varphi^k\partial^\mu\varphi^k]$ (i=1,2,...,k,...n), by resolving the constrant $\sum_i\varphi^i\varphi^i = 1$, to obtain

$$S = \frac{1}{2}\int d^2x g_{ij}(\varphi)\partial_\mu\varphi^i\partial^\mu\varphi^j, \tag{10}$$

with

$$g_{ij}(\varphi) = \delta_{ij} + (1+\lambda)\frac{\varphi_i\varphi_j}{1-\varphi^2}, \tag{11}$$

where $i, j \neq k$ and $\lambda > -1$.

By rewriting the original fields as $\tilde{\varphi} = \Big(\varphi_1, \varphi_2, ..., (1+\lambda)^{1/2}\varphi_k, ..., \varphi_n\Big)$, we get the surface S^{n-1} on an n-dimensional spheroid

$$\sum_{i \neq k} \tilde{\varphi}^i \tilde{\varphi}^i + \frac{\tilde{\varphi}^k \tilde{\varphi}^k}{b^2} = 1, \tag{12}$$

where $b^2 = (1+\lambda)$. From eqs.(11) and (12), we see that the anisotropy parameter, λ, deforms the usual metric on a sphere.

In two dimensions, the action(10) is not the most general one. Namely, assume that the target space carries, besides the given metric g, a given two-form ω. Them, the complete action is given by

$$S = \frac{1}{2}\int d^2x g_{ij}(\varphi)\partial_\mu \varphi^i \partial^\mu \varphi^j + \frac{1}{2}\int d^2x \epsilon^{\mu\nu} \omega_{ij}(\varphi)\partial_\mu \varphi^i \partial_\nu \varphi^j, \tag{13}$$

where we refer to $\omega_{ij}(\varphi)$ as the torsion potential. It is well-known that this term introduces torsion on the manifold [9].

We shall consider the following ansatz for the torsion:

$$T^a = \frac{1}{2}(1+\lambda)k f^a_{bc} e^b \wedge e^c, \tag{14}$$

where k is related to the torsion degree and $(1+\lambda)$ gives the dependence of the torsion on the parameter of anisotropy. It then follows that

$$\omega^a_b = -f^a_{bi} e^i - \frac{1}{2}(1+k+k\lambda) f^a_{bc} e^c. \tag{15}$$

This allows us to obtain the curvature and the Ricci tensor, respectively,

$$R^a_{bcd} = f^a_{bi} f^i_{cd} + \frac{1}{2}(1+k+k\lambda) f^a_{be} f^e_{cd} + \frac{1}{4}(1+k+k\lambda)^2 (f^a_{ec} f^e_{bd} - f^a_{ed} f^e_{bc}), \tag{16}$$

$$R_{ab} = f^c_{ai} f^i_{cb} + \frac{1}{4}[1 - (1+\lambda)^2 k^2] f^c_{ae} f^e_{cb}. \tag{17}$$

Now, our main idea is to try to understand how the fermion coupling to the torsion of G/H may probe the anomaly matching condition proposed in the work of ref. [11]. Since we have not yet found a general line of arguments that lead to find the geometrical counterpart of the condition quoted above, it is our idea to set the geometry of some non-symmetric spaces, namely, $Sp(4)/U(1) \times U(1)$, $SU(2) \times SU(2)/U(1)$, $G_2/SU(3)$, $SU(3)/U(1) \times U(1)$ and $Sp(4)/(SU(2) \times U(1))_{nonmax}$, to illustrate that the anomaly cancellation condition found in ref [11] has to do with the non-possibility of finding a Ricci-flattening connection with torsion.

For the latter three homogeneous spaces listed above, it is always possible to have a Ricci-flattening connection, whenever the coefficient k in eq.(7) and the anisotropy coefficient fulfill the condition:

$$(1+\lambda)k = \pm\sqrt{5}. \tag{18}$$

This is so because

$$R_{ab} = f^{c}{}_{ai} f^{i}{}_{cb} + \frac{1}{4}(1-(1+\lambda)^2k^2) f^{c}{}_{ad} f^{d}{}_{cb}, \tag{19}$$

can be taken vanishing by virtue of the results below

$$f^{c}{}_{ai} f^{i}{}_{cb} = f^{c}{}_{ad} f^{d}{}_{cb} = \begin{cases} \delta_{ab}, for \quad SU(3)/U(1) \times U(1); \\ \delta_{ab}, for \quad Sp(4)/(SU(2) \times U(1))_{nonmax}; \\ \frac{4}{3}\delta_{ab}, for \quad G_2/SU(3). \end{cases} \tag{20}$$

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However, for $Sp(4)/U(1) \times U(1)$ and $SU(2) \times SU(2)/U(1)$, Ricci- flatness cannot be achieved at the expenses of torsion. For $SU(2) \times SU(2)/U(1)$, this is so because $rank(SU(2) \times SU(2)/U(1)) \neq 1$; nevertheless, even though

$rank\ Sp(4) = rank(U(1) \times U(1))$, it is not possible to find a solution for k that enables to set $R_{ab} = 0$. This result can be understood wiht the help of the explicit calculations of the combination of structure constants of $Sp(4)$:

$$f^c{}_{ai} f^i{}_{cb} = \begin{cases} 1, for \quad a = b = 2, 4, 5, 8, 9; \\ 2, for \quad a = b = 3, 6, 7; \\ 0, for \quad a \neq b \end{cases} \tag{21}$$

whereas

$$f^d{}_{ac} f^c{}_{db} = \begin{cases} 2, for \quad a = b = 3, 4, 6, 7; \\ 4, for \quad a = b = 2, 5, 8, 9; \\ 0, for \quad a \neq b. \end{cases} \tag{22}$$

What then remains is to analyse the connection between the non-symmetry of the space, the non-existence of a Ricci-flattening connection with torsion and the fulfilment of the anomaly cancellation condition, as expressed in terms of the H-content of the fermions.

Our claim, by now only supported by explicit examples and not by a general approach, is that, if $rank\ G = rank\ H$, and if torsion is non-trivial and does not allow the vanishing of the Ricci tensor, then the isometry group anomaly does *not* show up. In the cases torsion yields a vanishing Ricci tensor, one notices that the anomalies are *not* cancelled, as it is the case of $SU(3)/U(1) \times U(1)$ and $Sp(4)/(SU(2) \times U(1))_{nonmax}$.

Also, for Grassmannians and projective spaces like CP^n, which appear in the framework of $N = 1 - D = 4$ supersymmetric σ-model, isometry anomalies persist as there is no torsion, once they are all symmetric spaces.

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