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CHIRAL ANOMALY, BOSONIZATION AND FRACTIONAL CHARGE*

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Abstract

We present a method to evaluate the Jacobian of chiral rotations, regulating determinants through the proper time method and using Seeley's asymptotic expansion. With this method we compute easily the chiral anomaly for $\nu=4,6$ dimensions, discuss bosonization of some massless two-dimensional models and handle the problem of charge fractionization. Besides, we comment on the general validity of Fujikawa's approach to regulate the Jacobian of chiral rotations with non-hermitean operators.

Key-words: Field theory; Path integral; Chiral anomaly.

I. INTRODUCTION

Chiral anomalies have been playing a role of increasing importance in field theory of elementary particles since their discovery some fifteen years ago[1].

More recently Fujikawa[2] developed a method which allows to study chiral anomalies in a path integral approach, independently of perturbation theory. He observed that the path integral fermionic measure is not invariant under a chiral change of variables and that the anomalous term comes from the Jacobian of the chiral rotation.

Afterwards his method was employed to implement a sort of path--integral version[3-8] of the bosonization technique[9] in two-dimensional models and recently Schaposnik has showed[10] how Fujikawa's method can be implemented to study the problem of charge fractionization[11-12] in two-dimensional models.

The method developed[6-8] by the group of La Plata (Gamboa-Saraví, Muschietti, Schaposnik and Solomin), to compute Jacobian of chiral rotations, makes use of the zeta function regularization of functional determinants[13] and the direct computation of Seeley's coefficients[14]. It is the purpose of this paper to use instead a method developed by Alvarez[15] to compute determinants, by means of the proper-time method and Seeley's asymptotic expansion[14], to study the chiral anomaly in space-time dimension $\nu=4,6$, bosonization of some massless two-dimensional models and charge fractionization.

There are some conveniences with this method, namely:

(i) For normal Dirac-like operators the computed Jacobian is directly identified with the regulated Jacobian of Fujikawa[2].

(ii) The asymptotic expansions are tabulated for all physically interesting examples we are considering.

The shortcoming of this method is that we cannot compute the Jacobian of a theory for all non-normal Dirac-like operators, as is done in ref. 6, for example in the physically important theory with a pseudo-vectorial coupling, unless we are able to analytically continue this operator for a region where it is normal.

In the next section we present Alvarez's method for computing determinants, give the Jacobian of chiral rotation by this method, and briefly discuss its direct identification with the regulated Jacobian of Fujikawa and possible consequences for the method developed by Fujikawa when the Dirac-like operator is non-hermitean[17].

In section III we compute easily the anomaly in $\nu=4,6$ space-time dimensions for Q.C.D. In section IV we apply this method to bosonization of the Schwinger, Thirring and massless two-dimensional QCD. And, finally, in section V we discuss the application of this method for the fractionization of fermion number.

II. THE JACOBIAN OF THE CHIRAL TRANSFORMATION

We start by considering the fermionic part of the generating functional of an Euclidean Dirac-like theory:

$$G = \int D\bar{\psi}D\psi \, \exp\{-\int \bar{\psi}D\psi \, d^{\nu}x\} \qquad (1)$$

and introduce a non-abelian local chiral transformation over the fermionic fields

$$\psi = e^{r\gamma_5\Phi}\,\eta_r$$
$$\bar{\psi} = \bar{\eta}_r\, e^{r\gamma_5\Phi} \tag{2}$$

with $\Phi = \Phi^a \lambda_a$, λ_a the generators of the group of interest, and r a real parameter ($0 \leq r \leq 1$).

The transformation (2) in the generating functional (1) introduces a Jacobian as:

$$G = \int D\bar{\eta}_r D\eta_r\, J(r) \quad \exp\{-\int \bar{\eta}_r D_r \eta_r\, d^\nu x\} \tag{3}$$

with

$$D_r = e^{r\gamma_5\Phi}\, D\, e^{r\gamma_5\Phi} \tag{4}$$

We integrate over the fermionic fields in (1) and (3), the result is formally the determinant of the Dirac operator:

$$G = \det D = J(r) \det D_r \tag{5}$$

so we may obtain a formal expression for the Jacobian of the transformation (2) in terms of functional determinants:

$$\ell n J(r) = \ell n \det D_{r=0} - \ell n \det D_r \tag{6}$$

The functional determinant in (6), as is well known, diverge and must be regularized. In order to regularize this determinant by the proper-time method we must construct a square operator, DD^+, and provided that $\ell n \det D^{+}$ is proportional to $\ell n \det D$ we have:

$$\ell n \det D_r^2 = \ell n \det D_r D_r^+ = \mathrm{Tr}\, \ell n\, D_r D_r^+ = -\int_{\varepsilon}^{\infty} \frac{ds}{s} \mathrm{Tr}[\exp(-sD_r D_r^+)] \tag{7}$$

with ε an ultraviolet cutoff on the proper-time integration.

But the operator given in (4) has the useful property that[15]:

$$\frac{d}{ds} D_r = fD_r + D_r f \tag{8}$$

with $f = \gamma_5 \Phi$. Then differentiating (7) with respect to r, using property (8) and the cyclic property of the functional trace we get[15]:

$$\frac{d}{dr} \ell n \det D_r D_r^+ = 4\mathrm{Tr}[f \exp(-\varepsilon D_r D_r^+)] \tag{9}$$

In deriving this last formulae (9) we are assuming that the operators D_r and D_r^+ satisfy:

$$\int_{\varepsilon}^{\infty} ds\, \mathrm{Tr}[D_r f D_r^+ \exp(-sD_r D_r^+)] = \int_{\varepsilon}^{\infty} ds\, \mathrm{Tr}[f D_r D_r^+ \exp(-s D_r D_r^+)] \tag{10}$$

which is valid when D_r is a normal operator.

Now, since f is a matrix function in order to compute the functional trace in (9) we integrate over the diagonal part of the heat kernel for $D_r D_r^+$. For this diagonal part Seeley has shown[14] that there is an asymptotic small ε expansion given by:

$$\langle x|\exp(-\varepsilon D_r D_r^+)|x\rangle \xrightarrow[\varepsilon\to 0]{} \frac{1}{(4\pi\varepsilon)^{\nu/2}} [a_0^r(x) + \varepsilon a_1^r(x) + \varepsilon^2 a_2^r(x) + \ldots] \tag{11}$$

with the coefficients of this expansion tabulated for physically interesting operators.

Integrating expression (9) over r from 0 to 1 we obtain the Jacobian of interest, i.e., for r=1:

$$\ell n J(r=1) = -2\int_0^1 dr \int d^\nu x \underset{c x \gamma}{\mathrm{tr}} [\gamma_5 \Phi(x) \langle x|\exp(-\varepsilon D_r D_r^+)|x\rangle] \qquad (12)$$

with $\underset{c x \gamma}{\mathrm{tr}}$ denoting the trace over γ-Dirac and color matrices*.

For the purpose of comparing this Jacobian with Fujikawa's regulated Jacobian[2] we consider abelian local infinitesimal chiral transformation. As the infinitesimal field $\Phi(x)$ appears directly in the integrand in (12), it is only necessary to consider the Φ independent term of the diagonal part of the heat kernel. Then, integrating over r and expanding over eigenfunctions of $\not{D}$ we get Fujikawa's expression[2] for the regulated Jacobian of an infinitesimal local chiral transformation:

$$\ell n J = -2\int d^\nu x \; \Phi(x) \underset{c x \gamma}{\mathrm{tr}} [\gamma_5 \sum_k \langle x|\lambda_x\rangle \exp(-\varepsilon\lambda_k^2)\langle\lambda_k|x\rangle] \qquad (13)$$

The expression (12) appears as a natural extension of Fujikawa's method for computing the Jacobian of a local finite chiral transformation.

However, in cases where the operator D is non-normal, as we have seen, we must be careful that $\ell n\det D$ is proportional to $\ell n\det D^+$ and that expression (10) be valid; this puts forward some questions concerning the general validity of Fujikawa's method to regulate non-hermitean operators[17].

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*Notice that we don't use the perturbative evaluation of the determinant[16].

III. CHIRAL ANOMALY IN 4 AND 6 DIMENSIONS

Let us consider the QCD lagrangean in an arbitrary dimension with SU(N) gauge group:

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu,a} F^{\mu\nu}_{a} + \bar{\psi} i (\not{\partial} + \not{A}) \psi \tag{14}$$

As it was said in the end of section II for the purpose of computing the chiral anomaly it is only necessary to consider the Φ independent term of the diagonal of the heat kernel in the Jacobian (12). For this case by a straightforward algebra we have:

$$D^2 = -D_\mu D_\mu + X \tag{15}$$

with

$$D_\mu = \partial_\mu + A_\mu$$

$$X = -\frac{1}{4} [\gamma_\mu, \gamma_\nu] [D_\mu, D_\nu] \tag{16}$$

Happily the coefficients of the asymptotic small ε expansion (11) are tabulated[18] with the values:

$$a^r_o = 1$$

$$a^r_1 = -X$$

$$a^r_2 = -\frac{1}{2} X^2 + \frac{1}{12} \partial_\mu \partial_\nu F_{\mu\nu} - \frac{1}{12} F_{\mu\nu} F_{\mu\nu} - \frac{1}{6} \partial^2 X$$

$$a^r_3 = -\frac{1}{45} (\partial_\alpha F_{\mu\nu})^2 - \frac{1}{180} \partial_\nu F_{\mu\nu} \partial_\alpha F_{\mu\alpha} - \frac{1}{60} \partial^2 (F_{\mu\nu} F_{\mu\nu}) + \frac{1}{30} F_{\mu\nu} F_{\nu\alpha} F_{\alpha\mu} -$$

$$-\frac{1}{60}\partial^4 X-\frac{1}{12}\partial^2 X^2-\frac{1}{12}(\partial_\mu X)^2-\frac{1}{6}X^3-\frac{1}{30}XF_{\mu\nu}F_{\mu\nu}-\frac{1}{60}F_{\mu\nu}XF_{\mu\nu}-$$

$$-\frac{1}{30}F_{\mu\nu}F_{\mu\nu}X+\frac{1}{60}\partial_\nu X\partial_\mu F_{\mu\nu}-\frac{1}{60}\partial_\mu F_{\mu\nu}\partial_\nu X \qquad (17)$$

By the use of the well known properties of γ-matrices we obtain with (11), (12) and (17) in four and six dimensions respectively:

$$\begin{aligned}\partial_\mu j_{\mu,5} &= \frac{i}{8\pi^2}\,\mathrm{tr}_c\, F_{\mu\nu}\tilde{F}_{\mu\nu}\\ \partial_\mu j_{\mu,7} &= -\frac{1}{3(4\pi)^3}\,\varepsilon_{\mu_1\ldots\mu_6}\,\mathrm{tr}_c(F_{\mu_1\mu_2}F_{\mu_3\mu_4}F_{\mu_5\mu_6})\end{aligned} \qquad (18)$$

which are the well known values of the anomaly in four and six dimensions[19].

IV. APPLICATION TO TWO-DIMENSIONAL MODELS

It was shown in several works[3-8] that by performing a chiral change of variables in massless Dirac like theory in two dimensions we decouple at classical level the fermions from other fields present.

The quantum aspect of this decoupling is given by the Jacobian of this transformation and we are going now to compute it by the method stated in section II for some field model theories in two dimensions.

Schwinger Model

The lagrangean of the theory is:

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}^2 + \bar{\Psi}(i\not\partial + e\not A)\Psi \tag{19}$$

with $\gamma_\mu^+=\gamma_\mu$, $A_\mu^+=A_\mu$ and $i\gamma_\mu\gamma_5=\varepsilon_{\mu\nu}\gamma_\nu$.

Performing the transformation (2) in this lagrangean and choosing the Lorentz gauge:

$$A_\mu = \frac{1}{e}\varepsilon_{\mu\nu}\partial_\nu\Phi \tag{20}$$

we obtain

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}^2 + \bar{\eta}_r(i\not\partial+e(1-r)\not A)\eta_r \ . \tag{21}$$

We see explicitly in (21) that for r=1 the fermion decouples from the gauge field. In this case

$$D_r^2 = D_\mu^r D_\mu^r + X^r \tag{22}$$

with

$$\begin{aligned} D_\mu^r &= \partial_\mu + ie(r-1)A_\mu \\ X^r &= -(1-r)\partial^2\Phi\gamma_5 \end{aligned} \tag{23}$$

Then by (11), (12), (17) and (23) we get for the Jacobian

$$\ell nJ = -\frac{e^2}{2\pi}\int d^2x\ A_\mu A_\mu \tag{24}$$

and the generating functional after chiral rotation is:

$$Z(\bar{\theta},\theta) = \int D\bar{\psi}D\psi DA \, \exp[-\int (-\frac{1}{4}F_{\mu\nu}^2 + \frac{e^2}{2\pi}A_\mu^2 + \bar{\psi}i\partial\!\!\!/\psi + \bar{\psi}e^{\gamma_5\Phi}\theta + \bar{\theta}e^{\gamma_5\Phi}\psi)\, d^2x] \tag{25}$$

with $\bar{\theta}$ and θ the fermion sources and any Green's function can be obtained from this generating functional[3].

Thirring Model

This model is a purely fermionic model with lagrangean

$$\mathcal{L} = \bar{\psi}i\partial\!\!\!/\psi - \frac{1}{2}g^2(\bar{\psi}\gamma_\mu\psi)^2 \tag{26}$$

but we can pass to an effective vector theory with generating functional as:

$$Z = \int D\bar{\psi}D\psi DA \, \exp\{-\int [\bar{\psi}(i\partial + gA\!\!\!/)\psi + \frac{1}{2}A_\mu^2]\, d^2x\} \tag{27}$$

We perform now the change of variables

$$\begin{aligned} \psi(x) &= e^{ir\eta(x)+r\gamma_5\Phi(x)}\chi_r(x) \\ \bar{\psi}(x) &= \bar{\chi}_r(x)\, e^{-ir\eta(x)+r\gamma_5(x)} \\ A_\mu(x) &= \frac{1}{g}\varepsilon_{\mu\nu}\partial_\nu\Phi(x) + \frac{1}{g}\partial_\mu\eta(x) \end{aligned} \tag{28}$$

Analogously to what we have in the Schwinger model the Jacobian relative to this change of variables (28) can be computed by the method developed in section II, and the result is:

$$\ell nJ = -\frac{1}{2\pi}\int d^2x(\partial_\mu\Phi)^2 \tag{29}$$

Then the generating functional after the change of variables (28) is:

$$Z(\bar{\theta},\theta) = \int D\Phi D\eta D\bar{\chi} D\chi \; \exp\{-\int d^2x [\frac{1}{2g^2}(1+\frac{g^2}{\pi})(\partial_\mu\Phi)^2 + \frac{1}{2g^2}(\partial_\mu\eta)^2 + \bar{\chi}\, e^{-i\eta+\gamma_5\Phi}\theta + \bar{\theta}\, e^{ir\eta+\gamma_5\Phi}\chi]\} \qquad (30)$$

with $\bar{\theta}$ and θ fermion sources and again any Green's function can be obtained from Z [5].

Two-Dimensional QCD

We consider now the QCD lagrangean in two dimensions with the SU(N) gauge group:

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu,a} F_a^{\mu\nu} + \bar{\psi}(i\partial\!\!\!/ + A\!\!\!/)\psi \qquad (31)$$

We choose the decoupling gauge [4,20] introduced by Gamboa-Saraví, Schaposnik, Solomin and Roskies, in this gauge the field A_μ reads:

$$A\!\!\!/ = i\, e^{-\gamma_5\Phi(x)}\, \partial\!\!\!/\, e^{-\gamma_5\Phi(x)} \qquad (32)$$

with $\Phi(x)$ taking values in the Lie algebra of SU(N).

Performing the non-abelian local chiral transformation (2) the lagrangean becomes

$$\mathcal{L} = \bar{\eta}_r [i\partial\!\!\!/ + i e^{\gamma_5(r-1)\Phi}\, \partial\!\!\!/\, e^{\gamma_5(r-1)\Phi}]\eta_r - \frac{1}{4} F_{\mu\nu,a} F_a^{\mu\nu} \qquad (33)$$

Again, for r=1 the fermion decouples from A_μ. Following reference 15 we define a vector V_μ^r and a pseudo-vector P_μ^r by:

$$e^{-\gamma_5(r-1)\Phi}\,\partial_\mu\, e^{\gamma_5(r-1)\Phi} = V^r_\mu + P^r_\mu \tag{34}$$

with $V^r_\mu = V^r_{\mu,a}\lambda_a$ and $P^r_\mu = P^r_{\mu,a}\gamma_5\lambda_a$.

The square of the Dirac operator D_r is given as usual:

$$D_r^2 = -(\partial_\mu + A^r_\mu)^2 - \frac{i}{2}\,\varepsilon_{\mu\nu}\gamma_5 F^r_{\mu\nu} \tag{35}$$

with

$$\begin{aligned} F^r_{\mu\nu} &= \partial_\mu A^r_\nu - \partial_\nu A^r_\mu + [A^r_\mu, A^r_\nu] \\ A^r_\mu &= V^r_\mu + i\varepsilon_{\mu\nu}\gamma_5 P^r_\nu \end{aligned} \tag{36}$$

The Jacobian relative to the chiral transformation (2) can be computed by using (11), (12), (17) and the property (8) with the result:

$$\ell n J = -\frac{1}{2\pi}\int d^2x\{\frac{1}{2}\,\mathop{Tr}_{c\times\gamma}(\not{A}\not{A}) - \int_0^1 dr\,\mathop{Tr}_{c\times\gamma}(\not{A}^r\not{A}^r\gamma_5\Phi)\} \tag{37}$$

which is the same result found in references 7,21. The first term is the non-abelian extension of the Schwinger mechanism, and the second can be shown[7] to correspond to the two-dimensional analog of the Wess-Zumino functional.

V. FRACTIONIZATION OF THE FERMION NUMBER

Recently[10] Schaponisk developed a method to study the charge fractionization[11-12] for fermions in a soliton field. The method consists in introducing a current source term in the generating functional and by performing a chiral change of variables to obtain directly the term responsible for fractionization from the Jacobian of this transformation.

We are going now to show that the method described in section II is also suitable to obtain charge fractionization in the models studied in ref. 10.

Let us consider the two-dimensional model of massless fermions interacting with the external soliton field ξ, the lagrangean is[12]:

$$\mathcal{L} = \bar{\Psi}(i\partial\!\!\!/ + ge^{\gamma_5 \xi})\psi \tag{38}$$

In order to compute the fermionic current we define the generating functional:

$$Z[s] = \int D\bar{\Psi}D\psi \; \exp[-\int \bar{\Psi}(i\partial\!\!\!/ + s\!\!\!/ + ge^{\gamma_5 \xi})\psi \, d^2x] \tag{39}$$

with the source term s_μ for the bilinear form $\bar{\Psi}\gamma_\mu\psi$.

We perform now the chiral rotation

$$\begin{aligned} \psi(x) &= \exp(-\gamma_5 \frac{\xi}{2} r)\eta(x) \\ \bar{\Psi}(x) &= \bar{\eta}(x) \; \exp(-\gamma_5 \frac{\xi}{2} r) \end{aligned} \tag{40}$$

where r is a real parameter varying from 0 to 1. The Jacobian relative to this transformation can be now computed by the method developed in section II

$$\ell n J = \int_0^1 dr \int d^2x \ \mathrm{tr}_\gamma \ [\gamma_5 \xi \langle x | \exp(-\varepsilon D_r^2) | x \rangle] \tag{41}$$

with

$$D_r = i\not{\partial} + \not{s} + \frac{r}{2} \gamma_\mu \varepsilon_{\mu\nu} \partial_\nu \xi + g e^{-(r-1)\xi\gamma_5} \ . \tag{42}$$

For an operator of the form:

$$A = -\partial^2 + P_\mu \partial_\mu + Q \tag{43}$$

with P_μ and Q matrix valued functions, following the standard steps[22] it is easy to tabulate the diagonal part of the asymptotic expansion, the result is

$$\langle x | e^{-\varepsilon A} | x \rangle \xrightarrow[\varepsilon \to 0]{} \frac{1}{4\pi\varepsilon} \{1 - \varepsilon [Q - \frac{1}{4}(2\partial_\mu P_\mu - P_\mu P_\mu)] + 0(\varepsilon^2)\} \tag{44}$$

Then computing D_r^2 and using (41), (43) and (44) we get:

$$\ell n J = \frac{1}{4\pi} \int d^2x [-\frac{1}{2} \xi \partial^2 \xi - 2 s_\nu \varepsilon_{\mu\nu} \partial_\mu \xi + g^2 (1 - \cosh 2\xi)] \tag{45}$$

Now, in terms of the new variables the generating functional is:

$$Z[s] = \exp\{ \frac{1}{4\pi} \int d^2x [-\frac{1}{2} \xi \partial^2 \xi - 2 s_\nu \varepsilon_{\alpha\mu} \partial_\alpha \xi + g^2 (1 - \cosh 2\xi)] \}$$

$$\int D\bar{\eta} D\eta \ \exp[-\int \bar{\eta} (i\not{\partial} + \not{s} + \not{a} + g) \eta d^2x] \tag{46}$$

with $a_\mu = \frac{1}{2} \varepsilon_{\mu\nu} \partial_\nu \xi$. Then differentiating with respect to s_μ and turning off s at the end we get

$$J_\mu = -\frac{1}{Z} \left.\frac{\delta Z}{\delta s_\mu}\right|_{s=0} = \frac{1}{2\pi}\,\varepsilon_{\mu\nu}\partial_\nu\xi + j_\mu \tag{47}$$

with

$$j_\mu = \frac{\delta}{\delta a_\mu}\,\ell n\det(i\not\partial + \not a + g)\ . \tag{48}$$

Considering a slow varying ξ-field it may be shown that[12]

$$j_\mu = \varepsilon_{\mu\nu}\partial_\nu[\text{const.}\ \frac{\partial^2\xi}{g^2} + \text{higher order terms in } \partial^2\xi]. \tag{49}$$

Then up to leading order in derivatives of ξ we obtain:

$$J_\mu \cong \frac{1}{2\pi}\,\varepsilon_{\mu\nu}\partial_\nu\xi\ . \tag{50}$$

Thus we see that for a soliton field ξ we get the fractionization of the fermion number from the Jacobian of the chiral transformation.

We could make this analysis for a non-abelian extension of the lagrangean (38) and to others two-dimensional models finding the well known results[10-12] on charge fractionization.

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