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ANISOTROPIC CUBIC LATTICE POTTS FERROMAGNET:
RENORMALISATION GROUP TREATMENT

by

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ABSTRACT

Within a real space renormalisation group framework, we discuss the criticality of the fully anisotropic (arbitrary J_x, J_y and J_z) q-state Potts ferromagnet in simple cubic lattice. Several already known exact results for the d=1 and d=2 particular cases are recovered. Furthermore we obtain: (i) the q-dependence of the d=3 correlation length critical exponent ν_3 (in particular, if $q \to 0$, $\nu_3(q) \sim \nu_3(0) + \nu_3'(0)q$ where the present approximate values are $\nu_3(0) \simeq 1.105$ and $\nu_3'(0) \simeq -0.66$; (ii) the q-dependence of the $d=2 \leftrightarrow d=3$ crossover critical exponent ϕ_{23} (in particular, $\phi_{23} \propto 1/\sqrt{q}$ if $q \to 0$); (iii) through a convenient numerical extrapolation, a quite accurate proposal for the critical temperatures corresponding to arbitrary ratios J_y/J_x and J_z/J_x and values of q.

Key-words: Anisotropic Potts, Criticality, Simple Cubic Lattice.

I. INTRODUCTION

During recent years many works have been devoted to the q-state Potts model, both because of its theoretical richness and its experimental utility (for an excellent review see Wu 1982). However most of these works have focused the two-dimensional (d=2) case (see Wu 1982, and de Oliveira and Tsallis 1982 and references therein). Some effort has also been dedicated to the *isotropic* d=3 ferromagnet (Blöte and Swendsen 1979), but we are not aware of any systematic study of the *anisotropic* d=3 case and its crossovers to lower dimensions. This is the purpose of the present work (restricted however to the discussion of the critical temperature T_c and correlation length and crossover critical exponents ν and ϕ) which follows along the real space renormalisation group (RG) lines of de Oliveira and Tsallis 1982 (which is herein recovered as particular case). By noting $q_c(d)$ the limiting value of q above which the phase transition is a first order one (we recall that $\lim_{d\to 1+0} q_c(d)=\infty$, $q_c(2)=4$ and $q_c(3)\lesssim 3$; see Wu 1982 and de Magalhães and Tsallis 1981 and references therein), the present work is restricted to $q \leq q_c(d)$. We present in Section II the model and the formalism, in Section III the RG results, and in Section IV the extrapolation procedure which provides accurate values for T_c corresponding to models with arbitrary anisotropy.

II. MODEL AND FORMALISM

Let us consider the q-state Potts ferromagnet whose Hamiltonian is given by

$$\mathcal{H} = -q \sum_{(i,j,k)} \left\{ J_x \delta_{\sigma_{i,j,k},\sigma_{i+1,j,k}} + J_y \delta_{\sigma_{i,j,k},\sigma_{i,j+1,k}} + J_z \delta_{\sigma_{i,j,k},\sigma_{i,j,k+1}} \right\} \qquad (J_x>0;\ J_y,\ J_z \geq 0) \tag{1}$$

where (i,j,k) runs over all sites of a simple cubic lattice and $\sigma_{i,j,k} = 1,2,\ldots,q, \forall (i,j,k)$. We briefly recall the present status of knowledge of the criticality (T_c, ν and ϕ) of this model:

i) for d=1 (i.e., $J_y=J_z=0$) the critical temperature T_c vanishes, and the correlation length critical exponent satisfies

$$\nu_1 = 1 \quad , \quad \forall q \quad ; \tag{2}$$

ii) for d=2 (i.e., $J_z=0$ and $J_y>0$) T_c satisfies

$$[1+(q-1)\, e^{-qJ_x/k_BT_c}]\,[1+(q-1)\, e^{-qJ_y/k_BT_c}] = q \tag{3}$$

and the corresponding critical exponent is given by (den Nijs 1979)

$$\nu_2 = \frac{2}{3}\{2 + \pi/[\arccos(\frac{1}{2}\sqrt{q}) - \pi]\}^{-1} \tag{4}$$

iii) the d=1 to d=2 and d=1 to d=3 crossover critical exponents (respectively ϕ_{12} and ϕ_{13}) are commonly believed to satisfy (Redner and Stanley 1979, de Oliveira and Tsallis 1982 and references therein)

$$\phi_{12} = \phi_{13} = 1, \quad \forall q \tag{5}$$

iv) for the isotropic $d=3$ case (i.e., $J_x=J_y=J_z$), T_c is given by

$$k_BT_c/qJ_x = \begin{cases} 3.52 \pm 0.05 & \text{for } q=1 \text{ (from Gaunt and Ruskin 1978)} \quad (6.a) \\ 2.2556 \pm 0.0002 & \text{for } q=2 \text{ (from Zinn-Justin 1979)} \quad (6.b) \\ 1.8169 & \text{for } q=3 \text{ (from Jensen and Mouritsen 1979)} \quad (6.c) \end{cases}$$

where the $q=1$ value has been obtained from $p_c=0.247 \pm 0.003$ by using the Kasteleyn and Fortuin 1969 isomorphism ($p=1-e^{-J_x/k_BT}$) with bond percolation, and where we recall that the $q=3$ case might be slightly first order; the corresponding critical exponent is given by

$$\nu_3 = \begin{cases} 0.88 & \text{for } q=1 \text{ (Heerman and Stauffer 1981)} \quad (7.a) \\ 0.630 \pm 0.0015 & \text{for } q=2 \text{ (Le Guillou and Zinn-Justin 1980)} \quad (7.b) \end{cases}$$

v) For the $d=2$ to $d=3$ crossover exponent ϕ_{23} the following results are available:

$$\phi_{23} = \begin{cases} 1.75 & \text{for } q=1 \text{ (Redner and Stanley 1979)} \quad (8.a) \\ 7/4 \text{ (exact)} & \text{for } q=2 \text{ (Liu and Stanley 1972, 1973, Citteur and Kasteleyn 1972, 1973)} \quad (8.b) \end{cases}$$

Before presenting our RG formalism, let us define a few convenient variables (Tsallis and Levy 1981, Tsallis 1981):

$$t_\alpha \equiv \frac{1-e^{-qJ_\alpha/k_BT}}{1+(q-1)e^{-qJ_\alpha/k_BT}} \in [0,1] \quad (\alpha=x,y,z) \qquad (9.a)$$

(referred to as *thermal transmissivity*) and

$$s_\alpha^{(d)} \equiv s^{(d)}(t_\alpha) \equiv \frac{\ln[1+(q-1)h(d)t_\alpha]}{\ln[1+(q-1)h(d)]} \in [0,1] \quad (\alpha=x,y,z) \quad (9.b)$$

where (Tsallis and de Magalhães 1981, de Magalhães and Tsallis 1981) the pure number $h(d)$ sensibly depends on dimensionality d and very slightly on the particular d-dimensional lattice ($h(2)=1$ for square lattice, and $h(3)=0.377 \pm 0.044$ for simple cubic lattice).

If we have a series (or parallel) array of two bonds with transmissivities t_1 and t_2, the overall transmissivities (respectively t_s and t_p) are given by

$$t_s = t_1 t_2 \quad \text{(series)} \quad (10)$$

and

$$t_p^D = t_1^D t_2^D \quad \text{(parallel)} \quad (11)$$

where

$$t_i^D \equiv \frac{1-t_i}{1+(q-1)t_i} \quad (i=1,2,p) \quad (12)$$

(D stands for dual). We can also verify that $h=1$ (square lattice) implies

$$s^{(2)}(t^D) = 1-s^{(2)}(t) \quad (13)$$

We can now introduce our RG framework. Following along the lines of the de Oliveira and Tsallis 1982 treatment of the square lattice case, we establish the RG recursive relations by renormalising the b=2 cell indicated in Fig.1 (g,h) into the b=1 cell in Fig. 1(d) (b denotes the size of the cell, and coincides with the linear scaling factor). The recurrence is based upon the preservation of the partition function, and can be economically established by using the Break-collapse method (Tsallis and Levy 1981). We obtain

$$t_x' = R_b(t_x, t_y, t_z; q) \tag{14}$$

where $R_b(t_x,t_y,t_z;q)=R_b(t_x,t_z,t_y;q)$ is a ratio of polynomials (in the t's) too lengthy to be reproduced herein (the numerator and denominator contain more than 1600 terms each). The sum of the coefficients of the numerator coincides with that corresponding to the denominator and is given (Tsallis and Levy 1981; Essam 1982) by q^κ where $\kappa \equiv$ cyclomatic number = [(number of bonds)-(number of sites)+1] (for the two-terminal graph of Fig. 1(h) it is $\kappa=20$). It is worthy to note that $R_b(t_x,t_y,0;q)$ recovers Eq. (12) of de Oliveira and Tsallis 1982.

The rest of the RG recursive relations is given by

$$t_y' = R_b(t_y, t_z, t_x; q) \tag{15}$$

$$t_z' = R_b(t_z, t_x, t_y; q) \tag{16}$$

where the equivalence of the x,y and z axes has been taken into account. By studying, for fixed q, the RG flow (in the (t_x,t_y,t_z)--space) determined by Eqs. (14)-(16) we can obtain the fixed points, the para-ferromagnetic separatrix, as well as the relevant Jacobians $\partial(t'_x,t'_y,t'_z)/\partial(t_x,t_y,t_z)$, which in turn determine the critical exponents ν and ϕ.

III. RESULTS

Our results are illustrated in Fig. 2. Eqs. (14)-(16) provide the following fixed points: (i) $(s_x^{(2)},s_y^{(2)},s_z^{(2)})=(0,0,0)$ and (1,1,1) are fully stable, and correspond respectively to the para-and ferromagnetic phases; (ii) (1,1,0), (1,0,1) and (0,1,1) are semistable ones, and belong to the ferromagnetic region; (iii) (1,0,0), (0,1,0) and (0,0,1) are fully unstable ones, and correspond to the d=1 case; (iv) (1/2,1/2,0), (1/2,0,1/2) and (0,1/2,1/2) are semistable ones, and correspond to the d=2 isotropic case; (v) $(s_c^{(3)},s_c^{(3)},s_c^{(3)})$ is a semi-stable one, and corresponds to the d=3 isotropic case ($s_c^{(3)}$ softly depends on q; see Fig. 3).

The RG critical surface contains the line $s_x^{(2)}+s_y^{(2)}=1$ at $s_z^{(2)}=0$ (and the equivalent ones), thus reproducing the *exact* d=2 result expressed in Eq. (3). The performance at the isotropic d=3 fixed point is not comparable to the d=2 case, as the RG provides, for q=1, $t_c\simeq 0.2260$ (instead of 0.247, corresponding to Eq. 6 (a)), for q=2, $t_c\simeq 0.1949$ (instead of 0.21811, corresponding to Eq. 6(b)), and, for q=3, $t_c\simeq 0.1750$ (instead of 0.1966, corresponding to Eq. 6(c)). The results obtained for T_c for arbitrary anisotropy ratios J_y/J_x and J_z/J_x are indicated in Table I.

The Jacobian at the d=1 fixed points is fully degenerate and its unique eigenvalue $\lambda^{(1)}$ equals 3. It can be shown that $\lambda^{(1)}=2b-1$

for arbitrary value of b, therefore $\nu_1 = \lim_{b\to\infty} \ln b/\ln(2b-1) = 1$, thus recovering the *exact* Eq. (2). The degeneracy of this Jacobian implies that both d=1 ↔ d=2 and d=1 ↔ d=3 crossover exponents ϕ_{12} and ϕ_{13} equal unity, thus confirming Eq. (5).

At the d=2 fixed points the Jacobians are as follows. Let us analyse for instance the $(s_x^{(2)}, s_y^{(2)}, s_z^{(2)}) = (1/2, 1/2, 0)$ fixed point (the others are analogous); its Jacobian has the following form

$$\begin{pmatrix} a(q) & b(q) & c(q) \\ b(q) & a(q) & c(q) \\ 0 & 0 & d(q) \end{pmatrix} \tag{17}$$

The eigenvalues are

$$\begin{aligned} \lambda_1^{(2)} = a(q) + b(q) = {} & (2025 + 11160\sqrt{q} + 26580q \\ & + 35792q^{3/2} + 29852q^2 + 15816q^{5/2} + 5207q^3 \\ & + 976q^{7/2} + 80q^4)/(2025 + 8820\sqrt{q} + 16804q \\ & + 18290q^{3/2} + 12444q^2 + 5424q^{5/2} + 1481q^3 \\ & + 232q^{7/2} + 16q^4) \end{aligned} \tag{18}$$

$$\sim \begin{cases} 1 + \dfrac{52}{45}\sqrt{q} & \text{if } q \to 0 \qquad (18') \\ 5\left(1 - \dfrac{23}{10}\dfrac{1}{\sqrt{q}}\right) & \text{if } q \to \infty \qquad (18'') \end{cases}$$

$$\lambda_2^{(2)} = a(q)-b(q) = (10125+88650\sqrt{q}+342860q + 781853q^{3/2} + 1178008q^2 + 1240724q^{5/2} + 939667q^3 + 516906q^{7/2} + 205408q^4 + 57611q^{9/2} + 10844q^5 + 1232q^{11/2} + 64q^6)/(91125 + 595350\sqrt{q} + 1782540q + 3234167q^{3/2} + 3960600q^2 + 3449388q^{5/2} + 2191343q^3 + 1023534q^{7/2} + 349008q^4 + 84773q^{9/2} + 3932q^5 + 1392q^{11/2} + 64q^6) \quad (19)$$

$$\sim \begin{cases} \frac{1}{9}(1+\frac{20}{9}\sqrt{q}) & \text{if} \quad q \to 0 \quad (19') \\ 1-\frac{5}{2}\frac{1}{\sqrt{q}} & \text{if} \quad q \to \infty \quad (19'') \end{cases}$$

and $\lambda_3^{(2)} = d(q)$ (too lengthy to be reproduced herein; it monotonically decreases from about 8.1 to about 3.3 when q increases from 0 to 3). The respective eigenvectors are (1,1,0) (1,-1,0) and $(1,1,(\lambda_3^{(2)}(q)-\lambda_1^{(2)}(q))/c(q))$. Eqs. (18) and (19) recover respectively Eqs. (13) and (14) of de Oliveira and Tsallis 1982. We verify that $\lambda_1^{(2)}(q) \geq 1 \geq \lambda_2^{(2)}(q) > 0, \forall q \geq 0$, and that $\lambda_3^{(2)}(q) \geq \lambda_1^{(2)}(q)$ $(\lambda_3^{(2)}(q) < \lambda_1^{(2)}(q))$ if $q \leq q^*$ $(q > q^*)$ where $q^* \simeq 5$. The coefficient $c(q)$ monotonically increases from roughly zero to roughly 10 when q varies from zero to infinity; consistently the eingenvector associated with $\lambda_3^{(2)}(q)$ is roughly along the (1,1,1) direction for q varying let us say between 1 and 3. Within the present b=2 RG approximation the critical exponents are given by $\nu_2 = \ell n 2/\ell n \lambda_1^{(2)}$ and

$\phi_{23} = \ell n\lambda_3^{(2)}/\ell n\lambda_1^{(2)}$: see Figs. 4 and 5 and Table II.

The Jacobian at the d=3 fixed point $(t_x=t_y=t_z=t_c^{(3)}(q))$ is as follows:

$$\begin{pmatrix} e(q) & f(q) & f(q) \\ f(q) & e(q) & f(q) \\ f(q) & f(q) & e(q) \end{pmatrix} \tag{20}$$

The eigenvalues are

$$\lambda_1^{(3)} = e(q) + 2f(q) \tag{21}$$

and

$$\lambda_2^{(3)} = \lambda_3^{(3)} = e(q) - f(q) \tag{22}$$

and the eigenvectors are respectively (1,1,1) and any vector perpendicular to (1,1,1). We verify $\lambda_1^{(3)}(q) \geq 1 \geq \lambda_2^{(3)}(q) > 0$, $\forall\, q \geq 0$. The corresponding approximate critical exponent is given by $\nu_3 = \ell n2/\ell n\lambda_1^{(3)}$ (see Fig. 6 and Table II); $\lambda_2^{(3)}(q)$ monotonically increases from roughly zero to 1 when q varies from zero to infinity.

IV. EXTRAPOLATION FOR THE CRITICAL POINT

In the present Section we describe an ad hoc extrapolation procedure for the critical temperature T_c for an arbitrary value of q. We take advantage from the fact that the anisotropic d=2 RG result is the *exact* one for all q, and that the isotropic d=3 RG result is not too bad (at least for q=1,2,3, where comparison with other results is possible). It essentially consists in "pushing" the center $(s_x^{(2)} = s_y^{(2)} = s_z^{(2)} = s_c^{(2)})$ of the RG critical surface in the $(s_x^{(2)}, s_y^{(2)}, s_z^{(2)})$ -space (see Fig. 2), until it coincides (by imposition) with the best value (noted s_0; usually from series) available in the literature for that particular value of q; the effects of this "pushing" monotonically and softly decrease while going from the center of the critical surface to its periphery, eventually vanishing on the anisotropic d=2 limiting case (i.e. $s_x^{(2)} = 0$ or $s_y^{(2)} = 0$ or $s_z^{(2)} = 0$) where, as said before, the exact result is reproduced by the RG. As no confusion can occur in the present Section, we use $s_\alpha \equiv s_\alpha^{(2)}$ $(\alpha = x,y,z)$, where $s_\alpha^{(2)}$ is given by Eq. (9b) with h(2)=1. Summarising, the input, for a given q, of the extrapolation procedure is the RG critical surface *and* the "exact" value for the isotropic d=3 critical point.

We consider, in the (s_x, s_y, s_z)-space (see Fig. 7.a), the point P (on the RG critical surface and not belonging to the trisectrix $s_x = s_y = s_z$) to be extrapolated; its coordinates are noted (s_x^P, s_y^P, s_z^P) and conventionally satisfy $1 \geq s_x^P \geq s_y^P \geq s_z^P \geq 0$ (every other region is directly associated with this one through trivial symmetry transformations). This point and the trisectrix determine a unique plane whose equation is given by

$$\frac{s_y - s_z}{s_x - s_z} = \frac{s_y^P - s_z^P}{s_x^P - s_z^P} \equiv g \in [0,1] \tag{23}$$

This plane and the plane

$$s_x + s_y + s_z = 1 \qquad (24)$$

(which contains all three *exact* d=2 critical lines, e.g., $s_x+s_y=1$ for $s_z=0$) determine a unique straight line. This line cuts the $s_z=0$ plane at the point $(s_x^{(z)}, s_y^{(z)}, 0)$ and the $s_x=0$ plane at the point $(0, s_y^{(x)}, s_z^{(x)})$, where

$$s_x^{(z)} = 1/(1+g) \qquad (25.a)$$

$$s_y^{(z)} = g/(1+g) \qquad (25.b)$$

$$s_y^{(x)} = (1-g)/(2-g) \qquad (25.c)$$

$$s_z^{(x)} = 1/(2-g) \qquad (25.d)$$

This line also cuts the trisectrix at the point T with coordinates (1/3,1/3,1/3). If we consider now the triangle determined by the points (0,0,0), $(s_x^{(z)}, s_y^{(z)}, 0)$ and $(0, s_y^{(x)}, s_z^{(x)})$ (see Fig. 7.b), we immediately obtain that

$$r_1 = + \left[(1/3)^2 + (1/3-s_y^{(x)})^2 + (1/3-s_z^{(x)})^2\right]^{1/2} \qquad (26.a)$$

$$r_2 = + \left[(1/3-s_x^{(z)})^2 + (1/3-s_y^{(z)})^2 + (1/3)^2\right]^{1/2} \qquad (26.b)$$

where r_1 and r_2 are defined in Fig. 7.b (r_1 and r_2 respectively correspond to $(s_x^{(z)}, s_y^{(z)}, 0)$ and $(0, s_y^{(x)}, s_z^{(x)})$). The angle θ defined in Fig. 7.b is determined by

$$\cos\theta = \frac{s_x^p + s_y^p + s_z^p}{\sqrt{3} \quad s^p} \tag{27}$$

where

$$s^p \equiv + \sqrt{(s_x^p)^2 + (s_y^p)^2 + (s_z^p)^2} \tag{28}$$

The quantity r^p defined in Fig. 7.b is given by

$$r^p = - \frac{\tan\theta}{\sqrt{3}} \in [-r_2, r_1] \tag{29}$$

Obviously $s^p \leq [1/3 + (r^p)^2]^{1/2}$.

The value s^p is going to be extrapolated into s^{ex} through the relation

$$s^{ex} = s^p [1 + F(r^p)] \tag{30}$$

where the extrapolating function F(r) is assumed to satisfy the following conditions:

(i) $$F(r_1) = F(-r_2) = 0 \quad ; \tag{31.a}$$

(ii) $$F(0) = \frac{\sqrt{3}\, s_o}{s^p} - 1 \quad ; \tag{31.b}$$

(iii) $$F(r) \text{ maximal at } r = 0 \quad . \tag{31.c}$$

The simplest polynomial which satisfies these conditions is

$$F(r) = F(0)[1 - A r^2 - B r^3] \tag{32}$$

where

$$A \equiv \frac{r_2^{\,3} + r_1^{\,3}}{r_1^{\,2} r_2^{\,3} + r_1^{\,3} r_2^{\,2}} \tag{33.a}$$

and

$$B \equiv \frac{1 - A\, r_1^{\,2}}{r_1^{\,3}} \tag{33.b}$$

Finally the coordinates of the extrapolated point are given by

$$s_\alpha^{ex} = \frac{s^{ex}}{s^p}\, s_\alpha^p \qquad (\alpha = x,y,z) \tag{34}$$

In spite of its apparent complexity, the implementation in computer of this extrapolating algorithm is very simple. The operational steps are as follows: (i) given (s_x^p, s_y^p, s_z^p), g is calculated through Eq. (23), and also $s_x^{(z)}$, $s_y^{(z)}$, $s_y^{(x)}$ and $s_z^{(x)}$ through Eqs. (25), hence r_1 and r_2 (through Eqs. (26)) and finally A and B (through Eqs. (33)); (ii) (s_x^p, s_y^p, s_z^p) also determine θ and s^p through Eqs. (27) and (28), which in turn determine r^p through Eq. (29); (iii) s_o (taken from the literature) and s^p determine $F(0)$ through Eq. (31.b); (iv) the knowledge of A, B, r^p and $F(0)$ determines $F(r^p)$ through Eq. (32), hence s^{ex} (through Eq. (30)) and finally $(s_x^{ex}, s_y^{ex}, s_z^{ex})$ through Eq. (34).

The results obtained by using the above algorithm are indicated in Table I. In order to test the confiability of our results we have compared them to series calculations available for q=1 (Fig. 8.a) and q=2 (Fig. 8.b) for the particular cases $0 \le J_z/J_x \le J_y/J_x = 1$ and $0 \le J_y/J_x = J_z/J_x \le 1$. The agreement is very satisfactory (the discrepancy in the t-variable is always smaller than 0.01).

V. CONCLUSION

We have discussed, within a real space renormalisation group framework, the q-state Potts ferromagnet in the fully anisotropic (arbitrary J_x, J_y and J_z) simple cubic lattice. The q-dependences of the critical temperature T_c, the one-, two- and three-dimensional correlation length critical exponents ν_1, ν_2 and ν_3, and the $d=1 \leftrightarrow d>1$ and $d=2 \leftrightarrow d=3$ crossover critical exponents ϕ_{1d} and ϕ_{23} are analysed in the second order phase transition region ($\forall q$ for d=1, $q \leq 4$ for d=2, and $q \leq q_c(3) \simeq 3$ for d=3).

The present renormalisation group reproduces a considerable amount of already known *exact* results such as $t_c^{(1)} = \nu_1 = \phi_{1d} = 1$, $\forall q$, for d=1, $t_c = 1/(\sqrt{q}+1)$ for d=2, etc; it also recovers, in the $q \to 0$ limit, the correct asymptotic behaviour $\nu_2 \propto 1/\sqrt{q}$. Whenever our numerical results do not coincide with available exact or series ones, the discrepancies are acceptable. Furthermore the universality classes we obtain are as commonly expected, i.e. the d=3 one for all values of J_x, J_y and J_z as long as none of them vanishes, and the d=2 one when only one among them vanishes. The general picture inspires reasonable confidence, and therefore we tend to believe that the $q \to 0$ d=3 results $\phi_{23} \propto 1/\sqrt{q}$, $t_c^{(3)}(q) \sim t_c^{(3)}(0) + t_c^{(3)'}(0)q$ and $\nu_3(q) \sim \nu_3(0) + \nu_3'(0)q$ (with finite values for $t_c^{(3)}(0)$, $t_c^{(3)'}(0)$, $\nu_3(0)$ and $\nu_3'(0)$) are correct.

We have also developed an extrapolation procedure for T_c which has proved to be quite satisfactory whenever comparison with other available results (typically from series) was possible, namely for the $0 \leq J_z/J_x \leq J_y/J_x = 1$ and $0 \leq J_y/J_x = J_z/J_x \leq 1$ particular cases of the q=1,2 models. Through this procedure we have calculated T_c for arbitrary ratios J_y/J_x and J_z/J_x and values of q (the q=3 results are probably almost unaffected by the fact that the transition might be slightly first order). A theory which enlarging the parameter space, would succeed in recovering the existence of first order phase transitions would be very wellcome. If alternatively the present RG is understood as refering to the hierarchical lattice defined by Fig. 1(h), then all the results it provides are exact for $q \geq 0$.

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CAPTION FOR FIGURES AND TABLES

Fig. 1 - RG cells and their equivalent two-rooted graphs; the arrows indicate the entrance and exit points of the cells; O and ● respectively denote terminal and internal nodes of the graphs; t_x, t_y and t_z are the transmissivities along the three crystal-axes. (a), (b) and (c) have been used (de Oliveira and Tsallis 1982) for the d=2 case (the cluster (c) is renormalized into the cluster (a)). (d)-(h) correspond to the d=3 case (the cluster (g), or equivalently the graph (h), is renormalized into the cluster (d)). Fig. (g) is the d=3 extension of the central cluster of Fig. (c); Fig. (h) is the d=3 extension of the right graph of Fig. (c); because of its complexity, we have omited the indication of the d=3 extension of the left cluster of Fig. (c).

Fig. 2 - Para(P)-ferro(F) magnetic critical surface in the $(s_x^{(2)}, s_y^{(2)}, s_z^{(2)})$ space. The arrows indicate the RG flow. The main fixed points are indicated: Δ(ferromagnetic) and ▲ (paramagnetic) attract all the points respectively above and below the critical surface; □, O and ● respectively are the d=1, d=2 and d=3 critical fixed points.

Fig. 3 - q-dependence of the RG critical point corresponding to the isotropic d=3 model (notice the ordinate scale). The dots are series results: q=1 (Gaunt and Ruskin 1978), q=2 (Zinn-Justin 1979) and q=3 (Jensen and Mouritsen 1979).

Fig. 4 - q-dependence of the d=2 correlation length critical exponent ν_2: RG (——) and exact (---;den Nijs 1979).

Fig. 5 - q-dependence of the d=2 ↔ d=3 RG crossover exponent ϕ_{23}. The dots are series (●; Redner and Stanley 1979) and

exact (O; Liu and Stanley 1979,1973, Citteur and Kasteleyn 1972,1973).

Fig. 6 - q-dependence of the d=3 correlation length critical exponent ν_3: RG (——) and series (●; Heerman and Stauffer 1981 for q=1; Le Guillou and Zinn-Justin 1980 for q=2).

Fig. 7 - Geometric constructions related to the extrapolation procedure (see Section IV): (a) the $(s_x^{(2)}, s_y^{(2)}, s_z^{(2)})$ space; (b) the triangle determined by the points O, P and T of (a).

Fig. 8 - Present extrapolated results (——) for the critical point corresponding to the particular anisotropic d=3 case where two coupling constants are assumed equal ($\equiv J_\perp$) and the third one ($\equiv J_\parallel$) eventually different. We have the isotropic d=1, d=2 and d=3 cases at the ordinate, abcissa and bissectrix respectively. (a) q=1; the dots are series results (Redner and Stanley 1979); (b) q=2; both dots (Oitmaa and Enting 1971) and circles (Paul and Stanley 1972) are series results.

Table I - Critical points (k_BT_c/qJ_x) for the anisotropic d=3 model: RG (top) and extrapolated (bottom) values. * indicates exact results (see for example Wu 1982) for the isotropic d=2 case; §, & and §§ are series results (see the text and Fig. 3) for the isotropic d=3 case.

Table II - Present RG and exact (or series) results for the critical point t_c and exponents ν and ϕ for the isotropic d-dimensional models. (a) Wu 1982 and references therein; (b) den Nijs MPM 1979; (c) Redner and Stanley 1979; (d) Liu and Stanley 1972,1973; Citteur and Kasteleyn 1972; 1973; (e) Gaunt and Ruskin 1978; (f) Zinn-Justin 1979; (g) Jensen and Mouritsen 1979; (h) Heerman and Stauffer 1981; (i) Le Guillou and Zinn-Justin 1980.

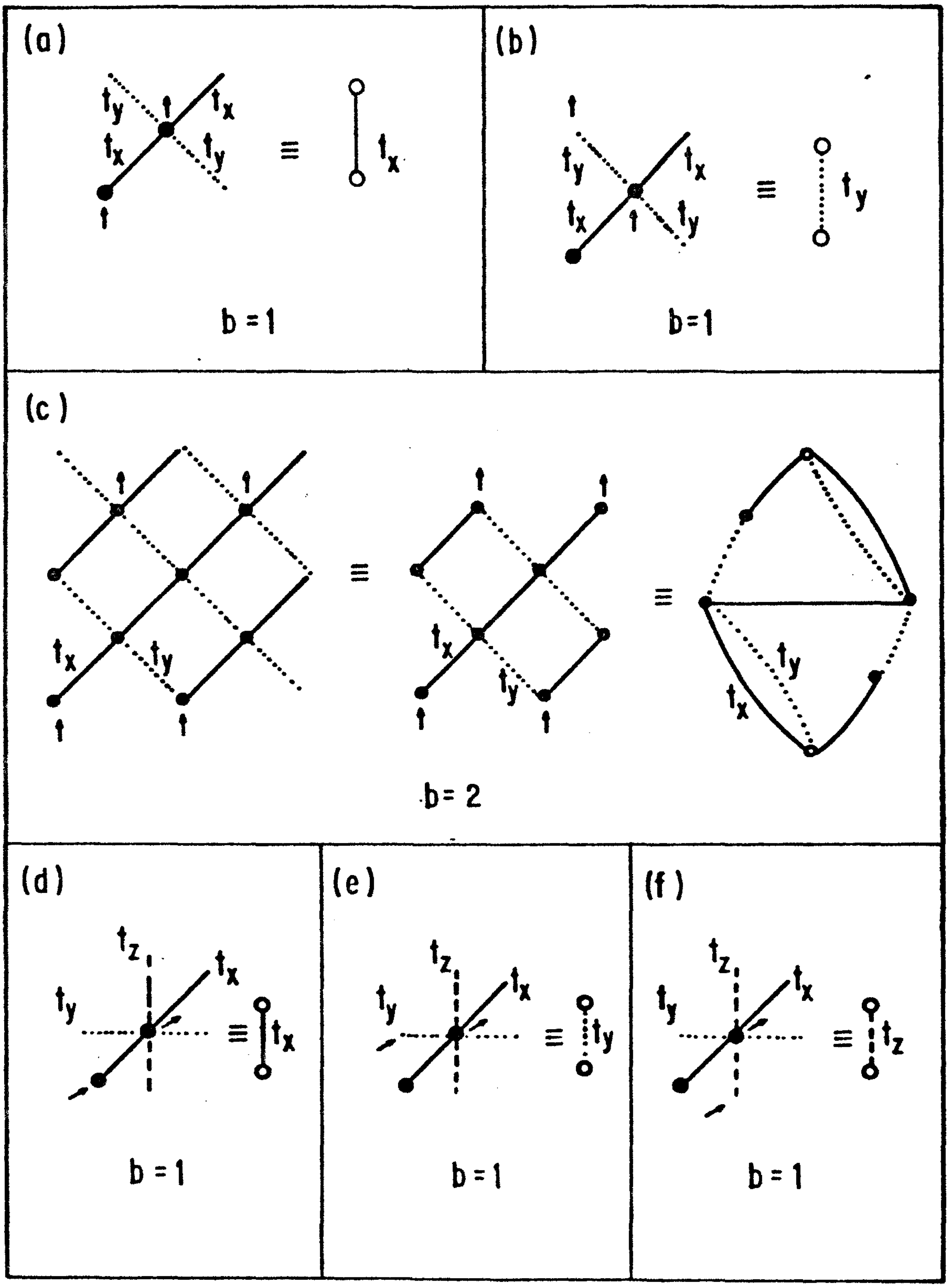
(a)
t_y
t_x
t_x
t_y
≡
t_x
b = 1
(b)
t_y
t_x
t_x
t_y
≡
t_y
b=1
(c)
t_x
t_y
≡
t_x
t_y
≡
t_y
t_x
b= 2
(d)
t_z
t_x
t_y
≡
t_x
b = 1
(e)
t_z
t_x
t_y
≡
t_y
b = 1
(f)
t_z
t_x
t_y
≡
t_z
b = 1

FIG.1

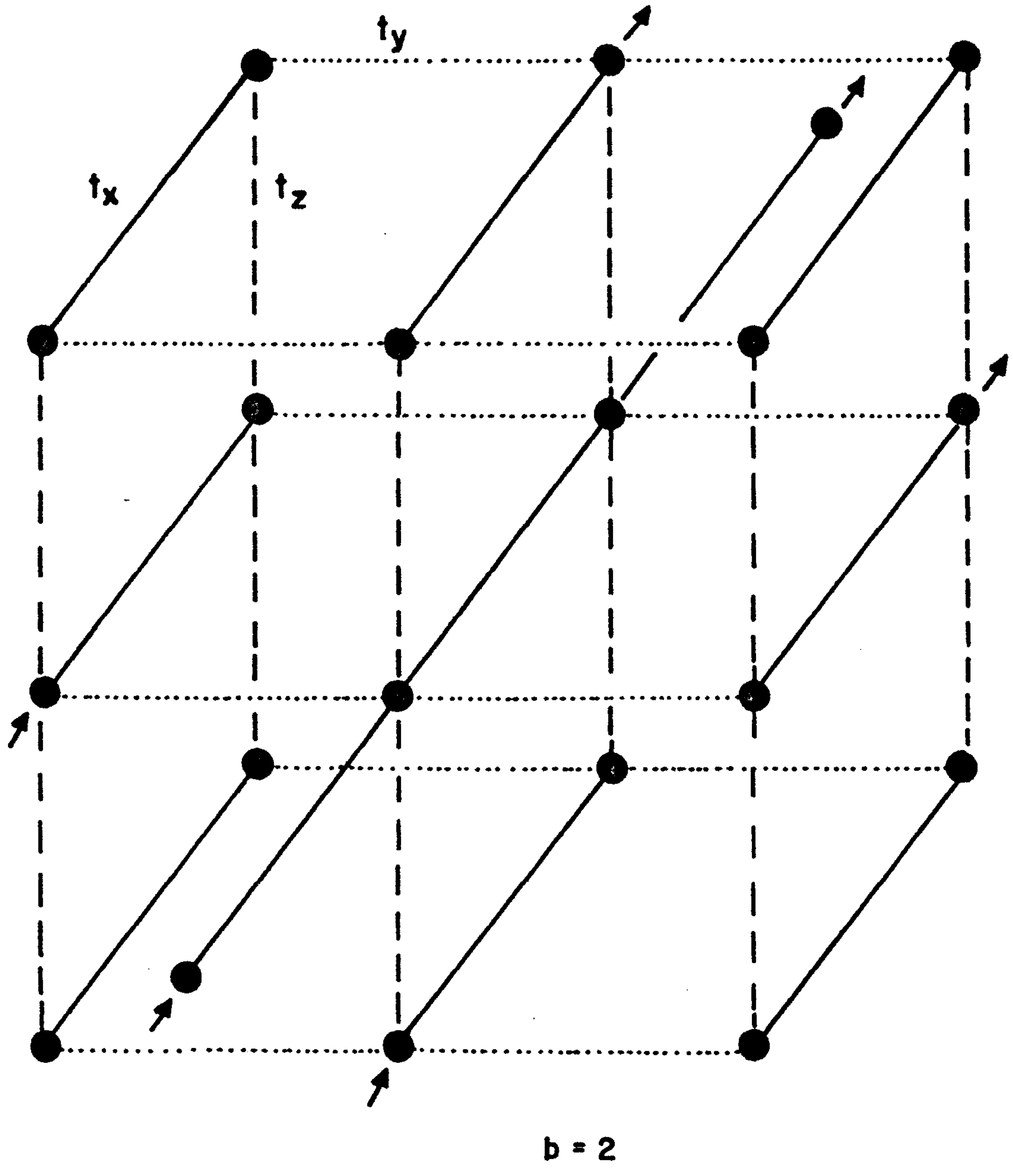

FIG .1g

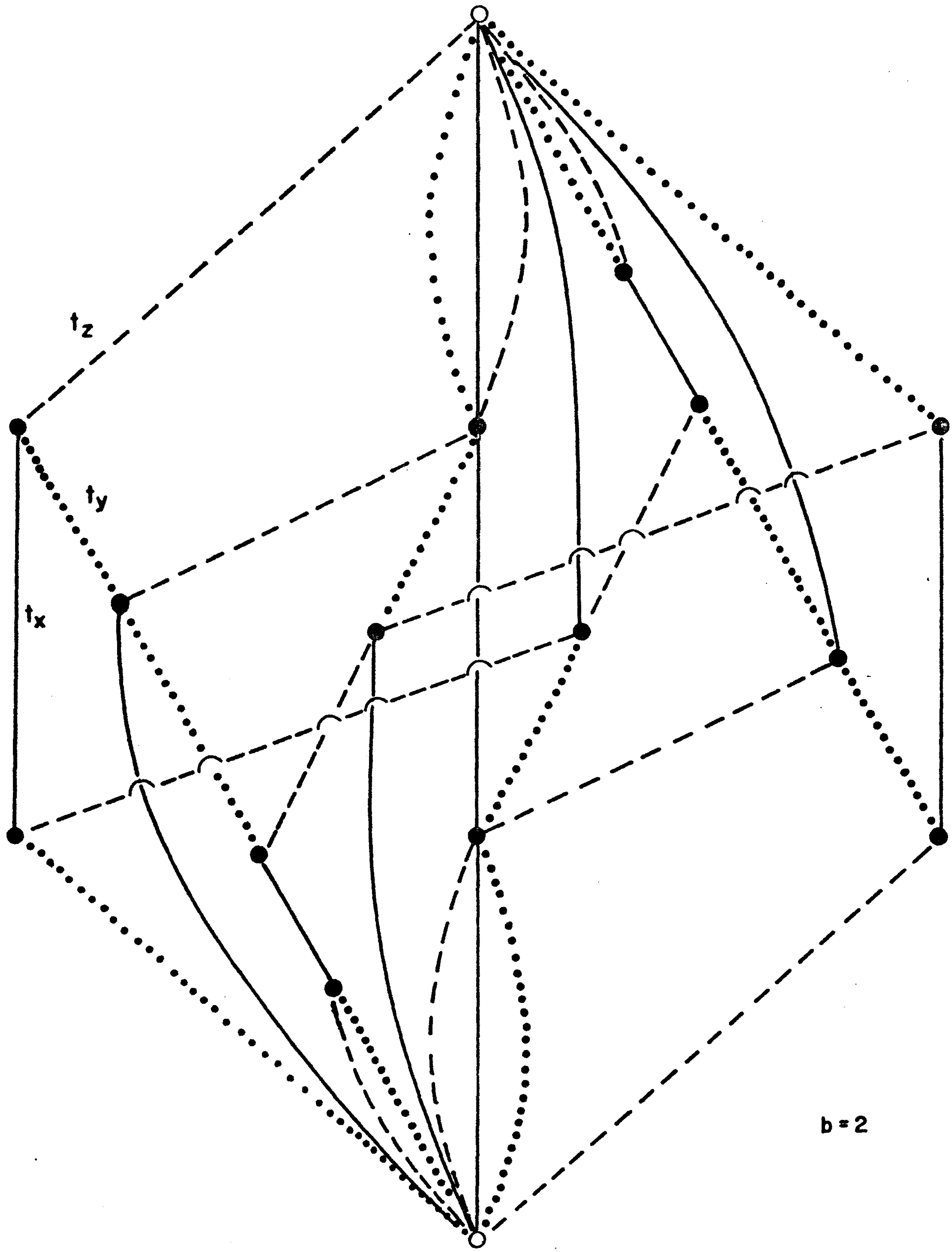

FIG . 1h

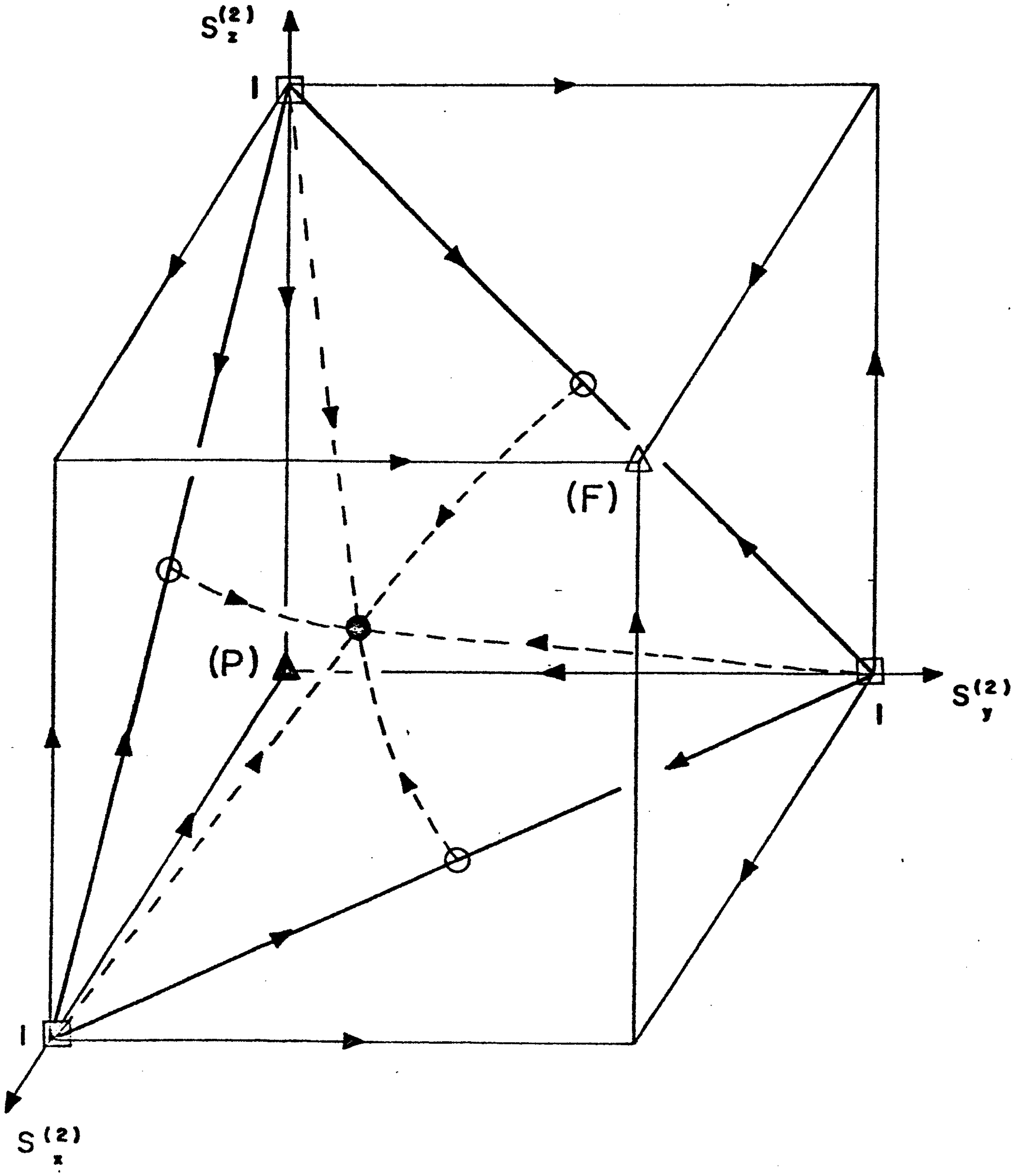

FIG. 2

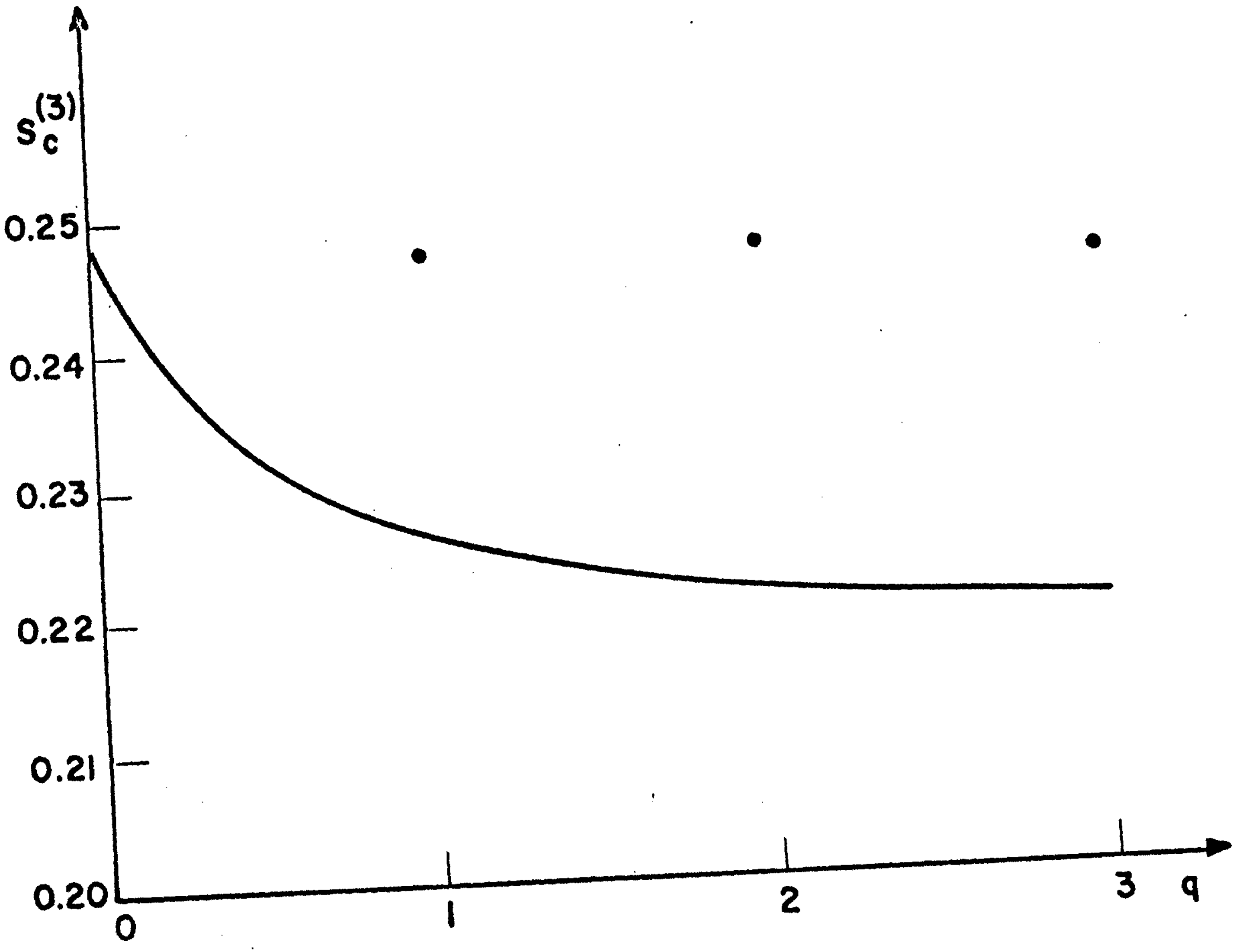

FIG. 3

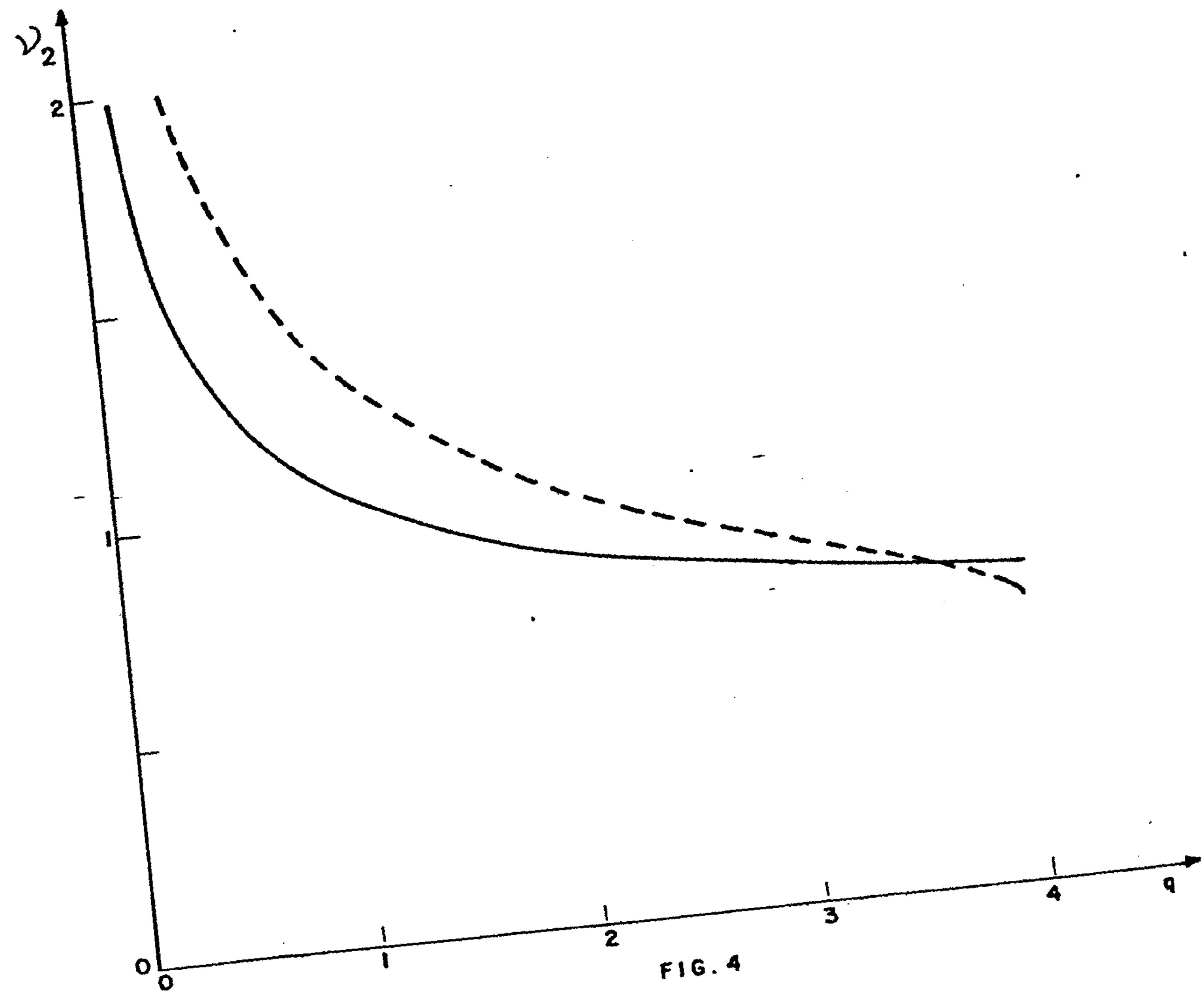

FIG. 4

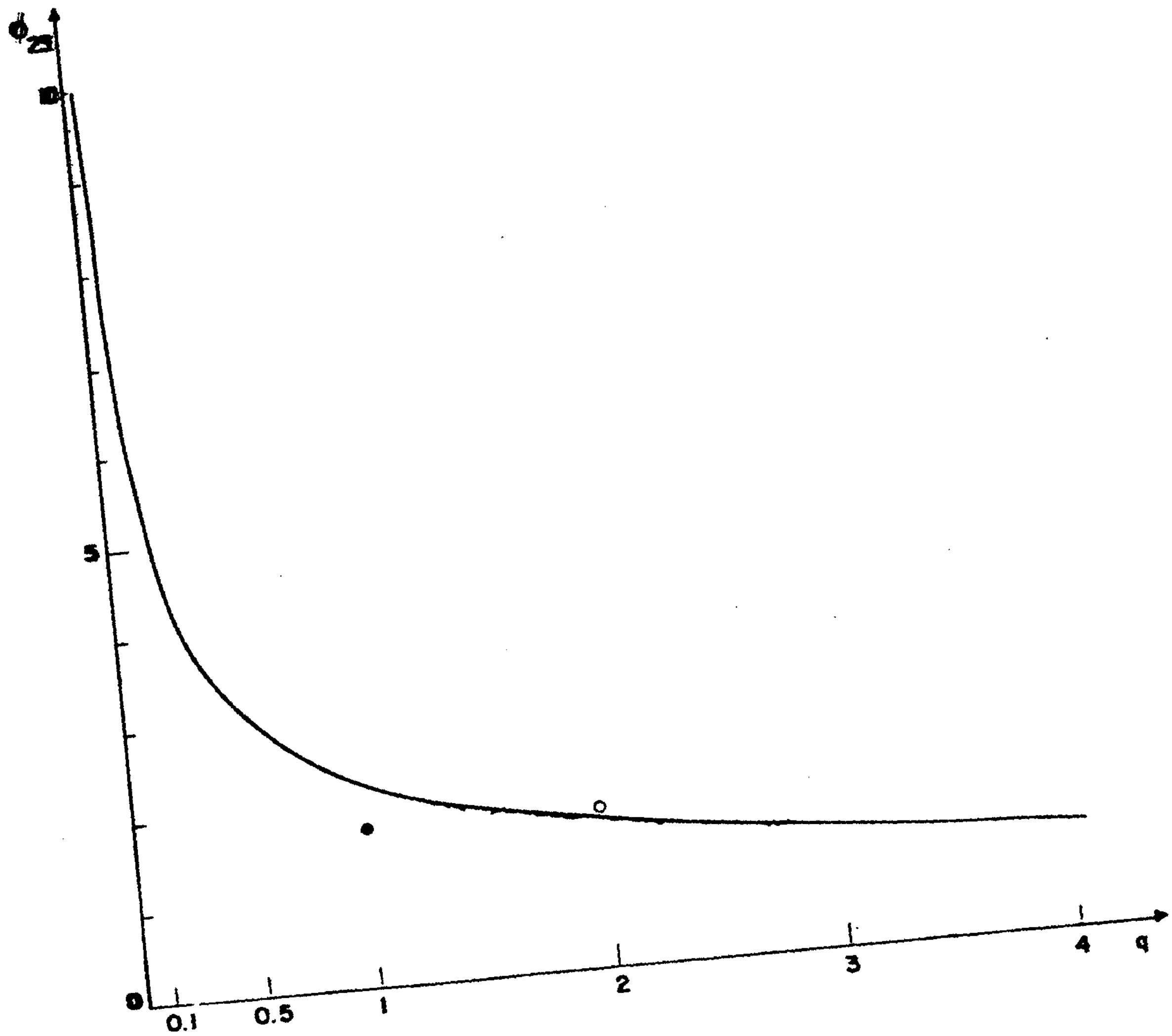

FIG.5

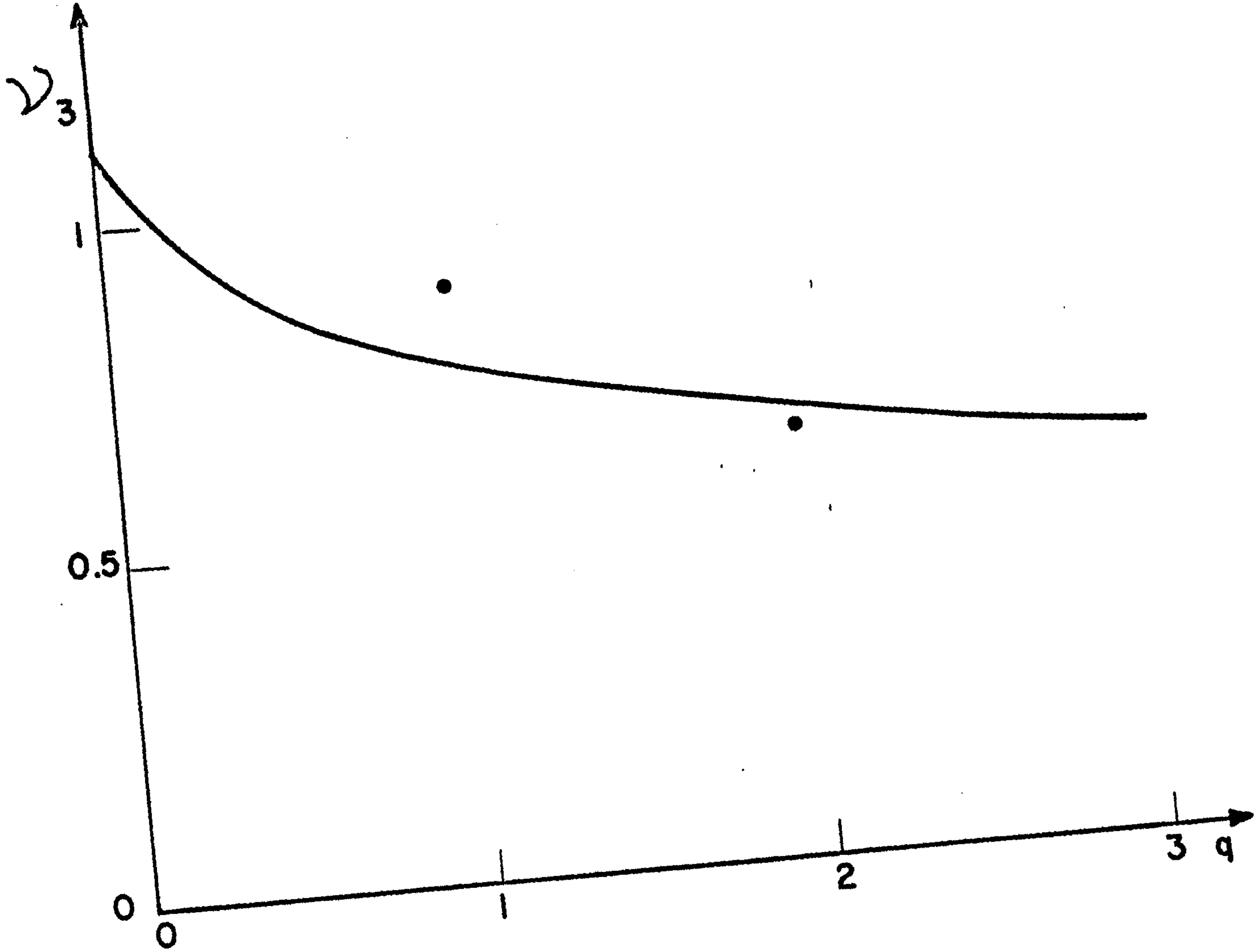

FIG. 6

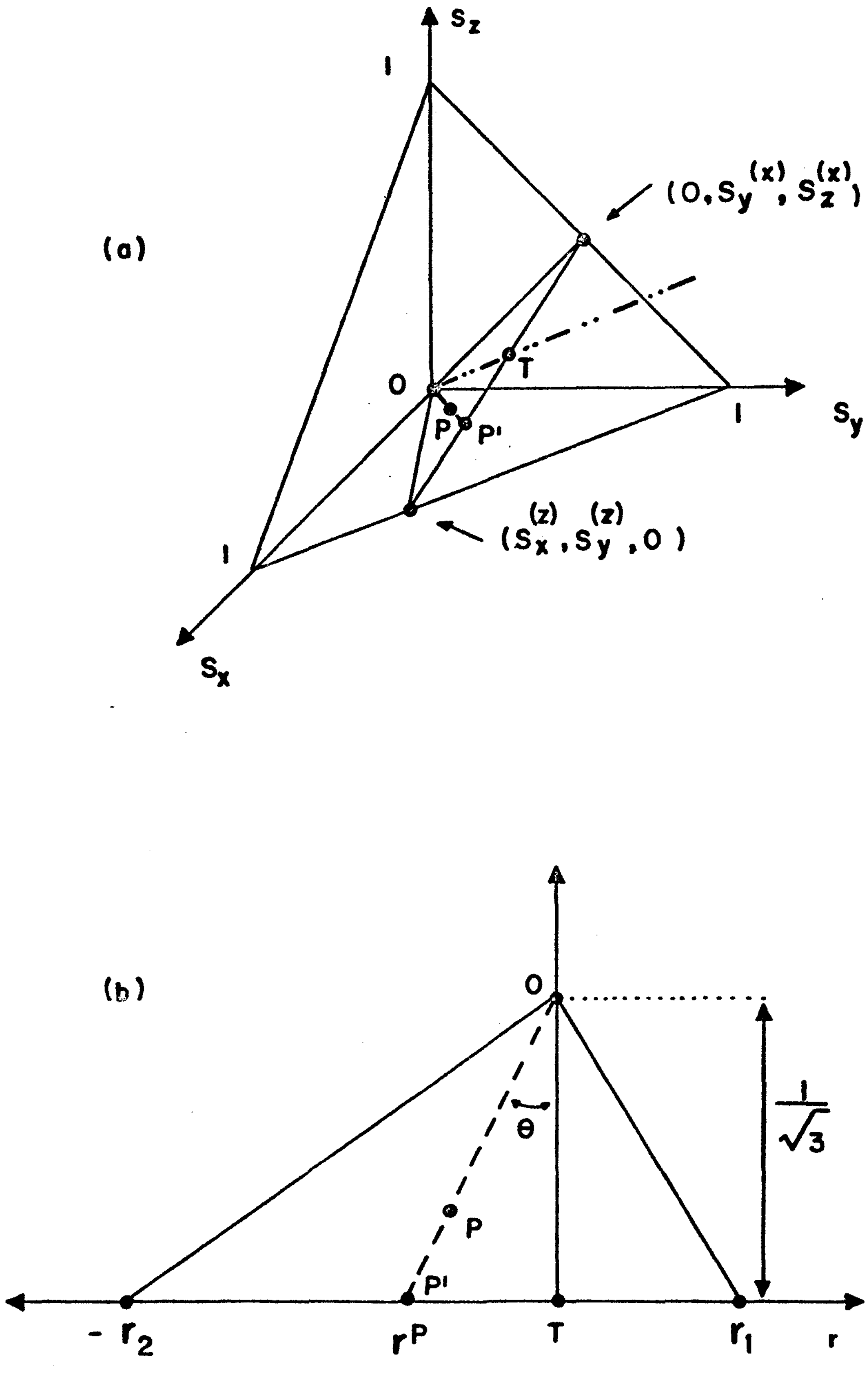
(a)
S_z
1
$(0, S_y^{(x)}, S_z^{(x)})$
T
0
P
P'
1
S_y
$(S_x^{(z)}, S_y^{(z)}, 0)$
1
S_x
(b)
0
θ
$\frac{1}{\sqrt{3}}$
P
P'
$-r_2$
r^P
T
r_1
r

FIG. 7

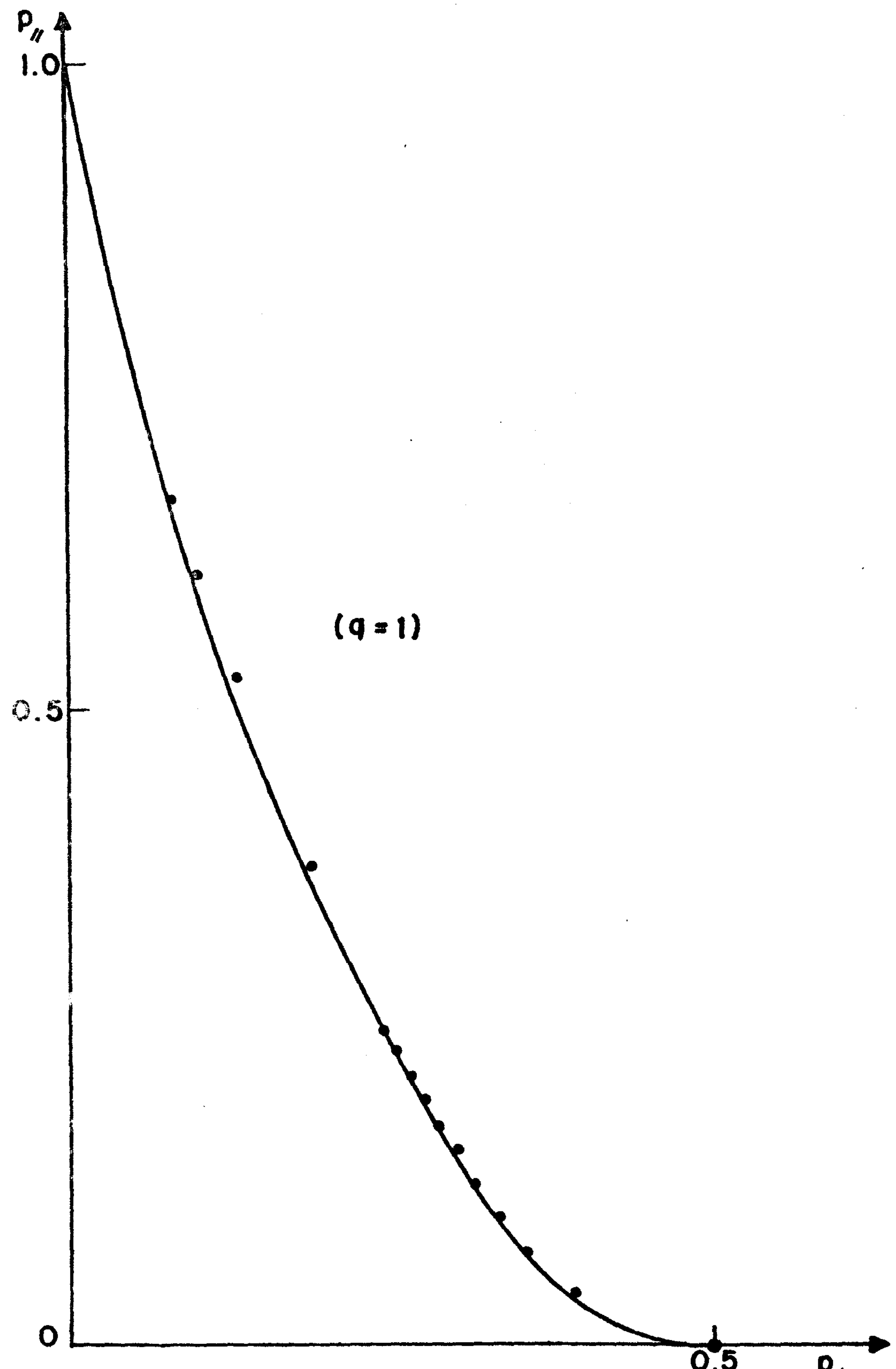

FIG. 8a

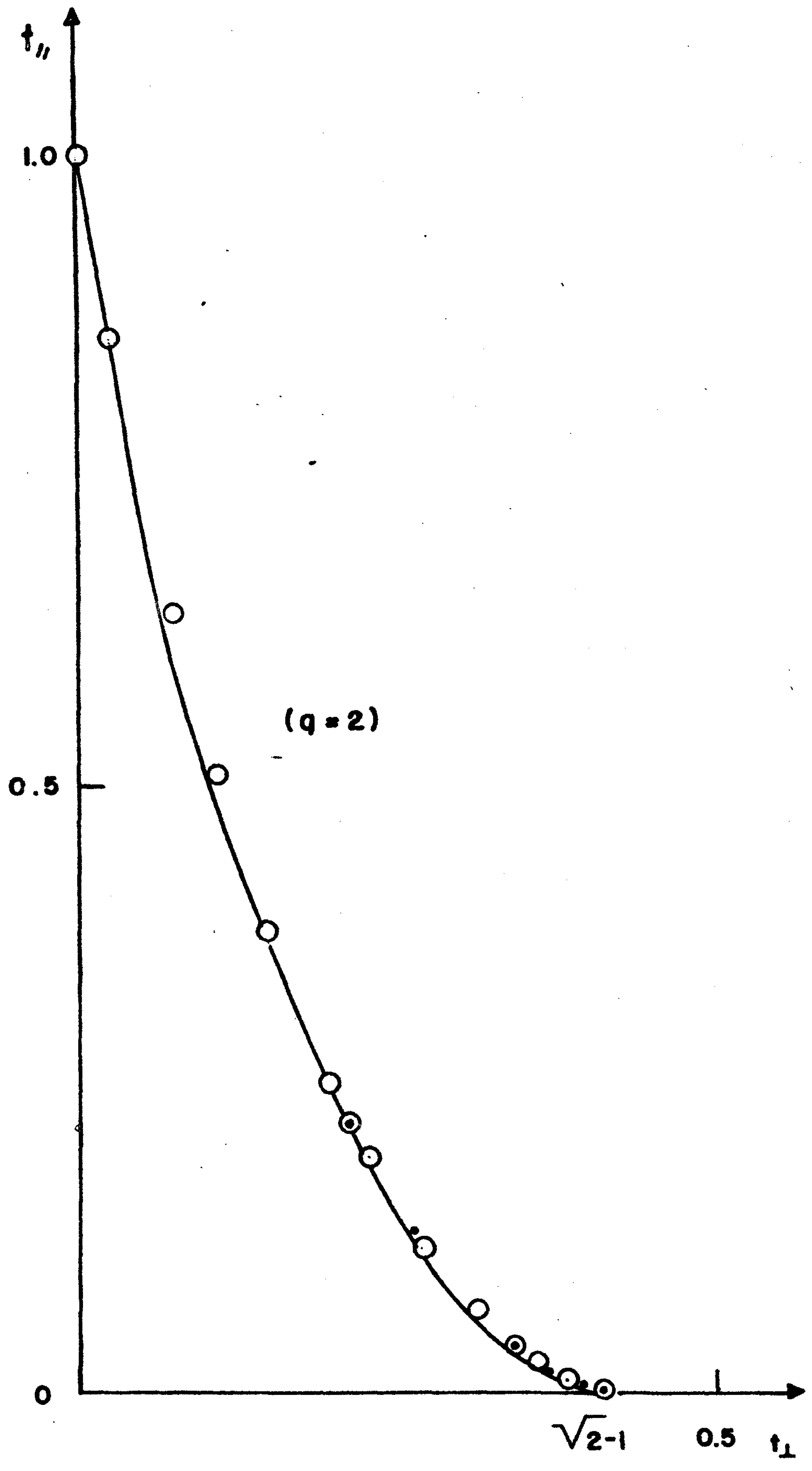

FIG. 8 b

TABLE I

(a) q = 1

J_z/J_x \ J_y/J_x	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
0	0 0	0.5548 0.5548	0.7112 0.7112	0.8349 0.8349	0.9421 0.9421	1.0390 1.0390	1.1287 1.1287	1.2129 1.2129	1.2928 1.2928	1.3692 1.3692	1.4427 1.4427*
0.1	– –	1.0592 1.0229	1.2643 1.2104	1.4255 1.3612	1.5646 1.4934	1.6896 1.6139	1.8047 1.7260	1.9125 1.8317	2.0145 1.9322	2.1117 2.0284	2.2049 2.1210
0.2	– –	– –	1.4912 1.4061	1.6689 1.5636	1.8216 1.7024	1.9584 1.8293	2.0840 1.9479	2.2013 2.0601	2.3121 2.1670	2.4174 2.2696	2.5183 2.3686
0.3	– –	– –	– –	1.8592 1.7259	2.0223 1.8690	2.1682 2.0002	2.3019 2.1231	2.4266 2.2394	2.5440 2.3506	2.6557 2.4575	2.7625 2.5607
0.4	– –	– –	– –	– –	2.1942 2.0158	2.3476 2.1504	2.4882 2.2766	2.6190 2.3963	2.7421 2.5107	2.8591 2.6209	2.9708 2.7273
0.5	– –	– –	– –	– –	– –	2.5078 2.2882	2.6543 2.4172	2.7905 2.5397	2.9186 2.6569	3.0402 2.7698	3.1563 2.8789
0.6	– –	– –	– –	– –	– –	– –	2.8061 2.5489	2.9472 2.6739	3.0798 2.7935	3.2055 2.9088	3.3256 3.0203
0.7	– –	– –	– –	– –	– –	– –	– –	3.0927 2.8012	3.2294 2.9232	3.3590 3.0406	3.4826 3.1543
0.8	– –	– –	– –	– –	– –	– –	– –	– –	3.3699 3.0472	3.5031 3.1667	3.6300 3.2824
0.9	– –	– –	– –	– –	– –	– –	– –	– –	– –	3.6395 3.2882	3.7696 3.4057
1.0	– –	– –	– –	– –	– –	– –	– –	– –	– –	– –	3.9026 3.5250§

TABLE I

(b) q = 2

J_z/J_x \ J_y/J_x	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
0	0 0	0.4529 0.4529	0.5708 0.5708	0.6644 0.6644	0.7462 0.7462	0.8205 0.8205	0.8897 0.8897	0.9550 0.9550	1.0172 1.0172	1.0769 1.0769	1.1346 1.1346[*]
0.1	– –	0.7276 0.6993	0.8578 0.8165	0.9631 0.9141	1.0555 1.0018	1.1398 1.0828	1.2183 1.1591	1.2925 1.2316	1.3631 1.3012	1.4309 1.3682	1.4964 1.4332
0.2	– –	– –	0.9967 0.9323	1.1090 1.0299	1.2075 1.1184	1.2972 1.2009	1.3806 1.2792	1.4593 1.3540	1.5341 1.4261	1.6059 1.4958	1.6751 1.5634
0.3	– –	– –	– –	1.2270 1.1275	1.3304 1.2164	1.4243 1.2998	1.5116 1.3791	1.5939 1.4553	1.6721 1.5289	1.7470 1.6002	1.8192 1.6696
0.4	– –	– –	– –	– –	1.4379 1.3058	1.5355 1.3898	1.6262 1.4700	1.7115 1.5472	1.7926 1.6219	1.8702 1.6944	1.9449 1.7652
0.5	– –	– –	– –	– –	– –	1.6365 1.4746	1.7301 1.5556	1.8181 1.6336	1.9018 1.7092	1.9818 1.7828	2.0588 1.8546
0.6	– –	– –	– –	– –	– –	– –	1.8264 1.6373	1.9169 1.7162	2.0029 1.7927	2.0851 1.8671	2.1642 1.9399
0.7	– –	– –	– –	– –	– –	– –	– –	2.0098 1.7958	2.0979 1.8732	2.1821 1.9485	2.2630 2.0221
0.8	– –	– –	– –	– –	– –	– –	– –	– –	2.1880 1.9513	2.2741 2.0274	2.3568 2.1019
0.9	– –	– –	– –	– –	– –	– –	– –	– –	– –	2.3619 2.1044	2.4463 2.1796
1.0	– –	– –	– –	– –	– –	– –	– –	– –	– –	– –	2.5323 2.2556[&]

TABLE I

(c) $q = 3$

J_z/J_x \ J_y/J_x	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
0	0 0	0.4060 0.4060	0.5066 0.5066	0.5869 0.5869	0.6572 0.6572	0.7213 0.7213	0.7813 0.7813	0.8380 0.8380	0.8923 0.8923	0.9445 0.9445	0.9950 0.9950*
0.1	– –	0.6044 0.5820	0.7072 0.6749	0.7918 0.7537	0.8670 0.8251	0.9360 0.8917	1.0008 1.9548	1.0622 1.0150	1.1211 1.0730	1.1778 1.1291	1.2327 1.1836
0.2	– –	– –	0.8144 0.7642	0.9028 0.8413	0.9814 0.9123	1.0537 0.9791	1.1214 1.0429	1.1857 1.1043	1.2472 1.1636	1.3065 1.2213	1.3638 1.2774
0.3	– –	– –	– –	0.9945 0.9174	1.0761 0.9878	1.1510 1.0547	1.2212 1.1187	1.2878 1.1807	1.3515 1.2408	1.4128 1.2993	1.4721 1.3565
0.4	– –	– –	– –	– –	1.1602 1.0580	1.2374 1.1249	1.3098 1.1392	1.3783 1.2515	1.4439 1.3122	1.5070 1.3714	1.5681 1.4293
0.5	– –	– –	– –	– –	– –	1.3168 1.1918	1.3911 1.2564	1.4614 1.3191	1.5287 1.3802	1.5934 1.4399	1.6560 1.4985
0.6	– –	– –	– –	– –	– –	– –	1.4671 1.3213	1.5391 1.3843	1.6080 1.4459	1.6742 1.5061	1.7382 1.5652
0.7	– –	– –	– –	– –	– –	– –	– –	1.6127 1.4478	1.6830 1.5098	1.7506 1.5705	1.8159 1.6301
0.8	– –	– –	– –	– –	– –	– –	– –	– –	1.7546 1.5723	1.8235 1.6335	1.8900 1.6936
0.9	– –	– –	– –	– –	– –	– –	– –	– –	– –	1.8936 1.6952	1.9613 1.7557
1.0	– –	– –	– –	– –	– –	– –	– –	– –	– –	– –	2.0300 1.8169§§

TABLE II

d			$q \to 0$	$q = 1$	$q = 2$	$q = 3$	$q = 4$
d=1	$t_c^{(1)}$	RG($\forall b$)	1	1	1	1	1
		exact	1 [a]	1 [a]	1 [a]	1 [a]	1 [a]
	ν_1	RG($\forall b$)	$\frac{\ell n\, b}{\ell n(2b-1)}$	$\frac{\ell n\, b}{\ell n(2b-1)}$	$\frac{\ell n\, b}{\ell n(2b-1)}$	$\frac{\ell n\, b}{\ell n(2b-1)}$	$\frac{\ell n\, b}{\ell n(2b-1)}$
		exact	1 [a]	1 [a]	1 [a]	1 [a]	1 [a]
	ϕ_{1d}	RG($\forall b$)	1	1	1	1	1
		exact	1 [a]	1 [a]	1 [a]	1 [a]	1 [a]
d=2	$t_c^{(2)}$	RG($\forall b$)	$\sim 1 - \sqrt{q}$	1/2	$\sqrt{2}-1$	$1/(\sqrt{3}+1)$	1/3
		exact	$\sim 1 - \sqrt{q}$ [a]	1/2 [a]	$\sqrt{2}-1$ [a]	$1/(\sqrt{3}+1)$ [a]	1/3 [a]
	ν_2	RG(b=2)	$\frac{45\ell n 2}{52\sqrt{q}} \simeq \frac{0.600}{\sqrt{q}}$	1.042	0.864	0.785	0.738
		exact	$\frac{\pi}{3\sqrt{q}} \simeq \frac{1.047}{\sqrt{q}}$ [b]	$4/3 \simeq 1.333$ [b]	1 [b]	$5/6 \simeq 0.833$ [b]	$2/3 \simeq 0.667$ [b]
	ϕ_{23}	RG(b=2)	$\simeq \frac{2}{\sqrt{q}}$	2.258	1.637	1.346	1.163
		exact or series	-	1.75 [c]	1.75 [d]	-	-
d=3	$t_c^{(3)}$	RG(b=2)	$\simeq 0.294 - 0.11q$	0.2260	0.1949	0.1750	-
		series	-	0.247 [e]	0.21811 [f]	0.1966 [g]	-
	ν_3	RG(b=2)	$\simeq 1.105 - 0.66q$	0.756	0.657	0.606	-
		series	-	0.88 [h]	0.630 [i]	-	-