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ON THE OPERATOR $p^4+V(r)$

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ABSTRACT

Eigenvalues and eigenfunctions of the operator $\frac{d^4}{dr^4} + V(r)$ are discussed for different examples. A generalization of the usual harmonic oscillator is discussed for fourth order equations in the Appendix.

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1 INTRODUCTION

For several reasons, it is interesting to consider differential equations of order higher than the second. For example, they appear in discussions of quantum gravity[1]. Also in an extension of supersymmetric Weiss-Zumino theories of higher dimensions[2,3] it was obtained a generalized Klein-Gordon equation of the form

$$\left[\Box^{\frac{d}{2}-1} + (m^2)^{\frac{d}{2}-1}\right]\phi = 0 \ . \tag{1}$$

This equation, for d=4, gives the usual Klein-Gordon equation and for d=6

$$\Box\Box\phi + m^4\phi = 0 \ . \tag{2}$$

Several arguments have been advanced to try theories in spaces of higher dimensions. Once they have been accepted into the game then, for the reasons mentioned above, higher order equations might also be considered in it.

Therefore, it is justified to try to gain some experience about the properties of systems obeying higher order equations.

To go from the higher order Lagrangians to a Hamiltonian system would require a detailed analysis. In order to simplify matters we will assume the following equation for stationary states:

$$H\psi = E\psi \tag{3}$$

with

$$H = \nabla^2\nabla^2 + V(r) \,, \tag{4}$$

and we will discuss this equation for different potentials V(r) and for spherically symmetric solutions.

In Section 2, in order to illustrate the main line of the method, we revisit the usual hydrogen atom.

In Section 3, we discuss the case in which V(r) is a δ function.

In Section 4, the potential $V(r) = -\alpha/r$ is considered. This is the Green's function of the bilaplacian operator in five dimensions.

In Section 5, boundary conditions are discussed. Finally, in two appendices we show briefly an equation of the fourth order with solutions parallel to that of the usual harmonic oscillator and elaborate on the conditions of self-adjointness.

2 THE HYDROGEN ATOM REVISITED

In order to illustrate the method we shall follow for the fourth order, we first consider the usual second order Schrödinger equation for the spherically symmetric solutions of the hydrogen atom

$$(-\nabla^2 - \frac{\alpha}{r})\psi = E\psi \tag{5}$$

which, with $\psi=\phi/r$, can be written as

$$\frac{d^2\phi}{dr^2} + \frac{\alpha}{r}\phi = -E\phi \,. \tag{6}$$

We shall use the Laplace transform (L)

$$\phi(r) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{pr}\phi(p)\,dp, \quad \phi(p) = \int_0^\infty e^{-pr}\phi(r)\,dr. \tag{7}$$

We look for a solution with physical boundary conditions

$$\phi(o) = 0, \quad \dot{\phi}(o) = 1 \quad . \tag{8}$$

Taking into account that (Ref. 5, p. 129)

$$L(\frac{1}{r}\ \phi(r)) = \int_p^\infty \phi(p')dp', \tag{9}$$

we get for $\phi(p)$:

$$p^2\phi(p) + \alpha\int_p^\infty \phi(p')dp' + E\phi(p) = 0 \tag{10}$$

and after taking the derivative,

$$(p^2+E)\ \frac{d\phi}{dp} + (2p-\alpha)\phi(p) = 0 \tag{11}$$

which on integration leads to

$$\phi(p) = \frac{1}{p^2+E}\left(\frac{p-i\sqrt{E}}{p+i\sqrt{E}}\right)^{\frac{\alpha}{2i\sqrt{E}}} , \tag{12}$$

which is valid for any real E.

For $E < 0$ it is convenient to write (12) in the form

$$\phi(p) = \frac{1}{p^2-|E|}\left(\frac{p-|E|^{1/2}}{p+|E|^{1/2}}\right)^{\frac{\alpha}{2|E|^{1/2}}} \tag{13}$$

where we used $E = -|E|$.

The Laplace anti-transform (L^{-1}) of $(p-\lambda_1)^{\beta_1}(p-\lambda_2)^{\beta_2}$ is (Ref. 5, p.238)

$$L^{-1}[(p-\lambda_1)^{\beta_1}(p-\lambda_2)^{\beta_2}] = \frac{r^{-\beta_1-\beta_2-1}e^{\lambda_1 r}}{\Gamma(-\beta_1-\beta_2)}$$

$$\times\ {}_1F_1(-\beta_2;-\beta_2-\beta_1;(\lambda_2-\lambda_1)r). \tag{14}$$

To find out which values of E correspond to eigenfunctions we must look for the behavior of ${}_1F_1$ as $r\to\infty$. We have, for the asymptotic behavior of ${}_1F_1$ (Ref. 5, p.278)

$${}_1F_1(-\beta_2;-\beta_2-\beta_1;2|E|^{1/2}r) = \frac{\Gamma(-\beta_2-\beta_1)}{\Gamma(-\beta_2)}\,(2|E|^{1/2}r)^{\beta_1}\,e^{2|E|^{1/2}r} \tag{15}$$

where we used $\lambda_1=-|E|^{1/2}$ and $\lambda_2=|E|^{1/2}$. Thus, for $r\to\infty$,

$$\phi(r)\to(2|E|^{1/2})^{\beta_1}\ \frac{r^{-\beta_2-1}}{\Gamma(-\beta_2)}\,e^{|E|^{1/2}r}\ . \tag{16}$$

This shows that $\phi(r)$ diverges unless we have $\beta_2=n$ where n is a positive integer. That is, from (13),

$$\beta_2 = \frac{\alpha}{2|E|^{1/2}} - 1 = n \tag{17}$$

or

$$E = -\frac{\alpha^2}{4(n+1)^2} \qquad \text{for} \qquad n=0,1,2,... \tag{18}$$

In short, the asymptotic behavior can be directly obtained by noting that the behavior for large r is dominated by the singularity $(p-\lambda)^{\beta}$ which has the largest real part of λ.

In fact,

$$L^{-1}(p-\lambda)^{\beta} = \frac{e^{\lambda r}}{\Gamma(-\beta)r^{\beta+1}} \quad , \tag{19}$$

which leads formally to (17) and (18).

With positive E, the singularities appear on the imaginary axis (see (12)) and this leads to scattering states for any value of E>0.

It is worth noting that equation (11) gives information about the locations of singularities and energy eigenvalues even without knowing its explicit solutions. We assume that near a singularity λ, $\phi(p)$ has the form

$$\phi(p) = (p-\lambda)^{\beta} + 0((p-\lambda)^{\beta+1}) \quad . \tag{20}$$

Replacing in (11) we have

$$\begin{aligned} &[(p-\lambda)^2 + 2\lambda(p-\lambda) + \lambda^2 + E]\beta(p-\lambda)^{\beta-1} \\ &+2[2(p-\lambda) + 2\lambda-\alpha](p-\lambda)^{\beta} = 0 . \end{aligned} \tag{21}$$

Disregarding $(p-\lambda)^{\beta+1}$ we get

$$\lambda^2 = -E \quad , \quad \lambda = \frac{\alpha}{2(\beta+1)} \quad . \tag{22}$$

When the energy is positive, these singularities are located on the imaginary axis giving the scattering states. When the energy is negative, the singularities lay on the real axis at $\lambda=\pm|E|^{1/2}$, In order to avoid the singularity on the right hand side plane, β should be chosen equal to a positive (or zero) integer leading to form (17).

Going back to (8) we want to point out that $\phi(p)$ goes to zero like $1/p^2$ when $|p|\to\infty$. As it has no poles on the right hand side of the p-plane, for r=0 one can close the integration path in (7) with a semicircle to the right. We then see that $\phi(0)=0$, but $\dot{\phi}(0)\neq 0$.

3 δ-FUNCTION POTENTIAL

We now want to find the spherically symmetric solution of the equation

$$\nabla^2\nabla^2\psi - \alpha\delta^3(\vec{r})\psi = E\psi \ . \tag{23}$$

We first look for a solution of (23) which outside the origin is a "free" equation. The boundary conditions at r=0 are left open so as to be able to adjust them to generate the δ-function potential.

We choose $\psi=\phi/r$ and obtain for $r\neq 0$

$$\frac{d^4\phi}{dr^4} = E\phi \ . \tag{24}$$

Taking the Laplace transform we get (Ref. 5, p.129)

$$(p^4-E)\phi(p) = p^3\phi(0) + p^2\phi'(0) + p\phi''(0) + \phi'''(0). \tag{25}$$

If $\phi(0)\neq 0$ then ψ has a 1/r singularity. Since $\nabla^2(\frac{1}{r}) = \delta(\vec{r})$ and $\nabla^2\nabla^2(\frac{1}{r}) = \nabla^2\delta$ this leads to a singularity not contained in equations (23) or (24); so we must impose $\phi(0)=0$, that is

$$\psi(0) = \phi'(0) \ . \tag{26}$$

Thus, in (25) we drop the p^3-term and obtain

$$\phi(p) = \frac{p^2\phi'(0)+p\phi''(0)+\phi'''(0)}{(p-\lambda_1)(p-\lambda_2)(p-\lambda_3)(p-\lambda_\psi)} \tag{27}$$

where the λ_i are the four roots of the equation

$$\lambda_i^4 = E \ . \tag{28}$$

Let us first look for solutions with negative $E=-|E|$. Explicitly,

$$\begin{aligned} \lambda_1 &= \frac{1+i}{\sqrt{2}}\,|E|^{1/4} \quad , \quad \lambda_2 = \lambda_1^* \\ \lambda_3 &= \frac{-1+i}{\sqrt{2}}\,|E|^{1/4} \quad , \quad \lambda_4 = \lambda_3^* \ . \end{aligned} \tag{29}$$

We have seen that singularities in the right hand plane generate solutions which increase exponentially for large r. So we must eliminate them in Eq. (27). These correspond to λ_1 and λ_2; then we choose the constants in such a way that

$$p^2\phi'(0) + p\phi''(0) + \phi'''(0) = \mathbb{C}(p-\lambda_1)(p-\lambda_2) \tag{30}$$

which leads to

$$\begin{aligned} \phi'(0) &= \mathbb{C} \quad , \quad \phi''(0) = -\,\mathbb{C}\sqrt{2}\,|E|^{1/4} \\ \phi'''(0) &= \mathbb{C}|E|^{1/2} \ . \end{aligned} \tag{31}$$

With these values we get

$$\phi(p) = \frac{\mathbb{C}}{(p-\lambda_3)(p-\lambda_4)} \tag{32}$$

whose antilaplace transform is (Ref.5, p.229)

$$\phi(r) = \frac{\mathbb{C}}{\lambda_3-\lambda_4}(e^{\lambda_3 r}-e^{\lambda_4 r}) \quad . \tag{33}$$

That is,

$$\psi(r) = \frac{\sqrt{2}\,\mathbb{C}}{|E|^{1/4} r}\, e^{-\frac{|E|^{1/4}}{\sqrt{2}} r} \sin \frac{|E|^{1/4}}{\sqrt{2}} r \quad . \tag{34}$$

Now we have

$$\nabla^2 \frac{\phi}{r} = (\nabla^2 \frac{1}{r})\, \phi + \frac{1}{r} \frac{d^2\phi}{dr^2} \; . \tag{35}$$

The first term of (35) drops out as $\phi(0)=0$ and we are left with

$$\nabla^2 (\frac{\phi}{r}) = \frac{1}{r} \frac{d^2\phi}{dr^2} \; . \tag{36}$$

From here,

$$\nabla^2\nabla^2 \frac{\phi}{r} = \nabla^2(\frac{1}{r} \frac{d^2\phi}{dr^2}) = (\nabla^2 \frac{1}{r}) \frac{d^2\phi}{dr^2} + \frac{1}{r} \frac{d^4\phi}{dr^4} \quad . \tag{37}$$

Comparing with (23) we have

$$\alpha = -4\pi \frac{\phi''(0)}{\psi(0)} = -4\pi \frac{\phi''(0)}{\phi'(0)} = -4\pi\sqrt{2}\,|E|^{1/4} \tag{38}$$

from which

$$|E|^{1/4} = \frac{\alpha}{4\pi\sqrt{2}} \tag{39}$$

which is possible only if $\alpha > 0$ (attractive potential). We also see that there is only one eigenvalue for each positive α:

$$E = -\frac{\alpha^4}{4(4\pi)^4} . \tag{40}$$

For $E > 0$, the singularities are at

$$\lambda_1 = E^{1/4} , \; \lambda_2 = -\lambda_1 , \; \lambda_3 = i\lambda_1 , \lambda_4 = -i\lambda_1 . \tag{41}$$

Now, if in (27) we choose the quadratic form in the numerator so as to eliminate a couple of roots we arrive at an inconsistency. Namely, if we eliminate the two real roots or the two imaginary ones, then

$$\phi''(0) = -\mathbb{C}(\lambda_a+\lambda_b) = 0 \tag{42}$$

and so it is not possible to get a δ-function like in (36). On the other hand, if we choose to eliminate one real and one imaginary root we are led to a complex value for α.

We cannot avoid, however, eliminating the positive root (as it will, otherwise, imply an exponentially increasing function), so we choose

$$p^2\phi'(0) + p\phi''(0) + \phi'''(0) = \mathbb{C}(p-a)(p-\lambda_1) \tag{43}$$

where the real parameter a should be different from λ_1 or λ_2. Now

$$\phi'(0) = \mathbb{C} \quad , \quad \phi''(0) = -\mathbb{C}(a+\lambda_1)$$
$$\phi'''(0) = \mathbb{C}a\lambda_1 \tag{44}$$

and

$$\phi(p) = \frac{\mathbb{C}(p-a)}{(p-\lambda_2)(p-\lambda_3)(p-\lambda_4)} , \tag{45}$$

whose Laplace transform is (Ref. 5, p.230)

$$\phi(r) = K\, e^{i\sigma}(e^{-\lambda_1 r} - e^{i\lambda_1 r}) + Ke^{i\delta}(e^{-\lambda_1 r} - e^{-i\lambda_1 r}) \tag{46}$$

where σ and δ are arbitrary phases. From (46) we get

$$\alpha = -\, 4\pi\, \frac{\phi''(0)}{\phi'(0)} = \frac{8\pi\lambda_1}{1-\tan\frac{\sigma-\delta}{2}} \quad . \tag{47}$$

We see from (47) that the strength of the δ-function potential determines the phase difference of both terms in (46). We also see that the solution exists for any sign of α.

4 THE OPERATOR $p^4-\alpha/r$

Note that $1/r$ is the Green's function in five dimensions of the operator $\nabla^2\nabla^2$. We now deal with the equation

$$\nabla^2\nabla^2\psi - \frac{\alpha}{r}\,\psi = E\psi\, . \tag{48}$$

With $\psi=\phi/r$ we obtain

$$\frac{d^4\phi}{dr^4} - \frac{\alpha}{r}\phi = E\phi \tag{49}$$

with $\phi(0) = \phi''(0) = 0$ to avoid δ-functions at the origin (see 3.).

For the time being, and for reasons of simplicity, we also take $\phi'(0)=0$. With these initial conditions the Laplace transform of (49) is (after taking a derivative d/dp)

$$(p^4-E)\,\frac{d\phi}{dp} + (4p^3+\alpha)\phi = 0\ . \tag{50}$$

Compare with (11) and note the differences in the sign of E and α. To get a qualitative idea of the problem we follow the analysis used at the end of 2. We assume

$$\phi = (p-\lambda)^{\beta} + 0((p-\lambda)^{\beta+1}) \tag{51}$$

and substitute in (50)

$$(p^4-E)\beta(p-\lambda)^{\beta-1} + (4p^3+\alpha)(p-\lambda)^{\beta-1} = 0\ . \tag{52}$$

So, near $p \approx \lambda$ we obtain the conditions

$$\lambda^4 = E$$

and

$$4\lambda^3 + \alpha + 4\lambda^3\beta = 0 \tag{53}$$

i.e.

$$\lambda^3 = -\frac{\alpha}{4(\beta+1)}\ . \tag{54}$$

Now, if $E<0$ we have

$$\lambda_1 = \frac{1+i}{\sqrt{2}} |E|^{1/4} , \lambda_2 = \lambda_1^* , \lambda_3 = \frac{-1+i}{\sqrt{2}} |E|^{1/4} , \lambda_4 = \lambda_3^* . \quad (55)$$

It is verified that λ_i^3 is another root λ_j which leads through (54) to a complex value of β. Then the exponential growth of $\phi(r)$ for $r\to\infty$ cannot be avoided. So there is no solution for Eq. (50) with negative E and normalizable $\phi(r)$ with the assumed initial conditions. On the other hand, if E is positive we have

$$\lambda_1 = E^{1/4} , \lambda_2 = -\lambda_1 , \lambda_3 = iE^{1/4} , \lambda_4 = -\lambda_3 . \quad (56)$$

The dominant singularity corresponds to λ_1 (largest real part). This singularity can be avoided only if $\alpha<0$ (see (54)) (repulsive potential) in which case Eq. (54) gives

$$E^{3/4} = \frac{|\alpha|}{4(\beta+1)} , \quad (57)$$

and choosing β integer (see discussion in 2)

$$E_n = \left(\frac{|\alpha|}{4(n+1)}\right)^{4/3} . \quad (58)$$

However, after this choice we still have λ_3 and λ_4 on the imaginary axis and the corresponding asymptotic behavior is now dominated by the waves $e^{\lambda_3 r}$ and $e^{\lambda_4 r}$. In the one dimensional case this corresponds to the states of "total reflexion" (see Ref.6).

As a matter of fact we can write the explicit solution of (50):

$$\phi(p) = \mathbb{C}(p-\lambda_1)^{\beta_1}(p-\lambda_2)^{\beta_2}(p-\lambda_3)^{\beta_3}(p-\lambda_4)^{\beta_4} \quad (59)$$

where

$$\beta_i = -\frac{\alpha\lambda_i}{4E} - 1 \ . \tag{60}$$

The inverse Laplace transform is (Ref.1, p.238)

$$\phi(r) = r^3\Phi_2(-\beta_1,-\beta_2,-\beta_3,-\beta_4;4;\lambda_1 r,\lambda_2 r,\lambda_3 r,\lambda_4 r) \ . \tag{61}$$

where Φ_2 is defined in Ref. 5 (p.235). By studying the asymptotic behavior of Φ_2, which is cumbersome and uninteresting, we arrive at the same conclusions already mentioned.

Formula (59) and (60) together with (55) and (56) show that as there are no poles on the right hand side in (7) we can close the integration by a semicircle on the right, giving $\phi(0)=0$. The same argument holds for $\phi'(0)=0$ and $\phi''(0)=0$ but not for $\phi'''(0)$. The reason is that the integrand vanishes like $1/p^4$.

5 MODIFIED BOUNDARY CONDITIONS

The condition $\phi(0)=0$ has to be imposed to avoid a $\nabla^2\delta(\vec{r})$ singularity. A similar argument is valid for $\phi''(0)=0$ which otherwise leads to δ singularities. On the other hand, $\phi'(0)$ and $\phi'''(0)$ should be left as arbitrary constants. In the previous paragraph we choose $\phi'(0)=0$ for simplicity reasons. We now drop this requeriment.

Taking into account Eq. (25) we now get instead of (50), the inhomogeneous equation

$$(p^4-E)\,\frac{d\phi}{dp} + (4p^3+\alpha)\phi = 2\phi'(0)p = 2Ap \tag{62}$$

which once solved will lead to the exact result. The general solution of (62) is (Ref.7, p.16)

$$\phi(p) = e^{-F(p)}\left(\eta + \int_{\xi}^{p} dx\; g(x)\, e^{F(x)}\right) \tag{63}$$

where

$$F(p) = \int_{\xi}^{p} \frac{4x^3+\alpha}{x^4-E}\, dx \quad \text{and} \quad g(p) = \frac{2Ap}{p^4-E} \; . \tag{64}$$

A careful study of the analytic properties of the function given by (63) is needed in order to fix the eigenvalues of this equation with the new boundary conditions.

6 DISCUSSION

The bilaplacian (p^4) equation has peculiar behavior when compared with the usual p^2 equation. The attractive δ-function potential has no negative eigenvalue, no bound state, in the p^2 case, while it has always one and only one for the p^4 case.

Furthermore, we get a solution for any positive value of E and for any sign of coupling constant α. It is a combination of imaginary exponentials plus exponentially decreasing functions as shown explicitly in Eq. (46). In order to produce a δ-function behavior at the origin it is essential to have $\phi''(0) \neq 0$.

The potential α/r is more involved. For the initial conditions $\phi(0) = \phi'(0) = \phi''(0) = 0$ there are no negative eigenvalues. On the other hand, if E is positive, with

$$E_n = \left(\frac{|\alpha|}{4(n+1)}\right)^{4/3} \tag{65}$$

(a discrete set of infinite eigenvalues!) then, the solutions behave like incoming and outgoing waves. They correspond to the solutions for "total reflexion" discussed in Ref. 6. For any other values of $E > 0$ one cannot avoid the exponential increase for $r \to \infty$.

Using a variational method, F. Perez[8] has proved that there are infinite negative eigenvalues provided $\phi'(0) \neq 0$. In order to find them it is necessary to know the analytical properties of the function $\phi(p)$ satisfying Eq. (63), so as to impose the correct asymptotic behavior to be satisfied by $\phi(r)$, which in turn will determine the eigenvalues. This study is under way.

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APPENDIX A

We want to mention a particular example of a fourth order self-adjoint equation whose eigensolutions are similar to those of the second order harmonic oscillator:

$$\frac{d^4\phi}{dr^4} - 2\,\frac{d}{dr}\left(r^2\,\frac{d\phi}{dr}\right) + r^4\phi = P\phi = E\phi . \tag{A1}$$

Defining

$$y_n = r^n\, e^{-r^2/2} \tag{A2}$$

we easily obtain

$$\begin{aligned} Py_n &= [4(n^2+n)+3]\,y_n - 2n(2n^2-3n+1)\,y_{n-2} \\ &\quad + n(n-1)(n-2)(n-3)\,y_{n-4} , \end{aligned} \tag{A3}$$

so we have

$$Py_n = \sum_{m=1}^{n} a_n^m\, y_m \tag{A4}$$

where

$$\begin{aligned} a_n^n &= E_n = 4n(n+1)+3 \\ a_n^{n-2} &= -\,2n(2n^2-3n+1) \;, \quad a_n^{n-4} = \frac{n!}{(n-4)!} \;. \end{aligned} \tag{A5}$$

We call ϕ_n the solution of

$$P\phi_n = E_n\phi_n \quad , \quad \text{where} \quad E_n = 4n(n+1)+3 \ . \tag{A6}$$

With

$$\phi_n = \sum_{\ell=1}^{n} A_n^{\ell} y_{\ell} \tag{A7}$$

we have

$$P\phi_n = \sum_{\ell=1}^{n} A_n^{\ell} \sum_{m=1}^{\ell} a_{\ell}^{m} y_m = E_n \sum_{m=1}^{n} A_n^{m} y_m \ . \tag{A8}$$

Then,

$$\sum_{\ell=1}^{n} \{ \sum_{s=\ell}^{n} A_n^{s} a_s^{\ell} - E_n A_n^{\ell} \} y_s = 0 \quad . \tag{A9}$$

For ℓ=n we obtain

$$A_n^n a_n^n - E_n A_n^n = 0 \qquad \text{(identity)} \tag{A10}$$

and thus

$$E_n = a_n^n \tag{A11}$$

and we can set arbitrarily $A_n^n=1$. For ℓ=n-2 we get

$$A_n^{n-2} = \frac{a_n^{n-2}}{E_n - E_{n-2}} \quad ; \tag{A12}$$

for ℓ=n-4,

$$A_n^{n-4}(E_n - E_{n-4}) = a_n^{n-4} + A_n^{n-2} a_{n-2}^{n-4} \quad , \tag{A13}$$

and so on.

APPENDIX B

The conditions at r=0 which must be imposed on any two arbitrary eigenfunctions ϕ_1, ϕ_2 of the Hamiltonian to secure the hermiticity of the operator d^4/dr^4 are

$$\phi_1(0)\phi_2'''(0) - \phi_1'''(0)\phi_2(0) + \phi_1''(0)\phi_2'(0) - \phi_1'(0)\phi_2''(0) = 0 . \quad \text{(B1)}$$

In the case of the δ-function we put $\phi_n(0)=0$, and the first two terms of (B1) vanish. The remaining two terms cancel each other in virtue of the conditions (38) and (47). So, it is not necessary to have $\phi''(0)=0$.

In the case of the Coulomb potential (B1) is automatically satisfied by imposing the physical conditions $\phi(0)=\phi''(0)=0$.

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