CBPF-NF-060/94

Virial Expansion for an ε-Deformed System

Marco A. R-Monteiro, Itzhak Roditi and Ligia M.C.S. Rodrigues

Centro Brasileiro de Pesquisas Físicas – CBPF/CNPq
Rua Dr. Xavier Sigaud, 150
22290-180 – Rio de Janeiro – RJ, Brasil

Abstract

We compute the virial expansion of an ideal quantum q-gas equation of state for small values of the deformation parameter.

Key-words: Statistical mechanics; Virial expansion; Quantum groups; q-oscillators.

The interest in q-oscillators comes specially from their connection with quantum algebras[1] and superalgebras [2]. Quantum groups [3–5] are non-trivial generalizations of Lie algebras and groups and have left their trace in several areas of physics[6]. Also, simple physical systems present quantum group symmetry [7, 8].

It has recently been-shown [9] that ideal quantum q-gases [10, 11] present interesting features: Bose-Einstein condensation, λ-point discontinuity for the specific heat and a general trend to favour criticality. These results were found for real $|q| > 1$ deformed systems and due to the method employed they are not extensible to $0 \leq |q| < 1$.

In this short note, we explore an ideal quantum q-gas for values of q smaller than one. We find the virial expansion for its equation of state in the high-temperature (or low density) regime.

A bosonic q-oscillator is the associative algebra generated by the elements A, A^+ and N satisfying the relations [1, 2, 12]:

$$\begin{gathered} [N, A^+] = A^+ \qquad [N, A] = -A \\ AA^+ - q^2 A^+ A = 1 \end{gathered} \tag{1}$$

where we are taking q a real parameter.

It is possible to construct representations of the relations (2.2) in the Fock space $\mathcal{F}$ spanned by the normalized eigenstates $|n>$ of the number operator N as

$$\begin{aligned} A|0> &= 0 \quad , \quad N|n>= n|n> \quad n = 0, 1, 2, ... \\ |n> &= \frac{1}{\sqrt{[n]!}} (A^+)^n |0> \end{aligned} \tag{2}$$

where $[n] = (q^{2n} - 1)/(q^2 - 1)$ and $[n]! = [n] \cdots [1]$.

In the Fock space $\mathcal{F}$ it is possible to express the deformed oscillators in terms of the standard bosonic ones b, b^+ as [12]

$$A = \left(\frac{[N+1]}{N+1} \right)^{1/2} b \ , \ A^+ = b^+ \left(\frac{[N+1]}{N+1} \right)^{1/2} ; \tag{3}$$

it can easily be shown in $\mathcal{F}$ that

$$AA^+ = [N+1] \quad , \quad A^+ A = [N] \, , \tag{4}$$

and as expected the standard bosonic algebra is obtained in the $q \to 1$ limit.

We have previously investigated highly deformed q-bosons [9] and we observed that in the limit $q = \infty$ the statistical properties are those of fermions while for $q = 1$ the usual bosonic behaviour is recovered [13].

Our ideal deformed system is described by the Hamiltonian

$$H = \sum_i \omega_i A_i^+ A_i = \sum_i \omega_i [N_i] \, , \tag{5}$$

where $A_i, A_i^\dagger$ and N_i obey algebra (1) and are the annihilation, creation and occupation number operators of particles in level i with energy ω_i. The total number operator is then $N = \sum_i N_i$. We note that different modes of bosonic q-oscillators commute among themselves being thus different from the alternative deformed commutation relations called quon algebra [14, 15]. With μ the chemical potential and Ω the grand canonical potential, our grand canonical partition function is given by

$$Z = Tr\ exp[-\beta(H - \mu N)] = exp(-\beta\Omega), \tag{6}$$

where $\beta = 1/kT$, with k the Boltzman constant.

As Z factorizes for the above system, the grand canonical potential is given by a sum over single level partition functions [11]

$$\Omega = -\frac{1}{\beta}\sum_i \ln Z_i^0(\omega_i, \beta, \mu)\ , \tag{7}$$

where

$$Z_i^0(\omega_i, \beta, \mu) = \sum_{n=0}^{\infty} e^{-\beta(\omega_i[n] - \mu n)}\ . \tag{8}$$

For a non-relativistic q-boson, the energy is $\omega_i = \vec{p}^{\,2}/2m$ and following the usual procedure we enclose our system in a large volume V, allowing for the sum over levels to be replaced by an integral over the p-space:

$$\sum_i \to \frac{V}{(2\pi h)^3}\int d^3p\ . \tag{9}$$

We shall now consider a small $q(q = \varepsilon)$ for which the partition function (8) is

$$Z_i^0 = 1 + ze^{-\beta\omega} + \frac{z^2}{1-z}e^{-\beta\omega}(1 - \beta\omega\varepsilon^2) + 0(\varepsilon^4) \tag{10}$$

where z is the fugacity, $z = e^{\beta\mu}$ and we have kept terms up to ε^2 order.

Assuming the fugacity z small compared to one and integrating by parts, we obtain the pressure $P = -\Omega/V$:

$$P = \beta^{-1}\Lambda^{-3}z\left[1 + z(-2^{-5/2} + 1 - \frac{3}{2}\varepsilon^2) + z^2\left(3^{-5/2} + 1 - 2^{-3/2} + \varepsilon^2(-\frac{3}{2} + 3\times 2^{-7/2})\right) + O(z^3)\right], \tag{11}$$

where $\Lambda = (h^2\beta/2\pi m)^{1/2}$ is the thermal wavelength.

The q-boson density $n = \frac{\partial P}{\partial\mu}|_{T,V}$ is then

$$n = \Lambda^{-3}z\left\{1 + 2z(-2^{-5/2} + 1 - \frac{3}{2}\varepsilon^2) + 3z^2\left[3^{-5/2} + 1 - 2^{-3/2} + \varepsilon^2\left(-\frac{3}{2} + 3\times 2^{-7/2}\right)\right] + \cdots\right\}. \tag{12}$$

Inverting the power series above and expanding z in powers of $n\Lambda^3$ in (11), we obtain the virial expansion of the equation of state

$$P = \frac{n}{\beta}\Big\{1 + (2^{-5/2} - 1 + \frac{3}{2}\varepsilon^2)n\Lambda^3 + 2\left[2^{-4} - 3^{-5/2} - 2^{-1/2} + 1 + \frac{3}{2}\varepsilon^2\left(3 + 2^{-5/2}\right)\right] n^2\Lambda^6 + \cdots\Big\}. \qquad (13)$$

We note that the approximations made here are valid for large V, $z << 1$ and $n\Lambda^3 << 1$, implying large-temperature (or low density) regime.

From expression (13) we see immediately that a small deformation increases the pressure relative to $q = 0$. Besides, a comparison of (13) with the virial expansion for a highly deformed system, namely [9]

$$P = \frac{n}{(2\pi)^3\beta}\left[1 + \left(\frac{1}{2^{5/2}} - \frac{1}{(q^2+1)^{3/2}}\right)n\Lambda^3 + \left(\frac{1}{2^3} - \frac{2}{3^{5/2}} - \frac{4}{(q^2+1)^3} - \frac{4}{(2q^2+2)^{3/2}} + \frac{2}{(q^2+2)^{3/2}}\right)n^2\Lambda^6 + \cdots\right], \qquad (14)$$

shows that: 1) P attains its minimum for $q = 0$; 2) P attains its maximum for $q = \infty$, which is the fermionic gas; 3) P has an intermediate value for $q = 1$. As a matter of fact, the pressure grows with q for the whole range $0 \le |q| < \infty$.

The authors thank C. Tsallis, and M.R-Monteiro is grateful to F. Gliozzi, for stimulating their interest in the $0 \le |q| < 1$ region.

References

[1] A.J. Macfarlane, J. Phys. **A22** (1989) 4581; L.C. Biedenharn, J. Phys. **A22** (1989) L873.

[2] M. Chaichian and P. Kulish, Phys. Lett. **B234** (1990) 72.

[3] V.G. Drinfeld, Sov. Math. Dokl. **32** (1985) 254.

[4] M. Jimbo, Lett. Math. Phys. **10** (1985) 63; **11** (1986) 247.

[5] L.D. Faddeev, N. Yu. Reshetikhin and L.A. Takhtadzhyan, Algebra and Analysis **1** (1987) 178.

[6] C. Zachos, Contemporary Mathematics **134** (1992) 351 (and references therein).

[7] F. Alcaraz, M. Barber, M. Batchelor, R. Baxter and G. Quispel, J. Phys. **A20** (1987) 6397; V. Pasquier and H. Saleur, Nucl. Phys. **B330** (1990) 523.

[8] A. Montorsi and M. Rasetti, Phys. Rev. Lett. **72** (1994) 1730.

[9] M.R-Monteiro, I. Roditi and L.M.C.S. Rodrigues, Mod. Phys. Lett. **B7** (1993) 1897; Phys. Lett. **A** 188 (1994) 11; "ν-Dimensional Ideal Quantum q-Gas: Bose-Einstein Condensation and λ-Point Transition", preprint CBPF-NF-020/94, (to appear in Int. J. Mod. Phys. B).

[10] M. Martin-Delgado, J. Phys. **A24** (1991) 1285; P. Něskovic and B. Urosševic, Int. J. of Mod. Phys. **A7** (1992) 3379.

[11] M. Chaichian, R. Gonzalez Felipe and C. Montonen, J. Phys. **A26** (1993) 4020.

[12] P. Kulish and E. Damaskinsky, J. Phys. **A23** (1990) L415; A. Polychronakos, Mod. Phys. Lett. **A5** (1990) 2325.

[13] S. Vokos and C. Zachos, Mod. Phys. Lett. **A9** (1994) 1133.

[14] O. Greenberg, Phys. Rev. Lett. **64** (1990) 705; Phys. Rev. **D43** (1991) 4111.

[15] R. Mohapatra, Phys. Lett. **B242** (1990) 407.