ISSN 0029 - 3865

CBPF-NF-060/87

ON THE QUANTIZATION OF CONSTRAINED GENERALIZED DYNAMICS

by

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£Partially supported by Conselho Nacional de Desenvolvimento Científico e Tecnológico, CNPq, Brasil.

ABSTRACT

A special class of degenerate second order Lagrangians, those which differ from a nondegenerate first order Lagrangian by a total time derivative (or a four divergence) of a function of both the coordinates and velocities, is studied in detail. Using Dirac's theory of constrained systems, we show that the canonical quantization starting from the second order Lagrangian leads to the same physical results as those obtained from the nondegenerate first order Lagrangian. We thus clarify some incorrect results and misleading arguments encountered in the literature on the subject.

Key-words: Quantum mechanics; Classical generalized mechanics; Constrained hamiltonians systems.

I - Introduction

In spite of the fact that most physical systems can be described by Lagrangians which depend at most on the first derivatives of the dynamical variables, there is a continuing interest in the so called generalized dynamics, that is, the study of physical systems described by Lagrangians containing derivatives of order higher than the first[1].

Besides its mathematical interest connected with general problems in the calculus of variations as first investigated by Ostrogradskii[2], higher order terms were used in the past as intended corrections to first order Lagrangians associated with certain physical theories. The attempts were to generalize them or to get rid of bad properties of those theories. To the best of our knowledge the earliest attempts in this direction were those by Weyl and Eddington[3] who added curvature squared terms to the Einstein-Hilbert Lagrangian so as to extend the theory of general relativity. Modifications to Maxwell's electromagnetic theory have been put forward by Bopp[4] and Podolski[5] with the goal of avoiding divergences such as the infinite self-energy of a point charge (which, in a certain sense, they succeeded to do). Stimulated by those findings Pais and Uhlenbeck[6] investigated whether the use of higher order field equations might conduce to the disappearance of the divergent quantities that plague quantum field theory. Their general conclusion was that it is impossible to reconcile finiteness, positivity of free field energy and causality. In other words, ghost states with negative norm and possibly unitarity violation are inherent to those theories and these facts turned out to be serious arguments responsible for a bad reputation of higher order theories.

However, higher order Lagrangians are endowed with nice properties too, and they have been the subject of some recent interest for many reasons. It has been shown[7], for instance, that curvature squared terms show up as small corrections in the effective action of superstring theories in the limit of zero slope. The same kind of corrections have been proposed in the quantum theory of gravitation to improve the ultraviolet behavior of the Einstein-Hilbert action[8], as higher order derivatives terms are known to improve the convergence of Feynman diagrams. As a mechanism for regularizing the ultraviolet divergences of gauge invariant supersymmetric theories it is the only available method which preserves both gauge invariance and supersymmetry[9]. Just to mention one more example, higher order Lagangians come forth naturally when one looks for a Hamiltonian description of certain nonlinear physical systems like those described by the equations associated with the names of Boussinesq or Korteweg and de Vries[10]. Last but not least, the Lagrangian of one of the most outstanding physical theories of our times, the Einstein theory of relativity, does in fact contain second order derivatives of the metric field. It is then clear that such theories deserve a deeper investigation.

On the other hand the kinetic term of any first order Lagrangian can be transformed into two terms, one of which is linear in the "accelerations" and the other is a divergence, thus generating a second order Lagrangian. Both Lagrangians are naively expected to describe the very same physical system and should not lead to different results even at the quantum level. But in the process of passing to the second order formulation one necessarily ends up with a degenerate (or singular) Lagrangian. It was precisely this fact that gave rise to some controversy in the literature concerning the quantization of higher order mechanical systems.

Hayes and Jankowski[11] analyzed a second order Lagrangian which generates the correct equation of motion for a harmonic oscillator but, they claimed, yields an energy spectrum different from the usual one upon quantization. Subsequently Hayes[12] proposed an unorthodox and peculiar quantization prescription to circunvent the difficulties encountered in his previous work. His quantization procedure was immediately criticized by Ryan[13] and Anderson[14], who, surprisingly enough, put the blame on the Lagrangian chosen by Hayes because it was singular. As a matter of fact they entirely missed the point, for the fundamental shortcoming in Hayes's approach was his lack of recognition that he was dealing with a constrained dynamical system to which Dirac's formalism[15] must be applied. This was perceived by Tesser[16] and also by Cognola, Vanzo and Zerbini[17]. However, having disregarded the need to substitute Dirac brackets for Poisson brackets before quantizing, Tesser did not apply in a fully transparent and systematic fashion Dirac's theory as developed for systems with second class constraints. This was partly done by Cognola et al., who considered only a very particular class of one-dimensional Lagrangians. Therefore they did not investigate the drawbacks of Hayes's treatment in their complete generality, and it is also worth mentioning that their reasoning did not furnish the thoroughly reduced phase space as witnessed by the fact that their

Hamiltonian and fundamental Dirac brackets retained a dependence on an arbitrary parameter. Furthermore, in a recent paper Tapia[18] does not employ correctly Dirac's formalism to constrained generalized mechanics and in his study of the quantum theory of the harmonic oscillator he arrives at an energy spectrum which again does not coincide with the usual one. His case is even worse than that of Hayes and Jankowski, because the spectrum he obtained is unbounded below, not to mention the misleading arguments that led to the above mentioned spectrum.

It is our purpose in the present paper to provide a general explanation why results such as those found by Hayes and Jankowski[11] or Tapia[18] for the harmonic oscillator are wrong, whereas those obtained by Barcelos-Neto and Braga[19] for the Klein-Gordon field are correct. It is also our aim to dismiss as unnecessary and groundless odd quantization procedures such as the one advanced by Hayes, and finally to characterize as misleading those arguments[13,14] to the effect that degenerate Lagrangians should be avoided in generalized dynamics.

The paper is organized as follows. In Section II we state and prove our main result for systems with a finite number of degrees of freedom, while its extension to field theory is briefly sketched in the Appendix. Section III is devoted to a few examples and a general conclusion.

II - The Main Result

The generalization of Hamilton's least action principle and of the Hamiltonian formulation to nondegenerate Lagrangians depending on higher order derivatives was first achieved by Ostrogradskii[2], and a more modern presentation of the canonical formalism is available in Whittaker's classical treatise[20]. With an eye to physical applications, and for the sake of simplicity, we shall consider only the second order case, although our reasoning may be extended in a very direct way to Lagrangians involving derivatives up to an arbitrarily high order.

Let $\bar{L}(x,\dot{x},\ddot{x},t)$ be a second order Lagrangian where we are using the notation $x = (x_1,...,x_N)$, $\dot{x} = (\dot{x}_1,...,\dot{x}_N)$, etc.. The action principle

$$\delta S \equiv \delta \int_{t_1}^{t_2} \bar{L}(\dot{x},\ddot{x},t)dt = 0 \quad , \tag{2.1}$$

where the variation is performed under the condition that the endpoints remain fixed, leads to the equations of motion

$$\frac{\partial\bar{L}}{\partial x_i} - \frac{d}{dt}\left(\frac{\partial\bar{L}}{\partial\dot{x}_i}\right) + \frac{d^2}{dt^2}\left(\frac{\partial\bar{L}}{\partial\ddot{x}_i}\right) = 0 \quad . \tag{2.2}$$

For future convenience it will be useful to introduce the notation

$$y_i \equiv \dot{x}_i \quad . \tag{2.3}$$

The canonical momenta are defined as

$$p_1{}^i = \frac{\partial\bar{L}}{\partial y_i} - \frac{d}{dt}\left(\frac{\partial\bar{L}}{\partial\dot{y}_i}\right) \tag{2.4}$$

and

$$p_2{}^i = \frac{\partial \bar{L}}{\partial \dot{y}_i} \quad . \tag{2.5}$$

Eqs. (2.5) can be solved for the $\dot{y}_i$ if and only if the Hessian matrix $\bar{W}$ whose elements are

$$\bar{W}_{ij} = \frac{\partial^2 \bar{L}}{\partial \dot{y}_i \partial \dot{y}_j} \tag{2.6}$$

is nonsingular. Assuming this is the situation, the Hamiltonian defined as (Einstein's summation convention over repeated indices is understood from now on)

$$\bar{H}(x,p_1;y,p_2;t) = y_i p_1{}^i + \dot{y}_i p_2{}^i - \bar{L}(x,y,\dot{y},t) \tag{2.7}$$

generates Hamilton's equations of motion

$$\dot{x}_i = \frac{\partial \bar{H}}{\partial p_1{}^i} \quad , \qquad \dot{p}_1{}^i = -\frac{\partial \bar{H}}{\partial x_i} \quad , \tag{2.8a}$$

$$\dot{y}_i = \frac{\partial \bar{H}}{\partial p_2{}^i} \quad , \qquad \dot{p}_2{}^i = -\frac{\partial \bar{H}}{\partial y_i} \quad . \tag{2.8b}$$

The pairs of canonically conjugate variables are (x,p_1) and (y,p_2), the Poisson brackets being defined as follows:

$$\{F,G\} = \frac{\partial F}{\partial x_i}\frac{\partial G}{\partial p_1{}^i} - \frac{\partial F}{\partial p_1{}^i}\frac{\partial G}{\partial x_i} + \frac{\partial F}{\partial y_i}\frac{\partial G}{\partial p_2{}^i} - \frac{\partial F}{\partial p_2{}^i}\frac{\partial G}{\partial y_i} \quad . \tag{2.9}$$

In terms of these brackets the equation of motion of any dynamical variable F becomes simply

$$\frac{dF}{dt} = \{F,\bar{H}\} + \frac{\partial F}{\partial t} \quad . \tag{2.10}$$

Suppose now $\bar{L}$ is of the form

$$\bar{L}(x,y,\dot{y},t) = L(x,y,t) + \frac{d}{dt} f(x,y) \qquad (2.11)$$

where f stands for an arbitrary function and L is a nondegenerate first order Lagrangian, that is, its Hessian matrix W whose elements are

$$W_{ij} = \frac{\partial^2 L}{\partial y_i \partial y_j} \equiv \frac{\partial^2 L}{\partial \dot{x}_i \partial \dot{x}_j} \qquad (2.12)$$

is nonsingular. Obviously, $\bar{L}$ and L generate the same equations of motion. The explicit form of $\bar{L}$ is

$$\bar{L}(x,y,\dot{y},t) = L(x,y,t) + \frac{\partial f}{\partial x_i} y_i + \frac{\partial f}{\partial y_i} \dot{y}_i \quad , \qquad (2.13)$$

from which it follows immediately that $\bar{W} = 0$ according to Eq. (2.6), so that $\bar{L}$ is singular. As a consequence, we are sure to meet relations of functional dependence among the canonical variables.

The canonical momenta are easily found to be

$$p_1{}^i = \frac{\partial \bar{L}}{\partial y_i} - \frac{d}{dt} \frac{\partial \bar{L}}{\partial \dot{y}_i} = \frac{\partial L}{\partial y_i} + \frac{\partial f}{\partial x_i} \quad , \qquad (2.14)$$

$$p_2{}^i = \frac{\partial \bar{L}}{\partial \dot{y}_i} = \frac{\partial f}{\partial y_i} \quad . \qquad (2.15)$$

Define the functions

$$g_i(x,y,t) = \frac{\partial L(x,y,t)}{\partial y_i} + \frac{\partial f(x,y)}{\partial x_i} \qquad (2.16)$$

and

$$h_i(x,y) = \frac{\partial f(x,y)}{\partial y_i} \quad . \qquad (2.17)$$

Then we have the following primary constraints:

$$\phi_i = p_1{}^i - g_i(x,y,t) \approx 0 \quad , \tag{2.18}$$

$$\psi_i = p_2{}^i - h_i(x,y,t) \approx 0 \quad . \tag{2.19}$$

The Poisson brackets of the constraints are easily calculated from definition (2.9). They are

$$\{\phi_i,\phi_j\} = \frac{\partial g_j}{\partial x_i} - \frac{\partial g_i}{\partial x_j} = \frac{\partial^2 L}{\partial x_i \partial y_j} - \frac{\partial^2 L}{\partial x_j \partial y_i} \equiv Y_{ij} \quad , \tag{2.20}$$

$$\{\phi_i,\psi_j\} = \frac{\partial h_j}{\partial x_i} - \frac{\partial g_i}{\partial y_j} = - \frac{\partial^2 L}{\partial y_i \partial y_j} = - W_{ij} \quad , \tag{2.21}$$

$$\{\psi_i,\psi_j\} = \frac{\partial h_j}{\partial y_i} - \frac{\partial h_i}{\partial y_j} = 0 \quad . \tag{2.22}$$

It is convenient to define

$$(\chi_1,\ldots,\chi_{2N}) = (\phi_1,\ldots,\phi_N,\psi_1,\ldots,\psi_N) \tag{2.23}$$

and agree that small Latin indices from the beginning of the alphabet always run from 1 to 2N. The 2N×2N matrix built up with the Poisson brackets of the constraints is, therefore,

$$\mathbb{C} = \|\{\chi_a,\chi_b\}\| = \begin{pmatrix} Y & -W \\ W & 0 \end{pmatrix} \quad , \tag{2.24}$$

where W and Y are the N×N matrices defined by Eqs. (2.12) and (2.20), respectively. It is readily shown that $\mathbb{C}$ is non-singular. In fact,

$$\det \|\{\chi_a,\chi_b\}\| = (\det W)^2 \neq 0 \tag{2.25}$$

since L is nonsingular by hypothesis. Thus, all of our constraints are of the second class, and no linear combination of the χ_a can become a first class constraint, so that the extended Hamiltonian is nothing but the usual one given by Eq. (2.7).

According to Dirac's formalism[15], one must replace the original Poisson brackets by the new Dirac brackets defined by

$$\{F,G\}^* = \{F,G\} - \{F,\chi_a\} \ (\mathbb{C}^{-1})^{ab} \ \{\chi_b,G\} \ , \tag{2.26}$$

where $\mathbb{C}^{-1}$ is the inverse of the matrix given by Eq. (2.24). It is an easy task to construct $\mathbb{C}^{-1}$ and check that it can be put in the form

$$\mathbb{C}^{-1} = \begin{pmatrix} 0 & W^{-1} \\ -W^{-1} & W^{-1}YW^{-1} \end{pmatrix} \quad . \tag{2.27}$$

The Hamiltonian (2.7) may be written as

$$\bar{H} = y_i p_1{}^i + \dot{y}_i p_2{}^i - L - \frac{\partial f}{\partial x_i} y_i - \frac{\partial f}{\partial y_i} \dot{y}_i \quad , \tag{2.28}$$

where use has been made of Eq. (2.13). Once we are working with Dirac brackets, we are allowed to regard the constraints (2.18) and (2.19) as strong equations. Therefore, taking

$$p_1{}^i = \frac{\partial L}{\partial y_i} + \frac{\partial f}{\partial x_i} \quad , \quad p_2{}^i = \frac{\partial f}{\partial y_i} \quad , \tag{2.29}$$

and inserting these equations into Eq. (2.28), we are left with

$$\bar{H} = \dot{x}_i \frac{\partial L}{\partial \dot{x}_i} - L \quad . \tag{2.30}$$

Defining

$$p_i = \frac{\partial L}{\partial \dot{x}_i} \quad , \tag{2.31}$$

these equations can be uniquely solved for the velocities $\dot{x}_i$ and Eq. (2.30) ensures that

$$\overline{H} = H(x,p,t) \tag{2.32}$$

where H is the ordinary Hamiltonian corresponding to the nonsingular first order Lagrangian L. Notice that the momenta p_1 and p_2 have been wholly removed from the theory. It remains to examine the fundamental Dirac brackets of the new presumably canonical pair (x,p). Performing a few straightforward computations one finds successively

$$\begin{aligned}\{x_i,p_j\}^* &= \{x_i,\frac{\partial L}{\partial y_j}\} - \{x_i,\chi_a\} \ (\mathbb{C}^{-1})^{ab} \ \{\chi_b,\frac{\partial L}{\partial y_j}\} = \\ &= -\{x_i,\phi_k\}(\mathbb{C}^{-1})^{kb} \ \{\chi_b,\frac{\partial L}{\partial y_j}\} = \\ &= -\delta_{ik}(\mathbb{C}^{-1})^{kb}\{\chi_b,\frac{\partial L}{\partial y_j}\} = - \ (W^{-1})^{i\ell}\{\psi_\ell,\frac{\partial L}{\partial y_j}\} \ ,\end{aligned} \tag{2.33}$$

where we have made use of the explicit form of $\mathbb{C}^{-1}$ given by Eq. (2.27). From Eqs. (2.9) and (2.19) it follows at once that

$$\{\psi_\ell,\frac{\partial L}{\partial y_j}\} = - \frac{\partial^2 L}{\partial y_\ell \partial y_j} = - W_{\ell j} \quad , \tag{2.34}$$

whence, after its insertion into Eq. (2.33), one finally gets

$$\{x_i,p_j\}^* = \delta_{ij} \quad . \tag{2.35}$$

With a little more effort, a similar kind of calculation allows us to show that

$$\{x_i,x_j\}^* = \{p_i,p_j\}^* = 0 \quad . \tag{2.36}$$

This establishes unequivocally that at the level of Dirac brackets (x,p) indeed constitute a pair of canonically conjugate variables.

So long as the canonical quantization then proceeds in a standard manner by requiring that the fundamental commutators be made equal to $i\hbar$ times the corresponding fundamental Dirac brackets, we have proved that the correct way of quantizing the classical theory associated with the singular second order Lagrangian $\bar{L}$ conduces to the same physical results as the quantum theory based upon the nonsingular first order Lagrangian L.

III - Examples and Conclusion

The Lagrangian introduced by Hayes and Jankowski[11,12] is

$$\bar{L} = -\frac{m}{2}\, x\ddot{x} - \frac{1}{2}\, kx^2 \quad , \tag{3.1}$$

which gives rise to the equation of motion of a harmonic oscillator:

$$m\ddot{x} + kx = 0 \quad . \tag{3.2}$$

The above Lagrangian is clearly of the form (2.11) with

$$L = \frac{m}{2}\, \dot{x}^2 - \frac{1}{2}\, kx^2 \tag{3.3}$$

and

$$f(x,\dot{x}) = -\frac{m}{2}\, x\dot{x} \quad . \tag{3.4}$$

Thus, the correct quantum theory of the harmonic oscillator based on the second order Lagrangian (3.1) is the same as the usual one whose starting point is the first order Lagrangian (3.3). The ambiguities encountered in Ref. 10 are just a consequence of an inaccurate treatment of the constraints.

Tapia's approach[18] corresponds to taking $\bar{L}$ of the form (2.11) with L the same as the one given by the above Eq. (3.3) and $f = 0$. His mistake stems from an incomplete treatment of the second class constraints in Section 5 of his paper. To go ahead to the quantum theory it would have been indispensable first to introduce Dirac brackets, which he did not. In the Appendix he falls into another error because he treats the canonical variables as independent, paying no attention to the constraints. Finally, it is not true that his energy spectrum differs from the standard one merely by an additive constant. From his method of obtaining the energy eigenvalues $E_n = n\hbar\omega$, it is plain to see that n may be any negative or positive integer, so that his spectrum is unbounded *below*, and this is unacceptable on physical grounds.

The results obtained for the Klein-Gordon field by Barcelos-Neto and Braga[19] can also be easily explained as a particular example of our general result (see the Appendix). Their Lagrangian is

$$\bar{L} = -\frac{1}{2}\,\phi\,\Box\,\phi + V(\phi) \quad , \tag{3.5}$$

which is of the form (A.1) with

$$L = \frac{1}{2}\,\partial_\mu\phi\partial^\mu\phi + V(\phi) \tag{3.6}$$

being the usual Klein-Gordon Lagrangian, and

$$\Omega_\mu = -\frac{1}{2}\,\phi\partial_\mu\phi \quad . \tag{3.7}$$

Since they treat the constraints correctly[19], it is no surprise that they recover the well-known quantum theory of the Klein--Gordon field.

As a conclusion, let us emphasize that degenerate Lagrangians are not to be rejected in generalized mechanics: all one has to do is apply correctly the formalism designed by Dirac to cope with constrained systems. In particular, Dirac's method can be applied to quantum gravity taking as the starting point the usual Einstein-Hilbert Lagrangian, it being apparent, then, that it is not necessary to change the standard action for the gravitational field through the addition of a surface term, as is commonly done in quantum cosmology[21].

APPENDIX: EXTENSION TO FIELD THEORY

The extension of the results of Section II to field theory is quite straightforward and for this reason we shall only briefly sketch the Hamiltonian approach. We denote a set of fields on Minkowski space-time by $\psi(x) \equiv \{\psi^A(x)\}$, $A = 1,2,\ldots,N$, where $x \equiv \{x^\mu\} \equiv \{x^0, x^i\}$, $i = 1,2,3$, are the space-time coordinates. Derivatives of the fields will be denoted by $\partial\psi \equiv \{\partial_\mu \psi^A(x)\}$, $\partial^2\psi \equiv \{\partial_\mu\partial_\nu\psi^A(x)\}$, etc., and, in particular, $\partial_0\psi^A \equiv \dot{\psi}^A \equiv \phi^A$.

The analogue of the Lagrangian (2.11) in field theory is

$$\begin{aligned}\bar{L} &= L(\psi,\partial\psi) + \partial_\mu\Omega^\mu(\psi,\partial\psi)\\ &= L(\psi,\partial\psi) + \frac{\partial\Omega^\mu}{\partial\psi^A}\,\partial_\mu\psi^A + \frac{\partial\Omega^\mu}{\partial(\partial_\alpha\psi^A)}\,\partial_\mu\partial_\alpha\psi^A \quad . \qquad \text{(A.1)}\end{aligned}$$

We shall assume that the Hessian matrix associated with L is nonsingular:

$$W_{AB} = \frac{\partial^2 L}{\partial\dot{\psi}^A\partial\dot{\psi}^B} \equiv \frac{\partial^2 L}{\partial\phi^A\partial\phi^B} \quad , \quad \det W \neq 0 \quad . \qquad \text{(A.2)}$$

It follows from (A.1) that

$$\bar{W}_{AB} = \frac{\partial^2\bar{L}}{\partial\dot{\phi}^A\partial\dot{\phi}^B} = 0 \quad ,$$

so that $\bar{L}$ is degenerate.

The canonical variables of the theory are $(\psi^A, \overset{(1)}{\Pi}_A)$ and $(\phi^A \equiv \dot{\psi}^A, \overset{(2)}{\Pi}_A)$, with the momenta defined[22] by

$$\overset{(1)}{\Pi}_A = \frac{\partial \bar{L}}{\partial \phi^A} - 2\partial_k\left(\frac{\partial \bar{L}}{\partial(\partial_0\partial_k\psi^A)}\right) - \partial_0\left(\frac{\partial \bar{L}}{\partial \dot{\phi}^A}\right) \quad ,$$

$$\overset{(2)}{\Pi}_A = \frac{\partial \bar{L}}{\partial \dot{\phi}^A} \quad .$$

Using (A.1) we obtain

$$\overset{(1)}{\Pi}_A = \frac{\partial L}{\partial \phi^A} + \frac{\partial \Omega^0}{\partial \psi^A} - \partial_k \left(\frac{\partial \Omega^0}{\partial(\partial_k\psi^A)}\right) \quad , \tag{A.3}$$

$$\overset{(2)}{\Pi}_A = \frac{\partial \Omega^0}{\partial \phi^A} \tag{A.4}$$

It is worth remarking that only $\Omega^0(\psi,\partial\psi)$ shows up in the above expressions. This is to be expected, as one can be easily convinced by looking at the action functional constructed with the Lagrangian (A.1).

Expressions (A.3) and (A.4) give us the primary constraints

$$\eta_A = \overset{(1)}{\Pi}_A - F_A(\psi,\partial_k\psi,\phi,\partial_k\phi) \approx 0 \quad , \tag{A.5}$$

$$\lambda_A = \overset{(2)}{\Pi}_A - G_A(\psi,\partial_k\psi,\phi,\partial_k\phi) \approx 0 \quad . \tag{A.6}$$

We have, after a straightforward calculation,

$$\{\eta_A,\eta_B\} = \frac{\partial F_B}{\partial \psi^A} - \frac{\partial F_A}{\partial \psi^B} \equiv \xi_{AB} \quad , \tag{A.7a}$$

$$\{\eta_A,\lambda_B\} = -\frac{\partial^2 L}{\partial \phi^A \partial \phi^B} = -W_{AB} \quad , \tag{A.7b}$$

$$\{\lambda_A,\lambda_B\} = 0 \quad , \tag{A.7c}$$

so that, as in the case of Section II, the constraints are second class. Since the matrix $\mathbb{C}$ constructed with the above Poisson brackets is non-singular, using its inverse, $\mathbb{C}^{-1}$, the Dirac brackets are now defined as

$$\{A(x),B(z)\}^{*} = \{A(x),B(z)\} - \int dw dy \{A(x),\chi_a(w)\}(\mathbb{C}^{-1})^{ab}(w,y)\{\chi_b(y),B(z)\} \tag{A.8}$$

where $\chi_a \equiv (\eta_A,\lambda_A)$.

In analogy with Section II, the constraints can be regarded as strong equations, thus removing from the theory the momenta $\overset{(1)}{\Pi}_A$ and $\overset{(2)}{\Pi}_A$. Defining

$$\Pi_A = \frac{\partial L}{\partial \dot{\psi}^A} \quad , \tag{A.9}$$

we find that the Hamiltonian $\overline{H}$ reduces to the ordinary Hamiltonian H associated with the first order Lagrangian L, and it may be easily verified that (ψ,Π) constitute a canonical pair in terms of the Dirac brackets (A.8).

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