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GAP ROAD TO CHAOS: LIAPUNOV AND UNCERTAINTY EXPONENTS AND MULTIFRACTALITY

by

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ABSTRACT

We study numerically the prototype of the gap road to chaos, namely $x_{t+1} = 1 - \varepsilon_i - a|x_t|^{z_i}$ ($i = 1,2$ respectively correspond to $x \geq 0$ and $x < 0$; $\varepsilon_1 \neq \varepsilon_2$). Intriguing properties are observed concerning the $(a, \varepsilon_1, \varepsilon_2)$-evolution of the attractors and the Liapunov and uncertainty exponents; also, multifractality is exhibited at the first entrance to chaos.

Key-words: Chaos; Multifractality; Liapunov exponent; Uncertainty exponent.

Chaotic behavior in one-dimensional continuous maps in the interval has been studied extensively[1]. These maps as stated until now, present only three types of roads to chaos, namely, period doubling, intermittency and quasiperiodicity. Recent papers[2,3] show that maps with an asymmetry at the extremum display a variety of new features in their dynamical behavior. Experiments related to this type of dynamics were performed in forced nonlinear oscilators[4]. Also, experiments for such maps were proposed for laser cavities[5]. Theoretical studies show that discontinuous maps at the extremum can be generated by appropriate Poincaré sections in the Lorenz model[6]. We have found that maps of such kind exhibit a new universal road to chaos[3]. The prototype map we consider is given by

$$x_{t+1} = f(x_t) \equiv \begin{cases} 1-\varepsilon_1 - a|x_t|^{z_1} & \text{if} \quad x_t > 0 \\ \\ 1-\varepsilon_2 - a|x_t|^{z_2} & \text{if} \quad x_t \leq 0 \end{cases} \tag{1}$$

with z_1, $z_2 \geq 1$. Other choices are of course possible for $f(0)$; however they are all expected to yield essentially the same dynamics. If $\varepsilon_1 = \varepsilon_2$ and $z_1 = z_2$ we recover the well known one-dimensional map whose road to chaos is via period-doubling. The *gap road to chaos* refers to $\varepsilon_1 \neq \varepsilon_2$. This is the case we study numerically in the present paper. Several intriguing properties are observed for the first time, which we now detail. Unless otherwise stated we shall focus the $z_1 = z_2$ case.

For fixed $(a,\varepsilon_1,\varepsilon_2)$ the iteration of the map drives the system to an attractor which typically is a finite cycle. The period of this cycle is a complex function of $(a,\varepsilon_1,\varepsilon_2)$ pres enting a (presumably) infinite number of discontinuites. We present in Fig. 1 a typical case: we shall refer to such "phase diagrams" as *bunchs of bananas*.

In spite of its complexity, the phase diagram can be described as follows. Let us fix ε_1 and vary a. We have *inverse cascades of attractors* whose periods grow *arithmetically* (e.g., ... $\leftarrow 26 \leftarrow 22 \leftarrow 18 \leftarrow 14 \leftarrow 10; \ldots \leftarrow 24 \leftarrow 20 \leftarrow 16 \leftarrow 12 \leftarrow 8 \leftarrow 4$; "inverse" refers to the fact that a is *decreasing*). Each inverse cascade accumulates on a value of a, immediately below which appears a cycle whose period precisely is the adding constant of that inverse cascade (4, in our examples). Furthermore, between any two "bananas" of the bunch exists another inverse cascade whose periods grow with the same rule (e.g., between periods 6 and 10, the cascade ... $\leftarrow 26 \leftarrow 16 \leftarrow 6$ exists). We therefore always have, between any two bananas, another banana, in a structure whose similarity with a devil's staircase is evident. The same kind of behavior is observed by fixing a and varying ε_1 (or ε_2 or both, with $\varepsilon_1 \neq \varepsilon_2$). The accumulation points of the cascades in turn accumulate (for *increasing* a if $(\varepsilon_1, \varepsilon_2)$ are fixed) on a point *which is the entrance to chaos*. In other words, we have (presumably) infinite number of *accumulation points where there is no chaos (negative* Liapunov exponents), *as this only emerges at the accumulation point of the accumulation points!*

For fixed $(\varepsilon_1, \varepsilon_2)$ a given banana exists between a minimal value a^m and a maximal value a^M. Within a given cascade of bananas (whose sequence is noted with $k = 1, 2, 3, 4, \ldots$) we verify

$$|a_k^m - a_{k+1}^m| \sim |a_{k-1}^m - a_k^m|^{z_1} \quad (k\to\infty) \tag{2}$$

as well as

$$|a_k^m - a_\infty^m| \sim |a_{k-1}^m - a_\infty^m|^{z_1} \quad (k\to\infty) \tag{3}$$

The same laws hold for $\{a_k^M\}$, for all cascades, for all values of $(\varepsilon_1,\varepsilon_2)$ such that $\varepsilon_1 \neq \varepsilon_2$, in the presence or absence of higher order terms in Eq. (1), and also if we fix a and vary $(\varepsilon_1,\varepsilon_2)$. Eqs. (2) and (3) replace the well known law $(a_k - a_{k-1})/(a_{k+1} - a_k) \sim \delta(z)$ valid for $\varepsilon_1 = \varepsilon_2$ and $z_1 = z_2 \equiv z$.

The Liapunov exponent λ characterizes the sensitivity to initial conditions ($\lambda > 0$ and $\lambda < 0$ respectively correspond to the sensitive and non-sensitive cases). In Fig. 2 we present a typical a-evolution for fixed gap. We remark: (i) The structure is roughly self-similar; (ii) the *fingers* corresponding to high periods are very narrow; for a given cascade they monotonously become narrower and shift towards negative values of λ, thus exhibiting (presumably) infinite periods with no chaos; the highest and largest finger of each cascade corresponds to the lowest period of that cascade; if we consider increasingly large lowest periods, the top of the fingers approach $\lambda = 0$, and drive the system into chaos; (iii) changement of periods occur for $\lambda \to -\infty$, in remarkable contrast with changements of periods in the doubling-period road which occur at $\lambda = 0$.

Let us now focus another interesting phenomenon concerning the basins of attraction. It is well established that continuous one-dimensional maps presenting an unique extremum, have at most one finite attractor. We verify that this picture is modified in the presence of a gap at the extremum. In such cases, *more than one finite attractors* (typically two attractors) appear when we cross from one banana (see Fig. 1) to a neighboring one (we observed this in several crossings, it might happen in *all* of them). The attractor which is chosen depends on the initial value x_o. Two examples are presented in Fig. 3 for $a = 1.3$ ($a = 1.540344$); the black and white regions respectively correspond to cycle periods 8 and 2 (25 and 21). We verify that the black and white regions are euclidean (dimensionality $D = 1$) *whereas the border-set between them is a fractal* with *capacity dimensionality* d. The uncertainty exponent [7] α_u is given by $\alpha_u = D - d$. The system is said to present *final-state sensitivity or non-sensitivity* according to be $0 \leq \alpha_u < 1$ or $\alpha_u = 1$. To calculate α_u we consider, in the interval of x_o corresponding to finite attractors (roughly $[-1,1]$), N randomly chosen values (typically $N = 10^4$). We then choose ε (say 10^{-3} and below) and check whether both attractors starting from $x_o \pm \varepsilon$ coincide with that of x_o. If not, that value of x_o is said *uncertain*. We note N_u the total number of uncertain points. The uncertainty ratio N_u/N varies as ε^{α_u}. We find $\alpha_u \simeq 0.85$ ($\alpha_u \simeq 0.22$) for $a = 1.3$ ($a = 1.540344$). α_u varies quite irregularly with a; we are presently studying whether crossings between long period bananas sysematically correspond to small α_u's. Numerical experiments based on forth and back variations of a might present hysteresis according to the initial values x_o retained for the various steps.

Let us finally focus the conection with multifractality. Fractal measure is a phenomenological characterization of many physical systems, in particular strange attractors of dynamical systems. The central goal of such characterization is to obtain the function $f(\alpha)$ [8]. Here α is the scaling index ($p_i \sim \ell_i^{\alpha}$) of the measure about a point on the multifractal and $f(\alpha)$ is the dimension of the set of points on the multifractal with the same value of α. Through a Legendre transformation $f(\alpha)$ is related to the generalized dimensionality D_q [8]. The minimal and maximal values of α respectively coincide with D_∞ and $D_{-\infty}$; the maximal value of $f(\alpha)$ coincides with Hausdorff dimensionality D_0. In Fig. (4) we present $f(\alpha)$ for the attractor characterizing the entrance to chaos in the presence of a gap. Its shape is different (more square-like) from that obtained without gap (period-doubling road to chaos), and the values we obtain are $D_0 \simeq 0.95$, $D_{-\infty} \simeq 5.7$ and $D_\infty \simeq 0.45$ (they do not satisfy the relation $D_{-\infty} = zD_\infty$ which holds in the absence of gap; here $z = 2$).

Summarizing we have exhibited that the presence of a gap in the extremum of a one-dimensional map drastically changes the main dynamical properties of the system. Indeed, a rich structure (bunch of bananas like) appears in the phase diagram; the Liapunov exponent λ presents, through a roughly self-similar scheme of fingers, an unexpected situation, namely accumulation points corresponding to infinite periods with *negative* values of λ; final-state sensitivity is observed, and an underlying fractal structure is exhibited for the border-set between basins of attraction; finally the attractor associated with the entrance to chaos is shown to be a multifractal with a function f(α) very different from that of the period-doubling road to chaos.

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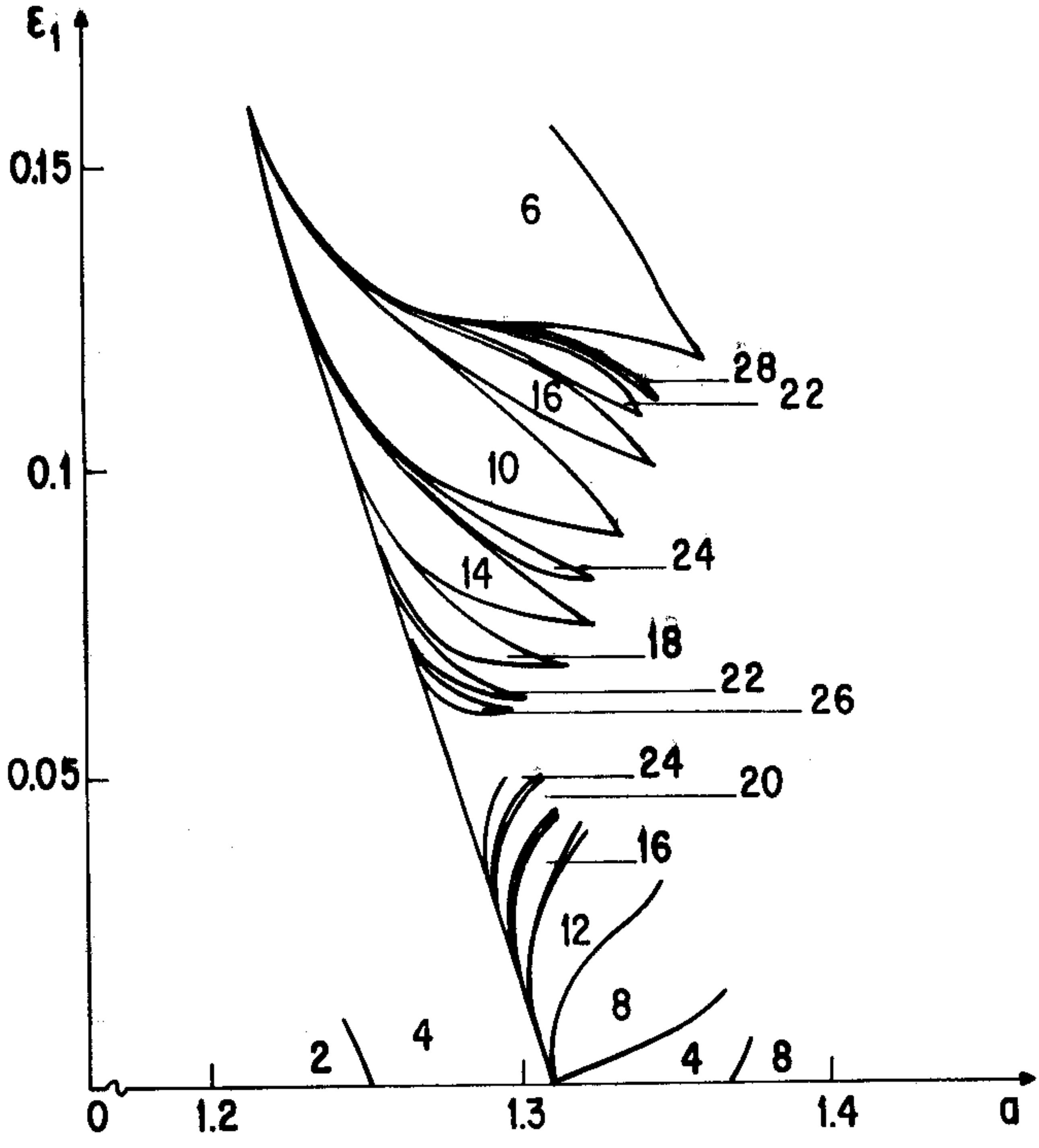
CAPTION FOR FIGURES

Fig. 1 - Phase diagram for $z_1 = z_2 = 2$ and $\varepsilon_2 = 0$. The numbers indicate the period of the attractor. For $\varepsilon_1 = 0$ we recover the well known doubling-period sequence. We used $x_o = 0.5$.

Fig. 2 - a-evolution of the Liapunov exponent for $\varepsilon_1 = 0$, $\varepsilon_2 = 0.1$, $z_1 = z_2 = 2$ and $x_o = 0.5$. The numbers in the fingers indicate the period of the attractor. (b) is the expansion of the small rectangle in (a).

Fig. 3 - Basins of attraction for two typical values of a and $\varepsilon_1 = 0$, $\varepsilon_2 = 0.1$ and $z_1 = z_2 = 2$ (see the text).

Fig. 4 - Multifractal function $f(\alpha)$ for $\varepsilon_1 = 0$, $\varepsilon_2 = 0.1$, $z_1 = z_2 = 2$ and $x_o = 0.5$. (chaos appears at $a^* = 1.5447398$)


ε1
0.15
0.1
0.05
0
1.2
1.3
1.4
a
6
28
22
16
10
24
14
18
22
26
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16
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8
2
4
4
8

Fig. 1

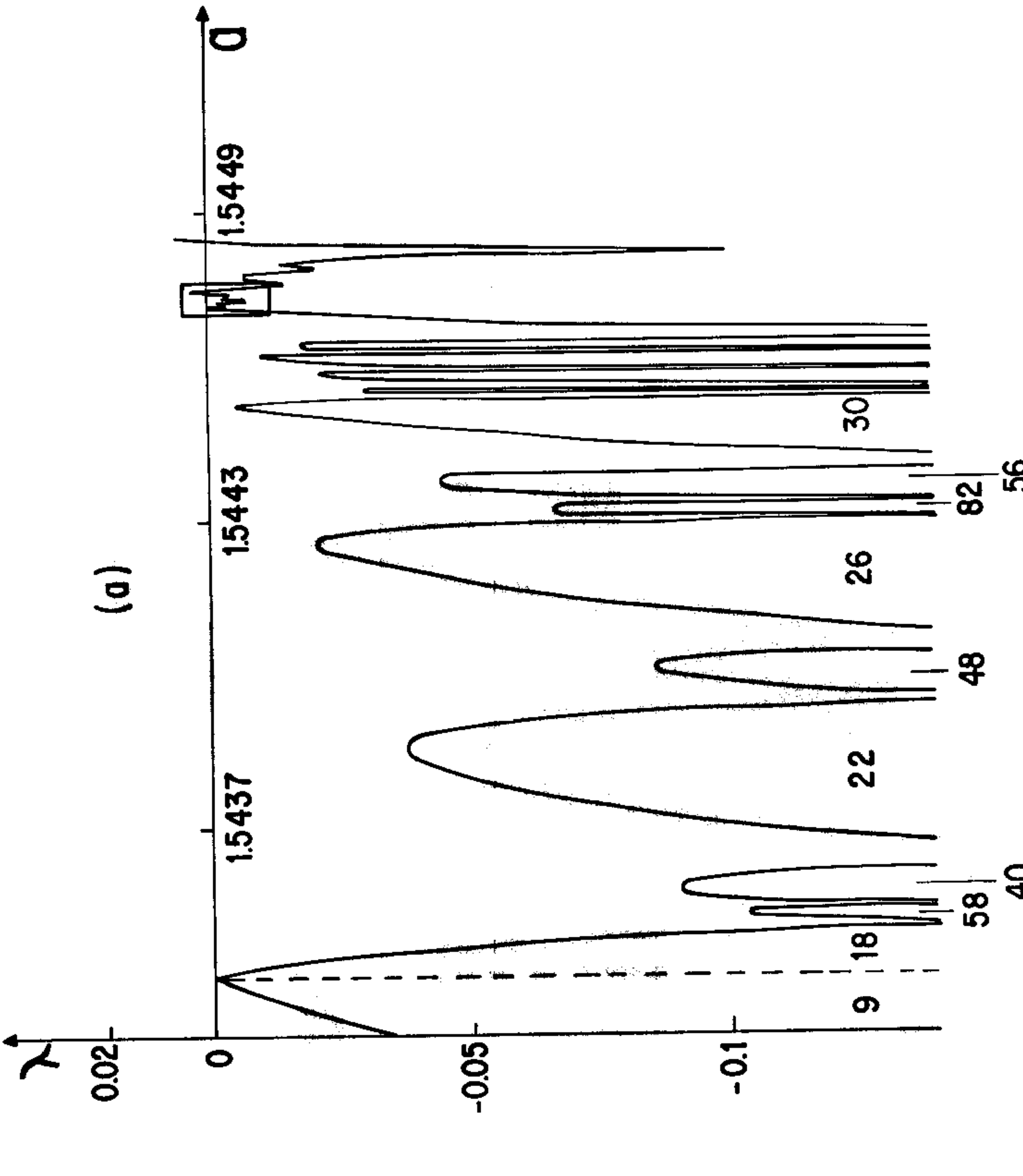

Fig. 2.a

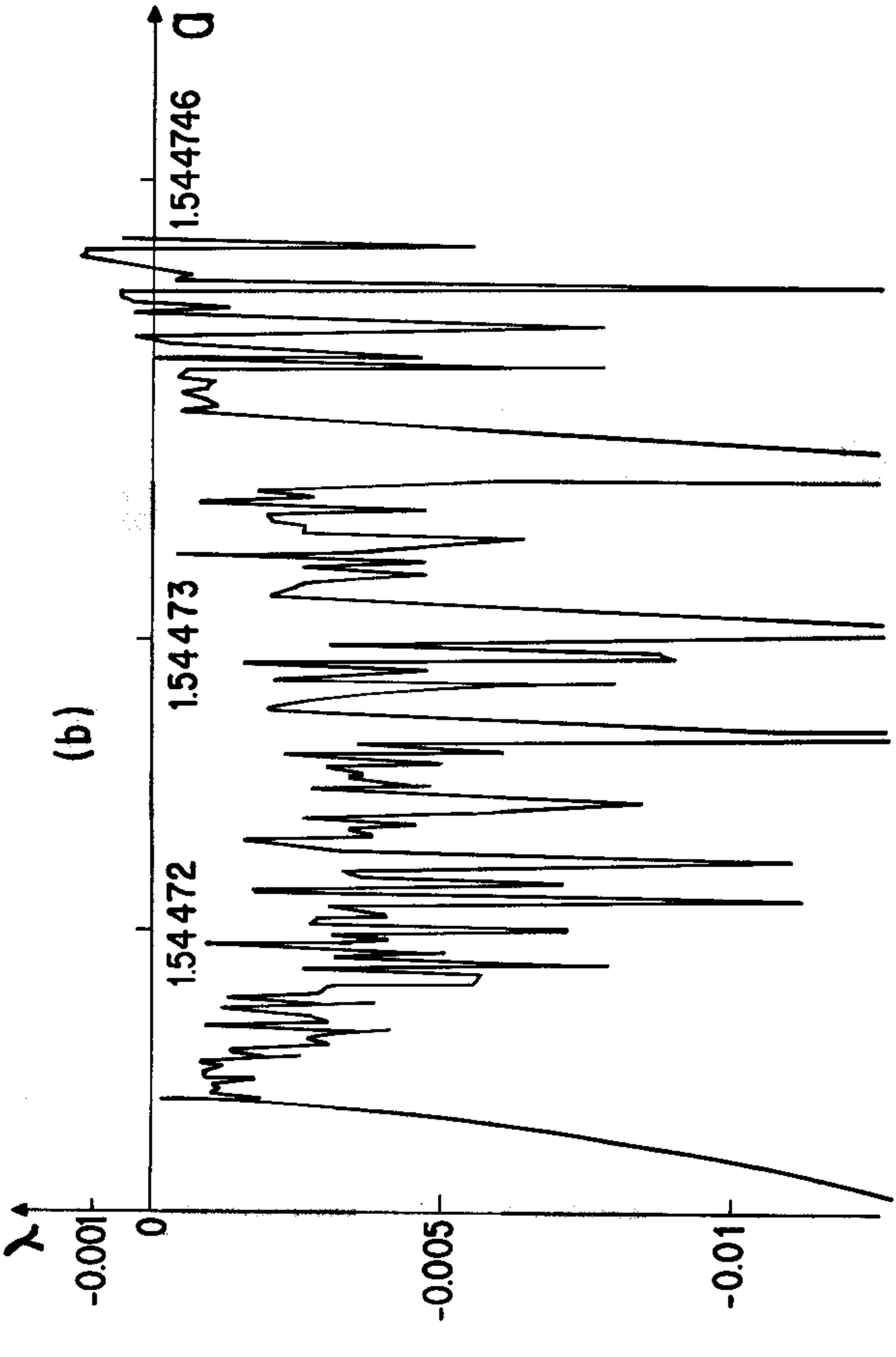

Fig. 2.b

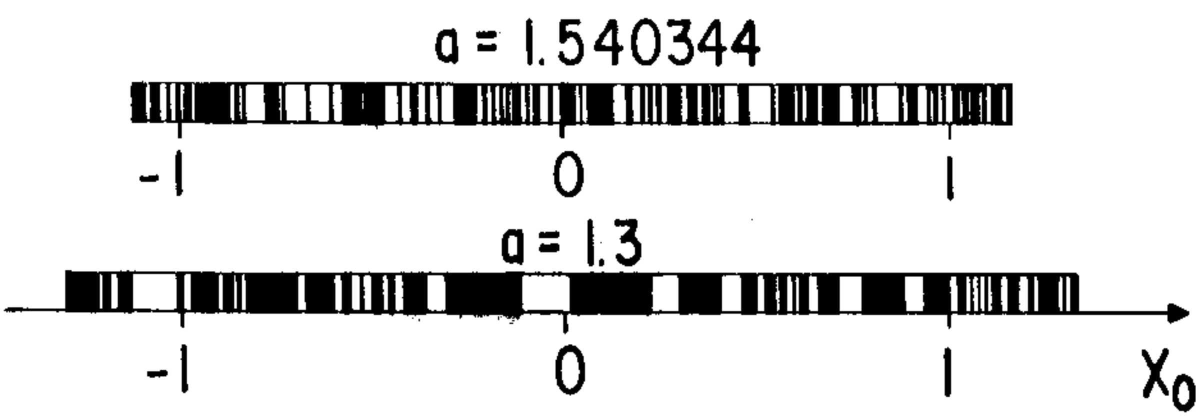

Fig. 3

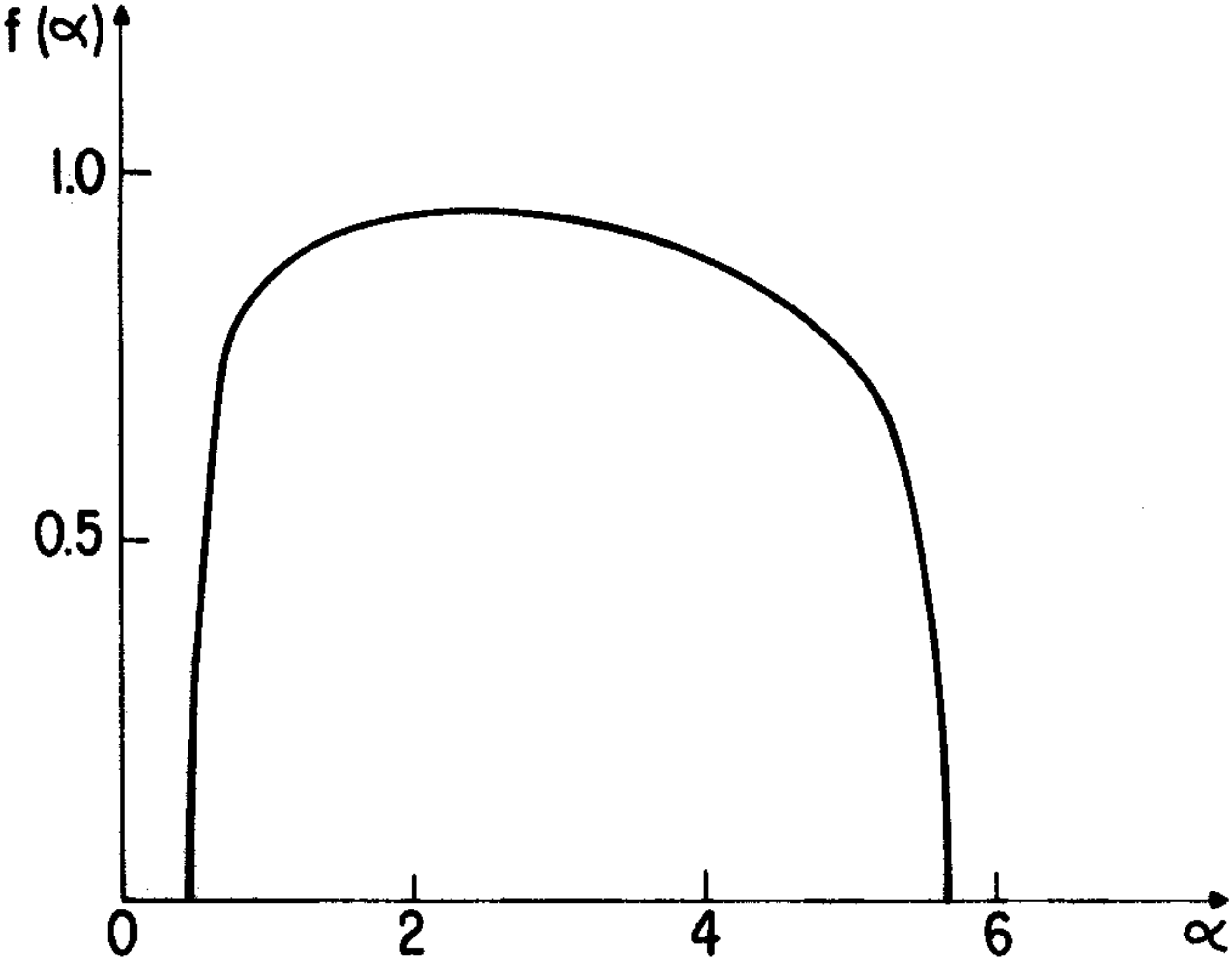

Fig. 4

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