SELF-SIMILAR PLANAR NEWTONIAN COLLAPSE

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ABSTRACT
In this work, two different classes of planar solutions for cold dust are analysed within the Newtonian framework. The first class contains planar collapsing solutions for a general set of self-similar initial conditions extending, in this way, Penston's original results. The second class consists of solutions without prescribing any initial conditions. In both classes, the solutions exhibit shell-crossing outside the plane of symmetry. The range of validity for both classes of solutions is then presented.

Key words: Self-similar collapse

1 INTRODUCTION

Planar symmetry occurs as the limiting case in a number of geometrical configurations. For instance, (i) the central region of a disc whose edge is at a very large distance from the center, can be considered a region with planar symmetry, (ii) a spherical surface with radius incomparably larger than typical distances of interest within the surface, can also be locally aproximated by planes. In these examples the planar approximation in the geometry can also be extended to symmetry in the dynamical laws. Indeed, it is well know that the gravitational potential generated by discs and spheres approach in the above well-defined limits the gravitational potential of a plane.

On the other hand, self-similar solutions are expected whenever the system under analysis does not have a characteristc scale, e. g., the basic constants and boundary conditions of the problem are insufficient to build up a scale. This may happen if the system looses at an intermediate stage of its evolution the memory of the intitial or boundary conditions (see e. g. Zeldovich and Raizer 1991 or Lynden-Bell 1986).

Self-similar collapsing solutions heve been widely studied. Self-similar spherically symmetric solutions for cold dust have been anlysed by Penston (1969), Dyer (1979), Lynden-Bell and Lemos (1989), Lemos and Lynden-Bell (1989), as well as for other types of fluid, Fillmore and Goldreich (1984) and, in general relativity, by Cahill and Taub (1971) among others. Self-similar collapse of flat cold axisymmetric Newtonian discs has been studied by Boily and Lynden-Bell (1993).

Self-similar gravitational collapse with planar symmetry, whitin the context of Newton's gravitational theory, has been mainly studied to understand the formation of structures, like stars, galaxies and clusters of galaxies (Penston 1969, Fillmore and Goldreich 1984). In particular, Penston (1969) has presented a study of the self-similar Newtonian collapse for cold dust. This work considered a quadratic profile in the density and included the spherical, cylindrical and planar symmetries. It was shown that the spherical collapse of nearby shells is well described by a planar equation for those shells. It was also shown that thermal forces start to be important earlier for planar collapse than for spherical collapse. Therefore, the plane flatten instability, which drives spheres into flatten systens (Lynden-Bell 1964, Lin, Mestel and Shu 1965), is opposite to the thermal pressure gradient appearing in these nearly planar regions. That phenomenon obstructs the formation of a plane with infinite density.

In the case of large scale structure formation, Fillmore and Goldreich (1984) investigated self-similar planar solutions. It was given time a prescription to continue the solutions past the shell-crossing singularities, confirming earlier numerical simulation results (Mellot 1980).

Planar collapse has also been studied in General Relativity within the context of singularities and black holes. Thorne (1972) has argued that the relativistic collapse of oblate spheroids of mass m with size bigger than $\frac{Gm}{c^2}$ (where G is the gravitational constant and c is the light velocity) is well approximated by Newtonians gravitation until the final stage when a singular disk forms. The collapse of the inner regions of these oblate spheroids can be approximated by collapse of planes. In the Newtonian framework the collapse to a disk was earlier analysed by Lin, Mestel and Shu (1965), whereas a full relativistic numerical analysis was done by Shapiro and Teukolsky (1991).

In this work we investigate two distinct cases. In section 2, we study self-similar planar systems that collapse from rest. In this case, we obtain a general class of self-similar initial conditions for the collapsing system. In section 3, we analyse self-similar planar systems that do not remember the initial conditions.

2 DUST SELF-SIMILAR SOLUTIONS WITH SELF-SIMILAR INITIAL CONDITIONS

In this section, we consider self-similar cold dust solutions in collapse with planar symmetry. This configuration was also treated in the Appendix 1 of Penston's paper. Here, we generalize Penston's solutions finding a general class of initial conditions for a self-similar collapse.

2.1 THE SOLUTIONS

Self-similar solutions, in this case, are such that the form of the density's profile distribuition remains unchanged with time. Consider a cold plane of gas with mean volumetric density $\bar{\rho}\,(z)$ at a distance z from the plane of symmetry. Then its interior effective mass $\sigma\,(z)$ (i. e., the mass per unit area within distance z) is given by $\sigma\,(z) = 2z\,\bar{\rho}\,(z)$. The equation of motion for such a plane is

$$\frac{\partial^2 z}{\partial t^2} = -2\pi G\sigma\,(z)\,, \tag{1}$$

We may analyse only the case $z \geq 0$. The $z \leq 0$ case has an analogous treatment.

Self-similarity requires $z = z_0\,(t)\, z_\star\,(\sigma_\star)$, where $\sigma_\star = \sigma/\sigma_0\,(t)$ is a time independent variable (see e. g. Lynden-Bell and Lemos 1989 for the spherical symmetric analogue). Let us go on now analysing one class of solutions for the plane distribuition of cold dust.

A solution of (1) is

$$z = z_i\,(\sigma) - \pi G\sigma t^2, \tag{2}$$

$$v = -2\pi G\sigma t, \tag{3}$$

where $z_i\,(\sigma)$ is an integration function. We have set the initial velocity of the planes equal to zero (we prove in the Appendix that the condition $v_i \neq 0$ is irrelevant). In order to obtain the functional dependence of z_i, we introduce the self-similar condition in equation (2). Then,

$$F\,(\sigma_\star) = f\,(t)\left[t^2 - \frac{z_i\,(\sigma_0\sigma_\star)}{\pi G\sigma_0\sigma_\star}\right], \tag{4}$$

where

$$f\,(t) = -\frac{\pi G\sigma_0}{z_0}, \tag{5}$$

$$F\,(\sigma_\star) = \frac{z_\star}{\sigma_\star}. \tag{6}$$

Differentiating (4) with respect to $\sigma_\star$, we obtain

$$F'\,(\sigma_\star) = \sigma_0 f\left[\frac{z_i}{(\sigma_0\sigma_\star)^2} - \frac{z_i'}{\sigma_0\sigma_\star}\right], \tag{7}$$

where the dash denotes a derivative with respect to its own argument. Next, we take logarithms in (7), differentiate the resulting equation with respect to σ_0, and multiply by σ_0 to obtain

$$\sigma_0^2 f\frac{d}{d\sigma_0}\left(\frac{1}{\sigma_0 f}\right) = \sigma\frac{d}{d\sigma}\log\left[\frac{z_i}{\sigma^2} - \frac{z_i'}{\sigma}\right]. \tag{8}$$

Since the functional dependence in both sides of (8) is different, we have that each side of this equation is a constant b, say. Therefore, integrating the right hand side of (8) we have

$$\frac{dz_i}{d\sigma} - \frac{z_i}{\sigma} = \pi G B_1\sigma^b, \tag{9}$$

where B_1 is also a constant. The general solution of (9), for $b \neq 0$, is

$$z_i = \pi G\sigma\left(B\sigma^b + t_{c0}^2\right) \tag{10}$$

with $B = bB_1$ and t_{c0}^2 a new integration constant. When $b \to 0$ we have $z_i \to \pi G\sigma\,(B_1\log\sigma + c_1)$, where c_1 is a new constant. These results give the general class of self-similar initial conditions for planar collapse.

If we now put

$$z = \pi G\sigma\left[t_c^2\,(\sigma) - t^2\right], \tag{11}$$

we may identify $t_c^2\,(\sigma) = z_i\,(\sigma)/\pi G\sigma$. So,

$$t_c^2\,(\sigma) = B\sigma^b + t_{c0}^2. \tag{12}$$

We see that $t_c\,(0) = t_{c0}$ which is the time the central plane collapses to the origin.With (12) and (11), we obtain:

$$z = \pi G\sigma\left(B\sigma^b + t_{c0}^2 - t^2\right). \tag{13}$$

We impose $b \geq 0$ to avoid initial shell-crossing in order that planes with larger interior masses start the collapse at greater distances.

Shell-crossing happens when two nearly planes with densities σ and $\sigma + d\sigma$ pile up into the same distance z, i. e., $dz = 0$ for these two planes. In this case the volumetric density diverges, $\rho = \frac{d\sigma}{dz} \to \infty$. The solution is no more valid when plane-crossing starts. To find a full solution one has to give a precise prescription for the continuation of the solution after the shell-crossing. Indeed, when $t = t_{c0}$, we have from equation (14) $\left.\frac{d\sigma}{dz}\right|_{\sigma\to 0} \to \infty$. For later times there will be shell-crossings in the region $z > 0$.

The density profile $\frac{d\sigma}{dz}$ is given by

$$\pi G\rho = \left[B\,(b+1)\,\sigma^b + t_{c0}^2 - t^2\right]^{-1}. \tag{14}$$

From (14) we see that there is shell-crossing for distances $z > z_l$ where,

$$z_l = \frac{\pi G b}{B^{1/b}}\left[\frac{t^2 - t_{c0}^2}{b+1}\right]^{\frac{b+1}{b}}. \tag{15}$$

Thus, the region with plane-crossing increases with time.

From equation (13), we can see that it is possible to choose the scale $\sigma_0\,(t)$ as

$$\sigma_0 = \left|\frac{t_{c0}^2 - t^2}{B}\right|^{1/b}, \tag{16}$$

yielding then the other sides,

$$z_\star = \sigma_\star\left(\sigma_\star^b \pm 1\right), \tag{17}$$

$$z_0 = \frac{\pi G}{B^{1/b}}\left|t_{c0}^2 - t^2\right|^{\frac{b+1}{b}}, \tag{18}$$

where $\pm$, in (17) is the signal of $t_{c0}^2 - t^2$. Immediately, we have the profile of $\rho_\star\,(\sigma_\star)$, $\rho_0\,(t)$, $v_\star\,(\sigma_\star)$ and $v_0\,(t)$, given by:

$$\rho_\star = \left[(b+1)\,\sigma_\star^b \pm 1\right]^{-1}, \tag{19}$$

$$\pi G \rho_0 = \left| t_{c0}^2 - t^2 \right|^{-1}, \tag{20}$$

$$v_* = -\frac{1}{2}\sigma_*, \tag{21}$$

$$v_0 = 4\pi G t \left| \frac{t_{c0}^2 - t^2}{B} \right|^{1/b}. \tag{22}$$

Therefore, we have two distincts forms for ρ_*. When the signal is positive, σ_* takes any value greater or equal to zero (being zero only in the centre of symmetry). When the signal is negative, i. e., $t^2 > t_{c0}^2$, than we have the condition $\sigma_* \geq 1$. However, the no-shell crossing condition imposes a stronger requirement:

$$z_* \geq \frac{b}{(b+1)^{(b+1)/b}}. \tag{23}$$

Note that the self-similar profiles are always valid in the $z_* > 1$ regions.

2.2 ILLUSTRATION OF THE SOLUTIONS

From equations (17), (19) and (21), one can check that before the central shell collapses, the $z_* \ll 1$ region can be described by the approximation

$$\rho_* \cong 1 - (b+1) z_*^b, \tag{24}$$

$$v_* \cong -\frac{1}{2} z_*. \tag{25}$$

On the other hand, the region with $z_* \gg 1$ can be approximated by

$$\rho_* \cong \frac{1}{(b+1)} z_*^{-b/(b+1)}, \tag{26}$$

$$v_* \cong -\frac{1}{2} z_*^{1/(b+1)}. \tag{27}$$

Using the time-scales $\sigma_0(t)$ and $z_0(t)$ given in equations (16) and (18) respectively, one can verify that the regions at large z have identical density and velocity profiles for all times, taking the following asymptotic forms: $\rho \propto z^{-\alpha}$,and $v \propto z^{\beta}$, where $\alpha = b/(b+1)$ and $\beta = 1/(b+1)$. Using (24) and (25), we have that, for $t < t_{c0}$, the regions with relatively small values of z have density and velocity profiles given by

$$\rho \cong \frac{1}{\pi G (t_{c0}^2 - t^2)} \left[1 - \frac{B(b+1)}{(\pi G)^b (t_{c0}^2 - t^2)^{b+1}} z^b \right] \tag{28}$$

and

$$v \cong -\frac{2t}{(t_{c0}^2 - t^2)} z. \tag{29}$$

The value of the constant b is constrained by the chosen density initial profile. Firstly, we analyse Penston's particular solution wich is characterized by $b = 2$. His results are linked to the initial condition

$$\rho(0, z) = \rho_i \left(1 - \frac{z^2}{A^2} \right), \tag{30}$$

where ρ_i and A are constants. For $t^2 < t_{c0}^2$, (i. e., before the central plane collapses) the time evolution of the density and velocity profiles is presented in the two first diagrams of Figure 1. We see that, when $z \longrightarrow \infty$, $\rho \longrightarrow z^{-2/3}$ and $v \rightarrow z^{1/3}$. In the other limit ($z \longrightarrow 0$), ρ presents a quadratic top. For $t^2 > t_{c0}^2$, there is plane-crossing but, the asymptotic dependence of ρ and v is invariant.

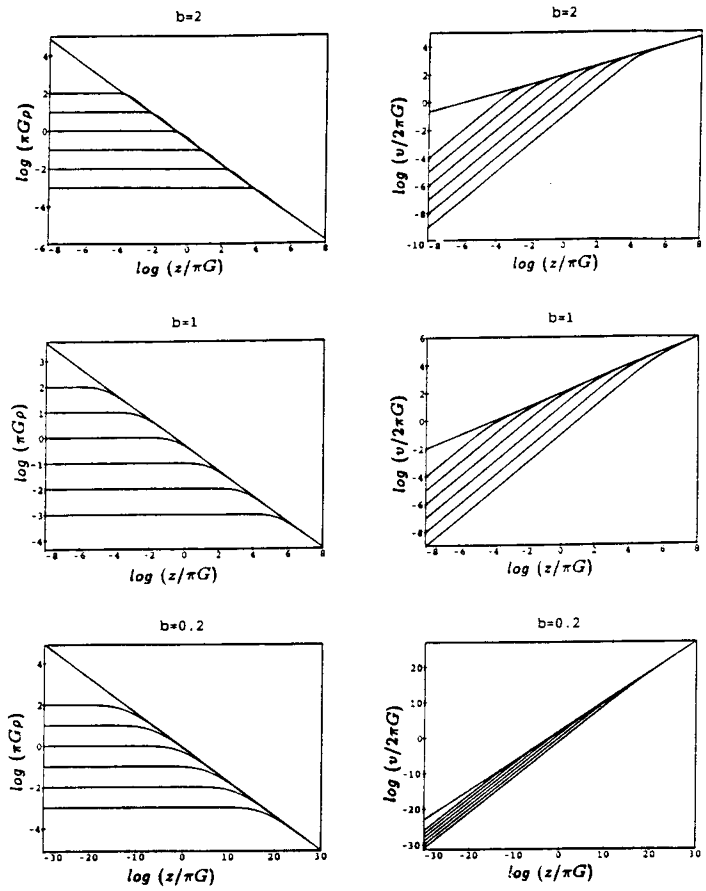

Figure 1: Diagram showing the time evolution of the density and the velocity as a function of the z coordinate for the penston, invertible and fillmore and Goldreich cases. Here $t_{c0}^2 - t^2 =$ .01, .1, 1, 10, 100, 1000.

Penston's ($b = 2$) case separates two types of solutions. For $b > 2$, the solutions present a smooth flat top in the $z \longrightarrow 0$ regions while for $b < 2$ the solutions have a cusp in the same small z region.

Another case of analytical interest is given by $b = 1$. This solution has the property of being invertible. Then, the density and velocity profiles can be described as explicit functions of z and t. This solution has the form:

$$\sigma = \frac{t^2 - t_{c0}^2}{2B} + \frac{1}{2B}\sqrt{(t_{c0}^2 - t^2)^2 + \frac{4B}{\pi G} z}, \tag{31}$$

$$\rho = \frac{(\pi G)^{3/2}}{\sqrt{\pi G (t_{c0}^2 - t^2)^2 + 4Bz}}$$

and

$$v = -\frac{\pi G t}{B}\left[t^2 - t_{c0}^2 + \sqrt{(t_{c0}^2 - t^2)^2 + \frac{4B}{\pi G} z} \right]. \tag{32}$$

The ρ and v profile are shown in the two intermediate diagrams of Figure 2, respectively. It is straightforward to note that when $z_* \longrightarrow 0$, the profile does present a cusp at $z = 0$.

A case of physical importance is $b = 0.2$. This exponent has been studied by Fillmore and Goldreich (1984) and illustrates qualitatively Mellot's numerical results (Mellot 1980).

This numerical calculations simulates the large scale planar collapse of massive neutrinos,which can be considered candidates to generate condensations for the formation of the large scale structures. For $b = 0.2$, we show in the last diagrams of Figure 1, the asymptotic forms of the ρ and v given by $\rho \to z^{-1/6}$ and $v \to z^{5/6}$.

3 SOLUTIONS FOR SELF-SIMILAR DUST WITHOUT INITIAL CONDITIONS

In the previous section, we obtained and illustrated one class of self-similar solutions with initial conditions. Now, we shall find another self-similar class of solutions for cold dust planar distribuition of matter. Here, we do not impose any initial conditions.

3.1 THE SOLUTIONS

First, we note that the equation of motion (1) has a first integral given by

$$E(\sigma) = \frac{1}{2}\dot{z}^2 + 2\pi G\sigma z. \tag{33}$$

Now, to obtain self-similar solutions without memory of initial conditions we first use the self–similar requirment and put Eq. (1) in the form:

$$\frac{\partial^2}{\partial t^2}\left[z_0(t)\, z_*(\sigma_*)\right]\bigg|_{\sigma} = -2\pi G\sigma_0(t)\,\sigma_*. \tag{34}$$

writing

$$\frac{\partial}{\partial t}\bigg|_{\sigma} = \frac{\partial}{\partial t}\bigg|_{\sigma_*} - \frac{\sigma_*\dot{\sigma}_0}{\sigma_0}\frac{\partial}{\partial\sigma_*}\bigg|_t,$$

eq. (34) may be rewritten as

$$\left[\frac{\ddot{L}}{\dot{L}^2}l' + l'^2 + l''\right] - \left[2l' + \frac{L}{L^2}\right]\frac{dl_*}{dL_*} + \left(\frac{dl_*}{dL_*}\right)^2 + \frac{d^2l_*}{dL_*^2} = = -\frac{2\pi G\sigma_0}{z_o\, L^2}\frac{\sigma_*}{z_*}, \tag{35}$$

where $L = \ln\sigma_0$, $l = \ln z_0$, $L_* = \ln\sigma_*$ and $l_* = \ln z_*$. The dash denotes derivation with respect to L and the dot with respect to time. Without gravitation (i. e., $G = 0$), the right hand side of (35) is zero. In this zero gravity case, the left hand side of (35) can be used without distinction for the three most used symmetries, planar, cylindrical and spherical.

One can find which constraints self-similarity imposes on the energy per unit mass $E(\sigma)$ and on the time function $t_c(\sigma)$. Noting that the difference to the spherically symmetric case is in the RHS of (35), one can follow closely the work of Lemos and Lynden-Bell 1989b, and without repeating here the whole calculation, we find (Ventura, 1992).

$$E = A\sigma^a \tag{36}$$

$$t_c(\sigma) = B\sigma^b + t_0 \tag{37}$$

where A, B, b are constants of integration and $a = 2(b+1)$. There is a special case, when $b \to 0$, given by

$$t_c(\sigma) = B_1 \log\left(\frac{\sigma}{\sigma_1}\right) + t_1 \tag{38}$$

where B_1, σ_1 and t_1 are constants. The equations (36), (37) and (38) give then a set of self-similar solutions without initial conditions.

One can write the distance z of the planes in the form

$$z = -\pi G\sigma\left(t + B\sigma^b\right)\left[t - \left(\frac{\sqrt{2A}}{\pi GB} - 1\right)B\sigma^b\right]. \tag{39}$$

3.2 ILLUSTRATION OF THE SOLUTIONS

We will present now a study of the solutions obtained in the previous sub-section. Unlike the spherical case, such solutions always have positive total energy, which follows a power-law in the interior effective mass, $E = A\sigma^a$. The other integration function is $t_c(\sigma) = B\sigma^a$, where, without loss of generality, we have set $t_0 = 0$. This, leads us to write the self-similar solutions as $\sigma = \sigma_0\sigma_*$ and $z = z_0 z_*$, where now

$$\sigma_0 = \left|\frac{t}{B}\right|^{1/b}, \tag{40}$$

$$z_0 = \pi GB^2\sigma_0^{2b+1} \tag{41}$$

and

$$z_* = \sigma_*\left(\sigma_*^b \pm 1\right)\left[\left(\frac{\sqrt{2A}}{\pi GB} - 1\right)\sigma_*^b \mp 1\right]. \tag{42}$$

Here the sign $\pm$ is the same as the one of t.We analyse the expansion and recollapse of the shells. To avoid initial plane-crossing, we impose $b \geq 0$. We also see that equation (42) represents a physical solution if $\frac{\sqrt{2A}}{\pi GB} > 1$.

Let us verify if these self-similar solutions have subsequent plane-crossing. From the equations (39), we have $\rho = \frac{d\sigma}{dz}$ given by

$$\pi G\rho = \left[\left(\frac{\sqrt{2A}}{\pi GB} - 1\right)(2b+1)\,B^2\sigma^{2b} + \right. \left. + \left(\frac{\sqrt{2A}}{\pi GB} - 2\right)(b+1)\,B\sigma^b t - t^2\right]^{-1}. \tag{43}$$

Then, there is shell-crossing for distances $z \leq z_l$, where

$$z_l = \pi GB(1+\beta_\pm)\left[1 + \beta_\pm - \frac{\sqrt{2A}}{\pi GB}\right]\left[\left|\frac{t}{B\beta_\pm}\right|\right]^{\frac{2b+1}{b}}, \tag{44}$$

and

$$\beta_\pm = \frac{1}{2}\left[(b+1)\left(\frac{\sqrt{2B}}{\pi GB} - 2\right) \pm \left[\frac{2A(b+1)^2}{\pi^2G^2B^2} + \right.\right. \left.\left. -4B^2\left(\frac{\sqrt{2A}}{\pi GB} - 1\right)\right]^{1/2}\right] \tag{45}$$

The validity of this solution is restricted to the regions with $|z| > |z_l|$. We analyse only the regions where there is no shell-crossing.

For large negative times the planes are expanding and the plane-crossing region decreases with time. For large positive times the planes are colapsing and the plane-crossing regime increases with time (see Figure 2). At $t = 0$ there is a plane expanding and recollapsing instantaneously. Since the planes reach their maximum distances when $t = \frac{B\sigma^b}{2}\left(\frac{\sqrt{2A}}{\pi GB} - 2\right)$, solutions with $\frac{\sqrt{2A}}{\pi GB} > 2$ have their maximum for $t > 0$ (see first diagram of Fig. 2), while solutions with $\frac{\sqrt{2A}}{\pi GB} < 2$ reach their maximum for $t < 0$. The marginal case $\left(\frac{\sqrt{2A}}{\pi GB} = 2\right)$ gives us that all the planes have their maximal expansion point at $t = 0$ (second diagram of Figure 2).

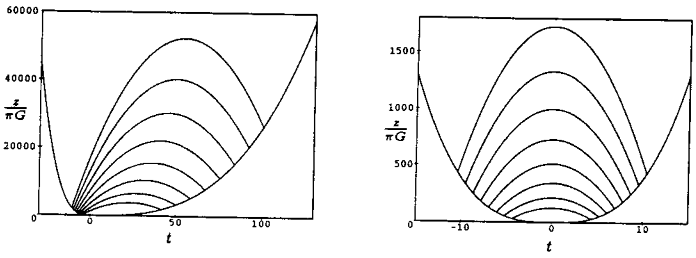

Figure 2: The z evolution of shells as a function of time the cases $b = 1$, $B = 1$, $\frac{\sqrt{2H}}{\pi GB} = 11$ and $b = 1$, $B = 1$, $\frac{\sqrt{zA}}{HGB} = 2$. Here, $\sigma = 5, 6, 7, 8, 9, 10, 11, 12$.

The sellf-similar form of the profile's density is given by:

$$\rho = \frac{1}{\pi GB^2\sigma_o^{2b}}\left[(2b+1)\left(\frac{\sqrt{2A}}{\pi GB} - 1\right)\sigma_*^{2b} + \pm\left(\frac{\sqrt{2A}}{\pi GB} - 2\right)(b+1)\sigma_*^b - 1\right]^{-1} \qquad (46)$$

and the velocity's profile by

$$v = \pi GB\sigma_0^{b+1}\sigma_*\left[\left(\frac{\sqrt{2A}}{\pi GB} - 2\right)\sigma_* \mp 2\right]. \qquad (47)$$

In the limit $z \to \infty$, the density has the form $\rho \to z^{-2b/(2b+1)}$, while the velocity goes as $v \to z^{(b+1)/(2b+1)}$. For large or intermediate z, such that $B\sigma^b\left(\frac{\sqrt{2A}}{\pi GB} - 2\right) \ll 2$ (see equation (47)), the velocity's profile is $v \to z^{1/(2b+1)}$. When $\frac{\sqrt{2A}}{\pi GB} = 2$, we have that this intermediate region is pushed to infinity.

For $t = 0$, the time-scale of the density, $\rho_0(t)$, diverges. The profiles of the density and velocity take the same form of the limiting $z \to \infty$ case, given by:

$$\rho = \frac{1}{(2b+1)\left[\pi GB^2\left(\frac{\sqrt{2A}}{\pi GB} - 1\right)\right]^{1/(2b+1)}} z^{-2b/(2b+1)}, \qquad (48)$$

$$v = \begin{cases} \dfrac{\left(\pi GB^2\right)^{\frac{b}{2b+1}}\left(\frac{\sqrt{2A}}{\pi GB} - 2\right)}{B\left(\frac{\sqrt{2A}}{\pi GB} - 1\right)^{\frac{b+1}{2b+1}}} z^{\frac{b+1}{2b+1}}, & \text{if } \frac{\sqrt{2A}}{\pi GB} \neq 2 \\ -\dfrac{2\left(\pi GB^2\right)^{\frac{2b}{2b+1}}}{B^2\left(\frac{\sqrt{2A}}{\pi GB} - 1\right)^{\frac{1}{2b+1}}} z^{1/(2b+1)}, & \text{if } \frac{\sqrt{2A}}{\pi GB} = 2. \end{cases} \qquad (49)$$

The solution with $b = 0$ and $B = 0$ is described by $z = \pi G\sigma\left[\frac{\sqrt{2A}}{\pi G}t - t^2\right]$, $v = \pi G\sigma\left[\frac{\sqrt{2A}}{\pi G} - 2t\right]$ and $\pi G\rho = \left[\frac{\sqrt{2A}}{\pi G}t - t^2\right]^{-1}$ where $0 < t < \frac{\sqrt{2A}}{\pi G}$. We verify that, in spite of the motion of planes, the volumetric density does not exhibit functional dependence on σ. This solution seem to have no direct physical interpretation.

APPENDIX

In Section 2 we analysed collapsing solutions for systems initialy at rest. Now, we will show that the dynamical description of the self-similar system is invariant when the initial velocity of the shells is non-vanishing. The general solution for such cases is:

$$z = z_i(\sigma) + v_i(\sigma)t - \pi G\sigma t^2 \qquad (A1)$$

and

$$v = v_i(\sigma) - 2\pi G\sigma t. \qquad (A2)$$

We impose the self-similarity condition, $v = v_0(t)v_*(\sigma_*)$ and $z = z_0(t)z_*(\sigma_*)$. From (A2), we have

$$F_1(\sigma_*) = f_1(t)\left[k_i(\sigma_o\sigma_*) - t\right], \qquad (A3)$$

where

$$F_1 = \frac{v_*(\sigma_*)}{\sigma_*}, \qquad (A4)$$

$$f_1 = \frac{2\pi G\sigma_0(t)}{v_0(t)}, \qquad (A5)$$

$$k_i = \frac{v_i(\sigma_0\sigma_*)}{2\pi G\sigma_0\sigma_*}. \qquad (A6)$$

Differentiating (A3) with respect to σ_*, and taking the logarithm of the resulting equation, we are lead to

$$\log F' = \log(\sigma_o f) + \log k_i', \qquad (A7)$$

where the dash denotes derivation with respect to its own argument. Differentiating (A7) with respect to σ_o, and multiplying afterwards by σ_o, we have

$$-\sigma_0\frac{d}{d\sigma_0}\log(\sigma_0 f) = \sigma\frac{d}{d\sigma}\log k_i' = b - 1 = \text{constant}. \qquad (A8)$$

For $b \neq 0$, the equation (A8) has the solution

$$k_i = \frac{1}{2}\left(B_1\sigma^b + B_2\right), \qquad (A9)$$

where B_1 and B_2 are constants. With (A6) and (A9), we have

$$v_i = \pi G\sigma\left(B_1\sigma^b + B_2\right). \qquad (A10)$$

Then (A10) and (A1) yield

$$F_2(\sigma_*) = f_2(t)\left[H(\sigma_o\sigma_*) + \left(B_1\sigma_o^b\sigma_*^b + B_2\right)t - t^2\right], \qquad (A11)$$

where,

$$H = \frac{z_i}{\pi G\sigma}, \qquad (A12)$$

$$F_2 = \frac{z_*}{\sigma_*} \qquad (A13)$$

and

$$f_2 = \frac{\pi G\sigma_0}{z_0}. \qquad (A14)$$

Differentiating (A11) with respect to σ_*, dividing by σ_*^{b-1} and differentiating again with respect to σ_*, we obtain:

$$\frac{d}{d\sigma_*}\left[\frac{F_2'}{\sigma_*^{b-1}}\right] = \sigma_0^b f_2\frac{d}{d\sigma}\left[\frac{H'}{\sigma^{b-1}}\right]. \qquad (A15)$$

Taking the logarithm of the equation (A15), differentiating with respect to σ_o and multiplying by σ_*, gives

$$\sigma_0 \frac{d}{d\sigma_0}\left[\sigma_0^{b+1} f_2\right] + \sigma \frac{d}{d\sigma}\left[\frac{H'}{\sigma^{b-1}}\right] = 0. \quad \text{(A16)}$$

Since the two terms of the equation (A16) are functions of different variables, each side must be a constant. In this case, the general solution for $H(\sigma)$ is

$$H(\sigma) = B_3\sigma^b + B_4 + \alpha I(\sigma), \quad \text{(A17)}$$

where B_3, B_4 and α are constants and $I(\sigma)$ is given by

$$I(\sigma) = \begin{cases} \sigma^b [\log\sigma - 1], & \text{for} \quad b \neq 1; \\ (\log\sigma)^2, & \text{for} \quad b = 1. \end{cases} \quad \text{(A18)}$$

Introducing (A10), (A12) and (A17) in (A1), we have

$$z = \pi G\sigma \left[(B_3 + B_1 t)\sigma^b + \alpha I + B_4 + + \frac{B_2^2}{4} - \left(t - \frac{B_2}{2}\right)^2\right] \quad \text{(A19)}$$

When we take $v_i = 0$, i. e., $B_1 = B_2 = 0$, the solutions obtained in Section 3 should be reproduced. But this occurs only if $\alpha = 0$. Therefore, (A19) is written as

$$z = \pi G\sigma \left(1 + \frac{B_1}{B_3}t\right)\left[B_3\sigma^b + - \frac{\left(t - \frac{B_2}{2}\right)^2 - \left(B_4 + \frac{B_2^2}{4}\right)}{1 + \frac{B_1}{B_3}t}\right]. \quad \text{(A20)}$$

From (A20), we see that the general form of the time scale for the surface density is

$$\sigma_0 = \gamma \left|\frac{\left(t - \frac{B_2}{2}\right)^2 - \left(B_4 + \frac{B_2^2}{4}\right)}{1 + \frac{B_1}{B_3}t}\right|^{1/b}, \quad \text{(A21)}$$

where γ is a constant which we set to one, without loss of generality, $\gamma = 1$. Then from (A20) and (A21) we have

$$z_0 = \pi G\left(1 + \frac{B_1}{B_3}t\right)\sigma_0^b. \quad \text{(A22)}$$

We know that $v_0(t) = \frac{dz_0}{dt}$. From (A21), (A22), (A2) and (A10) we have:

$$v_*(\sigma_*) = b\sigma_* J(t, \sigma_*) \quad \text{(A23)}$$

with

$$J = \left[(B_2 - 2t)\left(1 + \frac{B_1}{B_3}t\right) + B_1\left(t - \frac{B_2}{2}t\right)^2 \sigma_*^b\right] \times \times \left[(b+1)(B_2 - 2t)\left(1 + \frac{B_1}{B_3}t\right) - \frac{B_1}{B_3}\left\{\left(t - \frac{B_2}{2}\right)^2 + - \left(B_4 + \frac{B_2^2}{4}\right)\right\}\right]^{-1} \quad \text{(A24)}$$

From (A23) one finds that $J(t, \sigma_*)$ cannot depend on t. Then, we impose that the equation (A24) has dependence only on σ_*. In such case, we have only one possibility: $B_1 = 0$ and $J = \frac{1}{b+1}$. This condition, along with a time translation $\left(t \to t - \frac{B_2}{2}\right)$, allows us to write (A20) as

$$z = \pi G\sigma\left[B_3\sigma^b + t_{c0}^2 - t^2\right]. \quad \text{(A25)}$$

Similarly

$$v = -2\pi G\sigma t. \quad \text{(A26)}$$

The demonstration for the logarithmic ($b = 0$) case goes along the same steps.

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