CBPF-NF-056/95

An Alternative Dimensional Reduction Prescription

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Abstract

We propose an alternative dimensional reduction prescription which in respect with Green functions corresponds to drop the extra spatial coordinate. From this, we construct the dimensionally reduced Lagrangians both for scalars and fermions, discussing bosonization and supersymmetry in the particular 2-dimensional case. We argue that our proposal is in some situations more physical in the sense that it mantains the form of the interactions between particles thus preserving the dynamics correspondig to the higher dimensional space.

Key-words: Classical field theory; Quantum field theory; Wave equations

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Dimensional reduction is a well-honoured procedure to build theories in a given number of space-time dimensions starting from higher dimensional theories. Initiated with the Kaluza-Klein proposal for unifying electromagnetic and gravitational forces, there are many fields where it can be fruitfully exploited. Let us mention for example its wide application in Supersymmetry and Supergravity for constructing extended supersymmetric theories, studying spontaneous breaking of supersymmetry, etc [1]. Also in condensed matter problems where charged particles are constrained to move on a plane, the connection between the models in the higher $(3+1)$ and lower $(2+1)$ dimensional spaces is of relevance [2]-[3].

The dimensional reduction procedure, as it is usually applied, consists in dropping out extra coordinates at the Lagrangian level, this amounting to mantain unchanged the differential (kinetic energy) operator appearing in the Lagrangian. As a result, the Green function in momentum space preserves its form but in configuration space drastically changes. For example, if in the higher dimensional space the kinetic energy operator is a D'Alembertian, the same operator will appear in the lower dimensional theory. Evidently, the D'Alembertian Green function is different for different number of space-time dimensions. This, in a sense to be clarified below, implies that the dynamics of the interacting particles described by the dimensionally reduced Lagrangian is changed in configuration space.

The statement above can be clarified with an example which is relevant for the study of the Quantum Hall effect : electrodynamics of planar systems (see for example [2]). If one decides that electrons compelled to move in a plane are to be described by a $(2+1)$ gauge theory coupled to matter then, the resulting Coulomb potential is logarithmic instead of the $1/r$ potential to which electrons are actually subject even if they move on a plane. In passing from 3 to 2 spatial dimensions dynamics has changed.

The opposite attitude for implementing a dimensional reduction can also be thought of. Indeed, one can preserve the form of the Green function in configuration space dropping its extra space coordinate dependence. In the case of electrons in the plane, their interaction will then still be $1/r$ (with r depending only on planar coordinates). In this way one keeps the dynamics of the higher dimensional space. As we shall see below, for the case of planar electrons the Coulomb potential restricted to the plane, $1/\sqrt{x_1^2+x_2^2}$, is the Green function of the operator $(-\Delta)^{1/2}$:

$$(-\Delta)^{1/2}\frac{1}{\sqrt{x_1^2+x_2^2}} = \delta^{(2)}(x) \tag{1}$$

The previous expression can be interpreted using Riesz method of analytical continuation

[4] and distribution theory [5] for defining derivation of fractional order. Moreover, the pseudodifferential operator $(-\Delta)^{1/2}$ can be properly defined using the results of Seeley [6] on complex powers of elliptic operators.

We investigate in the present work this alternative dimensional reduction prescription which preserves higher dimensional dynamics in the restricted lower dimensional space-time manifold. As we shall see, the prescription changes the Green function in momentum space and leads to a change in the kinetic energy operator appearing in the Lagrangian. We analyse both the cases of scalar and fermionic theories and, in particular, we discuss bosonization and supersymmetry for the $3 \to 2$ dimensional reduction case.

Scalars

We start with a simple example. Consider the action for a scalar field ϕ in $(D+1)$ Minkowski space-time dimensions

$$S^{(D+1)} = \frac{1}{2} \int d^{D+1} x \phi \Box \phi + S_{int} \tag{2}$$

where S_{int} includes interactions. The corresponding Green function is defined by

$$\Box G^{(D+1)}(x) = \delta(x) \tag{3}$$

so that in momentum space one has

$$\tilde{G}^{(D+1)}(k) = \frac{1}{k_0^2 - k_1^2 - \ldots - k_D^2 - i0} \tag{4}$$

In configuration space one has [7]

$$\begin{aligned} G^{(D+1)}(x) \;\; &= \;\; -(i)^D 2^{D-1} \pi^{(D+1)/2} \Gamma \left[\frac{D-1}{2} \right] \times \\ & \qquad \left(t^2 - ((x^1)^2 + \ldots + (x^D)^2) + i0 \right)^{\frac{1-D}{2}} \end{aligned} \tag{5}$$

Let us first review how the usual dimensional reduction procedure manifests at the Green function level and then present our alternative prescription which, as we shall see, describes different Physics.

The usual way in which dimensional reduction is implemented corresponds, in this context, to drop k_D^2 in eq.(4). The Fourier transform, giving the configuration space Green function in one dimension less, will be

$$\begin{aligned} G^{(d+1)}(x) \;\; &= \;\; -(i)^d 2^{d-1} \pi^{(d+1)/2} \Gamma \left[\frac{d-1}{2} \right] \times \\ & \qquad \left(t^2 - ((x^1)^2 + \ldots + (x^d)^2) + i0 \right)^{\frac{1-d}{2}} \end{aligned} \tag{6}$$

$$d = D - 1 \tag{7}$$

Of course, this expression is different from the one we would have obtained just by dropping the extra space coordinate x_D in (5).

We are now ready to specify our dimensional reduction prescription: one starts from eq.(5) and drops the extra coordinate in the higher dimensional space Green function $G^{(D+1)}$. The resulting Green function $\mathcal{G}^{(d+1)}$ in the dimensionally reduced space ($d = D - 1$) is then defined as

$$\mathcal{G}^{(d+1)}(x) = -(i)^{d+1} 2^d \pi^{(d+2)/2} \Gamma[\frac{d}{2}] \left(t^2 - (x_1^2 + \ldots + x_d^2) + i0\right)^{-\frac{d}{2}} \tag{8}$$

The Fourier transform is given by

$$\tilde{\mathcal{G}}^{(d+1)}(k) = -2\pi i \frac{\Gamma[\frac{d}{2}]}{\Gamma[\frac{d-1}{2}]} \left(k_0^2 - (k_1^2 + \ldots + k_d^2)\right)^{-\frac{1}{2}} \tag{9}$$

with the appropriate condition for the pole. One can convince oneself that $\mathcal{G}^{(d+1)}$ is nothing but the Green function for the operator $\Box^{1/2}$ in $d+1$ dimensional space-time,

$$\Box^{1/2} \mathcal{G}^{(d+1)}(x) = \delta^{(d+1)}(x) \tag{10}$$

Then, the action for a scalar theory in the dimensionally reduced space leading to this Green function is

$$S^{(d+1)} = \frac{1}{2} \int d^{d+1}x \phi \Box^{1/2} \phi \tag{11}$$

In contrast, had we followed the habitual procedure consisting in dropping k_D^2 in eq.(4), we would had arrived to the Green function for the operator $\Box$ in $d+1$ dimensions and the action would correspond to the usual one, given by eq.(2) in the reduced space.

Let us end this discussion by analysing a $(3+1)$ case which, as mentionned above, is relevant for example in the study of electrons compelled to move in a plane. The retarded Green function for the $(3+1)$ dimensional classical system can be obtained from the general formula (5) as explained in ref.[8]

$$G^{(3+1)}(t, R) = \frac{1}{4\pi R} \delta(t - R) \tag{12}$$

where

$$R^2 = x_1^2 + x_2^2 + x_3^2 \tag{13}$$

Its Fourier transform is

$$\tilde{G}^{(3+1)} = \frac{1}{k_0^2 - K^2} \tag{14}$$

where

$$K^2 = k_1^2 + k_2^2 + k_3^2 \tag{15}$$

If, following our prescription, we drop in eq.(12) the x_3 coordinate, we still have for the wave equation Green function in the reduced $(2+1)$ space-time

$$\mathcal{G}^{(2+1)} = \frac{1}{4\pi r}\delta(t-r) \tag{16}$$

where

$$r^2 = x_1^2 + x_2^2 \tag{17}$$

One can easily see that $\mathcal{G}^{(2+1)}$ satisfies

$$\Box^{1/2}\mathcal{G}^{(2+1)}(x) = \delta^{(3)}(x) \tag{18}$$

Hence, there are two ways of doing dimensional reduction. The usual one, looked upon from the point of view of Green functions, is the one which drops the extra momentum space coordinate. The one we propose in the present work consists in dropping the extra spacial coordinate in the Green function and, from it, infer the resulting dimensionally reduced Lagrangian. This last prescription seems to be more physical in situations as that described for planar electrons which are suppose to be subject to $(3+1)$ interactions even when they are compelled to move in the plane.

It is important to stress that, independently of the number of dimensions, an action with the operator $\Box^{1/2}$ is always obtained if one starts from action (2) when one applies the dimensional reduction prescription advocated here. In summary, when one passes from D to $D-1=d$ spacial dimensions, the action changes as follows

$$\frac{1}{2}\int d^{D+1}x\phi\Box\phi \quad \stackrel{D\to d}{\longrightarrow} \quad \frac{1}{2}\int d^{d+1}x\phi\Box^{1/2}\phi \tag{19}$$

<u>Fermions</u>

Eq.(19) gives the rule for the change in the scalar field action under our dimensional reduction prescription. The case of fermions can be treated analogously. As an example we will describe here reduction from $(2+1)$ to $(1+1)$ dimensional space-times and then analyse how bosonization works for the reduced two-dimensional fermionic theory.

We start from (two-component) free Dirac fermions in Euclidean 3 dimensional space-time and take for the Dirac matrices $\gamma_0 = \sigma^1$, $\gamma_1 = \sigma^2$ and $\gamma_2 = \sigma^3$ with σ^a the Pauli matrices. To obtain the dimensionally reduced fermion action, we consider the fermion Green function,

$$G^{(3)}(X) = -\int \frac{d^3K}{(2\pi)^3}\exp(iKX)\frac{\not{K}}{K^2} \tag{20}$$

We use X (in general capital letters) for variables in 3 dimensional space time ($X = (X^0, X^1, X^2)$). If we make $X^2 = 0$ in (20) and integrate out over K_2 we obtain

$$\mathcal{G}^{(2)}(x) \equiv G^{(3)}(X)|_{X^2=0} = -\frac{1}{4\pi}\int \frac{d^2k}{(2\pi)^2}\exp(ikx)\frac{\not{k}}{k} \tag{21}$$

One can easily see that the resulting Green function $\mathcal{G}^{(2)}(x)$ in the reduced two-dimensional space satisfies

$$\frac{i\not{\partial}}{(-\Box)^{1/2}}\mathcal{G}^{(2)}(x,x') = \frac{1}{4\pi}\delta^{(2)}(x-x') \tag{22}$$

From this result we can infer the corresponding two-dimensional fermionic action

$$S^{(2)}_{fermion} = \int d^2x\bar{\psi}\frac{i\not{\partial}}{(-\Box)^{1/2}}\psi \tag{23}$$

As in the scalar case, we see that our dimensional reduction prescription reduces to change the operator A appearing in the original Lagrangian to $A/(-\Box)^{1/2}$. In the fermionic case we have arrived to a non-local expression where the Dirac operator $i\not{\partial}$ appears convolutionned with the Green function $((-\Box)^{-1/2})_{xy}$.

We shall now investigate how bosonization works when the fermionic action is given by eq.(23). To this end, we consider the two-dimensional partition function

$$Z_F = \int \mathcal{D}\bar{\psi}\mathcal{D}\psi\exp\left(-\int\bar{\psi}\frac{i\not{\partial}}{(-\Box)^{1/2}}\psi d^2x\right) \tag{24}$$

and follow the path-integral bosonization approach described in [9]. This approach starts by performing the change of variables

$$\psi \to \exp\left(i\theta(x)\right)\psi \tag{25}$$

$$\bar{\psi} \to \bar{\psi}\exp\left(-i\theta(x)\right) \tag{26}$$

with θ a real function. After this change, the partition function reads

$$Z_F = \int \mathcal{D}\bar{\psi}\mathcal{D}\psi\exp - \int\bar{\psi}\left(\frac{i\not{\partial}}{(-\Box)^{1/2}} + i\frac{i\not{\partial}}{(-\Box)^{1/2}}\theta\right)\psi d^2x \tag{27}$$

Being Z_F θ-independent, we can integrate out over θ both sides in eq.(27), this amounting to a trivial change in the normalization of the path-integral

$$Z_F = \mathcal{N}\int \mathcal{D}\bar{\psi}\mathcal{D}\psi\mathcal{D}\theta\exp - \int\bar{\psi}\left(\frac{i\not{\partial}}{\Box^{1/2}} + \not{\partial}\alpha\right)\psi d^2x \tag{28}$$

with

$$\alpha(x) \equiv -(-\Box)^{-1/2}\theta = -\int d^2y\big((-\Box)^{-1/2}\big)_{xy}\theta(y) \tag{29}$$

It is evident that $\partial_\mu\alpha$ in (28) can be thougth as a flat connection and hence it can be replaced by a "true" gauge field provided a constraint is introduced to assure its flatness. Hence, we can replace the θ integration by an integration over a flat connection b_μ by writing

$$Z_F = \mathcal{N}\int \mathcal{D}\bar{\psi}\mathcal{D}\psi Db_\mu\delta(\epsilon_{\mu\nu}f_{\mu\nu})\exp -\int\bar{\psi}\left(\frac{i\partial\!\!\!/}{(-\Box)^{1/2}}+ b\!\!\!/\right)\psi d^2x \tag{30}$$

where

$$f_{\mu\nu} = \partial_\mu b_\nu - \partial_\nu b_\mu \tag{31}$$

Here $\mu = 0, 1$ labels components in the reduced space-time. Now, performing the fermionic path integral, one has

$$Z_F = \int \mathcal{D}b_\mu\delta(\epsilon_{\mu\nu}f_{\mu\nu})\det\left(\frac{i\partial\!\!\!/}{(-\Box)^{1/2}}+ b\!\!\!/\right) \tag{32}$$

One can compute the two-dimensional fermionic determinant in (32) as a Fujikawa jacobian following the method described for the Schwinger model determinant in ref.[10]. The answer is

$$\log\det\left(\frac{i\partial\!\!\!/}{(-\Box)^{1/2}}- b\!\!\!/\right) = -\frac{1}{2\pi}\int d^2xb_\mu\left((-\Box)^{-1/2}\delta_{\mu\nu} - \partial_\mu(-\Box)^{-3/2}\partial_\nu\right)b_\nu \tag{33}$$

This, together with the representation

$$\delta[\epsilon_{\mu\nu}f_{\mu\nu}] = \int \mathcal{D}\phi\exp(-\frac{1}{\sqrt{\pi}}\int d^2x\phi\epsilon_{\mu\nu}f_{\mu\nu}) \tag{34}$$

leads, after a trivial gaussian integration over b_μ to the result:

$$Z_F = \int \mathcal{D}\phi\exp\left(-\frac{1}{2\pi}\int d^2x\phi(-\Box)^{1/2}\phi\right) \tag{35}$$

Hence, as one should expect, the two-dimensional fermion action (23), obtained within our dimensional reduction prescription, bosonizes to the two-dimensional scalar action (eq.(11)), precisely the one we obtained when applying the prescription to scalar fields.

Concerning bosonization rules for fermion currents, let us note that the addition of a fermion source s_μ in Z_F amounts to the inclussion of this source in the fermion determinant

$$Z_F[s] = \int \mathcal{D}b_\mu\det\left(\frac{i\partial\!\!\!/}{(-\Box)^{1/2}}+ b\!\!\!/ + s\!\!\!/\right)\delta[\epsilon_{\mu\nu}f_{\mu\nu}]. \tag{36}$$

Now, a trivial shift $b + s \to b$ in the integration variable b puts the source dependence into the constraint

$$Z_F[s] = \int \mathcal{D}b_\mu \det(\frac{i\partial\!\!\!/}{(-\Box)^{1/2}} + b\!\!\!/)\delta[\epsilon_{\mu\nu}(f_{\mu\nu} - 2\partial_\mu s_\nu)] \qquad (37)$$

so that, instead of (35) one ends with

$$Z_F[s] = \int \mathcal{D}\phi \exp(-\frac{1}{2}\int d^2x(\phi(-\Box)^{1/2}\phi + \frac{2}{\sqrt{\pi}}s_\mu\epsilon_{\mu\nu}\partial_\nu\phi)). \qquad (38)$$

By simple differentiation with respect to the source one infers from this expression the bosonization recipe for j_μ

$$\bar\psi\gamma^\mu\psi \to (1/\sqrt{\pi})\epsilon^{\mu\nu}\partial_\nu\phi \qquad (39)$$

which coincides with the usual one [11] except for the fact that the partition function with which one has to work in the scalar theory is given by eq.(35).

Another interesting issue where our alternative dimensional prescription can be investigated concerns supersymmetric models. The simplest supersymmetric action that one can write in 3-dimensional space is

$$S^{(3)} = \int d^3X(\phi^*\Box\phi - \bar\psi i\partial\!\!\!/\psi) \ . \qquad (40)$$

Here ϕ is a complex scalar and ψ a two component Dirac fermion. Action (40) is invariant under the supersymmetry transformation

$$\delta\phi = \bar\epsilon\psi \qquad (41)$$

$$\delta\psi = (i\partial\!\!\!/\phi)\epsilon \qquad (42)$$

Here ϵ is the real parameter associated with the supersymmetry transformation.

Using our dimensional reduction prescription one ends with a two dimensional action of the form

$$S^{(2)} = \int d^2x\phi\Box^{1/2}\phi - \int d^2x\bar\psi\frac{i\partial\!\!\!/}{\Box^{1/2}}\psi \qquad (43)$$

One can prove that this action is invariant under the supersymmetry transformations (41)-(42) now interpreted in two space-time dimensions. Hence, as in the case of bosonization, we see that our dimensional reduction prescription can be consistently applied in the case of supersymmetric models.

In summary, we have presented an alternative way for dimensional reduction which, looked upon from the point of view of Green functions, implies to be attached to the form that they take in configuration space and just drop extra coordinates. From the resulting reduced Green function one can infer the form that the Lagrangian takes in the dimensionally reduced space-time. This can be done in arbitrary number of dimensions. For the scalar theory, the answer is given in eq.(11). Concerning fermions, we have discused the $(2+1) \to (1+1)$ case (eq.(23) giving the reduced action) but more general cases can be envisaged. Of course, one has to take into account the appropriate number of spinor components in different number of space-time dimensions but this can be handled in the same way one does within the habitual dimensional reduction approach.

A comment is in order concerning the nonlocality arising from the $\Box^{-1/2}$ kernel appearing in the dimensionally reduced Lagrangians. As shown in [7]-[8], [12], by choosing the appropriate retarded or advanced prescriptions causality is in fact respected since the kernel $\Box^{-1/2}$ has support in the light-cone surface.

Our coments on the $(3+1) \to (2+1)$ case in connection with electrons compelled to move in the plane shows that in certain situations our prescription seems to be more physical than the usual one in the sense that it preserves the form of the interaction; the resulting reduced Lagrangian can be thought to give an effective description of this interaction. This is precisely the approach undertaken by Marino in ref.[3] in his study of QED for particles on a plane. In this work, the relation between the resulting effective gauge theory and the strictly $(2+1)$ Chern-Simons theory usually employed to describe fractional statistics is explored and the potential applications in condensed matter physics are discussed.

We have studied in some detail bosonization of free fermions in the reduced $(1+1)$ theory showing that bosonization recipe can be derived and are much the same as for the usual Dirac Lagrangian. This subject, as well as the issue of bosonization in $(2+1)$ space-time deserve a more thorough analysis. We hope to report on this aspects in a future work.

Acknowledgements: This work was supported in part by CICBA and CONICET, Argentina. F.A.S. thanks the Centro Brasileiro de Pesquisas Físicas for its kind hospitality.

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