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Wave Equations with Multiple Times
Classical and Quantum Solutions

J.J. Giambiagi

Centro Brasileiro de Pesquisas Físicas - CBPF

Rua Dr. Xavier Sigaud, 150

22.290-180 Rio de Janeiro/RJ, Brasil

Abstract

Several examples of classical solutions of $\Box$, $\Box\Box$ and $\Box\Box\Box$ with multiple times are discussed in 4,6 and 8 dimensions.

It is shown how to quantize the theory and also that all multiple time quantum solutions are contained in the euclidean one plus the appropriate analytical continuation.

Key-words: Wave equations; Iterated D'Alembertians; Classical field theory; Quantum field theory.

I Introduction

Multidimensionality is becoming fashion. But most of the interest goes to multidimensional space coordinates with only one time. This revewed interest is due essentially to supersymmetric theories but it has always been present in mathematical minded physicists (see refs. [1], [2], [3], [4]).

There are no compelling reasons why not to consider extra time dimensions, a subject which has attracted the attention of mathematicians see ref. [5].

In this note, we intend so discuss solutions of the homogeneous wave equations (outside the origin) in spaces with p spaces and q times and shall use the notation $(p+q)$ to label these spaces, $(p+q)$ being the total dimensionality.

In ref. [4] we discussed how to generate solutions of the homogeneous eqs. in $(p+2)$ or $(q+2)$ starting from solutions in $(p+q)$ and differentiating (resp) with respect to r^2 and t^2.

In refs. [3], [6] and [7] we discussed real powers of D'Alembertian and its relations with Huyghens' principle in even and odd dimensions. We also discussed its application to gravitational theories [8].

But always with only one time. It seems of interest to gain some experience by finding some specific solutions of D'Alembertians or power of D'Alembertians in multiple time dimensions.

Compatibility of high energy experimental results with multiple times were discussed in ref. [9].

We shall use the following notation

$$\Box = \frac{\partial^2}{\partial t_1^2} + \frac{\partial^2}{\partial t_2^2} + \cdots + \frac{\partial^2}{\partial t_q^2} - \frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial x_2^2} - \cdots \frac{\partial^2}{\partial x_p^2} \tag{1}$$

for solutions which depend only on

$$t = \sqrt{t_1^2 + t_2^2 + \cdots + t_q^2} \quad \text{and} \quad r = \sqrt{x_1^2 + \cdots + x_p^2} \tag{2}$$

We shall use

$$\Box = \left(\frac{\partial^2}{\partial t^2} + \frac{(q-1)}{t} \frac{\partial}{\partial t} \right) - \left(\frac{\partial^2}{\partial r^2} + \frac{(p-1)}{r} \frac{\partial}{\partial r} \right) \tag{3}$$

It is also attractive the idea of discussing the quantization of multiple time theories. The propagator is defined by the Green function and it will be shown that all quantum theories are contained in the euclidean theory of $(p+q)$ dimensions.

In §II we discuss solutions in four dimensions with two times, in six dimensions with two and three times for $\Box$ and $\Box\Box$, in eight dimensions, $\Box$, $\Box\Box$ and $\Box\Box\Box$ for two, three and four times.

In §III a discussion is given on how to quantize the theory, and it is shown that with the appropriate definition of the propagators, all quantum theories with multiple times are contained in the euclidean one.

All discussions, except at the end are given outside the origin (o.o).

II Higher Dimensions

Dimension 4

$2+2$.

$$\Box = \frac{\partial^2}{\partial t^2} + \frac{2}{t} + \frac{\partial}{\partial t} - \frac{\partial^2}{\partial r^2} - \frac{2}{r}\frac{\partial}{\partial r}$$

and it can easily be verified that

$$\varphi = \frac{\delta(t-r)}{r} \quad \text{with} \quad t = \sqrt{t_1^2 + t_2^2} \tag{4}$$

is a solution of $\Box\varphi = 0$ (o.o).

In verifying this solution, one has to use the property of the δ functions

$$x\delta(x) = 0 \qquad\qquad x\delta'(x) + \delta(x) = 0 \tag{5}$$

So, in (4) we cannot replace the δ function by an arbitrary function of $(t-r)$ as can be done in $3+1$. Solution (4) has no deformation. Starting from this solution in $2+2$ we obtain, by differentiating with respect to $r^2(t^2)$, solutions in $4+2$ $(2+4)$. We shall come back to this.

We see that causality can be violated with respect to any t_1, or t_2 but not with respect to the modulus.

(4) is not the Green (classical) function of $\Box$.

$3+1$. It is already well known.

The signal propagates without deformation.

$1+3$

$$\varphi = \frac{\delta(t-r)}{r} \tag{6}$$

$$t = \sqrt{t_1^2 + t_2^2 + t_3^3} \tag{7}$$

$$\Box\varphi = 0(o.o) \tag{8}$$

valid only for δ functions as use is made of $x\delta(x) = 0 \cdots$, etc.

Dimension 6

$5+1$ In six dimensions it is interesting to study not only the $\Box$ but also $\Box\Box$, as the Green function of the $\Box\Box$ here is equal to $\frac{\delta(t-r)}{r}$ which is the Green function in 4 dimensions of $\Box$.

It is also true that

$$\Box\Box\frac{f(t-r)}{r} = 0 \tag{9}$$

and we observe here an interesting fact, that for the simple $\Box$ there is distortion, for solutions depending only on r and t. This is not so for the $\Box\Box$ (see refs. [1][2]). The solution for the simple $\Box$ is

$$y = \frac{f(t-r)}{r^3} + \frac{f'(t-r)}{r^2} \tag{10}$$

and we see the distorsion when going from $r << 1$ to $r >> 1$.

$3+3$. Here we verify

$$\Box\Box\frac{f(t-r)}{r} = 0 \qquad f = \text{arbitrary} \tag{11}$$

$4+2$

$$\Box\Box\frac{\delta(t-r)}{r} = 0 \qquad (o.o) \tag{12}$$

but use has been made of $x\delta(x) = 0\cdots$, so (12) is not true when replacing the δ by an arbitrary $f(t-r)$.

As we have already said, any solution of $\Box\varphi = 0$ in p, q will generate solutions in $(p+2, q)$ and $(p, q+2)$ by simple differentiation. So, we have, starting from (4) that

$$y = \frac{\delta(t-r)}{r^3} + \frac{\delta'(t-r)}{r^2} \qquad t = \sqrt{t_1^2 + t_2^2} \tag{13}$$

is a solution of $\Box y = 0$ (o.o) but not the Green function.

$2+4$. Differentiating (4) with respect to t^2 have a solution in $2+4$.

$$\Box\frac{\delta'(t-r)}{tr} = 0 \qquad t = \sqrt{t_1^2 + \cdots + t_4^2} \tag{14}$$

(14) is not true for an arbitrary $f(t-r)$.

$3+3$ For the same reasons, starting from eq. (6) we obtain (o.o)

$$\Box \frac{\delta'(t-r)}{tr} = 0 \quad \text{with} \quad t = \sqrt{t_1^2 + t_2^2 + t_3^2} \tag{15}$$

this is true for an arbitrary function $f(t-r)$. It will be shown below that in fact $\frac{\delta'(t-r)}{tr}$ is the Green function in $3+3$.

Dimension 8

$7+1$ Here

$$\frac{\delta(t-r)}{r} \tag{16}$$

is the Green function of $\Box\Box\Box$ (see ref. 5). It is also true

$$\Box\Box\Box \frac{f(t-r)}{r} = 0 \quad (o.o)$$

$6+2$ Here, also

$$\Box\Box\Box \frac{\delta(t-r)}{r} = 0 \qquad (o.o) \tag{17}$$

but now $t = \sqrt{t_1^2 + t_2^2}$.

In order to prove (17) in $6+2$ use has to be made of the property $x\delta(x) = 0 \cdots$ etc. So, it is not valid if we change $\delta(t-r)$ by an arbitrary functions $f(t-r)$. $\frac{\delta(t-r)}{r}$ is not the Green function in $6+2$.

$4+4$ Again, it can easily be proved.

$$\Box\Box\Box \frac{\delta(t-r)}{r} = 0 \qquad (o.o) \qquad t = \sqrt{t_1^2 + \cdots + t_4^2} \tag{18}$$

Here specific use has to be made of the properties of the δ function, so it is not true for an arbitrary function $f(t-r)$.

It is to be observed that Green functions of $\Box$ and $\Box\Box$ can be obtained by differentiating with respect to r^2 and t^2.

Differentiating (13) we find that

$$y = \frac{3\delta'(t-r)}{r^4} + \frac{3\delta(t-r)}{r^5} + \frac{\delta''(t-r)}{r^3} \tag{19}$$

is the Green function up to a constant in $7+1$ (for proof see below) and

$$y = \frac{\delta'(t-r)}{tr} + \frac{\delta''(t-r)}{r^2 t} \tag{20}$$

satisfies

$$\Box y = 0 \qquad (o.o) \quad \text{in} 3+5 \text{with} t = \sqrt{t_1^2 + t_2^2 + t_3^2}. \tag{21}$$

It is the Green function in this space.

And so on $\cdots$

III Quantum Theory

To make it simple, we consider an ordinary D'Alembertian theory

$$\mathcal{L} = \varphi \Box \varphi \tag{22}$$

we consider first an euclidean theory. The Green function of the laplacian in 4 dimensions is

$$\Delta_4 G = \delta^4(x) \tag{23}$$

$$G = \frac{1}{t^2 + r^2} \tag{24}$$

which is the propagator of the theory.

In order to obtain the Feynman propagator (see ref. [10] we make $t' = at \quad a > 0$ which defines an analytic distribution in a. We perform the analytic continuation to $a = i + \varepsilon$, so (24) goes over to

$$G = \frac{1}{r^2 - t^2 + i\varepsilon} \tag{25}$$

and the analogous with the Fourier transform.

Observe that this is not the Wick rotation $t = i\tau$ which would lead to $(i\tau + \varepsilon)^2 = -\tau^2 + i\varepsilon\tau$ and this is not the Feynman propagator but the positive Schwinger Green function in momentum space.

This is exactly the procedure to be followed when we have multiple times.

$$t = \sqrt{t_1^2 + t_2^2 + \cdots + t_q^2} \tag{26}$$

$$G = \frac{1}{(t^2 + r^2)^{\frac{n}{2}-1}} \qquad n = p + q. \tag{27}$$

we do as before. $t'_i = at_i \quad a > 0$ (the same a for every t_i) and in the same way as before we are led to:

$$G = \frac{1}{(r^2 - t^2 + i\varepsilon)^{\frac{n}{2}-1}} \tag{28}$$

Formula (26) and (27) can be generalized for the Lagrangian

$$\mathcal{L} = \varphi \Box^{\lambda} \varphi \qquad \lambda < \frac{n}{2} \qquad \lambda = \text{integer} \tag{29}$$

when the generalizations of (27) and (28) will be:

$$G = \frac{1}{(t^2 + r^2)^{\frac{n}{2} - \lambda}} \tag{27'}$$

and the quantum propagator would be:

$$G = \frac{1}{(r^2 - t^2 + i\varepsilon)^{\frac{n}{2} - \lambda}}. \tag{28'}$$

See ref. [5] p. 275.

For $k \geq \frac{n}{2}$ see ref. [5] p. 276.

This is the quantum propagator for a massless particle. For a massive one, the procedure follows exactly parallel with the result that

$$(\Box + m^2)^{\lambda} G^k_m = \delta \tag{30}$$

leads to

$$G^k_m \simeq \frac{K\frac{n}{2} - \lambda(m\sqrt{t^2 - r^2 + i\varepsilon})}{\{m(t^2 - r^2 + io)^{1/2}\}^{\frac{n}{2} - \lambda}} \tag{31}$$

Using (28') and (31) all the calculations can be performed.

IV Discussion

The different solutions discussed in the second paragraph show that the modulus of the vector time, with components $t_1 t_2 t_3$ plays the role of a real time. Causality can be violated with respect to an individual time t_i but not with respect to the modulus t, as made explicit by the $\delta(t - r)$ in eq. (4).

We see also that the properties of solutions of $\Box\varphi = 0$ (or a δ) in four dimensions are analogous to those of $\Box\Box\varphi$ in six or to $\Box\Box\Box\varphi$ in eight. For instance, $\frac{\delta(t-r)}{r}$ is a solution of $\Box\varphi = 0$ in $2 + 2$; of $\Box\Box\varphi = 0$ in $4 + 2$ and of $\Box\Box\Box\varphi = 0$ in $6 + 2$ and in the three cases the $\delta(t - r)$ cannot be changed to an arbitrary function $f(t - r)$.

In all these cases we discussed-except when specifically mentioned-solutions of equations outside the origin (o.o) of space time where a δ function type singularity may be present. The specific appearance of a $\delta^n(x)$ in the second member will be covered by the following theorem.

If $G_\lambda(p,q;rt)$ is the Green function of

$$\Box^\lambda G_\lambda(p,q;rt=\delta) \tag{32}$$

then $\frac{1}{r}\frac{d}{dr}G\lambda(p,q;rt)$ will be up to a constant the Green function $G_\lambda(p+2,q;rt)$

$$(resp\frac{1}{t}\frac{\partial}{\partial t} \Longrightarrow G_\lambda(p,q+2;rt) \qquad\qquad k<\frac{n}{2}$$

In fact going over to the Fourier transform

$$G_\lambda(p,q;rt)=(2\pi)^\nu\int d^qk_0\ d^pk\ \frac{k}{(k_0^2-k^2)^\lambda}e^{ik_ix_i}e^{-ik_{oj}t_j} \tag{33}$$

Using now Bochner's theorem see ref. [11], (32) can be written

$$G_\lambda(p,q;rt)=\int_0^\infty dk_0\ \int_0^\infty dk\frac{k_0^{\frac{q}{2}}k^{\frac{p}{2}}}{(k_0^1-k^2)^\lambda}\frac{J_{\frac{q}{2}-1}(tk_0)}{t^{\frac{q}{2}-1}}\frac{J_{\frac{p}{2}-1}(rk)}{r^{\frac{p}{2}-1}} \tag{34}$$

Now we compute $\frac{1}{r}\frac{\partial G\lambda}{\partial r}$ and use the property of the Bessels functions

$$\left(\frac{1}{u}\frac{d}{du}\right)^m\ u^{-\nu}J_\nu(u)=(-1)^m u^{-\nu-m}J_{\nu+m}(u) \tag{35}$$

for $m=1$; and we get

$$\frac{1}{r}\frac{\partial G_\lambda(p,q;r,t)}{\partial r}=\int\int_0^\infty\frac{dk_0dk}{(k_0^2-k^2)^\lambda}k_0^{\frac{q}{2}}k^{\frac{p+2}{2}}\frac{J_{\frac{p}{2}}(kr)}{(kr)^{\frac{p}{2}}}\frac{J_{\frac{q}{2}-1}(k_0t)}{t^{\frac{q}{2}-1}} \tag{36}$$

From (36) and (33) we see that

$$\frac{1}{r}\frac{dG_\lambda^{(p,q,rt)}}{dr}=-G_\lambda(p+2,q;rt) \tag{37}$$

and using (35) for m integer, (37) can be generalized to

$$\left(\frac{1}{r}\frac{\partial}{\partial r}\right)^m G(p,q;rt)=(-1)^m G_\lambda(p+2m,q;rt) \tag{38}$$

Of course, in the same way

$$\left(\frac{1}{t}\frac{\partial}{\partial t}\right)^m G_\lambda(p,q;r,t)=G_\lambda(p,q+2m;rt)(-1)^m \tag{39}$$

So, starting from the wave eq. in $3+1$ we can generate the whole series of Green functions of $3+3$; $5+1$; $3+5\cdots$ etc. And the same for iterated D'Alembertians.

In this way, from (10) with $f = \delta(t-r)$ we see that is the Green function of $\Box$ in $5+1$. The same with (15) where

$$y = \frac{\delta'(t-r)}{tr} \quad \text{is the Green function} \tag{40}$$

in $3+3$ and form it we generate GF in higher time and space dimensions.

A completely similar discussion can be made for iterated D'Alembertians.

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