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The Quantum Algebraic Structure of the Twisted XXZ Chain

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ABSTRACT

We consider the Quantum Inverse Scattering Method with a new R-matrix depending on two parameters q and t. We find that the underlying algebraic structure is the two-parameter deformed algebra $SU_{q,t}(2)$ enlarged by introducing an element belonging to the centre. The corresponding Hamiltonian describes the spin-1/2 XXZ model with twisted periodic boundary conditions.

Key-words: Quantum groups; Statistical mechanics; Quantum field theory; Integrable models

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Many completely integrable one-dimensional quantum models have been treated [1] by the Quantum Inverse Scattering Method (QISM)[2, 3] which, among other achievements, led to the discovery of Quantum Groups [4] independently from Drinfeld and Jimbo[5]. In this letter we use this method to describe the relation between the spin-1/2 XXZ chain with twisted periodic boundary conditions and the two-parameter deformed algebra $SU_{q,t}(2)$ enlarged by introducing an element belonging to the centre, showing that the second parameter of the deformation, t, is linked to the twist.

The QISM introduces an auxiliary problem with the help of the so-called Lax operator. In our model this operator is

$$L_n(\lambda,t) = \begin{pmatrix} t^{Z_n - S_n^3}\, sh[\gamma(\lambda + iS_n^3)] & iS_n^-\, sin\gamma \\ iS_n^+ sin\gamma & t^{-Z_n - S_n^3}\, sh[\gamma(\lambda - iS_n^3)] \end{pmatrix}, \tag{1}$$

where $\vec{S}_n$ and Z_n are operators defined on the $n-th$ vectorial space of the periodic ($\vec{S}_{N+1} \equiv \vec{S}_1, Z_{N+1} \equiv Z_1$) chain, which in the fundamental representation are given by

$$Z_n = \frac{1}{2}1\!\!1_n \,, \vec{S}_n = \frac{1}{2}\vec{\sigma}_n \,, \tag{2}$$

where $\vec{\sigma}$ are the Pauli matrices and $1\!\!1$ is the identity operator.

The R-matrix associated to the Lax operator (1) is

$$R(\lambda,t) = \begin{pmatrix} a(\lambda) & 0 & 0 & 0 \\ 0 & c'(\lambda) & b(\lambda) & 0 \\ 0 & b(\lambda) & c''(\lambda) & 0 \\ 0 & 0 & 0 & a(\lambda) \end{pmatrix}, \tag{3}$$

where

$$\begin{aligned} a(\lambda) &= sh[\gamma(\lambda + i)] \\ b(\lambda) &= isin\gamma \\ c'(\lambda) &= tc(\lambda) \\ c''(\lambda) &= t^{-1}c(\lambda) \end{aligned} \tag{4}$$

and

$$c(\lambda) = sh\ \gamma\lambda\ ; \tag{5}$$

clearly, $R(\lambda) = R(\lambda, t = 1)$ is the appropriate matrix for the XXZ model [6]. It is easy to check that the matrix $R(\lambda, t)$ (eqs. (3-5)) satisfies the Yang-Baxter equation [7, 8]

$$R_{12}(\lambda_{12}, t)R_{13}(\lambda_{13}, t)R_{23}(\lambda_{23}, t) = R_{23}(\lambda_{23}, t)R_{13}(\lambda_{13}, t)R_{12}(\lambda_{12}, t) \tag{6}$$

and that $L_n(\lambda, t)$ (eq. (1)) obeys the Fundamental Commutation Relations (FCR)

$$R_{12}(\lambda_{12}, t)L_n^1(\lambda_1, t)L_n^2(\lambda_2, t) = L_n^2(\lambda_2, t)L_n^1(\lambda_1, t)R_{12}(\lambda_{12}, t)\ . \tag{7}$$

In (6) and (7), $\lambda_{ij} = \lambda_i - \lambda_j$ and

$$\begin{aligned} R_{12} &= \sum_i a_i \otimes b_i \otimes 1\!\!1 \quad , \quad R_{13} = \sum_i a_i \otimes 1\!\!1 \otimes b_i\ , \\ R_{23} &= \sum_i 1\!\!1 \otimes a_i \otimes b_i\ , \end{aligned} \tag{8}$$

with the $R(\lambda, t)$ matrix written as

$$R(\lambda, t) = \sum_i a_i \otimes b_i \tag{9}$$

and the upper indices in eq. (7) follow

$$L^1 = L \otimes 1\!\!1 \quad , \quad L^2 = 1\!\!1 \otimes L. \tag{10}$$

We notice that the R-matrix is defined on the tensor product of two auxiliary spaces $\mathbb{C}^2 \otimes \mathbb{C}^2$ and the L-matrix is defined on the tensor product of the auxiliary space $\mathbb{C}^2$ and the internal space $\mathbb{C}^d$, with d the dimension of the representation of the associated algebra satisfied by the operators in the elements of the matrix L.

The reason for $R(\lambda, t)$ to satisfy eq. (6) is that it can be written in terms of $R(\lambda)$ as[1]

$$R_{12}(\lambda, t) = g^1(g^2)^{-1}R_{12}(\lambda)g^1(g^2)^{-1}\ , \tag{11}$$

[1]Equivalently one could write

$$g^1(g^2)^{-1} = t^{S^3 \otimes Z - Z \otimes S^3}\ ,$$

with S^3 and Z given by eq. (2). This form is more appropriate if one wishes to compare the algebraic structure here presented with ref. [9]. In a forthcoming paper we shall discuss this subject, as well as the relationship of our approach with the one in ref. [10].

where

$$g^1 = g \otimes 1\!\!1 \quad , \quad g^2 = 1\!\!1 \otimes g \quad , \quad g = t^{\frac{1}{2}S^3} \tag{12}$$

and

$$[g^1 g^2, R_{12}(\lambda)] = 0. \tag{13}$$

Moreover, eq. (7) follows from eq. (6), because (for d=2)

$$L_n(\lambda, t) = R_{o,n}(\lambda - \frac{i}{2}, t), \tag{14}$$

where "o" labels the auxiliary space. We also observe that

$$R(0, t) = P, \tag{15}$$

where P is the permutation operator on the tensor product of the two spaces where the R-matrix is defined.

According to the standard procedure of the QISM, eq. (7) allows one to build an infinite set of commuting operators

$$F(\lambda, t) = Tr[L_N(\lambda, t) \cdots L_2(\lambda, t) L_1(\lambda, t)] \;, \tag{16}$$

where both the matrix product and the trace are performed in the auxiliary space.

In our case the Bethe Ansatz equations for the fundamental representation are given by

$$\left(\frac{\alpha(\lambda_\beta)}{\delta(\lambda_\beta)}\right)^N = t^{-N} \prod_{\substack{\alpha=1 \\ \alpha \neq \beta}}^{M} \left\{ \frac{a(\lambda_\beta - \lambda_\alpha)}{a(\lambda_\alpha - \lambda_\beta)} \frac{c(\lambda_\alpha - \lambda_\beta)}{c(\lambda_\beta - \lambda_\alpha)} \right\} \; ; \; \beta = 1, \cdots, M \leq N \;, \tag{17}$$

with

$$\begin{aligned} \alpha(\lambda) &= sh\left[\gamma\left(\lambda + \frac{i}{2}\right)\right] \\ \delta(\lambda) &= sh\left[\gamma\left(\lambda - \frac{i}{2}\right)\right] \;, \end{aligned} \tag{18}$$

explicitly showing the contribution due to the parameter t, as α, δ, a and c are the same functions appearing in the XXZ model.

In order to show the algebraic structure underlying the R and L matrices defined in eqs. (1) and (3-5) we perform a suitable similarity transformation [3] on (6) and (7) which permits us to have the following decomposition:

$$\tilde{L}_n(\lambda,t) = \frac{1}{2}(e^{\lambda\gamma}L_+ - e^{-\lambda\gamma}L_-) \tag{19}$$
$$\tilde{R}(\lambda_{ij},t) = e^{\gamma\lambda_{ij}}R_+ - e^{-\gamma\lambda_{ij}}R_- \quad ;$$

where

$$L_+ = \begin{pmatrix} q^{S^3}t^{Z-S^3} & \Omega S^- \\ 0 & q^{-S^3}t^{-Z-S^3} \end{pmatrix} \tag{20}$$

$$L_- = \begin{pmatrix} q^{-S^3}t^{Z-S^3} & 0 \\ -\Omega S^+ & q^{S^3}t^{-Z-S^3} \end{pmatrix}$$

and

$$R_+ = \begin{pmatrix} q & 0 & 0 & 0 \\ 0 & t & \Omega & 0 \\ 0 & 0 & t^{-1} & 0 \\ 0 & 0 & 0 & q \end{pmatrix} \quad , \tag{21}$$

with $\Omega = q - q^{-1}$ and $R_- = PR_+^{-1}P$.

Substituting eq. (19) in the Y-B equation (6) and in the FCR, eq. (7), one gets the following independent equations:

$$R_+L_\epsilon^1L_\epsilon^2 = L_\epsilon^2L_\epsilon^1R_+ \qquad (\epsilon = \pm 1) \tag{22}$$
$$R_+L_+^1L_-^2 = L_-^2L_+^1R_+ \ ,$$

which imply that the operators in the entries of L must satisfy

$$[S^3, Z] = [S^\pm, Z] = 0 \tag{23.a}$$

$$[S^3, S^\pm] = \pm S_\pm \tag{23.b}$$

$$t^{-1}S^+S^- - tS^-S^+ = t^{-2S^3}[2S^3]_q \quad , \tag{23.c}$$

where $[x]_q = (q^x - q^{-x})/(q - q^{-1})$ with $q = exp(i\gamma)$. Eqs. (23) are the commutation relations of the two-parametric deformed SU(2) [9, 11, 12] with Z, an element of the center of the resulting algebra. The coproduct is obtained by considering the product of two L_ε acting on two internal spaces and we find:

$$\Delta S^3 = S^3 \otimes 1\!\!1 + 1\!\!1 \otimes S^3 \tag{24.a}$$

$$\Delta Z = Z \otimes 1\!\!1 + 1\!\!1 \otimes Z \tag{24.b}$$

$$\Delta S^\pm = q^{S^3} t^{\mp Z - S^3} \otimes S^\pm + S^\pm \otimes q^{-S^3} t^{\pm Z - S^3}. \tag{24.c}$$

The coproduct (24c) is related to the one in ref. [12] by a similarity transformation generated by the operator $t^{S^3 \otimes Z - Z \otimes S^3}$.

Following the QISM, a local Hamiltonian can be written as

$$H \propto \frac{\partial}{\partial\lambda} \ell g F(\lambda, t)|_{\lambda=\frac{i}{2}} \tag{25}$$

and thanks to eqs. (14-16), it becomes for the fundamental representation of the algebra

$$H = \sum_{i=1}^{N} \quad H_{i,i+1} \quad (N + 1 \equiv 1) \tag{26}$$

$$H_{i,i+1} = \frac{J sin\gamma}{i\gamma} \frac{\partial}{\partial\lambda} \hat{R}_{i,i+1}(\lambda, t)|_{\lambda=0} \, ,$$

where $\hat{R} = PR$ and $R(\lambda, t)$ is given by eqs. (3-5). In the above equation $R_{i,i+1}(\lambda, t)$ acts on the two internal spaces $(i, i + 1)$ instead of acting on two auxiliary spaces.

Substituting $R(\lambda, t)$ given by eqs. (3-5) in eq. (26), apart from an additive constant, we get

$$H = \frac{J}{2} \sum_{i=1}^{N} \left[2t^{-1} \sigma_i^+ \sigma_{i+1}^- + 2t \sigma_i^- \sigma_{i+1}^+ + \frac{q + q^{-1}}{2} \sigma_i^z \sigma_{i+1}^z \right] \, , \tag{27}$$

where $\sigma^\pm = (\sigma^x \pm i\sigma^y)/2$.

Such a Hamiltonian is very similar to the XXZ model with periodic boundary conditions but for each pair of sites $(i, i + 1)$, the site $(i + 1)$ is rotated of an angle α $(t = e^{i\alpha})$ in the $x - y$ plane with respect to the site i.

The similarity transformation generated by $exp\{-i\frac{\alpha}{2}\sum_{\ell=1}^{N}(\ell-1)\sigma_\ell^z\}$ takes the Hamiltonian (eq. (27)) to

$$H=\frac{J}{2}\left[\sum_{n=1}^{N-1}\left(\sigma_n^x\sigma_{n+1}^x+\sigma_n^y\sigma_{n+1}^y+cos\gamma\;\sigma_n^z\sigma_{n+1}^z\right)+\right.$$
$$\left.cos\gamma\;\sigma_N^z\sigma_1^z+2t^{-N}\sigma_N^+\sigma_1^-+2t^N\sigma_N^-\sigma_1^+\right]\quad, \tag{28}$$

which is the well-known [13] Hamiltonian for the XXZ chain with twisted periodic boundary conditions.

It is amusing to observe that, thanks to eq. (13) and following the procedure of ref. [10], the Hamiltonian (eq. (27)) could also be obtained from the R-matrix of the XXZ model $R(\lambda)=R(\lambda,t=1)$, using $L_n'(\lambda,t)=t^{S^3}L_n(\lambda,t=1)\neq L_n(\lambda,t)$. Conversely, the untwisted XXZ model can be built from $R(\lambda,t)$ (eqs. (3-5)) and $L_n''(\lambda,t)=t^{-S^3}L_n(\lambda,t)$. All these topics will be discussed in detail in a forthcoming paper.

Finally, we would like to point out that by introducing a central element Z which enlarges the $SU_{q,t}(2)$ algebra, we make appear the underlying algebraic structure of the so-called twisted XXZ model.

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