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LANCZOS POTENTIAL AND JORDAN THEORY OF GRAVITY

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Abstract

The hamiltonian formulation of Jordan's theory of gravity is presented by means of the Lanczos' potential. The consequences of using this potential in gravity and in its interactions with other fields is discussed.

1 INTRODUCTION

At the end of the decade of 50 Jordan and collaborators[1,2] made the very interesting remark that Einstein's theory of General Relativity (GR) could exhibit a great formal analogy with Maxwell's electrodynamics. At that time such discovery appeared to be a happy evidence that gravitational and electromagnetic fields could be unified following the line proposed and started by A. Einstein.

For many reasons such alternative program of description of the gravitational field remained almost completely abandoned during the next decade. Then, at the end of the sixties and thanks to the high status of the so called "gauge theories" the Jordan Program was revived. The main lines of research developped at the end of the sixties and which continued to be examined afterwards follow two directions: the fundamentalist and the pragmatic one.

The first one deals basically with the Jordan's theory either as a candidate for a true gauge theory of gravity [Camenzind[3], Carmeli[4], Fairchild[5], Novello[6,7], Yang[8]] or as a modification of Einstein's GR due to processes related to quantum fluctuations. The second one is nothing but the tentative (firstly suggested by S. Hawking[9]) to analyse the evolution of small perturbations of the metric of space-times. Such method which has been used in many cosmological solutions (e.g. Friedmann, Kasner, Gödel) is particularly useful in those geometries which are conformally flat in the umperturbed stage. In this case, we are not dealing with a new alternative theoretical proposal of the equations of evolution of the gravitational field but instead, using the power of the Jordan's formalism in the alternative investigation of the perturbation metrics. This has been exhaustively reviewed recently by Novello and Salim[10] through the examination of the modes of vibration of Friedmann Universes for almost generic kind of perturbation. Thus, we limit our investigation here only to the first case. In section 2 we present Jordan's theory and its relation with Einstein's GR. Section 3 presents the Lanczos' potential and its consequence on the Hamiltonian for the gravitational field in Jordan's formulation. In Section 4 we present a new formulation of a general covariant theory of a massive gravity. In Section 5 we introduce a new lagrangian for the dynamics of an electron interacting with gravity using Lanczo's potential and we compare this with the equation of an electron in a Cartan geometry. We end with section 6 in which some conclusions and suggestions for future investigation are presented.

2 JORDAN'S THEORY OF GRAVITY

Bianchi identities, satisfied by Riemannian geometries can be written in terms of Weyl conformal tensor $W_{\alpha\beta\mu\nu}$ under the form

$$W^{\alpha\beta\mu\nu}{}_{;\nu} = \frac{1}{2} R^{\mu[\alpha;\beta]} - \frac{1}{12} g^{\mu[\alpha} R^{,\beta]} \qquad (1)$$

in which ; means covariant derivative and the bracket [,] means antisymmetrization $f_{[\mu\nu]} = f_{\mu\nu} - f_{\nu\mu}$. Any theory, for instance Einstein's GR, which requires an algebraic relation between $R_{\mu\nu}$ and a non-geometric tensor (say, the energy-momentum tensor $T_{\mu\nu}$) induces through equation (1) an equation of the type

$$W^{\alpha\beta\mu\nu}{}_{;\nu} = J^{\alpha\beta\mu} \tag{2}$$

It seens that Jordan and co-workers were the first who have considered expressions (1) and (2) as the basis of a theory of gravity, which from now on will be called Jordan theory.

It is however clear that taken together, equations (1) and (2), which can be interpreted as a set of metric equations, cannot in general be identified with GR.

The recognition of this fact has led many authors to propose distinct alternative theories of gravity with different leitmotiv. Among these we can quote those proposals made by Camenzind, Carmeli, Fairchild, Novello, Yang and others. Some of these theories have been previously criticized for different reasons. For instance, Fairchild[5] argues that the main objection to Camenzind's Theory is one's inability to put it into a variational form; in another paper Pavelle[11] shows how to find unphysical solutions of Yang's gravitational field equations. Although I do not intend to give arguments to support any of these theories, we will show, as a by product of our formalism, that both criticisms can be overcome by the use of Lanczos potential approach and the analysis of initial value problem for the proposed extended equations of gravity.

Coming back to the restricted point of view of GR one should like to answer the question: under what conditions will equations (1) and (2) be equivalent to Einstein's theory? The answer to this question was contained in the work of Lichnerowicz, in the early sixties, which, after a thorough examination of the Cauchy problem has shown that if we admit Einstein's equation to be valid in an space-like hypersurface $\sum$, that is $R_{\mu\nu}(\sum) = 0$ as initial conditions of Jordan's equations, then this system is completely equivalent to GR. The generalization for the case in which matter is present is straightforward. Thus, the use of Einstein's equation under the conventional form

$$R_{\mu\nu} = -T_{\mu\nu} + \frac{1}{2} Tg_{\mu\nu} \tag{3}$$

or in Jordan's formalism (eq. 1,2) is just a matter of taste and/or simplicity which should be dictated by the examination of the problem being investigated. The main formal advantage of Jordan's theory rests upon the great formal similitude which it has with Maxwell's electrodynamics. This allows us to undertake an investigation of its properties along lines very similar to this theory. With the proposal to exhibit more clearly such resemblance and for later reference (to our calculations) we present here in a compact manner the representation of Jordan's theory in the quasi-Maxwellian form as has been made by Truper and others. This means that instead of dealing with expression (2) we will project it in the rest space of an arbitrary observer which moves in space time with velocity v^{μ} (normalized: $V_{\mu}V_{\nu}g^{\mu\nu}=1$). The electric ($E_{\mu\nu}$) and the magnetic ($B_{\mu\nu}$) tensors are defined as

$$E_{\mu\nu} = -W_{\mu\alpha\nu\beta} V^{\alpha}V^{\beta} \tag{4a}$$

$$B_{\mu\nu} = -\overset{*}{W}_{\mu\alpha\nu\beta} V^{\alpha}V^{\beta} \tag{4b}$$

in wich a star * means the dual:

$$: f^{*}_{\mu\nu} = \frac{1}{2}\,\eta_{\mu\nu\rho\sigma}\, f^{\rho\sigma} = \frac{1}{2}\,\sqrt{-g}\,\varepsilon_{\mu\nu\rho\sigma}\, f^{\rho\sigma}$$

and $\varepsilon_{\mu\nu\rho\sigma}$ is Levi-Civita symbol.

These definitions imply the properties $E_{\mu\nu} = E_{\nu\mu}$, $E_{\mu\nu}g^{\mu\nu} = 0$, $E_{\mu\nu}V^{\nu} = 0$, $B_{\mu\nu} = B_{\nu\mu}$, $B_{\mu\nu}g^{\mu\nu} = 0$, $B_{\mu\nu}V^{\nu} = 0$. Thus the 5+5 independent components of $E_{\mu\nu}$ and $B_{\mu\nu}$ specifies completely, for each observer, the components of $W_{\alpha\beta\mu\nu}$ which can then be written as

$$W_{\alpha\mu\beta\nu} = (\eta_{\alpha\mu\lambda\sigma}\,\eta_{\beta\nu\tau\varepsilon} - g_{\alpha\mu\lambda\sigma}\,g_{\beta\nu\tau\varepsilon})\, V^{\lambda}\, V^{\tau}\, E^{\sigma\varepsilon} +$$

$$+ (\eta_{\alpha\mu\lambda\sigma}\, g_{\beta\nu\tau\varepsilon} + g_{\alpha\mu\lambda\sigma}\,\eta_{\beta\nu\tau\varepsilon})\, V^{\lambda}\, V^{\tau}\, B^{\sigma\varepsilon} \qquad (5)$$

in which $g_{\alpha\beta\mu\nu} \equiv g_{\alpha\mu}g_{\beta\nu} - g_{\alpha\nu}g_{\beta\mu}$.

Using the projective properties of $h_{\mu}{}^{\nu} = \delta_{\mu}{}^{\nu} - V_{\mu}V^{\nu}$ (e.g. $h_{\mu}{}^{\nu}\, h_{\nu}{}^{\lambda} = h_{\mu}{}^{\lambda}$, and so on) we can re-write equation (2) in the form of a set of equations envolving "div" and "curl" operators acting on $E_{\mu\nu}$ and $B_{\mu\nu}$ as in Maxwell's theory (following Truper et al.). We then obtain

$$h^{\varepsilon\alpha}\, h^{\lambda\gamma}\, E_{\alpha\lambda;\gamma} + \eta^{\varepsilon}{}_{\beta\mu\nu}\, V^{\beta}\, B^{\nu\lambda}\, \sigma^{\mu}{}_{\lambda} + 3B^{\varepsilon\nu}\,\omega_{\nu} = 0 \qquad (6a)$$

$$h^{\varepsilon\alpha}\, h^{\lambda\gamma}\, B_{\alpha\lambda;\gamma} - \eta^{\varepsilon}{}_{\beta\mu\nu}\, V^{\beta}\, E^{\nu\lambda}\, \sigma^{\mu}{}_{\lambda} - 3E^{\varepsilon\nu}\,\omega_{\nu} = 0 \qquad (6b)$$

which are the "div E" and "div B" equations. The equations of evolution involving time derivatives and "curl" operators are:

$$\dot{E}^{\mu\nu}\, h_{\mu}{}^{\rho}\, h_{\nu}{}^{\tau} + \theta\, E^{\rho\tau} - \frac{1}{2}\, E_{\nu}{}^{(\rho}\, \sigma^{\tau)\nu} - \frac{1}{2}\, E_{\nu}{}^{(\rho}\, \omega^{\tau)} +$$

$$+ \eta^{\tau\nu\mu\varepsilon}\, \eta^{\rho\lambda\alpha\beta}\, V_{\mu}V_{\lambda}\, E_{\varepsilon\alpha}\,\sigma_{\beta\nu} + a_{\alpha}\, B_{\beta}{}^{(\rho}\, \eta^{\tau)\lambda\alpha\beta}\, V_{\lambda} -$$

$$- \frac{1}{2}\, B_{\beta}{}^{\mu}{}_{;\alpha}\, h_{\mu}{}^{(\tau}\, \eta^{\rho)\lambda\alpha\beta}\, V_{\lambda} = 0 \qquad (6c)$$

$$\dot{B}^{\mu\nu} h_{\mu}{}^{\rho} h_{\nu}{}^{\tau} + \theta\, B^{\rho\tau} - \frac{1}{2}\, B_{\nu}{}^{(\rho}\, \sigma^{\tau)\nu} - \frac{1}{2}\, B_{\nu}{}^{(\rho}\, \omega^{\tau)} + \qquad \text{(6d)}$$

$$+ \eta^{\tau\nu\mu\varepsilon}\, \eta^{\rho\lambda\alpha\beta}\, V_{\mu} V_{\lambda} B_{\varepsilon\alpha} \sigma_{\beta\mu} - a_{\alpha} E_{\beta}\, {}^{(\rho}\, \eta^{\tau)\lambda\alpha\beta}\, V_{\lambda} +$$

$$+ \frac{1}{2}\, E_{\beta}{}^{\mu}{}_{;\alpha}\, h_{\mu}{}^{(\tau}\, \eta^{\rho)\lambda\alpha\beta}\, V_{\lambda} = 0$$

Note that these equations are invariant under the dual map $W_{\alpha\beta\mu\nu} \to W'_{\alpha\beta\mu\nu} = \cos\psi\, W_{\alpha\beta\mu\nu} + \sin\psi\, \overset{*}{W}_{\alpha\beta\mu\nu}$ for an arbitrary constant angle ψ. In these equations (6) $\sigma_{\mu\nu}$ is the shear, a^{μ} is the acceleration and ω^{μ} the rotation vector of the congruence of the curves generated by V^{μ}; the covariant derivative of V^{μ} has been decomposed in the conventional way

$$V_{\mu;\nu} = \sigma_{\mu\nu} + \frac{\theta}{3}\, h_{\mu\nu} + \omega_{\mu\nu} + a_{\mu} V_{\nu} \qquad (7)$$

and

$$\omega^{\tau} = \frac{1}{2}\, \eta^{\alpha\beta\rho\tau}\, \omega_{\alpha\beta}\, V_{\tau} \qquad (8)$$

$$\theta = V^{\mu}{}_{;\mu} \qquad (9)$$

It seems worthwhile to remark that this set (6), which is called the quasi-Maxwellian equations of gravity, has to be complemented by the equations of evolution of the observer, contained in the irreducible quantities θ, $\sigma_{\nu\mu}$, a_{α} and ω_{α}. In our case, choosing an irrotational geodesic observer these equations reduce to the set

$$\frac{d\theta}{dt} + \frac{1}{3}\, \theta^{2} + 2\sigma^{2} = 0 \qquad (10)$$

$$h_\alpha{}^\mu h_\beta{}^\nu \dot{\sigma}_{\mu\nu} - \frac{2}{3} h_{\alpha\beta}\sigma^2 + \frac{2}{3}\theta\sigma_{\alpha\beta} + \sigma_{\alpha\mu}\sigma^\mu{}_\beta = R_{\alpha\varepsilon\beta\nu}V^\varepsilon V^\nu - \frac{1}{3} R_{\mu\nu}V^\mu V^\nu h_{\alpha\beta} \tag{11}$$

in which $\dot{\sigma}_{\mu\nu} = \dfrac{D\sigma_{\mu\nu}}{Dt} = \dfrac{d\sigma_{\mu\nu}}{dt} - \Gamma^\alpha_{\mu\rho}\sigma_{\alpha\nu}V^\rho - \Gamma^\alpha_{\nu\rho}\sigma_{\mu\nu}V^\rho$

3 LANCZOS POTENTIAL

The remarkable similitude of equations (6) with those which describe Maxwell's electrodynamics led me to conceive the idea that an alternative hamiltonian formulation of gravity could be naturally developped in the framework of Jordan's theory. To do this, we should, first of all, be able to construct a third order tensor potential from which - by first order derivatives- one can construct the Weyl tensor. Fortunatelly this problem has indeed been solved by Cornelius Lanczos[12] some twenty years ago. Quite surprisingly such potential theory of the conformal tensor remained almost forgotten by the scientific community. As far as I know very few works have been published dealing with Lanczos' potentials. This seems to have its origin in the general inability of realizing how fair was Lanczos' formulation, due to the lack of a mathematical rigorous demonstration of such result. This situation was recently remedied with the publication of the work of Bampi and Caviglia[13] which have shown a theorem which guarantees the existence of such potential in any manifold endowed with a Riemannian structure.

Following Lanczos we set:

$$-W_{\alpha\beta\mu\nu} = A_{\alpha\beta\mu;\nu} - A_{\alpha\beta\nu;\mu} + A_{\mu\nu\alpha;\beta} - A_{\mu\nu\beta;\alpha} +$$

$$+\frac{1}{2}(A_{\nu\alpha} + A_{\alpha\nu})g_{\beta\mu} + \frac{1}{2}(A_{\mu\beta} + A_{\beta\mu})g_{\alpha\nu} - \frac{1}{2}(A_{\alpha\mu} + A^{\mu\alpha})\ g_{\beta\nu} -$$

$$-\frac{1}{2}(A_{\beta\nu} + A_{\nu\beta})\ g_{\mu\alpha} + \frac{2}{3}\ A^{\sigma\lambda}{}_{\sigma;\lambda}\ g_{\alpha\beta\mu\nu} \tag{12}$$

in which

$$A_{\alpha\mu} \equiv A_{\alpha}{}^{\sigma}{}_{\mu;\sigma} - A_{\alpha}{}^{\lambda}{}_{\lambda;\mu}$$

Lanczos' tensor $A_{\alpha\beta\mu}$ has the properties

$$A_{\alpha\beta\mu} = -A_{\beta\alpha\mu} \tag{13a}$$

$$A^{*}_{\alpha\beta\mu}\, g^{\alpha\mu} = 0 \tag{13b}$$

Such tensor has then 20 independent components. This implies that equation (12) has a gauge symmetry. Lanczos' proposed to specify from the beginning a gauge (which we will call from now on the Lanczos' gauge) by specifying the 10 conditions

$$A_{\alpha\beta\mu}\, g^{\alpha\mu} = 0 \tag{14a}$$

$$A_{\alpha\beta}{}^{\sigma}{}_{;\sigma} = 0 \tag{14b}$$

Bampi and Caviglia have shown that there always exists a tensor which satisfies (13) and with which one can construct $W_{\alpha\beta\mu\nu}$. Let me remark that although one can look at equation (12) as a functional definition of $A_{\alpha\beta\mu}$ in terms of the metric tensor, it is not an easy task to obtain the explicit local relation between them. Further, such local dependence does not even exist in general, but only the global structural dependence represented in the integral function of $A_{\alpha\beta\mu}$ on $g_{\mu\nu}$.

However, there is a special case of interest in which such explicit dependence of $A_{\alpha\beta\mu}$ on $g_{\mu\nu}$ in a local basis, can be exhibited: the case of weak gravitational field. Indeed, as Lanczos shown in his paper if we have $g_{\mu\nu} \simeq \eta_{\mu\nu} + \epsilon\, \psi_{\mu\nu}$ with $\epsilon^2 << \epsilon$ then we can write (in the first order approximation):

$$A_{\alpha\beta\mu} = \frac{1}{4}[\psi_{\alpha\mu,\beta} - \psi_{\mu\beta,\alpha} + \frac{1}{6}\psi_{,\alpha}\eta_{\mu\beta} - \frac{1}{6}\psi_{,\beta}\eta_{\mu\alpha}] \qquad (15)$$

in which we have chosen to work in Lanczos' gauge.

Another useful gauge which one can use is what I will call the reducible gauge. In this case, instead of (14) I impose

$$A_{\mu\nu} + A_{\nu\mu} = -R_{\mu\nu} \qquad (16)$$

Then $R=-4\, A^{\alpha\beta}{}_{\alpha;\beta}$ and consequently

$$-W_{\alpha\beta\mu\nu} = A_{\alpha\beta\mu;\nu} - A_{\alpha\beta\nu;\mu} + A_{\mu\nu\alpha;\beta} - A_{\mu\nu\beta;\alpha} + \qquad (17)$$

$$+ \frac{1}{2}[R_{\alpha\mu}g_{\beta\nu} + R_{\beta\nu}g_{\alpha\nu} - R_{\alpha\nu}g_{\beta\mu} - R_{\beta\nu}g_{\alpha\nu}] - \frac{1}{6} R\, g_{\alpha\beta\mu\nu}$$

Now, the expression of Weyl tensor can be written as:

$$-W_{\alpha\beta\mu\nu} = -R_{\alpha\beta\mu\nu} + \frac{1}{2}[R_{\alpha\mu}g_{\beta\nu} + R_{\beta\nu}g_{\alpha\mu} - \qquad (18)$$

$$-R_{\alpha\nu}g_{\beta\mu} - R_{\beta\mu}g_{\alpha\nu}] - \frac{1}{6} R\, g_{\alpha\beta\mu\nu}$$

Thus comparison of expressions (17) and (18) yields

$$-R_{\alpha\beta\mu\nu} = A_{\alpha\beta\mu;\nu} - A_{\alpha\beta\nu;\mu} + A_{\mu\nu\alpha;\beta} - A_{\mu\nu\beta;\alpha} \tag{19}$$

Now, Bampi and Caviglia have shown that expression (19) is not true in general but only for some class of Riemannian geometries. We conclude then that the reducible gauge is admissible only for those cases and is not valid in general.

Let me remark that the fact that $W_{\alpha\beta\mu\nu}$ is completely trace-free implies $W^{*}_{\alpha\beta\mu\nu}={W_{\alpha\beta}}^{*}_{\mu\nu}$, a property which is not manifestly exhibited by the decomposition (12). In order to make such important property explicit we re-write eq. (12) in a more convenient form.

We have:

$$-{A^{*}_{\alpha\beta\mu}}^{*}_{;\nu} = \frac{1}{4}\, {\eta_{\alpha\beta}}^{\rho\sigma}\, {\eta_{\mu\nu}}^{\varepsilon\tau}\, A_{\rho\sigma\varepsilon;\tau} = -\frac{1}{4}\, \delta^{\rho\sigma\varepsilon\tau}_{\alpha\beta\mu\nu}\, A_{\rho\sigma\varepsilon;\tau}$$

Developing the right-hand side we obtain

$$-{A^{*}_{\alpha\beta\mu}}^{*}_{;\nu} = \frac{1}{2}[A_{\mu\nu\alpha;\beta} - A_{\mu\nu\beta;\alpha} + A_{\mu\beta}g_{\alpha\nu} + A_{\alpha\nu}g_{\mu\beta} - A_{\mu\alpha}g_{\beta\nu} - A_{\beta\nu}g_{\mu\alpha}] \tag{20}$$

Then, we arrive at the identity

$${A^{*}_{\alpha\beta\mu}}^{*}_{;\nu} + {A^{*}_{\mu\nu\alpha}}^{*}_{;\beta} = -\frac{1}{2}[A_{\alpha\beta\mu;\nu} - A_{\alpha\beta\nu;\mu} + \tag{21}$$

$$+ A_{\mu\nu\alpha;\beta} - A_{\mu\nu\beta;\alpha} + 2A_{\mu\beta}g_{\alpha\nu} + 2A_{\alpha\nu}g_{\mu\beta} - 2A_{\beta\nu}g_{\mu\alpha} - 2A_{\mu\alpha}g_{\beta\nu}]$$

which allows us to write an alternative equivalent expression for the Weyl conformal tensor as:

$$W_{\alpha\beta\mu\nu} = +2(A^{*}_{\alpha\beta\mu;\nu}{}^{*} + A^{*}_{\mu\nu\alpha;\beta}{}^{*}) + A_{\mu\beta}\, g_{\alpha\nu} + A_{\alpha\nu}\, g_{\beta\mu} - A_{\alpha\mu}\, g_{\beta\nu} - A_{\beta\nu}\, g_{\alpha\mu} \tag{22}$$

Although it is not essential, we are using the L-gauge in order to simplify those expressions.

4 HAMILTONIAN FORMULATION OF JORDAN'S THEORY OF GRAVITY

The Lagrangian of Jordan's equation (2) is given by

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_1 \tag{23}$$

with

$$\mathcal{L}_0 = \frac{1}{8} \sqrt{-g}\; W^{\alpha\beta\mu\nu}\; W_{\alpha\beta\mu\nu} \tag{23a}$$

$$\mathcal{L}_1 = \sqrt{-g}\; J^{\alpha\beta\mu}\; A_{\alpha\beta\mu} \tag{23b}$$

Let us first examine $\mathcal{L}_0$.

We have

$$\mathcal{L}_0 = \sqrt{-g}(E^{\mu\nu}E_{\mu\nu} - B^{\mu\nu}B_{\mu\nu}) \equiv \sqrt{-g}(E^2 - B^2)$$

In order to simplify our calculations we will work in a gaussian system of coordinates associated to an irrotational congruence generated by vector V^{μ}. That is, we set for the fundamental length the expression

$$ds^2 = dt^2 - g_{ij}(x^\mu)\, dx^i\, dx^j \tag{24}$$

with i,j,k... = 1,2,3.

In this case we have $V^\mu = \delta^\mu_o$ and $V_{\mu;\nu} = \sigma_{\mu\nu} + \frac{\theta}{3} h_{\mu\nu}$. We emphasize that the choice of this system of coordinates is not essential but has the great advantage to simplify our formulas. It is not a hard task to generalize all our expressions to an arbitrary non-gaussian system of coordinates. Due to this choice we conclude immediately that

$$\theta_{\mu\nu} \equiv \sigma_{\mu\nu} + \frac{\theta}{3} h_{\mu\nu} = \frac{1}{2} h_{(\mu}{}^{\alpha} h_{\nu)}{}^{\beta} V_{\alpha;\beta}$$

implies that the unique non-vanishing terms are:

$$\theta_{ij} = \frac{1}{2} g_{ij,0}$$

Further, we have $\omega_{\mu\nu}=0$.

Consequently the mixed Christoffel symbols are

$$\Gamma^o_{ij} = -\theta_{ij}$$

$$\Gamma^i_{o\,j} = \theta^i{}_j = \theta_{kj} g^{ki}$$

We can then re-write our Lagrangian to the form

$$\mathcal{L}_o = \sqrt{-g}\,[W^{\ell oko} W_{\ell oko} + \frac{1}{2} W^{\ell iko} W_{\ell iko}]$$

The fundamental variables of our theory are A_{ioo}, A_{iko}, A_{ijo} and $C_{iok} = A_{iok} + A_{koi}$. Through the standard procedure we obtain the momenta cannonically conjugated, respectivelly:

$$\Pi^{iok} = \frac{\delta \mathcal{L}}{\delta C_{iok,o}} = -\sqrt{-g}\; W^{ioko} = \sqrt{-g}\; E^{ik} \tag{25a}$$

$$\Pi^{ijk} = \frac{\delta \mathcal{L}}{\delta A_{ijk,o}} = -\sqrt{-g}\; W^{ijko} = \sqrt{-g}\; \eta^{ijom} B_m{}^{k} \tag{23b}$$

$$\Pi^{ioo} = \frac{\delta \mathcal{L}}{\delta A_{ioo,o}} = 0 \tag{23c}$$

$$\Pi^{iko} = \frac{\delta \mathcal{L}}{\delta A_{iko,o}} = 0 \tag{25d}$$

We remark that these momenta c.c. are given by the electric and the magnetic parts of Weyl tensor, in contradistinction to Maxwell's theory in which the momenta is the electric vector uniquely. This, of course, is a direct consequence of the higher number of degrees of freedom of gravity (represented by $W_{\alpha\beta\mu\nu}$). Equations (25c,d) are constraints. They have to be viewed as weak equations in Dirac's terminology[14]. The Hamiltonian of our system is given by

$$\mathcal{H} = \mathcal{H}_c + \mathcal{H}_M + \mathcal{H}_N$$

where

$$\mathcal{H}_c = \frac{1}{\sqrt{-g}} [\Pi^{iok}\, \Pi_{iok} + \frac{1}{2} \Pi^{ijk}\, \Pi_{ijk}] \tag{26a}$$

$$\mathcal{H}_M = -4 A_{ioo} \overset{(3)}{\nabla}_k \Pi^{iok} - \frac{3}{2} A_{ijo} \overset{(3)}{\nabla}_k \Pi^{ijk} + C_{koi} \overset{(3)}{\nabla}_j \Pi^{ijk} - 2 A_{i\ell k} \overset{(3)}{\nabla}_\ell \Pi^{iok} \tag{26b}$$

$$\mathcal{H}_N = -3\, A_{i\ell o}\, \theta^{\ell k}\, \Pi^{i}{}_{ok} - \theta_{i\ell}\, C^{\ell}{}_{ok}\, \Pi^{iok} + \qquad (26c)$$

$$+ \theta\, C^{iok}\, \Pi_{iok} + 2(A_{mjk} + A_{mkj})\, \theta^{m}{}_{i}\, \Pi^{ijk} - 2\, A_{joo}\, \theta_{ik}\, \Pi^{ijk}$$

in which the canonical part is $\mathcal{H}_c = E^2 - B^2$; $\mathcal{H}_M$ and $\mathcal{H}_N$ depend, besides the fundamental objects A's and Π's, on the 3-geometry of Σ and on the expansion factor θ_{ij}, respectively. In order to preserve the constraints (25c,d) we must impose that their Poisson brackets with the Hamiltonian vanishes. We obtain then the equations of divergence (6a) and (6b). The evolution of Π_{iok} and Π_{ijk} gives, through the Poisson bracket with $\mathcal{H}$, equations (6c) and (6d) - as it should be.

Let me point out that the set of fundamental dynamical objects are the A_{ijk} and C_{iok} and its momenta Π_{ijk} and Π_{iok}. However, as in any theory in a curved Riemannian space-time, we must know the values of θ and σ_{ij} in the hypersurface Σ. This gives a complete knowledge of $\mathcal{H}$ on Σ, which allows us to propagate the initial values of $W_{\alpha\beta\mu\nu}$ (on Σ) to the future of Σ, on another hypersurface, say $\tilde{\Sigma}$. To describe the system on $\tilde{\Sigma}$ and to propagate it to the future of $\tilde{\Sigma}$ we must know θ and σ_{ij} on $\tilde{\Sigma}$. This is given by the equations of evolution (10) and (11), once we know the Weyl tensor on $\tilde{\Sigma}$ - which is provided by the knowledge of $\mathcal{H}(\Sigma)$.

4 FINITE RANGE GRAVITY-LIKE FORCES: A GENERAL COVARIANT TREATMENT

Recently, some authors have suggested that the present scientific description of our cosmos could be incomplete just because we are not taking into account the whole system of forces responsible for the structure of space-time. This has led to the emergence of theories in which new long-range forces are supposed to exist and also to the investigation of modifications of the behavior of classical fields in new unknown regimes (Okum[15]). Among these we can quote the idea, which has been set up some years ago, of a possible short-range distinct component of the gravitational field which could modify the local structure of space-time. Such idea has been developped in the framework of a massive gravity. The reason for considering this theory here may be related to the recent success of the investigation of massive electrodynamics (the *mass* of the photon being a consequence of the non--minimal coupling with gravity) which admits as a basic solution an eternal Universe Friedmann-like without singularity[16]. The general inability to produce an equation of motion of massive gravity in the context of Einstein's general relativity that is, retaining the manifold mapping group as a group of symmetry of the theory, has led in the seventies to the belief that such a theory does not exist which as we will show here, was a mistake. Thus, different types of bi-metric theories which exhibit explicitly the break of general covariance have appeared and has been considered as the natural way of introducing massive gravity-like forces[17,18,19,20].

Although nowadays it seems more fashionable to introduce matter in a field theory through some sort of symmetry breaking mechanism, the old fashioned way to consider matter terms directly in a conformally non-invariant field equation still has an interest by its own. Besides, the fact that mass is still a sort of misterious property which could be related - in a Machian sense -to the structure of space-time led us to believe that the presence of a mass-term in Newton-Einstein's theory of gravity should have a deeper appeal, once one is dealing precisely with the kind of force on which the structure of space-time is supposed to depend.

We will construct such massive gravity in the context of Jordan's formulation of Einstein's general relativity. To do this, we set up the Lagrangian of the theory as given by

$$L = \frac{1}{2}\sqrt{-g}\,[\frac{1}{4} W^{\alpha\beta\mu\nu} W_{\alpha\beta\mu\nu} + \mu^2 A^{\alpha\beta\mu} A_{\alpha\beta\mu}] \tag{27}$$

which implies

$$W^{\alpha\beta\lambda\nu}{}_{;\nu} + \mu^2 A^{\alpha\beta\lambda} = 0 \tag{28}$$

We remark that in case $\mu\neq 0$, Lanczos' gauge $A^{\alpha\beta\mu}{}_{;\mu} = 0$ is a necessary condition of compatibility of the equation for $A_{\alpha\beta\mu}$. Indeed, taking the co-variant derivative of equation (28) we obtain:

$$W^{\alpha\beta\lambda\nu}{}_{;\nu;\lambda} + \mu^2 A^{\alpha\beta\lambda}{}_{;\lambda} = 0$$

which can be reduced to:

$$R_{\alpha\varepsilon\mu\nu} W^{\beta\varepsilon\mu\nu} - R_{\beta\varepsilon\mu\nu} W^{\alpha\varepsilon\mu\nu} + 2\mu^2 A^{\alpha\beta\sigma}{}_{;\sigma} = 0$$

Now, using the identity

$$R_{\alpha\varepsilon\mu\nu} W^{\beta\varepsilon\mu\nu} \equiv \frac{1}{4} W_{\rho\varepsilon\mu\nu} W^{\rho\varepsilon\mu\nu} \delta^{\beta}{}_{\alpha} + W_{\alpha\rho}{}^{\beta\sigma} R^{\rho}{}_{\sigma}$$

we obtain

$$A^{\alpha\beta\mu}{}_{;\mu} = 0$$

Let us finally comment that equation (28) reduces to the good limit of Pauli-Fierz spin-two theory in case of weak gravity. To prove this we should be able to solve the functional dependence of tensor $A_{\alpha\beta\mu}$ in terms of the metric $g_{\mu\nu}$. Although this can not be done for a generic Riemannian space-time, we can do this is case of weak field as we have previously discussed. We can then re-write equation (28) in the form:

$$\begin{aligned} &\Box A^{\alpha\beta\mu} + W^{\alpha}{}_{\varepsilon}{}^{\mu}{}_{\nu} A^{\beta\nu\varepsilon} - W^{\beta}{}_{\varepsilon}{}^{\mu}{}_{\nu} A^{\alpha\nu\varepsilon} - W^{\alpha}{}_{\varepsilon}{}^{\beta}{}_{\nu} A^{\nu\varepsilon\mu} + \\ &+ W^{\alpha}{}_{\varepsilon\lambda\nu} A^{\varepsilon\lambda\nu} g^{\beta\mu} - W^{\beta}{}_{\varepsilon\lambda\nu} A^{\varepsilon\lambda\nu} g^{\alpha\mu} + 2R^{\mu}{}_{\varepsilon} A^{\alpha\beta\varepsilon} + R_{\varepsilon\nu} g^{\alpha\mu} A^{\beta\nu\varepsilon} \\ &- R_{\varepsilon\nu} g^{\beta\mu} A^{\alpha\nu\varepsilon} - R^{\alpha}{}_{\nu} A^{\beta\mu\nu} + R^{\beta}{}_{\nu} A^{\alpha\mu\nu} - \left(\frac{1}{2} R - \mu^2\right) A^{\alpha\beta\mu} = 0 \end{aligned} \tag{29}$$

Which is the fundamental (non-linear) equation for the gravitational potential. One should ask if the mass μ should be constructed from known constants of nature or if it should be given only on an empirical basis. Recently there has been some experiments performed to check the inverse square law of gravitation on a macroscopic scale - that means beyond $\mu^{-1} = 1\text{m}$, with very restrictive limits. However, one should wonder if for much smaller values of μ^{-1}, a new property of gravity-like forces should not manifest.

5 INTERACTION OF A SPINOR FIELD WITH GRAVITY

The common procedure of generalyzing Dirac's equation of the electron - represented by a spinor $\psi(x)$ - through the use of the minimal coupling principle (MCP) in a curved Riemannian space--time is given by

$$i\,\gamma^{\mu}\,\nabla_{\mu}\psi - m\,\psi = 0 \qquad (30)$$

in which the generalized γ's, as space-time dependent objects, o-bey $\{\gamma^{\mu}(x),\ \gamma^{\nu}(x)\}_{+} = 2g^{\mu\nu}(x)I_4$, and I_4 is the identity. One can interpret such γ's in term of tetrad field $e^{\mu}_{A}(x)$ which is related to the constant Dirac matrices γ^{A} $(\gamma^{A}\gamma^{B}+\gamma^{B}\gamma^{A} = 2\eta^{AB}\,I_4)$ through the relation $\gamma^{\mu}(x) = e^{\mu}_{A}(x)\ \gamma^{A}$.

The co-variant internal derivative ∇_{μ} is defined by

$$\nabla_{\mu}\psi = \partial_{\mu}\psi - \tau_{\mu}\psi$$

and the internal connection $\tau_{\mu\ B}^{\ A}$, firstly calculated by Fock and Ivanenko is given by

$$\tau_{\mu}^{(F.I.)} = \frac{1}{8}\left[\gamma^{\lambda}(x)\ \gamma_{\lambda}(x)_{,\mu} - \gamma_{\lambda}(x)_{,\mu}\ \gamma^{\lambda}(x) + \{^{\ \varepsilon}_{\lambda\mu}\}\ \Sigma_{\varepsilon}^{\ \lambda}\right] \qquad (31)$$

in which

$\{^{\ \varepsilon}_{\lambda\,\mu}\}$ is the Christoffel symbol and $\Sigma_{\mu\nu} = \gamma_{\mu}\gamma_{\nu} - \gamma_{\nu}\gamma_{\mu}$.

The existence of Lanczos' potential introduces a new possible type of interaction between gravity and the electron. Indeed, we can add to the usual Lagrangian, $\mathcal{L}_0 = \overline{\psi}(i\gamma^{\mu}\nabla_{\mu}-m)\psi$ the parity-conserving term

$$\Delta \mathcal{L} = i \bar{\psi} \gamma^{\mu} \Sigma^{\alpha\beta} \psi A_{\alpha\beta\mu} \tag{32}$$

Remark that there is no need to introduce a new dimensional constant in the theory, once $\dim[A_{\alpha\beta\mu}] = (\text{length})^{-1}$.

We obtain thus a new proposal for the equation of an electron interacting with a gravitational field:

$$i \gamma^{\mu} \nabla_{\mu} \psi - m \psi + i A_{\alpha\beta\mu} \gamma^{\mu} \Sigma^{\alpha\beta} \psi = 0 \tag{33}$$

We can equivalently, interpret such equation (33) as a direct consequence of the change of the internal connection

$$\tau_{\mu}^{(F.I.)} \rightarrow \tau_{\mu} = \tau_{\mu}^{(F.I.)} - A_{\alpha\beta\mu} \Sigma^{\alpha\beta} \tag{34}$$

Remark that this new term is compatible with the Riemannian structure of the metric. Indeed, expression (34) implies

$$\nabla_{\mu} \gamma_{\nu} \equiv \partial_{\mu} \gamma_{\nu} - \left\{ {\varepsilon \atop \mu\nu} \right\} \gamma_{\varepsilon} + [\tau_{\mu}, \gamma_{\nu}] = [U_{\mu}, \gamma_{\nu}] \tag{35}$$

with $U_{\mu} = A_{\alpha\beta\mu} \Sigma^{\alpha\beta}$. Now, it has been known since many years ago [7,21,5] that equation (35) for any element U_{μ} of the Clifford algebra of the γ's implies the Riemannian condition on the metric:

$$g_{\mu\nu;\lambda} = 0 \tag{36}$$

How could one envisage an experiment to decide between equation

(33) or the usual equation (30)? Trying to answer to this question we find another additional difficulty which comes from the indeterminacy of a direct measure of τ_μ. Indeed, should the space have a non-null torsion $\tau^{\alpha}_{\mu\nu} \equiv \Gamma_{\mu}{}^{\alpha}{}_{\nu} - \Gamma_{\nu}{}^{\alpha}{}_{\mu}$ for a non-symmetric connection $\Gamma_{\mu}{}^{\alpha}{}_{\nu}$, the motion of an electron could not distinguish between the presence of such torsion or the minimal coupling with gravity through the additional Lagrangian (32). The equation of motion of an electron in a Cartan non-symmetric geometry is given by (30) with the internal connection

$$\tau_\mu = \frac{1}{8}[\gamma^\lambda \gamma_{\lambda,\mu} - \gamma_{\lambda,\mu}\gamma^\lambda + \Gamma^{\varepsilon}_{\lambda\mu} \Sigma_\varepsilon{}^\lambda] \tag{37}$$

in which

$$\Gamma^{\varepsilon}_{\lambda\mu} = \{^{\varepsilon}_{\lambda\mu}\} + K^{\varepsilon}_{\lambda\mu} \tag{38}$$

and the contortion $K^{\varepsilon}_{\lambda\mu}$ is given by:

$$K^{\varepsilon}{}_{\lambda\mu} = g^{\varepsilon\nu} g_{\sigma\mu} \tau^{\sigma}{}_{\nu\lambda} + g^{\varepsilon\nu} g_{\sigma\lambda} \tau^{\sigma}{}_{\nu\mu} + \tau^{\varepsilon}_{\lambda\mu}$$

Thus, a comparison of (37), (38) and (34) shows that an electron cannot distinguish between the coupling (32) with gravity or (37,38) due to the possible new coupling (of an electron) with Lanczos' potential. Remark that in this case $A_{\rho\lambda}{}^{\mu} = -\tau^{\mu}{}_{\rho\lambda}$, that is, torsion must be pseudo-traceless.

This last equality could induce us to speculate on the possible relation between Lanczos' potential and torsion.

6. CONCLUSION

There have been many works dealing with quadratic action recently. Although the subject is not new the main reason for this renewal of interest is due to the hope that the speculative theory of quantum conformal gravity is renormalizable and asymptotically free[22,23]. However, all these programs deal basically with the problem of fourth-derivative in the equation of motion of the field, once one considers that curvature is constructed as a second derivative of a symmetric tensor (the metric field). Our treatment, which has its root in Lanzos formulation of Riemannian geometry, deals with Lanczos potential $A_{\alpha\beta\mu}$ and changes the situation drastically. We have seen, further that by a choice of initial conditions Jordan's theory (with a quadratic action) is precisely equivalent to Einstein's theory. We have explored such a result in order to present a new alternative hamiltonian formulation of the gravitational field. Then we have presented in a direct and simple way how it is possible to construct a theory of massive gravity which exhibits general covariance.

All these results should be deeply explored in the future.

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