CBPF-NF-053/94

ALLOWED GRAVITATIONAL LAGRANGIANS IMPOSING $\frac{\delta(t-r)}{r}$ AS WEAK FIELD LIMIT GREEN FUNCTION

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ABSTRACT

A discussion is given on the restrictions imposed on generalized gravitational Lagrangians by requiring that the Green function, for their corresponding weak field limit be of the form $\frac{\delta(t-r)}{r}$, which implies the very physical fact that waves propagate with only one velocity.

It is shown that in four dimensions only a Lagrangian proportional to R is allowed and for six dimensions it should be proportional to $R^{\mu\nu}R_{\mu\nu}$. No powers of combinations the curvature tensors give a satisfactory theory for dimensions $n > 6$.

PACS number(s): 04.20.Cv, 04.50.+h

Key-words: Field theory; Gravitation Huyghens' principle.

I. INTRODUCTION

Modifications of the Einstein-Hilbert action, considering increasing powers of the Ricci scalar, the Ricci tensor and the Riemann tensor, have a long history [1]. They are of interest, among others, because some of these theories can be renormalized when quantized [2], pure gravity inflationary models emerge on adding an R^2 term to the usual gravitational Lagrangian [3]. On the other hand, the effective gravitational actions predicted by closed bosonic, heterotic and supersymmetric strings, in their corresponding higher dimensional space, contain higher power terms in the curvatures [4]. More recently, some non-linear Lagrangians have been chosen with the property that the field equations for the metric are second order, these are the so-called Lovelock actions [5] which can be regarded as formed by the dimensional continuation of the Euler characteristics of lower dimensions [6, 7, 8]. These Gauss-Bonnet terms [9,10] seem to be of importance for the quantization of these theories.

On one hand, dimensions, other than four correspond to the non-linear actions predicted by strings and to the Lovelock Lagrangians. And, on the other, each one of these theories can be considered as an alternative to classical general relativity. Therefore, the physical consequences of these kind of theories have been extensively studied [see references 15 to 22]. In this work we will pay attention to the weak field limit (w.f.l.) of these theories and, in particular, will impose, on each of them, the very physical fact that waves propagate with only one velocity (Huygen's Principle (HP) [11,12]. This condition is satisfied by requiring that, in the corresponding number of dimensions, the Green function of the field equations, of the gravity theory under consideration, in the w.f.l. is given by [12]

$$\frac{\delta(t-r)}{r}. \tag{1.1}$$

Among the Lagrangians already mentioned, we will study some toy models and will show that the imposed Green function (1.1) severely restricts the functional form of those Lagrangians.

In section 2 we will see that for the imposed Green function, the power of the D'Alembertian λ is related to the number of space-time dimensions n by $n = 2\lambda + 2$. In section 3 we choose as toy models $\mathcal{L} \sim R^2, \mathcal{L} \sim R_{\mu\nu}R^{\mu\nu}, \mathcal{L} \sim \alpha R^2 + \beta R$, and $\mathcal{L} \sim R^m (m > 2)$. We will show that the first two Lagrangians give as w.f.l. a squared D'Alembertian and consequently satisfy in six dimensions the required condition. However, $\mathcal{L} \sim R^2$ has an inconsistent w.f.l. in six dimensions (this is also true in four dimensions), the w.f.l. of the third one results in a linear combination of two operators not giving the desired Green function and $\mathcal{L} \sim R^m (m > 2)$ does not have a w.f.l. Then only the second one gives an acceptable theory in six-dimensions. In section four, dedicated to final remarks, we venture to outline some very preliminar ideas to construct gravity theories in any number of dimensions whose w.f.l. should be compatible with the Green function (1.1) imposed in this paper.

II. RELATION BETWEEN THE DIMENSIONS AND THE POWER OF THE D'ALEMBERTIAN.

We use the results concerning the Green function of λ power of the D'Alembertian following Marcel Riesz [13]

$$G_{Rz}^{(\lambda)} = \frac{2Q_+^{\lambda - \frac{n}{2}} \theta(-t)}{4^\lambda \pi^{\frac{n}{2}-1} \Gamma(1 + \lambda - \frac{n}{2}) \Gamma(\lambda)}, \tag{2.1}$$

where $Q_+ = t^2 - R^2$ for $t^2 > R^2$ and 0 otherwise, θ is the usual step function. In addition we make use of the following result by Guelfand [14];

$$Res Q_+^\gamma |_{\gamma=-1} \sim \frac{\delta(t + r) + \delta(t - r)}{r}. \tag{2.2}$$

We see then, that the power of $Q_+^{\lambda - \frac{n}{2}}$ has to be equal to -1 in order to have as residue the usual retarded (or advanced) potential. The presence of $\Gamma(1 + \lambda - \frac{n}{2})$ in the denominator of formula (2.1) sweep away the singularity of the pole, leaving us with the residue. In order to have this situation, the following relation must be valid

$$n = 2\lambda + 2, \tag{2.3}$$

between the number of dimensions n and the power of the D'Alembertian λ (observe that if $n = odd$, λ is a seminteger). From (2.3) we find

$$
\begin{array}{lll}
n=4, & \lambda=1, & \Box \\
n=6, & \lambda=2, & \Box^2 \\
n=8, & \lambda=3, & \Box^3 \\
n=10, & \lambda=4, & \Box^4 \\
\cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot
\end{array} \tag{2.4}
$$

where $\Box$ denotes D'Alembertian. This means that for each space- time dimension there is *only* one power of the D'Alembertian for which we get the demanded Green function (1.1).

III. LAGRANGIANS WITH INCREASING POWERS ON THE CURVATURES AND THEIR WEAK FIELD APPROXIMATION.

As mentioned in the introduction string-theory provides us with an effective Lagrangian containing higher powers in the curvature Riemann tensors [4]. On the other hand, similar theories arise following the proposal due to Lovelock [5] which seem to be of relevance in connection with higher dimensions than four and Gauss-Bonnet terms [9,10]. The field equations following from this effective action should replace the classical equations of general relativity and their properties and physical consequences have been studied in connection with exact solutions [15], plane waves [16], cosmology [17], blackholes [18], inflation [3], wormholes [19] and also quantum cosmolgy [20] among other aspects.

Let us begin by considering the following Lagrangians [21]

$$\mathcal{L} = \sqrt{-g}\; R^2, \tag{3.1}$$

$$\mathcal{L} = \sqrt{-g}\ R^{\mu\nu} R_{\mu\nu}, \tag{3.2}$$

$$\mathcal{L} = \sqrt{-g}\ R^{\mu\nu\alpha\beta} R_{\mu\nu\alpha\beta}. \tag{3.3}$$

These three actions are, in four dimensions, not independent due to the well known Gauss-Bonnet theorem [9,10]

$$\delta \int \sqrt{-g}\ [R_{\mu\nu\alpha\beta} R^{\mu\nu\alpha\beta} - 4 R_{\mu\nu} R^{\mu\nu} + R^2]\ d^4x = 0. \tag{3.4}$$

In more dimensions similar relations hold. We will consider only the first two theories. Following the same linearization procedure as in General Relativity the corresponding equations to (3.1) and (3.2) are then given, in any number of dimensions n, respectively, by

$$\Box\ (\partial_\mu \partial_\nu - \eta_{\mu\nu} \Box)h = -kT_{\mu\nu}, \tag{3.5}$$

$$\frac{1}{2} \Box [(\partial_\mu \partial_\nu - \frac{1}{2}\eta_{\mu\nu} \Box)h - \Box h_{\mu\nu}] = -kT_{\mu\nu}, \tag{3.6}$$

where $k = -8\pi$.

Contracting these equations, we get

$$(n-1) \Box\ \Box h = kT \tag{3.7}$$

$$\frac{n}{4} \Box\ \Box h = kT. \tag{3.8}$$

In both theories, as operator, we have $\Box^2$. Then according with the previous section the demanded Green function (1.1) is obtained for six space-time dimensions.

Now we consider the static case for the space dimension under consideration, for a point particle at the origin $T^{\mu\nu}$ is in this case given by

$$T^{\mu\nu} = \delta_0^\mu \delta_0^\nu M \delta(\vec{r}). \tag{3.9}$$

From (3.7) and (3.9) we get

$$h \sim -\frac{M}{r} \ . \tag{3.10}$$

To obtain this result we have used that the Green function for $\nabla^2\nabla^2$ is $\sim \frac{1}{r}$. From (3.10) we get $\nabla^2 h_{,12} \neq 0$ while from (3.5) one has $\nabla^2 h_{,12} = 0$. Consequently the linearized equations (3.5) of the theory (3.1) for a mass point at rest at the origin do not have any solution at all (this is also true in four dimensions for which the Green function $\sim r$ [21]).

For the theory (3.2) the corresponding linearized equations (3.6) result to be compatible with a "Newtonian limit" in the non-relativistic regime and, as stated, have the desired Green function (1.1) which guarantees the fact that waves propagate with only one velocity [12].

Another theory which has been frequently considered in the literature [22] is

$$\mathcal{L} = \sqrt{-g}\ (\alpha R^2 + \beta R), \tag{3.11}$$

Leading to the following equation for the corresponding Green function

$$a \Box\ \Box G + b \Box G = \delta, \tag{3.12}$$

We have for de Fourier transform $\overline{G}$

$$\overline{G} = \frac{1}{ap^4 + bp^2}. \tag{3.13}$$

When we add both operators $\Box^2$ and $\Box$ the desired Green function is not obtained. For $a = 0$ one gets in four dimensions the adequate Green function, for $b = 0$ the theory does not give a "Newtonian limit" as already shown.

For models $\mathcal{L} \sim R^m$ for $m > 2$ there does not exist a weak field approximation, their action

$$I = \int \sqrt{-g}\ R^m d^n x \tag{3.14}$$

give as equations of motion

$$R^{m-1}\left(R^{\alpha\beta}-\frac{R}{2m}g^{\alpha\beta}\right)-(R^{m-1})^{;\alpha;\beta}+g^{\alpha\beta}(R^{m-1})^{;\lambda}_{;\lambda}=0. \qquad (3.15)$$

Contracting this equation, we get

$$R^{m}\left(1-\frac{n}{2m}\right)+(n-1)(R^{m-1})^{;\lambda}_{;\lambda}=0. \qquad (3.16)$$

As $R\sim\ \Box\, h$, for $m>2$ none of the two terms in this equation results to be linear. For any other combination of three or more curvature tensors $(R, R_{\mu\nu}$ and $R_{\mu\nu\alpha\beta})$ there is also no w. f. l.

IV. FINAL REMARKS

It is clear from the previous discussion that for dimensions higher than six it is not possible to have combinations of the curvature tensors leading, in the w. f. l., to equations whose Green functions would satisfy the required form.

As a consequence of these results it follows that only the Lagrangians $\mathcal{L}\sim R$ in four dimensions and $\mathcal{L}\sim R_{\mu\nu}R^{\mu\nu}$ in six dimensions are allowed.

The question arises what kind of geometrical theories may satisfy our requirement i. e. to provide $\Box^{\lambda}h$ with $\lambda>2$ and $n>6$. What one should get as a w. f. l. in n dimensions is

$$\eta^{\mu_1\nu_1}...\eta^{\mu_\lambda\nu_\lambda}\ \frac{\partial^{2\lambda}}{\partial x^{\mu_1}\partial x^{\nu_1}...\partial x^{\mu_\lambda}\partial x^{\nu_\lambda}}=0, \qquad (4.1)$$

where $n=2\lambda+2$.

This is an indication that one could as a possibility generalize the metric in the following way

$$ds=(G_{\mu_1...\mu_{2\lambda}}dx^{\mu_1}...dx^{\mu_{2\lambda}})^{\frac{1}{2\lambda}}. \qquad (4.2)$$

The generalization of the Riemann tensor should contain higher derivatives of $G_{\mu_1\ldots\mu_{2\lambda}}$. Extensions of general relativity using multiple indices metrics have already been considered [23]. However, in a particular dimension of interest there is not a unique way to construct these generalized gravity theories, all these possibilities would deserve further study.

Acknowledgements

This work has been supported in part by CONACyT, through a Cátedra Patrimonial and Grants No. 1683-E9209 and No. F246-E9207 and by Coordinación de Investigación Científica de la UMSNH. The authors gratefully acknowledge helpful conversations with J. Benítez.

References

[1] Lanczos C. 1938 *Annals of Math.* **39** 842.

[2] Utiyama R. and DeWitt B. S. 1962 *J. Math. Phys.* **3** 608.
Stelle K.S. 1977 *Phys. Rev. D* **16**, 953.

[3] Mijic M. B., Morris M. S. and Suen W. 1986 *Phys. Rev.* D **34** 2934.
Maeda K. 1989 *Phys. Rev.* D **37** 858.

[4] Deser S. and Redlich N 1986 *Phys. Lett.* **176B** 350.

[5] Chern S. S. and Simons J. 1974 *Ann. Math.* **99** 78.
Lovelock D. 1971 *J. Math. Phys.* **12**, 498.

[6] Mena Marugán G. A. 1990 *Phys. Rev.* D **42** 2607.

[7] Bañados M., Teitelboim C. and Zanelli J 1992 *Phys. Rev. Lett.* **69** 184.

[8] Bañados M., Henneaux M., Teitelboim C. and Zanelli J. 1993 *Phys. Rev.* D **48** 2797.

[9] Bach R., *Math. Z* **9**, 110 (1921).

[10] Lanczos C., *Ann. of Math.*, **39**, 842 (1938).

[11] Schimming R. 1978 *Proc. of the IMU Symposium "Group Theoretical Methods in Mechanics"*, Novosibirsk, URSS.

[12] Baker B. B. and Copson E. T., in *The Mathematical Theory of Huygens' Principle* (Clarendon Press, 1939).

[13] Riesz M. *Math. Ann.* **107**, 400 (1933).
Comptes R. **208**, 1776 (1939).

[14] Guelfand I. M. and Chilov G. E., *Les Distributions* (DUNOD, 1962).

[15] Candelas P., Horowitz G., Strominger A., Witten E., 1985 *Nucl. Phys.* **B258**, 46.

[16] Güven R., *Phys. Lett.* **B191**, 275.
Amati D. and Klimcik C. *Phys. Lett.* **B210**, 443.

[17] Antoniadis I., Bachas C., Ellis J., and Nanopoulos D.V. (1988) *Phys. Letts.* **B211**, 393.
Antoniadis I., Bachas C., Ellis J. and Nanopoulos D.V. (1989) *Nucl. Phys.* **B328**, 117.

[18] Horowitz G.T. and Steif A.R. (1990a), *Phys. Rev. Letts.* **64**, 260.

[19] Bhawal B. and Kar S. (1992) *Phys. Rev. D* **46**, 2464.

[20] Hawking S.W. and Luttrell J.C., 1984 *Nucl. Phys. B* **247** 250.
Vilenkin A. 1985 *Phys. Rev. D* **32** 2511.
Pimentel L.O. and Obregón O. To be published in *Class. Quantum Grav.* **11** (1994).

[21] Accioly A.J. *Rev. Bras. Fís* **18**, 593 (1988)
Ruzmaikina T.V. and Ruzmaikin A.A. *Sov. Phys. JETP* **30**, 372 (1970).
Bollini C.G., Giambiagi J.J. and Obregón O. *Mod. Phys. Lett. A* **7**, (1992), 593.

[22] Berkin A.L., (1990), *Phys. Rev. D* **42**, 1016.

[23] Tapia V. *"Fourth-Rank Gravity"*, ICTP preprint (1992).
Tapia V. *Motion in fourth-rank Gravity", ICTP preprint (1992).*