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DISORDERED D-VECTOR MODEL: MEAN FIELD RENORMALIZATION GROUP TREATMENT*

by

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ABSTRACT

The mean field renormalization group is applied to study critical temperatures of the quenched (F,AF and SG phases) an annealed (F and AF phases) disordered D-vector model. The phase diagrams, critical temperature against concentration are analytically calculated, plotted and analyzed for the Ising ($D=1$), Heisenberg ($D=3$) and $D \to \infty$ cases for several values of the competing parameter and coordination number and for a particular distribution function.. Reentrancies and limiting slope at p_c of the ferro - and antiferromagnetic boundaries are studied in details. There is a critical value of the competing parameter for the existence of reentrancies in these lines. The limiting slope at p_c is lower bounded by the $D \to \infty$ value. A mapping is applied to study the phase diagram of the antiferromagnetic Ising model with random decorating competing D-vector bond spins. It is found that for a fixed competing parameter there is a lower critical dimensionality D_c for the stability of the ferromagnetic and spin glass phases. In all diagrams reentrancies are more prominent for higher dimensionalities. The present results are strictly equivalent to the exact results on the Bethe lattice.

Key-words: D-vector model; Mean field renormalization group; Reentrant phases; Decorated model; Spin glass on the Bethe lattice.

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1 INTRODUCTION

The D-vector model where the spins are described as "classical" vectors with continuous symmetry has been widely studied since it was formally formulated and solved exactly in one-dimension by Stanley (1968, 1969) (see Stanley (1974) for general reference). However the effects of disorder in the physical properties of this model have not received the same attention as for the pure system. In this paper we study the disordering effects on the critical temperature of the D-vector model in the framework of the mean field renormalization group (MFRG) proposed by Indekeu et al.(1982). The MFRG method has been successfuly applied to study a very large variety of problems like geometrical critical phenomena and percolation, ordered and disordered classical and quantum spin models, dynamical critical phenomena in Glauber models and in quantum spin systems, bulk and surface critical behavior of spin models (see Indekeu et al. (1987) and references there in), and also Ising model with anisotropy (Plascak and Silva (1986)), Ashkin-Teller model (Plascak and Sá Barreto (1986)), ANNNI model (Valadares and Plascak (1987)), compressible Ising systems Plascak and Figueiredo (1987)) and mixed spin systems (Verona de Rezende et al. (1987)).

In the MFRG the interactions within the cluster are treated exactly while the effects of the neighboring spins is taken into account by a mean field acting on the spins of the cluster boundary. This approach has been already applied for dilute, random fields and random bond Ising systems by Droz et al. (1982). Actually we have generalized this scheme by considering D-vector spins with continuous symmetry (the Ising model is the particular case for $D = 1$) and assymmetric competing bond disorder. We consider the simplest choice for the clusters to be renormalized which gives strictly the critical coupling given by the Bethe-Peierls approximation as stressed by Indekeu et al. (1987). The results can be improved to some extent if large-size cells are considered but with cost of lengthy calculations with slow convergence. In this paper we make use of this "phenomenological" renormalization group approach to study the critical temperature of the D-vector model with bond disorder in a d-dimensional hypercubic lattice with random competing coupling constants J and $-\alpha J (J < 0)$ between nearest neighbours where $\alpha (\alpha > 0)$ is the competing parameter. We both consider the quenched and the annealed disorder. We also apply the results to study the antiferromagnetic Ising model on a hypercubic lattice decorated with random quenched diluted competing D-vector bond spins. This model is equivalent to an Ising model with random bond coupling constants J_{eff} and $-\gamma J$ where J_{eff} is the effective coupling constant produced by the decorating D-vector bond spins which is temperature dependent (dos Santos et al. (1986)). These latter results are analyzed and compared with the exact ones obtained by dos Santos and Coutinho (1987) for the annealed disorder version of considered model on a square lattice. We also study the critical temperature phase diagram of the bond dilute limit case $(\alpha = 0)$ of the original model.

In section §2 we review the basic features of MFRG for disordered systems and calculate the phase diagrams of the quenched and annealed bond disordered systems (subsections § 2.A and § 2.B). In subsection § 2.C we apply the results to calculate the phase diagram of the antiferromagnetic Ising model with random decorating competing D-vector bond spins. Section §3 is devoted to present and to discuss the phase diagrams features. The conclusions are summarized in section §4.

2 MEAN FIELD RENORMALIZATION GROUP FOR QUENCHED AND ANNEALED COMPETING DISORDERED SYSTEMS.

In this section the MFRG approach is applied to investigate the phase diagram of both quenched and annealed bond disordered magnetic systems.

To start with we consider a system described by a nearest neighbour spin hamiltonian on a d-dimensional hypercubic lattice with bond disorder given by

$$-\beta\mathcal{H}(\vec{S}) \equiv H(\vec{S}) = \sum_{<ij>} K_{ij}\vec{S}_i.\vec{S}_j \qquad (1)$$

where $K_{ij} = \beta J_{ij}$ is the reduced coupling constant of the disordered exchange interaction between the pair of nearest neighbour spins $<i,j>$ and $\vec{S}$ is a D-dimensional vector spin with cartesian components $S^{\nu}(\nu = 1,2...D)$ which are subjected to the normalizing condition

$$\sum_{\nu}(S^{\nu})^2 = \lambda^2 \qquad (2)$$

In what follows we will assume that $\lambda^2 = D$ to renormalize the exchange coupling constants relative to the spin dimensionality.

We consider two finite clusters with N and N' interacting spins (N' < N) and compute the order parameters associated with the ordered phases of the systems.

We assume that the there exist an effective symmetry breaking field acting on the boundary of each cell for each order parameter. The main step in the MFRG is to self-consistently impose a scaling relation between the order parameter and the effective symmetry breaking field of the two cells, (Indekeu et al. (1987)), that is

$$\overline{f_{N'}(K',b',p')} = \zeta\, \overline{f_N(K,b,p)} \tag{3}$$

$$\overline{b'} = \zeta\, \overline{b} \tag{4}$$

where $\overline{f_n(k,b,p)}$ is the configurational average of the order parameter (quenched or annealed), $k(k')$, $b(b')$ and $p(p')$, being the coupling constant, the effective field and the disorder parameter of the N(N') cluster and ζ is the scaling factor. If we are dealing with second order phase transitions, eqs. (3) and (4) can be expanded for small b and b' if the system is close to the transition, which results that

$$\left.\frac{\partial \bar{f}_{N'}}{\partial b'}\right|_{b'=0} = \left.\frac{\partial \bar{f}_N}{\partial b}\right|_{b=0} \tag{5}$$

We note that eq. (5) is independent of the scaling factor ζ. There is one recursion equation, like eq. (5), to each order parameter. The critical lines can be obtained from these equation by assuming appropriated boundary condition for the corresponding phases.

In the present case the average magnetization and the staggered magnetization per spin are the order parameters for the ferro - and the antiferromagnetic phases respectively and we assume the Edwards-Anderson order parameter (Edwards and Anderson (1975)) for the spin-glass quenched disorder phase. As we mentioned in the introduction we have chose the simplest clusters to be renormalized, that is, the ones with $N'=1$ and $N=2$ as shown in Fig. (1).

A. Quenched disordered systems

The reduced hamiltonian for the $N'=1$ and $N=2$ cells are given respectively by

$$H' = \sum_{i=1}^{z} K'_i \vec{S}' . \vec{b}'_i \tag{6}$$

and

$$H = K_{12} \vec{S}_1 . \vec{S}_2 + \sum_{j=1}^{q} (K_{1j} \vec{S}_1 . \vec{b}_{1j} + K_{2j} \vec{S}_2 . \vec{b}_{2j}) \tag{7}$$

where $\vec{b}'_i$ and $\vec{b}_{\ell j}$ $(\ell=1,2)$ are the effective fields acting on the boundary of the $N'=1$ and $N=2$ cells respectively, $q = (z-1)$, $z=2d$ being coordination number of the d-dimensional hypercubic lattice. The effective field associated with the magnetization and the staggered magnetization are subjected to the following symmetry conditions:

$$\overline{\vec{b}'_i} = b'\hat{e}_1 \qquad i = 1\ldots z \tag{8}$$

$$\left.\begin{aligned} \overline{\vec{b}_{1j}} &= b\,\hat{e}_1 \\ \overline{\vec{b}_{2j}} &= \pm\, b\,\hat{e}_1 \end{aligned}\right\} \quad j = 1\ldots q \tag{9}$$

where the sign +(-) holds for the ferromagnetic (antiferromagnetic) order parameters and $\hat{e}_1$ is the unit vector in the direction 1. For the spin-glass order parameter the symmetry conditions are

$$\overline{\vec{b}'_i} = 0 \quad \text{and} \quad \overline{b'^{\mu}_i b'^{\nu}_j} = h'\delta_{ij}\delta_{\mu\nu} \tag{10}$$

$$\left.\begin{aligned} \overline{\vec{b}_{mj}} &= 0 \\ \overline{b^{\mu}_{mj} b^{\nu}_{nk}} &= h\,\delta_{mn}\delta_{jk}\delta_{\mu\nu} \end{aligned}\right\} \tag{11}$$

with $i = 1\ldots z$, $J, k = 1\ldots 9$, $m,n = 1,2$ and $\mu,\nu = 1\ldots D$.

The magnetization for the $N' = 1$ and $N = 2$ cells can be evaluated straighforwardly from

$$\langle S'^1 \rangle = \frac{Tr_{\vec{S}'}\{S'^1 e^{H'}\}}{Tr_{\vec{S}'}\{e^{H'}\}} \tag{12}$$

and

$$<S_m^1> = \frac{Tr_{\vec{S}_1} Tr_{\vec{S}_2} \{S_m^1 \, e^H\}}{Tr_{\vec{S}_1} Tr_{\vec{S}_2} \{e^H\}} \tag{13}$$

respectively where

$$Tr_{\vec{S}}\{\ldots\} \equiv \int d\vec{S} \, \delta(D - |\vec{S}|^2) \{\ldots\} \tag{14}$$

By substituting eqs. (6) and (7) in (12) and (13) respectively and expanding the integrand for small effective fields we get as shown in the Appendix that

$$<S'^1> \simeq \sum_{i=1}^{z} K_i b_i^1 \tag{14}$$

$$<S_i^1> \simeq \left[\sum_{j=1}^{q} K_{1j} b_{ij}^1 + th_n(DK_{12}) \sum_{j=1}^{q} K_{2j} b_{2j}^1\right] \tag{15}$$

with $n = \frac{1}{2} D - 1$ where $th_n(x)$ is the generalized hyperbolic tangent given by

$$th_n(x) = \frac{I_{n+1}(x)}{I_n(x)} \tag{16}$$

where $I_n(x)$ is the modified Bessel function of the first kind of order n.

To perform the configurational average (c-average) we assume that the K_i parameters and the b_i effective fields are independent random variables according to a probability distribution. Therefore, by using the symmetry breaking condition given by eqs. (8) and (9) we get

$$\overline{\langle S'^1 \rangle} \simeq b' z \overline{K'_i} \qquad (17)$$

$$\overline{\langle S_1^1 \rangle} \simeq bq \overline{K_{ij}} \pm bq \overline{k_{2j}} \; \overline{th_n(DK_{12})} \qquad (18)$$

where sign +(-) holds for the ferromagnetic (antiferromagnetic) order parameters.

The Edwards-Anderson spin-glass order parameter for the $N'=1$ and $N=2$ cells be obtained by squaring eqs. (14) and (15) respectively and performing the c-average following the appropriated symmetry breaking conditions given by eqs. (10) and (11), that is

$$\overline{\langle S'^1 \rangle^2} \simeq h' z \overline{K_i'^2} \qquad (19)$$

$$\overline{\langle S_1^1 \rangle^2} \simeq hq \overline{K_{ij}^2} + hq \overline{K_{2j}^2} \; \overline{th_n^2(DK_{12})} \qquad (20)$$

Now, by imposing the scaling relation for each order parameter we obtain from eq. (5) that

$$z\overline{K'} = q\overline{K}(1 \pm \overline{t}_n) \qquad (21)$$

for the ferromagnetic (sign,+) and antiferromagnetic (sign) order parameter, and

$$z\overline{K'^2} = q\overline{K^2}(1 + \overline{t_n^2}) \qquad (22)$$

for the spin-glass order parameter where $\overline{t_n^m} = \overline{th_n^m(DK)}$. In eqs. (21) and (22) the subscript indices are dropped out for cleaness.

We assume for simplicity that the independent probability distributions for each bond are like

$$P(K_i) = p\,\delta(K_i - K) + (1-p)\,\delta(K_i + \alpha K) \tag{23}$$

and we force the renormalization distribution to have the same form for both cells but parametrized with different concentrations. Therefore from eqs. (21) and (22) we get

$$zK'\left[p'(1+\alpha')-\alpha'\right] = qK\left[p(1+\alpha)-\alpha\right]\left[1 \pm (pt-(1-p)\,t_\alpha)\right] \tag{24}$$

$$zK_1'^2\left[p'(1-\alpha'^2) + \alpha'^2\right] = qK\left[p(1-\alpha^2)+\alpha^2\right]\left[1 + pt^2 + (1-p)\,t_\alpha^2\right] \tag{25}$$

where $t = th_n(DK)$ and $t_\alpha = th_n(\alpha DK)$.

The complete renormalization flow in the (K,p,α) space can not be fully determined by the eqs. (24) and (25) except for symmetric distributions ($\alpha = 1$ in eq. (23)) or for some invariant sets which are known a priori on the basis of other symmetries. For instance for the pure case distribution ($p = 1$) we always expect to have $p' = 1$ after a scaling step (Droz et al. (1982)). Therefore we restrict ourselves study the fixed point solutions in the subspace $\alpha = \alpha'$ and $p = p'$, that is, the solutions of

$$zK_c = qK_c\left[1 \pm (pt - (1-p)\,t_\alpha\right] \tag{26}$$

$$zK_c^2 = qK_c^2\left[1 + pt^2 + (1-p)t_\alpha^2\right] \tag{27}$$

which is $K_c = 0$ or the solutions of

$$1 = \pm\, q\left[pt - (1-p)\,t_\alpha\right] \tag{28}$$

$$1 = q\left[pt^2 + (1-p)\,t_\alpha^2\right] \tag{29}$$

We note that for $p=1$ one recovers the Bethe-Peierls critical temperature of the ferromagnetic and the spin-glass phases for the pure system, $th_n(DK_c)=1/q$ (Stanley (1969) and $th_n(DK_c) = 1/\sqrt{q}$ (for $D=1$, $\tanh K_c = 1/\sqrt{q}$ as obtained by Katsura et al (1979) respectively. On the other hand for the symmetric bond disorder case ($\alpha=1$) we have

$$1 = \pm\, q(2p-1)t \tag{30}$$

$$1 = qt^2 \tag{31}$$

while for the bond dilute case ($\alpha = 0$) we have

$$1 = \pm\, qpt \tag{32}$$

$$1 = qpt^2 \tag{33}$$

For $D=1$ eqs. (30) and (32) recover the results of Droz et al, (1982) for the ferromagnetic boundary of the bond disorder and bond diluted Ising system cases. However for the Edwards-Anderson order parameter critical line our result given by eq. (31) differs from the one obtained by Droz et al. (1982) since the spin glass random boundary conditions used by these authors *do include* correlations between the symmetry breaking fields. We argue that correlations should not be present in this case as expressed in eq. (11). For the Ising systems (D = 1) our spin-glass critical line given by eq. (31) is the same as the one* achieved by Thouless (1986). We note that eq. (33) does not represent the critical line between the spin-glass and the paramagnetic phases since it is actually located in the ferromagnetic region (under the ferromagnetic line in the (T x p) diagram.

B. Annealed disordered systems

For annealed distributions the disorder variables are not independent any more and the randomness of the disorder is allowed to adjust itself so that the system achieves genuine thermodynamic equilibrium. This means that the configurational average should be done over the partition function (Thorpe and Beemann (1976)).

The reduced hamiltonian of the system should be rewritten as

$$H(\vec{S}) = \sum_{<ij>} \left[(t_{ij}(1+\alpha) - \alpha) K \vec{S}_i \vec{S}_j \right] + \Delta_{ij} t_{ij} \qquad (34)$$

where t_{ij} is the disorder variable for the <ij> - bond, i.e.

t_{ij} = 1(0) if the bond is ferromagnetic with reduced coupling constant K(antiferromagnetic with reduced coupling constant $-\alpha K$) and α is the ratio between the antiferromagnetic and ferromagnetic coupling constants. For $\alpha=1$ the annealed bond disorder is symmetric with respect to the ferro- and antiferromagnetic bonds while for $-\alpha = 0$ the system is annealed ferromagnetic bond diluted. For $\alpha = -1$ we should recover the pure system case. In (34) $\Delta_{ij} = -\beta\mu_{ij}$ is the reduced chemical potential associated with the annealed disorder that should be chosen as to make $\langle t_{ij}\rangle$ temperature independent since the thermodynamic average of the disorder variable t_{ij} should gives the concentration of ferromagnetic bonds. Therefore the chemical potential μ_{ij} will be temperature dependent (Thorpe and Beemann (1976)).

The reduced hamiltonian for the N' =1 and N =2 cells are given by

$$H' = \sum_{i=1}^{z}\left[(t_i'(1+\alpha') - \alpha')K'\vec{S}'.\vec{b}_i' + \Delta_i' t_i'\right] \tag{35}$$

$$H = \sum_{j=1}^{q}\sum_{m=1,2}\left[(t_{mj}(1+\alpha) - \alpha)K\vec{S}_m.\vec{b}_{mj} + \Delta_{mj}t_{mj} + \right.$$

$$\left. + (t_{12}(1+\alpha) - \alpha)K\vec{S}_1.\vec{S}_2 + \Delta_{12}t_{12}\right] \tag{36}$$

where the $\vec{b}$'s are the symmetry breaking fields acting on the boundary sites, Δ_i' and Δ_{ms} are the effective reduced chemical potential associated with the surface bonds of the N' and N cells respectively and Δ_{12} is related with $\langle 1,2\rangle$ bond.

The ferromagnetic (antiferromagnetic) order parameter for both cells should be calculated under the following boundary conditions

$$\vec{b}_i' = b'\hat{e}_1 \qquad i = 1 \ldots z \tag{37}$$

$$\left.\begin{aligned} \vec{b}_{1j} &= b\hat{e}_1 \\ b_{2j} &= +(-)b\hat{e}_1 \end{aligned}\right\} \quad j = 1 \ldots q \tag{38}$$

Furthermore the effective chemical potential should satisfy the self-consistent condition

$$\langle t_i' \rangle = p' \tag{39}$$

$$\langle t_{mj} \rangle = \langle t_{12} \rangle = p \tag{40}$$

where <...> means the thermodynamic average and p(p') the average concentration of ferromagnetic bonds for the N(N') cell.

The magnetization for both cells can be evaluated from

$$\langle S'^1 \rangle = \frac{Tr_{\vec{S}'}\, Tr_{\{t'\}}\{S'^1 e^{H'}\}}{Tr_{\vec{S}'}\, Tr_{\{t'\}}(e^{H})} \tag{41}$$

and

$$\langle S_m^1 \rangle = \frac{Tr_{\{\vec{S}\}} Tr_{(t)}\{S_m^1 e^{H}\}}{Tr_{\{\vec{S}\}} Tr_{\{t\}}\{e^{H}\}} \tag{42}$$

where $Tr_{\{t\}}$ is the trace over the disorder variables.

The average concentration for both cells can be straighforwardly calculated from

$$p' = \langle t'_i \rangle = \frac{\partial}{\partial \Delta_i} \ell n \left[Tr_{\vec{S}'} \; Tr_{\{t'\}} e^{H'} \right] \tag{43}$$

and

$$p = \langle t_{12} \rangle = \frac{\partial}{\partial \Delta_{12}} \ell n \left[Tr_{\{\vec{S}\}} Tr_{\{t\}} e^{H} \right] \tag{44}$$

$$p = \langle t_{mj} \rangle = \frac{\partial}{\partial \Delta_{mj}} \ell n \left[Tr_{\{\vec{S}\}} Tr_{\{t\}} e^{H} \right] \tag{45}$$

From eqs. (35-45) and with help of the Appendix integrals we get

$$\langle S'^1 \rangle = b'k' \sum_i \left(\frac{\eta'_i - \alpha}{\eta'_i + 1} \right) \tag{46}$$

$$\langle S' \rangle = bk \left\{ \sum_i \left(\frac{\eta_{1i} - \alpha}{\eta_{1i} + 1} \right) + \sum_j \left(\frac{\eta_{2j} - \alpha}{\eta_{2j} + 1} \right) \left[\frac{\eta_{12} I_{n+1}(DK) + I_{n+1}(-\alpha DK)}{\eta_{12} I_n(DK) + I_n(-\alpha DK)} \right] \right\} \tag{47}$$

$$p' = \frac{\eta'_i - \alpha}{\eta_i + 1} \qquad i = 1 \ldots z \tag{48}$$

$$p = \frac{\eta_{mi} - \alpha}{\eta_{mi} + 1} \qquad m = 1,2 \quad i = 1 \ldots q \tag{49}$$

$$p = \left[1 + \frac{I_n(-\alpha DK)}{\eta_{12} I_{n+1}(DK)} \right]^{-1} \tag{50}$$

From eqs. (48-50) the unwanted fugacities $\eta_i' = e^{\Delta_i'}$, $\eta_{mi} = e^{\Delta_{mi}}$ and $\eta_{12} = e^{\Delta_{12}}$ can be eliminated by writting them as functions of the average concentrations and by substituting in eqs. (46) and (47) we get

$$\langle S'^{1}\rangle = z b' k' \left[p'(1+\alpha') - \alpha'\right] \tag{50}$$

$$\langle S^{1}\rangle = q b k \left[p(1+\alpha) - \alpha\right]\left[1 \pm (pt - (1-p)t_{\alpha})\right] \tag{51}$$

where sign +(-) holds for the ferromagnetic (antiferromagnetic) order parameter.

Now by imposing the MFRG scaling condition given by (5) we get

$$z k' \left[p'(1+\alpha') - \alpha'\right] = q k \left[p(1+\alpha) - \alpha\right]\left[1 \pm (pt - (1-p)t_{\alpha}\right] \tag{52}$$

We note that eq. (52) is identical to eq. (24) therefore the ferromagnetic (antiferromagnetic) critical lines for the annealed bond disordered system are the same as for the quenched bond disordered one with the probability distribution given by (23). This is an expected result and can be generalized for an arbitrary distribution as shown by Thorpe and Beeman (1976)(see also Thorpe (1978)) in agreement with an earlier work by Matsubara (1974) for the Ising model on the Bethe lattice.

C. Random decorating D-vector systems

This approach can be straighforwardly applied to study a generalization of the Syozi model (Syozi 1966) in which decorating D-vector bond spins are randomly diluted in the system. In this model the decorating spins are randomly diluted on the bonds with probability p, with an exchange coupling $J_1 (J_1 > 0)$ with the Ising spins at either end of the bond. We assume that there is an antiferromagnetic coupling $J_2 (J_2 < 0)$ between the Ising site spins. This model has been already exactly solved for annealed dilution on the square lattice by dos Santos and Coutinho (1987). However for quenched (or annealed) dilution on the Bethe lattice we can make use of the above results of subsection A and B by substituting the previous ferromagnetic reduced coupling constant K by the effective exchange coupling K_{eff} mediated by the decorating bond spin if it is present or by an antiferromagnetic reduced coupling constant $K_2 = \beta J_2$ if it is absent. This effective exchange coupling has been already evaluated by dos Santos et al. (1986) and reads

$$K_{eff} = K_2 + \frac{1}{2} \ln G_n(2D^{1/2} K_1) \tag{53}$$

with

$$G_n(y) = 2^n \Gamma(n+1) \alpha_m y^{-n} I_n(y) \tag{54}$$

where $K_i = \beta J_i (i=1,2)$, $n = \frac{1}{2} D - 1$, $\alpha_n = 1 + \delta_{n,-1/2}$

($\delta_{n,n'}$ being the Kronecker delta function), $\Gamma(n)$ being the gamma function and I_n being the modified Bessel function of first kind of order n. The expression of $G_n(y)$ for D=1,2,3 and ∞ are given in Table 1. Finally the critical lines for the ferromagnetic (anti-ferromagnetic) and spin-glass phase transitions can be obtained from eqs. (21) and (22) by using an independent probability distribution given by

$$P(K_i) = p\,\delta(k_i - k_{eff}) + (1-p)\,\delta(k_i - k_2) \tag{55}$$

Therefore from eqs. (28) and (29) we can writte straighforwardly that

$$1 = \pm q\left[p\, t_{eff} - (1-p)\, t_2\right] \tag{56}$$

$$1 = q\left[p\, t_{eff}^2 + (1-p)\, t_2^2\right] \tag{57}$$

with $t_{eff} = \tanh K_{eff}$ and $_2 = \tanh k_2$ for the ferromagnetic (sign +), antiferromagnetic (sign −) and spin glass critical temperature as a function of the concentration of decorating spins respectively.

3. THE PHASE DIAGRAMS AND DISCUSSION

A. Quenched and annealed disordered systems

The phase diagrama ($\tilde{T}_c$ xp) for both quenched and annealed disorder systems can be plotted from eqs. (28) to (33), where

$\tilde{T}_c = K_c^{-1}$ is the reduced critical temperature. Note that the spin glass lines (eqs. (29),(31) and (33) are valid only for quenched systems. The corresponding generalized hyperbolic tangent $th_D(x)$ for Ising $(D=1)$, XY$(D=2)$, Heisenberg $(D=3)$ and $D\to\infty$ limit cases are given in the table 1.

In figure 2 we show the diagram $(\tilde{T}_c\, x\, p)$ of the ferromagnetic diluted system for $D = 1,3$ and ∞ and for $z=6$. We observe that the critical concentration at $T = 0$ (percolation threshold) is D independent as expected. Also, the slope at $p=1$ is almost independent of spin dimensionality (Note that all lines are normalized with respect to the spin dimensionality with $\lambda^2 = D$ as mentioned in the beginning of section 2). However, the slope at $p=p_c$ is given by $d\tilde{T}_c/dp|_{p_c} = 2Dq/(D-1)$ being infinite only for the Ising case $(D=1)$ and equal to $2q$ for $D\to\infty$ (lower bound). Figure 3 shows the $(\tilde{T}_c\, x\, p)$-diagram for the symmetric disordered system $(\alpha=1)$ for $D=1$ and $D\to\infty$, and for $z=6$. The diagram is symmetric with respect of $p=0.5$ showing the presence of the ferro-, the antiferromagnetic and the spin-glass phases. The dependence of the limiting slope with the spin dimensionality at $p=p_c$ and $p=1(0)$ have the same features as for the diluted system (fig. 2). We note that the spin glass line is p independent in agreement with the results of Katsura et al. (1979), Katsura and Shimada (1980), Thouless (1986) and Carlson et al. (1987) but in contrast with the one obtained by Droz et al. (1982) (see fig. 2(b) of this reference) that have, erroneously, included correlations between the symmetry breaking fields boundary conditions for spin glass phase. In the figure 4 we show the $(\tilde{T}_c\, x\, p)$-diagram for an assymmetric disordered system with strong competing parameter $\alpha = 2.5$ and for $z = 6$. We call the reader's attention for the appearance of reentrancies in the antiferromagnetic line and crossing between the all lines with different D for small T. These features are dictated by the limiting slope of critical line at $p = p_c$, that is given by

$$\left.\frac{d\tilde{T}_c}{dp}\right|_{p_c} \simeq \frac{8D\alpha q}{(D-1)(q \pm 1)(\alpha - \alpha_c^{\pm})} \qquad (58)$$

where $\alpha_c^{\pm} = (q \mp 1)/(q \pm 1)$ with the up (down) sign holding for the ferromagnetic (antiferromagnetic) line. We note that $d\tilde{T}_c/dp|_{p_c} = \infty$ for the Ising case $(D = 1)$ whatever are the values of α and q as it should be expected, but it is also infinite for $\alpha = \alpha_c$. We also note that the limiting slope at p_c is bounded by $8\alpha q/(q \pm 1)(\alpha - \alpha_c)$ for $D \to \infty$ (lower bound). From eq. (58) we can also see that reentrancies appears in the ferromagnetic (antiferromagnetic) boundary for $\alpha < \alpha_c$ $(\alpha > \alpha_c)$.

B. Random decorating D-vector systems

The phase diagrams for the random decorating D-vector model are given by eqs. (56) and (57) with help of (53) and (54). In the present model the existence of ferromagnetic and spin-glass phases will be dependent of the sign of random effective interaction since there is an uniform antiferromagnetic coupling through the system. At $T = 0$ the effective coupling is given from eq. (53) by

$$J_{eff} = (\gamma_c - \gamma)J_1 \tag{59}$$

where $\gamma = -J_2/J_1$ is the actual competing parameter and $\gamma_c=D^{1/2}$. Therefore for $\gamma < \gamma_c$ competing interactions will be present in the system and the ferromagnetic and the spin-glass phases will be stable. In other words we can say that for fixed values of the original coupling constants $J_1 > 0$ and $J_2 < 0$ there is a lower critical dimensionality of the decorating bond spin $D_c = (J_2/J_1)^2$ for the stability of the ferromagnetic and the spin glass phases. In figure 5 we show the normalized $(t_c \, x p)$ - diagram where $t_c = T_c(p)/T_c(0)$ with $\gamma = 1$, $z = 6$ and for $D = 1,3$ and ∞. We note the absence of the ferromagnetic and the spin-glass phases for $D \leq 1$ while they appear for $D=3$ and ∞. In figure 6 we plot the $(t_c \, x p)$-diagram with $\gamma = .5$, $z = 8$ for $D = 1,3$ and ∞. In this diagram we note the presence of several reentrancies which are more prominent for higher dimensionalities. This behavior has also been observed by dos Santos and Coutinho (1987) in the annealed decorating D-vector model on the square lattice (exact result). As it has been pointed out by these authors the existence of re-entrant behavior is a consequence of the presence of local diluted competing effects in the system introduced by the decoration. In figure 7 we show the $(t_c \, x p)$ - diagrams with $\gamma = .5$, $D=1$ and for $z=6$ and 12. We observe that the domain of the antiferromagnetic phase increases with the coordination number. In the $q \to \infty$ limit the system is antiferromagnetic for all region of the normalized $(t_c \, x p)$ - diagram.

Actually the antiferromagnetic reduced critical temperature K_BT_c/qJ_2 is independent of the concentration and it is equal to one with qJ_2 = constant in the $q \to \infty$ limit, which corresponds to the mean field behavior. Another particular feature of the random decorating system is that the limiting slope at p_c is always infinite and independent of the spin dimensionality, competing parameter and coordination number since we have considered site Ising spins variables. This behavior has also been observed by dos Santos and Coutinho (1987)).

In all diagrams shown in figures 3-7 the critical concentration p_c at $T = 0$ for both ferro - and antiferromagnetic critical lines are independent of the spin dimensionality and competing parameter, that is, $p_c^{\pm} = (q \pm 1)/2q$ where sign +(-) holds for the ferromagnetic (antiferromagnetic) lines. We note that $p^+ - p^- = 1/q$ which is the well known percolation threshold for the Bethe lattice (Fisher and Essam (1961)). We also note that the critical lines for the D = 2(XY) case, omited in all diagrams shown in figures 2-7, have the same features of the critical lines for D > 1. For the size of clusters choose in the present work the MFRG method is strictly equivalent to the Bethe-Peierls approximation and we should not expect to have topological transitions characteristic of the XY model on two dimensional Bravais lattices.

Finally we comment that the critical frontiers between the spin-glass phase and the ferro - and antiferromagnetic phases shown in figures 3-7 are not the actual exact ones for the Bethe-lattice but they represent the zero order approximation in the moment analysis. These lines has been cal culated under the symmetry breaking fields boundary conditions for the ferromagnetic (antiferromagnetic) to paramagnetic transition where both ferromagnetic and spin-glass order parameters goes to zero. The actual transition line between the ferromagnetic and spin-glass phase cannot be evaluated exactly by the MFRG approach and should be determined by moment analysis. In a very recent letter of Carlson et al. (1987, preprint) they found a magnetized spin glass (MSG) phase between the ferromagnetic and the spin glass phases for the Bethe lattice Ising spin glass model where the transition should be marked by the onset of a small but non-zero magnetization. The stability of the spin glass phase as well as the existence of the MSG phase in the disordered D-vector model on Bethe lattice is now being in study.

4 CONCLUSIONS

We have studied the critical temperature of the quenched and annealed bond disordered D-vector model in the framework of the mean field renormalization group approach. For the simplest choice of clusters (fig. 1) used here the results are equivalent to the exact solution on the Bethe lattice (or Bethe-Peierls approximation). The results has been applied to study an antiferromagnetic Ising model decorated with random quenched diluted competing D-vector bond spins. The phase diagrams, critical temperature against concentration has been studied for $D = 1,3$ and ∞ values of the spin dimensionality and several values of the competing parameter, and for a particular distribution function. Figures 2 to 4 shows these diagrams for the quenched and annealed bond disordered D-vector model. The quenched and annealed ferromagnetic (and antiferromagnetic) critical lines are found equal in agreement with general previous results of Thorpe and Beeman (1976). The spin glass phase is present only for quenched disorder as expected. The limiting slope at $p=p_c(T=0)$ of the ferro - and antiferromagnetic lines are found to be lower bounded by the $D\to\infty$ limit value and infinite for $D=1$. The lower bound limiting slope is also dependent upon the competing parameter and the coordination number. The appearance of reentrancies in the ferromagnetic (antiferromagnetic) boundary occurs for certain values of the competing parameter $\alpha < \alpha_c^+ (>\alpha_c^-)$ when the limiting slope becomes negative (positive), where α_c^+ depends of the coordination number only. We also observe crossing between critical lines for different spin dimensionalities when reentrancies occurs. For $\alpha = \alpha_c^\pm$ the limiting slope at p_c are infinite irrespective to the value of the spin dimensionality.

The phase diagram for the random decorating D-vector bond spin model, which is mapped in the quenched bond disordered Ising model with random bonds J_{eff} and $J_2(<0)$ where J_{eff} is the effective exchange interaction induced by the decoration. In this model the stability of the ferromangetic and the spin glass phases is dependent on the dimensionality of the decorating bond spin and the competing parameter. For $\gamma \leq \gamma_c = D^{1/2}$ these phases are stable. Therefore for a given competing parameter there is a lower critical dimensionality $D_c = \gamma^2$ for the stability of these phases. We also found that the reentrancies of the ferromagnetic (and antiferromagnetic) phase are more prominent for higher dimensionalities. We also found that the domain of the antiferromagnetic phase increases with the coordination number for a fixed competing parameter reflecting the mean field behavior. In all phase diagrams we observe that the critical concentration $p_c^{\pm}$ at $T=0$ for the ferro and the antiferromagnetic lines depends only on the coordination number as expected from the geometric character of transition at $T = 0$. However as we comment in the end of section 3 the point $T=0$, $p=p_c^{+}$ represents the zero order approximation in moment analysis for ferromagnetic to spin glass boundary at $T=0$. We expect that the ferromagnetic - spin glass boundary should shifts to higher concentrations when high order moments is considered as obtained by Carlson et al. (1987) for the symmetric Ising spin glass on the Bethe lattice. However we believe that will persist the re-entrant and limiting slope behavior if higher order moments is included in the moment analysis.

APPENDIX: Magnetization for the N' = 1 and N = 2 cells.

From eq. (12) and (13) we have

$$<S'^1> = \frac{\int D\vec{S}' S'^1 \exp\{\sum_{i=1}^{z} K_i' \vec{S}'.\vec{b}_i'\}}{\int D\vec{S}' \exp\{\sum_{i=1}^{z} K_i' \vec{S}'.b_i'\}} \quad (A1)$$

$$<S_1^1> = \frac{\int D\vec{S}_1 \int D\vec{S}_2 S_1^1 \exp\{K_{12}\vec{S}_1.\vec{S}_2 + \sum_{j=1}^{q}(K_{1j}\vec{S}_1\vec{b}_{1j} + K_{2j}\vec{S}_2.\vec{b}_{2j})\}}{\int D\vec{S}_1 \int D\vec{S}_2 \exp\{K_{12}\vec{S}_1.\vec{S}_2 + \sum_{j=1}^{q}(K_{1j}\vec{S}_1.\vec{b}_{1j} + K_{2j}\vec{S}_2.\vec{b}_{2j}\}} \quad (A2)$$

where $\int D\vec{S}\{\ldots\} \equiv \int d\vec{S}\ \delta(D-|\vec{S}|^2)$.

Now, expanding both the numerator and the denominator of eqs. (A1) and (A2) for small b's up to first order, we have

$$<S'^1> \simeq \frac{\int D\vec{S}' S'^1 \{1 + \sum_{i=1}^{z} K_i' \vec{S}'.\vec{b}_i'\}}{\int D\vec{S}' \{1 + \sum_{i=1}^{z} K_i' \vec{S}'.\vec{b}_i'\}} \quad (A3)$$

$$<S_1^1> \simeq \frac{\int D\vec{S}_1 \int D\vec{S}_2 S_1^1 \left[1 + \sum_{j=1}^{q}(K_{1j}\vec{S}_1.\vec{b}_{1j} + K_{2j}\vec{S}_2.\vec{b}_{2j})\right] \exp\left[K_{12}\vec{S}_1.\vec{S}_2\right]}{\int D\vec{S}_1 \int D\vec{S}_2 \left[1 + \sum_{j=1}^{q}(k_{1j}\vec{S}_1.\vec{b}_{1j} + K_{2j}\vec{S}_2.\vec{b}_{2j})\right] \exp\left[K_{12}\vec{S}_1.\vec{S}_2\right]} \quad (A4)$$

The D-dimensional integrals in the eqs. (A3) and (A4) can be evaluated from particular cases of the integral

$$I = \int d\vec{S}\ \delta(D - |\vec{S}|^2) f(S^\nu) \exp\{K\vec{S}.\vec{R}\} \quad (A5)$$

Integral I can be written as

$$I = \frac{k}{2\pi i}\int_{\alpha-i\infty}^{\alpha+i\infty} du \exp\left[uDK\right] \prod_{\nu=1}^{D}\int_{-\infty}^{\infty} dS^{\nu} f(S^{\nu}) \exp\left\{-K\left[u(S^{\nu})^2 - S^{\nu}R^{\nu}\right]\right\} \quad (A6)$$

where we make use of the integral representation of de δ-function

$$\delta(D - |\vec{S}|^2) = \frac{k}{2\pi i}\int_{-i\infty}^{i\infty} dv \exp\left\{Kv\left[D - \sum_{\gamma=1}^{D}(S^{\nu})^2\right]\right\} \quad (A7)$$

and multiply the integrand by an unitary factor $\exp\{K\alpha\left[D - \sum_{\nu=1}^{D}(S^{\nu})^2\right]\}$ following the approach used by Stanley (1969).

If $f\{S^{\nu}\} = \prod_{\nu=1}^{D} F(S^{\nu})$ the integral I contains a product of D independent guassian integrals of the type of

$$I_{\nu} = \int_{-\infty}^{+\infty} dS^{\nu} f(S^{\nu}) \exp\left\{-K\left[u(S^{\nu})^2 - S^{\nu}R^{\nu}\right]\right\} \quad (A8)$$

For $F(S^1) = (S^1)^n$ and $F(S^{\nu}) = 1 \ \forall \ \nu = 2,3\ldots D$ we have

$$I_1 = \left[\int_{-\infty}^{\infty} dx \left[x + \frac{1}{2}\,\frac{R^1}{u}\right]^n \exp\{-Kux^2\}\right] \exp\left\{\frac{KR_1^2}{4u}\right\} \quad (A9)$$

which equals

$$I_1 = \frac{1}{2}\,\frac{R^1}{u}\left(\frac{\pi}{Ku}\right)^{1/2} \exp\left\{\frac{K(R^1)^2}{4u}\right\} \qquad \text{for} \quad n = 1 \quad (A10)$$

$$I_1 = \left[\frac{1}{2Ku} + \left(\frac{R^1}{2u}\right)^2\right]\left(\frac{\pi}{Ku}\right)^{1/2} \exp\left\{\frac{K(R^1)^2}{4u}\right\} \qquad \text{for} \quad n = 2 \quad (A11)$$

and

$$I_\nu = \left(\frac{\pi}{Ku}\right)^{1/2} \exp\left\{\frac{K(R^\nu)^2}{4u}\right\} \quad \text{for} \quad n = 0 \quad \text{and} \quad \nu = 1\ldots D \qquad \text{(A12)}$$

By substituting eqs. (A10)-(A12) in (A6) we have

$$I = \pi\left(\frac{2\pi}{|K|}\right)^{\frac{1}{2}D-1} \mathbb{I}_{\frac{1}{2}D-1}(D|K|) \quad \text{for} \quad n = 0 \qquad \text{(A13)}$$

$$I = \pi\left(\frac{2\pi}{|k|}\right)^{\frac{1}{2}D-1} \mathbb{I}_{\frac{1}{2}D}(D|k|)R^1 \quad \text{for} \quad n = 1 \qquad \text{(A14)}$$

and

$$I = \pi\left(\frac{2\pi}{|k|}\right)^{\frac{1}{D}D-1} \left[\mathbb{I}_{\frac{1}{2}D+1}(D|k|)(R^1)^2 + \frac{1}{K}\,\mathbb{I}_{\frac{1}{2}D}(D|K|)\right] \qquad \text{(A15)}$$

Furthermore, integrals of type

$$\int D\vec{S}\, f\{S^\nu\} = \lim_{K\to 0} \int D\vec{S}\, f\{S^\nu\}\exp\{k\vec{S}.\vec{R}\} \qquad \text{(A16)}$$

can be evaluated straighforwardly from eqs. (A13)-(A15). For example, for $f\{S^\nu\} = (S^1)^n (n \geq 0)$ we get

$$W_n = \int D\vec{S}(S^\nu)^n = \begin{cases} \dfrac{D^{n/2}\Gamma(\frac{n}{2}+\frac{1}{2})\Gamma(\frac{D}{2})}{\pi^{1/2}\Gamma(\frac{n}{2}+\frac{D}{2})}\,\Omega_D & \text{for n even} \\ \\ 0 & \text{for n odd} \end{cases} \qquad \text{(A17)}$$

where $\Gamma(x)$ is the gamma function and $\Omega_D = \pi^{\frac{D}{2}} D^{\frac{D}{2}-1}/\Gamma(D/2)$ is the surface of the hypersphere of radius $D^{1/2}$ in the D-dimensional space.

Therefore with help of eqs. (A10)-(A12) and (A17) we can get.

$$\int D\vec{S}_1 S_1^1 = 0 \qquad \text{(A18)}$$

$$\int D\vec{S}_1 S_1^1 S_1^\nu = \begin{cases} \Omega_D & \nu = 1 \\ 0 & \nu \neq 1 \end{cases} \qquad \text{(A19)}$$

$$\int D\vec{S}_1 \int D\vec{S}_2 \exp\{K_{12}\vec{S}_1 . \vec{S}_2\} = \pi\left(\frac{2\pi}{|K_{12}|}\right)^{\frac{1}{2}D-1} I_{\frac{1}{2}D-1}(D|K_{12}|)\,\Omega_D \qquad \text{(A20)}$$

$$\int D\vec{S}_1 \int D\vec{S}_2 S_1^1 \exp\{K_{12}\vec{S}_1 . \vec{S}_2\} = 0$$

$$\int D\vec{S}_1 \int D\vec{S}_2 S_1^1 S_1^\nu \exp\{K_{12}\vec{S}_1 . \vec{S}_2\} = \begin{cases} \pi\left(\frac{2\pi}{|K_{12}|}\right)^{\frac{1}{2}D-1} I_{\frac{1}{2}D-1}(D|k_{12}|)\Omega_D & \nu = 1 \\ 0 & \nu \neq 1 \end{cases} \qquad \text{(A21)}$$

$$\int D\vec{S}_1 \int D\vec{S}_2 S_1^1 S_2^\nu \exp\{K_{12}\vec{S}_1 . \vec{S}_2\} = \begin{cases} \pi\left(\frac{2\pi}{|k_{12}|}\right)^{\frac{1}{2}D-1} I_{\frac{D}{2}}(D|K_{12}|)\,\Omega_D & \nu = 1 \\ 0 & \nu \neq 1 \end{cases} \qquad \text{(A22)}$$

Finally eqs. (13) and (14) can be obtained be using eqs. (A18)-(A22) in eqs. (A3) and (A4) and expanding up to first order in the b's.

TABLE 1

Table 1. Functions $th_n(x)$ and $G_n(y)$ for $n = \frac{1}{2}D - 1$ and $x = DK$ and $y = 2D^{1/2}K_1$

D	$th_n(x)$	$G_n(y)$
1	tanh x	2 cosh y
2	$I_1(x)/I_o(x)$	$I_o(y)$
3	$\mathcal{L}(x)^{+}$	senh(y)/y
∞	$\dfrac{2K}{1+(1+4K^2)^{1/2}}$	$\exp(2k_1^2)$

(+) $\mathcal{L}(x) = (\coth x - 1/x)$ is the Langevin function.

FIGURE CAPTIONS

Fig. 1 - Schematic representation of the simplest clusters with (a) $N' = 1$ and $i = 1 \ldots z$ and (b) $N = 2$ and $j = 1 \ldots (z-1)$.

Fig. 2 - Reduced critical temperature $\tilde{T}_c = (k_B T_c/J)$ against concentration p for the diluted ferromagnetic D-vector model with $z = 6$, for $D = 1$ (Ising), $D = 3$ (Heisenberg) and $D \to \infty$.

Fig. 3 - Reduced critical temperature $\tilde{T}_c = (k_B T_c/J)$ against concentration p for the symmetric ($\alpha = 1$) bond disordered D-vector model with $z = 6$ and for $D = 1$ (Ising) $D \to \infty$.

Fig. 4 - Reduced critical temperature $\tilde{T}_c = (k_B T_c/J)$ against concentration p for the assymetric bond disordered D-vector model with $z = 6$, $\alpha = 2.5$ and for $D = 1$ (Ising), $D = 3$ (Heisenberg) and $D \to \infty$.

Fig. 5 - Normalized critical temperature $\tilde{t} = T_c(p)/T_c(0)$ against concentration p for the antiferromagnetic Ising model decorated with random D-vector bond spins with $z = 6$, $\gamma = 1$ and $D = 1$ (Ising), $D = 3$ (Heisenberg) and $D \to \infty$.

Fig. 6 - Normalized critical temperature $\tilde{t} = T_c(p)/T_c(0)$ against concentration p for the antiferromagnetic Ising model decorated with random D-vector bond spins with $z = 8$, $\gamma = 0.5$ and for $D = 1$ (Ising), $D = 3$ (Heisenberg) and $D \to \infty$.

Fig. 7 - Normalized critical temperature $\tilde{t} = T_c(p)/T_c(0)$ against concentration p for the antiferromagntic Ising model decorated ith random D-vector bond spins with $D = 1$, $\gamma = 0.5$ and for $z = 6$ and $z = 12$.

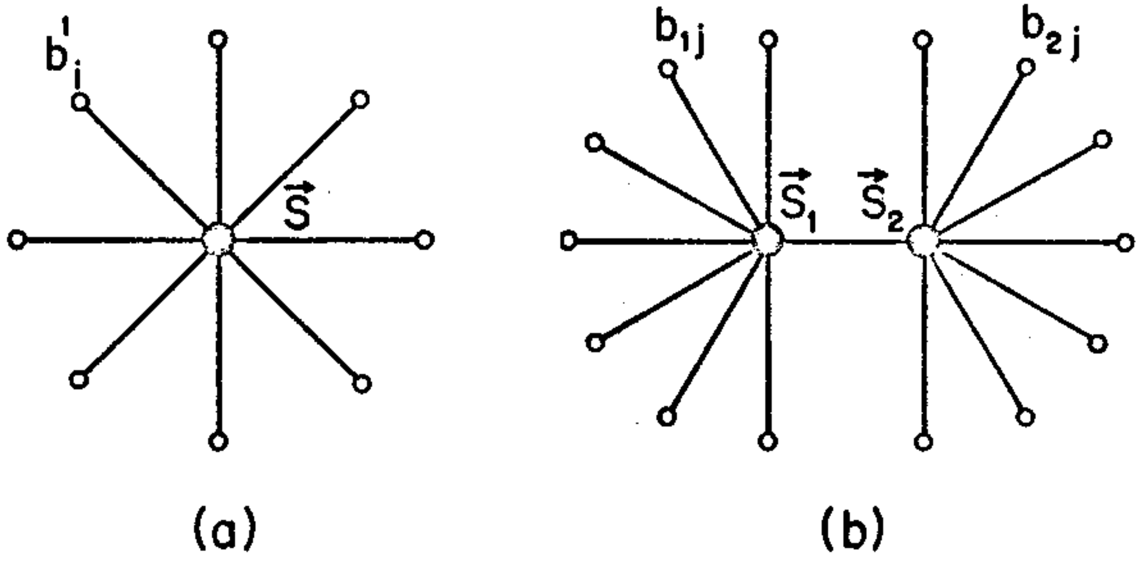

Figure 1

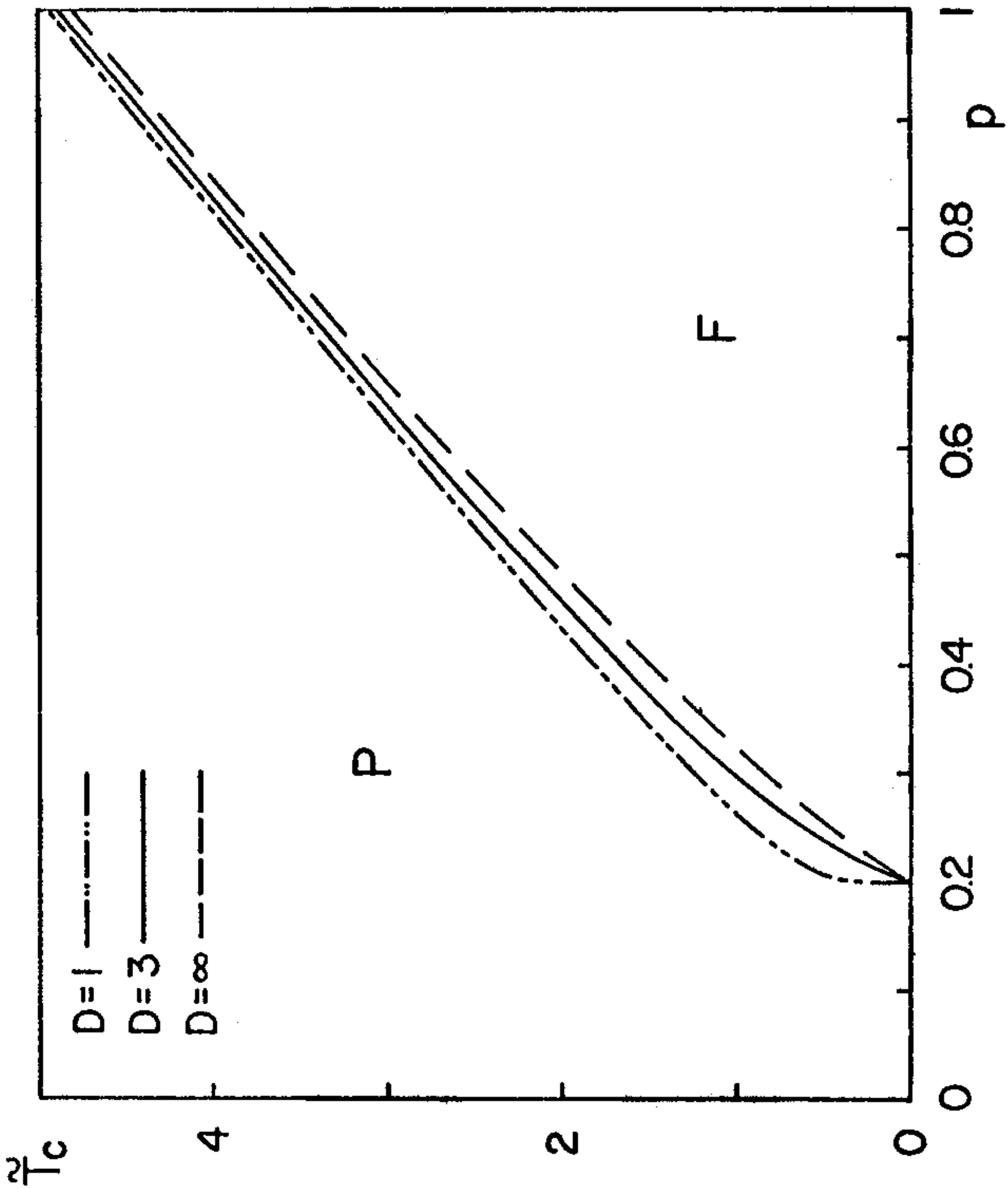

Figure 2

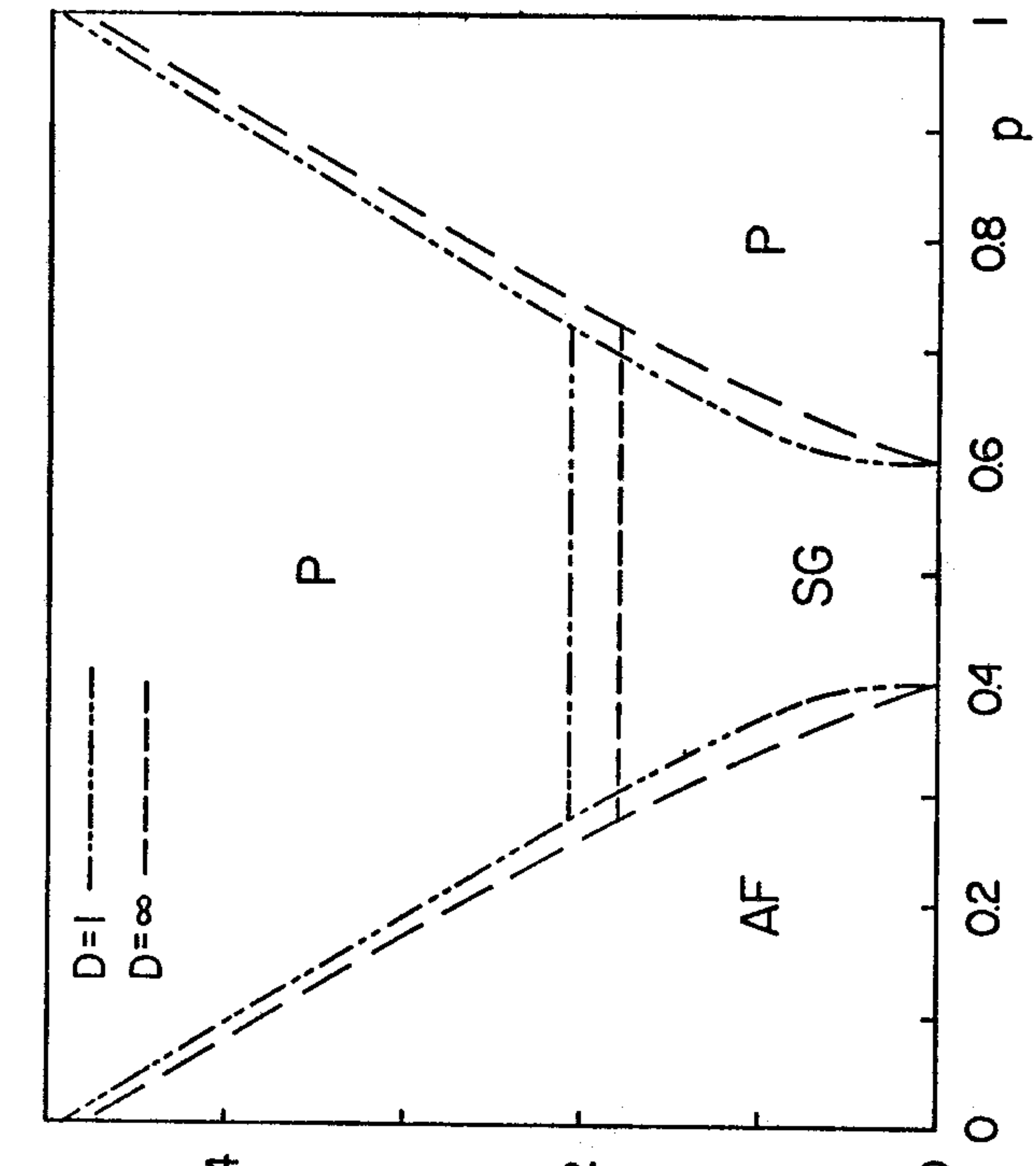

Figure 3

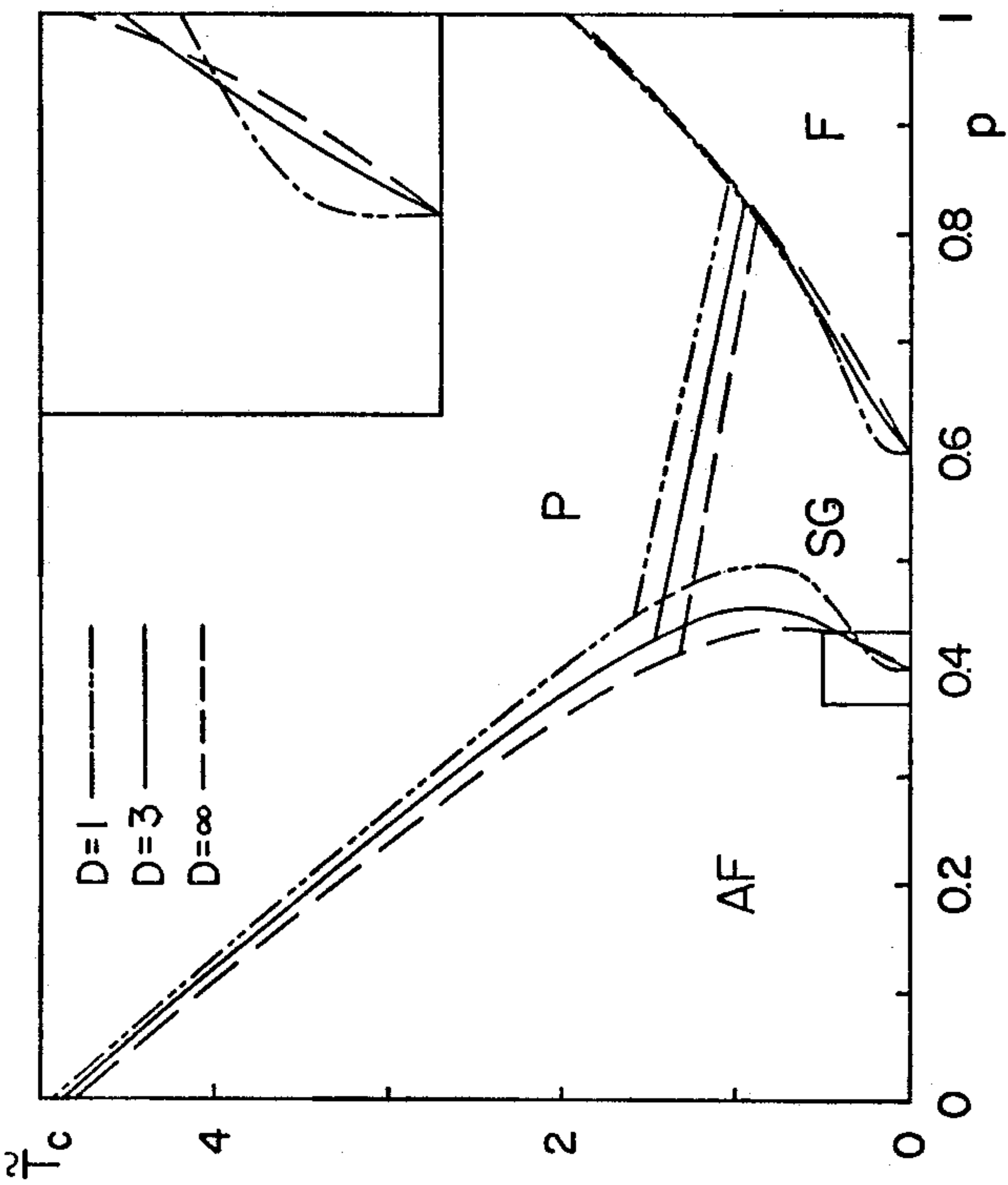

Figure 4

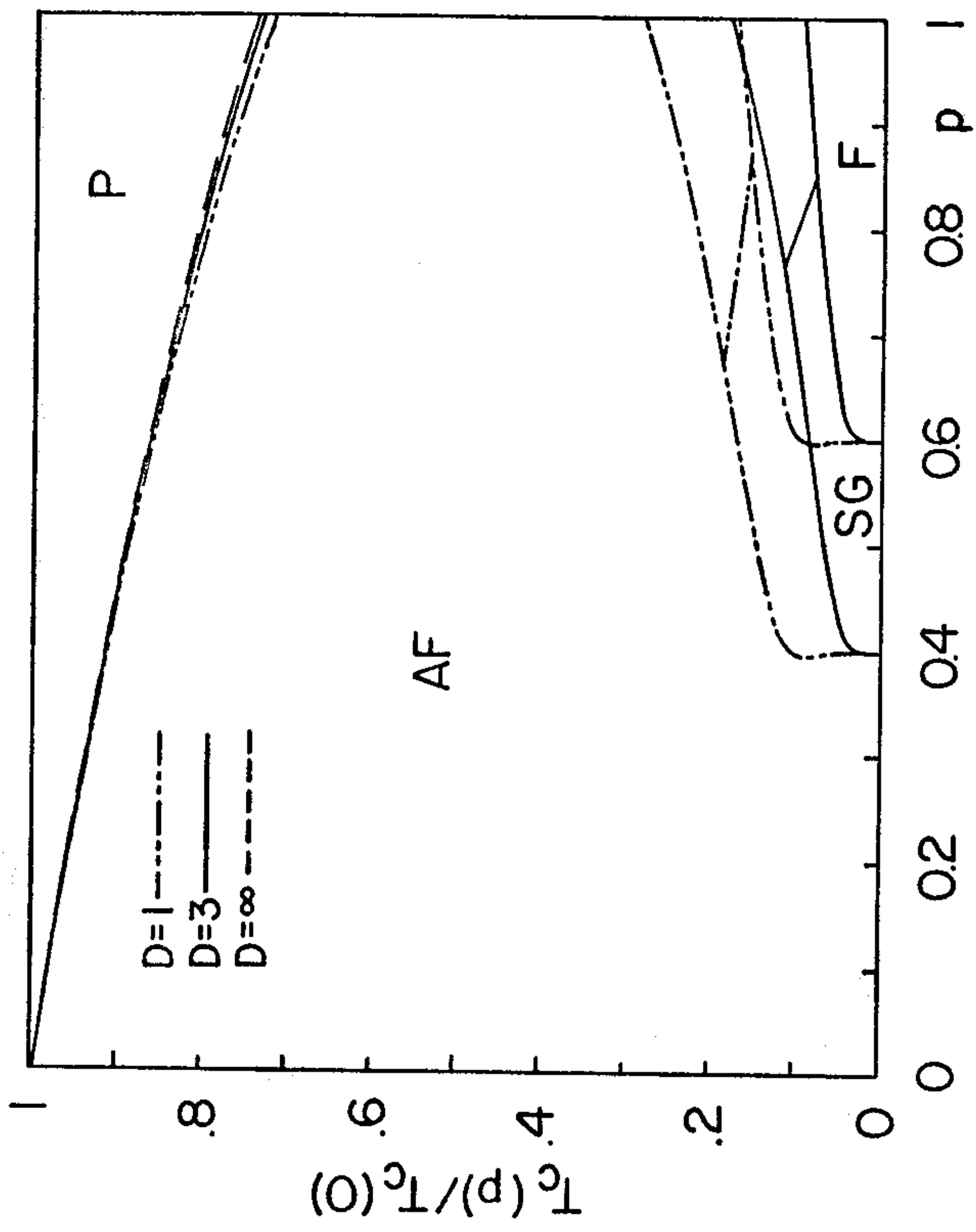

Figure 5

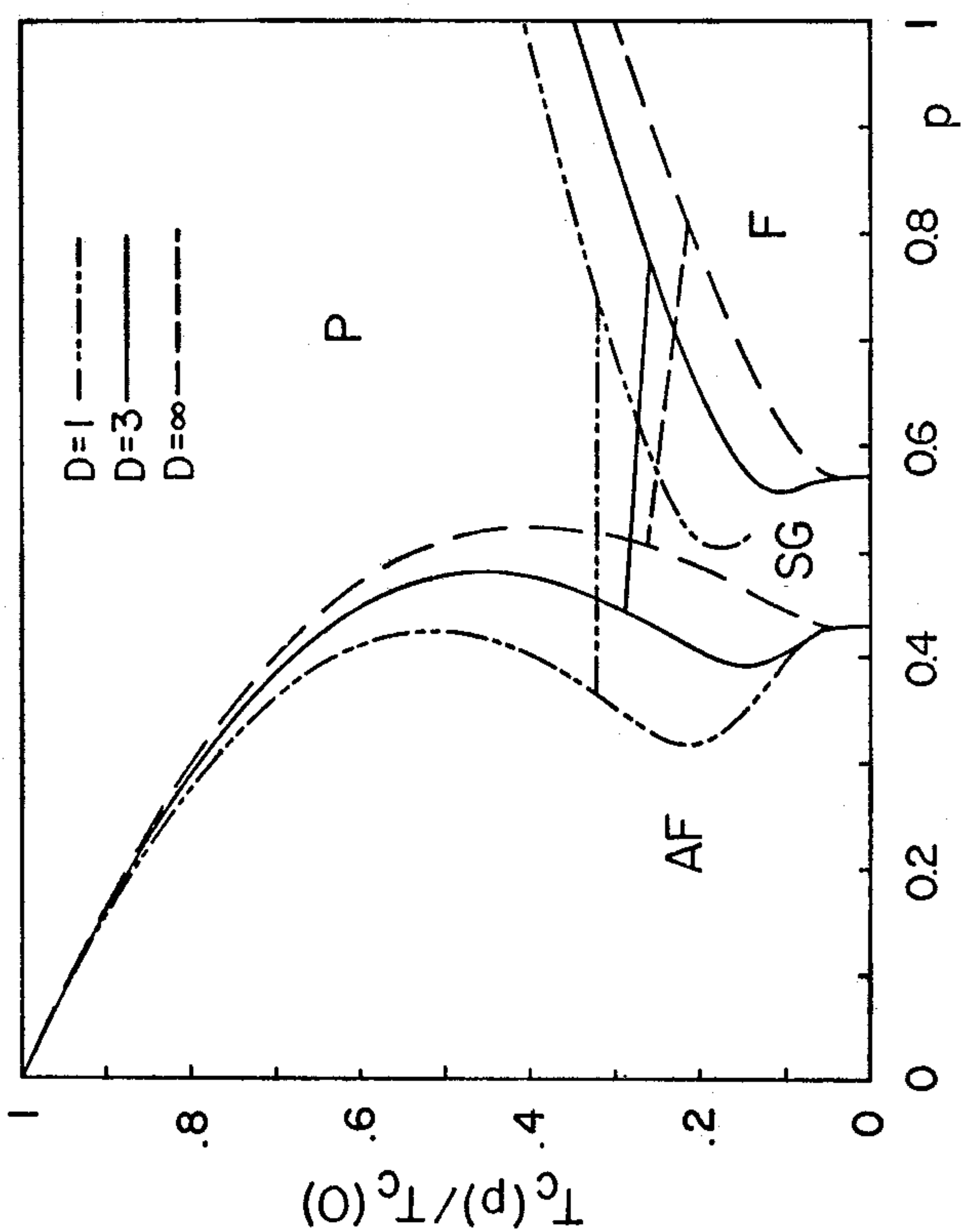

Figure 6

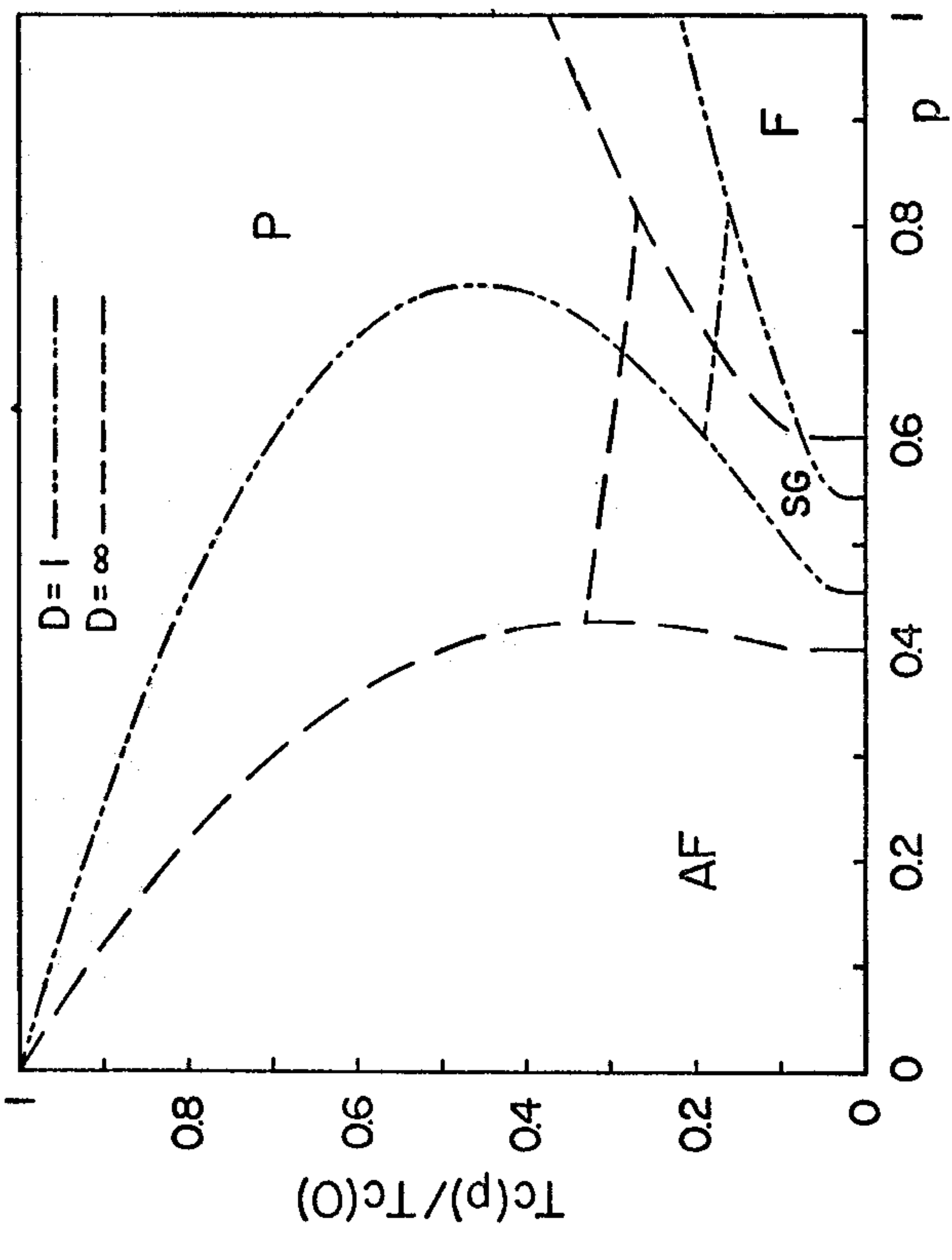

Figure 7

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