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A MAJORITY-RULE MODEL: REAL-SPACE RENORMALIZATION-GROUP SOLUTION AND FINITE SIZE SCALING

by

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ABSTRACT

Through a simple majority rule a statistical geometrical d-dimensional model(d can even be a fractal dimensionality) is formulated which presents a continuous phase transition as a function of a certain independent occupancy probability p.Both critical point p_c and "correlation length" exponent ν are exactly calculated through real space renormalization group(with linear scaling factor b). The well known finite size scaling hypothesis $\nu(b)-\nu \propto 1/\ln b$ (in the limit $b\to\infty$) is analytically exhibited on a non trivial example presumably for the first time. A new and more rapidly convergent finite size extrapolation procedure is suggested.

The simple majority rule (see [1]) has been frequently used during recent years within real space renormalization group (RG) approximate treatments of several thermal models (mainly Ising-like). We shall use it herein as the basis for constructing a geometrical model which presents a continuous phase transition. Let us consider a macroscopic square checkerboard (yet uncolored) which is going to be randomly and independently (quenched model) occupied by black and white plaquettes (whose respective occupancy probabilities are p and (1-p)). We arbitrarily choose an elementary square (hereafter referred to as the center) and assume, for convenience, that it is always black (this weak restriction in the ensemble of possible macroscopic occupancy configurations has clearly no relevance in the thermodynamic limit). For a given macroscopic configuration associated to $p<1/2$ we proceed as follows: by starting from the black center (which corresponds to a degree of consultation $n=0$) we check if the inclusion of its immediate neighborhood (which corresponds to $n=1$; the sub-system under consideration contains now N_1 elementary squares; $N_1=9$ in Fig.1.a) preserves the black majority; if this is the case we expand even more our subsystem ($n=2$; the new subsystem contains N_2 elements; $N_2=25$ in Fig. 1.a); we keep on through this procedure up to the point where the black majority is reversed (the reversal point is characterized by a consultation degree n_r; $n_r=2$ in Fig. 1.a). When this change occurs (and necessarily occurs as we have assumed $p<1/2$) we consider a new random configuration (still associated to the same value of p) and go through the same operational sequence, thus obtaining a new value for n_r; we note ξ the (arithmetic) mean value of the $\{n_r\}$ associated with a thermody namically great number of configurations; clearly ξ diverges (presumably as $(p_c-p)^{-\nu}$ with $p_c=1/2$) when p approaches p_c. If $p>1/2$ the procedure to be followed is exactly the same excepting for the fact that only configurations leading to finite n_r enter into the calculation of ξ, which is now expected to diverge as $(p-p_c)^{-\nu}$ in the vicinity of p_c.

Now that the majority model (MM) has been introduced we intend to calculate ν within a RG framework which renormalizes finite squares (with side length b) into smaller ones (side length $b'<b$); both b and b' are odd numbers in order to avoid majority ambiguities. For analytical simplicity we shall work, in the RG framework, with no color restriction on the center of the squares (if we impose the black color for the centers, the final asymptotic results in the $b\to\infty$ region are exactly the same). Let us first consider the simplest case, i.e. b=3 and b'=1. The RG recursion is is given by

$$p' = R_3(p) \equiv p^9+9p^8(1-p)+36p^7(1-p)^2+84p^6(1-p)^3+126p^5(1-p)^4 \quad (1)$$

which presents (besides the two trivial stable fixed points p=0 and p=1) an unstable fixed point at p=1/2 (exact answer) and leads to $\lambda_3 \equiv dR_3(p)/dp|_{p=1/2} = 315/2^7$, therefore to the approximate critical exponent $\nu_{3,1} = \ln 3/\ln\lambda_3 \simeq 1{,}2199$. For general values of (b,b') and considering now a d-dimensional hypercubic checkerboard, Eq. (1) becomes

$$R_{b'}(p') = R_b(p) \qquad (b=3,5,7,\dots;b'=1,3,\dots,b-2) \qquad (2)$$

with

$$R_b(p) \equiv \sum_{i=0}^{(b^d-1)/2} \binom{b^d}{i} p^{b^d-i}(1-p)^i \qquad (3)$$

This recursion admits, for all d and (b,b'), the unstable fixed point p= 1/2 (exact answer) and leads to

$$\nu_{b,b'} = \frac{\ln(b/b')}{\ln(\lambda_b/\lambda_{b'})} \tag{4}$$

with

$$\lambda_b \equiv \left.\frac{dR_b(p)}{dp}\right|_{p=1/2} = \frac{b^d}{2^{b^d-1}} \begin{pmatrix} b^d-1 \\ \frac{b^d-1}{2} \end{pmatrix} \quad (b=1,3,5,\ldots) \tag{5}$$

By using Stirling's approximation ($\ln N! = N\ln N - N + \frac{1}{2}\ln(2\pi N) + \frac{1}{12N} + o(1/N^3)$ in the limit $N\to\infty$) we straightforwardly obtain (in the limit $b\to\infty$)

$$\nu_{b,1} = \frac{2/d}{1-\frac{\ln(\pi/2)}{d\ \ln b}} + o(1/b^d \ln b) \tag{6}$$

and

$$\nu_{b,b-2} = \frac{2/d}{1-\frac{1}{2b^d}} + \sup\left[o(1/b^{d+1}), o(1/b^{2d})\right] \tag{7}$$

and finally $\nu = \lim_{\substack{b\to\infty \\ b'<b}} \nu_{b,b'} = \frac{2}{d}$.

This result deserves a few comments. First of all, how is it we obtain a phase transition for any dimensionality, even for d=1?; this is clear once we realize that the majority rule implies long range (as a matter of fact, infinite range) interactions as, at any given degree of consultation n, even very distant elements enter into consideration on equal standing as the neighboring ones. Our second comment refers to standard site percolation to which the MM might, at first sight, look isomorphic; this is clearly wrong as the question "can our region be spanned through (let us say first neighboring) black plaquettes?" is very different from "in our region, are the black plaquettes majority?" (e.g. in the subsystem n=2 of Fig. 1.a, the answer to the first question is yes, where it is no to the second one; the opposite answers are possible as well). The MM might also look like similar to the Ising one: although both binary, these models are quite different as, at any finite temperature, neighboring plaquettes are likely of the same (different) "color" in the ferromagnetic (antiferromagnetic) Ising model whereas there is no such correlation in the MM. We might also think about more sophisticated models, like the triangular lattice quenched bond-mixed Ising one with a distribution law for the coupling constant J given by $P(J)=(1-x)\delta(J-J_0)+x\delta(J+J_0)$ with $J_0>0$; if we consider an elementary triangular plaquette and associate "black" to "frustated" we obtain $p= x^3+3(1-x)^2x$ (x= 0, 1/2,1 respectively imply p= 0,1/2,1), but we can easily verify that the probability for two first-neighboring plaquettes being "frustrated" does not equal p^2 as in the MM. Our last comment concerns a possible generalization of the present results. There is clearly no reason for the consultation sequence N_n to grow(while the consultation degree n runs over all natural numbers) as that of an hypercubic lattice (i.e. $N_n= (2n+1)^d$) and any regular lattice can be used as well (see Fig. 1.b); furthermore N_n has not to be related to any regular (or even irregular) lattice: it can be an arbitrary one. In that case we can define a fractal dimensionality (see [2]) $d_f \equiv \lim_{n\to\infty} \frac{\ln N_n}{\ln (2n+1)}$ (it is clear that all d-dimensional regular lattices lead to $d_f=d$); for this MM we expect $p_c=1/2$ and $\nu= 2/d_f$.

Let us now turn back to Eqs. (4) and (5) (and the asymptotic behaviors (6) and (7)) in order to discuss the well known finite size scaling hypothesis [3] frequently used in pure [4] and RG [5] Monte Carlo treatments of statistical models, and herein analytically exhibited (as far as we know for the first time on a non trivial, at least non one-dimensional, model). The critical value $p_c=1/2$ is exactly obtained within the present RG for all (b,b'), therefore we confine our discussion onto the behavior of $\nu_{b,b'}$ for large values of b. We shall make three remarks:

i) It is systematically assumed in the RG Monte Carlo treatments of statistical models that $(\nu_{b,1}-\nu)$ (therefore $(1/\nu_{b,1}-1/\nu)$) is proportional to $1/\ln b$ in the limit $b\to\infty$: Eq.(6) confirms this assumption which presumably for most models is still true for $(\nu_{b,b'}-\nu)$ with <u>any</u> fixed value of b'(or values of b' growing, with b, in a sufficiently slow manner);

ii) As seen from Eq.(6) the asymptotic regime is more rapidly attained by $1/\nu_{b,1}$ than by $\nu_{b,1}$; consequently, for practical purposes, extrapolations $1/\nu_{b,1}$ vs. $1/\ln b$ are expected to provide higher numerical precision than that obtained in extrapolations $\nu_{b,1}$ vs. $1/\ln b$; no such <u>a priori</u> expectation exists related to Eq.(7), therefore a $\nu_{b,b-2}$ vs. $1/b^d$ extrapolation can lead to results practically as good as those obtained through a $1/\nu_{b,b-2}$ vs. $1/b^d$ extrapolation (the MM d=2 case is illustrated in Fig.2).

iii) As seen from Eqs.(6) and (7) the asymptotic regime is more rapidly attained if the sequence (b,b-2), rather than the (b,1) one, is used (the MM d=2 case is illustrated in Fig.2); in more general terms, sequences (b,b') such that $b/b'\to 1$ are, for a possibly very large class of extrapolations, to be a priori preferred to those which inply $b/b' \to \infty$.

To synthetize let us say that the present work suggests, for the numerical calculation of the "correlation length" critical exponent ν associated to a possibly large variety of statistical systems, the use of extrapolations $1/\nu_{b,b'}$ vs. $1/b^d$ with b' running as close to b as possible, rather than the standard ones (namely $\nu_{b,1}$ vs. $1/\ell n b$ or $1/\nu_{b,1}$ vs. $1/\ell n b$). If we focuse the majority model (which admits, in fact, several natural extension on which we are presently working) introduced herein, let us recall that its exact critical point and "correlation lenght" exponent respectively are $p_c = 1/2$ and $\nu = 2/d$ for the d-dimensional hypercubic lattice (for an arbitrary sequence N_n we expect $\nu = 2/d_f$ where $d_f \equiv \lim_{n\to\infty} \frac{\ell n N_n}{\ell n\, n}$ is a fractal dimensionality). A straightforward application of the present majority model could clearly be within mathematical models for political sciences (see also page 432 of [1]); it should be interesting to search for physical applications within the standard thermal and /or geometrical statistical problems (see, for example, [6] and references therein).

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REFERENCES

[1] Th. Niemeijer and J.M.J. van Leeuwen, "Renormalization Theory for Ising-like Spin Systems" from "Phase Transitions and Critical Phenomena" ed. by C.Domb and M.S. Green, vol 6, 425 (Academic Press, 1976)

[2] B.B. Mandelbrot, "Fractals:Form, Chance and Dimension" (W.H. Freeman, San Francisco, 1977)

[3] M.E. Fisher, in "Proceedings of the International Summer School Enrico Fermi 1970", Course 51, Varenna, Italy, ed. by M.S.Green (Academic, New York, 1971); M.E. Fisher and M.N. Barber, Phys. Rev. Lett. 28, 1516(1972); A. Sur, J.L. Lebowitz, J.Marro, M. L. Kalos and S. Kirkpatrick, J.Stat. Phys. 15, 345(1976); E. Brézin, preprint (1981).

[4] D.P. Landau, Phys. Rev. B 13, 2997(1976), 14, 255(1976) and 22, 2450(1980); J.Hoshen, D.Stauffer, G.H.Bishop , R.J. Harrison and G.D. Quinn, J.Phys. A 12, 1285(1979).

[5] M.Suzuki, Prog. Theor. Phys. 58, 1142(1977); P.J.Reynolds, H.E. Stanley and W. Klein, Phys. Rev.B 21, 1223(1980); P.D. Eschbach, D.Stauffer and H.J. Herrmann, Phys. Rev. B 23, 422(1981)

[6] C.Tsallis and S.V.F.Levy, Phys. Rev. Lett. 47, 950(1981); H. O.Martin and C.Tsallis, J.Phys. C 14, 5645(1981); N.C.Chao, G.Schwachheim and C.Tsallis, Z.Physik B43, 305(1981); E.M.F. Curado, C.Tsallis, S.V.F.Levy and M.J. de Oliveira, Phys.Rev. B 23, 1419(1981).

CAPTION FOR FIGURES

Fig.1 - Examples of occupancy configurations. (a) $N_n=(2n+1)^2$ and $n_r=2$; (b) $N_n= 3n(n+1)+1$ and $n_r=1$.

Fig.2 - Critical exponent ν finite size scaling exhibited for the d=2 MM. Full lines represent the $b\to\infty$ asymptotic behaviours; dashed lines are guides to the eye and run over the actual approximate values of ν (represented by dots). The ordinate scale for the ν-lines (ν^{-1}-lines) is the outside (inside)one; the abcissa scale for the $\nu_{b,1}$- and $\nu^{-1}_{b,1}$- lines ($\nu_{b,b-2}$- and $\nu^{-1}_{b,b-2}$-lines) is the bottom (top) one. Concerning numerical procedures remark that: (a) the standard $\nu^{-1}_{b,1}$ vs. $1/\ell n b$ one is superior to the standard $\nu_{b,1}$ vs. $1/\ell n b$ one; (b) both $\nu_{b,b-2}$ vs. b^{-d} and $\nu^{-1}_{b,b-2}$ vs. b^{-d} (numerically slightly better) procedures are superior to the standard ones (using $1/\ell n b$). A numerical illustration: $\nu_{13,1}= 1.09583$, $\nu_{15,1}= 1.09047$, $\nu_{13,11}= 1.00353$ and $\nu_{15,13}= 1.00258$;the <u>two points</u> linear extrapolation of these values provide the following errors for ν: -0.56% in the case $\nu_{b,1}$ vs. $1/\ell n b$, 0.26% in the case $1/\nu_{b,1}$ vs. $1/\ell n b$, -0.027% in the case $\nu_{b,b-2}$ vs. $1/b^2$ and -0.026% in the case $1/\nu_{b,b-2}$ vs. $1/b^2$.

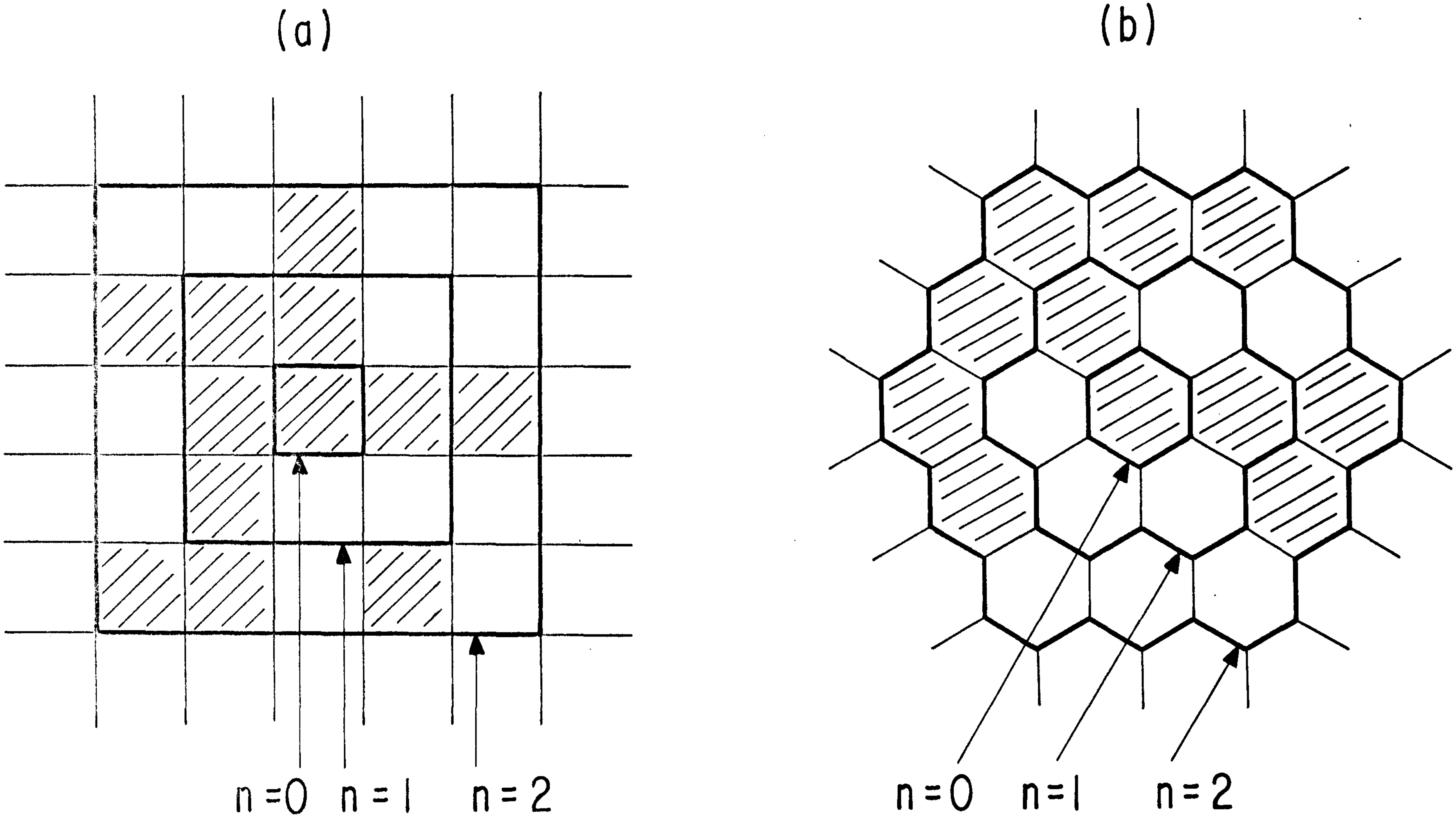

FIG.1

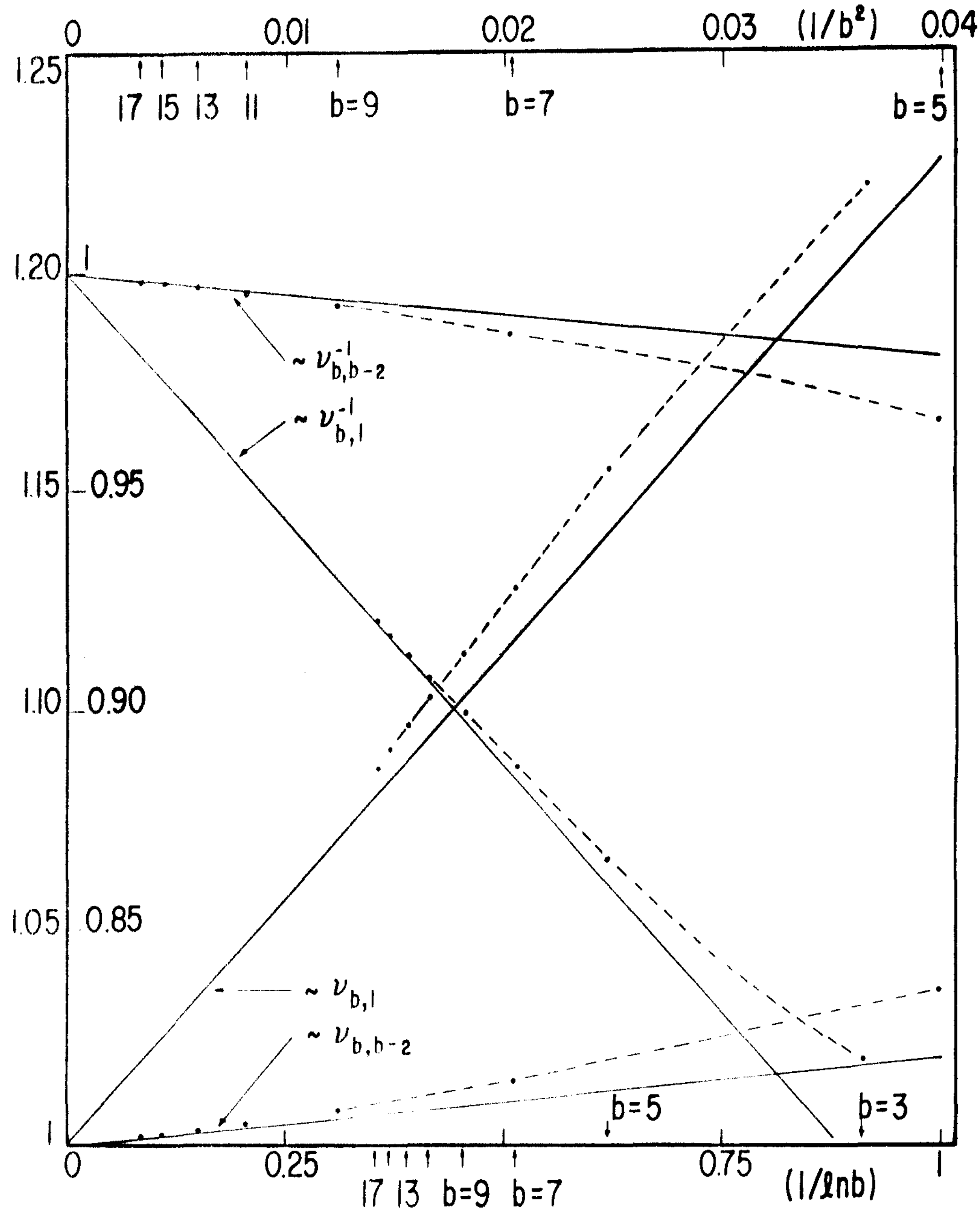
0
0.01
0.02
0.03
(1/b²)
0.04
1.25
17 15 13 11 b=9 b=7 b=5
1.20
1
~ $\nu^{-1}_{b,b-2}$
~ $\nu^{-1}_{b,1}$
1.15 0.95
1.10 0.90
1.05 0.85
~ $\nu_{b,1}$
~ $\nu_{b,b-2}$
b=5
b=3
1
0
0.25
0.75
1
(1/ℓnb)
17 13 b=9 b=7

FIG.2