Spherically Symmetric Dissipative Anisotropic Fluids: A General Study

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Abstract

The full set of equations governing the evolution of self–gravitating spherically symmetric dissipative fluids with anisotropic stresses is deployed and used to carry out a general study on the behaviour of such systems, in the context of general relativity. Emphasis is given to the link between the Weyl tensor, the shear tensor, the anisotropy of the pressure and the density inhomogeneity. In particular we provide the general, necessary and sufficient, condition for the vanishing of the spatial gradients of energy density, which in turn suggests a possible definition of a gravitational arrow of time. Some solutions are also exhibited to illustrate the discussion.

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1   Introduction

This work is devoted to the study of dissipative, locally anisotropic, spherically symmetric self–gravitating fluids, with particular emphasis in a set of physical and geometrical variables which appear to play a fundamental role in the evolution of such systems. These variables are the Weyl tensor, the shear tensor, the local anisotropy of the presssure and the density inhomogeneity.

The Weyl tensor [1] or some functions of it [2], have been proposed to provide a gravitational arrow of time. The rationale behind this idea being that tidal forces tend to make the gravitating fluid more inhomogeneous as the evolution proceeds, thereby indicating the sense of time. However, some works have thrown doubts on this proposal [3]. Further evidence about the relevance of the Weyl tensor in the evolution of self–gravitating systems may be found in [4].

The role of density inhomogeneities in the collapse of dust [5] and in particular in the formation of naked singularities [6], has been extensively discussed in the literature.

On the other hand, the assumption of local anisotropy of pressure, which seems to be very sensible to describe matter distribution under a variety of circumstances, has been proved to be very useful in the study of relativistic compact objects (see [7] and references therein).

A hint pointing to the relevance of the above mentioned three factors in the fate of spherical collapse is also provided by the expression of the active gravitational mass in terms of those factors [8].

Finally, the relevance of the shear tensor in the evolution of self–gravitaing systems has been brought out by many authors (see [9] and references therein).

Now, in the study of self–gravitating compact objects it is usually assumed that deviations from spherical symmetry are likely to be incidental rather than basic features of the process involved (see however the discussion in [10]). Thus, since the seminal paper by Oppenheimer and Snyder [11], most of the work dedicated to the problem of general relativistic gravitational collapse, deal with spherically symmetric fluid distribution. Accordingly we shall consider spherically symmetric fluid distributions.

Also, the fluid distribution under consideration will be assumed to be dissipative. Indeed, dissipation due to the emission of massless particles (photons and/or neutrinos) is a characteristic process in the evolution of massive stars. In fact, it seems that the only plausible mechanism to carry away the bulk of the binding energy of the collapsing star, leading to a neutron star or black hole is neutrino emission [12]. Consequently, in this paper, the matter distribution forming the self–gravitating object will be described as a

dissipative fluid.

In the diffusion approximation, it is assumed that the energy flux of radiation (as that of thermal conduction) is proportional to the gradient of temperature. This assumption is in general very sensible, since the mean free path of particles responsible for the propagation of energy in stellar interiors is in general very small as compared with the typical length of the object. Thus, for a main sequence star as the sun, the mean free path of photons at the centre, is of the order of $2\,cm$. Also, the mean free path of trapped neutrinos in compact cores of densities about $10^{12}\,g.cm.^{-3}$ becomes smaller than the size of the stellar core [13, 14].

Furthermore, the observational data collected from supernova 1987A indicates that the regime of radiation transport prevailing during the emission process, is closer to the diffusion approximation than to the streaming out limit [15].

However in many other circumstances, the mean free path of particles transporting energy may be large enough as to justify the free streaming approximation. Therefore we shall include simultaneously both limiting cases of radiative transport (diffusion and streaming out), allowing for describing a wide range situations.

It is also worth mentioning that although the most common method of solving Einstein's equations is to use commoving coordinates (e.g. [16]), we shall use noncomoving coordinates, which implies that the velocity of any fluid element (defined with respect to a conveniently chosen set of observers) has to be considered as a relevant physical variable ([17]).

The paper is organized as follows: In the next section we introduce the notation and write all relevant equations. Section 3 is devoted to the analysis of different special cases. Finally the results are discussed in the last section.

2   The Basic equations

In this section we shall deploy the relevant equations for describing a dissipative self–gravitating locally anisotropic fluid. In spite of the fact that not all these equations are independent (for example the field equations and the conservation equations (Bianchi identities)) we shall present them all, since depending on the problem under consideration, it may be more advantageous using one set instead of the other.

2.1   Einstein equations

We consider spherically symmetric distributions of collapsing fluid, which for sake of completeness we assume to be locally anisotropic, undergoing dissipation in the form of heat flow and/or free streaming radiation, bounded by a spherical surface Σ.

The line element is given in Schwarzschild–like coordinates by

$$ds^2 = e^\nu dt^2 - e^\lambda dr^2 - r^2 \left(d\theta^2 + sin^2\theta d\phi^2 \right), \tag{1}$$

where $\nu(t, r)$ and $\lambda(t, r)$ are functions of their arguments. We number the coordinates: $x^0 = t;\ x^1 = r;\ x^2 = \theta;\ x^3 = \phi$.

The metric (1) has to satisfy Einstein field equations

$$G_\mu^\nu = 8\pi T_\mu^\nu, \tag{2}$$

which in our case read [18]:

$$-8\pi T_0^0 = -\frac{1}{r^2} + e^{-\lambda}(\frac{1}{r^2} - \frac{\lambda'}{r}) \tag{3}$$

$$-8\pi T_1^1 = -\frac{1}{r^2} + e^{-\lambda}(\frac{1}{r^2} + \frac{\nu'}{r}) \tag{4}$$

$$-8\pi T_2^2 = -8\pi T_3^3 = -\frac{e^{-\nu}}{4}(2\ddot{\lambda} + \dot{\lambda}(\dot{\lambda} - \dot{\nu}))$$
$$+\frac{e^{-\lambda}}{4}(2\nu'' + \nu'^2 - \lambda'\nu' + 2\frac{\nu' - \lambda'}{r}) \tag{5}$$

$$-8\pi T_{10} = -\frac{\dot{\lambda}}{r}, \tag{6}$$

where dots and primes stand for partial differentiation with respect to t and r, respectively. In order to give physical significance to the T_ν^μ components we apply the Bondi approach [18].

Thus, following Bondi, let us introduce purely locally Minkowski coordinates (τ, x, y, z)

$$d\tau = e^{\nu/2}dt\,; \qquad dx = e^{\lambda/2}dr\,; \qquad dy = rd\theta\,; \qquad dz = rsin\theta d\phi.$$

Then, denoting the Minkowski components of the energy tensor by a bar, we have

$$\bar{T}_0^0 = T_0^0\,; \qquad \bar{T}_1^1 = T_1^1\,; \qquad \bar{T}_2^2 = T_2^2\,; \qquad \bar{T}_3^3 = T_3^3\,; \qquad \bar{T}_{01} = e^{-(\nu+\lambda)/2}T_{01}.$$

Next, we suppose that when viewed by an observer moving relative to these coordinates with proper velocity ω in the radial direction, the physical content of space consists of

an anisotropic fluid of energy density ρ, radial pressure P_r, tangential pressure $P_\perp$, radial heat flux q and unpolarized radiation of energy density ϵ traveling in the radial direction. Thus, when viewed by this moving observer the covariant tensor in Minkowski coordinates is

$$
\begin{pmatrix}
\rho + \epsilon & -q - \epsilon & 0 & 0 \\
-q - \epsilon & P_r + \epsilon & 0 & 0 \\
0 & 0 & P_\perp & 0 \\
0 & 0 & 0 & P_\perp
\end{pmatrix}.
$$

Then a Lorentz transformation readily shows that

$$
T_0^0 = \bar{T}_0^0 = \frac{\rho + P_r \omega^2}{1 - \omega^2} + \frac{2\omega q}{1 - \omega^2} + \frac{\epsilon(1 + \omega)}{1 - \omega} \tag{7}
$$

$$
T_1^1 = \bar{T}_1^1 = -\frac{P_r + \rho \omega^2}{1 - \omega^2} - \frac{2\omega q}{1 - \omega^2} - \frac{\epsilon(1 + \omega)}{1 - \omega} \tag{8}
$$

$$
T_2^2 = T_3^3 = \bar{T}_2^2 = \bar{T}_3^3 = -P_\perp \tag{9}
$$

$$
T_{01} = e^{\frac{\nu + \lambda}{2}} \bar{T}_{01} = -\frac{(\rho + P_r)\omega e^{\frac{\nu + \lambda}{2}}}{1 - \omega^2} - \frac{q e^{\frac{\lambda + \nu}{2}}}{1 - \omega^2}(1 + \omega^2) - \frac{e^{\frac{\lambda + \nu}{2}} \epsilon(1 + \omega)}{1 - \omega} \tag{10}
$$

Note that the coordinate velocity in the (t, r, θ, ϕ) system, dr/dt, is related to ω by

$$
\omega = \frac{dr}{dt} e^{(\lambda - \nu)/2}. \tag{11}
$$

Feeding back (7–10) into (3–6), we get the field equations in the form

$$
\frac{\rho + P_r \omega^2}{1 - \omega^2} + \frac{2\omega q}{1 - \omega^2} + \frac{\epsilon(1 + \omega)}{1 - \omega} = -\frac{1}{8\pi}\left\{ -\frac{1}{r^2} + e^{-\lambda}\left(\frac{1}{r^2} - \frac{\lambda'}{r}\right) \right\}, \tag{12}
$$

$$
\frac{P_r + \rho \omega^2}{1 - \omega^2} + \frac{2\omega q}{1 - \omega^2} + \frac{\epsilon(1 + \omega)}{1 - \omega} = -\frac{1}{8\pi}\left\{ \frac{1}{r^2} - e^{-\lambda}\left(\frac{1}{r^2} + \frac{\nu'}{r}\right) \right\}, \tag{13}
$$

$$
P_\perp = -\frac{1}{8\pi}\left\{ \frac{e^{-\nu}}{4}\left(2\ddot{\lambda} + \dot{\lambda}(\dot{\lambda} - \dot{\nu})\right) \right.
$$
$$
\left. -\frac{e^{-\lambda}}{4}\left(2\nu'' + \nu'^2 - \lambda'\nu' + 2\frac{\nu' - \lambda'}{r}\right) \right\}, \tag{14}
$$

$$
\frac{(\rho + P_r)\omega e^{\frac{\lambda + \nu}{2}}}{1 - \omega^2} + \frac{q e^{\frac{\lambda + \nu}{2}}}{1 - \omega^2}(1 + \omega^2) + \frac{e^{\frac{\lambda + \nu}{2}} \epsilon(1 + \omega)}{1 - \omega} = -\frac{\dot{\lambda}}{8\pi r}. \tag{15}
$$

The four–velocity vector is defined as

$$
u^\alpha = \left(\frac{e^{-\frac{\nu}{2}}}{(1 - \omega^2)^{\frac{1}{2}}}, \frac{\omega e^{-\frac{\lambda}{2}}}{(1 - \omega^2)^{\frac{1}{2}}}, 0, 0 \right). \tag{16}
$$

From which we can calculate the four acceleration $a^\alpha = u^\alpha_{;\beta} u^\beta$ to obtain

$$\omega a_1 = -a_0 e^{\frac{\lambda-\nu}{2}} = -\frac{\omega}{1-\omega^2}[(\frac{\omega\omega'}{1-\omega^2} + \frac{\nu'}{2}) + e^{\frac{\lambda-\nu}{2}}(\frac{\omega\dot\lambda}{2} + \frac{\dot\omega}{1-\omega^2})]. \tag{17}$$

For the exterior of the fluid distribution, the spacetime is that of Vaidya, given by

$$ds^2 = \left(1 - \frac{2M(u)}{R}\right) du^2 + 2dudR - R^2\left(d\theta^2 + sin^2\theta d\phi^2\right), \tag{18}$$

where u is a coordinate related to the retarded time, such that $u = constant$ is (asymptotically) a null cone open to the future and R is a null coordinate ($g_{RR} = 0$).

The two coordinate systems (t, r, θ, ϕ) and (u, R, θ, ϕ) are related at the boundary surface and outside it by

$$u = t - r - 2M \, ln\left(\frac{r}{2M} - 1\right), \tag{19}$$

$$R = r. \tag{20}$$

In order to match smoothly the two metrics above on the boundary surface $r = r_\Sigma(t)$, we require the continuity of the first and the second fundamental forms across that surface, yielding (see [19] for details)

$$e^{\nu_\Sigma} = 1 - \frac{2M}{R_\Sigma}, \tag{21}$$

$$e^{-\lambda_\Sigma} = 1 - \frac{2M}{R_\Sigma}, \tag{22}$$

$$[P_r]_\Sigma = [q]_\Sigma. \tag{23}$$

Where, from now on, subscript Σ indicates that the quantity is evaluated on the boundary surface Σ, and (23) expresses the discontinuity of the radial pressure in the presence of heat flow, which is a well known result [20].

Eqs. (21), (22), and (23) are the necessary and sufficient conditions for a smooth matching of the two metrics (1) and (18) on Σ.

2.2 Conservation laws $\left(T^\mu_{\nu;\mu} = 0\right)$

The energy–momentum tensor (7)-(10) may be written as :

$$T^\mu_\nu = \tilde\rho u^\mu u_\nu - \hat{P} h^\mu_\nu + \Pi^\mu_\nu + \tilde{q}(s^\mu u_\nu + s_\nu u^\mu), \tag{24}$$

with

$$h^\mu_\nu = \delta^\mu_\nu - u^\mu u_\nu,$$

$$\Pi^\mu_\nu = \Pi(s^\mu s_\nu + \frac{1}{3}h^\mu_\nu),$$

$$\hat{P} = \frac{\tilde{P}_r + 2P_\perp}{3},$$

$$\tilde{\rho} = \rho + \epsilon,$$

$$\tilde{P}_r = P_r + \epsilon,$$

$$\tilde{q} = q + \epsilon,$$

$$\Pi = \tilde{P}_r - P_\perp$$

and s^μ is defined as

$$s^\mu = (\frac{\omega e^{-\frac{\nu}{2}}}{(1-\omega^2)^{\frac{1}{2}}}, \frac{e^{-\frac{\lambda}{2}}}{(1-\omega^2)^{\frac{1}{2}}}, 0, 0) \tag{25}$$

with the properties $s^\mu u_\mu = 0$, $s^\mu s_\mu = -1$, and $\tilde{q}^\mu = \tilde{q}s^\mu$.

We may write for the shear tensor

$$\sigma_{\alpha\beta} = \frac{1}{2}\sigma(s_\alpha s_\beta + \frac{1}{3}h_{\alpha\beta}) \tag{26}$$

with

$$\sigma = -\frac{1}{(1-\omega^2)^{\frac{1}{2}}}[e^{-\frac{\nu}{2}}(\dot{\lambda} + \frac{2\omega\dot{\omega}}{1-\omega^2}) + e^{-\frac{\lambda}{2}}(\omega\nu' + \frac{2\omega'}{1-\omega^2} - \frac{2\omega}{r})]. \tag{27}$$

Then from $T^\mu_{\nu;\mu} = 0$, using (24), we find:

$$\tilde{\rho}_{;\alpha}u^\alpha + (\tilde{\rho} + \hat{P})\theta + \tilde{q}^\alpha_{;\alpha} = \Pi_{\alpha\beta}\sigma^{\alpha\beta} + \tilde{q}a^\nu s_\nu \tag{28}$$

and

$$(\tilde{\rho} + \hat{P})a_\alpha + h^\beta_\alpha(\tilde{q}_{;\nu}u^\nu s_\beta + \tilde{q}s_{\beta;\nu}u^\nu - \hat{P}_{,\beta} + \Pi^\mu_{\beta;\mu}) + \sigma_{\alpha\beta}\tilde{q}s^\beta + \frac{4}{3}\theta\tilde{q}s_\alpha = 0. \tag{29}$$

Or, contracting (29) with s^α:

$$\tilde{P}_{r;\mu}s^\mu + (\tilde{P}_r - P_\perp)s^\mu_{;\mu} - (\tilde{\rho} + P_\perp)a_\mu s^\mu + \frac{4}{3}\theta\tilde{q} + \tilde{q}_{;\nu}u^\nu - \tilde{q}s^\mu s^\nu\sigma_{\mu\nu} = 0.$$

2.3   Ricci identities

Ricci identitites for the vector u_α read:

$$u_{\alpha;\beta;\nu} - u_{\alpha;\nu;\beta} = R^\mu_{\alpha\beta\nu} u_\mu, \tag{30}$$

or using

$$u_{\alpha;\beta} = a_\alpha u_\beta + \sigma_{\alpha\beta} + \frac{1}{3}\theta h_{\alpha\beta}, \tag{31}$$

we have

$$\frac{1}{2} R^\rho_{\alpha\beta\mu} u_\rho = a_{\alpha;[\mu} u_{\beta]} + a_\alpha u_{[\beta;\mu]} + \sigma_{\alpha[\beta;\mu]} + \frac{1}{3}\theta_{,[\mu} h_{\beta]\alpha} + \frac{1}{3}\theta h_{\alpha[\beta;\mu]}. \tag{32}$$

2.3.1   Raychaudhuri equation

Contracting (32) with u^β and then the indices α and μ, we find the Raychaudhuri equation for the evolution of the expansion:

$$\theta_{;\alpha} u^\alpha + \frac{\theta^2}{3} + \sigma_{\alpha\beta}\sigma^{\alpha\beta} - a^\alpha_{;\alpha} = -u_\rho u^\beta R^\rho_\beta = -4\pi(\tilde\rho + 3\hat P), \tag{33}$$

where

$$\sigma_{\alpha\beta}\sigma^{\alpha\beta} = \frac{1}{6}\sigma^2.$$

2.3.2   Constraint equation

If in (32) we contract first α and μ and then contract with $h^{\alpha\beta}$, we obtain the constraint equation expressing a direct relation between expansion θ, shear $\sigma^{\alpha\beta}$ and the heat flux q:

$$R^\rho_\beta u_\rho h^{\alpha\beta} = h^\alpha_\beta(\sigma^{\beta\mu}_{;\mu} - \frac{2}{3}\theta^{;\beta}) + \sigma^{\alpha\beta} a_\beta = 8\pi\tilde q s^\alpha. \tag{34}$$

2.3.3   Propagation equation of the shear

Contracting (32) with $u^\beta h^\alpha_\gamma h^\mu_\nu$ we have

$$u_\rho u^\beta R^\rho_{\alpha\beta\mu} h^\alpha_\gamma h^\mu_\nu = h^\alpha_\gamma h^\mu_\nu (a_{\alpha;\mu} - \sigma_{\alpha\mu;\beta} u^\beta) - a_\gamma a_\nu - u^\beta_{;\mu} h^\mu_\nu(\sigma_{\gamma\beta} + \frac{\theta}{3} h_{\gamma\beta}) - \frac{\theta_{;\alpha} u^\alpha}{3} h_{\gamma\nu}. \tag{35}$$

On the other hand we know that the Riemann tensor may be expressed through the Weyl tensor $C^\rho_{\alpha\beta\mu}$, the Ricci tensor $R_{\alpha\beta}$ and the scalar curvature R, as:

$$R^\rho_{\alpha\beta\mu} = C^\rho_{\alpha\beta\mu} + \frac{1}{2} R^\rho_\beta g_{\alpha\mu} - \frac{1}{2} R_{\alpha\beta}\delta^\rho_\mu + \frac{1}{2} R_{\alpha\mu}\delta^\rho_\beta$$

$$-\frac{1}{2} R^\rho_\mu g_{\alpha\beta} - \frac{1}{6} R(\delta^\rho_\beta g_{\alpha\mu} - g_{\alpha\beta}\delta^\rho_\mu). \tag{36}$$

Contracting (36) with $u_\rho u^\beta h_\gamma^\alpha h_\nu^\mu$ and using Einstein equation (2), we find:

$$R^\rho_{\alpha\beta\mu} u_\rho u^\beta h_\gamma^\alpha h_\nu^\mu = E_{\gamma\nu} + 4\pi\Pi_{\gamma\nu} + \frac{4\pi}{3} h_{\gamma\nu}(\tilde{\rho} + 3\hat{P}), \tag{37}$$

where $E_{\gamma\nu}$ denotes the "electric" part of the Weyl tensor defined by the equation (42) below.

From (35) and (37) taking into account (33) it follows:

$$E_{\gamma\nu} + 4\pi\Pi_{\gamma\nu} = h_\gamma^\alpha h_\nu^\mu(a_{\alpha;\mu} - \sigma_{\alpha\mu;\beta}u^\beta) - a_\gamma a_\nu - \sigma_\nu^\beta\sigma_{\gamma\beta} - \frac{2}{3}\theta\sigma_{\gamma\nu} - \frac{1}{3}(a_{;\alpha}^\alpha - \frac{1}{6}\sigma^2)h_{\gamma\nu} \tag{38}$$

2.4 Evolution equations for the Weyl tensor

According to Kundt and Trümper [21], Bianchi identitites

$$R_{\mu\nu\kappa\delta;\lambda} + R_{\mu\nu\lambda\kappa;\delta} + R_{\mu\nu\delta\lambda;\kappa} = 0, \tag{39}$$

may be written as:

$$C_{\mu\nu\kappa}{}^\lambda{}_{;\lambda} = R_{\kappa[\mu;\nu]} - \frac{1}{6}g_{\kappa[\mu}R_{,\nu]}. \tag{40}$$

Then taking into account Einstein equations (2), (40) reads:

$$C_{\mu\nu\kappa}{}^\lambda{}_{;\lambda} = 8\pi T_{\kappa[\mu;\nu]} - \frac{8\pi}{3}g_{\kappa[\mu}T_{,\nu]}. \tag{41}$$

In the spherically symmetric case the "magnetic" part of the Weyl tensor vanishes($H_{\alpha\beta} = 0$), then we have:

$$C_{\mu\nu\kappa\lambda} = (g_{\mu\nu\alpha\beta}g_{\kappa\lambda\gamma\delta} - \epsilon_{\mu\nu\alpha\beta}\epsilon_{\kappa\lambda\gamma\delta})u^\alpha u^\gamma E^{\beta\delta}, \tag{42}$$

with $g_{\mu\nu\alpha\beta} = g_{\mu\alpha}g_{\nu\beta} - g_{\mu\beta}g_{\nu\alpha}$, $\epsilon_{\mu\nu\alpha\beta}$ is the Levi-Civita symbol multiplied by $\sqrt{-g}$ and $E^{\beta\gamma}$, the "electric" part of Weyl tensor, may be written as:

$$E_{\alpha\beta} = E(s_\alpha s_\beta + \frac{1}{3}h_{\alpha\beta}) \tag{43}$$

with

$$E = \frac{e^{-\nu}}{4}(\ddot{\lambda} + \frac{\dot{\lambda}(\dot{\lambda} - \dot{\nu})}{2}) \tag{44}$$

$$-\frac{e^{-\lambda}}{4}(\nu'' + \frac{\nu'^2 - \lambda'\nu'}{2} - \frac{\nu' - \lambda'}{r} + \frac{2(1 - e^\lambda)}{r^2})$$

Contracting (42) with u^ν we obtain:

$$u^\nu C_{\mu\nu k\lambda} = E_{\mu k}u_\lambda - E_{\mu\lambda}u_k, \tag{45}$$

from where it follows that:

$$u^\nu C_{\mu\nu\kappa}{}^\lambda{}_{;\lambda} + u^\nu_{;\lambda} C_{\mu\nu\kappa}{}^\lambda = \theta E_{\mu\kappa} + u^\alpha E_{\mu\kappa;\alpha} - u_{\kappa;\lambda} E_\mu^\lambda - u_\kappa E_{\mu;\lambda}^\lambda. \tag{46}$$

Also, from (31) and (42), we obtain:

$$u^\nu_{;\lambda} C_{\mu\nu\kappa}{}^\lambda = u_\mu u_\kappa \sigma_{\delta\beta} E^{\delta\beta} - a_\beta u_\mu E_\kappa^\beta - h_{\mu\kappa} \sigma^{\alpha\beta} E_{\alpha\beta} + \sigma_{\kappa\alpha} E_\mu^\alpha + \sigma_{\mu\alpha} E_\kappa^\alpha - \frac{\theta}{3} E_{\mu\kappa}. \tag{47}$$

Replacing (47) into (46), it results:

$$u^\nu C_{\mu\nu\kappa}{}^\lambda{}_{;\lambda} = \quad \frac{4\theta}{3} E_{\mu\kappa} + u^\alpha E_{\mu\kappa;\alpha} - u_{\kappa;\lambda} E_\mu^\lambda - u_\kappa E_{\mu;\lambda}^\lambda - u_\mu u_\kappa \sigma_{\delta\beta} E^{\delta\beta} +$$
$$+ \quad a_\beta u_\mu E_\kappa^\beta + h_{\mu\kappa} \sigma^{\alpha\beta} E_{\alpha\beta} - \sigma_{\kappa\alpha} E_\mu^\alpha - \sigma_{\mu\alpha} E_\kappa^\alpha. \tag{48}$$

Contracting (48) with $h_\alpha^\mu h_\beta^\kappa$ we have:

$$h_\alpha^\mu h_\beta^\kappa u^\nu C_{\mu\nu\kappa}{}^\lambda{}_{;\lambda} = \quad \frac{4\theta}{3} E_{\alpha\beta} - u_{\beta;\lambda} E_\alpha^\lambda + u^\nu E_{\mu\kappa;\nu} h_\alpha^\mu h_\beta^\kappa +$$
$$+ \quad h_{\alpha\beta} \sigma^{\kappa\nu} E_{\kappa\nu} - \sigma_{\kappa\alpha} E_\beta^\kappa - \sigma_{\kappa\beta} E_\alpha^\kappa. \tag{49}$$

On the other hand

$$h_\alpha^\mu h_\beta^\kappa u^\nu T_{\kappa\mu;\nu} = -u^\nu \hat{P}_{;\nu} h_{\alpha\beta} + u^\nu \Pi_{\kappa\mu;\nu} h_\alpha^\mu h_\beta^\kappa + \tilde{q}_\alpha a_\beta + \tilde{q}_\beta a_\alpha$$

$$h_\alpha^\mu h_\beta^\kappa u^\nu T_{\kappa\nu;\mu} = (\tilde{\rho} + \hat{P})(\sigma_{\alpha\beta} + \frac{\theta}{3} h_{\alpha\beta}) - \Pi_{\beta\nu}(\sigma_\alpha^\nu + \frac{\theta}{3} h_\alpha^\nu) + \tilde{q}_{\kappa;\mu} h_\alpha^\mu h_\beta^\kappa \tag{50}$$

$$h_\alpha^\mu h_\beta^\kappa u^\nu g_{\kappa[\mu} T_{,\nu]} = \tfrac{1}{2} u^\nu (\tilde{\rho}_{;\nu} - 3\hat{P}_{;\nu}) h_{\alpha\beta}$$

Feeding back (49) and (50) into (41) we find:

$$\theta E_{\alpha\beta} + (u^\nu E_{\mu\kappa;\nu} - 4\pi u^\nu \Pi_{\mu\kappa;\nu} + 4\pi \tilde{q}_{\kappa;\mu}) h_\alpha^\mu h_\beta^\kappa + \frac{4\pi}{3} u^\nu \tilde{\rho}_{;\nu} h_{\alpha\beta} + E\sigma_{\alpha\beta} = \tag{51}$$
$$-4\pi(\tilde{\rho} + \hat{P})(\sigma_{\alpha\beta} + \frac{\theta}{3} h_{\alpha\beta}) + 4\pi(\tilde{q}_\alpha a_\beta + \tilde{q}_\beta a_\alpha) + 4\pi \Pi_{\nu\beta}(\sigma_\alpha^\nu + \frac{\theta}{3} h_\alpha^\nu)$$

Next, contracting (48) with u^k we have:

$$u^\kappa u^\nu C_{\mu\nu k}{}^\lambda{}_{;\lambda} = -E_{\mu;\lambda}^\lambda - a^\lambda E_{\mu\lambda} - \sigma_\lambda^\nu E_\nu^\lambda u_\mu. \tag{52}$$

The following expressions can also be easily calculated:

$$u^k u^\nu T_{k\mu;\nu} = u^\nu \tilde{\rho}_{;\nu} u_\mu + (\tilde{\rho} + \hat{P}) a_\mu - a^k (\tilde{q}_k u_\mu + \Pi_{\mu k}) + u^\nu \tilde{q}_{\mu;\nu}$$

$$u^k u^\nu T_{k\nu;\mu} = \tilde{\rho}_{;\mu} - 2\tilde{q}_k (a^k u_\mu + \sigma_\mu^k + \frac{\theta}{3} h_\mu^k) \tag{53}$$

$$g_{k[\mu} T_{;\nu]} u^\nu u^k = -\tfrac{1}{2} T_{,\nu} h_\mu^\nu = -\tfrac{1}{2} (\tilde{\rho} - 3\tilde{P})_{,\nu} h_\mu^\nu$$

Finally, feeding back (52) and (53) into (41) and contracting with h_α^μ we have:

$$E_{\mu;\lambda}^\lambda h_\alpha^\mu + a^\lambda E_{\alpha\lambda} = -8\pi\tilde{q}_\kappa(\sigma_\alpha^\kappa + \frac{\theta}{3}h_\alpha^\kappa) + \frac{4\pi}{3}(2\tilde{\rho} + 3\hat{P})_{;\nu}h_\alpha^\nu \tag{54}$$

$$-4\pi(\tilde{\rho} + \hat{P})a_\alpha + 4\pi a^\kappa\Pi_{\alpha\kappa} - 4\pi u^\nu\tilde{q}_{\mu;\nu}h_\alpha^\mu$$

2.5   Weyl tensor, mass function and anisotropy

For the line element (1) we have:

$$R_{232}^3 = 1 - e^{-\lambda} = \frac{2m}{r} \tag{55}$$

Where the mass function $m(r,t)$ is defined as

$$m = 4\pi\int_0^r r^2 T_0^0 dr, \tag{56}$$

Then from (36), (43) and Einstein equations (2) it follows:

$$\frac{3m}{r^3} = 4\pi\tilde{\rho} + 4\pi(P_\perp - \tilde{P}_r) + E \tag{57}$$

which in tensorial form reads

$$E_{\alpha\beta} - 4\pi\Pi_{\alpha\beta} = (\frac{3m}{r^3} - 4\pi\tilde{\rho})(s_\alpha s_\beta + \frac{1}{3}h_{\alpha\beta}) \tag{58}$$

2.6   Summary

Equations (28), (29), (33), (34), (38), (52), (55) and (58) are

$$\tilde{\rho}_{;\alpha}u^\alpha + (\tilde{\rho} + \hat{P})\theta + \tilde{q}_{;\alpha}^\alpha = \Pi_{\alpha\beta}\sigma^{\alpha\beta} + \tilde{q}a^\nu s_\nu \tag{59}$$

$$(\tilde{\rho} + \hat{P})a_\alpha + h_\alpha^\beta(\tilde{q}_{,\nu}u^\nu s_\beta + \tilde{q}s_{\beta;\nu}u^\nu - \hat{P}_{,\beta} + \Pi_{\beta;\mu}^\mu) + \sigma_{\alpha\beta}\tilde{q}s^\beta + \frac{4}{3}\theta\tilde{q}s_\alpha = 0 \tag{60}$$

$$\theta_{;\alpha}u^\alpha + \frac{\theta^2}{3} + \sigma_{\alpha\beta}\sigma^{\alpha\beta} - a_{;\alpha}^\alpha = -u_\rho u^\beta R_\beta^\rho = -4\pi(\tilde{\rho} + 3\hat{P}) \tag{61}$$

$$R_\beta^\rho u_\rho h^{\alpha\beta} = h_\beta^\alpha(\sigma_{;\mu}^{\beta\mu} - \frac{2}{3}\theta^{;\beta}) + \sigma^{\alpha\beta}a_\beta = 8\pi\tilde{q}s^\alpha \tag{62}$$

$$E_{\gamma\nu} + 4\pi\Pi_{\gamma\nu} = h_\gamma^\alpha h_\nu^\mu(a_{\alpha;\mu} - \sigma_{\alpha\mu;\beta}u^\beta) - a_\gamma a_\nu - \sigma_\nu^\beta\sigma_{\gamma\beta} - \frac{2}{3}\theta\sigma_{\gamma\nu} - \frac{1}{3}(a_{;\alpha}^\alpha - \frac{1}{6}\sigma^2)h_{\gamma\nu} \quad (63)$$

$$\theta E_{\alpha\beta} + (u^\nu E_{\mu\kappa;\nu} - 4\pi u^\nu\Pi_{\mu\kappa;\nu} + 4\pi\tilde{q}_{\kappa;\mu})h_\alpha^\mu h_\beta^\kappa + \frac{4\pi}{3}u^\nu\tilde{\rho}_{;\nu}h_{\alpha\beta} + E\sigma_{\alpha\beta} = \quad (64)$$
$$-4\pi(\tilde{\rho} + \hat{P})(\sigma_{\alpha\beta} + \frac{\theta}{3}h_{\alpha\beta}) + 4\pi(\tilde{q}_\alpha a_\beta + \tilde{q}_\beta a_\alpha) + 4\pi\Pi_{\nu\beta}(\sigma_\alpha^\nu + \frac{\theta}{3}h_\alpha^\nu)$$

$$E_{\mu;\lambda}^\lambda h_\alpha^\mu + a^\lambda E_{\alpha\lambda} = -8\pi\tilde{q}_\kappa(\sigma_\alpha^\kappa + \frac{\theta}{3}h_\alpha^\kappa) + \frac{4\pi}{3}(2\tilde{\rho} + 3\hat{P})_{;\nu}h_\alpha^\nu \quad (65)$$

$$-4\pi(\tilde{\rho} + \hat{P})a_\alpha + 4\pi a^\kappa\Pi_{\alpha\kappa} - 4\pi u^\nu\tilde{q}_{\mu;\nu}h_\alpha^\mu$$

$$E_{\alpha\beta} - 4\pi\Pi_{\alpha\beta} = (\frac{3m}{r^3} - 4\pi\tilde{\rho})(s_\alpha s_\beta + \frac{1}{3}h_{\alpha\beta}) \quad (66)$$

In each of equations (59)–(66) there is only one scalar independent component, thus contracting with s^α we may write the equivalent set:

$$\tilde{\rho}^* + (\tilde{\rho} + \tilde{P}_r)\theta = \frac{2}{3}(\theta + \frac{\sigma}{2})\Pi - \tilde{q}^\dagger - 2\tilde{q}a - \frac{2s^1}{r}\tilde{q} \quad (67)$$

$$\tilde{P}_r^\dagger + (\tilde{\rho} + \tilde{P}_r)a + \frac{2s^1}{r}\Pi = \frac{\sigma}{3}\tilde{q} - \tilde{q}^* - \frac{4\theta}{3}\tilde{q} \quad (68)$$

$$\theta^* + \frac{\theta^2}{3} + \frac{\sigma^2}{6} - a^\dagger - a^2 - \frac{2as^1}{r} = -4\pi(\tilde{\rho} + 3\tilde{P}_r) + 8\pi\Pi \quad (69)$$

$$(\frac{\sigma}{2} + \theta)^\dagger = -\frac{3\sigma s^1}{2r} + 12\pi\tilde{q} \quad (70)$$

$$E + 4\pi\Pi = -a^\dagger - a^2 - \frac{\sigma^*}{2} - \frac{\theta\sigma}{3} + \frac{as^1}{r} + \frac{\sigma^2}{12} \quad (71)$$

$$(4\pi\tilde{P}_r + \frac{3m}{r^3})(\theta + \frac{\sigma}{2}) + (E - 4\pi\Pi + 4\pi\tilde{\rho})^* = -\frac{12\pi s^1}{r}\tilde{q} \quad (72)$$

$$(E + 4\pi\tilde{\rho} - 4\pi\Pi)^\dagger = \frac{3s^1}{r}(4\pi\Pi - E) + 4\pi\tilde{q}(\frac{\sigma}{2} + \theta) \quad (73)$$

$$\frac{3m}{r^3} = 4\pi\tilde{\rho} + 4\pi(P_\perp - \tilde{P}_r) + E \tag{74}$$

with $f^\dagger = f_{,\alpha}s^\alpha$, $f^* = f_{,\alpha}u^\alpha$ and $a^\alpha = as^\alpha$, and where the expansion is given by

$$\theta = u^\mu_{;\mu} = \frac{1}{2(1-\omega^2)^{\frac{1}{2}}}[e^{-\frac{\nu}{2}}(\dot{\lambda} + \frac{2\omega\dot{\omega}}{1-\omega^2}) + e^{-\frac{\lambda}{2}}(\omega\nu' + \frac{2\omega'}{1-\omega^2} + \frac{4\omega}{r})] \tag{75}$$

Then from (27) and (75) it follows at once that:

$$\frac{\sigma}{2} + \theta = \frac{3\omega s^1}{r} \tag{76}$$

3   Special cases

We shall now apply equations (67)–(74) to analyze different particular cases.

3.1   Geodesic fluids

If the fluid is geodesic, non–dissipative and locally isotropic, then for bounded configurations, it follows at once from (68) and the vanishing of the pressure at the boundary, that it should be dust. In this case the vanishing of the Weyl tensor implies the shear free condition as it follows from (72). On the other hand the shear free condition implies conformally flat as it follows from (71). Thus in this special case both conditions are equivalent. For non–geodesic fluids this equivalence is not generally true (see below).

3.2   Locally isotropic perfect fluids

Let us now consider locally isotropic and non–dissipative fluids ($\Pi = q = \epsilon = 0$) and find the relations linking the Weyl tensor, the shear and the local density inhomogeneity. Although almost all results in this case are known, we think that it is worth while to present them, in order to illustrate the general method that will be used later to study more complicated situations.

From (73), we obtain after some rearrengements (with $\Pi = \tilde{q} = 0$)

$$[r^3 E]^\dagger + r^3 4\pi\rho^\dagger = 0. \tag{77}$$

Next, it is convenient to write (72), with the help of (11), (67), (74), and (76), as

$$[r^3 E]^\cdot + \frac{dr}{dt}[r^3 E]' = -2\pi\sigma r^3(\rho + P_r)\sqrt{1-\omega^2}e^{\nu/2}. \tag{78}$$

Implying that the vanishing of the Weyl tensor results in the vanishing of spatial gradients of energy density and the shear tensor.

Let us now assume $\rho^\dagger = 0$, then we obtain from (77)

$$[r^3 E]^\dagger = 0. \tag{79}$$

Implying, since the Weyl tensor should be regular inside the fluid distribution, $E = 0$. Thus $E = 0$ and $\rho^\dagger = 0$ are equivalent, and either one of them implies $\sigma = 0$. These results were already known (see [22] and references therein)

Next, if $\sigma = 0$ it follows from (70) that

$$\theta^\dagger = 0. \tag{80}$$

Observe that from the above it follows, using equation (29), that if the fluid is conformally flat and satisfies a barotropic equation of state of the form $P_r = P_r(\rho)$, then the fluid is geodesic ($a^\alpha = 0$).

Also, assuming the shear–free condition alone ($\sigma = 0$) it follows from (78) that the convective derivative of Er^3 vanishes, which in turn means that such quantity remains constant for any fluid element along the fluid lines.

3.3   Locally anisotropic non–dissipative fluids

We shall now relax the condition of local isotropy of the pressure, and shall assume $\Pi \neq 0$. Then from (76) and (73), it follows that

$$[r^3(E - 4\pi\Pi)]^\dagger + r^3 4\pi\rho^\dagger = 0. \tag{81}$$

Implying that the vanishing of $E - 4\pi\Pi$ results in the vanishing of spatial gradients of energy density.

On the other hand if we assume the vanishing of $\rho^\dagger$, then assuming that all physical variables are regular within the fluid distribution it follows at once that

$$E - 4\pi\Pi = 0. \tag{82}$$

Thus $E - 4\pi\Pi = 0$ and $\rho^\dagger = 0$ are equivalent, but neither one of them implies $\sigma = 0$.

Therefore if we assume the spacetime to be conformally flat ($E = 0$), then the local anisotropy produces inhomogeneity in the energy density according to the equation

$$(r^3\Pi)^\dagger = r^3\rho^\dagger. \tag{83}$$

Next, it follows from (72), with the help of (11), (67), (74) and (76) that

$$[r^3(E-4\pi\Pi)]^{\cdot} + \frac{dr}{dt}[r^3(E-4\pi\Pi)]' + 8\pi\Pi\omega e^{(\nu-\lambda)/2}r^2 = -2\pi\sigma r^3(\rho+P_r)\sqrt{1-\omega^2}e^{\nu/2}. \quad (84)$$

implying thereby that the convective derivative of $r^3(E-4\pi\Pi)$ is controlled not only by σ, but also by Π.

If $E = 4\pi\Pi$, then the following link between the shear and the anisotropy results

$$4\Pi\omega e^{-\frac{\lambda}{2}} = -\sigma r\sqrt{1-\omega^2}(\rho+P_r). \quad (85)$$

If the fluid is shear-free and $E = 4\pi\Pi$ then it is either static or locally isotropic. Of course in this last case the fluid is also conformally flat.

3.4   Locally isotropic dissipative fluids in the quasi–static evolution

We shall now relax the condition of non–dissipation by allowing $q \neq 0$ (for simplicity we put $\epsilon = 0$), but assuming that the evolution is slow, which means that $\omega^2 = \dot{\omega} = \dot{\tilde{q}} = \ddot{\lambda} = \ddot{\nu} = 0$ and $q \approx O(\omega)$ (see [19]). Then from (76) and (73), we obtain (in the quasi–static approximation)

$$E'e^{-\lambda/2} + \frac{3Ee^{-\lambda/2}}{r} = -4\pi\rho^{\dagger}, \quad (86)$$

taking into account that in the quasi–static approximation

$$\rho^{\dagger} = \rho'e^{-\frac{\lambda}{2}}, \quad (87)$$

we have

$$E' + \frac{3E}{r} = -4\pi\rho'. \quad (88)$$

Next, it follows from (72), with the help of (67)

$$\dot{E}e^{-\nu/2} + E'\omega e^{-\lambda/2} + \omega e^{-\lambda/2}\frac{3E}{r} - 4\pi e^{-\frac{\lambda}{2}}q' + 4\pi q e^{-\frac{\lambda}{2}}\left(\frac{1}{r} - \nu'\right) = -2\pi\sigma(\rho+P_r), \quad (89)$$

or, equivalently

$$\frac{e^{\lambda/2}}{4\pi r^3}(r^3 E)^* = q' + q\left(\nu' - \frac{1}{r}\right) - \frac{1}{2}e^{\lambda/2}\sigma(\rho+P_r). \quad (90)$$

From (88) it follows at once that conformally flat and $\rho' = 0$ are equivalent conditions, which is also true in the perfect (non–dissipating) fluid, in the quasi–static approximation.

Also, from (89) or (90) it follows that $E = 0$ does not implies shear–free. Indeed, assuming $E = 0$ in (90) we have

$$q' + q\left(\nu' - \frac{1}{r}\right) = \frac{1}{2}e^{\lambda/2}\sigma(\rho+P_r), \quad (91)$$

yielding

$$q = re^{-\nu}\left[\int \frac{e^{\lambda/2+\nu}}{2r}\sigma(\rho + P_r)dr + \beta(t)\right]. \tag{92}$$

If, we further impose the shear–free condition then, one obtains from (92)

$$q = r\beta(t)e^{-\nu} \tag{93}$$

leading to a condition on the temperature, which may be obtained using the Landau-Eckart equation

$$q_\mu = \kappa h_\mu^\nu(T_{,\nu} - Ta_\nu), \tag{94}$$

or

$$q = -\kappa e^{-\frac{\lambda}{2}}(T' + \frac{T\nu'}{2}). \tag{95}$$

Using (95) in (93) we obtain

$$T = e^{-\nu/2}\left[C(t) - \frac{\beta(t)}{\kappa}\int_0^r re^{(\lambda-\nu)/2}dr\right]. \tag{96}$$

The two functions $\beta(t)$ and $C(t)$ are simply related to the total luminosity of the sphere and the central temperature, through (93) and (96), respectively. A simple model satisfying $E = \sigma = 0$ wil be next presented.

3.5   A conformally flat, shear–free sphere, dissipating in the quasi–static regime (with $\epsilon = 0$)

From (5) and $E = 0$ we have

$$8\pi P_\perp = \frac{e^{-\lambda}}{r}\left(\nu' - \lambda' - \frac{1}{r}\right) + \frac{1}{r^2}, \tag{97}$$

Then substracting (4) from (97), and considering $\Pi = 0$, we obtain

$$e^{-\lambda} = r^2 c_1 + 1, \tag{98}$$

where $c_1(t)$ is an arbitrary function of time. Substituting (98) into (3) yields

$$8\pi\rho = -3c_1. \tag{99}$$

Considering $E = 0$ with (98) we obtain

$$\nu'' + \frac{\nu'^2}{2} - \frac{\nu'}{r(r^2 c_1 + 1)} = 0, \tag{100}$$

which has the solution

$$e^{\nu/2} = (r^2 c_1 + 1)^{1/2} c_2 + c_3, \tag{101}$$

where $c_2(t)$ and $c_3(t)$ are arbitrary functions of t.

Substituting (98) and (101) into (4) we obtain

$$8\pi P_r = c_1 \left[1 + 2c_2 e^{-(\lambda+\nu)/2} \right]. \tag{102}$$

From (27), condition $\sigma = 0$ can be rewriten as

$$\left(\frac{\omega e^{\nu/2}}{r} \right)' = -\frac{\dot{\lambda} e^{\lambda/2}}{2r}, \tag{103}$$

which after integration becomes

$$\omega = \left(c_4 - \frac{\dot{c_1}}{2c_1} e^{\lambda/2} \right) r e^{-\nu/2}, \tag{104}$$

where $c_4(t)$ is an arbitrary function of t.

From (98), (99), (101), (102) we have

$$8\pi(\rho + P_r)\omega = - \left(2c_1 c_4 - \dot{c_1} e^{\lambda/2} \right) r c_3 e^{-\nu}. \tag{105}$$

Now substituting (98), (101), (105) into (6) we obtain

$$8\pi q = \left(2c_1 c_3 c_4 + \dot{c_1} c_2 \right) r e^{-\nu}. \tag{106}$$

Next, using (21) and (22) in (98) and (101) we obtain

$$c_1 = -\frac{2M}{r_\Sigma^3}, \tag{107}$$

and

$$c_3 = \sqrt{1 - \frac{2M}{r_\Sigma}} (1 - c_2). \tag{108}$$

Also, from the junction condition (23) and from (104) evaluated at the boundary surface, it follows that

$$c_1(1 + 2c_2) = (2c_1 c_3 c_4 + \dot{c_1} c_2) \frac{r_\Sigma}{(1 - \frac{2M}{r_\Sigma})}, \tag{109}$$

and

$$\dot{c_1} = 2c_1 \sqrt{1 - \frac{2M}{r_\Sigma}} \left(c_4 - \frac{\omega_\Sigma}{r_\Sigma} \sqrt{1 - \frac{2M}{r_\Sigma}} \right). \tag{110}$$

Solving algebraically the system (108)–(110) for c_2, c_3 and c_4, we can express these functions in terms of $M, r_\Sigma, \omega_\Sigma$ and $\dot{M}$.

We shall further specify our model by assuming

$$c_2 = 1, \tag{111}$$

$$c_3 = 0, \tag{112}$$

implying

$$e^{-\lambda} = e^{\nu} = 1 - \frac{2Mr^2}{r_\Sigma^3}. \tag{113}$$

Using (107), (111), (112) and (113) in (99) and (102) we obtain

$$8\pi\rho = -8\pi P_r = \frac{6M}{r_\Sigma^3}. \tag{114}$$

Then using (111) and (112) in (106) we obtain

$$8\pi q = r\dot{c}_1 e^{-\nu}, \tag{115}$$

which is our equation (93) with $\frac{\dot{c}_1}{8\pi} = \beta(t)$. By virtue of (107) and (109), the expression for q becomes

$$8\pi q = -\frac{6Mr}{r_\Sigma^4} \frac{(1 - \frac{2M}{r_\Sigma})}{(1 - \frac{2Mr^2}{r_\Sigma^3})}. \tag{116}$$

Next, using (107), (109), (113) in (96), we obtain

$$T = \frac{1}{\sqrt{1 - \frac{2Mr^2}{r_\Sigma^3}}} \left[T_c - \frac{3}{16\pi\kappa r_\Sigma}(1 - \frac{2M}{r_\Sigma}) \log\left(1 - \frac{2Mr^2}{r_\Sigma^3}\right) \right], \tag{117}$$

with T_c denoting the temperature at $r = 0$, and from (104)

$$\omega = \frac{r}{r_\Sigma} \sqrt{\frac{1 - \frac{2M}{r_\Sigma}}{1 - \frac{2Mr^2}{r_\Sigma^3}}} \left[\frac{3}{2} \left(1 - \sqrt{\frac{1 - \frac{2M}{r_\Sigma}}{1 - \frac{2Mr^2}{r_\Sigma^3}}} \right) + \omega_\Sigma \right]. \tag{118}$$

Thus our model represents a conformally flat sphere of fluid evolving slowly and shear free, with homogeneous energy density and pressure, satisfying the "inflationary" equation of state $\rho + P_r = 0$, with an inward ($q < 0$) heat flux.

3.6   General case

In the most general case, we obtain from (76) and (73), after some rearrengements,

$$[r^3(E - 4\pi\Pi - 4\pi\tilde{q}\omega)]^\dagger + r^3 4\pi\tilde{\rho}^\dagger = -4\pi(\tilde{q}\omega)^\dagger r^3, \tag{119}$$

or, equivalently

$$[r^3(E - 4\pi\Pi)]^\dagger + 4\pi r^3 \tilde{\rho}^\dagger = 4\pi r^3 \tilde{q}(\frac{\sigma}{2} + \theta), \tag{120}$$

if $\tilde{q} = 0$ we recover (81).

Also, from (72), with the help of (67) we get

$$[r^3(E - 4\pi\Pi)]^\cdot + \frac{dr}{dt}[r^3(E - 4\pi\Pi)]' + 8\pi\Pi\omega e^{(\nu-\lambda)/2}r^2$$

$$+4\pi r^3\sqrt{1 - \omega^2}e^{\nu/2}[\tilde{q}(\frac{s^1}{r} - 2a) - \tilde{q}^\dagger] = -2\pi\sigma r^3(\tilde{\rho} + \tilde{P}_r)\sqrt{1 - \omega^2}e^{\nu/2}, \tag{121}$$

which yields (84) in the non–dissipative case.

From (120) it is clear that the appearance of inhomogeneities in the energy–density are controlled by the Weyl tensor, the anisotropy of the pressure and the dissipation. If we express the heat flux through the temperature, the relaxation time and the heat conduction coefficient, by means of some transport equation (e.g. ([23]) then the aforementioned parameters may be related with the creation of such density inhomogeneities.

A similar conclusion applies to the shear of the fluid, as it follows from (121).

With the sole purpose of bringing out the role of dissipation in the formation of density inhomogeneities, let us present a simple and highly idealized model.

3.7   A dissipative model with $E - 4\pi\Pi = 0$

Since we want to exhibit the role of dissipation in the formation of ihomogeneities we shall assume $E - 4\pi\Pi = 0$, then from (55) and (58) it follows that

$$E - 4\pi\Pi = 0 \Leftrightarrow m = \frac{4\pi}{3}\tilde{\rho}r^3 \Leftrightarrow e^{-\lambda} = 1 - \frac{8\pi}{3}\tilde{\rho}r^2 \tag{122}$$

From (72), (120) and (76) we obtain

$$\tilde{\rho}^\dagger = \tilde{q}\frac{3\omega s^1}{r}, \tag{123}$$

and

$$3s^1 r^2(\tilde{q} + \tilde{P}_r\omega) + (\tilde{\rho}r^3)^* = 0, \tag{124}$$

then combining the Einstein equations (12) and (13) with (122), it follows that

$$\nu' = \frac{8\pi(\tilde{\rho} + \tilde{\rho}'r + 3\tilde{P}_r)r}{3(1 - \frac{8\pi}{3}\tilde{\rho}r^2)}. \tag{125}$$

We shall further assume the equation of state

$$\tilde{P}_r = \frac{1}{3}\tilde{\rho} \tag{126}$$

then, replacing (126) into (125) we obtain

$$e^\nu = \frac{(1 - \frac{8\pi}{3}\tilde{\rho}_\Sigma r_\Sigma^2)^2}{1 - \frac{8\pi}{3}\tilde{\rho}r^2}. \tag{127}$$

Next, using (124) and (123) it follows that:

$$\tilde{\rho}^\dagger = \frac{3\omega\tilde{q}}{r}\sqrt{\frac{1 - \frac{8\pi}{3}\tilde{\rho}r^2}{1 - \omega^2}} \tag{128}$$

and

$$\omega = \frac{-\dot{\tilde{\rho}}r \pm \sqrt{(\dot{\tilde{\rho}}r)^2 - r\tilde{\rho}'(r\tilde{\rho}' + 4\tilde{\rho})(1 - \frac{8\pi}{3}\tilde{\rho}_\Sigma r_\Sigma^2)^2}}{(r\tilde{\rho}' + 4\tilde{\rho})(1 - \frac{8\pi}{3}\tilde{\rho}_\Sigma r_\Sigma^2)} \tag{129}$$

In the particular case $\tilde{\rho} = \tilde{\rho}(t)$ (but $\tilde{\rho}^\dagger \neq 0$), these last equations reduce to

$$\omega = \frac{-\dot{\tilde{\rho}}r}{2\tilde{\rho}(1 - \frac{8\pi}{3}\tilde{\rho}_\Sigma r_\Sigma^2)} \tag{130}$$

and

$$\tilde{q} = \frac{1}{3}\dot{\tilde{\rho}}r e^{\frac{\lambda - \nu}{2}} \tag{131}$$

From junction conditions it is very simple to express $\tilde{\rho}$ and $\dot{\tilde{\rho}}$, and thereof all physical and metric variables in terms of the total mass M, the velocity of the surface ω_Σ, the radius r_Σ and the luminosity of the sphere. Equation (128) shows how dissipation produces density inhomogeneity. It is worth noticing that both kinds of dissipative processes (ϵ and q) may produce such inhomogeneity.

4   Conclusions

We have established the set of equations governing the structure and evolution of self–gravitating spherically symmetric dissipative anisotropic fluids. Emphasis has been put on the role played by the Weyl tensor, the anisotropy of the pressure, dissipation, density inhomogeneity and the shear tensor.

The particular simple relation between the Weyl tensor and density inhomogeneity, for perfect fluids (also valid for locally isotropic, dissipative fluids in the quasistatic regime), is at the origin of the Penrose's proposal to provide a gravitational arrow of time. However the fact that such relationship is no longer valid in the presence of local anisotropy of the pressure and/or dissipative processes, explains its failure in scenarios where the abovementioned factors are present.

From (120) it is apparent that the production of density inhomogeneities is related to a quantity involving all those factors ($[r^3(E - 4\pi\Pi)]^\dagger - 4\pi r^3\tilde{q}(\frac{\sigma}{2} + \theta)$). This situation is

illustrated in the example provided in the last section, where density inhomogeneity is produced by dissipative processes alone. Thus if following Penrose we adopt the point of view that self–gravitating systems evolve in the sense of increasing of density inhomogeneity, then the absolute value of the quantity above (or some function of it) should increase, providing an alternative definition for an arrow of time.

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